Maxwell's Equations.

$\nabla \cdot \vec{D} = \rho_v$

$\nabla \cdot \vec{B} = 0$

$\nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$

$\nabla \times \vec{H} = \vec{J} + \dfrac{\partial \vec{D}}{\partial t}$.

Continuity Eqn:- $\nabla \cdot \vec{J} = -\dfrac{\partial \rho_v}{\partial t}$

In linear, homogeneous, isotropic medium,

$\vec{D} = \epsilon \vec{E}$

$\vec{B} = \mu \vec{H}$

$\vec{J} = \sigma \vec{E}$

Boundary conditions:-

$(\vec{E}_1 - \vec{E}_2) \times \hat{a}_n = 0$

$(\vec{H}_1 - \vec{H}_2) \times \hat{a}_n = \vec{K}$

$(\vec{D}_1 - \vec{D}_2) \cdot \hat{a}_n = \rho_s$

$(\vec{B}_2 - \vec{B}_1) \cdot \hat{a}_n = 0$

Potentials:- $\vec{B} = \nabla \times \vec{A}$

$\nabla \times \vec{E} = -\dfrac{\partial}{\partial t}(\nabla \times \vec{A})$

or $\nabla \times \left(\vec{E} + \dfrac{\partial \vec{A}}{\partial t}\right) = 0$

$\Rightarrow \vec{E} + \dfrac{\partial \vec{A}}{\partial t} = -\nabla V$

$\Rightarrow \vec{E} = -\nabla V - \dfrac{\partial \vec{A}}{\partial t}$.

Time - Harmonic Fields

$\vec{A} = \mathrm{Re}(A_s e^{j\omega t})$

$\dfrac{\partial \vec{A}}{\partial t} = \mathrm{Re}(j\omega A_s e^{j\omega t})$

$= j\omega A_s$.

$\int \vec{A}\, dt = \dfrac{A_s}{j\omega}$.

$\nabla \cdot \vec{D_s} = \rho_{vs}$

$\nabla \cdot \vec{B_s} = 0$

$\nabla \times \vec{E_s} = -j\omega \vec{B_s}$

$\nabla \times \vec{H_s} = \vec{J_s} + j\omega \vec{D_s}$

$$\nabla \cdot \vec{E} = \frac{\rho_v}{\varepsilon} \Rightarrow -\nabla^2 V - \frac{\partial}{\partial t}(\nabla \cdot \vec{A}) = \rho_v/\varepsilon.$$

$$\Rightarrow \nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \vec{A}) = -\rho_v/\varepsilon$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\Rightarrow \nabla \times \nabla \times \vec{A} = \mu \vec{J} + \varepsilon\mu \frac{\partial}{\partial t}\left(-\nabla V - \frac{\partial \vec{A}}{\partial t}\right)$$

$$= \mu \vec{J} - \mu\varepsilon \nabla\left(\frac{\partial}{\partial t} V\right) - \mu\varepsilon \frac{\partial^2 \vec{A}}{\partial t^2}$$

Using, $\nabla \times \nabla \times \vec{A} = \nabla(\nabla \cdot \vec{A}) - \vec{\nabla^2 A}$

$$\Rightarrow -\nabla(\nabla \cdot \vec{A}) + \vec{\nabla^2 A} = -\mu \vec{J} + \mu\varepsilon \nabla\left(\frac{\partial}{\partial t} V\right) + \mu\varepsilon \frac{\partial^2 \vec{A}}{\partial t^2}$$

Lorentz guage condition:—

$$\boxed{\nabla \cdot \vec{A} = -\mu\varepsilon \frac{\partial V}{\partial t}}$$

Thus, $\nabla^2 V - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = -\rho_v/\varepsilon$ & $\vec{\nabla^2 A} - \mu\varepsilon \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{J}$ } Wave Equations

$$V(t) = \iiint_V \frac{\rho_v(t')\, dv}{4\pi\varepsilon R} \quad \& \quad A_{x,y,z} = \iiint_V \frac{\mu J_{x,y,z}(t')\, dv}{4\pi R}$$

$$t' = t - \frac{|\vec{r} - \vec{r}'|}{u}, \quad u = \frac{1}{\sqrt{\mu\varepsilon}}$$

Lossy Dielectrics.

$\nabla \cdot \vec{E_s} = 0$

$\nabla \cdot \vec{H_s} = 0$

$\nabla \times \vec{E_s} = -j\omega\mu \vec{H_s}$

$\nabla \times \vec{H_s} = (\sigma + j\omega\varepsilon)\vec{E_s}$

$= \underbrace{j\omega\varepsilon\left[1 - \frac{j\sigma}{\omega\varepsilon}\right]}\vec{E_s}$

$\nabla \times \nabla \times \vec{E_s} = -j\omega\mu \, \nabla \times \vec{H_s}$

$= -j\omega\mu(\sigma + j\omega\varepsilon)\vec{E_s}$

$\Rightarrow \nabla(\underbrace{\cancel{\nabla \cdot \vec{E_s}}}_{0}) - \nabla^2 \vec{E_s} = \underbrace{-j\omega\mu(\sigma + j\omega\varepsilon)}_{\gamma^2}\vec{E_s}$

$\varepsilon\left[1 - j\frac{\sigma}{\omega\varepsilon}\right]$

$= \varepsilon' - j\varepsilon''$

(complex permittivity)

$\Rightarrow \nabla^2 \vec{E_s} - \gamma^2 \vec{E_s} = 0$, Similarly $\nabla^2 \vec{H_s} - \gamma^2 \vec{H_s} = 0$ } Helmholtz's Eqns.

$\gamma = \alpha + j\beta$, $\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]}$, $\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right]}$, $v = \frac{\omega}{\beta}$

(Propagation const)

For wave propagating along $+\hat{a}_z$, $\vec{E_s}$ with x-component,

$\frac{\partial^2 E_{xs}(z)}{\partial z^2} - \gamma^2 E_{xs}(z) = 0 \Rightarrow E_{xs}(z) = E_0 e^{-\gamma z} + \underbrace{E_0' e^{+\gamma z}}_{\text{wave travelling } -\hat{a}_z}$

Thus, $E_{xs}(z) = E_0 e^{-\gamma z} = E_0 e^{-\alpha z} e^{-j\beta z}$.

$\boxed{\eta_0 = 120\pi}$ in free-space

$\vec{E}(z,t) = E_0 e^{-\alpha z} \cos(\omega t - \beta z)\, \hat{a}_x$

Using $\nabla \times \vec{E_s} = -j\omega\mu \vec{H_s}$, $\vec{H}_{ys}(z) = H_0 e^{-\gamma z}$, $H_0 = \frac{E_0}{\eta}$, $\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}$ (Intrinsic wave-impedance).

Ratio of conduction current density $\vec{J_c}$ to ~~the~~ displacement current density $\vec{J_d}$.

$$\frac{|\vec{J_c}|}{|\vec{J_d}|} = \frac{|\sigma \vec{E_s}|}{|j\omega\varepsilon \vec{E_s}|} = \frac{\sigma}{\omega\varepsilon} = \tan\theta \quad \text{(Loss Tangent)}.$$

A good dielectric (low loss) $\Rightarrow$ $\sigma \ll \omega\varepsilon$, $\tan\theta \ll 1$.

Good Conductors

$\sigma \to \infty$, $\varepsilon = \varepsilon_0$, $\mu = \mu_0\mu_r$. , $\frac{\sigma}{\omega\varepsilon} \gg 1$

$\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f \mu\sigma}$.

$\vec{E} = E_0 e^{-\alpha z} \cos(\omega t - \beta z)$

magnitude reduces by $\frac{1}{e}$ i.e. 37%.

$v = \frac{\omega}{\beta} = \sqrt{\frac{2\omega}{\mu\sigma}}$

Over a distance 'δ', $|\vec{E}| = E_0 e^{-\alpha\delta} = E_0 e^{-1}$

$\eta = \sqrt{\frac{j\omega\mu}{\sigma}} = \sqrt{\frac{\omega\mu}{\sigma}} \angle 45^\circ$

$\Rightarrow \delta = \frac{1}{\alpha}$ (Skin depth)

$= \frac{1}{\sqrt{\pi f \mu\sigma}}$

For a width 'w', length 'l',

$R_{ac} = \frac{1}{\sigma}\frac{l}{\delta w} = R_s \frac{l}{w}$, R_s (Surface Resistance) $= \frac{1}{\sigma\delta} = \sqrt{\frac{\pi f\mu}{\sigma}}$.

Poynting Vector

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

$$\nabla \times \vec{H} = \sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t}$$

$$\vec{E} \cdot (\nabla \times \vec{H}) = \sigma E^2 + \vec{E} \cdot \varepsilon \frac{\partial \vec{E}}{\partial t}$$

Using, $\nabla \cdot (\vec{H} \times \vec{E}) = \vec{E} \cdot (\nabla \times \vec{H}) - \vec{H} \cdot (\nabla \times \vec{E})$

$$\Rightarrow \underbrace{\vec{H} \cdot (\nabla \times \vec{E})}_{} + \nabla \cdot (\vec{H} \times \vec{E}) = \sigma E^2 + \frac{1}{2} \varepsilon \frac{\partial}{\partial t} E^2$$

$$\vec{H} \cdot \left(-\mu \frac{\partial \vec{H}}{\partial t}\right) = -\frac{\mu}{2} \frac{\partial}{\partial t} H^2$$

$$\Rightarrow \nabla \cdot (\vec{E} \times \vec{H}) = -\frac{\mu}{2} \frac{\partial}{\partial t} H^2 - \sigma E^2 - \frac{1}{2} \varepsilon \frac{\partial E^2}{\partial t}.$$

$$\underbrace{\iiint_V \nabla \cdot (\vec{E} \times \vec{H})}_{\oiint_S (\vec{E} \times \vec{H}) \cdot \vec{ds}} = \underbrace{-\frac{\partial}{\partial t} \iiint_V \left[\frac{1}{2} \varepsilon E^2 + \frac{1}{2} \mu H^2\right] dv}_{} - \underbrace{\iiint_V \sigma E^2 dv.}_{}$$

POYNTINGS Theorem.

$\oiint_S (\vec{E} \times \vec{H}) \cdot \vec{ds}$: Total power leaving the volume

rate of decrease in energy stored in electric & magnetic fields.

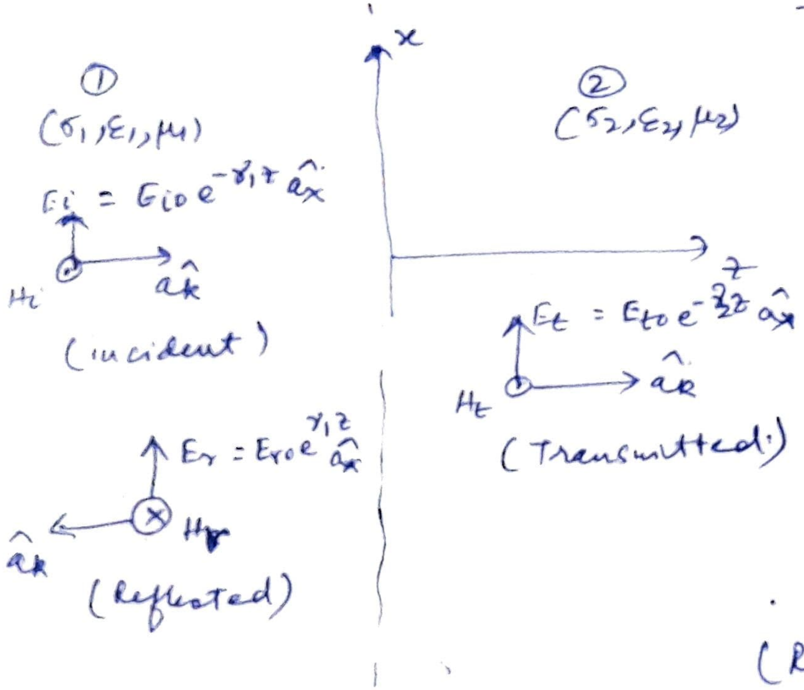
ohmic power dissipated

Poynting vector
$\vec{P} = \vec{E} \times \vec{H}$

$$\vec{P}_{ave}(z) = \frac{1}{T} \int_0^T \vec{P}(z,t)\, dt = \frac{1}{2} \mathrm{Re}\{E_s \times H_s^*\}$$

Total time-avg. power crossing a surface 'S' is $P_{ave} = \iint_S \vec{P}_{ave} \cdot \vec{ds}$.

REFLECTION



At interface $(z=0)$

$E_i(0) + E_r(0) = E_t(0) \Rightarrow E_{i0} + E_{r0} = E_{t0}$

$H_i(0) + H_r(0) = H_t(0) \Rightarrow \frac{1}{\eta_1}(E_{i0} - E_{r0}) = \frac{E_{t0}}{\eta_2}$

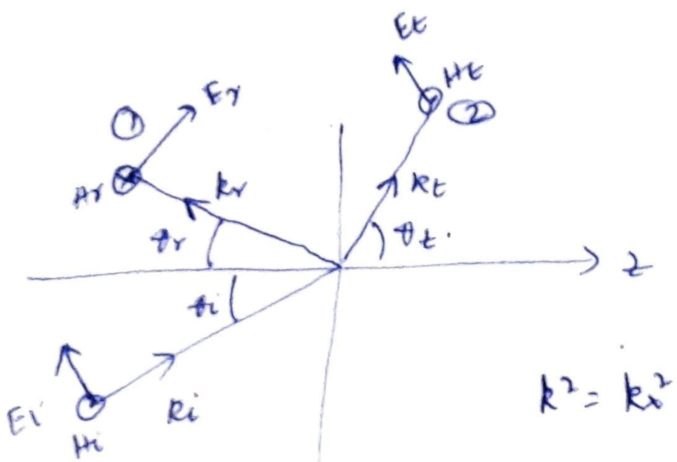
$\Rightarrow E_{r0} = E_{i0} \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}, \quad E_{t0} = E_{i0} \frac{2\eta_2}{\eta_1 + \eta_2}.$

$\Gamma = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$, (Reflection coeff.)

$T = \frac{2\eta_2}{\eta_1 + \eta_2}$ (Transmission coeff.)

$\boxed{1 + \Gamma = T}$

$\boxed{0 \le |\Gamma| \le 1}$



$\vec{E}(r,t) = \vec{E}_0 \cos(\vec{k} \cdot \vec{r} - \omega t)$ (uniform plane wave)

$\vec{r} = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$

$\vec{k} = k_x \hat{a}_x + k_y \hat{a}_y + k_z \hat{a}_z$ (propagation vector)

$k^2 = k_x^2 + k_y^2 + k_z^2 = \omega^2 \mu \varepsilon$

$E_i = E_{i0} \cos(k_{ix} x + k_{iy} y + k_{iz} z - \omega t)$, $E_t = E_{t0} \cos(k_{tx} x + k_{ty} y + k_{tz} z - \omega t)$

$E_r = E_{r0} \cos(k_{rx} x + k_{ry} y + k_{rz} z - \omega t)$

For B.C.) $E_i(z=0) + E_r(z=0) = E_t(z=0)$

$\Rightarrow$ $k_{ix} = k_{rx} = k_{tx} = k_x$
$k_{iy} = k_{ry} = k_{ty} = k_y$ } phase-matching condition.

$\Rightarrow \vec{k_i}, \vec{k_r}, \vec{k_t}$ must all lie in the plane of incidence.

$\Rightarrow$ $k_i \sin\theta_i = k_r \sin\theta_r$, $k_i = k_r = \beta_1 = \omega\sqrt{\mu_1\varepsilon_1}$ $\Rightarrow$ $\theta_i = \theta_r$

$k_i \sin\theta_i = k_t \sin\theta_t$, $k_t = \beta_2 = \omega\sqrt{\mu_2\varepsilon_2}$ $\Rightarrow$ $\dfrac{\sin\theta_t}{\sin\theta_i} = \dfrac{k_i}{k_t} = \sqrt{\dfrac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}}$. Snell's Law.

$\Rightarrow \cos\theta_t = \sqrt{1 - \frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}\sin^2\theta_i}$

Parallel Polarizn:- $\vec{E}$ lies in x-z plane of incidence.

$$\vec{E_{is}} = E_{io}(\cos\theta_i\,\hat{a_x} - \sin\theta_i\,\hat{a_z})\,e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\vec{H_{is}} = \frac{E_{io}}{\eta_1}\,e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}\,\hat{a_y}$$

$$\vec{E_{rs}} = E_{ro}(\cos\theta_i\,\hat{a_x} + \sin\theta_i\,\hat{a_z})\,e^{-j\beta_1(x\sin\theta_i - z\cos\theta_i)}$$

$$\vec{H_{rs}} = -\frac{E_{ro}}{\eta_1}\,e^{-j\beta_1(x\sin\theta_i - z\cos\theta_i)}\,\hat{a_y}$$

$$\vec{E_{ts}} = E_{to}(\cos\theta_t\,\hat{a_x} - \sin\theta_t\,\hat{a_z})\,e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

$$\vec{H_{ts}} = \frac{E_{to}}{\eta_2}\,e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)}\,\hat{a_y}$$

Tangential at boundary

$$E_{io}\cos\theta_i + E_{ro}\cos\theta_i = E_{to}\cos\theta_t$$

$$\frac{1}{\eta_1}(E_{io} - E_{ro}) = \frac{1}{\eta_2}E_{to}$$

$$\Gamma_{\parallel} = \frac{E_{ro}}{E_{io}} = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}$$

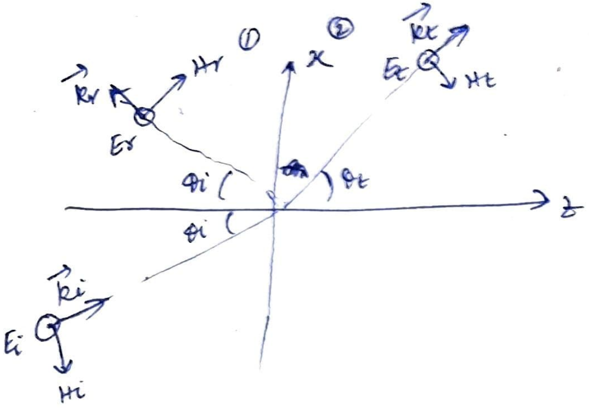
$$T_{\parallel} = \frac{E_{to}}{E_{io}} = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}$$

$$1 + \Gamma_{\parallel} = T_{\parallel}\left(\cos\theta_t/\cos\theta_i\right)$$

$\Gamma_{\parallel} = 0$ (i.e. $E_{ro} = 0$) at an angle θ_B (Brewster angle / polarizing angle)

$$\Rightarrow \eta_2 \cos\theta_t = \eta_1 \cos\theta_B \quad \Rightarrow \quad \sin^2\theta_B = \frac{1 - \frac{\mu_2 \epsilon_1}{\mu_1 \epsilon_2}}{1 - \left(\frac{\epsilon_1}{\epsilon_2}\right)^2}.$$

Perpendicular Polarizn. - $\vec{E}$ field is perpendicular to the plane of incidence.



$$\vec{E}_{is} = E_{io}\, e^{-j\beta_1 (x \sin\theta_i + z\cos\theta_i)}\, \hat{a}_y$$

$$\vec{H}_{is} = \frac{E_{io}}{\eta_1} (-\cos\theta_i\, \hat{a}_x + \sin\theta_i\, \hat{a}_z)\, e^{-j\beta_1 (x\sin\theta_i + z\cos\theta_i)}$$

$$\vec{E}_{rs} = E_{ro}\, e^{-j\beta_1 (x \sin\theta_i - z\cos\theta_i)}\, \hat{a}_y$$

$$\vec{H}_{rs} = \frac{E_{ro}}{\eta_1} (\cos\theta_i\, \hat{a}_x + \sin\theta_i\, \hat{a}_z)\, e^{-j\beta_1 (x\sin\theta_i - z\cos\theta_i)}.$$

$$\vec{E}_{ts} = E_{to}\, e^{-j\beta_2 (x\sin\theta_t + z\cos\theta_t)}\, \hat{a}_y$$

$$\vec{H}_{ts} = \frac{E_{to}}{\eta_2} (-\cos\theta_t\, \hat{a}_x + \sin\theta_t\, \hat{a}_z)\, e^{-j\beta_2 (x\sin\theta_t + z\cos\theta_t)}$$

B.C:-

- $E_{io} + E_{ro} = E_{to}$
- $\frac{1}{\eta_1}(E_{io} - E_{ro})\cos\theta_i = \frac{1}{\eta_2} E_{to} \cos\theta_t$

$$\Rightarrow \Gamma_{\perp} = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$$

$$\& \; T_{\perp} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}.$$

$$1 + \Gamma_{\perp} = T_{\perp}$$

Brewster angle (θ_B) for $\Gamma_{\perp} = 0 \Rightarrow \eta_2 \cos\theta_B = \eta_1 \cos\theta_t \Rightarrow \sin^2\theta_B = \dfrac{1 - \frac{\mu_1 \epsilon_2}{\mu_2 \epsilon_1}}{1 - \left(\frac{\mu_1}{\mu_2}\right)^2}$