Poisson Process :

$X(t) \rightarrow$ no. of occurrences in an interval of length t

$$P_n(t) = P(X(t) = n)$$

$= P($there are n occurrences in interval $(0, t])$

$$P_1(h) = \lambda h + o(h) \quad \cdots \quad (A)$$

$$P_2(h) + P_3(h) + \cdots = o(h) \quad \cdots \quad (B)$$

$$\Rightarrow \quad 1 - P_0(h) - P_1(h) = o(h)$$

$$\Rightarrow \quad P_0(h) = 1 - \lambda h + o(h) \quad \cdots (c)$$

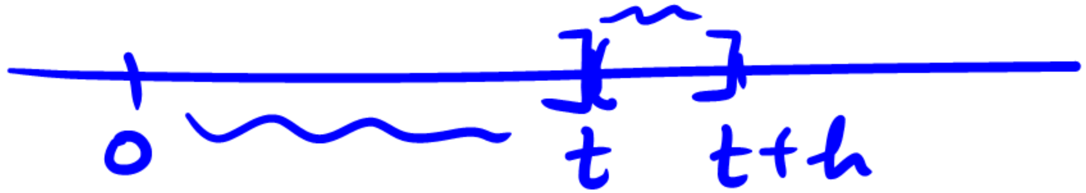
Under the three assumptions, we have to show that

$$P_n(t) = \frac{e^{-\lambda t} (\lambda t)^n}{n!}, \quad n = 0, 1, 2, \cdots \quad \cdots (1)$$

We will prove Relation (1) using induction.

First we prove for $n = 0$

$$P_0(t+h) = P(\text{no occurrence in } (0,t+h])$$



$$= P(\{\text{no occurrence in } (0,t]\} \cap \{\text{no occurrence in } (t,t+h]\})$$

independent

$$= P(\text{no occurrence in } (0,t))\,P(\text{no occurrence in } (t,t+h])$$

$$= P_0(t)\,P_0(h) = P_0(t)\left(1-\lambda h + o(h)\right)$$

$$\Rightarrow \frac{P_0(t+h) - P_0(t)}{h} = -\lambda P_0(t) + \frac{o(h)}{h} P_0(t)$$

Take limit as $h \to 0$ on both sides, to get

$$P_0'(t) = -\lambda P_0(t)$$

This is a first order ODE (in variable-separable form) and has solution

$$P_0(t) = c\, e^{-\lambda t}$$

Using initial condition $P_0(0) = 1$, we get $c = 1$.

So we get $P_0(t) = e^{-\lambda t}$.

So we have established (1) for $n=0$

Now consider $n=1$

$$P_1(t+h) = P(\text{one occurrence in } (0, t+h])$$

$$= P(\{\text{one occurrence in } (0,t]\} \cap \{\text{no occurrence in } (t,t+h]\})$$

$$+ P(\{\text{no occurrence in } (0,t]\} \cap \{\text{one occurrence in } (t,t+h]\})$$

$$= P(\text{one occurrence in } (0,t])\,P(\text{no occurrence in } (t,t+h])$$

$+ P($ no occurrence in $(0,t]) \, P($ one occurrence in $(t,t+h])$

$= P_1(t) P_0(h) + P_0(t) P_1(h)$

$= P_1(t)\left(1 - \lambda h + o(h)\right) + e^{-\lambda t}\left(\lambda h + o(h)\right)$

So we can write

$$\frac{P_1(t+h) - P_1(t)}{h} = -\lambda P_1(t) + \lambda e^{-\lambda t} + \frac{o(h)}{h} P_1(t) + \frac{o(h)}{h} e^{-\lambda t}$$

Take limit as $h \to 0$ on both the sides, to get

$$P_1'(t) = -\lambda P_1(t) + \lambda e^{-\lambda t}$$

This is a first order linear ODE. It has solution

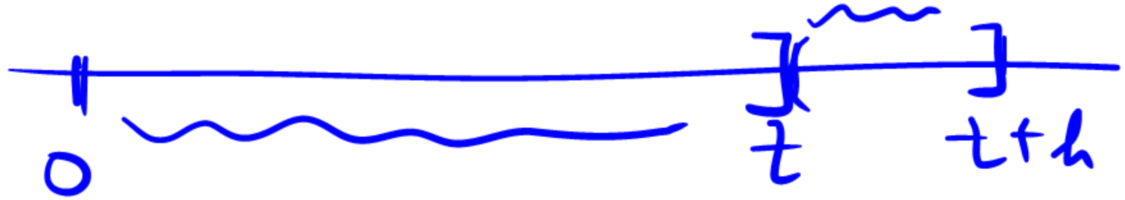
$$P_1(t) = \lambda t e^{-\lambda t} + c_1$$

Using initial condition $P_1(0) = 0$, we get $c_1 = 0$. So $P_1(t) = \lambda t e^{-\lambda t}$.

So (1) is established for $n = 1$.

Next we assume it to be true for $n \leq k$.

Now consider $n = k+1$.



$$P_{k+1}(t+h) = P\big((k+1) \text{ occurences in } (0, t+h]\big)$$

$$= P_{k+1}(t)\, P_0(h) + P_k(t)\, P_1(h) + \sum_{j=1}^{k} P_{k-j}(t)\, P_{j+1}(h)$$

$$= P_{k+1}(t)\big(1 - \lambda h + o(h)\big) + \frac{e^{-\lambda t}(\lambda t)^k}{k!}\big(\lambda h + o(h)\big)$$

$$+ \sum_{j=1}^{k} \left\{ \frac{e^{-\lambda t}(\lambda t)^{k-j}}{(k-j)!} \right\} o(h)$$

So $$\frac{P_{k+1}(t+h) - P_{k+1}(t)}{h} = -\lambda P_{k+1}(t)$$

$$+ \frac{\lambda^{k+1} t^k}{k!} e^{-\lambda t} + \frac{o(h)}{h}\left(\cdots \right)$$

Taking limit as $h \to 0$ on both the sides, to get

$$P'_{k+1}(t) = -\lambda P_{k+1}(t) + \frac{\lambda^{k+1} t^k}{k!} e^{-\lambda t}$$

This is again a first order linear ODE and has solution :

$$P_{k+1}(t) = \frac{(\lambda t)^{k+1} e^{-\lambda t}}{(k+1)!} + c_2 .$$

Using initial condition $P_{k+1}(0) = 0$, we get

$c_2 = 0$. So the solution is

$$P_{k+1}(t) = \frac{e^{-\lambda t} (\lambda t)^{k+1}}{(k+1)!} .$$

This establishes (1) for all $n = 0, 1, 2, \cdots$

Example: Suppose students enter the class at the rate of 10 per minute.

(i) What is the prob that no student entered in one minute period?

(ii) What is the prob that 20 students entered in a 5 min. period?

Sol^n Here unit of time is minute. Then $\lambda = 10$

$$P_0(1) = e^{-\lambda t} = e^{-10} \approx 0.000045$$

P(no student entered in 6 seconds)

$$= P_0(1/10) = e^{-10 \times 1/10} = e^{-1} \approx 0.37$$

$$P_{20}(5) = \frac{e^{-\lambda t}(\lambda t)^{20}}{20!} = \frac{e^{-50}(50)^{20}}{20!} \approx \cdots$$

Example: Suppose natural disasters take place in an area at the rate of 3 per year. What is the probability of no disaster in 6 months? One disaster in 4 months?

Sol^n Unit : 1 year , $\lambda = 3$

$$P_0\left(\tfrac{1}{2}\right) = e^{-\lambda t} = e^{-3/2} = e^{-1.5}$$
$$\simeq 0.22$$

$$P_1\left(\frac{1}{3}\right) = \lambda t\, e^{-\lambda t} = e^{-1} \approx 0.37$$

When our time frame / area / space is fixed then we can consider λt to be fixed constant (Let us rename it λ)

Then

$$P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!}, \quad x = 0, 1, 2, \ldots$$

This is classical Poisson distribution.

$$\sum_{x=0}^{\infty} \frac{e^{-\lambda} \lambda^{x}}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^{x}}{x!} = e^{-\lambda} \cdot e^{\lambda} = 1$$

$$\mu_1' = E(X) = \sum_{x=1}^{\infty} x \cdot \frac{e^{-\lambda} \lambda^{x}}{x!}$$

$$= e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x}}{(x-1)!} = \lambda e^{-\lambda} \left(\sum_{y=0}^{\infty} \frac{\lambda^{y}}{y!} \right) = \lambda$$

$$\mu_2' = E(X^2) = E\,X(X-1) + E(X) = \lambda^2 + \lambda$$

$$\mu_2 = Var(X) = \mu_2' - \mu_1'^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

So in a Poisson distⁿ mean and variance are always same.

$$E\,X(X-1)(X-2) = \lambda^3, \quad E\,X(X-1)(X-2)(X-3) = \lambda^4$$

$$\mu_3' = \lambda^3 + 3\lambda^2 + \lambda, \quad \mu_3 = \lambda$$

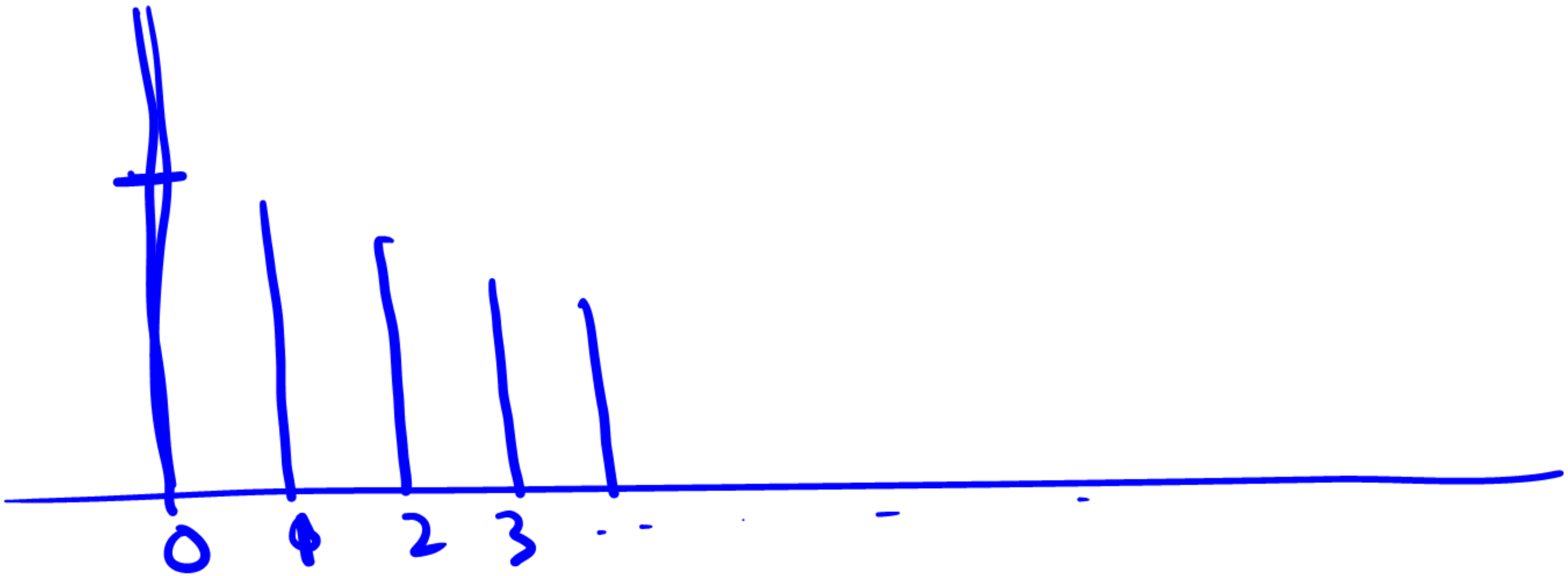
$$\mu_4' = \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda, \quad \mu_4 = \lambda + 3\lambda^2$$

* **Prove**

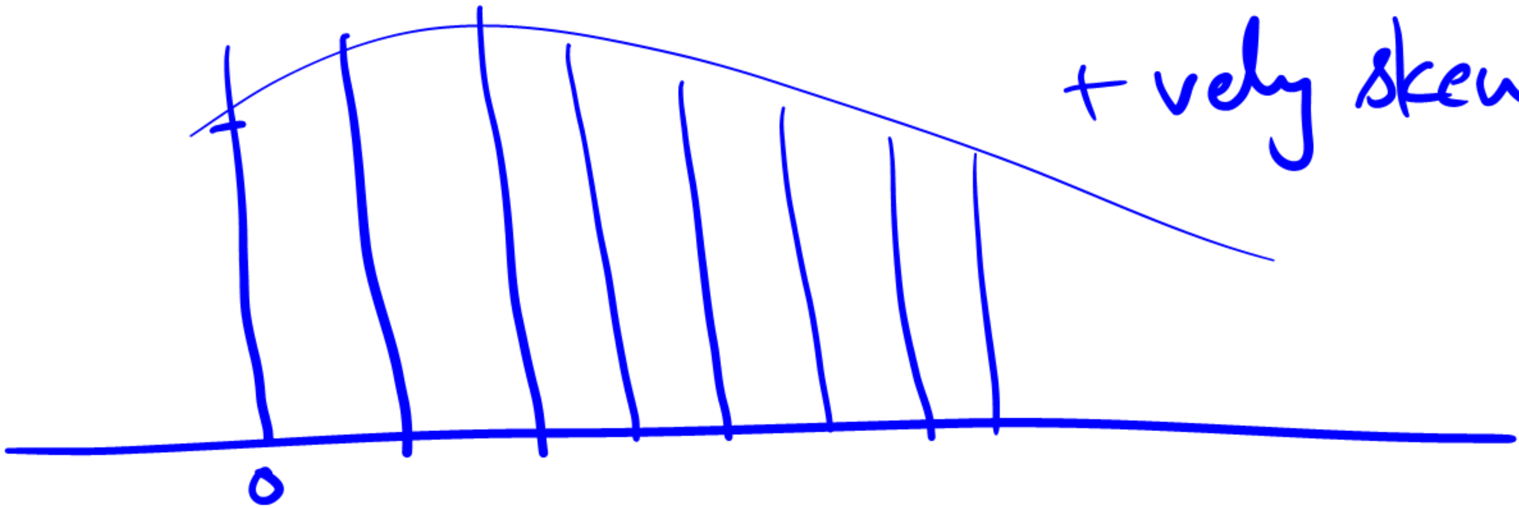
$$\beta_1 = \frac{\mu_3}{\mu_2^{3/2}} = \frac{\lambda}{\lambda^{3/2}} = \frac{1}{\lambda^{1/2}} > 0$$

So Poisson dist^n is positively skewed

$\lambda < 1$



$\lambda > 1$



+vely skewed.

$$\beta_2 = \frac{\mu_4}{\mu_2^2} - 3 = \frac{\lambda + 3\lambda^2}{\lambda^2} - 3 = \frac{1}{\lambda} > 0$$

so leptokurtic

<u>MGF</u>: $M_X(t) = E\left(e^{tX}\right)$

$$= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda} \lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t - 1)}$$

Poisson $dist^n$ as a limiting form of Binomial $dist^n$

Theorem: Let $X \sim Bin(n, p)$. As $n \to \infty$,

$p \to 0 \ni np \to \lambda$,

$$p_X(x) \to \frac{e^{-\lambda} \lambda^x}{x!}, \quad x = 0, 1 \ldots$$

Proof: $p_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$

$$\simeq \frac{n!}{x!\,(n-x)!} \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\lambda^x}{x!} \cdot \left\{ \frac{n(n-1)\cdots(n-x+1)}{n^x} \right\} \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-x}$$

$$\longrightarrow \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, 2 \cdots$$

<u>Aliter</u>: (Using mgf)

$$M_X(t) = (q+pe^t)^n = \left(1+p(e^t-1)\right)^n$$

$$\cong \left\{1 + \frac{\lambda}{n}(e^t - 1)\right\}^n \rightarrow e^{\lambda(e^t - 1)}$$

which is mgf of $P(\lambda)$ distn.

Due to uniqueness property of mgf it follows that binomial distn converges to Poisson distn.

Special Continuous Distributions

1. **Uniform Distribution**: If the density function is uniform/constant over an interval, it is called continuous uniform $dist^n$.

$$f_X(x) = \begin{cases} k, & a < x < b \\ 0, & \text{otherwise} \end{cases}$$

To determine k,

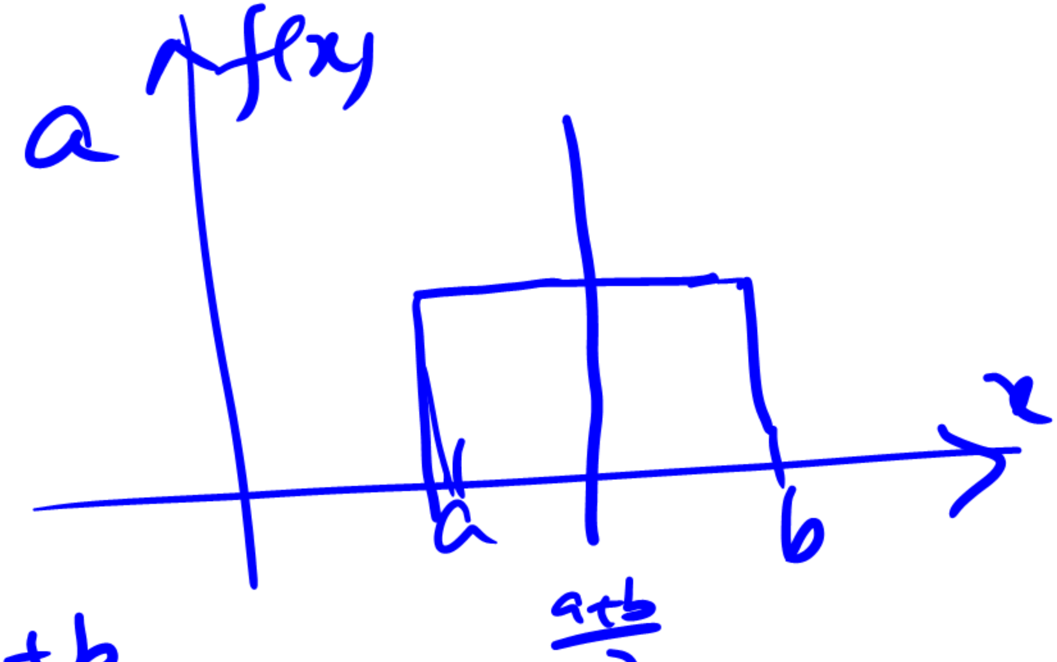
$$\int_a^b k\,dx = 1 \Rightarrow k(b-a) = 1$$

$$\Rightarrow k = 1/(b-a)$$

So the pdf of a continuous uniform r.v. X is given by

$$f_X(x) = \begin{cases} \dfrac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise.} \end{cases}$$

This is also called a rectangular distⁿ



$$E(x) = \int_a^b \frac{x}{b-a}\,dx = \frac{a+b}{2}$$

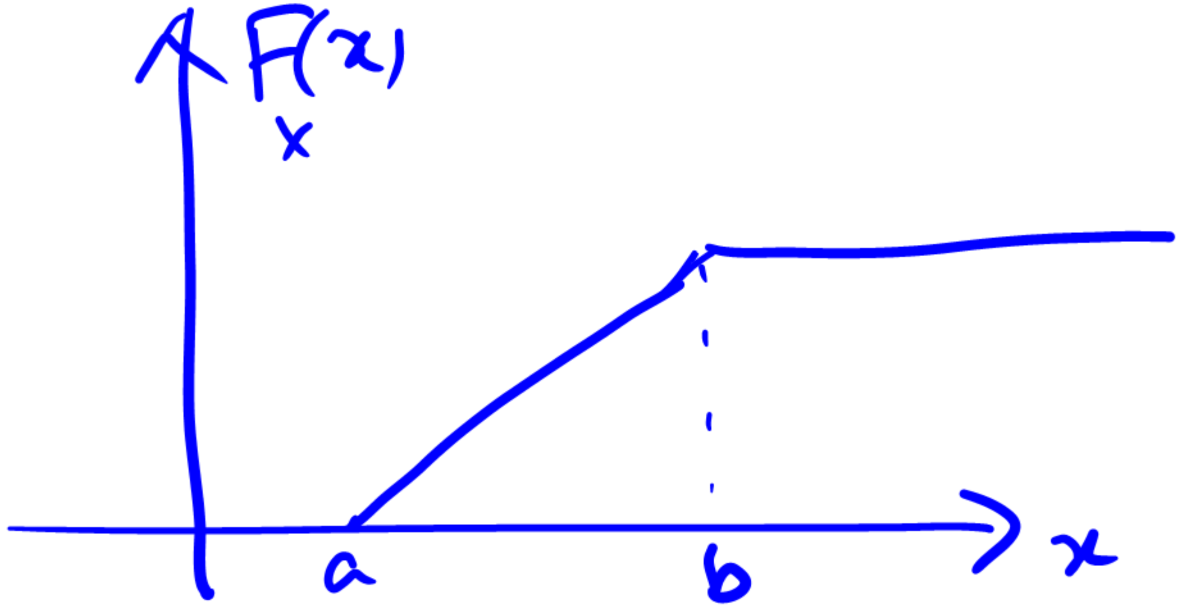
$$\mu_k' = \int_a^b \frac{x^k}{b-a}\,dx = \frac{b^{k+1} - a^{k+1}}{(k+1)(b-a)}$$

$$\mu_2' = \frac{b^3 - a^3}{3(b-a)} = \frac{a^2 + b^2 + ab}{3}$$

$$\mu_2 = Var(X) = \mu_2' - {\mu_1'}^2 = \frac{(b-a)^2}{12} = \sigma^2$$

$$s.d.(X) = \sigma = \frac{b-a}{2\sqrt{3}}$$

$$F_X(x) = \begin{cases} 0, & x \leq a \\ \dfrac{x-a}{b-a}, & a < x < b \\ 1, & x \geq b \end{cases}$$



$$M_X(t) = E(e^{tX}) = \int_a^b \frac{e^{tx}}{b-a}\,dx$$

$$= \frac{e^{bt} - e^{at}}{t(b-a)}, \quad t \neq 0$$

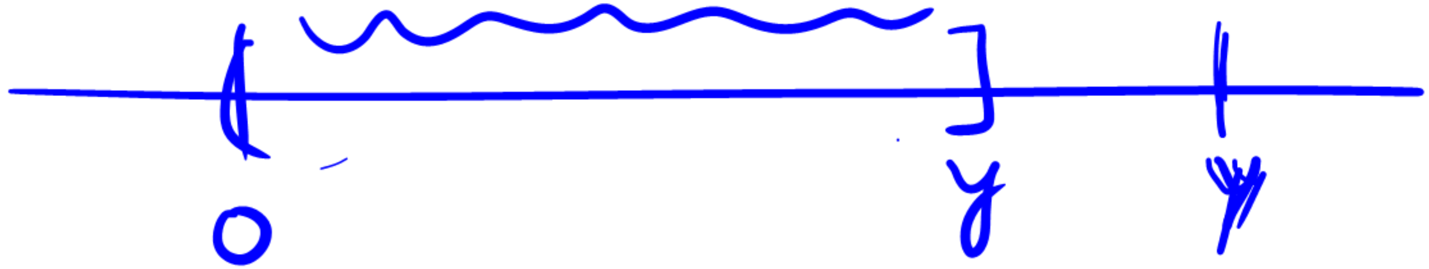
$= 1$ if $t = 0$

<u>Special Case</u>: $a = 0, \ b = 1$

$$X \sim U(0, 1)$$

$$f_X(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{ew} \end{cases}, \quad M_X(t) = \begin{cases} \dfrac{e^t - 1}{t}, & t \neq 0 \\ 1, & t = 0 \end{cases}$$

Consider a Poisson process $X(t)$ with rate λ. Let Y be the time for the first occurrence. What is the dist^n of Y?



$$P(Y > y) = P(\text{no occurrence in } (0, y])$$

$$= P(X(y) = 0) = \begin{cases} e^{-\lambda y}, & y > 0 \\ 1, & y \le 0 \end{cases}$$

$$F_Y(y) = 1 - P(Y > y) = \begin{cases} 0, & y \leq 0 \\ 1 - e^{-\lambda y}, & y > 0 \end{cases}$$

So the pdf of Y is

$$f_Y(y) = \begin{cases} 0, & y < 0 \\ \lambda e^{-\lambda y}, & y \geq 0 \end{cases}$$

This is called a Negative Exponential Distn.

$$\mu_k' = E(Y^k) = \int_0^\infty y^k \cdot \lambda e^{-\lambda y} \, dy$$

$$= \frac{\lambda \cdot \Gamma(k+1)}{\lambda^{k+1}} = \frac{k!}{\lambda^{k}}, \quad k=1,2,\dots$$

$$\mu_1' = \frac{1}{\lambda} = E(Y), \quad \mu_2' = \frac{2}{\lambda^2}, \quad \mu_2 = \frac{1}{\lambda^2}$$

$$s.d.(Y) = \frac{1}{\lambda}$$

$$\mu_3 = \frac{2}{\lambda^3}, \quad \mu_4 = \frac{9}{\lambda^4}$$

$$\beta_1 = \frac{\mu_3}{\mu_2^{3/2}} = 2 > 0$$ So exponential diff^n is always +vely skewed.

$$\beta_2 = \frac{\mu_4}{\mu_2^2} - 3 = 6 > 0$$ always leptokurtic