Lecture 10

Fourier Transform

(Lathi 7.1-7.3)

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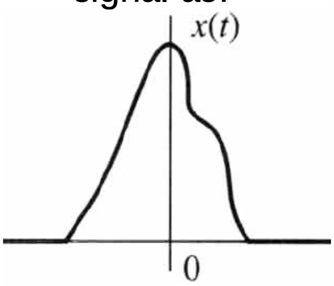
Definition of Fourier Transform

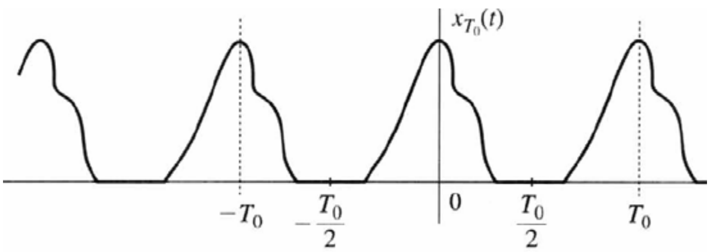
- The forward and inverse **Fourier Transform** are defined for **aperiodic** signal as:



$$X(\omega) = \mathcal{F}[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$x(t) = \mathcal{F}^{-1}[X(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

- Already covered in Year 1 Communication course (Lecture 5).
- **Fourier series** is used for **periodic** signals



$$x_{T_0}(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$

$$D_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x_{T_0}(t)e^{-jn\omega_0 t}dt$$

$$\omega_0 = \frac{2\pi}{T_0}$$

L7.1 p678

Connection between Fourier Transform and Laplace Transform

- Compare Fourier Transform: $$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$
- With Laplace Transform: $$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$
- Setting $s = j\omega$ in this equation yield:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt \qquad \text{where } X(j\omega) = X(s)|_{s=j\omega}$$

- Is it true that: $X(j\omega) = X(\omega)$?
- Yes only if x(t) is absolutely integrable, i.e. has finite energy:

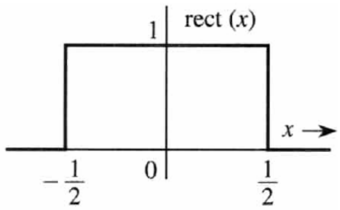
$$\int_{-\infty}^{\infty} |x(t)|\,dt < \infty$$

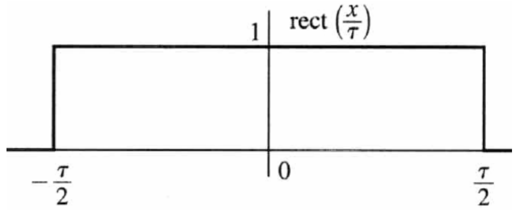
L7.2-1 p697

Define three useful functions

- A unit rectangular window (also called a unit gate) function **rect(x)**:

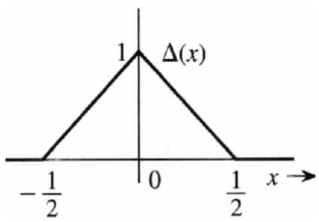
$$\text{rect}(x) = \begin{cases} 0 & |x| > \frac{1}{2} \\ \frac{1}{2} & |x| = \frac{1}{2} \\ 1 & |x| < \frac{1}{2} \end{cases}$$

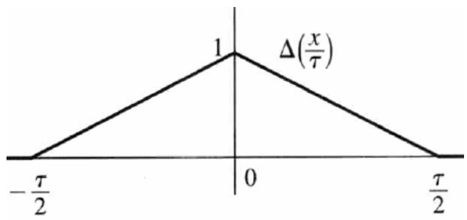




- A unit triangle function **Δ(x)**:

$$\Delta(x) = \begin{cases} 0 & |x| \geq \frac{1}{2} \\ 1 - 2|x| & |x| < \frac{1}{2} \end{cases}$$





- Interpolation function **sinc(x)**:

$$\text{sinc}(x) = \frac{\sin x}{x} \qquad \text{or} \qquad \text{sinc}(x) = \frac{\sin \pi x}{\pi x}$$

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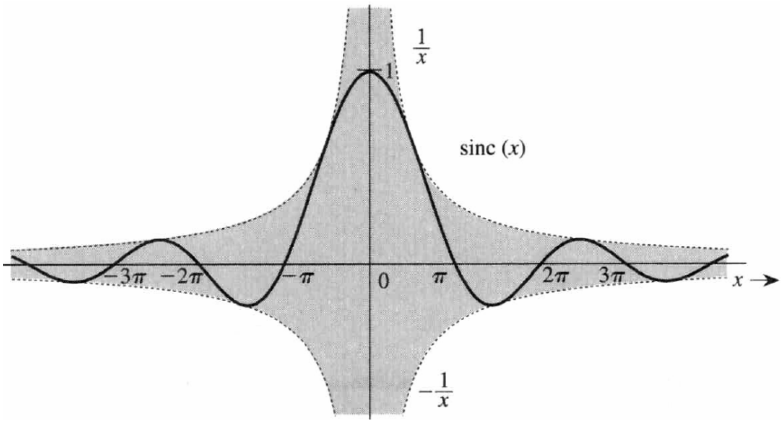
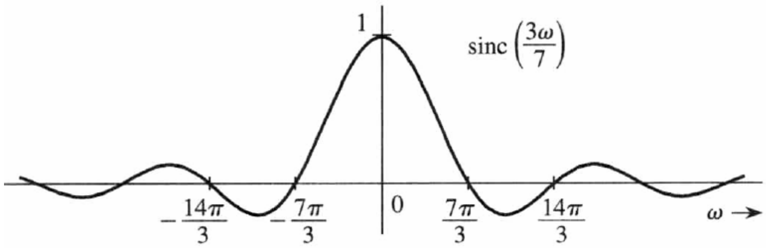
More about sinc(x) function

- sinc(x) is an even function of x.
- sinc(x) = 0 when sin(x) = 0 except when x=0, i.e. x = ±π, ±2π, ±3π.....
- sinc(0) = 1 (derived with L'Hôpital's rule)
- sinc(x) is the product of an oscillating signal sin(x) and a monotonically decreasing function 1/x. Therefore it is a damping oscillation with period of 2π with amplitude decreasing as 1/x.





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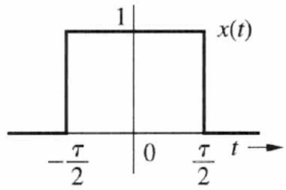
Fourier Transform of x(t) = rect(t/τ)

- Evaluation:

$$X(\omega) = \int_{-\infty}^{\infty} \mathrm{rect}\left(\frac{t}{\tau}\right) e^{-j\omega t} dt$$

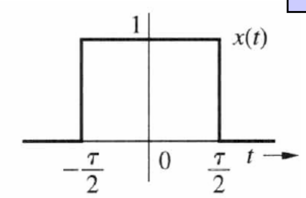


- Since rect(t/τ) = 1 for -τ/2 < t < τ/2 and 0 otherwise

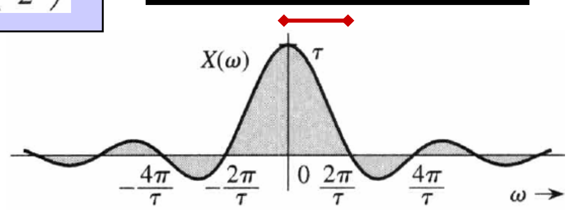
$$X(\omega) = \int_{-\tau/2}^{\tau/2} e^{-j\omega t} dt = -\frac{1}{j\omega}(e^{-j\omega\tau/2} - e^{j\omega\tau/2}) = \frac{2\sin\left(\frac{\omega\tau}{2}\right)}{\omega} = \tau\frac{\sin\left(\frac{\omega\tau}{2}\right)}{\left(\frac{\omega\tau}{2}\right)} = \tau\,\mathrm{sinc}\left(\frac{\omega\tau}{2}\right)$$

$$\boxed{\mathrm{rect}\left(\frac{t}{\tau}\right) \Longleftrightarrow \tau\ \mathrm{sinc}\left(\frac{\omega\tau}{2}\right)}$$

Bandwidth ≈ 2π/τ



$\Leftrightarrow$



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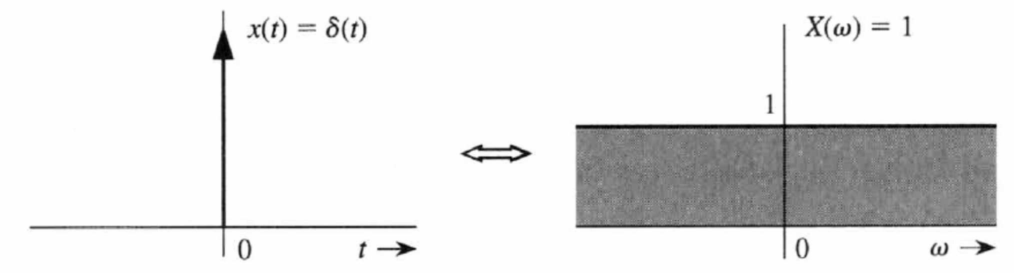
Fourier Transform of unit impulse $x(t) = \delta(t)$

- Using the sampling property of the impulse, we get:

$$\mathcal{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

- IMPORTANT – Unit impulse contains COMPONENT AT EVERY FREQUENCY.

$$\delta(t) \Longleftrightarrow 1$$



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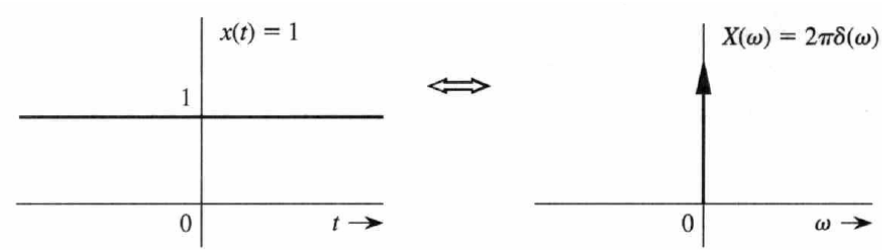
Inverse Fourier Transform of $\delta(\omega)$

- Using the sampling property of the impulse, we get:

$$\mathcal{F}^{-1}[\delta(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} d\omega = \frac{1}{2\pi}$$

- Spectrum of a constant (i.e. d.c.) signal x(t)=1 is an impulse $2\pi\delta(\omega)$.

$$\frac{1}{2\pi} \Longleftrightarrow \delta(\omega) \quad \text{or} \quad 1 \Longleftrightarrow 2\pi\delta(\omega)$$



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Inverse Fourier Transform of $\delta(\omega - \omega_0)$

- Using the sampling property of the impulse, we get:

$$\mathcal{F}^{-1}[\delta(\omega-\omega_0)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(\omega-\omega_0)e^{j\omega t}\,d\omega \;=\frac{1}{2\pi}e^{j\omega_0 t}$$

- Spectrum of an everlasting exponential $e^{j\omega_0 t}$ is a single impulse at $\omega=\omega_0$.

$$\frac{1}{2\pi}e^{j\omega_0 t} \Longleftrightarrow \delta(\omega-\omega_0)$$

or

$$e^{j\omega_0 t} \Longleftrightarrow 2\pi\,\delta(\omega-\omega_0)$$

and

$$e^{-j\omega_0 t} \Longleftrightarrow 2\pi\,\delta(\omega+\omega_0)$$

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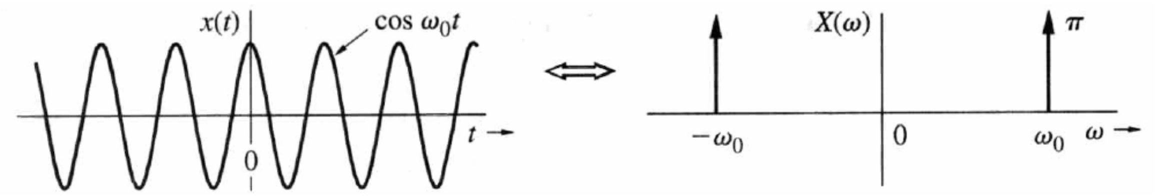
Fourier Transform of everlasting sinusoid cos $\omega_0 t$

- Remember Euler formula: $\cos\,\omega_0 t = \frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})$

- Use results from slide 9, we get:

$$\cos\,\omega_0 t \Longleftrightarrow \pi[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)]$$

- Spectrum of cosine signal has two impulses at positive and negative frequencies.



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Fourier Transform of any periodic signal

- Fourier series of a periodic signal x(t) with period T_0 is given by:

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T_0}$$

- Take Fourier transform of both sides, we get:

$$X(\omega) = 2\pi \sum_{n=-\infty}^{\infty} D_n \delta(\omega - n\omega_0)$$

- This is rather obvious!

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Fourier Transform of a unit impulse train

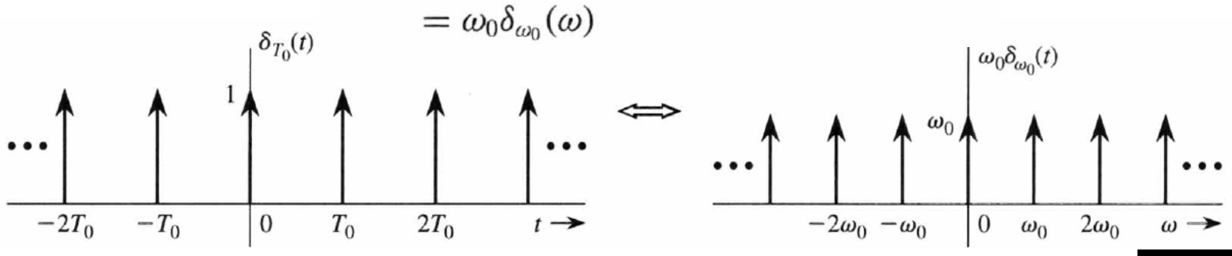
- Consider an impulse train $\delta_{T_0}(t) = \sum_{-\infty}^{\infty} \delta(t - nT_0)$
- The Fourier series of this impulse train can be shown to be:

$$\delta_{T_0}(t) = \sum_{-\infty}^{\infty} D_n e^{jn\omega_0 t} \quad \text{where} \quad \omega_0 = \frac{2\pi}{T_0} \quad \text{and} \quad D_n = \frac{1}{T_0}$$

- Therefore using results from the last slide (slide 11), we get:

$$X(\omega) = \frac{2\pi}{T_0} \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0) \qquad \omega_0 = \frac{2\pi}{T_0}$$

$$= \omega_0 \delta_{\omega_0}(\omega)$$



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Fourier Transform Table (1)

No.	$x(t)$	$X(\omega)$	
1	$e^{-at}u(t)$	$\dfrac{1}{a+j\omega}$	$a>0$
2	$e^{at}u(-t)$	$\dfrac{1}{a-j\omega}$	$a>0$
3	$e^{-a\|t\|}$	$\dfrac{2a}{a^2+\omega^2}$	$a>0$
4	$te^{-at}u(t)$	$\dfrac{1}{(a+j\omega)^2}$	$a>0$
5	$t^n e^{-at}u(t)$	$\dfrac{n!}{(a+j\omega)^{n+1}}$	$a>0$
6	$\delta(t)$	1	
7	1	$2\pi\delta(\omega)$	
8	$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$	

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Fourier Transform Table (2)

No.	$x(t)$	$X(\omega)$	
9	$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$	
10	$\sin\omega_0 t$	$j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$	
11	$u(t)$	$\pi\delta(\omega)+\dfrac{1}{j\omega}$	
12	$\operatorname{sgn} t$	$\dfrac{2}{j\omega}$	
13	$\cos\omega_0 t\, u(t)$	$\dfrac{\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]+\dfrac{j\omega}{\omega_0^2-\omega^2}$	
14	$\sin\omega_0 t\, u(t)$	$\dfrac{\pi}{2j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]+\dfrac{\omega_0}{\omega_0^2-\omega^2}$	
15	$e^{-at}\sin\omega_0 t\, u(t)$	$\dfrac{\omega_0}{(a+j\omega)^2+\omega_0^2}$	$a>0$
16	$e^{-at}\cos\omega_0 t\, u(t)$	$\dfrac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$	$a>0$

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Fourier Transform Table (3)

No.	$x(t)$	$X(\omega)$	
16	$e^{-at}\cos\omega_0 t\, u(t)$	$\dfrac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$	$a>0$
17	$\text{rect}\left(\dfrac{t}{\tau}\right)$	$\tau\,\text{sinc}\left(\dfrac{\omega\tau}{2}\right)$	
18	$\dfrac{W}{\pi}\,\text{sinc}\,(Wt)$	$\text{rect}\left(\dfrac{\omega}{2W}\right)$	
19	$\Delta\left(\dfrac{t}{\tau}\right)$	$\dfrac{\tau}{2}\,\text{sinc}^2\left(\dfrac{\omega\tau}{4}\right)$	
20	$\dfrac{W}{2\pi}\,\text{sinc}^2\left(\dfrac{Wt}{2}\right)$	$\Delta\left(\dfrac{\omega}{2W}\right)$	
21	$\displaystyle\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\displaystyle\omega_0\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_0)$	$\omega_0=\dfrac{2\pi}{T}$
22	$e^{-t^2/2\sigma^2}$	$\sigma\sqrt{2\pi}\,e^{-\sigma^2\omega^2/2}$	

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