FSM minimisation

Limitations of FSMs

Computing $2^p \times 2^p$

- Result: 2^{2p} i.e. 1 followed by $2p$ zeroes, length $2p + 1$
- Let the FSM have n states
- Input length: $p + 1$
- As the numbers are input (serially) $2p + 1$ zeroes should be output
- Next, in $p - 1$ zeroes should be output and finally a 1
- Let $p > n$, the m/c must loop through the states to generate the $p - 1$ zeroes
- The required 1 will never be generated

State equivalence

Basic notions

Distinguishable states
: S_i and S_j of M are distinguishable if and only if a finite input sequence applied to M produce distinct output sequence, depending of whether M is in S_i or S_j

Distinguishing sequence for S_i and S_j
: A sequence that allows S_i and S_j to be distinguished

k-distinguishable
: S_i and S_j are k-distinguishable is there exists a sequence of length k to distinguish S_i and S_j

k-equivalent
: No sequence of length k to distinguish S_i and S_j

Equivalent states
: S_i and S_j are equivalent if they are k-equivalent for all k; for n state m/c sufficient if k-equivalent for all $k \leq n - 1$ (in view of the limited memory of FSMs)

Minimisation of completely specified FSMs

Sample m/c and partition table

FSM specification

PS	NS, output	
	$x = 0$	$x = 1$
A	E,0	D,1
B	F,0	D,0

Equivalence partition table

k	Partitions
P_0	(ABCDEF)
P_1	
P_2	

C	E,0	B,1
D	F,0	B,0
E	C,0	F,1
F	B,0	C,0

P_3	
P_4	

Equivalence partition is unique

- Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two distinct equivalence partitions
- There there must be two states s_i and s_j in the same partition of $\mathcal{P}_1$ but in different partitions of $\mathcal{P}_2$
- Since those are in different partitions in $\mathcal{P}_1$, those are distinguishable and cannot be in the same equivalence partition of $\mathcal{P}_2$
- Thus, the two equivalence partitions cannot be distinct
- By corollary, the equivalence partition $\mathcal{P}$ is minimum — a distinct equivalence partition $\mathcal{P}'$ will not exist

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Minimisation of incompletely specified FSMs

Sample incompletely specified FSMs

FSM specification: M_A

PS	NS, output	
	$x=0$	$x=1$
A	C,1	E,—
B	C,—	E,1
C	B,0	A,1
D	D,0	E,1
E	D,1	E,0

FSM specification: M_B

PS	NS, output	
	$x=0$	$x=1$
A	B,1	—
B	—,0	C,0
C	A,1	B,0

FSM specification: M_C

PS	NS, output	
	$x=0$	$x=1$
A	B,1	T,—
B	T,0	C,0
C	A,1	B,0
T	T,—	T,—

FSM specification: M_D

PS	NS, output			
	I_1	I_2	I_3	I_4

FSM specification: M_E

PS	NS, output	
	I_1	I_2

FSM specification: M_F

PS	NS, output			
	I_1	I_2	I_3	I_4

A	—	C,1	E,1	B,1
B	E,0	—	—	—
C	F,0	F,1	—	—
D	—	—	B,1	—
E	—	F,0	A,0	D,1
F	C,0	—	B,0	C,1

A	E,0	B,0
B	F,0	A,0
C	E,—	C,0
D	F,1	D,0
E	C,1	C,0
F	D,—	B,0

A	—	D,0	D,0	C, —
B	A,1	—	—	D, —
C	B,1	C,0	E,0	—
D	E,1	C,0	—	D,0
E	—	—	—	A,1

Basic notions

- For an completely specified m/c two states s_i and s_j of a machine M are compatible if and only if, for every input sequence the same output sequence will be produced regardless of whether s_i or s_j is the initial state
- However, for incompletely specified m/cs the next state many not be defined
- An input sequence is said to be applicable to a m/c if its state transitions can be definitely determined for a given input sequence (next state for each input symbol is specified)
- Behaviour of the m/cs may be observed when applicable sequences are applied when the m/c starts with either s_i or s_j as the initial state
- Outputs need not be specified for an applicable sequence; they need to match when specified

Compatible states

Two states s_i and s_j of a machine M are compatible if and only if, for every input sequence applicable to both s_i and s_j, the same output sequence will be produced whenever both output symbols are specified and regardless of whether s_i or s_j is the initial state

Compatibility issues

- Let s_i, s_j and s_k be states whose compatibilities are to be determined
- Let s_i^x and s_j^x be their x—successors on application of applicable sequence x
- What to do if the output of s_i^x is specified (as o_a) but that of s_j^x is not specified?
  - Since it is unspecified, it can be set to the specified output (o_a) so that s_i^x and s_j^x may be merged (consider merging of B and C of M_E)
- What to do if the output of s_k^x is specified (as o_b, $o_a \neq o_b$) but that of s_j^x is not specified, assuming that it is decided to merge s_i^x and s_j^x?
  - State s_j^x may be split; let s' be resultant of splitting s_j^x; now the output of s' can be set to the specified output (o_b) so that s_k^x and s' may be merged (consider merging of D and C of M_E)
- What to do if, on input a, the next state of s_i^x is specified (as s_a) but that of s_j^x is not specified?
  - Since it is unspecified, it can be set to the specified state (s_a) so that s_i^x and s_j^x may be merged
- What to do if, on input a, the next state of s_k^x is specified (as s_a', not compatibile with s_a) but that of s_j^x is not specified, assuming that it is decided to merge s_i^x and s_j^x?

- State s_j^x may be split; let s' be resultant of splitting s_j^x; now the next state of s' can be set to the specified state (s'_a) so that s_k^x and s' may be merged

Merger table

Table construction

- If s_i and s_j have output conflict on the next state, they are incompatible, indicated by a × in M_{ij}
  - States A and D have output conflict on I_1
  - States B and E have output conflict on I_1
- For each pair of states $\langle s_i, s_j \rangle$, M_{ij} indicates the required compatibilities of states
  - Compatibility of states A and B contingent of compatibility of E and F, also compatibility of A and B
  - Compatibility of states B and F contingent of compatibility of F and F, also compatibility of A and B
- Merger table is symmetric
  - Sufficient to have entries for s_i and s_j for $i > j$
- May be equivalently represented by a merger graph

FSM specification: M_E

PS	NS, output	
	I_1	I_2
A	E,0	B,0
B	F,0	A,0
C	E,—	C,0
D	F,1	D,0
E	C,1	C,0
F	D,—	B,0

Merger table

	A	B	C	D	E
B	EF, AB				
C	BC, EE	AC,EF			
D	×	×	EF, CD		
E	×	×	CE, CC	CD,CF	
F	DE, BB	~~AB,DF~~	BC,DE	~~BD, FD~~	BC,CD

Table simplification

Unsatisfiable dependencies (direct)

DF depending of compatibility of BD

Entries with (direct) unsatisfiable dependencies are crossed

Unsatisfiable dependencies (transitive)

AF depending of compatibility of DF

Entries with (transitive) unsatisfiable dependencies are crossed

Self dependencies

AB depending on AB, CD depending on CD, FD depending on FD

Self dependencies are dropped

Reflexive dependencies

Depending on compatibility of E with E, B with B
Reflexive dependencies are dropped

Merger table after simplification

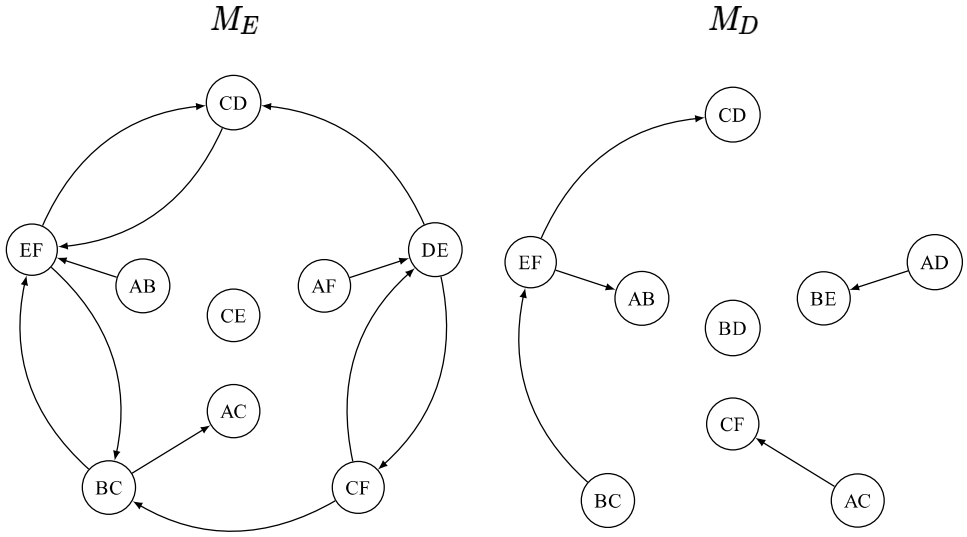
Merger table

	A	B	C	D	E
B	EF				
C	BC	AC,EF			
D	×	×	EF		
E	×	×	✓	CD,CF	
F	DE	×	BC,DE	×	BC,CD

Compatibility graph

- Compatibility dependencies indicated in the merger table are depicted in the compatibility graph
- Each for each M_{ij} in the reduced merger table that is not crossed out, there is a node in compatibility graph
- If $\langle s_p, s_q \rangle \in M_{ij}$, there is a directed edge from the node for $\langle s_i, s_j \rangle$ to the node for $\langle s_p, s_q \rangle$, note that $\langle s_i, s_j \rangle \equiv \langle s_j, s_i \rangle$

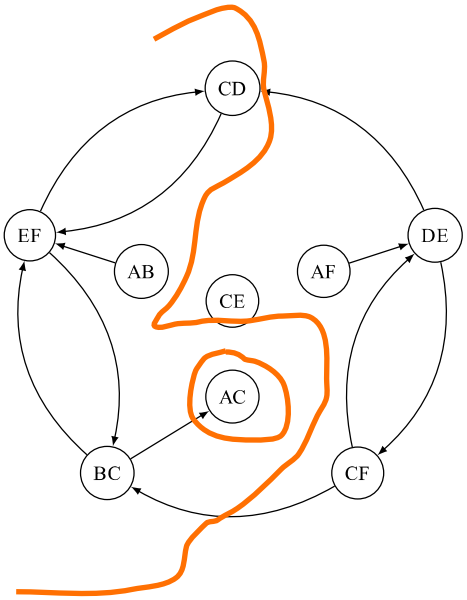
Compatibility graphs



Closed set of compatibiles

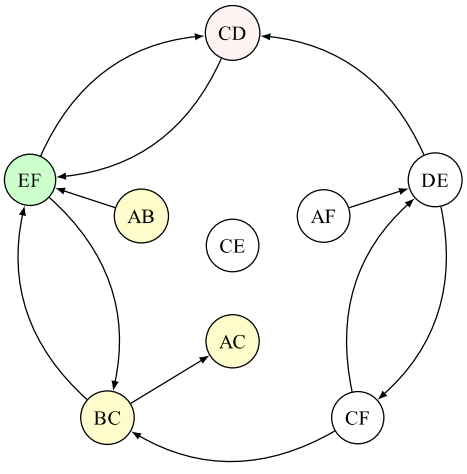
- It is desirable to merge compatibile states
- For M_E, states C and E are compatibile and may be merged

- Similarly, states A and C are compatibile and may be merged
- However, states A and B are compatibile subject to E and F being compatibile; to merge A and B, E and F must also be merged
- States E and F are compatibile subject to compatibility of BC and CD; to merge EF, BC and CD must also be merged
- States B and C are compatibile subject to compatibility of AC; to merge BC, AC must also be merged
- Just CE may be merged without having to merge any other states, then there will be 5 states: A, B, CE, D, F so that all states are covered
- AB, EF, CD, BC and AC may be merged without having to merge any other states, then there will be states: AB, EF, CD, BC and AC, covering all states
    - However, AB, BC and AC together form a clique and may be merged to ABC
    - Then after, merging there will be only 3-states: ABC, CD and EF
    - Note that C had to be split between CD and ABC
- Thus, closed sets of compatibles need to be systematically (through branch and bound) explored and costed
- Cliques in a closed set of compatibiles may be combined to a single state
- Each closed set of compatible needs to be costed to retain the closed set of compatibles (after clique reduction) of least cosst
- The reduced FSM (NS and output function tables) need to be constructed



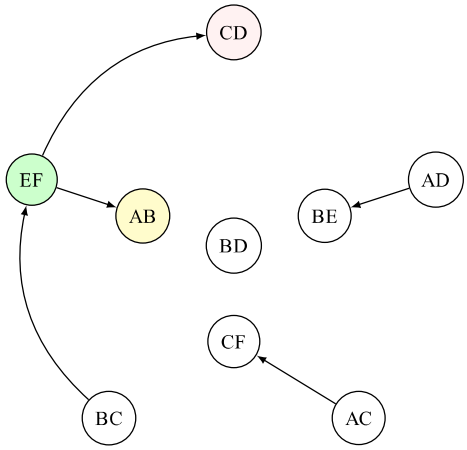
Costing of closed set of compatibiles for M_E

Sl	Closed set of compatibiles	Cost
1	A, B, CE, D, F	5
2	AB, EF, CD, BC, AC	4
3	ABC, EF, CD	3
4	DE, CD, EF, BC, AC, CF	6
5	DE, CD, EF, ABC, CF	5
6	CDE, EF, ABC, CF	4



Reduced FSM for M_E

PS		NS, output	
Members	Symbol	I_1	I_2
ABC	α	δ,0	α,0
CD	γ	δ,1	γ,0
EF	δ	γ,1	α,0



Reduced FSM for M_D

PS		NS, output			
Members	Symbol	I_1	I_2	I_3	I_4
AB	α	γ,0	β,1	γ,1	α,1
CD	β	γ,0	γ,1	α,1	—
EF	γ	β,0	γ,0	α,0	β,1

Exercises

- Present the pseudocode for finding the minimum cost closed set of compatibles
- Work out the state minimisation for M_D and M_F