1. $P(X=x) = k\binom{n}{x}, \quad x = 0, 1, \dots n,$

$M_X(t) = ?$

$$\sum_{x=0}^{n} P(X=x) = 1 \Rightarrow \sum_{x=0}^{n} k\binom{n}{x} = 1 \Rightarrow \boxed{k = \left(\tfrac{1}{2}\right)^n}$$

$$\Rightarrow P(X=x) = \binom{n}{x}\left(\tfrac{1}{2}\right)^x\left(\tfrac{1}{2}\right)^{n-x}, \quad x = 0, 1, 2, \dots, n.$$

$$\Rightarrow X \sim Bin\left(n, \tfrac{1}{2}\right).$$

$$M_X(t) = \left(\tfrac{1}{2} + \tfrac{1}{2}e^t\right)^n = \frac{(1+e^t)^n}{2^n} \quad \underline{Ans}.$$

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2. $f(x) = \alpha e^{-x^2 - \beta x}, \quad -\infty < x < \infty.$

$E(X) = -\frac{1}{2}, \quad \alpha, \beta = ??$

<u>We have</u> $\int_{-\infty}^{\infty} \alpha e^{-x^2-\beta x} dx = 1$

$$\Rightarrow \alpha e^{\beta^2/4} \cdot \sqrt{\pi} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,(1/\sqrt{2})} e^{-\frac{1}{2}\left(\frac{x+\beta/2}{1/\sqrt{2}}\right)^2} dx = 1$$

$$\Rightarrow \alpha e^{\beta^2/4} \cdot \sqrt{\pi} = 1. \quad \text{—— (1)}$$

<u>and</u> $E(X) = -\frac{1}{2} \Rightarrow \int_{-\infty}^{\infty} x f(x) dx = -\frac{1}{2}$

$$\Rightarrow \alpha e^{\beta^2/4} \cdot \sqrt{\pi} \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}\,(1/\sqrt{2})} e^{-\frac{1}{2}\left(\frac{x+\beta/2}{1/\sqrt{2}}\right)^2} dx = -\frac{1}{2}$$

$$\Rightarrow 1 \cdot \left(-\frac{\beta}{2}\right) = -\frac{1}{2} \Rightarrow \boxed{\beta = 1}$$

using (1) $\boxed{\alpha = \frac{1}{\sqrt{\pi}} e^{-1/4}}$.

3. $X \sim \text{Geom}(0.4)$

$P(X) = 0.4 \times (0.6)^{x-1}, \quad x = 1, 2, 3, \ldots$

$$P(X=5 / X \geq 2) = \frac{P(X=5, X \geq 2)}{P(X \geq 2)}$$

$$= \frac{P(X=5)}{1-P(X=1)} = \frac{0.4 \times 0.6^4}{1-0.4} = \underline{0.0864} \text{ Ans.}$$

---

4. $f(x) = \frac{1}{4} e^{-|x|/2}, \quad -\infty < x < \infty.$

$$E(|X|) = \int_{-\infty}^{\infty} |x| \frac{1}{4} e^{-|x|/2} dx$$

$$= 2 \cdot \int_0^{\infty} x \frac{1}{4} e^{-x/2} dx$$

$$= \int_0^{\infty} \frac{x}{2} e^{-x/2} dx$$

$$= \underline{2} \text{ Ans.}$$

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5. Let $X \sim N(2, 4)$.

$$g(a) = P(a \leq X \leq a+2)$$

$$g(a) = P\left(\frac{a-2}{2} \leq Z \leq \frac{a}{2}\right) = \Phi\left(\frac{a}{2}\right) - \Phi\left(\frac{a-2}{2}\right).$$

Let $\varphi(\cdot)$ denote the pdf of $N(0,1)$.

$$\Rightarrow g'(a) = \frac{1}{2}\varphi\left(\frac{a}{2}\right) - \frac{1}{2}\varphi\left(\frac{a}{2}-1\right)$$

Now $g'(a) = 0 \Rightarrow \boxed{a = 1}$

Now

$$g''(a) = \frac{1}{4}\psi'\left(\frac{a}{2}\right) - \frac{1}{4}\psi'\left(\frac{a}{2}-1\right)$$

we know that $\psi'(x) = -x\psi(x)$

$$\Rightarrow g''(a) = -\frac{a}{8}\psi\left(\frac{1}{2}\right) + \frac{1}{4}\left(\frac{a}{2}-1\right)\psi\left(\frac{a}{2}-1\right)$$

$$\Rightarrow g''(1) = -\frac{1}{8}\psi\left(\frac{1}{2}\right) - \frac{1}{8}\psi\left(\frac{1}{2}\right) = -\frac{1}{4}\psi\left(\frac{1}{2}\right) < 0.$$

$\Rightarrow$ ~~a=1 is the~~ $\Rightarrow a=1$ maximizes $g(a)$.

---

6. The required probability is given by

$$P\left(-\frac{1}{2} \leq X \leq 2\right) = \int_{-\frac{1}{2}}^{2} f(x)\,dx$$

$$= \int_{-\frac{1}{2}}^{1} \frac{1}{4}\,dx + \int_{1}^{2} \frac{1}{4x^2}\,dx$$

$$= 0.5$$

7. The required probability is given by

$$P(X=0 \mid 0 \leq X < 1) = \frac{P(X=0,\, 0 \leq X < 1)}{P(0 \leq X < 1)} = \frac{P(X=0)}{P(0 \leq X < 1)}$$

$$P(X=0) = F(0) - F(0-) = F(0) - \lim_{h\to 0+} F(0-h) = \frac{1}{4} - \lim_{h\to 0+} 0 = \frac{1}{4},$$

$$P(0 \leq X < 1) = F(1-) - F(0-)$$

$$= \lim_{h\to 0+} F(1-h) - \lim_{h\to 0+} F(0-h)$$

$$= \lim_{h\to 0+}\left[\frac{1}{4} + \frac{1}{6}\left(4(1-h) - (1-h)^2\right)\right] - \lim_{h\to 0+} 0$$

$$= \frac{3}{4}.$$

Thus, $P(X=0 \mid 0 \le X < 1) = \frac{1/4}{3/4} = \frac{1}{3} = 0.33.$

8\. Given that the probability density function $f(x)$ is symmetric about 0, i.e., $f(-x) = f(x)$ for all $x \in \mathbb{R}$. Let $F(x) = \int_{-\infty}^{x} f(u)\,du$ be the cumulative distributio function of random variable X. Then, it is easy to verify that $F(-x) = 1 - F(x)$ for all $x \in \mathbb{R}$. Now,

$$\int_{-2}^{2}\int_{-\infty}^{x} f(u)\,du\,dx = \int_{-2}^{0}\int_{-\infty}^{x} f(u)\,du\,dx + \int_{0}^{2}\int_{-\infty}^{x} f(u)\,du\,dx$$

$$= \int_{-2}^{0} F(x)\,dx + \int_{0}^{2} F(x)\,dx$$

$$= -\int_{2}^{0} F(-x)\,dx + \int_{0}^{2} F(x)\,dx$$

$$= \int_{0}^{2} F(-x)\,dx + \int_{0}^{2} F(x)\,dx$$

$$= \int_{0}^{2} (F(-x) + F(x))\,dx$$

$$= \int_{0}^{2} 1\,dx$$

$$= 2.$$

9\. The required probability is

$$P(\pi R^2 < 1) = P(R^2 < \tfrac{1}{\pi})$$

$$= P\left(-\tfrac{1}{\sqrt{\pi}} < R < \tfrac{1}{\sqrt{\pi}}\right)$$

$$= \int_{0}^{1/\sqrt{\pi}} 1\,dR \quad (\text{since } R \sim U(0,1))$$

$$= \frac{1}{\sqrt{\pi}}.$$

10. We know that if $Y \sim N(\mu, \sigma^2)$, then the mgf of Y is $M_Y(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$, $t \in \mathbb{R}$. Given that

$$M_X(t) = e^{2t(t+1)} = e^{2t + \frac{1}{2}4t^2}, t \in \mathbb{R}.$$

Using the uniqueness property of mgf, it follows that $X \sim N(2,4)$. Now,

$$P(X \leq 2) = P\left(\frac{X-2}{2} \leq \frac{2-2}{2}\right)$$

$$= P(Z \leq 0) \left(\text{where } Z = \frac{X-2}{2} \sim N(0,1)\right)$$

$$= \frac{1}{2}.$$

11. The required probability is given by

$$P(\tfrac{1}{4} \leq X \leq 1) = F(1) - F(\tfrac{1}{4}-)$$

$$= F(1) - \lim_{h \to 0+} F(\tfrac{1}{4} - h)$$

$$= \frac{1+3}{5} - \lim_{h \to 0+} (\tfrac{1}{4})$$

$$= \frac{4}{5} - \frac{1}{4}$$

$$= \frac{11}{20}.$$

12. By the uniqueness property of moment generating functions, we get that X has poisson distribution with parameter 0.5, and therefore,

$$P(X \leq 1) = P(X=0) + P(X=1)$$

$$= \frac{e^{-0.5}(0.5)^0}{0!} + \frac{e^{-0.5}(0.5)^1}{1!} = \frac{3}{2}e^{-\frac{1}{2}}.$$

13. $$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= \int_{-\infty}^{\infty} \frac{x^2}{(2+x^2)^{3/2}}\,dx$$

$$= 2\int_0^{\infty} \frac{x^2}{(2+x^2)^{3/2}} dx \quad \text{(since the integrand is an even function)}$$

$$= \int_2^{\infty} \frac{(t-2)^{1/2}}{t^{3/2}} dt \quad \text{(putting } 2+x^2=t\text{)}$$

$$= \int_2^{\infty} \left(1-\frac{2}{t}\right)^{1/2} \frac{1}{t} dt$$

$$= \int_2^{\infty} \left(1-\frac{2}{t}\right)^{1/2} \frac{t}{2} \frac{2}{t^2} dt$$

$$= \int_0^1 y^{1/2}(1-y)^{-1} dy \quad \text{(putting } 1-\frac{2}{t}=y\text{)}$$

which is of the form $\int_0^1 y^{m-1}(1-y)^{n-1} dy$ with $m=\frac{3}{2}$ and $n=0$. Since the Beta function $\int_0^1 y^{m-1}(1-y)^{n-1} dy$ converges if, and only if, $m>0$ and $n>0$, it follows that $E(X^2)$ does not exist.

14. Given that $Y = \log_e X^{-2\alpha}$. Then, $X = e^{-\frac{Y}{2\alpha}}$ and the Jacobian of the transformation is given by $J = \frac{dx}{dy} = -\frac{1}{2\alpha} e^{-\frac{y}{2\alpha}}$. The pdf of Y is given by

$$g(y) = f\left(e^{-\frac{y}{2\alpha}}\right)|J|$$

$$= \begin{cases} \alpha\left(e^{-\frac{y}{2\alpha}}\right)^{\alpha-1} \frac{1}{2\alpha} e^{-\frac{y}{2\alpha}}, & \text{if } 0<e^{-\frac{y}{2\alpha}}<1, \\ 0, & \text{otherwise,} \end{cases}$$

$$= \begin{cases} \frac{1}{2} e^{-\frac{y}{2}}, & \text{if } y>0, \\ 0, & \text{otherwise,} \end{cases}$$

which is the pdf of χ^2_2 random variable.

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15. It is easy to see that F is continuous on $\mathbb{R}$ and $F(0)=0$, which implies that X is a non-negative continuous random variable. Then,

$$E(X)=\int_0^\infty [1-F(x)]\,dx$$

$$=\int_0^2 [1-\tfrac{x}{8}]\,dx+\int_2^4 [1-\tfrac{x^2}{16}]\,dx+\int_4^\infty [1-1]\,dx$$

$$=\frac{31}{12}.$$

16. Let E and E_i, respectively, denote the events that a randomly selected laptop has lifetime more than two years and that the laptop was supplied by vendor V_i, $i=1,2$. Also, let X_i denote the lifetime (in years) of laptops supplied by vendor V_i, $i=1,2$. Given that $X_1 \sim U(0,4)$ and $X_2 \sim Exp(1/2)$. Then, using the given information, we obtain

$$P(E_1)=\frac{1}{2}=P(E_2),$$

$$P(E|E_1)=P(X_1>2)=\int_2^4 \frac{1}{4}\,du=\frac{4-2}{4}=\frac{1}{2},$$

and $$P(E|E_2)=P(X_2>2)=\int_2^\infty \frac{1}{2}e^{-x/2}\,du$$

$$=e^{-1}.$$

Now, using the Bayes' theorem, the required probability is given by

$$P(E_2|E)=\frac{P(E|E_2)P(E_2)}{P(E|E_1)P(E_1)+P(E|E_2)P(E_2)}=\frac{e^{-1}\times\frac{1}{2}}{\frac{1}{2}\times\frac{1}{2}+e^{-1}\times\frac{1}{2}}=\frac{2}{e+2}$$

18. Clearly, $E(Y) = 72 \times \frac{1}{3} = 24$ and $Var(Y) = 72 \times \frac{1}{3} \times \frac{2}{3}$

$= 16.$

An approximate value of $P(22 \leq Y \leq 28)$ is given by $P(22 \leq Y \leq 28) = P(21.5 < Y < 28.5)$ (using the continuity correction)

$$= P\left(\frac{21.5 - 24}{4} < \frac{Y - 24}{4} < \frac{28.5 - 24}{4}\right)$$

$$= P(-0.625 < Z < 1.125)$$

$$= \phi(1.125) - \phi(-0.625)$$

$$= \phi(1.125) - (1 - \phi(0.625))$$

$$= 0.8697 - (1 - 0.7341)$$

$$= 0.6038.$$

17. We have

$$E(X) = \int_0^3 x \frac{2x}{9} dx = \frac{2}{9}[x^3/3]_0^3 = \frac{2}{9} \times \frac{27}{3} = 2.$$

$$E(X^2) = \int_0^3 x^2 \frac{2x}{9} dx = \frac{2}{9}[x^4/4]_0^3 = \frac{9}{2}$$

$$Var(X) = E(X^2) - (E(X))^2$$

$$= \frac{9}{2} - 4$$

$$= \frac{1}{2}.$$

Now, using chebyshev's inequality, the upper bound of $P(|X - 2| > 1)$ is given by

$$P(|X - 2| > 1) \leq \frac{Var(X)}{1^2} = \frac{1}{2} = 0.5$$

19.

Given that $X \sim Bin(2,p)$ and $P(X \geq 1) = \frac{5}{9}$.
Therefore,

$$\frac{4}{9} = P(X=0) = (1-p)^2 \Rightarrow p = \frac{1}{3}.$$

Since $Y \sim Bin(4,p)$, we have

$$P(Y \geq 1) = 1 - P(Y=0)$$

$$= 1 - (1-\tfrac{1}{3})^4 = 1 - \frac{16}{81} = \frac{65}{81} = 0.8024$$

20. The required probability is given by

$$P(\tfrac{1}{4} < X^2 < \tfrac{1}{2}) = P(-\tfrac{1}{\sqrt{2}} < X < -\tfrac{1}{2}) + P(\tfrac{1}{2} < X < \tfrac{1}{\sqrt{2}})$$

$$= \int_{-\frac{1}{\sqrt{2}}}^{-\frac{1}{2}} \frac{x+1}{2}\,dx + \int_{1/2}^{\frac{1}{\sqrt{2}}} \frac{x+1}{2}\,dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2} + x\right]_{-\frac{1}{\sqrt{2}}}^{-\frac{1}{2}} + \frac{1}{2}\left[\frac{x^2}{2} + x\right]_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}}$$

$$= \frac{1}{\sqrt{2}} - \frac{1}{2}$$

$$= 0.207$$

21. The required probability is given by

$$P(\min(X, 1-X) \leq \tfrac{1}{4}) = 1 - P(\min(X, 1-X) > \tfrac{1}{4})$$

$$= 1 - P(\tfrac{1}{4} < X < \tfrac{3}{4})$$

$$= 1 - (\tfrac{3}{4} - \tfrac{1}{4}) \text{ (since } X \sim U(0,1))$$

$$= \frac{1}{2} = 0.5$$

22. The required probability is given by

$$P(\tfrac{1}{2} < X < 2) = \int_{1/2}^{2} f(x)\,dx$$

$$= \int_{1/2}^{1} x^3\,dx + \int_{1}^{2} \frac{3}{x^5}\,dx$$

$$= \left[\frac{x^4}{4}\right]_{1/2}^{1} + \left[-\frac{3}{4x^4}\right]_{1}^{2}$$

$$= \frac{15}{16}.$$

23. The given moment generating function can be written as

$$M_X(t) = \frac{1}{216}(5+e^t)^3$$

$$= \left(\frac{5}{6} + \frac{1}{6}e^t\right)^3, \; t \in \mathbb{R}$$

which implies that $X \sim \text{Bin}(3, \frac{1}{6})$.

Then

$$P(X > 1) = 1 - P(X \le 1)$$

$$= 1 - [P(X=0) + P(X=1)]$$

$$= 1 - \binom{3}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^3 - \binom{3}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^2$$

$$= \frac{2}{27}.$$

24. Since $p(x)$ is the pmf, we have

$$1=\sum_{x=-2}^{2} p(x)$$

$$=k\sum_{x=-2}^{2}(1+|x|)^2$$

$$=27k$$

which implies that $k=\frac{1}{27}$.

Therefore, $P(X=0)=p(0)$

$$=\frac{1}{27}(1+|0|)^2$$

$$=\frac{1}{27}.$$

25. The pdf of X is given by

$$f_X(x)=\begin{cases}\frac{1}{\frac{\pi}{2}-\frac{\pi}{6}}, & \frac{\pi}{6}<x<\frac{\pi}{2},\\ 0, & \text{otherwise}\end{cases}$$

$$=\begin{cases}\frac{3}{\pi}, & \frac{\pi}{6}<x<\frac{\pi}{2}\\ 0, & \text{otherwise}\end{cases}$$

Then,

$$P(\cos X>\sin X)=P(\tan X<1)$$

$$=P(X<\tan^{-1}(1))$$

(since $\tan^{-1}x$ is increasing in $x\in(\frac{\pi}{6},\frac{\pi}{2})$)

$$=P(X<\frac{\pi}{4})$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{3}{\pi}\, dx = \frac{1}{4}$$

26. Since $X \in (0,1)$ with probability 1, it follows that $Y = -2\log_e X$ takes values in $(0, \infty)$ with positive probabilities. Let $F_Y(y)$ denote the cumulative distribution function of Y at point y. clearly, for $y < 0$, $F_Y(y) = 0$, and for $y \geq 0$,

$$F_Y(y) = P(Y \leq y)$$

$$= P(-2\log_e X \leq y)$$

$$= P\left(\log_e X \geq -\frac{y}{2}\right)$$

$$= P\left(\exp(\log_e X) \geq e^{-y/2}\right)$$

(since e^x is an increasing function)

$$= P(X \geq e^{-y/2})$$

$$= \int_{e^{-y/2}}^{1} 1\, dx$$

$$= 1 - e^{-y/2}$$

Thus, Y has an exponential distribution with mean 2. Hence, $E(Y) = 2$.

27. Given that $Y = \log_{10} X \sim N(\mu, \sigma^2)$ and $M_Y(t) = e^{5t+2t^2} = e^{5t+\frac{1}{2}4t^2}$, $t \in (-\infty, \infty)$.
Using the uniqueness property of mgf, we get that $\mu = 5$ and $\sigma^2 = 4$. Now,

$$P(X < 1000) = P(\log_{10} X < \log_{10}(1000))$$

$$= P(Y < 3)$$

$$= P\left(\frac{Y-5}{2} < \frac{3-5}{2}\right)$$

$$= P(Z < -1)$$

$$= P(Z \leq -1) \text{ (since } Z \text{ is a continuous r.v)}$$

$$= \Phi(-1)$$

$$= 1 - \Phi(1) \text{ (since } \Phi(-x) = 1 - \Phi(x))$$

$$= 1 - 0.8413$$

$$= 0.1587.$$

28. Since $f(x)$ is a probability density function, it follows that

$$1=\int_{-\infty}^{\infty} f(x)\,dx$$

$$=\int_{0}^{2}\frac{x}{8}\,dx+\int_{2}^{4}\frac{k}{8}\,dx+\int_{4}^{6}\frac{6-x}{8}\,dx$$

$$=\frac{k+2}{4},$$ which implies that $k=2$.

Now, the required probability is

$$P(1<X<5)=\int_{1}^{2}\frac{x}{8}\,dx+\int_{2}^{4}\frac{2}{8}\,dx+\int_{4}^{5}\frac{6-x}{8}\,dx$$

$$=\frac{7}{8}=0.875.$$

Optional solution: The required probability is

$$P(1<X<5)=1-P(0<X\leq 1)-P(5\leq X<6)$$

$$=1-\int_{0}^{1}\frac{x}{8}\,dx-\int_{5}^{6}\frac{6-x}{8}\,dx$$

$$=0.875.$$

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29. It is easy to see that X has gamma distribution with parameters r and λ. Then $E(X) = \frac{r}{\lambda}$ and $Var(X) = \frac{r}{\lambda^2}$.

Given that $E(X) = \frac{r}{\lambda} = 2$ and $Var(X) = \frac{r}{\lambda^2} = 2$. On solving these equations, we get $r = 2$ and $\lambda = 1$. Now, putting these values in the given pdf of X, we get

$$f(x) = xe^{-x}, \quad x \geq 0.$$

Now, the required probability is

$$P(X<1) = \int_0^1 f(x)\,dx$$

$$= \int_0^1 xe^{-x}\,dx$$

$$= 1 - 2e^{-1}$$

$$= 0.264$$