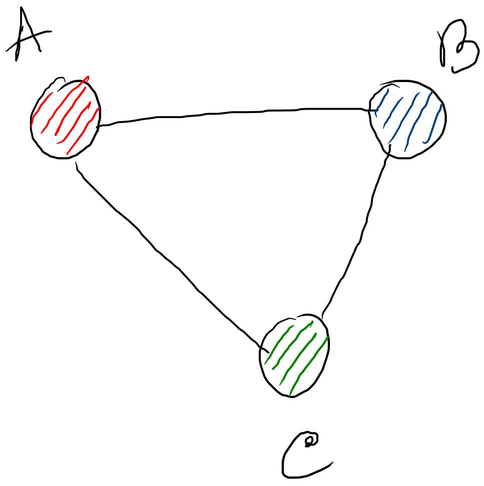
Graph coloring

K-coloring

The concept of coloring vertices is a very important application in graph theory.

A K-coloring is a partition of V into K sets such that each of the K sets are disjoint and no two vertices in the same set are adjacent to each other. A graph which has a K-coloring but no (K-1)-coloring is called K-colorable

$G = \langle V, E \rangle$



an odd cycle.

This graph cannot be colored using less than 3 colors.

Anecdotal example

A special kind of social graph

$V = \{ \text{man}, \text{woman} \}$ $\quad E = \{ \text{a sexual contact} \}$

If we are not able to color this network using just two colors (Man = , Woman =) then this network has same-sex relationships.

Odd cycle in the graph then → conclude that there exists same-sex relationships in the graph.

Consider a non-null bipartite graph. The two partitions be denoted as V_1 & V_2. In such bipartite graphs V_1 can have one color (///) and V_2 can have another color (///). A bipartite is a 2-colorable graph.

whether a graph is 2-colorable $\rightarrow$ whether the graph is bipartite $\rightarrow$ whether the graph has an odd cycle or not.

If there is an odd cycle then we will need at least three colors to color the graph and therefore such a graph cannot be 2-colorable & therefore cannot be a bipartite graph.

Theorem: G is bipartite iff it has no odd cycle.

Direct: Let us say $G = \langle V, E \rangle$ is a bipartite graph. Also let $V = A \cup B$ s.t. $A \cap B = \phi$ and edges run from nodes in A to nodes in B only.

Suppose G has at least one odd cycle. Let the length of the cycle be 'n' and let $C = (v_1, v_2, \ldots, v_n, v_1)$.

Without any loss of generality,
let $v_1 \in A$, $v_2 \in B$, $v_3 \in A$ ... ...

$$\forall k \in \{1, \ldots, n\}$$

$$v_k \in \begin{cases} A & : k \text{ odd} \\ B & : k \text{ even} \end{cases}$$

But as n is odd, $v_n \in A$ and also we started with $v_1 \in A$. Now since $v_n v_1 \in C$ therefor $(v_n, v_1) \in E$ which the contradicts with our assumption that G is bipartite.

Reverse: G has even cycles.
Let v_0 be any vertex. For each vertex $v \in G$ let $d(v)$ denote the length of the shortest path from v_0 to v. Color red every vertex of G whose shortest distance from v_0 is even. Color the other vertices with blue. Now if there is an edge between any pair of red nodes or any pair of blue nodes then an odd cycle will be formed. Thus G must be bipartite and the blue & red nodes defines the two partitions. $\square$

Use BFS to identify if odd cycles are present in a graph (very simple modification of the original BFS pseudocode that work)

→ homework.

↓ no odd cycles in the graph then the graph is bipartite.