T-61.5100 Digital image processing, Exercise 1/12

demo 1. The *Mach bands* phenomenon is an example of how the human perception of brightness is not a simple function of of intensity (see Fig. 2.7 in the course book). How could the "Mexican-hat" function (see Fig. 1(a) below) be used to "explain" this phenomenon? As a simplification you can use the approximated function in Fig. 1(b) in one dimension.

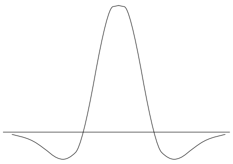
Figure 1(a)

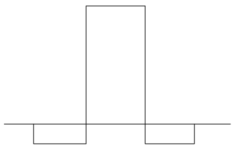
Figure 1(b)

demo 2. Suppose that a flat area with centre at (x_0, y_0) is illuminated by a light source with intensity distribution

$$i(x, y) = Ke^{-[(x-x_0)^2+(y-y_0)^2]}.$$

The reflectance $r(x, y)$ of the area is 1 and $K = 255$. If the resulting image is digitized using n bits of intensity resolution, and the eye can detect an abrupt change of eight shades of intensity between adjacent pixels, what value of n will cause visible false contouring?

3. Consider the two image subsets S_1 and S_2 shown below. For $V = \{1\}$, determine how many (a) 4-connected, (b) 8-connected, and (c) m-connected components there are in S_1 and S_2. Are S_1 and S_2 adjacent?

	S_1				S_2				
0	0	0	0	0	0	0	1	$\boxed{1}$	0
1	0	0	1	0	0	1	0	0	1
1	0	0	1	0	1	1	0	0	0
0	0	$\boxed{1}$	1	1	0	0	1	1	1
0	0	1	1	1	0	0	1	1	1

4. In the image above, compute the D_4- and D_8-distances between the two points marked with rectangles. Also compute the D_m-distance given $V = \{1\}$.

5. Assume that we have many noisy versions $g_i(x, y)$ of the same image $f(x, y)$, i.e.

$$g_i(x, y) = f(x, y) + \eta_i(x, y)$$

where the noise η_i is zero-mean and all point-pairs $(\eta_i(x, y), \eta_j(x, y))$ are uncorrelated between each image version. Then we can reduce noise by taking the mean of all the noisy images

$$\bar{g}(x, y) = \frac{1}{M} \sum_{i=1}^{M} g_i(x, y).$$

Prove that

$$E\{\bar{g}(x, y)\} = f(x, y)$$

and

$$\sigma^2_{\bar{g}(x,y)} = \frac{1}{M} \sigma^2_{\eta(x,y)}$$

where $\sigma^2_{\eta(x,y)}$ is the variance of η and $\sigma^2_{\bar{g}(x,y)}$ the variance of $\bar{g}(x, y)$.

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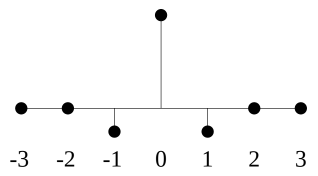
1.

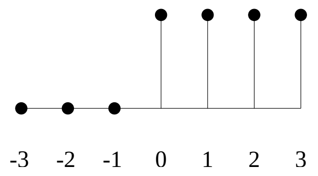
In the early visual system there are cells with an excitatory effect in the centre and an inhibitory effect in the surrounding area. This can be roughly approximated by the "Mexican hat"-function in one dimension. For this problem we use the simplest possible "hat":

$$h(i) = \begin{cases} -0.25, & \text{when } i = \pm 1 \\ 1, & \text{when } i = 0 \\ 0, & \text{otherwise} \end{cases} \tag{1}$$



An edge is modelled by a step function:

$$f(i) = \begin{cases} 0, & \text{when } i < 0 \\ 1, & \text{otherwise} \end{cases} \tag{2}$$

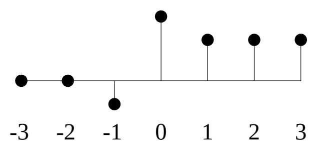


Convolution is used:

$$g(x) = \sum_{i=-\infty}^{\infty} h(i) f(x-i) = \sum_{i=-1}^{1} h(i) f(x-i) \tag{3}$$

And we have

$$g(-3) = 0. \quad g(-2) = 0, \quad g(-1) = -0.25, \quad g(0) = 0.75, \quad g(1) = 0.5, \quad g(2) = 0.5 \tag{4}$$



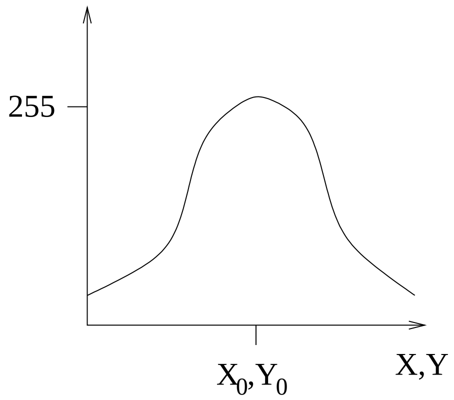
Thus, we see a darker and a lighter bands near the edge. In two dimensions these would be seen as a moat and a wall.

2.

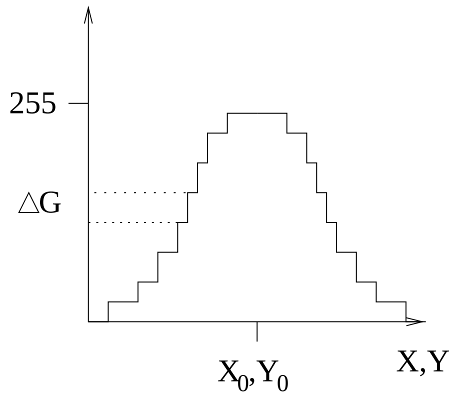
In the simple image formation model (book section 2.3.4) the image is the product of the illumination $i(x, y)$ and the reflectance $r(x, y)$, which is now given as:

$$f(x, y) = i(x, y)r(x, y) = 255e^{-[(x-x_0)^2+(y-y_0)^2]}, \tag{5}$$

and the cross section of the image looks like this



Quantisation with n bits means that we discretise the intensity so that the smallest change is $\Delta G = \frac{255+1}{2^n}$. This is what the quantised picture might look like:



Since the eye is able to perceive a sudden change of 8 intensity levels, we see a false contour, when ΔG equals 8 or is larger, i.e.

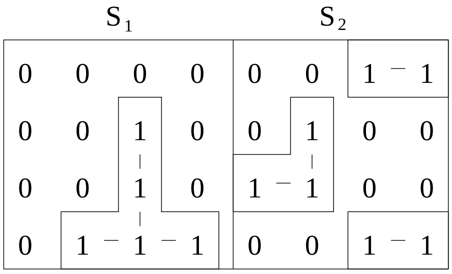
$$\begin{aligned} \Delta G &\geq 8 \\ \frac{255+1}{2^n} &\geq 8 \\ \frac{256}{8} &\geq 2^n \\ \log_2 32 &\geq \log_2 2^n \\ 5 &\geq n. \end{aligned} \tag{6}$$

So when $n \leq 5$ a false contour can be seen.

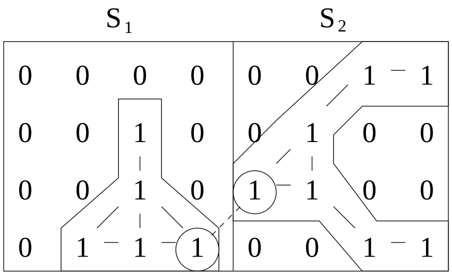
3.

A *connected component* (see book section 2.5.2) is a set in which all its pixels are *connected* with each other. Two pixels are connected if there is a *path* between them, i.e. a chain of pixels which all belong to the set and are *adjacent* in each step of the chain. Adjacency, in turn, means that two pixels are neighbours (e.g. 4-neighbours) and both belong to the set V. Here $V = \{1\}$.

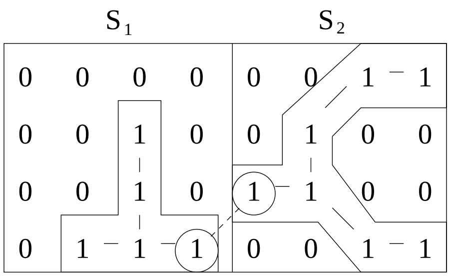
a) 4-connected components, i.e. where we use 4-neighbourhood to define adjacency:
S_1: 1, S_2: 3



b) 8-connected components, i.e. with 8-adjacency:
S_1: 1, S_2: 1



c) M-connected components, i.e. with mixed adjacency:
S_1: 1, S_2: 1



The answer is the same as in b), but now there exists only one possible path connecting each pixel-pair.

Adjacency of subsets

Two *subsets* are adjacent if there are two pixels (one in each set) which are adjacent (as pixels).

a): S_1 and S_2 are not adjacent, since no pixel of S_2 that belongs to V is a 4-neighbour of any pixel in S_1 that belongs to V.

b) and c): In both cases S_1 and S_2 are adjacent, thanks to the pixels that have been circled in the figures.

4.

D_4 ("City-block") and D_8 ("Chess-board") distances do not depend on V but only on the coordinates of the two points.

$$
\begin{aligned}
D_4(p,q) &= |x_p - x_q| + |y_p - y_q| = 6 + 3 = 9 && (7)\\
D_8(p,q) &= \max(|x_p - x_q|, |y_p - y_q|) = \max(6,3) = 6 && (8)
\end{aligned}
$$

The D_m distance is defined as the shortest m-path (path defined with m-adjacency) between the two points. In this case, the distance will depend on the values of the pixels along the path, as well as the values of their neighbours because only pixels in V can be adjacent.

$$
\begin{array}{c|ccccccc|ccccccc|c}
\cline{2-15}
0 & 0 & 0 & & 0 & & 0 & & 0 & & 0 & & 1 & \rightarrow & \boxed{1} & 0\\
 & & & & & & & & & & & \nearrow & & & & \\
1 & 0 & 0 & & 1 & & 0 & & 0 & & 1 & & 0 & & 0 & 1\\
 & & & & & & & & & & \uparrow & & & & & \\
1 & 0 & 0 & & 1 & & 0 & & 1 & \rightarrow & 1 & & 0 & & 0 & 0\\
 & & & & & & & \nearrow & & & & & & & & \\
0 & 0 & \boxed{1} & \rightarrow & 1 & \rightarrow & 1 & & 0 & & 0 & & 1 & & 1 & 1\\
\cline{2-15}
\multicolumn{1}{c}{0} & \multicolumn{1}{c}{0} & 1 & & 1 & & 1 & \multicolumn{1}{c}{} & 0 & & 0 & & 1 & & \multicolumn{1}{c}{1} & 1
\end{array}
$$

$$D_m(p,q) = 7 \qquad (9)$$

We could also calculate the lengths of the 4- and 8-paths between p and q. Note however that these are not the same as the D_4 and D_8 distances. In fact, what would happen in we calculated the 4- and 8-path lengths for the current example?

5.

The noisy image is now given as

$$\underbrace{g_i(x,y)}_{\text{noisy image}} = \underbrace{f(x,y)}_{\text{original image}} + \underbrace{\eta_i(x,y)}_{\text{noise}},$$

where noise has zero mean and it is uncorrelated.

When we average the noisy images

$$\bar{g}(x,y) = \frac{1}{M}\sum_{i=1}^{M} g_i(x,y)$$

we get for the expectation

$$E\{\bar{g}(x,y)\} = E\left\{\frac{1}{M}\sum_{i=1}^{M} g_i(x,y)\right\} = E\left\{\frac{1}{M}\sum_{i=1}^{M} f(x,y) + \frac{1}{M}\sum_{i=1}^{M}\eta_i(x,y)\right\} =$$
$$\frac{1}{M}\sum_{i=1}^{M} E\{f(x,y)\} + \frac{1}{M}\sum_{i=1}^{M}\underbrace{E\{\eta_i(x,y)\}}_{=0\text{ (zero mean)}} = f(x,y)$$

and for the variance

$$\sigma^2_{\bar{g}(x,y)} = E\left\{(\bar{g} - E\{\bar{g}\})^2\right\} = E\left\{\left(\frac{1}{M}\sum_{i=1}^{M}(f+\eta_i) - f\right)^2\right\} = E\left\{\left(\frac{1}{M}\sum_{i=1}^{M}\eta_i\right)^2\right\} =$$
$$\frac{1}{M^2}E\left\{\left(\sum_{i=1}^{M}\eta_i\right)^2\right\} = \frac{1}{M^2}E\left\{\sum_{i=1}^{M}\left(\eta_i^2 + \sum_{j=1,\ j\neq i}^{M}\eta_i\eta_j\right)\right\} =$$
$$\frac{1}{M^2}\left[\sum_{i=1}^{M}\left(\underbrace{E\{\eta_i^2\}}_{=\sigma_\eta^2} + \sum_{j=1,\ j\neq i}^{M}\underbrace{E\{\eta_i\eta_j\}}_{=0}\right)\right] = \frac{1}{M}\sigma_\eta^2,$$

since the noise had zero mean and it was uncorrelated.