7 - Colour theorem

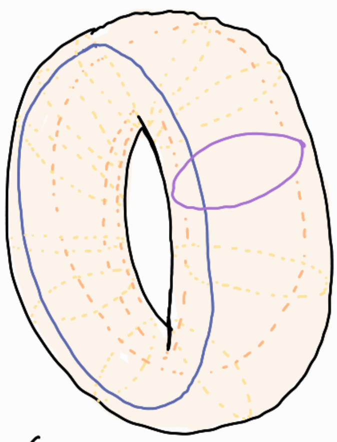
We know about 4 - colour theorem :

Any planar graph is four - colourable (vertex).

$\Rightarrow$ (considering dual of the graph) : no more than four colours are required to colour map on a plane

There is an analougas version for maps on torus

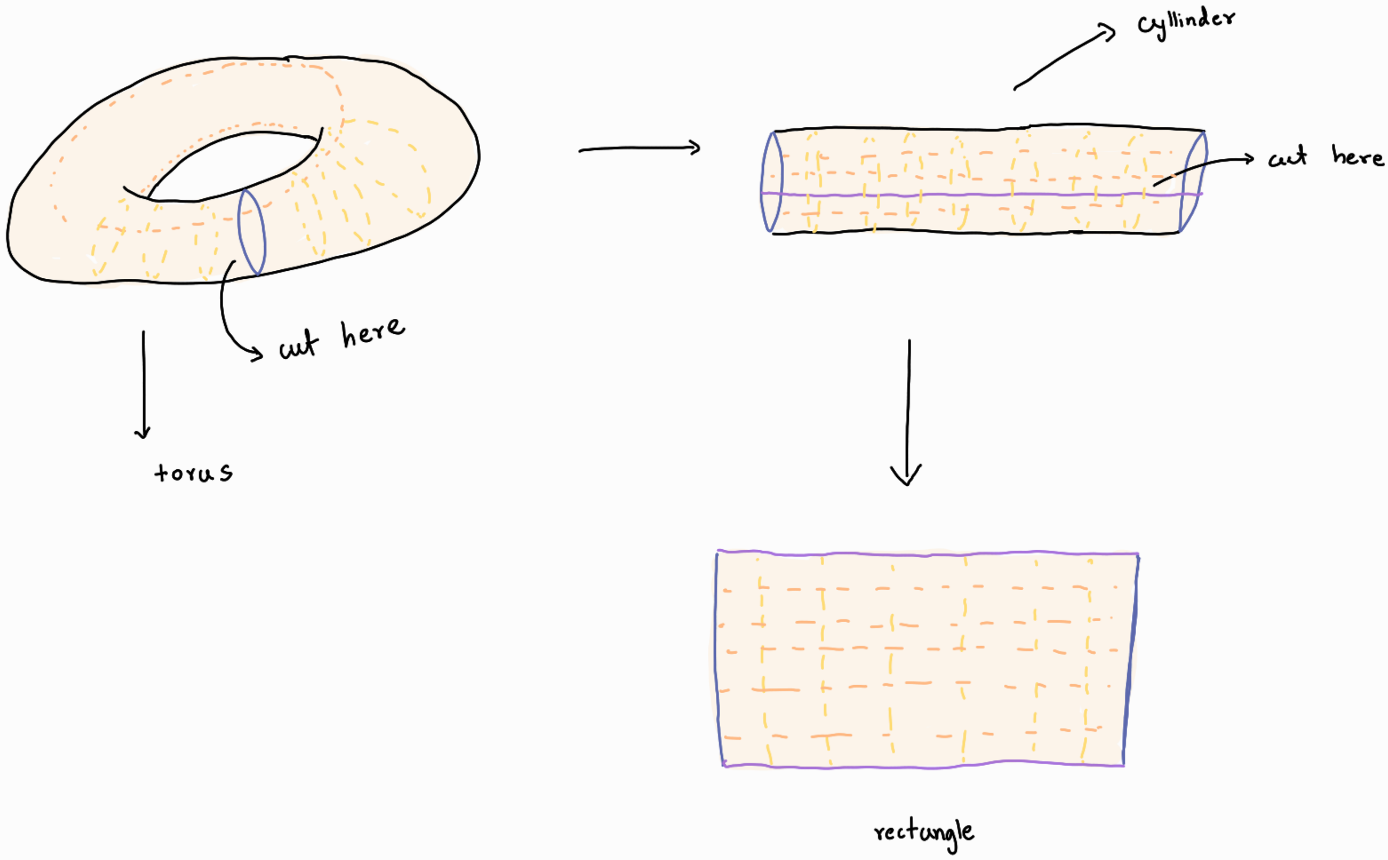
Def : A ring toras is homeomorphic to cartesian product of two
isomorphism , continuous inverse
circles $S^1 \times S^1$. It has genus 1 .
holes in the surface

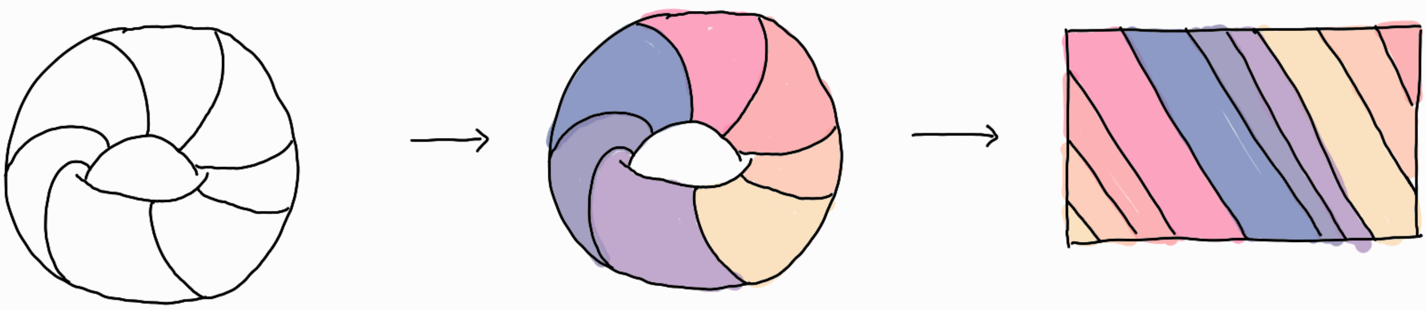


toras

Thm [Seven colour theorem] : We only need 7 colours to colour a graph on torus.

A conversion is needed :





We consider the surface as vertices. There is an edge

iff the two surfaces are neighbours.

Note that this graph would be a connected graph.

Note that, since each edge borders atmost two faces & each face borders at least 3 edges $\Rightarrow$ $3F \leq 2E$

Heawood

Thm : $V - E + F = 2 - 2g$, g is the no of holes in polyhedron
$\geq \chi$

proof: Induction to the planar case. ■

Main proof: let $\chi(G) = k$. Note $\delta(G) \geq k-1$.

Note $|V| \geq k$. Note $3f \leq 2e$

Note $V - E + F = \chi$ $\Rightarrow$ $V - \chi = E - F$ $\Rightarrow$ $V - \chi \geq \frac{E}{3}$

Note $\frac{E}{3} \geq \frac{1}{6}(k-1)n$ $\Rightarrow$ $1 - \frac{\chi}{n} \geq \frac{1}{6}(k-1)$ $\Rightarrow$ $1 - \frac{\chi}{k} \geq \frac{1}{6}(k-1)$

$\Rightarrow$ $k^2 - k \leq 6(k - \chi)$

$\Rightarrow$ $0 \geq k^2 - 7k + 6\chi$ $\Rightarrow$ $k \leq \left\lfloor \frac{7 + \sqrt{49 - 24\chi}}{2} \right\rfloor$

Torus has $\chi = 2-2g = 0$, so $k = 7$!

In Torus, bound 7 is minimum.