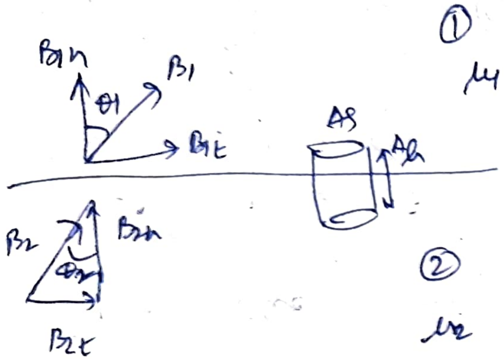
Magnetic Boundary Conditions

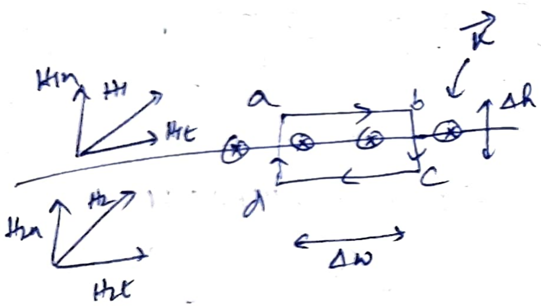


$$\oint \vec{B}\cdot d\vec{s} = 0 \Rightarrow B_{1n}\,\Delta s - B_{2n}\,\Delta s = 0$$

$$\Rightarrow \boxed{B_{1n} = B_{2n}} \quad \text{or} \quad \boxed{\mu_1 H_{1n} = \mu_2 H_{2n}}$$

$$\oint \vec{H}\cdot d\vec{l} = I \Rightarrow K\cdot \Delta w = H_{1t}\cdot\Delta w + H_{1n}\cdot\frac{\Delta h}{2} + H_{2n}\cdot\frac{\Delta h}{2}$$

$$- H_{2t}\cdot\Delta w - H_{2n}\cdot\frac{\Delta h}{2} - H_{1n}\cdot\frac{\Delta h}{2}$$



$$\Rightarrow H_{1t} - H_{2t} = K$$

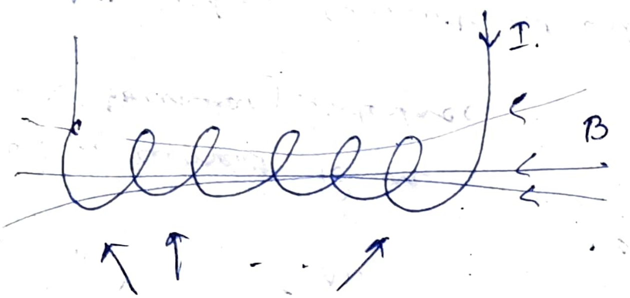
i.e. $(\vec{H}_1 - \vec{H}_2)\times \hat{a}_{n12} = \vec{K}$, $\hat{a}_{n12}$: Unit vector normal to the interface, directed from medium-1 to medium-2.

Surface current $\vec{K}$ on the boundary interface.

For no surface currents at interface,

$$\left.\begin{aligned} B_1\cos\theta_1 &= B_2\cos\theta_2 \\ K \quad \frac{B_1}{\mu_1}\sin\theta_1 &= \frac{B_2}{\mu_2}\sin\theta_2 \end{aligned}\right\} \Rightarrow \boxed{\frac{\tan\theta_1}{\tan\theta_2} = \frac{\mu_1}{\mu_2}}$$

Inductors.



N - identical turns.

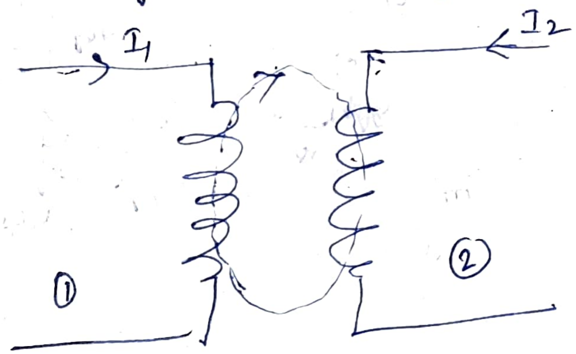
Current I produces a magnetic field $\vec{B}$.

Flux $\psi = \iint \vec{B} \cdot \vec{ds}$ through each turn

Flux linkage $\lambda = N\psi$.

(Self) Inductance $= \dfrac{\lambda}{I} = \dfrac{N\psi}{I}$. (Henry / webers/Amp).

Magnetic Energy stored in inductor $W_m = \frac{1}{2} L I^2$.



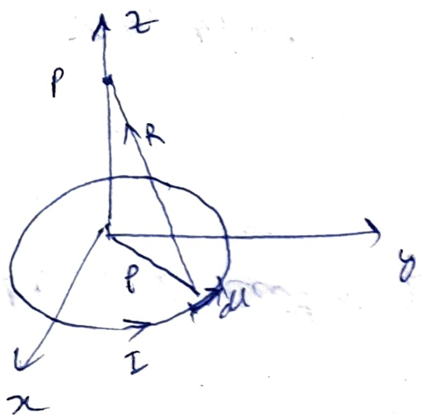
$\psi_{12} \triangleq$ Flux passing through circuit-1 due to current I_2 in circuit-2.

$= \iint_{S_1} \vec{B_2} \cdot \vec{ds}$

(For linear medium) Also, $M_{12} = M_{21}$.

(Mutual) Inductance $M_{12} = \dfrac{\lambda_{12}}{I_2} = \dfrac{N_1 \psi_{12}}{I_2}$, $M_{21} = \dfrac{\lambda_{21}}{I_1}$

SOLENOID



A current-loop

$\vec{dl} = \rho\, d\phi\, \hat{a}_\phi \qquad \vec{R} = (0,0,h) - (x,y,0) = -\rho \hat{a}_\rho + h\hat{a}_z$

$P(0,0,h)$.

$$\vec{dl} \times \vec{R} = \begin{vmatrix} \hat{a}_\rho & \hat{a}_\phi & \hat{a}_z \\ 0 & \rho\, d\phi & 0 \\ -\rho & 0 & h \end{vmatrix} = \rho h\, d\phi\, \hat{a}_\rho + \rho^2 d\phi\, \hat{a}_z$$

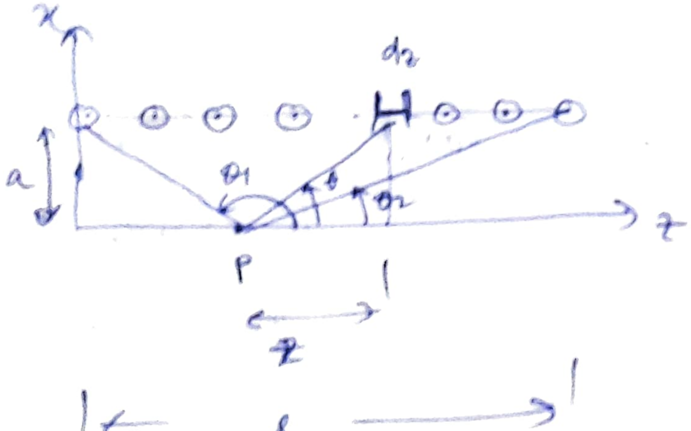
$$\vec{dH} = \frac{I\, \vec{dl} \times \vec{R}}{4\pi R^3} = \frac{I}{4\pi [\rho^2 + h^2]^{3/2}} (\rho h\, d\phi\, \hat{a}_\rho + \rho^2 d\phi\, \hat{a}_z)$$

$$= dH_\rho\, \hat{a}_\rho + dH_z\, \hat{a}_z.$$

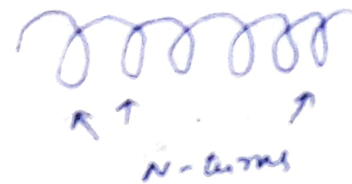
Note, $\hat{a}_\rho = \cos\phi\, \hat{a}_x + \sin\phi\, \hat{a}_y$

Integrating, $\cos\phi$ or $\sin\phi$ over $0 \le \phi \le 2\pi$ gives zero, thereby $H_\rho = 0$.

Thus, $\vec{H} = \int dH_z\, \hat{a}_z = \dfrac{I\rho^2 \cdot \cancel{2\pi}}{\cancel{4\pi}_2 [\rho^2 + h^2]^{3/2}}\, \hat{a}_z$



cross-section of solenoid



$n = \frac{N}{l}$ (turns/length)

Also $\tan\theta = a/z$

$dz = -a\,\mathrm{cosec}^2\theta\, d\theta$

$$dH_z = \frac{a^2 I (n\,dz)}{2[a^2+z^2]^{3/2}}$$

$$= -\frac{nI}{2}\sin\theta\, d\theta$$

$$H_z = -\frac{nI}{2}\int_{\theta_1}^{\theta_2} \sin\theta\, d\theta = \frac{nI}{2}(\cos\theta_2 - \cos\theta_1)$$

At the center of the solenoid, $\cos\theta_2 = -\cos\theta_1 \Rightarrow \vec{H} = \frac{I n l}{2[a^2 + l^2/4]^{1/2}}\hat{a}_z$

$$= \frac{l/2}{[a^2+l^2/4]^{1/2}}$$

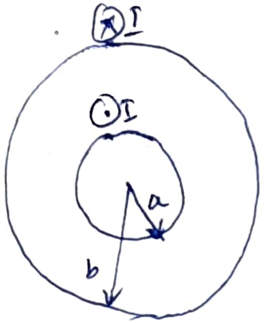
If $l \gg a$, $\theta_2 \simeq 0°$, $\theta_1 \simeq 180°$, $\boxed{\vec{H} = nI\,\hat{a}_z}$

$B = \mu H = \mu I n$, $\psi = BS = \mu I n(\pi a^2)$,

Inductance/length $(L') = \frac{\lambda'}{I} = \boxed{\mu n^2(\pi a^2)}$ H/m

Linkage/length $(\lambda') = \frac{\lambda}{l} = n\psi = \mu n^2 I(\pi a^2)$

Co-axial cable.



For $0 \le \rho \le a$, $\vec{B}_1 = \dfrac{\mu I \rho}{2\pi a^2}\, \hat{a}_\phi$

For $a \le \rho \le b$, $\bar{B}_2 = \dfrac{\mu I}{2\pi\rho}\, \hat{a}_\phi$

Internal Inductance :- Inductance produced by the flux internal to the conductor.
(Lint)

External Inductance :- Inductance produced by the flux external to the conductor.
(Lext)

Total inductance $L = L_{int} + L_{ext}$.

$$d\psi_1 = B_1\, d\rho\, dz.$$

$$= \frac{\mu I \rho}{2\pi a^2}\, d\rho\, dz.$$

(Int.) Flux linkage $d\lambda_1 = d\psi_1 \cdot \dfrac{I_{encl.}}{I}$

$$= d\psi_1 \cdot \frac{\pi\rho^2}{\pi a^2}.$$

$$= \frac{\mu I \rho}{2\pi a^2}\,\frac{\rho^2}{a^2}\, d\rho\, dz.$$

$$\lambda_1 = \int_0^a \int_{z=0}^{l} \frac{\mu I \rho^3}{2\pi a^4}\, d\rho\, dz = \frac{\mu I l}{8\pi} \quad \text{(for length '}l\text{' of the cable)}$$

$$L_{int} = \frac{\lambda_1}{I} = \boxed{\frac{\mu l}{8\pi}}$$

Internal Inductance/length $= \mu/8\pi$ H/m.

$$d\psi_2 = B_2\, d\rho\, dz = \frac{\mu I}{2\pi\rho}\, d\rho\, dz.$$

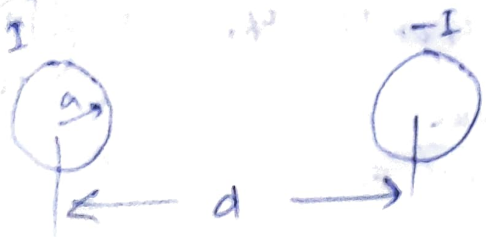
In this case the total current I is enclosed within the path enclosing the flux.

Extn. Flux linkage $\lambda_2 = \psi_2 = \displaystyle\int_{\rho=a}^{b}\int_{z=0}^{l} \frac{\mu I}{2\pi\rho}\, d\rho\, dz = \frac{\mu I l}{2\pi} \ln(b/a)$

$$L_{ext} = \frac{\lambda_2}{I} = \boxed{\frac{\mu l}{2\pi} \ln b/a.}$$

$$L_{tot}/\text{length} = \frac{\mu}{8\pi} + \frac{\mu}{2\pi} \ln b/a.$$

Two-wire Transmission line (length l)



For region $0 \le \rho \le a$, $\lambda_1 = \frac{\mu I l}{8\pi}$.

For region $a \le \rho \le d-a$, $\lambda_2 = \Psi_2 = \int_{\rho=a}^{d-a} \int_{z=0}^{l} \frac{\mu I}{2\pi\rho} \, d\rho \, dz = \frac{\mu I l}{2\pi} \ln \frac{d-a}{a}$.

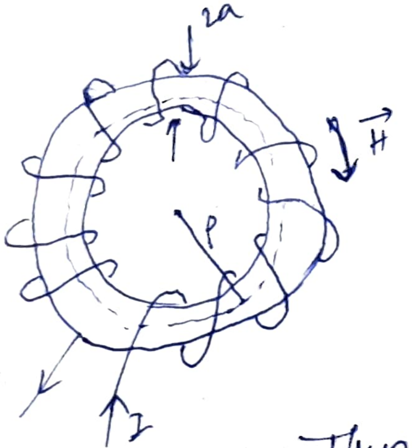
Flux linkages produced by wire 1 are $\lambda_1 + \lambda_2$.

By symmetry, the same amount of flux produced by current $-I$ in wire 2

The total linkages are $\lambda = 2(\lambda_1 + \lambda_2)$

For $d \gg a$, self-inductance / length $= \frac{\mu}{\pi}\left[\frac{1}{4} + \ln \frac{d}{a}\right]$ H/m.

TOROID



N-turns, carrying current I., mean radius of toroid $= \rho_0$.

$\oint \vec{H} \cdot \vec{dl} = I_{enclosed} \Rightarrow H_\phi \cdot 2\pi\rho = NI$

(As N-wires cut through the Amperian path each carrying current I)

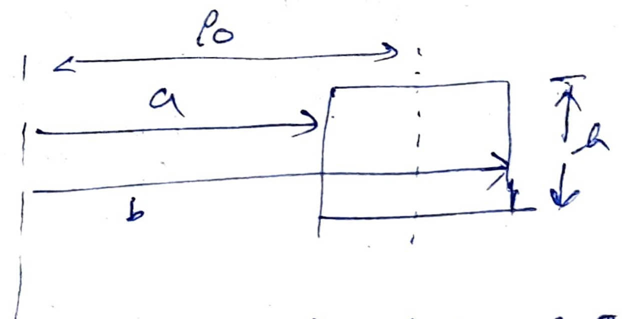
Thus, $H_\phi = \frac{NI}{2\pi\rho}$, $\rho_0 - a < \rho < \rho_0 + a$.

At the center of toroid, $H_\phi = \frac{NI}{2\pi\rho_0}$.

Flux through a single turn is $\iint \vec{B} \cdot \vec{da} = \frac{\mu_0 NI}{2\pi} h \int_a^b \frac{1}{\rho} d\rho = \frac{\mu_0 NIh}{2\pi} \ln(b/a)$

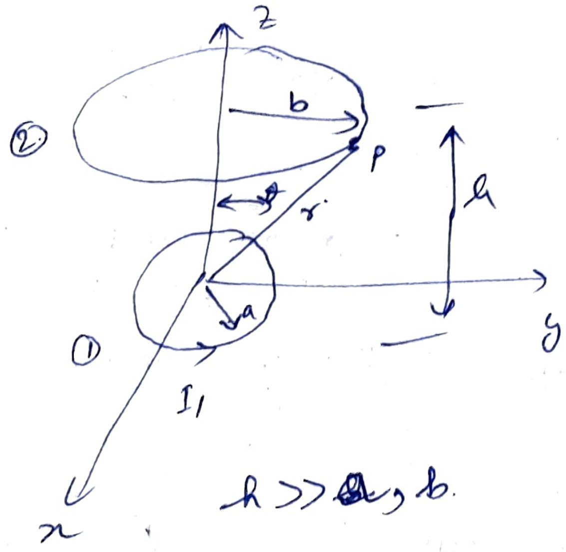
Flux linkage $\lambda = N \cdot \left(\frac{\mu_0 NIh}{2\pi} \ln(b/a) \right)$

Self-inductance $L = \frac{\mu_0 N^2 h}{2\pi} \ln(b/a)$



Toroid-cross-section

Coaxial circular wires.



$h >> a, b.$

let I_1 current flow in wire-1.

At P on wire-2, $\vec{A_1} = \frac{\mu I_1 a^2 \sin\theta}{4r^2} \hat{a_\phi}$

$= \frac{\mu I_1 a^2 b}{4[h^2+b^2]^{3/2}} \hat{a_\phi} \simeq \frac{\mu I_1 a^2 b}{4h^3} \hat{a_\phi}$

$\psi_{12} = \oint \vec{A_1} \cdot \vec{dl_2} \simeq \frac{\mu I_1 a^2 b}{4h^3} 2\pi b = \frac{\mu \pi I_1 a^2 b^2}{2h^3}.$

Mutual Inductance $(M_{12}) = \frac{\psi_{12}}{I_1} = \boxed{\frac{\mu \pi a^2 b^2}{2h^3}}.$

Magnetic Circuits

- Used to analyze magnetic devices — toroids, transformers, motors, generators, relays.

- Magnetomotive force (m.m.f.) $\mathcal{F} = NI = \oint \vec{H} \cdot \vec{dl}$ (Ampere-turns). $\longleftrightarrow$ $V = El$ (e.m.f.)

Reluctance ($\mathcal{R}$) $= \dfrac{l}{\mu S}$, (Ampereturns/Weber) $\longleftrightarrow$ R (Resistance)

Permeance ($\mathcal{P}$) $= \dfrac{1}{\mathcal{R}}$ $\longleftrightarrow$ G (Conductance)

Ohm's Law :- $\mathcal{F} = \psi \mathcal{R}$ $\longleftrightarrow$ $V = IR$

KCL :- $\Sigma \psi = 0$ $\longleftrightarrow$ $\Sigma I = 0$

KVL :- $\Sigma \mathcal{F} - \Sigma \mathcal{R} \psi = 0$ $\longleftrightarrow$ $\Sigma V - \Sigma IR = 0$

For n-magnetic circuits in series

$\psi_1 = \psi_2 = \cdots \psi_n$.

$\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2 + \cdots + \mathcal{F}_n$.

For n-magnetic circuits in parallel

$\psi = \psi_1 + \psi_2 + \cdots + \psi_n$.

$\mathcal{F}_1 = \mathcal{F}_2 = \cdots = \mathcal{F}_n$.

Energy in Magnetic Fields.

Magnetic Energy stored in an inductor $= \frac{1}{2} L I^2$
(W_m)

flux ϕ through the loop $= \iint_S \vec{B} \cdot \vec{da} = \iint_S (\nabla \times \vec{A}) \cdot \vec{da} = \oint_L \vec{A} \cdot \vec{dl} = LI$

$$\therefore \; W_m = \frac{1}{2} I \oint \vec{A} \cdot \vec{dl} = \frac{1}{2} \oint (\vec{A} \cdot \vec{I}) \, dl$$

$$= \frac{1}{2} \iiint (\vec{A} \cdot \vec{J}) \, dv$$

$$= \frac{1}{2\mu_0} \iiint \vec{A} \cdot (\nabla \times \vec{B}) \, dv$$

Using, $\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$

$$= \frac{1}{2\mu_0} \left[\iiint_V \vec{B} \cdot \vec{B} \, dv - \oint_S (\vec{A} \times \vec{B}) \cdot \vec{ds} \right]$$

'S' is the surface bounding the volume 'V'.

Choose the volume $V \to \infty$, $\vec{A} \propto \frac{1}{r}$, $\vec{B} \propto \frac{1}{r^2}$, $\vec{ds} \propto r^2$, thus $\oint_S (\vec{A} \times \vec{B}) \cdot \vec{ds} \longrightarrow 0$,

$$W_m = \frac{1}{2\mu_0} \iiint_{\text{All space}} |\vec{B}|^2 dv$$

Energy density $= \frac{B^2}{2\mu_0} = \frac{1}{2} (\vec{B} \cdot \vec{H})$