Mixed Random Variable

$X \rightarrow$ waiting time at a traffic signal.

$P(X=0) = \frac{1}{4} \longrightarrow$ pmf part

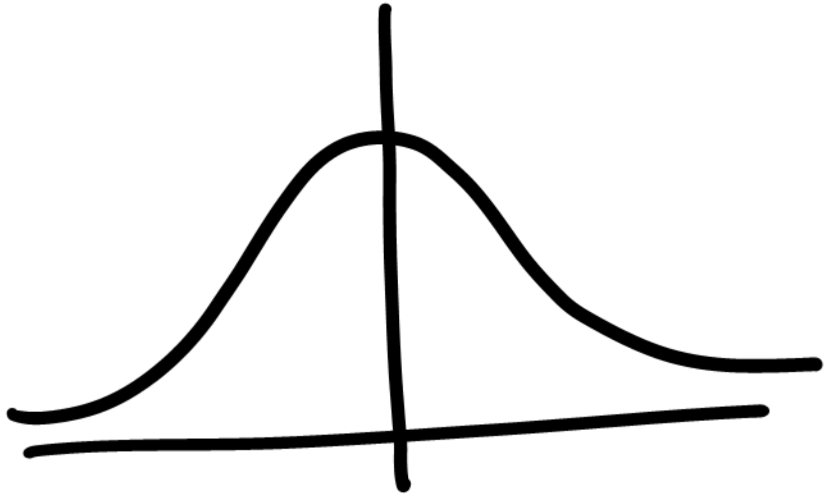
$$f_X(x) = \begin{cases} \frac{3}{4}, & 0 \le x < 1 \\ 0, & \text{otherwise} \end{cases} \Bigg\} \text{ pdf part}$$

$$F_X(x) = \begin{cases} 0, & x < 0 \\ 1/4 & x = 0 \\ \frac{1}{4} + \frac{3}{4}\int_0^x dt, & 0 < x < 1 \\ 1, & x \geq 1 \end{cases}$$

$$= \begin{cases} 0 & x < 0 \\ \frac{1}{4} + \frac{3x}{4}, & 0 \leq x < 1 \\ 1, & x \geq 1 \end{cases}$$

$$E(X) = 0 \cdot \frac{1}{4} + \int_0^1 \frac{3}{4} x \, dx = \frac{3}{8}$$

Symmetric Distribution

A r.v. X is symmetric about a point c if

$$P(X \geq c+x) = P(X \leq c-x)$$

for all $x \in \mathbb{R}$

If $c = 0$, the condition becomes

$$P(X \geq x) = P(X \leq -x)$$

Expectation is a linear operator

ie $$E(aX + b) = aE(X) + b$$

$$Eg(X) = \sum_{x_i \in \mathcal{X}} g(x_i)\, p_X(x_i)$$

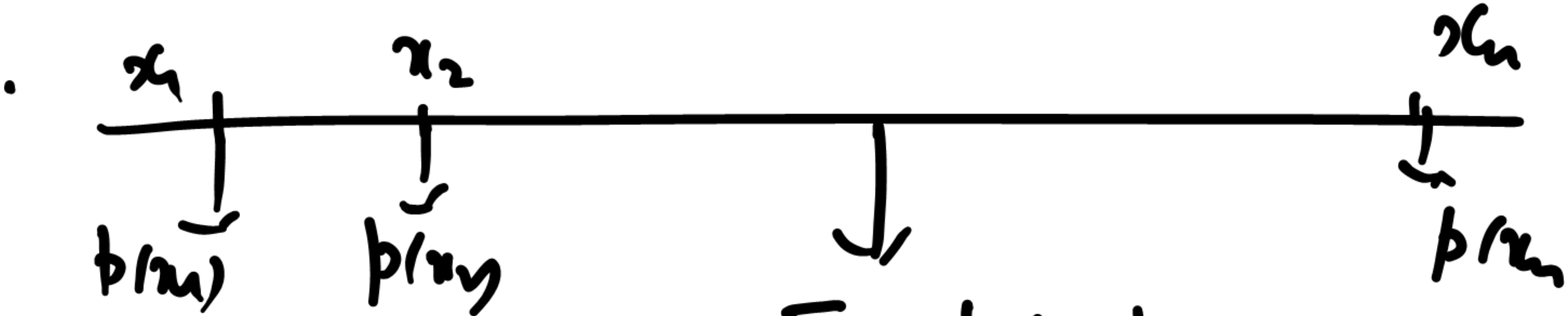
if X is discrete

$$= \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx \quad \text{if } X \text{ is continuous}$$

provided the right hand series/ integral is absolutely convergent.

<u>Moments</u>: Let $g(x) = x^k$

$\mu_k' = E(X^k) \longrightarrow k^{th}$ moment about origin or k^{th} non-central moment



$$E(X) = \sum x_i \, p(x_i)$$

$$E(X) = \int_{-\infty}^{\infty} x f(x) \, dx$$

We usually denote $\mu_1' = E(X) = \mu$ called mean of r.v. X

$$\mu_k = E(X-\mu)^k ,$$

→ k^{th} moment about mean

→ k^{th} central moment

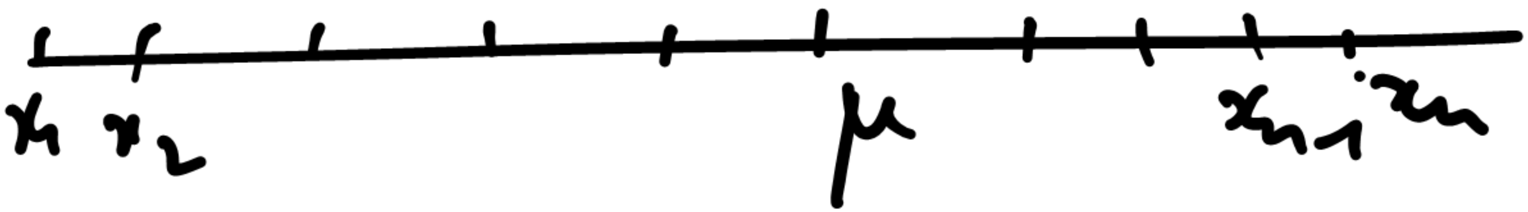
$$\mu_1 = E(X-\mu) = E(X) - \mu = 0$$

So the first central moment is always 0.

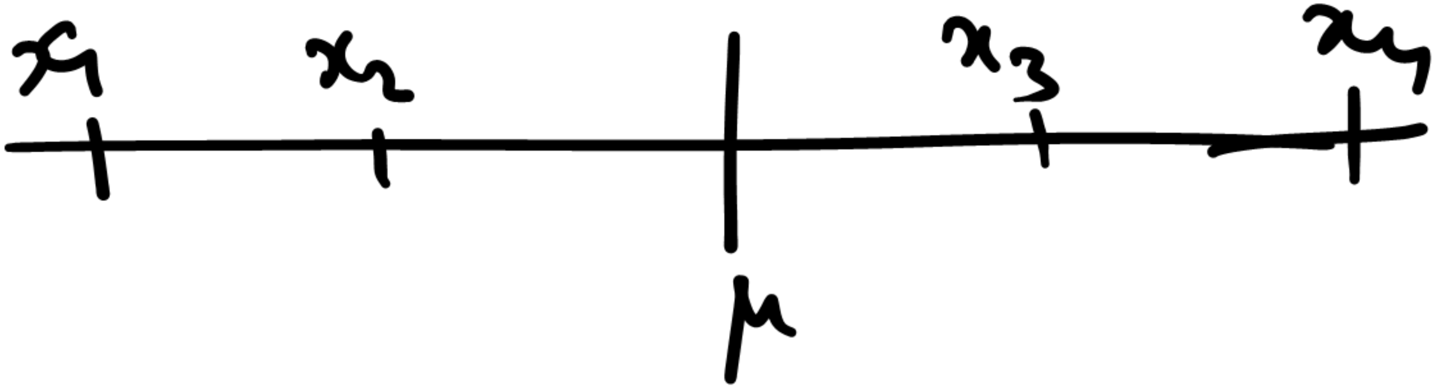
$X-\mu$ $\quad x_i-\mu, \quad i=1,2,\ldots.$

$$\sum_{x_i \in \mathcal{X}} (x_i-\mu)\, p_X(x_i) = 0$$

$$\int_{-\infty}^{\infty} (x-\mu)\, f_X(x)\, dx = 0$$



1



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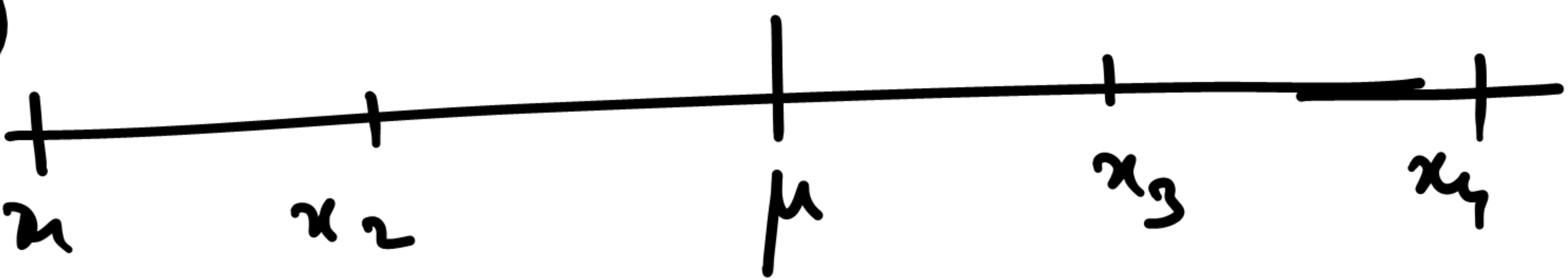


$$E|X-\mu| \rightarrow$$ Absolute Mean deviation from mean

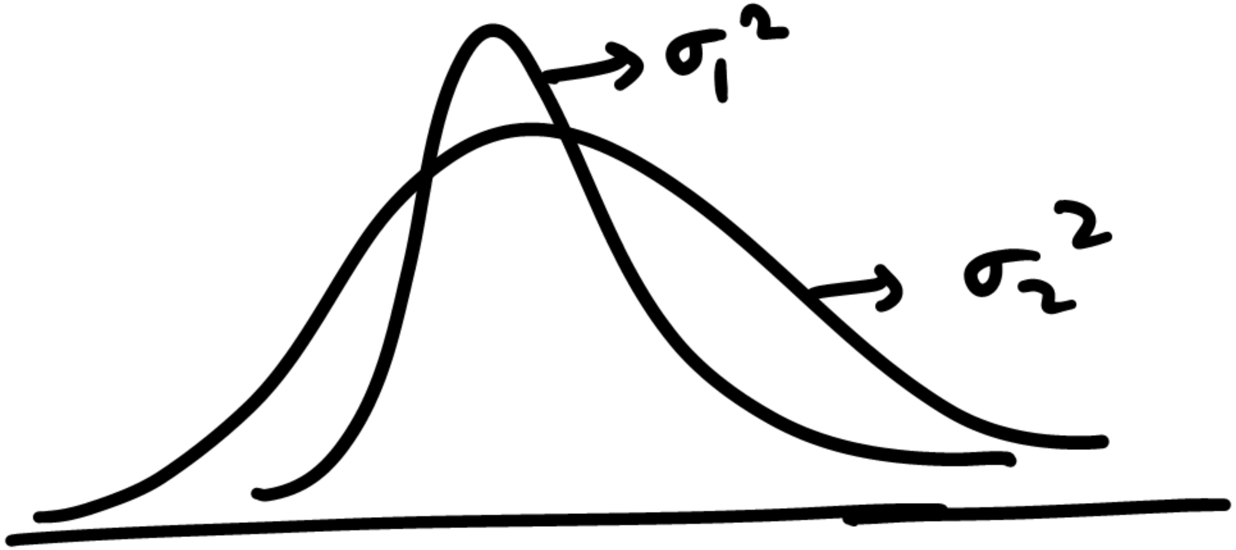
$$\mu_2 = E(X-\mu)^2 \rightarrow$$ Variance of X

$$= Var(X)$$

$$= \sigma^2$$

(usual notation)

$$\sigma = \sqrt{Var(X)}$$ = standard deviation of X.



$\sigma_1^2 < \sigma_2^2$

Relationship between μ_k & μ_k'

$\mu_k = E(X-\mu)^k$ (k +ve integer)

$= E\left[X^k - \binom{k}{1} X^{k-1}\mu + \binom{k}{2} X^{k-2}\mu^2 - \cdots\cdots + (-1)^{k+1}\mu^k\right]$

$$= \mu_k' - \binom{k}{1}\mu_{k-1}'\mu + \binom{k}{2}\mu_{k-2}'\mu^2 - \cdots + (-1)^{k+1}\mu^k$$

$$\mu_2 = \mu_2' - 2\mu^2 + \mu^2 = \mu_2' - \mu^2$$

$$\mathrm{Var}(X) = E(X^2) - \{E(X)\}^2 \geq 0$$

$$\Rightarrow E(X^2) \geq \{E(X)\}^2$$

$$\mu_k' = E(X^k) = E(\overline{X-\mu} + \mu)^k$$

$$= E\left[(X-\mu)^k + \binom{k}{1}(X-\mu)^{k-1}\mu + \binom{k}{2}(X-\mu)^{k-2}\mu^2 + \cdots + \mu^k\right]$$

$$= \mu_k + \binom{k}{1}\mu_{k-1}\mu + \binom{k}{2}\mu_{k-2}\mu^2 + \cdots + \mu^k.$$

$$\beta'_k = E|X|^k \rightarrow k^{th} \text{ absolute moment about origin}$$

$$\beta_k = E|X-\mu|^k \rightarrow k^{th} \text{ absolute moment about mean}$$

$$\beta_1 = E|X-\mu| \rightarrow \text{mean absolute deviation about mean}$$

Factorial Moments:

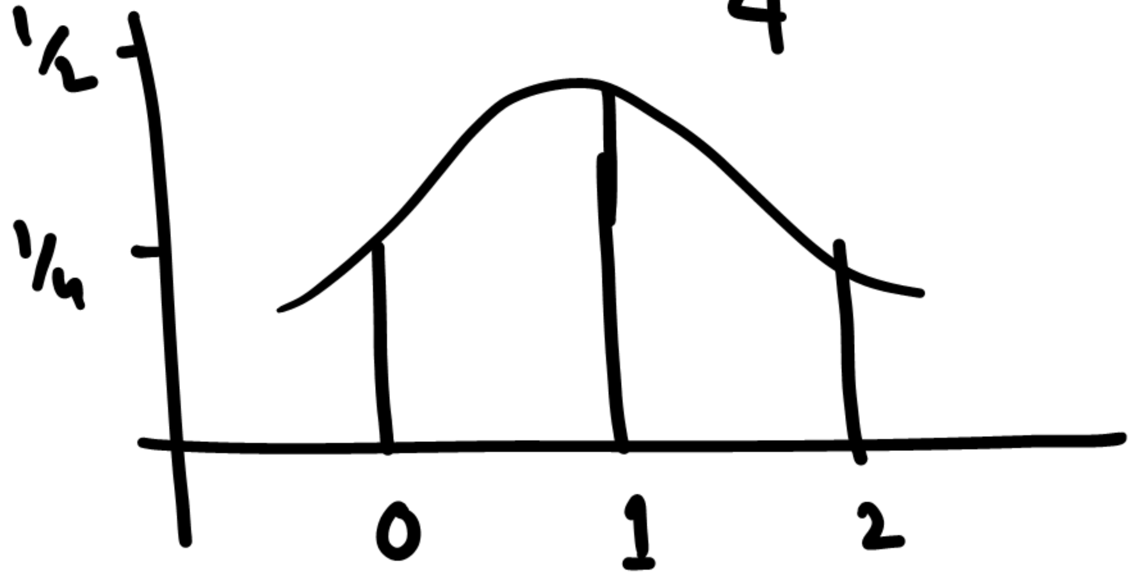
$$\alpha_k = E[X(X-1)\cdots(X-k+1)] \quad k=1,2$$

$$\alpha_1 = E(X), \alpha_2 = E\{X(X-1)\} \quad \cdots$$

1. Tossing of Two fair coins

$X \to$ no of heads

$P(X=0) = \frac{1}{4}$, $P(X=1) = \frac{1}{2}$, $P(X=2) = \frac{1}{4}$



$E(X) = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2}$

$+ 2 \cdot \frac{1}{4}$

$= 1$

$E(X^2) = \frac{1}{2} + 1 = \frac{3}{2}$, $V(X) = \frac{3}{2} - 1 = \frac{1}{2}$

$$\sigma = \frac{1}{\sqrt{2}} \simeq 0.71$$

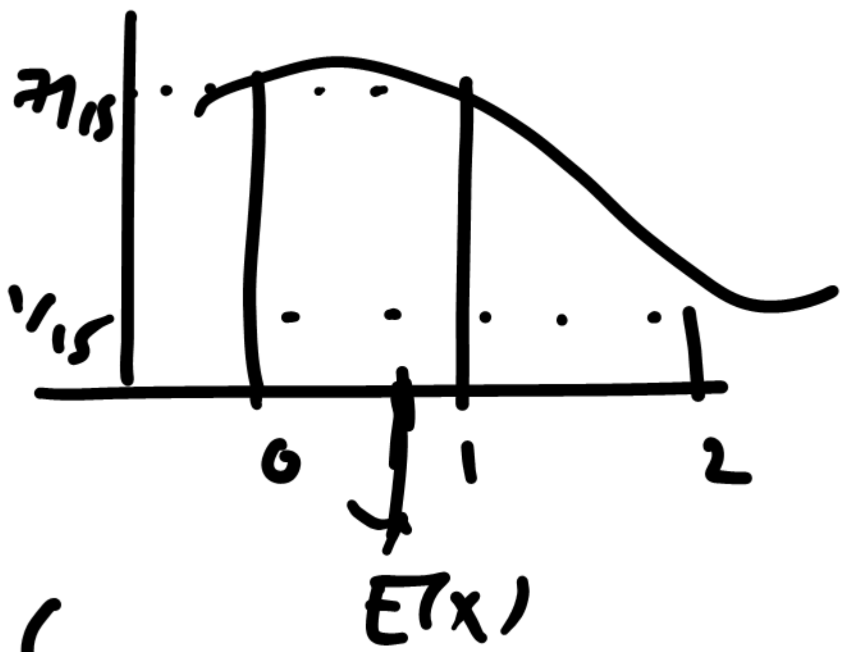
2. Suppose a store has 10 AC's & 3 are defective. A consumer buys 2 at random.

X → no. of defectives in his selection

X → 0, 1, 2

$$p_X(0) = \frac{{}^7C_2}{{}^{10}C_2} = \frac{7}{15}, \quad p_X(1) = \frac{{}^7C_1 \times {}^3C_1}{{}^{10}C_2} = \frac{7}{15}$$

$$p_X(2) = \frac{{}^3C_2}{{}^{10}C_2} = \frac{1}{15}$$



$$E(X) = 0\cdot\frac{7}{15} + 1\cdot\frac{7}{15} + 2\cdot\frac{1}{15}$$

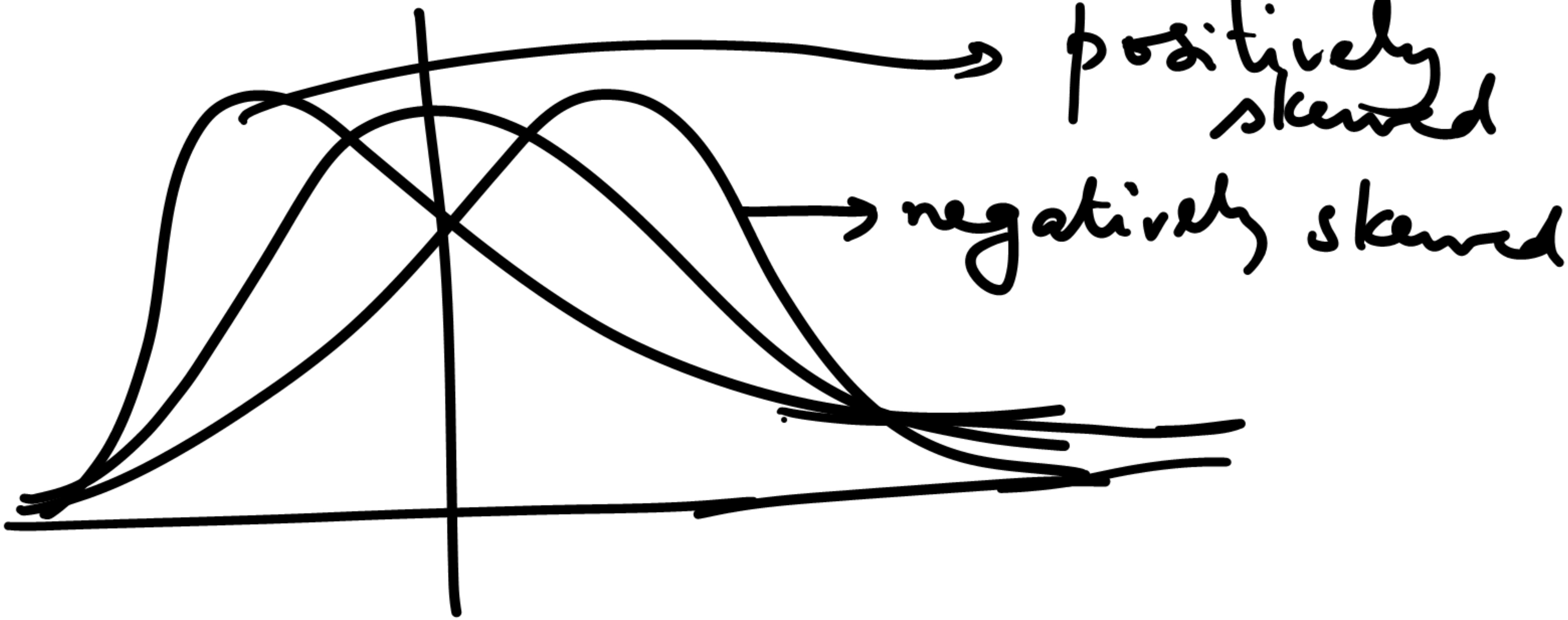
$$= \frac{9}{15} = 3/5 = 0.6$$

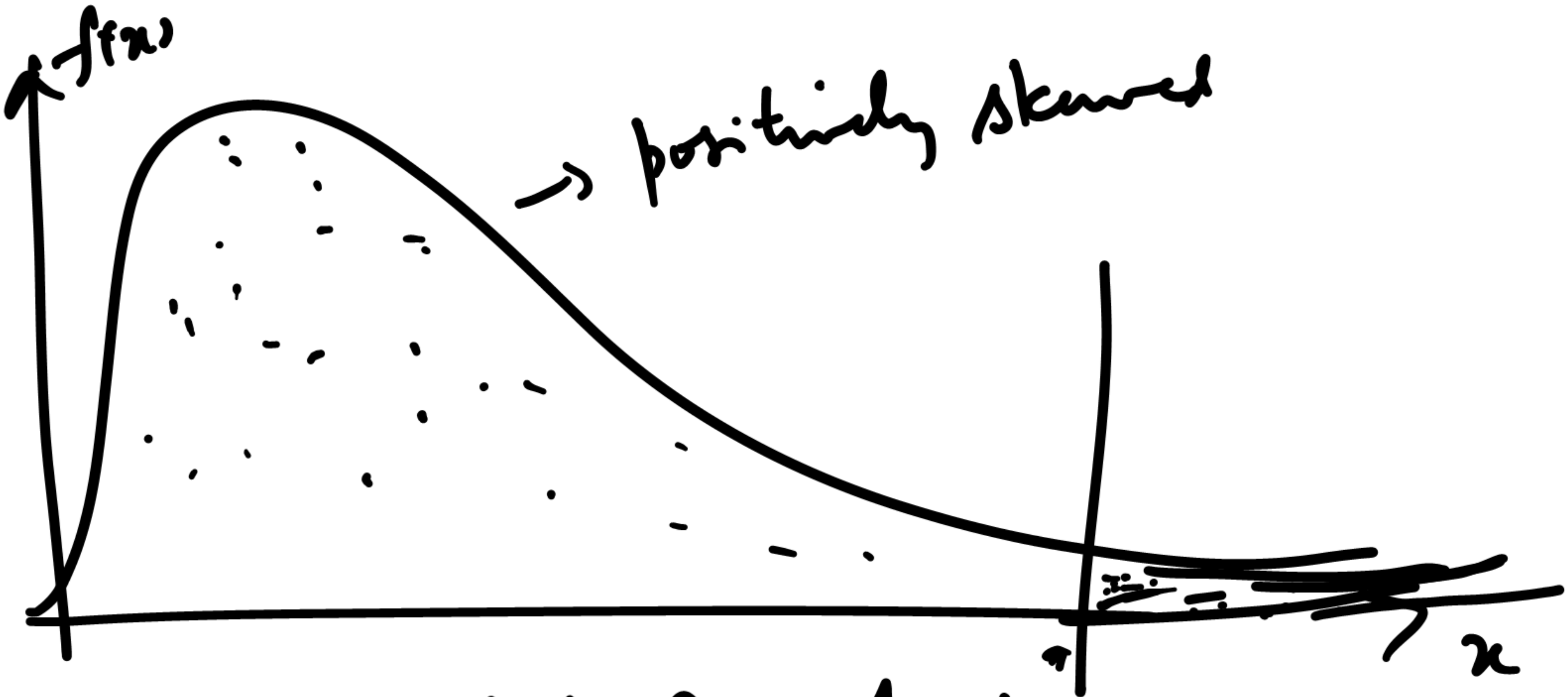
$$E(X^2) = 0\cdot\frac{7}{15} + 1\cdot\frac{7}{15} + 4\cdot\frac{1}{15} = \frac{11}{15}$$

$$\mathrm{Var}(X) = \mu_2' - \mu^2 = \frac{11}{15} - \frac{9}{25} \simeq \boxed{0.37}$$

$\sigma \simeq 0.61$

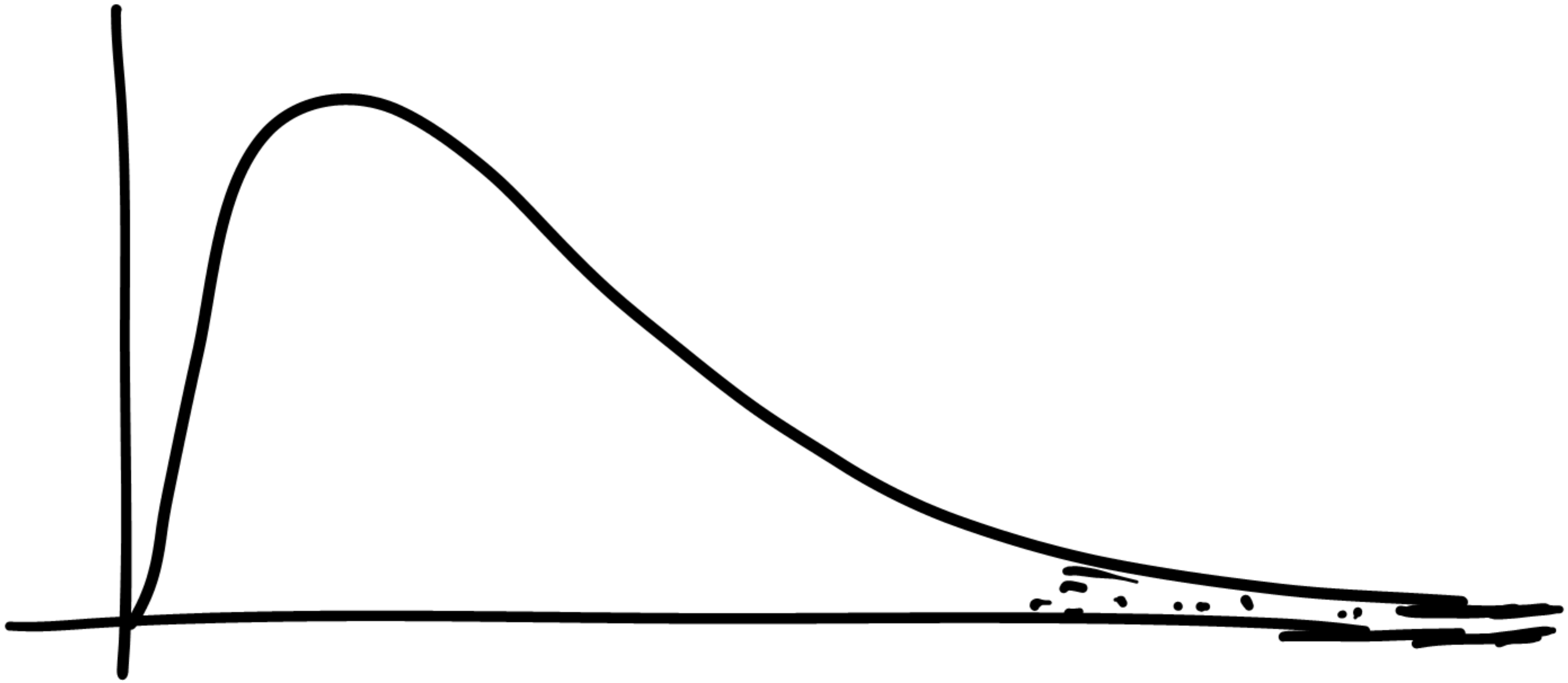
Asymmetry or Skewness





X → distribution of marks in a competetive exam

Y → income distⁿ in India



Age / longevity

negatively skewed distⁿ

Measuring of Skewness

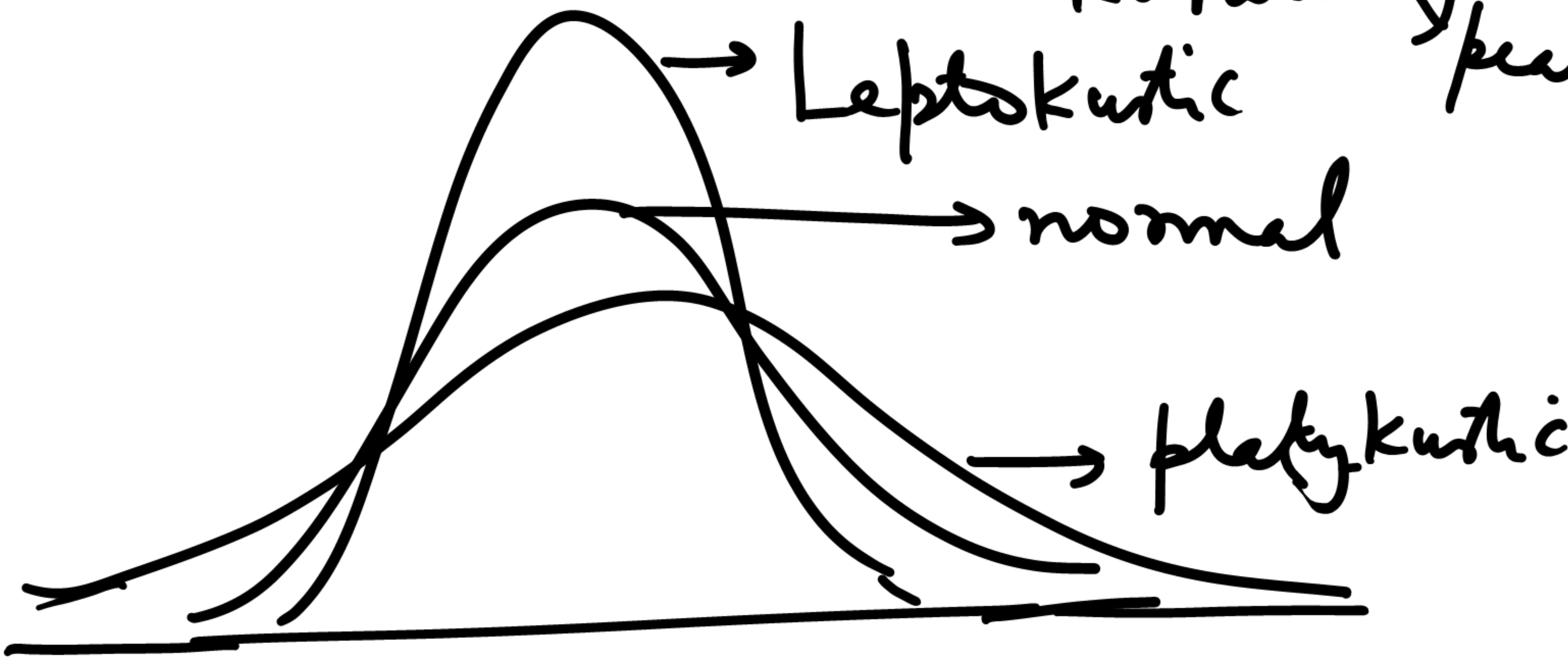
$$\beta_1 = \frac{\mu_3}{\sigma^3} = \frac{\mu_3}{\mu_2^{3/2}}$$

$\beta_1 = 0 \longrightarrow$ symmetric distⁿ

$\quad > 0 \longrightarrow$ for +vely skewed

$\quad < 0 \longrightarrow$ for -vely skewed

Kurtosis → peakedness



Measure of Kurtosis

$$\beta_2 = \frac{\mu_4}{\mu_2^2} - 3 \quad \begin{aligned} &= 0 \text{ for normal} \\ &> 0 \text{ Leptokurtic} \\ &< 0 \text{ platykurtic} \end{aligned}$$