MA20104 Probability and Statistics

Hints/Solutions to Assignment No. 1

1. One empty cell can be selected in 7 ways. Now one cell will have two balls. That cell can be selected in 6 ways from the remaining 6 cells. Two balls can be selected out of 7 in $\binom{7}{2}$ ways. Remaining 5 balls can be placed in 5 cells (one in each cell) in 5! ways.

   Hence

   $$\text{Required probability } = \frac{7\times 6\times \binom{7}{2}\times 5!}{7^7} = \frac{2160}{16807} \approx 0.1285.$$

2. It is given that

   $$P(E\cap F) = P(E)P(F) \qquad \cdots(1)$$
   $$P(E\cap (F\cap G)) = P(E)P(F\cap G) \qquad \cdots(2)$$

   and

   $$P(E\cap (F\cup G)) = P(E)P(F\cup G) \qquad \cdots(3)$$

   Using distributive property of union and intersection in the left hand term of (3), we get

   $$P(E\cap F)\cup(E\cap G)) = P(E)P(F\cap G).$$

   Now applying the addition rule on both the sides and using equations (1) and (2) we get

   $$P(E\cap G) = P(E)P(G).$$

3. Let $A \rightarrow$ 4 appears, $B \rightarrow$ odd sum appears.

   Then $A = \{(1,\ 3),\ (3,1),\ (2,\ 2)\}$ has three elements.

   Also B will have 18 elements. So

   $$\text{P}(4 \text{ appears first}) = \frac{1}{12} + \frac{15}{36}.\frac{1}{12} + \left(\frac{15}{36}\right)^2.\frac{1}{12} + ... = \frac{1}{7}.$$

4. Let $E \to$ owns a desktop and $F \to$ owns a laptop

Given $\mathrm{P(E) = 0.5, P(F) = 0.25, P(E \cap F) = 0.1.}$

The required probability $= \mathrm{P(E \cap F^C) + P(E^C \cap F)}$
$= \mathrm{P(E) - P(E \cap F) + P(F) - P(E \cap F) = 0.55.}$

5. Let A $\to$ person is smoker, D $\to$ death due to lung cancer.

Given P(A) = 0.2, P(D) = 0.006. Let $P(D \mid A^C) = \alpha$. Then $P(D \mid A) = 10\,\alpha$.

Using the theorem of total probability

$P(D) = P(D \mid A)P(A) + P(D \mid A^C)P(A^C).$

This implies that $0.006 = 10\alpha \times 0.2 + \alpha \times 0.8.$

This gives $10\,\alpha = \dfrac{3}{140} = 0.0214.$

6. If the first number is between 2 and $n-1$, then the second number can be chosen in two ways for the two to be consecutive. If the first number is either 1 or n, then the second number can be chosen in one way only. So the total number of favourable cases $= (n-2) \times 2 + 2 \times 1 = 2(n-1).$

Hence the required probability is $\dfrac{2(n-1)}{n^2}$.

7. $P(A \mid B) = 1 \Rightarrow P(B) = P(A \cap B).$ Now by the Theorem of Total Probability $P(B) = P(A \cap B) + P(A^C \cap B)$ and so $P(A^C \cap B) = 0.$ Once again by the Theorem of Total Probability $P(A^C) = P(A^C \cap B) + P(A^C \cap B^C).$ Hence $P(A^C) = P(A^C \cap B^C).$ This gives $P(B^C \mid A^C) = 1.$

8. $P(B \mid A \cup B^c) = \dfrac{P(B \cap (A \cup B^c))}{P(A \cup B^c)} = \dfrac{P((B \cap A) \cup (B \cap B^C))}{P(A) + P(B^C) - P(A \cap B^C)} = \dfrac{P(A \cap B)}{(0.7 + 0.6 - 0.5)}.$

Also $P(A) = P(A \cap B) + P(A \cap B^C)$ gives $P(A \cap B) = 0.2.$

So the required probability $= 0.25.$

9. (i) Use Bayes theorem, Reqd prob. $= \dfrac{2(1-\alpha)}{2(1-\alpha) + 2\beta + 3\gamma}$.

(ii) Use theorem of total probability. Reqd prob. $= (\alpha + 2\beta + 3\gamma)/6.$

(iii) Use theorem of total probability,

P(digit 1 was received) = (2 - 2α + 2β + 3γ)/ 12,

P(digit 2 was received) = (α + 4 - 4β + 3γ)/ 12,

P(digit 3 was received) = (α + 2β + 6 - 6γ)/ 12.

10. Apply laws of probability to get

(i) False (ii) True (iii) False (iv) False

11. $\binom{13}{4}\Big/\binom{52}{4} = \frac{11}{4165} \cong 0.0026$.

12. P(getting at least 'A' in one semester in all subjects) = $\frac{1}{2^5} = \frac{1}{32}$.

So the reqd. prob. $= 1 - \left(\frac{31}{32}\right)^4 = 0.1193.$.

13. Use definition of the conditional probability to prove the result.

14. $P(X = i) = \binom{n}{i}\Big/2^n,\ i = 0, 1, ..., n.\ \ P(A \subset B \mid X = i) = 2^{i-n}.$

$$P(A \subset B) = \left(\frac{3}{4}\right)^n.$$

Also $P(A \cap B = \phi) = P(A \subset B^C).$

15. We consider the possible cases as follows for scoring at least 8 marks:

P(student scores at least 8 marks)

= P(scores 8 marks) + P(scores 9 marks) + P(scores 10 marks).

$$P(\text{scores 8 marks}) = \left(\frac{1}{2}\right)^6 \binom{4}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^2 + \binom{6}{1}\left(\frac{1}{2}\right)^6\binom{4}{3}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right) + \binom{6}{2}\left(\frac{1}{2}\right)^6\left(\frac{1}{4}\right)^4.$$

$$P(\text{scores 9 marks}) = \left(\frac{1}{2}\right)^6\binom{4}{3}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right) + \binom{6}{1}\left(\frac{1}{2}\right)^6\left(\frac{1}{4}\right)^4.$$

$$P(\text{scores 10 marks}) = \left(\frac{1}{2}\right)^6\left(\frac{1}{4}\right)^4.$$

Using these we get the required probability $\frac{5}{512} \approx 0.0098$.

16. Let X $\rightarrow$ number of girls qualifying, Y$\rightarrow$ number of boys qualifying.

$$p = P(X > Y) = P(X = n+1) + P(X = n+2) + \ldots + P(X = 2n)$$
$$= \left(\frac{1}{2}\right)^{2n}\left[\binom{2n}{n+1} + \ldots + \binom{2n}{2n}\right] = \left(\frac{1}{2}\right)^{2n}\left[\binom{2n}{0} + \ldots + \binom{2n}{n-1}\right]$$
$$= P(Y > X)$$

$$r = P(X = Y) = \binom{2n}{n}\left(\frac{1}{2}\right)^{2n}.$$

As $2p + r = 1$, we get $p = \frac{1}{2}\left\{1 - \left(\frac{1}{2}\right)^{2n}\binom{2n}{n}\right\}.$

17. Define, $A_i = i^{th}$ person gets back his own hat. So we need to find

$P($ No one gets back his own hat $)$

$$= P\left(\bigcap_{i=1}^{n} A_i^C\right) = 1 - P\left(\bigcup_{i=1}^{n} A_i\right) = 1 - \sum_{i=1}^{n}(-1)^{i-1}S_i\ ,$$

where $S_i = \binom{n}{i}\frac{(n-i)!}{n!}$ = Prob. that i persons will get back their own hats.

Expanding, we get the required probability as

$$1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n\frac{1}{n!}.$$

Taking limit as $n \to \infty$, the probability converges to $e^{-1} \approx 0.3679$.

$P($ At least one gets back his own hat $) = 1 - \left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n\frac{1}{n!}\right).$

Taking limit as $n \to \infty$, the probability converges to $1 - e^{-1} \approx 0.6321$.

18. We can mark the boxes as

1	2	......	r	r+1	......	R

Now a ball will be placed randomly in first r boxes out of total R boxes with probability $\frac{r}{R}$ and will be placed in one of the remaining $R-r$ boxes with probability $1-\frac{r}{R}$. So the required probability is $\binom{n}{k}\left(\frac{r}{R}\right)^k\left(1-\frac{r}{R}\right)^{n-k}$.

19. $$\frac{\binom{26}{3}\binom{26}{10}}{\binom{52}{13}} \approx 0.0217.$$

20. $$\frac{\binom{13}{3}\binom{13}{4}\binom{13}{4}\binom{13}{2}}{\binom{52}{13}} \approx 0.018.$$

21. Define Q_i = Queen of the ith suit drawn i= c, h, d, s.

Define K_i = King of the ith suit drawn i= c, h, d, s.

(a) $$\frac{1}{52^4}(4!) = \frac{4}{52}\frac{3}{52}\frac{2}{52}\frac{1}{52}$$

(b) $$\left(\frac{4}{52}\right)^4$$

22. 1st box will contain no ball. So we remove the first box.

From the remaining, one box must contain 2 balls in $\binom{n-1}{1}\binom{n}{2}$ ways.

Left (n-2) ball can be distributed in (n-2) boxes in (n-2)! ways. So the probability of interest is

$$\frac{\binom{n-1}{1}\binom{n}{2}(n-2)!}{n^n}$$

23. (a) $$\frac{n(n-1)^{r-1}}{n^r}$$

(b) Choose r people from n people and spread the rumour in any order. So the probability of interest is $\dfrac{{}^{n}C_{r}}{n^r}$.