Types of Random Variables

A random variable is said to be discrete if it takes finite or countable number of values.
For example, number of wins in a series of games, number of children in a family, no of deaths in a hospital,

no of wickets taken etc.

If the random variable takes value over an interval it is said to be a continuous r.v. For example, length of a string, age of a person, life of a bulb etc.

To describe prob. $dist^n$ of a r.v. we have following cases:

1. **Probability Mass Function** :

If a r.v. X is discrete and it takes values $x_1, x_2, \ldots \in \mathcal{X}$, its prob. distⁿ is described by a fn. called prob. mass fn. (pmf) $p_X(\cdot)$ satisfying :

1. $P(X = x_i) = p_X(x_i)$, $x_i \in \mathcal{X}$

2. $0 \leq p_X(x_i) \leq 1 \quad \forall\, x_i \in \mathcal{X}$

3. $$\sum_{x_i \in \mathcal{X}} p_X(x_i) = 1$$

Example: Consider tossing of two fair dice. Let X = sum on the two dice

$X \to 2, 3, \ldots\ldots, 12$

The pmf of X

$p_X(2) = P(X=2) = \frac{1}{36}$, $p_X(3) = \frac{2}{36}$

$p_X(4) = \frac{3}{36}$, $p_X(5) = \frac{4}{36}$, $p_X(6) = \frac{5}{36}$, $p_X(7) = \frac{6}{36}$

$p_X(8) = \frac{5}{36}$, $p_X(9) = \frac{4}{36}$, $p_X(10) = \frac{3}{36}$, $p_X(11) = \frac{2}{36}$

$p_X(12) = \frac{1}{36}$

<u>Example</u>: Suppose there are 5 ATMs

in an office & two are working. A person randomly selects three machines. Let X be the no of working machines in his selection. Find the prob distn of X.

$X \rightarrow 0, 1, 2$

W	NW
2	3

$$p_X(0) = P(X=0) = \frac{{}^3C_3}{{}^5C_3}$$

$$= \frac{1}{10}$$

$$p_X(1) = \frac{{}^3C_2 \cdot {}^2C_1}{{}^5C_3} = \frac{6}{10}.$$

$$p_X(2) = \frac{{}^3C_1 \cdot {}^2C_2}{{}^5C_3}$$

$$= 3/10$$

$p_X(0) = \frac{1}{10}$, $p_X(1) = \frac{6}{10}$, $p_X(2) = \frac{3}{10}$

<u>Continuous R.V.</u> : The prob. distn of a continuous r.v. X is described by a prob. density function $f_X(x)$ satisfying

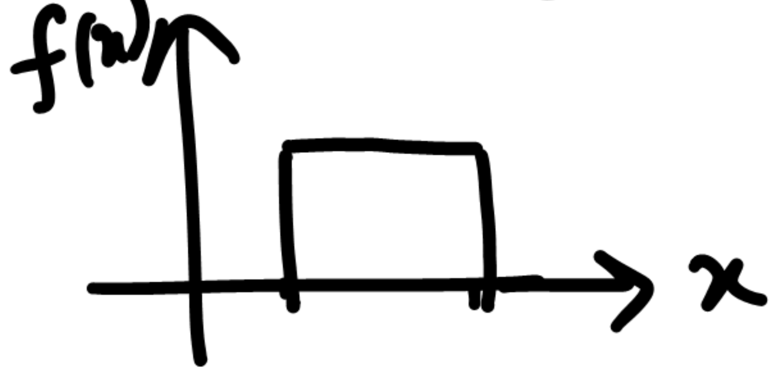
(i) $f_X(x) \geq 0 \quad \forall \ x \in \mathbb{R}$

(ii) $\displaystyle\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

(iii) $P(a < X < b) = \int_a^b f_X(x)\,dx$

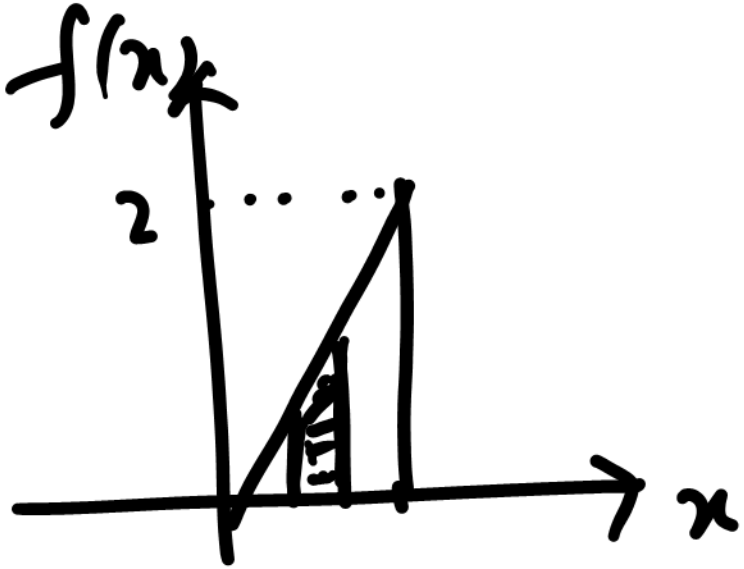
Examples: $f_X(x) = \begin{cases} \frac{1}{3}, & 1 < x < 4 \\ 0, & \text{otherwise} \end{cases}$

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_1^4 \frac{1}{3}\,dx = \frac{4-1}{3} = 1$$



$$P(2<X<3) = \int_2^3 \frac{1}{3}\,dx = \frac{1}{3}$$

2. $$f_X(x) = \begin{cases} 2x, & 0<x<1 \\ 0, & \text{otherwise} \end{cases}$$



$$\int_0^1 2x\,dx = 1$$

$$P\left(\frac{1}{4}<X<\frac{1}{3}\right) = \int_{1/4}^{1/3} 2x\,dx = \frac{1}{9} - \frac{1}{16} = \frac{7}{16}.$$

For a continuous r.v. the prob of a point is zero ie

$$P(X = c) = 0 \quad \forall\ c \in \mathbb{R}$$

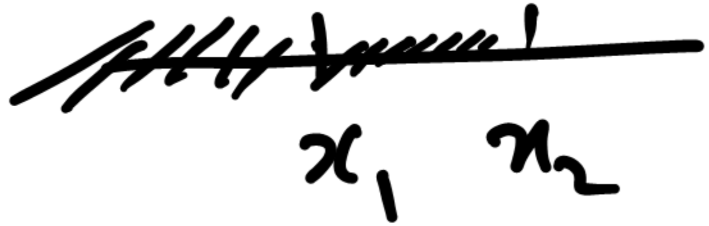
<u>Cumulative Distribution Function of a R.V.</u> :

For a r.v. X, cdf $F_X(x)$ is defined as

$$F_X(x) = P(\{\omega : X(\omega) \le x\})$$

$$= P(X \le x)$$

Properties of CDF :



(i) $\lim\limits_{x \to -\infty} F(x) = 0$

(ii) $\lim\limits_{x \to \infty} F(x) = 1$

(iii) If $x_1 < x_2$, $F(x_1) \le F(x_2)$

ie F is a monotonic non-decreasing fn.

(iv) F is continuous from right at every point ie.

$$\lim_{h \to 0} F(x+h) = F(x)$$

Conversely if a function F satisfies the above four properties, then it is cdf of a r.v. X.

If X is a discrete r.v. then the relationship between pmf & cdf is

$$F_X(x) = \sum_{x_i \le x} p_X(x_i)$$

$$p_X(x_i) = F_X(x_i) - F_X(x_{i-1})$$

If X is a continuous r.v. then

the relationship between pdf & cdf is

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt \qquad \&$$

$$\frac{d}{dx} F_X(x) = f_X(x) \quad \text{for almost all } x.$$

For dice problem $X \to$ sum

$$
F_X(x) = \begin{cases} 0, & x < 2 \\ \frac{1}{36}, & 2 \leq x < 3 \\ \frac{3}{36}, & 3 \leq x < 4 \\ 6/36, & 4 \leq x < 5 \\ 10/36, & 5 \leq x < 6 \\ 15/36, & 6 \leq x < 7 \\ 21/36, & 7 \leq x < 8 \end{cases}
$$

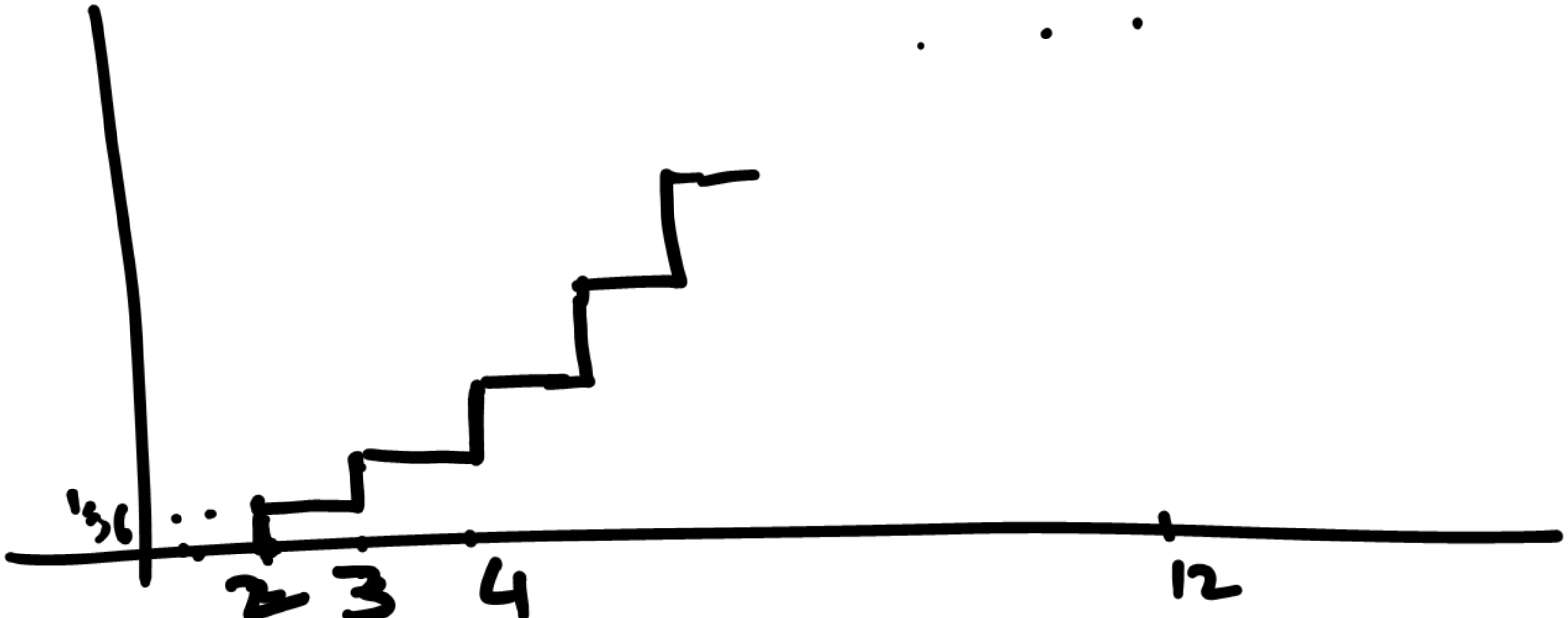
$= 26/36, \qquad 8 \leq x < 9$

$= 30/36, \qquad 9 \leq x < 10$

$= 33/36, \qquad 10 \leq x < 11$

$= 35/36, \qquad 11 \leq x < 12$

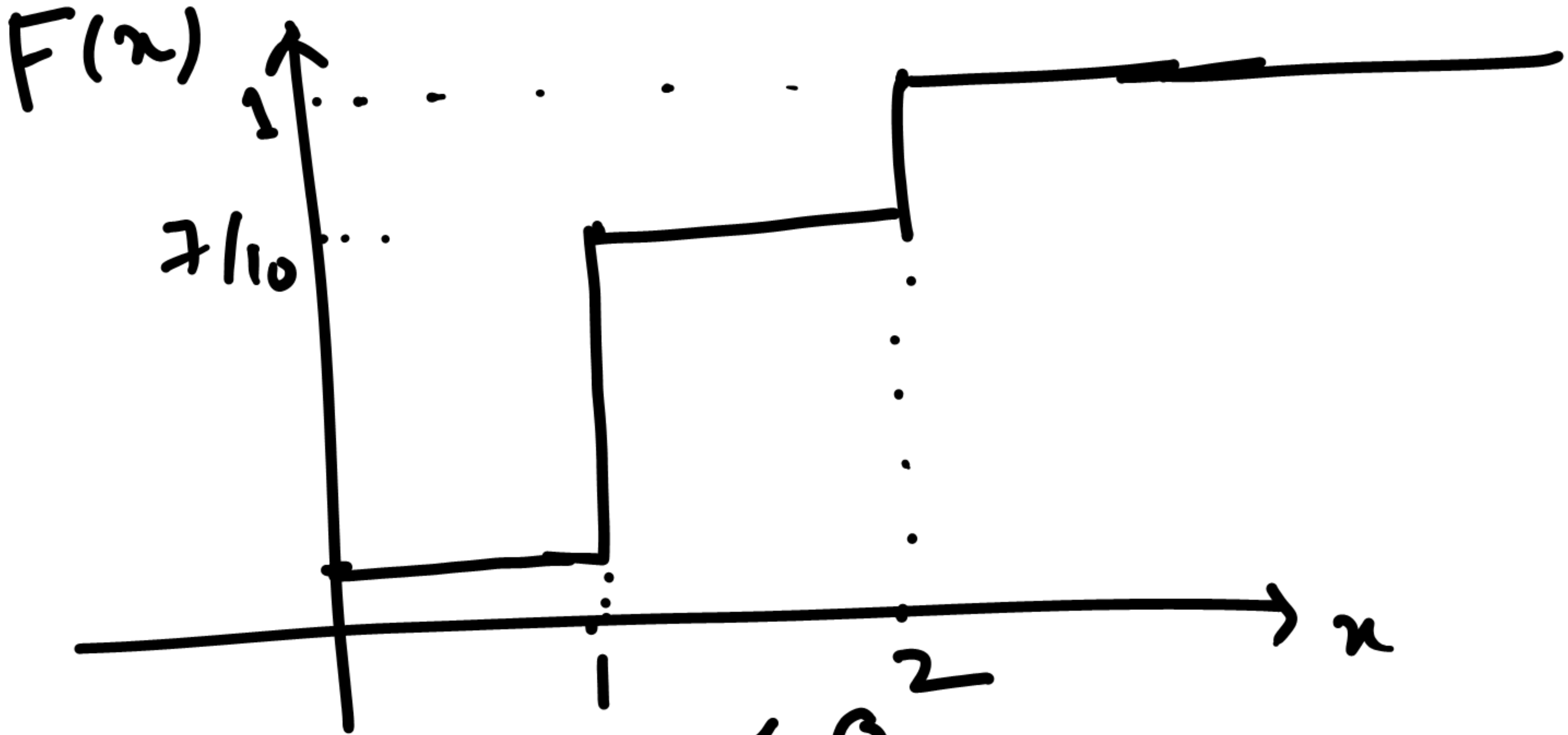
$= 1, \qquad x \geq 12$



So the cdf of a discrete r.v. is a step-function and the size of discontinuity at finite or countably infinite no. of points determine the pmf.

2 → no of working machines

$p_X(0) = 1/10, \quad p_X(1) = 6/10, \quad p_X(2) = 3/10$



$$F_X(x) = 0, \quad x < 0$$
$$= 1/10, \quad 0 \le x < 1$$

$$= 7/10, \quad 1 \leq x < 2$$

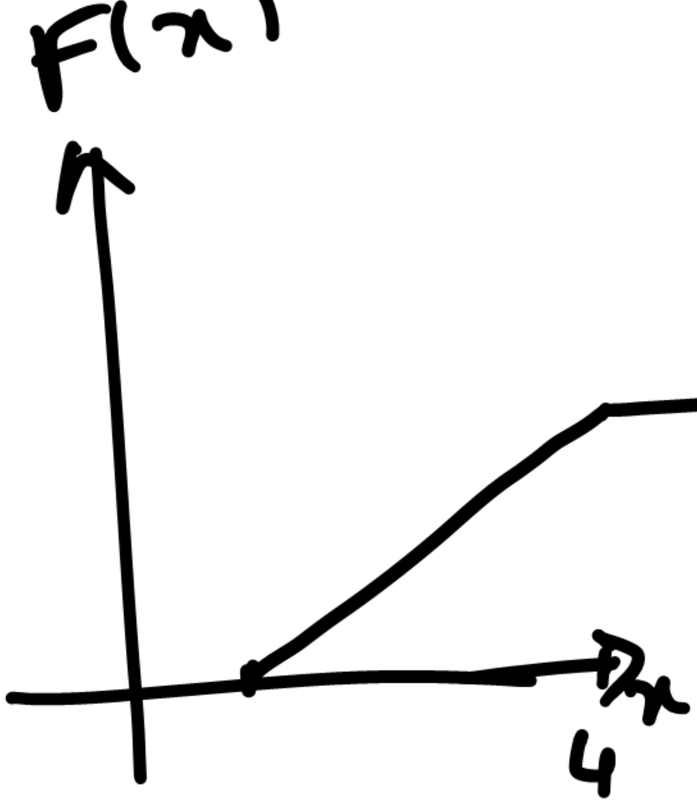
$$= 1, \quad x \geq 2$$

$$f_X(x) = \begin{cases} \frac{1}{3}, & 1 < x < 4, \\ 0, & \text{otherwise}. \end{cases}$$

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt$$

$$= 0, \quad x < 1$$

$$= \int_1^x \frac{1}{3}\,dt \qquad 1 \le x < 4$$



$$= \frac{x-1}{3}, \qquad 1 \le x < 4$$

$$= \int_1^4 \frac{1}{3}\,dt = 1. \qquad x \geq 4$$

Thus

$$F_X(x) = \begin{cases} 0, & x < 1 \\ (x-1)/3, & 1 \le x < 4 \\ 1, & x \geq 4 \end{cases}$$

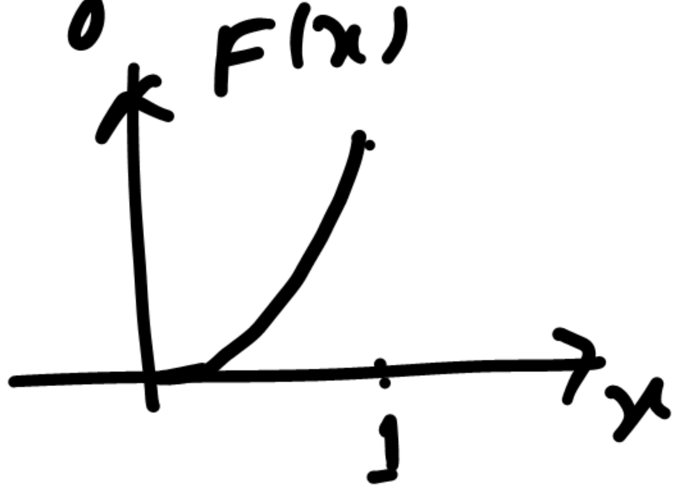
$$\frac{d}{dx} F_X(x) = \begin{cases} 0, & x < 1 \\ 1/3, & 1 < x < 4 \\ 0, & x > 4 \end{cases}$$

$$f_X(x) = \begin{cases} \frac{1}{3}, & 1 < x < 4 \\ 0, & \text{otherwise} \end{cases}$$

$$f_X(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \begin{cases} 0, & x \leq 0 \\ \int_0^x 2t\,dt, & 0 < x \leq 1 \\ \int_0^1 2t\,dt, & x \geq 1 \end{cases}$$

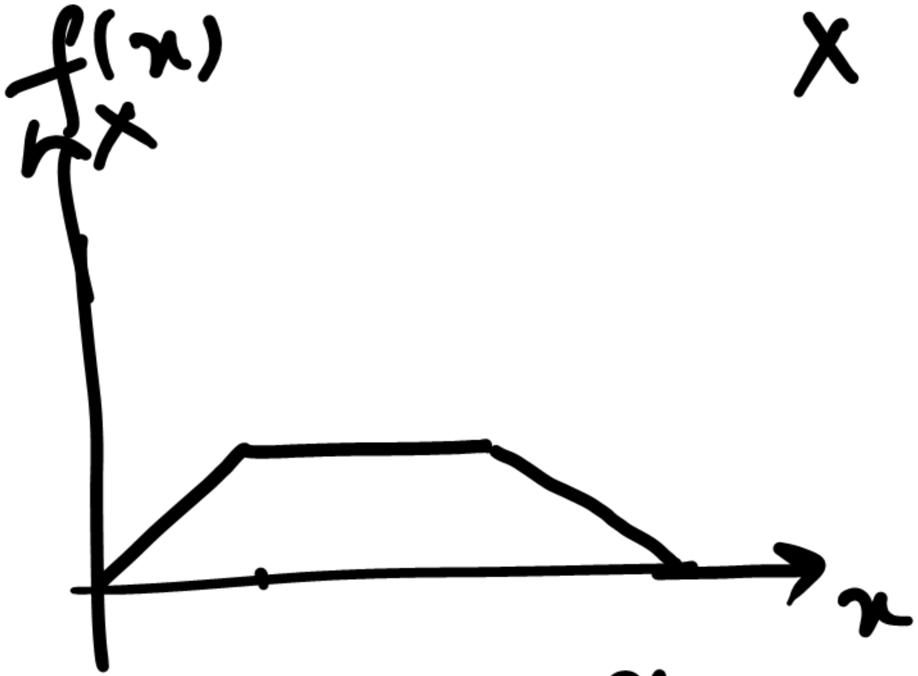
$$= \begin{cases} 0, & x \leq 0 \\ x^2, & 0 \leq x < 1 \\ 1, & x \geq 1 \end{cases}$$



$$\frac{d}{dx} F_X(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Example:

$$f_X(x) = \begin{cases} x/2, & 0 < x < 1 \\ 1/2, & 1 \le x < 2 \\ (3-x)/2, & 2 \le x < 3 \\ 0, & \text{otherwise} \end{cases}$$



$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = 0, \quad x \le 0$$

$$= \int_0^x \frac{t}{2}\,dt \qquad 0 < x \le 1$$

$$= \int_0^1 \frac{t}{2}\,dt + \int_1^x \frac{1}{2}\,dt \quad , \quad 1 \leq x \leq 2$$

$$= \int_0^1 \frac{t}{2}\,dt + \int_1^2 \frac{1}{2}\,dt + \int_2^x \frac{(3-t)}{2}\,dt \quad , \quad 2 \leq x < 3$$

$$= \int_0^1 \frac{t}{2}\,dt + \int_1^2 \frac{1}{2}\,dt + \int_2^3 \frac{(3-t)}{2}\,dt, \quad x \geq 3$$

So

$$F_X(x) = \begin{cases} 0, & x \le 0 \\ x^2/4, & 0 < x \le 1 \\ \dfrac{1}{4} + \dfrac{x-1}{2}, & 1 \le x < 2 \\ \dfrac{3}{4} + \left\{ -\dfrac{(3-t)^2}{4} \right\} \Big|_2^x, & 2 \le x < 3 \\ 1, & x \geq 3 \end{cases}$$

$$f(x) = \begin{cases} 0, & x \le 0 \\ x^2/4, & 0 < x \le 1 \\ (2x-1)/4, & 1 \le x < 2 \\ 1 - (3-x)^2/4, & 2 \le x < 3 \\ 1, & x \geq 3 \end{cases}$$

$$\frac{d}{dx} F_X(x) = f_X(x)$$

$\forall$ x except $x = 0, 1, 2, 3$

<u>Examples</u>: $F_X(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \le x \le \frac{1}{2} \\ 1, & x > 1/2 \end{cases}$

This is not continuous from right at $x = 1/2$. So not a cdf.

$$F_X(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \le x < 1/2 \\ 1, & x \geq 1/2 \end{cases}$$

$$\left\{\begin{array}{l} f_X(x) = 1, \quad 0 < x < \frac{1}{2} \\ P(X = \frac{1}{2}) = \frac{1}{2} \end{array}\right\} \text{Mixed r.v.}$$

If a r.v. is partly discrete & partly continuous, it is said to be a mixed r.v.

Concept of Expectation

Let X be discrete with pmf $p_X(x_i)$, $x_i \in \mathcal{X}$. We define the expected value of X as

$$E(X) = \sum_{x_i \in \mathcal{X}} x_i \, p_X(x_i)$$

provided the series on the right is absolutely convergent.

In case of continuous r.v.

$$E(X) = \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

provided the integral on the right is absolutely convergent.

<u>Examples</u>: Sum on two dice

$$E(X) = 2 \cdot \frac{1}{36} + 3 \cdot \frac{2}{36} + 4 \cdot \frac{3}{36}$$

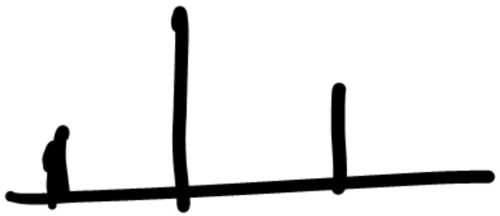
$$+5\cdot\frac{4}{36}+6\cdot\frac{5}{36}+7\cdot\frac{6}{36}+8\cdot\frac{5}{36}+9\cdot\frac{4}{36}$$

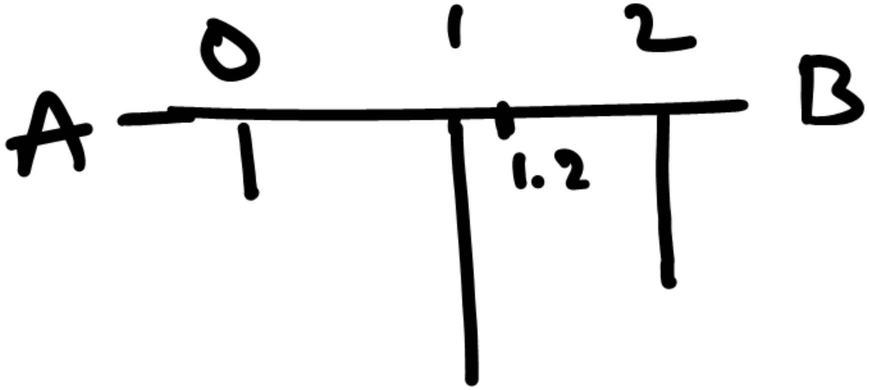
$$+10\cdot\frac{3}{36}+11\cdot\frac{2}{36}+12\cdot\frac{1}{36}=\frac{252}{36}=7$$

working ATM

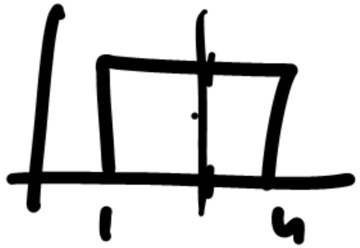
$$E(X)=0\cdot\frac{1}{10}+1\cdot\frac{6}{10}+2\cdot\frac{3}{10}$$

$$=\frac{6}{5}=\boxed{1.2}$$





$$f_X(x) = \begin{cases} \frac{1}{3}, & 1 < x < 4 \\ 0, & \text{ow} \end{cases}$$



$$E(X) = \int_1^4 \frac{x}{3}\,dx = \frac{1}{6}(16-1)$$

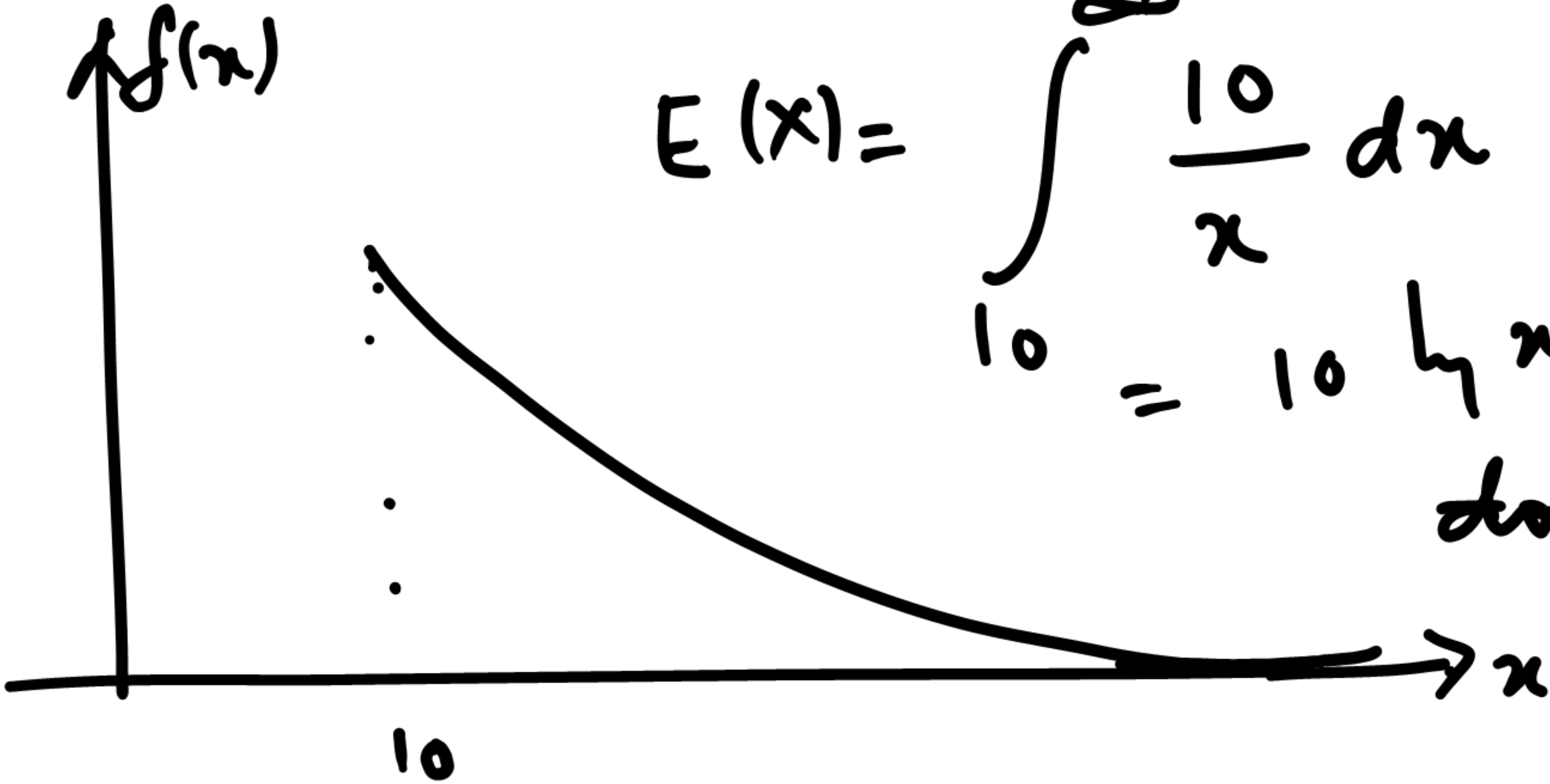
$$= \frac{5}{2}$$

$$f_X(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{ew} \end{cases}$$

$$E(X) = \int_0^1 2x^2 \, dx = \frac{2}{3}$$

$$f_X(x) = \begin{cases} \dfrac{10}{x^2}, & x > 10 \\ 0, & \text{ew} \end{cases}$$

$E(X)$ does not exist



$$E(X) = \int_{10}^{\infty} \frac{10}{x}\,dx$$

$$= 10 \ln x \Big|_{10}^{\infty}$$

does not converge

$$f_X(x) = \begin{cases} \dfrac{K}{x^3}, & x > 1 \\ 0, & \text{ew} \end{cases}$$

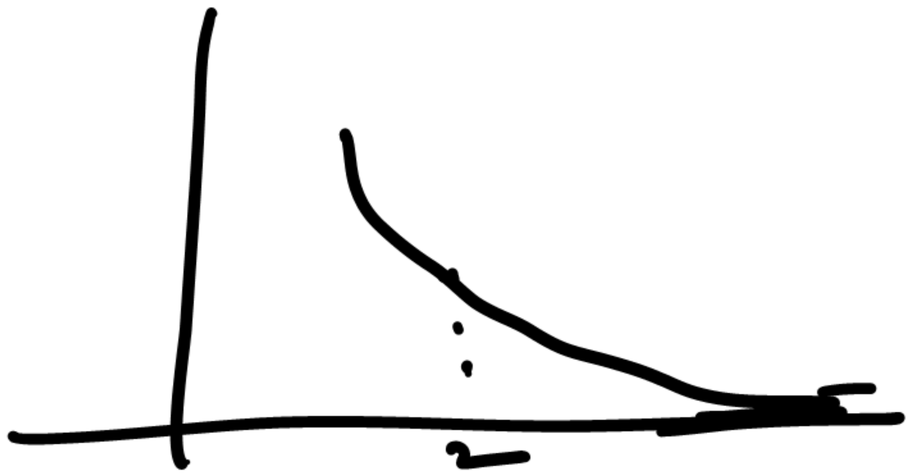
Find K, $E(X)$ & $P(1<X<2)$.

$$\int_{1}^{\infty} \frac{k}{x^3}\,dx = 1$$

$$\Rightarrow k \cdot \left.\frac{x^{-2}}{-2}\right|_{1}^{\infty} = \frac{k}{2} = 1$$

$$\Rightarrow k = 2$$

$$f_X(x) = \begin{cases} \frac{2}{x^3}, & x > 1 \\ 0, & \text{ew} \end{cases}$$



$$E(X) = \int_1^{\infty} \frac{2}{x^2}\,dx = 2\left(-\frac{1}{x}\right)\Big|_1^{\infty}$$

$$= 2$$

$$P(1 < X < 2) = \int_1^{2} \frac{2}{x^3}\,dx = \cdots\cdots$$