5 The Discrete–Time Fourier Transform

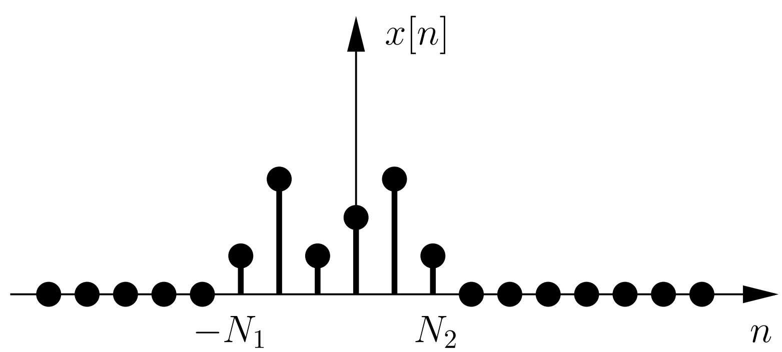
Fourier (or frequency domain) analysis — the last

- Complete the introduction and the development of the methods of Fourier analysis
- Learn frequency-domain methods for discrete-time signals and systems

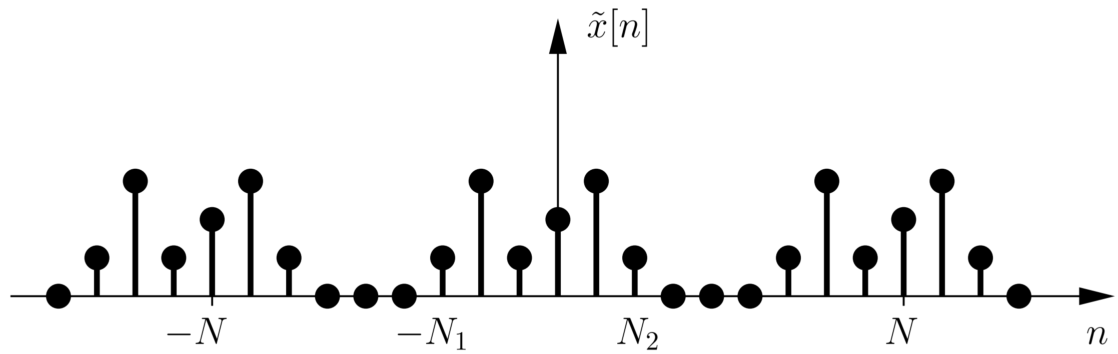
Outline

5.1 The Discrete–Time Fourier Transform

- Development of the Fourier Transform Representation
  - Analogous to continuous–time case
    * Aperiodic signal $x[n]$
    * Construct periodic signal $\tilde{x}[n]$ with $\tilde{x}[n] = x[n]$ over one period
    * Period $\rightarrow \infty \Rightarrow \tilde{x}[n] = x[n]$ over any finite time interval
    * Fourier series representation of $\tilde{x}[n]$ converges to Fourier transform representation of $x[n]$
- General sequence $x[n]$ with $x[n] = 0$ outside $-N_1 \leq n \leq N_2$



  - "Corresponding" periodic signal $\tilde{x}[n]$



  - Obviously: $\tilde{x}[n] \rightarrow x[n]$ for $N \rightarrow \infty$

– Frequency domain:

$$\tilde{x}[n] = \sum_{k=<N>} a_k \mathrm{e}^{\mathrm{j}k(2\pi/N)n} \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k = \frac{1}{N} \sum_{n=<N>} \tilde{x}[n] \mathrm{e}^{-\mathrm{j}k(2\pi/N)n}$$

or

$$\begin{aligned} N a_k &= \sum_{n=-N_1}^{N_2} x[n] \mathrm{e}^{-\mathrm{j}k(2\pi/N)n} = \sum_{n=-\infty}^{\infty} x[n] \mathrm{e}^{-\mathrm{j}k(2\pi/N)n} \\ &= \sum_{n=-\infty}^{\infty} x[n] \mathrm{e}^{-\mathrm{j}\omega n} \Big|_{\omega=k\omega_0} = X(\mathrm{e}^{\mathrm{j}\omega}) \Big|_{\omega=k\omega_0} \end{aligned}$$

Sample spacing: $\omega_0 = 2\pi/N$

– Therefore

$$\tilde{x}[n] = \sum_{k=<N>} \frac{1}{N} X(\mathrm{e}^{\mathrm{j}k\omega_0}) \mathrm{e}^{\mathrm{j}k(2\pi/N)n} = \frac{1}{2\pi} \sum_{k=<N>} X(\mathrm{e}^{\mathrm{j}k\omega_0}) \mathrm{e}^{\mathrm{j}k\omega_0 n} \omega_0$$

– Now: $N \to \infty$ ($\omega_0 \to 0$)

* Summation $\to$ integration
* $k\omega_0 \to \omega$, $\omega_0 \to d\omega$
* Summation over N intervals of width $\omega_0 = 2\pi/N$
  $\Rightarrow$ integration interval 2π
* $X(\mathrm{e}^{\mathrm{j}\omega})\mathrm{e}^{\mathrm{j}\omega n}$ periodic with period 2π
  $\Rightarrow$ any interval of length 2π

- *Discrete-time Fourier transform*

$$\boxed{X(\mathrm{e}^{\mathrm{j}\omega}) = \sum_{n=-\infty}^{\infty} x[n]\mathrm{e}^{-\mathrm{j}\omega n}}$$

and *inverse discrete-time Fourier transform*

$$\boxed{x[n] = \frac{1}{2\pi}\int_{2\pi} X(\mathrm{e}^{\mathrm{j}\omega})\mathrm{e}^{\mathrm{j}\omega n}\,d\omega}$$

- $X(\mathrm{e}^{\mathrm{j}\omega})$: referred to as the *Fourier transform* or the *spectrum* of $x[n]$
- Short-hand notation

$$\boxed{x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(\mathrm{e}^{\mathrm{j}\omega})}$$

$$\boxed{X(\mathrm{e}^{\mathrm{j}\omega}) = \mathcal{F}\{x[n]\}}$$

$$\boxed{x[n] = \mathcal{F}^{-1}\{X(\mathrm{e}^{\mathrm{j}\omega})\}}$$

- Note: Fourier series coefficients a_k of periodic signal $\tilde{x}[n]$ are equally spaced samples of Fourier transform $X(\mathrm{e}^{\mathrm{j}\omega})$ of aperiodic signal $x[n]$, where $x[n] = \tilde{x}[n]$ over one period and zero otherwise.
- Differences to continuous–time Fourier transform
  - Periodicity of discrete–time Fourier transform $X(\mathrm{e}^{\mathrm{j}\omega})$
  - Finite interval of integration in inverse Fourier transform
- Slowly varying signals: nonzero spectrum around $2\pi k$, $k \in \mathbb{Z}$

  Fast varying signals: nonzero spectrum around $\pi + 2\pi k$, $\in \mathbb{Z}$

Example:

1. Signal

$$x[n] = a^n u[n], \quad |a| < 1$$

– Fourier transform

$$\begin{aligned} X(\mathrm{e}^{\mathrm{j}\omega}) &= \sum_{n=-\infty}^{\infty} x[n]\mathrm{e}^{-\mathrm{j}\omega n} = \sum_{n=-\infty}^{\infty} a^n u[n]\mathrm{e}^{-\mathrm{j}\omega n} = \sum_{n=0}^{\infty} (a\mathrm{e}^{-\mathrm{j}\omega})^n \\ &= \frac{1}{1 - a\mathrm{e}^{-\mathrm{j}\omega}} \end{aligned}$$

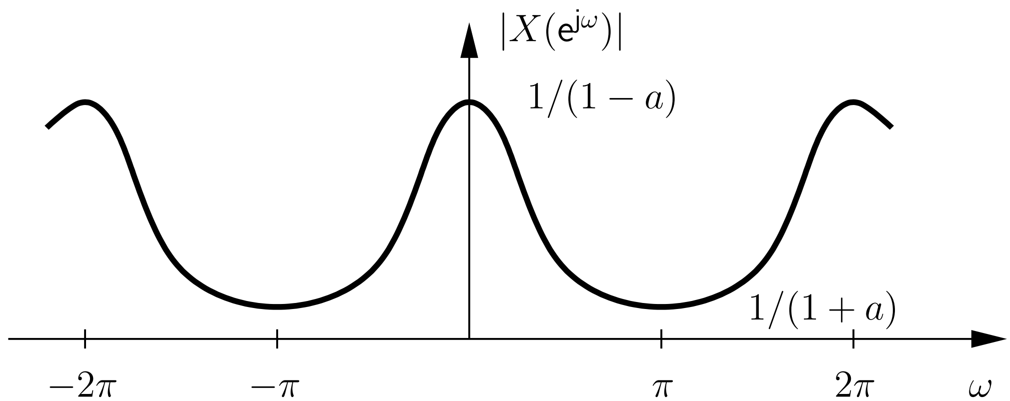
– Discrete-time Fourier transform pair

$$\boxed{a^n u[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1 - a\mathrm{e}^{-\mathrm{j}\omega}}}$$

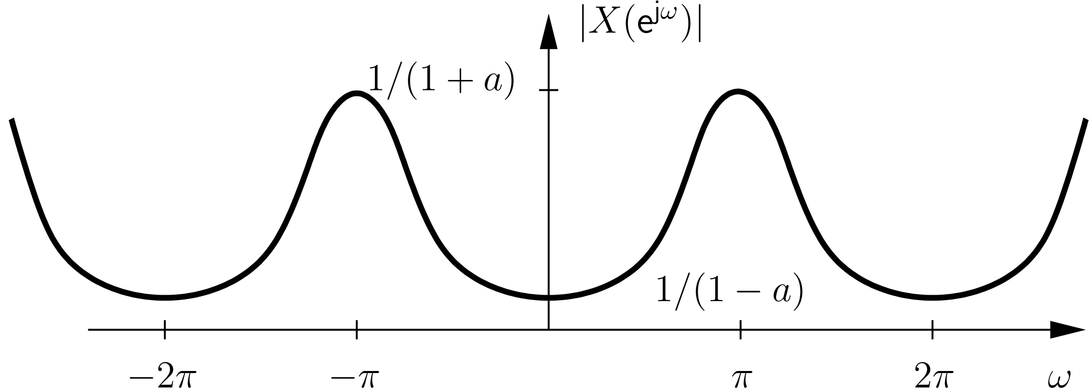
– Magnitude of spectrum

$$|X(\mathrm{e}^{\mathrm{j}\omega})| = \left|\frac{1}{1 - a\mathrm{e}^{-\mathrm{j}\omega}}\right| = \frac{1}{\sqrt{1 + a^2 - 2a\cos\omega}}$$

– $a \approx 1$: $x[n] \approx u[n]$
  $\Rightarrow$ very slowly varying
  $\Rightarrow$ spectrum concentrated around $2\pi k$, $k \in \mathbb{Z}$



– $a \approx -1$: $x[n] \approx (-1)^n u[n]$
$\Rightarrow$ very fast varying
$\Rightarrow$ spectrum concentrated around $\pi + 2\pi k$, $k \in \mathbb{Z}$



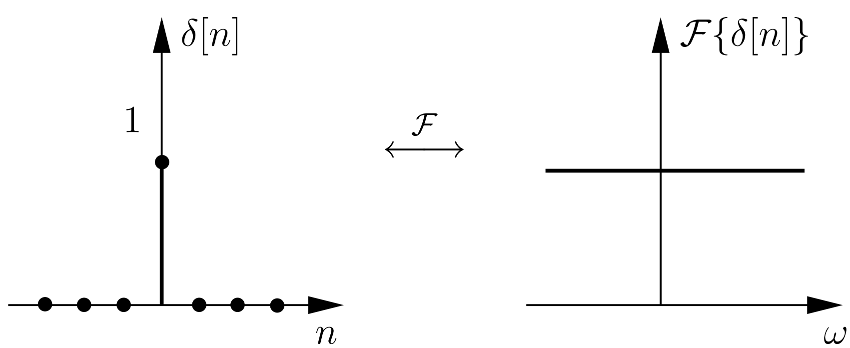
2. Unit impulse

$$x[n] = \delta[n]$$

Fourier transform

$$X(\mathrm{e}^{\mathrm{j}\omega}) = \sum_{n=-\infty}^{\infty} \delta[n]\mathrm{e}^{-\mathrm{j}\omega n} = 1$$

$$\boxed{\delta[n] \overset{\mathcal{F}}{\longleftrightarrow} 1}$$



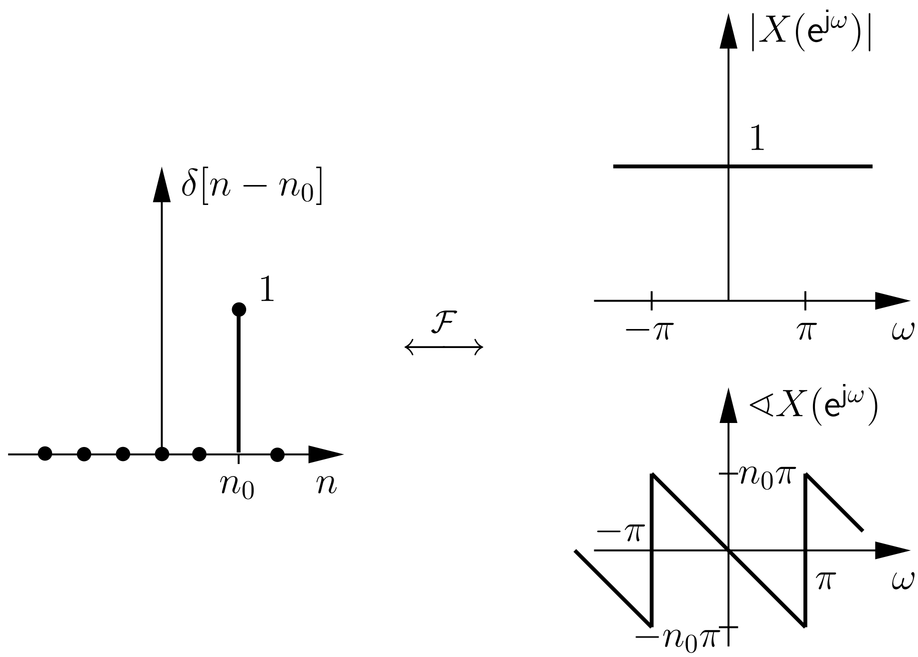
3. Shifted unit impulse

$$x[n] = \delta[n - n_0]$$

Fourier transform

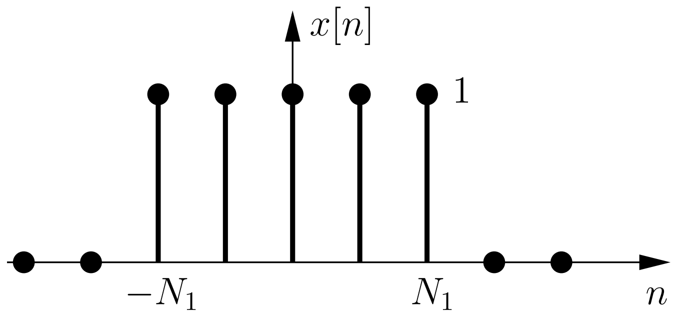
$$X(\mathrm{e}^{\mathrm{j}\omega}) = \sum_{n=-\infty}^{\infty} \delta[n - n_0]\mathrm{e}^{-\mathrm{j}\omega n} = \mathrm{e}^{-\mathrm{j}\omega n_0}$$

$$\boxed{\delta[n - n_0] \overset{\mathcal{F}}{\longleftrightarrow} \mathrm{e}^{-\mathrm{j}\omega n_0}}$$



4. Rectangular pulse

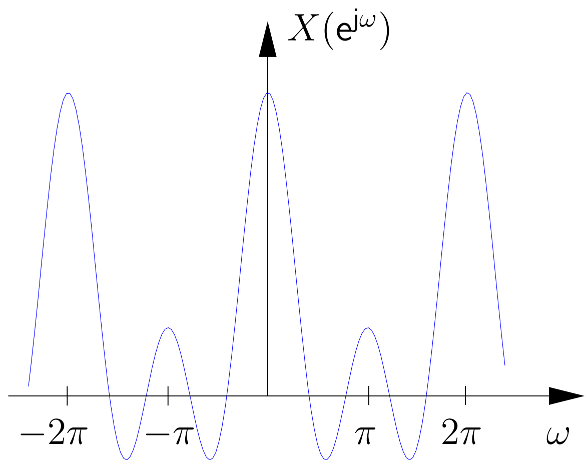
$$x[n] = \begin{cases} 1, & |n| \le N_1 \\ 0, & |n| > N_1 \end{cases}$$



Fourier transform

$$\begin{aligned} X(\mathrm{e}^{\mathrm{j}\omega}) &= \sum_{n=-N_1}^{N_1} \mathrm{e}^{-\mathrm{j}\omega n} = \mathrm{e}^{\mathrm{j}N_1\omega} \sum_{n=0}^{2N_1} \mathrm{e}^{-\mathrm{j}\omega n} = \mathrm{e}^{\mathrm{j}N_1\omega} \frac{1-\mathrm{e}^{-\mathrm{j}(2N_1+1)\omega}}{1-\mathrm{e}^{-\mathrm{j}\omega}} \\ &= \frac{\mathrm{e}^{\mathrm{j}N_1\omega} - \mathrm{e}^{-\mathrm{j}(N_1+1)\omega}}{\mathrm{e}^{-\mathrm{j}\omega/2}(\mathrm{e}^{\mathrm{j}\omega/2} - \mathrm{e}^{-\mathrm{j}\omega/2})} = \frac{\sin\omega(N_1+1/2)}{\sin(\omega/2)} \end{aligned}$$

$$\boxed{x[n] = \begin{cases} 1, & |n| \le N_1 \\ 0, & |n| > N_1 \end{cases} \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad \frac{\sin(\omega(N_1+1/2))}{\sin(\omega/2)}}$$



- *Convergence* of discrete–time Fourier transform
  - Fourier transform also valid for signals with infinite duration if the associated infinite sum converges.
  - Sufficient conditions for convergence
    * $x[n]$ is absolutely summable

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

or

    * $x[n]$ has finite energy

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

  - No convergence issues associated with inverse discrete-time Fourier transform
  - No Gibbs phenomenon

5.2 The Fourier Transform for Periodic Signals

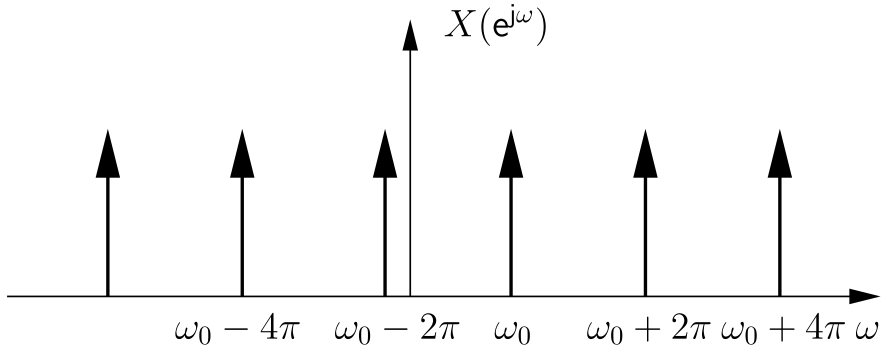
- *Periodic* discrete-time signals
  - do not meet the Convergence Conditions of Section 5.1,
  - but also have discrete-time Fourier transforms,
  - which can directly be constructed from their Fourier series.

Example:

Consider $x[n]$ with Fourier transform

$$X(\mathrm{e}^{\mathrm{j}\omega}) = \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l)$$

$X(\mathrm{e}^{\mathrm{j}\omega})$: Impulse train



Inverse Fourier transform

$$x[n] = \frac{1}{2\pi} \int_{2\pi} \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l)\mathrm{e}^{\mathrm{j}\omega n}\, d\omega = \mathrm{e}^{\mathrm{j}\omega_0 n}$$

$$\Rightarrow \quad \boxed{\mathrm{e}^{\mathrm{j}\omega_0 n} \stackrel{\mathcal{F}}{\longleftrightarrow} \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l)}$$

- Accordingly: periodic signal with period N

$$x[n] = \sum_{k=<N>} a_k \mathsf{e}^{\mathsf{j}k(2\pi/N)n} \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k$$

$$\stackrel{\mathcal{F}}{\longleftrightarrow} X(\mathsf{e}^{\mathsf{j}\omega}) = \sum_{k=<N>} a_k \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - 2\pi k/N - 2\pi l)$$

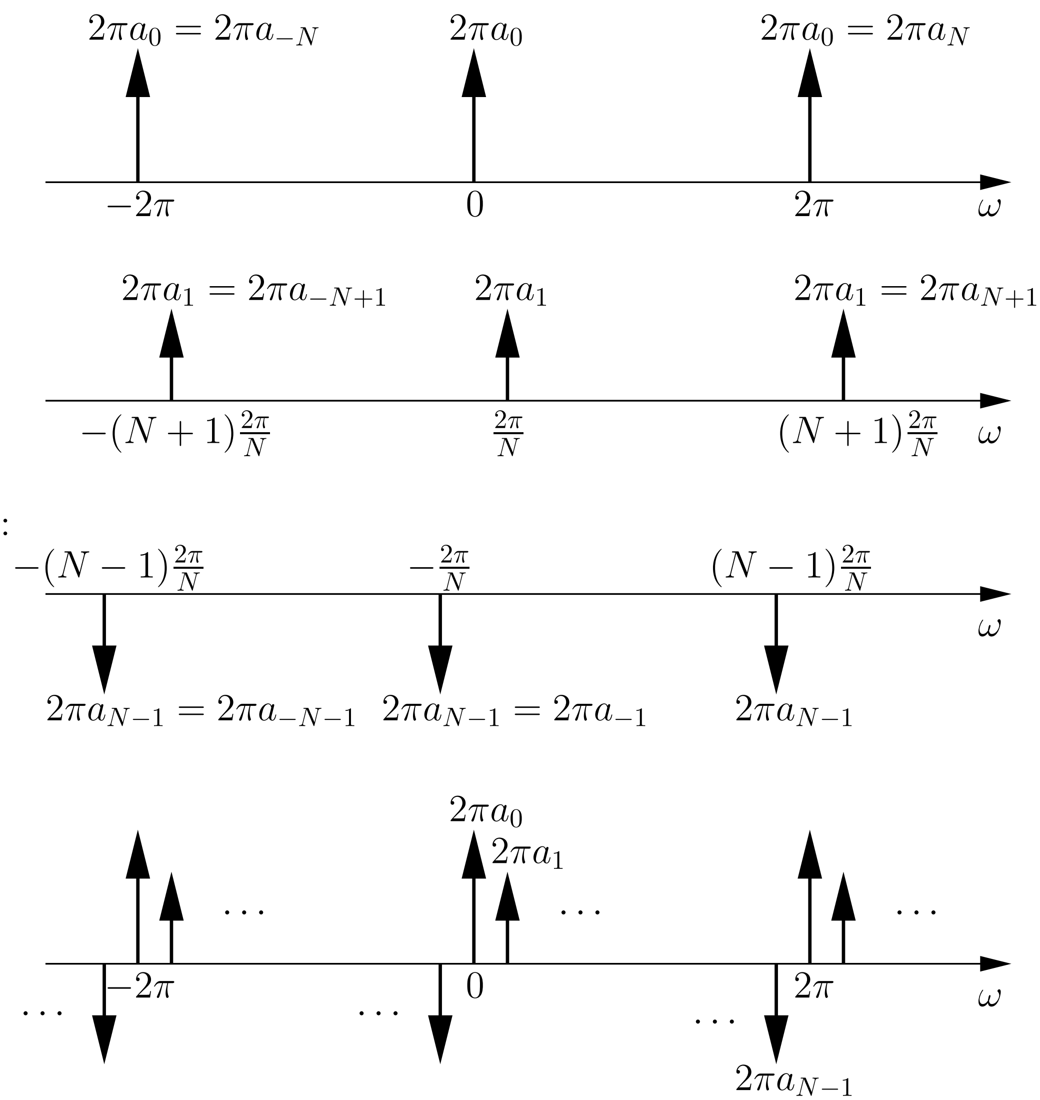
$$\stackrel{\mathcal{F}}{\longleftrightarrow} \boxed{X(\mathsf{e}^{\mathsf{j}\omega}) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - 2\pi k/N)}$$

$k = 0:$

$k = 1:$

$k = N-1:$

$\sum_{\langle N \rangle}:$



- Fourier transform of a periodic signal
  - train of impulses occurring at multiples of the fundamental frequency $2\pi/N$
  - area of impulse at $k2\pi/N$ is 2π times the kth Fourier series coefficient

Example:

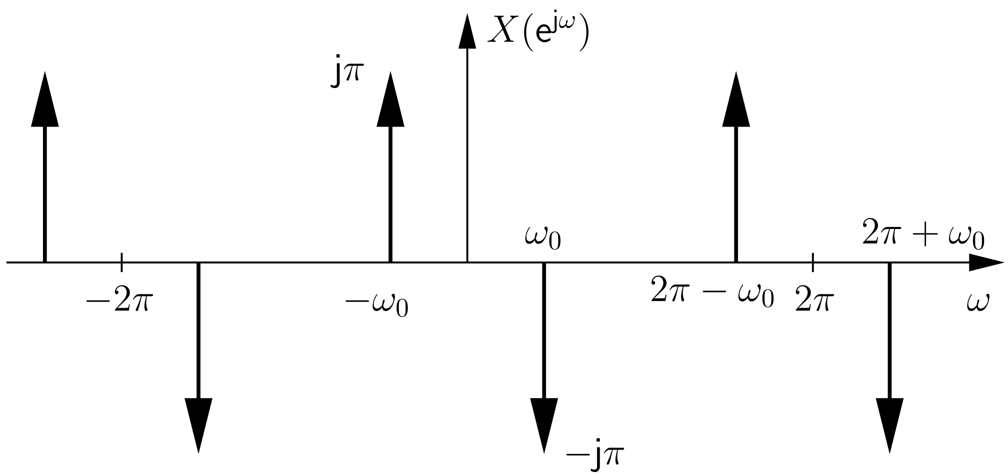
1. Sine function

$$x[n] = \sin(\omega_0 n) = \frac{1}{2\mathrm{j}}\left(\mathrm{e}^{\mathrm{j}\omega_0 n} - \mathrm{e}^{-\mathrm{j}\omega_0 n}\right)$$

Periodic if $\omega_0 = 2\pi m/N$

Fourier transform

$$X(\mathrm{e}^{\mathrm{j}\omega}) = -\mathrm{j}\pi \sum_{l=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi l) + \mathrm{j}\pi \sum_{l=-\infty}^{\infty} \delta(\omega + \omega_0 - 2\pi l)$$



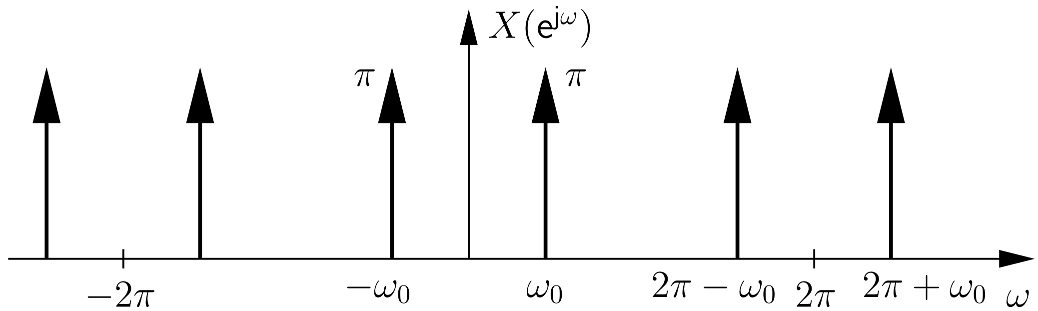
2. Cosine function

$$x[n] = \mathsf{cos}\omega_0 n = \frac{1}{2}\mathsf{e}^{\mathsf{j}\omega_0 n} + \frac{1}{2}\mathsf{e}^{-\mathsf{j}\omega_0 n}$$

Periodic if $\omega_0 = 2\pi m/N$

Fourier transform

$$X(\mathsf{e}^{\mathsf{j}\omega}) = \pi \sum_{l=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi l) + \pi \sum_{l=-\infty}^{\infty} \delta(\omega + \omega_0 - 2\pi l)$$



3. Impulse train sequence

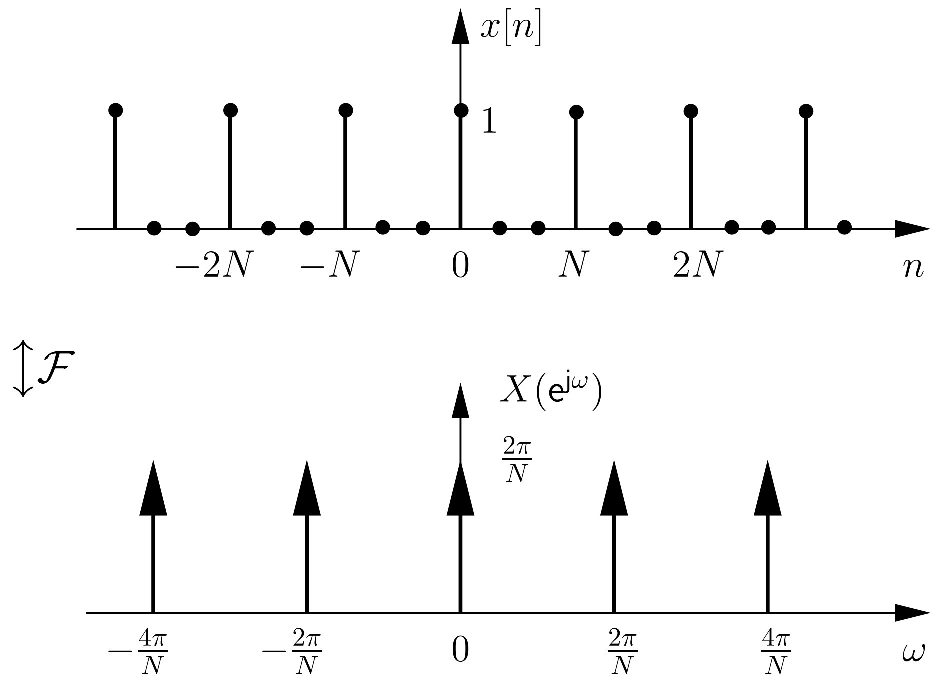
$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n - kN]$$

Fourier series coefficients

$$\begin{aligned} a_k &= \frac{1}{N} \sum_{n=<N>} x[n]\mathsf{e}^{-\mathsf{j}k(2\pi/N)n} \\ &= \frac{1}{N} \sum_{n=<N>} \sum_{l=-\infty}^{\infty} \delta[n - lN]\mathsf{e}^{-\mathsf{j}k(2\pi/N)n} \\ &= \frac{1}{N} \end{aligned}$$

Fourier transform

$$X(\mathsf{e}^{\mathsf{j}\omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k/N)$$



$x[n]$
1
$-2N$
$-N$
0
N
$2N$
n
$\updownarrow \mathcal{F}$
$X(e^{j\omega})$
$\frac{2\pi}{N}$
$-\frac{4\pi}{N}$
$-\frac{2\pi}{N}$
0
$\frac{2\pi}{N}$
$\frac{4\pi}{N}$
ω

5.3 Properties of the Discrete–Time Fourier Transform

- *Periodicity*
  - Discrete–time Fourier transform is *always* periodic in ω with period 2π
$$\boxed{X(\mathrm{e}^{\mathrm{j}(\omega+2\pi)}) = X(\mathrm{e}^{\mathrm{j}\omega})}$$
  - Different from continuous–time Fourier transform!

- *Linearity*
  - Fourier transform
$$\boxed{\mathcal{F}\{ax[n] + by[n]\} = a\mathcal{F}\{x[n]\} + b\mathcal{F}\{y[n]\}}$$
  - inverse Fourier transform
$$\boxed{\mathcal{F}^{-1}\{cX(\mathrm{e}^{\mathrm{j}\omega}) + dY(\mathrm{e}^{\mathrm{j}\omega})\} = c\mathcal{F}^{-1}\{X(\mathrm{e}^{\mathrm{j}\omega})\} + d\mathcal{F}^{-1}\{Y(\mathrm{e}^{\mathrm{j}\omega})\}}$$

- *Time Shifting*
  - If
$$x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(\mathrm{e}^{\mathrm{j}\omega})$$
    then
$$\boxed{x[n-n_0] \stackrel{\mathcal{F}}{\longleftrightarrow} \mathrm{e}^{-\mathrm{j}\omega n_0} X(\mathrm{e}^{\mathrm{j}\omega})}$$
  - Proof:
$$\begin{aligned}
\mathcal{F}\{x[n-n_0]\} &= \sum_{n=-\infty}^{\infty} x[n-n_0]\mathrm{e}^{-\mathrm{j}\omega n} = \sum_{n'=-\infty}^{\infty} x[n']\mathrm{e}^{-\mathrm{j}\omega(n'+n_0)} \\
&= \mathrm{e}^{-\mathrm{j}\omega n_0} \sum_{n'=-\infty}^{\infty} x[n']\mathrm{e}^{-\mathrm{j}\omega n'} = \mathrm{e}^{-\mathrm{j}\omega n_0} X(\mathrm{e}^{\mathrm{j}\omega})
\end{aligned}$$

■

- *Frequency Shifting*

  – If

$$x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(\mathrm{e}^{\mathrm{j}\omega})$$

then

$$\boxed{\mathrm{e}^{\mathrm{j}\omega_0 n} x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(\mathrm{e}^{\mathrm{j}(\omega-\omega_0)})}$$

  – Proof:

$$\begin{aligned} \mathcal{F}^{-1}\{X(\mathrm{e}^{\mathrm{j}(\omega-\omega_0)})\} &= \frac{1}{2\pi} \int\limits_{2\pi} X(\mathrm{e}^{\mathrm{j}(\omega-\omega_0)}) \mathrm{e}^{\mathrm{j}\omega n}\, d\omega \\ &= \frac{1}{2\pi} \int\limits_{2\pi} X(\mathrm{e}^{\mathrm{j}\omega'}) \mathrm{e}^{\mathrm{j}(\omega'+\omega_0)n}\, d\omega' \\ &= \mathrm{e}^{\mathrm{j}\omega_0 n} \frac{1}{2\pi} \int\limits_{2\pi} X(\mathrm{e}^{\mathrm{j}\omega'}) \mathrm{e}^{\mathrm{j}\omega' n}\, d\omega' \\ &= \mathrm{e}^{\mathrm{j}\omega_0 n} x[n] \end{aligned}$$

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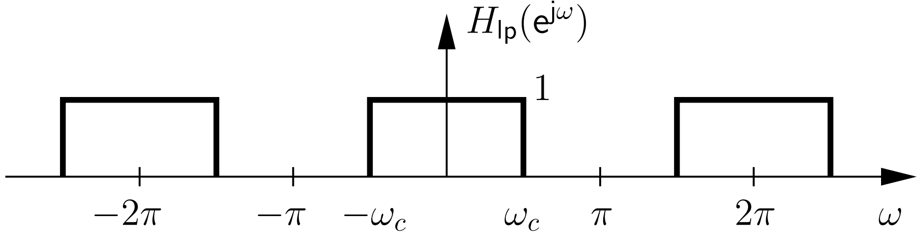
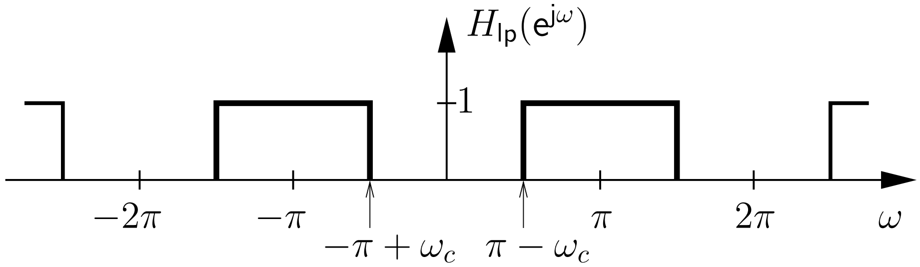
  – Periodicity property + frequency–shifting property
    ⇒ special relationship between ideal lowpass and ideal highpass discrete–time filters

Example:

– Lowpass filter with cutoff frequency ω_c
Frequency response $H_{\mathsf{lp}}(\mathsf{e}^{\mathsf{j}\omega})$



– Highpass filter with cutoff frequency $\pi - \omega_c$
Frequency response $H_{\mathsf{hp}}(\mathsf{e}^{\mathsf{j}\omega})$



– Frequency domain relation

$$H_{\mathsf{hp}}(\mathsf{e}^{\mathsf{j}\omega}) = H_{\mathsf{lp}}(\mathsf{e}^{\mathsf{j}(\omega-\pi)})$$

– Impulse response = inverse Fourier transform of frequency response of an LTI system

$$h_{\mathsf{hp}}[n] = \mathsf{e}^{j\pi n} h_{\mathsf{lp}}[n] = (-1)^n h_{\mathsf{lp}}[n]$$

- *Time Reversal*
  - If
$$x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(\mathrm{e}^{\mathrm{j}\omega})$$
    then
$$\boxed{x[-n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(\mathrm{e}^{-\mathrm{j}\omega})}$$
  - Proof:
$$\begin{aligned} Y(\mathrm{e}^{\mathrm{j}\omega}) &= \sum_{n=-\infty}^{\infty} y[n]\mathrm{e}^{-\mathrm{j}\omega n} = \sum_{n=-\infty}^{\infty} x[-n]\mathrm{e}^{-\mathrm{j}\omega n} \\ &\stackrel{m=-n}{=} \sum_{m=-\infty}^{\infty} x[m]\mathrm{e}^{\mathrm{j}\omega m} = \sum_{m=-\infty}^{\infty} x[m]\mathrm{e}^{-\mathrm{j}(-\omega)m} \\ &= X(\mathrm{e}^{-\mathrm{j}\omega}) \end{aligned}$$

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- *Conjugation and Conjugate Symmetry*
  - If
$$x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(\mathrm{e}^{\mathrm{j}\omega})$$
    then
$$x^*[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X^*(\mathrm{e}^{-\mathrm{j}\omega})$$
  - Real $x[n]$: *conjugate symmetry*
$$X(\mathrm{e}^{\mathrm{j}\omega}) = X^*(\mathrm{e}^{-\mathrm{j}\omega})$$

– Conjugation + time reversal properties

$$x[n] = \mathrm{Re}\{\mathrm{Ev}\{x[n]\}\} + \mathrm{Re}\{\mathrm{Od}\{x[n]\}\} + \mathrm{jIm}\{\mathrm{Ev}\{x[n]\}\} + \mathrm{jIm}\{\mathrm{Od}\{x[n]\}\}$$

$$X(\mathrm{e}^{\mathrm{j}\omega}) = \mathrm{Re}\{\mathrm{Ev}\{X(\mathrm{e}^{\mathrm{j}\omega})\}\} + \mathrm{Re}\{\mathrm{Od}\{X(\mathrm{e}^{\mathrm{j}\omega})\}\} + \mathrm{jIm}\{\mathrm{Ev}\{X(\mathrm{e}^{\mathrm{j}\omega})\}\} + \mathrm{jIm}\{\mathrm{Od}\{X(\mathrm{e}^{\mathrm{j}\omega})\}\}$$

Example:

Consider

$$x[n] = a^{|n|}\,, \quad |a| < 1$$

– Observe

$$x[n] = z[n] + z[-n] - \delta[n]$$

where

$$z[n] = a^n u[n]$$

– With

$$z[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1 - a\mathrm{e}^{-\mathrm{j}\omega}}$$

and time reversal property

$$z[-n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1 - a\mathrm{e}^{\mathrm{j}\omega}}$$

– Finally

$$\begin{aligned} X(\mathrm{e}^{\mathrm{j}\omega}) &= \frac{1}{1 - a\mathrm{e}^{-\mathrm{j}\omega}} + \frac{1}{1 - a\mathrm{e}^{\mathrm{j}\omega}} - 1 \\ &= \frac{1 - a^2}{1 - 2a\cos(\omega) + a^2} \end{aligned}$$

– $x[n]$ real and even $\Rightarrow X(\mathrm{e}^{\mathrm{j}\omega})$ real and even

- First order *differencing*
  - If
    $$x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(\mathrm{e}^{\mathrm{j}\omega})$$
    then
    $$\boxed{x[n] - x[n-1] \stackrel{\mathcal{F}}{\longleftrightarrow} (1 - \mathrm{e}^{-\mathrm{j}\omega})X(\mathrm{e}^{\mathrm{j}\omega})}$$
  - Follows immediately from linearity and time-shifting properties
- *Accumulation*
  - If
    $$x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(\mathrm{e}^{\mathrm{j}\omega})$$
    then
    $$\boxed{\sum_{m=-\infty}^{n} x[m] \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{1-\mathrm{e}^{-\mathrm{j}\omega}} X(\mathrm{e}^{\mathrm{j}\omega}) + \pi X(\mathrm{e}^{\mathrm{j}0}) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)}$$
  - Impulse train $\cdot\, X(\mathrm{e}^{\mathrm{j}0})$: dc component
  - Zero-mean $x[n]$ ($X(\mathrm{e}^{\mathrm{j}0}) = 0$): $\dfrac{X(\mathrm{e}^{\mathrm{j}\omega})}{1-\mathrm{e}^{-\mathrm{j}\omega}}$
    (see differencing property)

Example:

Unit step $x[n] = u[n]$

– Known

$$\delta[n] \overset{\mathcal{F}}{\longleftrightarrow} 1 = G(\mathrm{e}^{\mathrm{j}\omega})$$

and

$$u[n] = \sum_{m=-\infty}^{n} \delta[m]$$

– Accumulation property

$$\begin{aligned} X(\mathrm{e}^{\mathrm{j}\omega}) &= \frac{1}{1-\mathrm{e}^{-\mathrm{j}\omega}} G(\mathrm{e}^{\mathrm{j}\omega}) + \pi G(\mathrm{e}^{\mathrm{j}0}) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k) \\ &= \frac{1}{1-\mathrm{e}^{-\mathrm{j}\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k) \end{aligned}$$

$$\boxed{u[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1-\mathrm{e}^{-\mathrm{j}\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)}$$

- *Decimation and Time Expansion*
  - Similar property(ies) to time and frequency scaling in continuous–time case
  - Consider $x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(\mathrm{e}^{\mathrm{j}\omega})$
  - *Decimation:* $x[an]\ ,\ a \in \mathbb{N}$

$$\boxed{x[an] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{a}\sum_{k=0}^{a-1} X(\mathrm{e}^{\mathrm{j}(\omega/a - 2\pi k/a)})}$$

  - Proof:

$$\begin{aligned}
\frac{1}{a}\sum_{k=0}^{a-1} X(\mathrm{e}^{\mathrm{j}(\omega/a-2\pi k/a)}) &= \frac{1}{a}\sum_{k=0}^{a-1}\sum_{n=-\infty}^{\infty} x[n]\mathrm{e}^{-\mathrm{j}(\omega/a-2\pi k/a)n} \\
&= \sum_{n=-\infty}^{\infty} x[n]\mathrm{e}^{-\mathrm{j}\omega n/a} \quad \underbrace{\frac{1}{a}\sum_{k=0}^{a-1}\mathrm{e}^{-\mathrm{j}2\pi kn/a}}_{=\begin{cases} 1\ , & \text{if } n/a \in \mathbb{Z} \\ 0\ , & \text{else}\end{cases}} \\
&\overset{n'=n/a}{=} \sum_{n'=-\infty}^{\infty} x[an']\mathrm{e}^{-\mathrm{j}\omega n'} \\
&= \mathcal{F}\{x[an']\}
\end{aligned}$$

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– *Time Expansion* ($a \in \mathbb{N}$)

$$x_{(a)}[n] = \begin{cases} x[n/a]\,, & \text{if } n \text{ is a multiple of } a \\ 0\,, & \text{otherwise} \end{cases}$$

$$\boxed{x_{(a)}[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(\mathrm{e}^{\mathrm{j}a\omega})}$$

– Proof:

$$\begin{aligned} X_{(a)}(\mathrm{e}^{\mathrm{j}\omega}) &= \sum_{n=-\infty}^{\infty} x_{(a)}[n]\mathrm{e}^{\mathrm{j}\omega n} \stackrel{r=n/a}{=} \sum_{r=-\infty}^{\infty} x_{(a)}[ra]\mathrm{e}^{\mathrm{j}\omega ra} \\ &= \sum_{r=-\infty}^{\infty} x[r]\mathrm{e}^{\mathrm{j}(a\omega)r} = X(\mathrm{e}^{\mathrm{j}a\omega}) \end{aligned}$$

■

Example:

1. Decimation

   – Rectangular pulse of length $2N_1 = 5$

$$x[n] = \begin{cases} 1, & |n| \le 2 \\ 0, & |n| > 2 \end{cases}$$

   and Fourier transform

$$X(\mathrm{e}^{\mathrm{j}\omega}) = \frac{\sin(5\omega/2)}{\sin(\omega/2)}$$

   – Decimation by 2

$$x[2n] = \begin{cases} 1, & |n| \le 1 \\ 0, & |n| > 1 \end{cases}$$

and Fourier transform

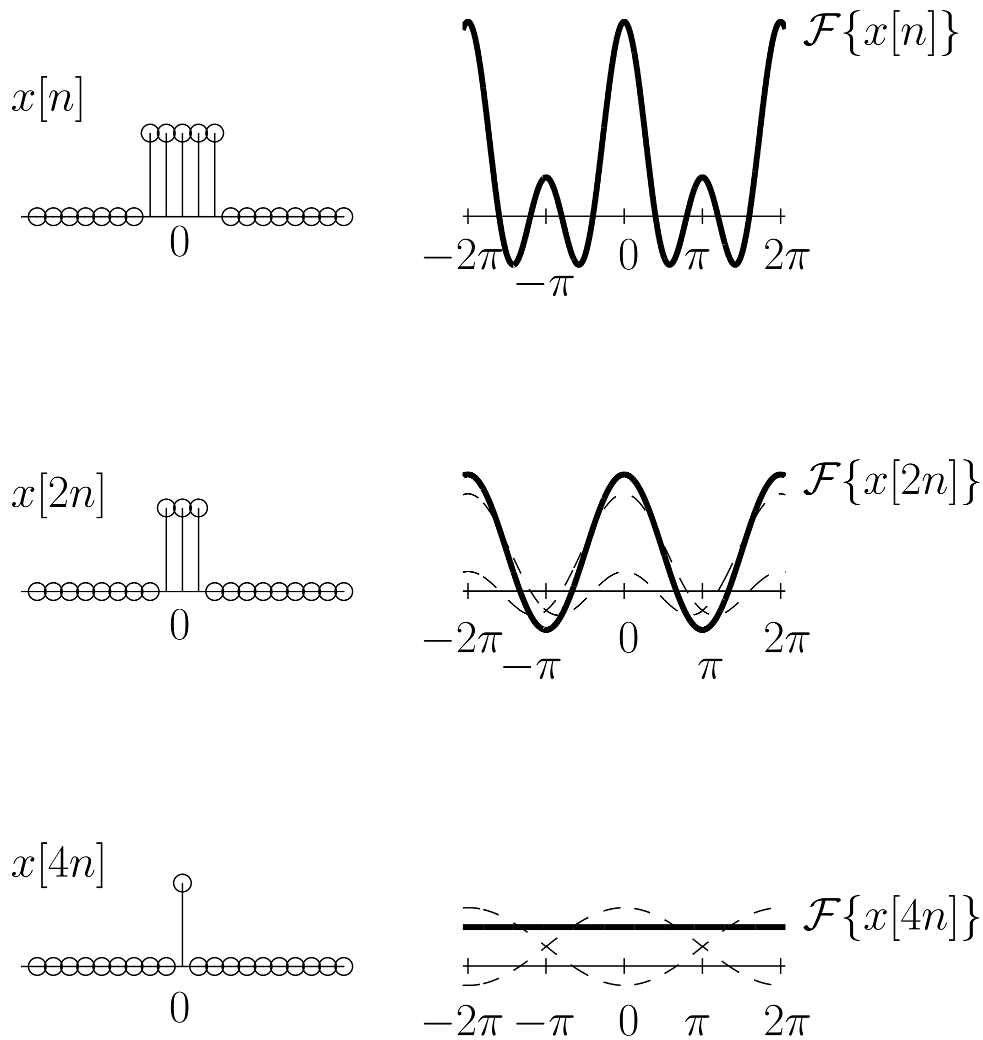
$$\mathcal{F}\{x[2n]\} = \frac{1}{2}\left(\frac{\sin(5\omega/4)}{\sin(\omega/4)} + \frac{\sin(5(\omega-\pi)/4)}{\sin((\omega-\pi)/4)}\right)$$

– Decimation by 4

$$x[4n] = \delta[n]$$

and Fourier transform

$$\mathcal{F}\{x[4n]\} = 1$$



2. Expansion

– Rectangular pulse of length $2N_1 = 5$

$$x[n] = \begin{cases} 1, & |n| \leq 2 \\ 0, & |n| > 2 \end{cases}$$

and Fourier transform

$$X(\mathsf{e}^{\mathsf{j}\omega}) = \frac{\sin(5\omega/2)}{\sin(\omega/2)}$$

– Expansion by 2

$$x_{(2)}[n] = \begin{cases} 1, & |n| = 0, 2, 4 \\ 0, & \text{else} \end{cases}$$

and Fourier transform

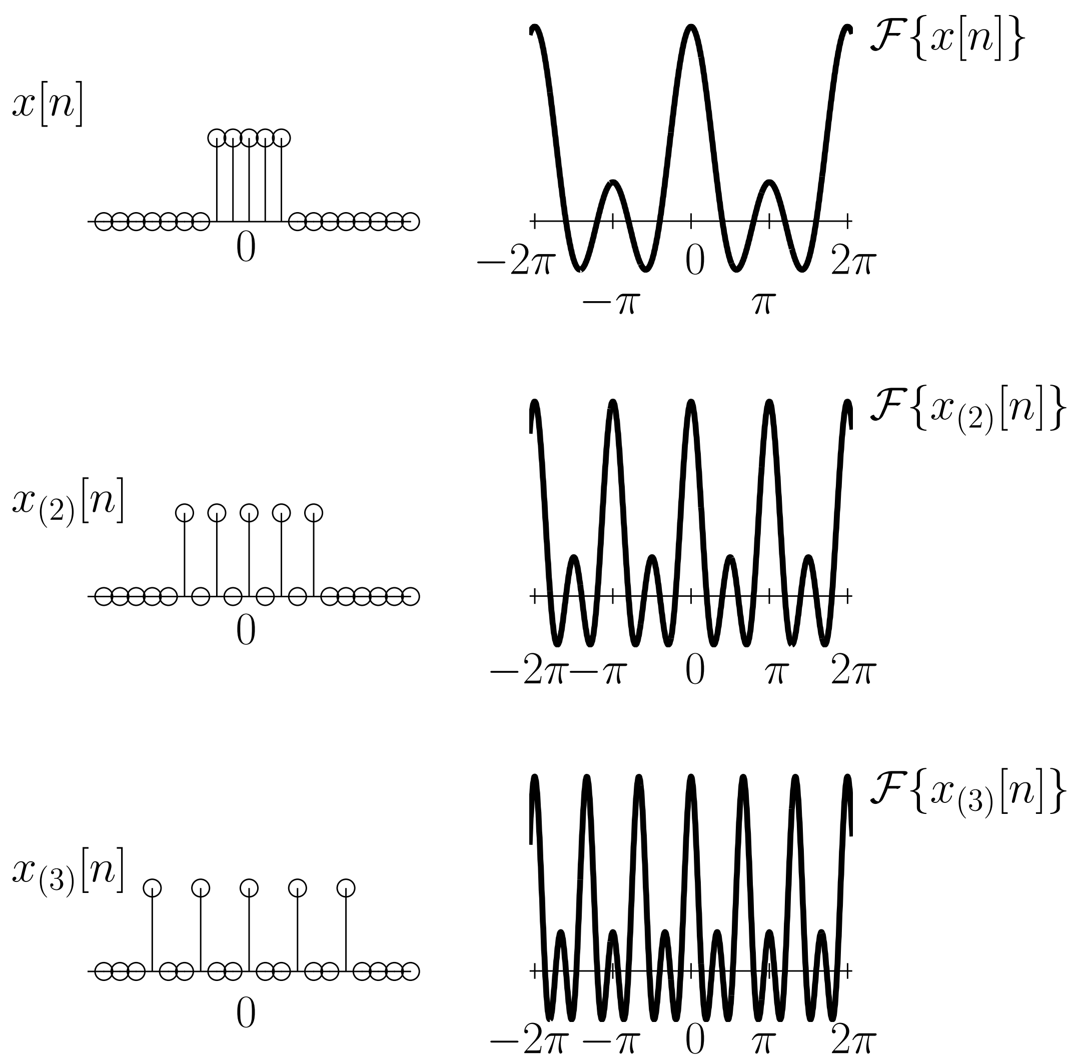
$$X(\mathsf{e}^{\mathsf{j}\omega}) = \frac{\sin(5\omega)}{\sin(\omega)}$$

– Expansion by 3

$$x_{(3)}[n] = \begin{cases} 1, & |n| = 0, 3, 6 \\ 0, & \text{else} \end{cases}$$

and Fourier transform

$$X(\mathsf{e}^{\mathsf{j}\omega}) = \frac{\sin(15\omega/2)}{\sin(3\omega/2)}$$



3. Fourier transform of sequence

$$x[n] = \begin{cases} 1\,, & n = 2k \\ 2\,, & n = 2k+1 \end{cases}\,, \quad 0 \leq k < 5$$

– Observe

$$x[n] = y_{(2)}[n] + 2y_{(2)}[n-1]$$

with shifted rectangular sequence

$$y[n] = \begin{cases} 1, & 0 \leq n < 5 \\ 0, & \text{else} \end{cases}$$

– Time-shifting and time-expansion properties

$$y_{(2)}[n] \xleftrightarrow{\mathcal{F}} \mathrm{e}^{-\mathrm{j}4\omega} \frac{\sin(5\omega)}{\sin(\omega)}$$

– Linearity and time-shifting properties

$$2y_{(2)}[n-1] \xleftrightarrow{\mathcal{F}} 2\mathrm{e}^{-\mathrm{j}5\omega} \frac{\sin(5\omega)}{\sin(\omega)}$$

– Finally

$$X(\mathrm{e}^{\mathrm{j}\omega}) = \mathrm{e}^{-\mathrm{j}4\omega}(1 + 2\mathrm{e}^{-\mathrm{j}\omega}) \frac{\sin(5\omega)}{\sin(\omega)}$$

- *Differentiation in Frequency*

  – If

  $$x[n] \xleftrightarrow{\mathcal{F}} X(\mathrm{e}^{\mathrm{j}\omega})$$

  then

  $$\boxed{nx[n] \xleftrightarrow{\mathcal{F}} \mathrm{j}\frac{dX(\mathrm{e}^{\mathrm{j}\omega})}{d\omega}}$$

  – Proof:

  $$\frac{dX(\mathrm{e}^{\mathrm{j}\omega})}{d\omega} = \frac{d}{d\omega}\left(\sum_{n=-\infty}^{\infty} x[n]\mathrm{e}^{-\mathrm{j}\omega n}\right) = \sum_{n=-\infty}^{\infty} -\mathrm{j}nx[n]\mathrm{e}^{-\mathrm{j}\omega n}$$

  ■

Example:

1. Fourier transform of

$$x[n] = na^n u[n] , \quad |a| < 1$$

   – Already known

$$z[n] = a^n u[n] \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{1 - a\mathrm{e}^{-\mathrm{j}\omega}} = Z(\mathrm{e}^{\mathrm{j}\omega})$$

   – Using differentiation-in-frequency property

$$x[n] = nz[n] \stackrel{\mathcal{F}}{\longleftrightarrow} \mathrm{j}\frac{dZ(\mathrm{e}^{\mathrm{j}\omega})}{d\omega} = \mathrm{j}\frac{d}{d\omega}\left(\frac{1}{1 - a\mathrm{e}^{-\mathrm{j}\omega}}\right)$$

   we find

$$\boxed{na^n u[n] \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{a\mathrm{e}^{-\mathrm{j}\omega}}{(1 - a\mathrm{e}^{-\mathrm{j}\omega})^2}}$$

2. Fourier transform of

$$x[n] = (n+1)a^n u[n] , \quad |a| < 1$$

   – Since

$$x[n] = na^n u[n] + a^n u[n]$$

   we find from linearity property

$$\boxed{(n+1)a^n u[n] \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{(1 - a\mathrm{e}^{-\mathrm{j}\omega})^2}}$$

- *Parseval's Relation*

  – If

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(\mathsf{e}^{\mathsf{j}\omega})$$

  then

$$\boxed{\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int\limits_{2\pi} |X(\mathsf{e}^{\mathsf{j}\omega})|^2 \, d\omega}$$

  – Proof:

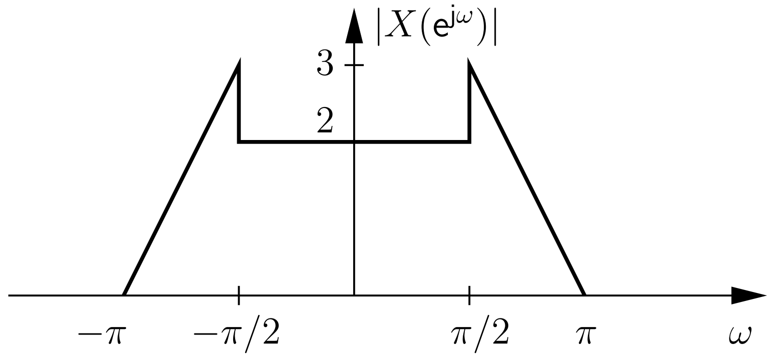
$$\begin{aligned}
\sum_{n=-\infty}^{\infty} |x[n]|^2 &= \sum_{n=-\infty}^{\infty} x[n] \frac{1}{2\pi} \int_{2\pi} X^*(\mathsf{e}^{\mathsf{j}\omega}) \mathsf{e}^{-\mathsf{j}\omega n} \, d\omega \\
&= \frac{1}{2\pi} \int_{2\pi} X^*(\mathsf{e}^{\mathsf{j}\omega}) \sum_{n=-\infty}^{\infty} x[n] \mathsf{e}^{-\mathsf{j}\omega n} \, d\omega \\
&= \frac{1}{2\pi} \int_{2\pi} |X(\mathsf{e}^{\mathsf{j}\omega})|^2 \, d\omega
\end{aligned}$$

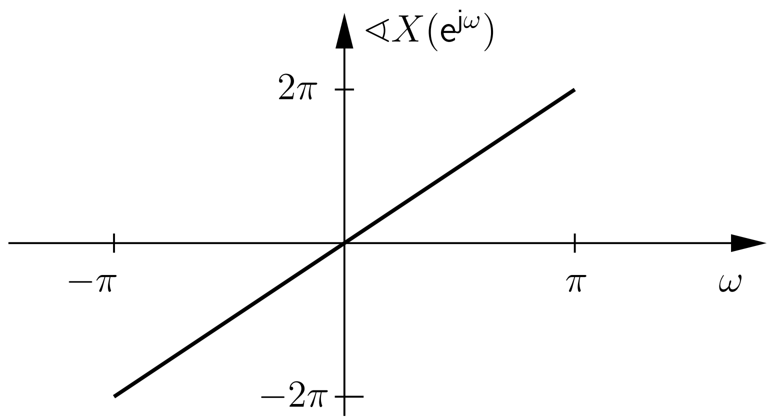
$\blacksquare$

  – In analogy with continuous–time case
  $|X(\mathsf{e}^{\mathsf{j}\omega})|^2$: *Energy–density spectrum* of $x[n]$

Example:

– Given: Fourier transform $X(\mathrm{e}^{\mathrm{j}\omega})$ for $-\pi \leq \omega \leq \pi$ of sequence $x[n]$





– Determine

* $x[n]$ periodic? Fourier transform has no impulses $\Rightarrow$ $x[n]$ is aperiodic.
* $x[n]$ real? Even magnitude and odd phase function, $X(\mathrm{e}^{\mathrm{j}\omega}) = X^*(\mathrm{e}^{-\mathrm{j}\omega}) \overset{\text{Symmetry}}{\Rightarrow} x[n]$ is real.
* $x[n]$ even? $X(\mathrm{e}^{\mathrm{j}\omega})$ is not real–valued $\overset{\text{Symmetry}}{\Rightarrow} x[n]$ is not even.
* $x[n]$ of finite energy? Integral of $|X(\mathrm{e}^{\mathrm{j}\omega})|^2$ over $-\pi \leq \omega \leq \pi$ is finite $\overset{\text{Parseval}}{\Rightarrow} x[n]$ has finite energy.

5.4 The Convolution Property

- Consider discrete-time LTI system with impulse response $h[n]$

$$x[n] \longrightarrow y[n] = h[n] * x[n]$$

- $\mathrm{e}^{\mathrm{j}\omega n}$ is eigenfunction with eigenvalue $H(\mathrm{e}^{\mathrm{j}\omega}) = \mathcal{F}\{h[n]\}$ (frequency response)

$$\mathrm{e}^{\mathrm{j}\omega n} \longrightarrow H(\mathrm{e}^{\mathrm{j}\omega})\mathrm{e}^{\mathrm{j}\omega n}$$

- Guess:

$$Y(\mathrm{e}^{\mathrm{j}\omega}) = H(\mathrm{e}^{\mathrm{j}\omega})X(\mathrm{e}^{\mathrm{j}\omega})$$

Proof:

$$\begin{aligned}
Y(\mathrm{e}^{\mathrm{j}\omega}) &= \mathcal{F}\{\sum_{k=-\infty}^{\infty} x[k]h[n-k]\} \\
&= \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} x[k]h[n-k]\mathrm{e}^{-\mathrm{j}\omega n} \\
&= \sum_{k=-\infty}^{\infty} x[k] \sum_{n=-\infty}^{\infty} h[n-k]\mathrm{e}^{-\mathrm{j}\omega n} \\
&= \sum_{k=-\infty}^{\infty} x[k] \sum_{n'=-\infty}^{\infty} h[n']\mathrm{e}^{-\mathrm{j}\omega(n'+k)} \\
&= \sum_{k=-\infty}^{\infty} x[k]\mathrm{e}^{-\mathrm{j}\omega k} \sum_{n'=-\infty}^{\infty} h[n']\mathrm{e}^{-\mathrm{j}\omega n'} \\
&= H(\mathrm{e}^{\mathrm{j}\omega})X(\mathrm{e}^{\mathrm{j}\omega})
\end{aligned}$$

■

- *Convolution property*

$$\boxed{y[n] = h[n] * x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} Y(\mathrm{e}^{\mathrm{j}\omega}) = H(\mathrm{e}^{\mathrm{j}\omega})X(\mathrm{e}^{\mathrm{j}\omega})}$$

- Important to memorize
  - Convolution in time domain $\stackrel{\mathcal{F}}{\longleftrightarrow}$ multiplication in frequency domain
  - Frequency response $H(\mathrm{e}^{\mathrm{j}\omega})$ of discrete-time LTI system is discrete-time Fourier transform of impulse response $h[n]$.
  - Impulse response completely characterizes LTI system.
    Frequency response completely characterizes LTI system.

- Existence of frequency response $\leftrightarrow$ convergence of discrete-time Fourier transform
  - LTI system is BIBO stable

$$\sum_{-\infty}^{\infty} |h[n]| < \infty,$$

$\Rightarrow$ Discrete-time Fourier transform converges

$\Rightarrow$ Stable LTI systems have frequency response $H(\mathrm{e}^{\mathrm{j}\omega}) = \mathcal{F}\{h[n]\}$.

Example:

1. Discrete-time LTI system with impulse response

$$h[n] = \delta[n - n_0]$$

– Frequency response

$$H(\mathrm{e}^{\mathrm{j}\omega}) = \sum_{n=-\infty}^{\infty} \delta[n - n_0]\mathrm{e}^{-\mathrm{j}\omega n} = \mathrm{e}^{-\mathrm{j}\omega n_0}$$

– Input-output relation in frequency domain

$$Y(\mathrm{e}^{\mathrm{j}\omega}) = \mathrm{e}^{-\mathrm{j}\omega n_0} X(\mathrm{e}^{\mathrm{j}\omega})$$

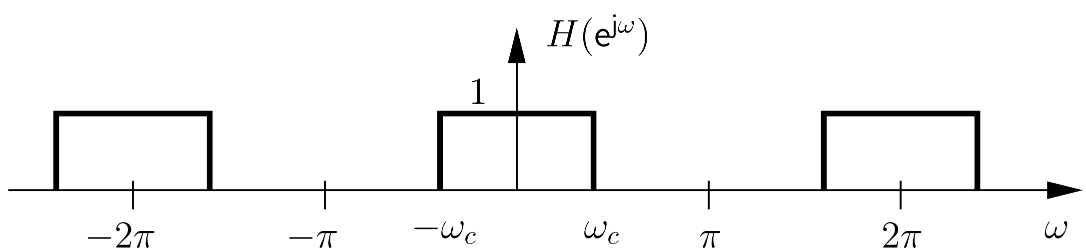
– Time–shifting property of Fourier transform

$$y[n] = x[n - n_0]$$

– Note: time shift n_0 in time domain $\stackrel{\mathcal{F}}{\longleftrightarrow}$ unit magnitude and linear phase characteristic $-\omega n_0$ in frequency domain

2. Frequency response of an ideal discrete–time lowpass filter defined for $-\pi \leq \omega \leq \pi$

$$H(\mathrm{e}^{\mathrm{j}\omega}) = \begin{cases} 1, & |\omega| \leq \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$



– Impulse response

$$h[n] = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} H(\mathsf{e}^{\mathsf{j}\omega})\mathsf{e}^{\mathsf{j}\omega n}\,d\omega = \frac{1}{2\pi}\int\limits_{-\omega_c}^{\omega_c} \mathsf{e}^{\mathsf{j}\omega n}\,d\omega = \frac{\sin(\omega_c n)}{\pi n}$$

– Discrete-time Fourier pair

$$\boxed{\frac{\omega_c}{\pi}\mathrm{sinc}\left(\frac{\omega_c n}{\pi}\right) \overset{\mathcal{F}}{\longleftrightarrow} H(\mathsf{e}^{\mathsf{j}\omega}) = \begin{cases} 1, & |\omega| \leq \omega_c \\ 0, & |\omega| > \omega_c \end{cases}}$$

$H(\mathsf{e}^{\mathsf{j}\omega})$ periodic with period 2π

– Observe:

 * Impulse response $h[n]$ is not causal.
 * Impulse response $h[n]$ has infinite duration.

$\Rightarrow$ ideal lowpass not realizable

- As in the continuous-time case:

 Computing response of LTI system via frequency domain

$$\begin{tikzcd}
x[n] \arrow[r] \arrow[d, "\mathcal{F}"] & *h[n] \arrow[r] \arrow[d, "\mathcal{F}"] & y[n] \\
X(\mathsf{e}^{\mathsf{j}\omega}) \arrow[r] & \cdot H(\mathsf{e}^{\mathsf{j}\omega}) \arrow[r] & Y(\mathsf{e}^{\mathsf{j}\omega}) \arrow[u, "\mathcal{F}^{-1}"']
\end{tikzcd}$$

 – "Computational bottleneck": inverse Fourier transform

 – Solutions:

  * Look up table of (basic) Fourier transform pairs (e.g., Table 5.2 in text book)
  * *Partial fraction expansion* for ratio of polynomials

Example:

1. Discrete-time LTI system with impulse response

$$h[n] = \alpha^n u[n] \; , \quad |\alpha| < 1$$

and input signal

$$x[n] = \beta^n u[n] \; , \quad |\beta| < 1$$

Output $y[n]$?

– Discrete-time Fourier transforms of $x[n]$ and $h[n]$

$$X(\mathsf{e}^{\mathsf{j}\omega}) = \frac{1}{1 - \beta \mathsf{e}^{-\mathsf{j}\omega}}$$

and

$$H(\mathsf{e}^{\mathsf{j}\omega}) = \frac{1}{1 - \alpha \mathsf{e}^{-\mathsf{j}\omega}}$$

– Discrete-time Fourier transform of $y[n]$

$$Y(\mathsf{e}^{\mathsf{j}\omega}) = H(\mathsf{e}^{\mathsf{j}\omega}) X(\mathsf{e}^{\mathsf{j}\omega}) = \frac{1}{(1 - \alpha \mathsf{e}^{-\mathsf{j}\omega})(1 - \beta \mathsf{e}^{-\mathsf{j}\omega})}$$

– Inverse discrete-time Fourier transform

(a) $\alpha \neq \beta$

Write

$$Y(\mathsf{e}^{\mathsf{j}\omega}) = \frac{A}{1 - \alpha \mathsf{e}^{-\mathsf{j}\omega}} + \frac{B}{1 - \beta \mathsf{e}^{-\mathsf{j}\omega}}$$

and consider generalized function

$$Y(v) = \frac{1}{(1 - \alpha v)(1 - \beta v)} = \frac{A}{1 - \alpha v} + \frac{B}{1 - \beta v}$$

$$
\begin{aligned}
A &= (1-\alpha v)Y(v)\bigg|_{v=1/\alpha} = \frac{\alpha}{\alpha-\beta} \\
B &= (1-\beta v)Y(v)\bigg|_{v=1/\beta} = \frac{\beta}{\beta-\alpha}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \qquad Y(\mathrm{e}^{\mathrm{j}\omega}) &= \frac{1}{\alpha-\beta}\left(\frac{\alpha}{1-\alpha\mathrm{e}^{-\mathrm{j}\omega}} - \frac{\beta}{1-\beta\mathrm{e}^{-\mathrm{j}\omega}}\right) \\
\Rightarrow \qquad y[n] &= \frac{\alpha}{\alpha-\beta}\alpha^n u[n] - \frac{\beta}{\alpha-\beta}\beta^n u[n] \\
&= \frac{1}{\alpha-\beta}\left(\alpha^{n+1}-\beta^{n+1}\right)u[n]
\end{aligned}
$$

(b) $\alpha = \beta$

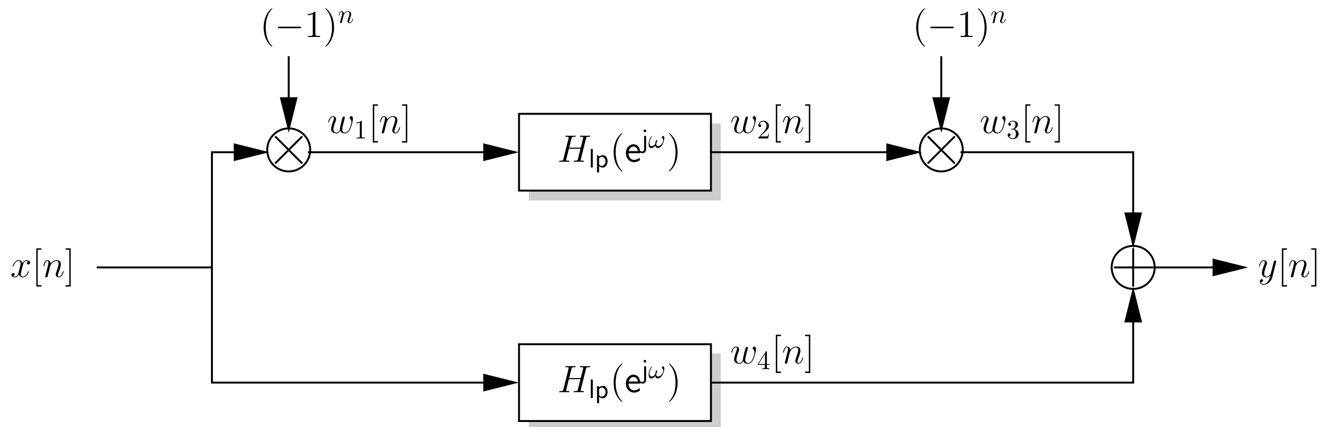
We have

$$
Y(\mathrm{e}^{\mathrm{j}\omega}) = \frac{1}{(1-\alpha\mathrm{e}^{-\mathrm{j}\omega})^2}
$$

$$
\Rightarrow \quad y[n] = (n+1)\alpha^n u[n]
$$

(Fourier pair derived in an example in Section 5.3, alternatively, write $Y(\mathrm{e}^{\mathrm{j}\omega}) = \frac{\mathrm{j}}{\alpha}\mathrm{e}^{\mathrm{j}\omega}\frac{d}{d\omega}\left(\frac{1}{1-\alpha\mathrm{e}^{-\mathrm{j}\omega}}\right)$ and use frequency-differentiation and time-shifting properties)

2. Consider LTI system with ideal lowpass filters $H_{\mathsf{lp}}(\mathsf{e}^{\mathsf{j}\omega})$ with cut-off frequency $\pi/4$



Relation between $x[n]$ and $y[n]$?

(a) Top path

– Since $(-1)^n = \mathsf{e}^{\mathsf{j}\pi n}$

$$w_1[n] = \mathsf{e}^{\mathsf{j}\pi n} x[n]$$

from frequency–shifting property

$$W_1(\mathsf{e}^{\mathsf{j}\omega}) = X(\mathsf{e}^{\mathsf{j}(\omega-\pi)})$$

– Convolution property

$$W_2(\mathsf{e}^{\mathsf{j}\omega}) = H_{\mathsf{lp}}(\mathsf{e}^{\mathsf{j}\omega}) X(\mathsf{e}^{\mathsf{j}(\omega-\pi)})$$

– Again frequency–shifting property

$$W_3(\mathsf{e}^{\mathsf{j}\omega}) = W_2(\mathsf{e}^{\mathsf{j}(\omega-\pi)}) = H_{\mathsf{lp}}(\mathsf{e}^{\mathsf{j}(\omega-\pi)}) X(\mathsf{e}^{\mathsf{j}(\omega-2\pi)})$$

– Periodicity of discrete–time Fourier transforms with period 2π,

$$W_3(\mathsf{e}^{\mathsf{j}\omega}) = H_{\mathsf{lp}}(\mathsf{e}^{\mathsf{j}(\omega-\pi)}) X(\mathsf{e}^{\mathsf{j}\omega})$$

(b) Lower path

– Convolution property

$$W_4(\mathrm{e}^{\mathrm{j}\omega}) = H_{\mathrm{lp}}(\mathrm{e}^{\mathrm{j}\omega})X(\mathrm{e}^{\mathrm{j}\omega})$$

(c) Combining top and lower path

– Linearity property

$$Y(\mathrm{e}^{\mathrm{j}\omega}) = W_3(\mathrm{e}^{\mathrm{j}\omega}) + W_4(\mathrm{e}^{\mathrm{j}\omega}) = (H_{\mathrm{lp}}(\mathrm{e}^{\mathrm{j}(\omega-\pi)}) + H_{\mathrm{lp}}(\mathrm{e}^{\mathrm{j}\omega}))X(\mathrm{e}^{\mathrm{j}\omega})$$

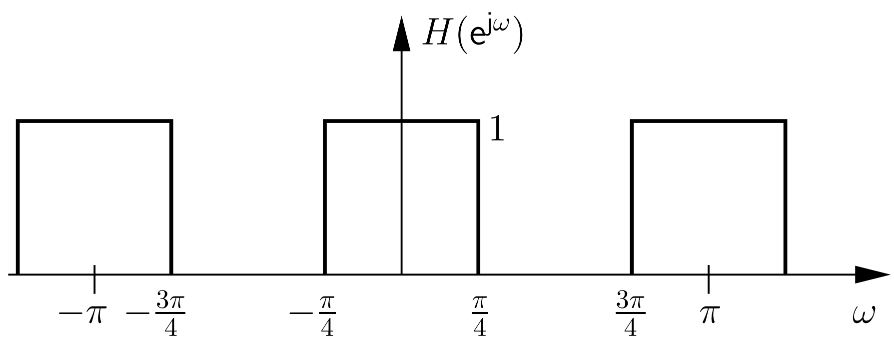
$\Rightarrow$ Frequency response of overall system

$$H(\mathrm{e}^{\mathrm{j}\omega}) = H_{\mathrm{lp}}(\mathrm{e}^{\mathrm{j}(\omega-\pi)}) + H_{\mathrm{lp}}(\mathrm{e}^{\mathrm{j}\omega})$$

– From an example in Section 5.3:
Ideal lowpass shifted by π in frequency is ideal highpass filter

– $\Rightarrow$ Overall system passes both low and high frequencies and stops frequencies between these two passbands

– $\Rightarrow$ Filter with *ideal bandstop characteristic*, here stopband region $\pi/4 < |\omega| < 3\pi/4$



5.5 The Multiplication Property

- Analogous to continuous-time case: *multiplication property*

$$\boxed{y[n] = x_1[n]x_2[n] \stackrel{\mathcal{F}}{\longleftrightarrow} Y(\mathrm{e}^{\mathrm{j}\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(\mathrm{e}^{j\Theta})X_2(\mathrm{e}^{\mathrm{j}(\omega-\Theta)})\,d\Theta}$$

Right–hand side integral corresponds to *periodic convolution* $\Rightarrow$ can be evaluated over any interval of length 2π.

Proof:

$$\begin{aligned} Y(\mathrm{e}^{\mathrm{j}\omega}) &= \sum_{n=-\infty}^{\infty} y[n]\mathrm{e}^{-\mathrm{j}\omega n} \\ &= \sum_{n=-\infty}^{\infty} x_1[n]x_2[n]\mathrm{e}^{-\mathrm{j}\omega n} \\ &= \sum_{n=-\infty}^{\infty} \frac{1}{2\pi}\left(\int_{2\pi} X_1(\mathrm{e}^{j\Theta})\mathrm{e}^{j\Theta n}\,d\Theta\right) x_2[n]\mathrm{e}^{-\mathrm{j}\omega n} \\ &= \frac{1}{2\pi}\int_{2\pi} X_1(\mathrm{e}^{j\Theta})\left(\sum_{n=-\infty}^{\infty} x_2[n]\mathrm{e}^{-\mathrm{j}(\omega-\Theta)n}\right) d\Theta \\ &= \frac{1}{2\pi}\int_{2\pi} X_1(\mathrm{e}^{j\Theta})X_2(\mathrm{e}^{\mathrm{j}(\omega-\Theta)})\,d\Theta \end{aligned}$$

$\blacksquare$

Example:

Spectrum of signal

$$x[n] = \frac{\mathsf{sin}(\pi n/2)\mathsf{sin}(3\pi n/4)}{\pi^2 n^2}$$

– Write $x[n]$ as product of two sinc functions

$$x[n] = \frac{1}{2}\mathsf{sinc}\left(\frac{n}{2}\right) \cdot \frac{3}{4}\mathsf{sinc}\left(\frac{3n}{4}\right)$$

– From multiplication property

$$X(\mathsf{e}^{\mathsf{j}\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\mathsf{e}^{j\Theta})X_2(\mathsf{e}^{\mathsf{j}(\omega-\Theta)})\, d\Theta$$

with periodic rectangular spectra defined in $-\pi \leq \omega \leq \pi$ as

$$X_1(\mathsf{e}^{\mathsf{j}\omega}) = \begin{cases} 1, & |\omega| \leq \pi/2 \\ 0, & |\omega| > \pi/2 \end{cases}$$

and

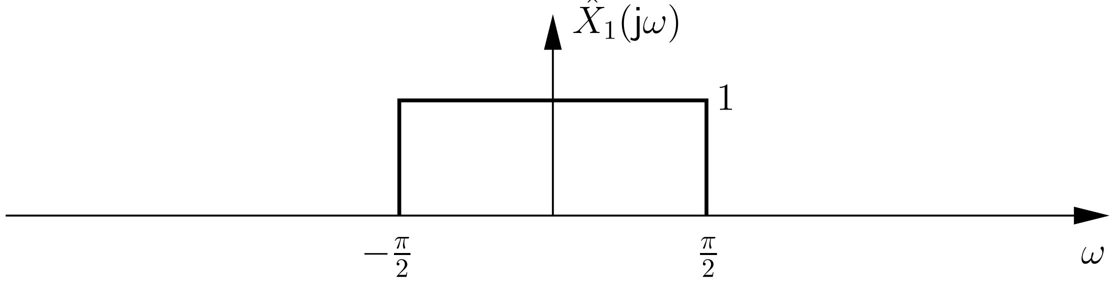
$$X_2(\mathsf{e}^{\mathsf{j}\omega}) = \begin{cases} 1, & |\omega| \leq 3\pi/4 \\ 0, & |\omega| > 3\pi/4 \end{cases}$$

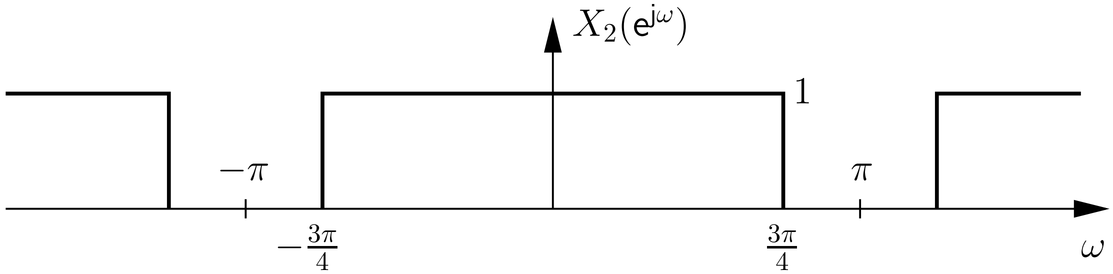
– Simplification of above *periodic convolution* by defining *aperiodic* lowpass

$$\hat{X}_1(\mathsf{j}\omega) = \begin{cases} X_1(\mathsf{e}^{\mathsf{j}\omega}), & -\pi < \omega \leq \pi \\ 0, & \text{otherwise} \end{cases}$$

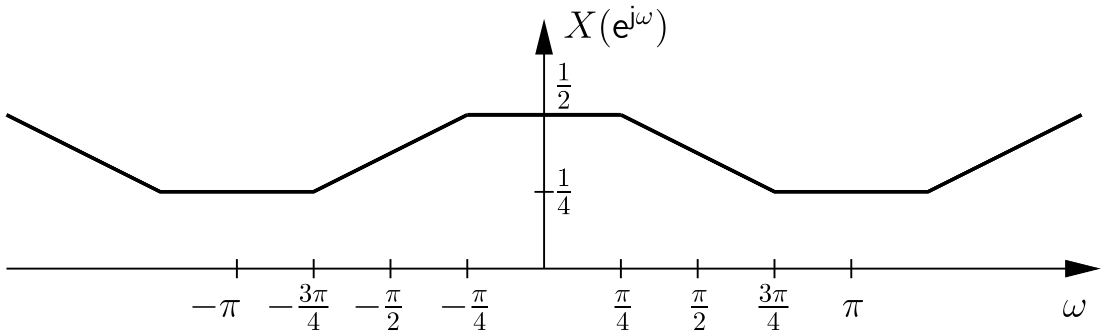
$\Rightarrow$ *aperiodic* convolution

$$X(\mathrm{e}^{\mathrm{j}\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{X}_1(\mathrm{j}\Theta) X_2(\mathrm{e}^{\mathrm{j}(\omega-\Theta)})\, d\Theta$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{X}_1(\mathrm{j}\Theta) X_2(\mathrm{e}^{\mathrm{j}(\omega-\Theta)})\, d\Theta$$





– Result of convolution is Fourier transform $X(\mathrm{e}^{\mathrm{j}\omega})$



5.6 Duality

- *No* formal similarity between discrete-time Fourier transform and its inverse

$$X(\mathrm{e}^{\mathrm{j}\omega}) = \sum_{n=-\infty}^{\infty} x[n]\mathrm{e}^{-\mathrm{j}\omega n} \text{ and } x[n] = \frac{1}{2\pi}\int_{2\pi} X(\mathrm{e}^{\mathrm{j}\omega})\mathrm{e}^{\mathrm{j}\omega n}\, d\omega$$

  $\Rightarrow$ *No* duality property as in continuous-time case

- However: Formal similarity between *discrete–time Fourier series equations*

$$x[n] = \sum_{k=\langle N\rangle} a_k \mathrm{e}^{\mathrm{j}k(2\pi/N)n} \text{ and } a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]\mathrm{e}^{-\mathrm{j}k(2\pi/N)n}$$

  – Formally define two periodic sequences

$$g[n] \overset{\mathcal{FS}}{\longleftrightarrow} f[k]$$

  – Changing roles of k and n in Fourier coefficients series equation

$$f[n] = \frac{1}{N}\sum_{k=\langle N\rangle} g[k]\mathrm{e}^{-\mathrm{j}k(2\pi/N)n} = \frac{1}{N}\sum_{k=\langle N\rangle} g[-k]\mathrm{e}^{\mathrm{j}k(2\pi/N)n}$$

  we find

$$\boxed{f[n] \overset{\mathcal{FS}}{\longleftrightarrow} \frac{1}{N}g[-k]}$$

– Duality of properties of discrete–time Fourier series

  * *Time and Frequency Shift*

$$x[n-n_0] \overset{\mathcal{FS}}{\longleftrightarrow} a_k \mathrm{e}^{-\mathrm{j}k(2\pi/N)n_0}$$

and

$$\mathrm{e}^{\mathrm{j}m(2\pi/N)n} x[n] \overset{\mathcal{FS}}{\longleftrightarrow} a_{k-m}$$

  * *Convolution and Multiplication*

$$\sum_{r=\langle N\rangle} x[r]y[n-r] \overset{\mathcal{FS}}{\longleftrightarrow} N a_k b_k$$

and

$$x[n]y[n] \overset{\mathcal{FS}}{\longleftrightarrow} \sum_{l=\langle N\rangle} a_l b_{k-l}$$

– Duality often useful in reducing complexity of calculations involved in determining Fourier series representations

Example:

– Periodic signal with a period of $N=9$:

$$x[n] = \begin{cases} \dfrac{1}{9}\dfrac{\sin(5\pi n/9)}{\sin(\pi n/9)}, & n \neq \text{ multiple of } 9 \\ \frac{5}{9}, & n = \text{ multiple of } 9 \end{cases}$$

– Recall: Fourier series coefficients b_k of periodic square wave $g[n]$ of length N_1 and a period N (Chapter 3, page 84)

$$g[n] \;=\; \begin{cases} 1\,, & |n| \le N_1 \\ 0\,, & |n| > N_1 \end{cases}$$

$$b_k \;=\; \begin{cases} \dfrac{1}{N}\dfrac{\sin(2\pi k(N_1+1/2)/N)}{\sin(\pi k/N)}\,, & k \neq 0,\, \pm N,\, \pm 2N,\, \ldots \\[2ex] \dfrac{2N_1+1}{N}\,, & k = 0,\, \pm N,\, \pm 2N,\, \ldots \end{cases}$$

– With $N = 9$ and $N_1 = 2$

$$b_k = \begin{cases} \dfrac{1}{9}\dfrac{\sin(5\pi k/9)}{\sin(\pi k/9)}\,, & k \neq \text{ multiple of } 9 \\[2ex] \dfrac{5}{9}\,, & k = \text{ multiple of } 9 \end{cases}$$

– Duality

$$g[n] \overset{\mathcal{FS}}{\longleftrightarrow} b_k \quad \text{and} \quad b_n = x[n]$$

$$\Rightarrow \quad x[n] \overset{\mathcal{FS}}{\longleftrightarrow} a_k = \frac{1}{9}g[-k]$$

– Sequence of Fourier series coefficients with a period $N = 9$

$$a_k = \begin{cases} 1/9, & |k| \le 2 \\ 0, & 2 < |k| \le 4 \end{cases}$$

- Furthermore: Formal similarity between *discrete–time Fourier transform*

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\mathrm{e}^{\mathrm{j}\omega}) \mathrm{e}^{\mathrm{j}\omega n}\, d\omega \;, \quad X(\mathrm{e}^{\mathrm{j}\omega}) = \sum_{n=-\infty}^{\infty} x[n] \mathrm{e}^{-\mathrm{j}\omega n}$$

and *continuous–time Fourier series*

$$g(t) = \sum_{k=-\infty}^{\infty} a_k \mathrm{e}^{\mathrm{j}k\omega_0 t} \;, \quad a_k = \frac{1}{T} \int_{T} x(t) \mathrm{e}^{-\mathrm{j}k\omega_0 t}\, dt$$

  – Fourier transform $X(\mathrm{e}^{\mathrm{j}\omega})$ periodic with period 2π

  – Formally replace argument $\mathrm{e}^{\mathrm{j}\omega}$ by t: $X(t)$, periodic with period $T = 2\pi$, $\omega_0 = 2\pi/T = 1$

  – "continuous-time" Fourier series

$$X(t) = \sum_{n=-\infty}^{\infty} x[n] \mathrm{e}^{-\mathrm{j}tn} = \sum_{k=-\infty}^{\infty} x[-k] \mathrm{e}^{\mathrm{j}k\omega_0 t}$$

  – Observe:
  If $x(t)$ periodic with period $T = 2\pi$ and

$$g(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k$$

  then

$$\boxed{x[n] = a_{-n} \stackrel{\mathcal{F}}{\longleftrightarrow} X(\mathrm{e}^{\mathrm{j}\omega}) = g(\omega)}$$

Example:

– Desired: discrete–time Fourier transform of sequence

$$x[n] = \frac{\sin(\pi n/2)}{\pi n} = \frac{1}{2}\mathsf{sinc}\left(\frac{n}{2}\right)$$

– Continuous–time signal $g(t)$ with period $T = 2\pi$ and Fourier coefficients $a_k = x[-k]$ (Chapter 3, page 74):

$$g(t) = \begin{cases} 1, & |t| \le \pi/2 \\ 0, & \pi/2 < |t| \le \pi \end{cases}$$

$$a_k = \frac{1}{2}\mathsf{sinc}\left(\frac{n}{2}\right) = a_{-k} = x[k]$$

– Duality $\Rightarrow$ Fourier transform

$$X(\mathrm{e}^{\mathrm{j}\omega}) = g(\omega) = \begin{cases} 1, & |\omega| \le \pi/2 \\ 0, & \pi/2 < |\omega| \le \pi \end{cases}$$

– Verification:

We have

$$g(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k = \frac{\sin(\pi k/2)}{k\pi} = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)\mathrm{e}^{-\mathrm{j}kt}\,dt = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}(1)\mathrm{e}^{-\mathrm{j}kt}\,dt$$

or

$$\frac{\sin(\pi n/2)}{\pi n} = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}(1)\mathrm{e}^{-\mathrm{j}n\omega}\,d\omega = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}(1)\mathrm{e}^{\mathrm{j}n\omega}\,d\omega$$

or

$$x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(\mathrm{e}^{\mathrm{j}\omega})$$

Summary of Fourier series and transform expressions

	Continuous time		Discrete time	
	Time domain	Frequency domain	Time domain	Frequency domain
Fourier Series	$x(t) = \sum_{k=-\infty}^{\infty} a_k \mathrm{e}^{\mathrm{j}k\omega_0 t}$ continuous time periodic in time	$a_k = \frac{1}{T} \int_T x(t)\mathrm{e}^{-\mathrm{j}k\omega_0 t}$ discrete frequency aperiodic in frequency	$x[n] = \sum_{k=\langle N\rangle} a_k \mathrm{e}^{\mathrm{j}k(2\pi/N)n}$ discrete time periodic in time	$a_k = \frac{1}{N} \sum_{k=\langle N\rangle} x[n]\mathrm{e}^{-\mathrm{j}k(2\pi/N)n}$ discrete frequency periodic in frequency
Fourier Transform	$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\mathrm{j}\omega) e^{\mathrm{j}\omega t}\, d\omega$ continuous time aperiodic in time	$X(\mathrm{j}\omega) = \int_{-\infty}^{\infty} x(t)\mathrm{e}^{-\mathrm{j}\omega t}\, dt$ continuous frequency aperiodic in frequency	$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\mathrm{e}^{\mathrm{j}\omega})\mathrm{e}^{\mathrm{j}\omega n}\, d\omega$ discrete time aperiodic in time	$X(\mathrm{e}^{\mathrm{j}\omega}) = \sum_{k=-\infty}^{\infty} x[n]\mathrm{e}^{-\mathrm{j}\omega n}$ continuous frequency periodic in frequency

Note:

– continuous $\longleftrightarrow$ aperiodic

– discrete $\longleftrightarrow$ periodic

5.7 Frequency Response and Linear Constant–Coefficient Difference Equations

- Recall: Discrete-time system description by linear constant-coefficient difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

- Frequency response $H(\mathsf{e}^{\mathsf{j}\omega})$?
  - Apply $x[n] = \mathsf{e}^{\mathsf{j}\omega n}$ as input $\rightarrow$ output $y[n] = H(\mathsf{e}^{\mathsf{j}\omega})\mathsf{e}^{\mathsf{j}\omega n}$
  - Alternatively
    * Convolution property $H(\mathsf{e}^{\mathsf{j}\omega}) = Y(\mathsf{e}^{\mathsf{j}\omega})/X(\mathsf{e}^{\mathsf{j}\omega})$
    * Apply Fourier transform to difference equation + linearity and time-shifting properties

$$\begin{aligned}
\mathcal{F}\left\{\sum_{k=0}^{N} a_k y[n-k]\right\} &= \mathcal{F}\left\{\sum_{k=0}^{M} b_k x[n-k]\right\} \\
\sum_{k=0}^{N} a_k \mathcal{F}\left\{y[n-k]\right\} &= \sum_{k=0}^{M} b_k \mathcal{F}\left\{x[n-k]\right\} \\
\sum_{k=0}^{N} a_k \mathsf{e}^{-\mathsf{j}k\omega} Y(\mathsf{e}^{\mathsf{j}\omega}) &= \sum_{k=0}^{M} b_k \mathsf{e}^{-\mathsf{j}k\omega} X(\mathsf{e}^{\mathsf{j}\omega})
\end{aligned}$$

    * Consequently

$$\boxed{H(\mathsf{e}^{\mathsf{j}\omega}) = \frac{Y(\mathsf{e}^{\mathsf{j}\omega})}{X(\mathsf{e}^{\mathsf{j}\omega})} = \frac{\sum\limits_{k=0}^{M} b_k \mathsf{e}^{-\mathsf{j}k\omega}}{\sum\limits_{k=0}^{N} a_k \mathsf{e}^{-\mathsf{j}k\omega}}}$$

- Observe
  - $H(\mathrm{e}^{\mathrm{j}\omega})$ obtained from difference equation by inspection
  - $H(\mathrm{e}^{\mathrm{j}\omega})$ is a rational function in $\mathrm{e}^{-\mathrm{j}\omega} \rightarrow$ inverse Fourier transform by partial-fraction expansion

Example:

1. Causal discrete-time LTI system characterized by

$$y[n] - ay[n-1] = x[n] \ , \quad |a| < 1$$

   - Frequency response

$$H(\mathrm{e}^{\mathrm{j}\omega}) = \frac{1}{1 - a\mathrm{e}^{-\mathrm{j}\omega}}$$

   - Impulse response

$$h[n] = a^n u[n]$$

2. Causal discrete-time LTI system characterized by

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n]$$

   - Frequency response

$$H(\mathrm{e}^{\mathrm{j}\omega}) = \frac{2}{1 - \frac{3}{4}\mathrm{e}^{-\mathrm{j}\omega} + \frac{1}{8}\mathrm{e}^{-j2\omega}}$$

   - Partial-fraction expansion

$$H(\mathrm{e}^{\mathrm{j}\omega}) = \frac{2}{(1 - \frac{1}{2}\mathrm{e}^{-\mathrm{j}\omega})(1 - \frac{1}{4}\mathrm{e}^{-\mathrm{j}\omega})} = \frac{4}{1 - \frac{1}{2}\mathrm{e}^{-\mathrm{j}\omega}} - \frac{2}{1 - \frac{1}{4}\mathrm{e}^{-\mathrm{j}\omega}}$$

   - Impulse response

$$h[n] = 4\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{4}\right)^n u[n]$$

3. System of previous example and input

$$x[n] = \left(\frac{1}{4}\right)^n u[n]$$

– Spectrum of output

$$\begin{aligned} Y(\mathrm{e}^{\mathrm{j}\omega}) &= H(\mathrm{e}^{\mathrm{j}\omega})X(\mathrm{e}^{\mathrm{j}\omega}) \\ &= \left(\frac{2}{1-\frac{3}{4}\mathrm{e}^{-\mathrm{j}\omega}+\frac{1}{8}\mathrm{e}^{-\mathrm{j}2\omega}}\right)\left(\frac{1}{1-\frac{1}{4}\mathrm{e}^{-\mathrm{j}\omega}}\right) \\ &= \left(\frac{2}{(1-\frac{1}{2}\mathrm{e}^{-\mathrm{j}\omega})(1-\frac{1}{4}\mathrm{e}^{-\mathrm{j}\omega})^2}\right) \end{aligned}$$

– Partial–fraction expansion

$$Y(\mathrm{e}^{\mathrm{j}\omega}) = \frac{B_{11}}{1-\frac{1}{4}\mathrm{e}^{-\mathrm{j}\omega}} + \frac{B_{12}}{(1-\frac{1}{4}\mathrm{e}^{-\mathrm{j}\omega})^2} + \frac{B_{21}}{1-\frac{1}{2}\mathrm{e}^{-\mathrm{j}\omega}}$$

with $B_{11} = -4$, $B_{12} = -2$, and $B_{21} = 8$

– Inverse Fourier transform

$$y[n] = \left(-4\left(\frac{1}{4}\right)^n - 2(n+1)\left(\frac{1}{4}\right)^n + 8\left(\frac{1}{2}\right)^n\right)u[n]$$