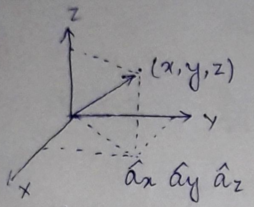
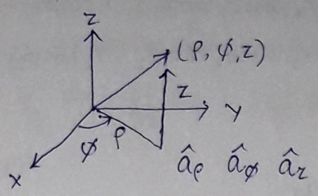
# COORDINATE SYSTEMS

08 Jan '20

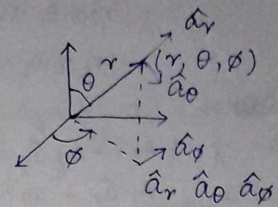
Cartesian




Cylindrical




Spherical




Azimuthal plane :- taken as XY, and angle of projected vector $\vec{P}$ :- $\phi$.

$\phi = 0 \Rightarrow \hat{x} \quad \phi = 90^\circ \Rightarrow \hat{y}$

all unit vectors are orthogonal (orthonormal basis) :-

$\hat{a}_x \times \hat{a}_y = \hat{a}_z \qquad \hat{a}_\rho \times \hat{a}_\phi = \hat{a}_z \qquad \hat{a}_r \times \hat{a}_\theta = \hat{a}_\phi$

→ ~~For~~ Elevation plane :- two principal elevation planes – $\underset{\phi=0^\circ}{XZ}$, $\underset{\phi=90^\circ}{YZ}$.
can be characterized by angle $\theta$

$\theta = 0 \Rightarrow \hat{z}$ axis

$\hat{a}_\phi \rightarrow$ direction of increment of $\phi$.

$\hat{a}_\theta \rightarrow$ direction of increment of $\theta$ at a constant non-zero $\phi$.

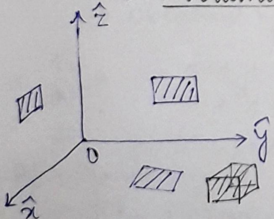
$0 \le \phi \le 360^\circ \qquad 0^\circ \le \theta \le 180^\circ$ by convention.

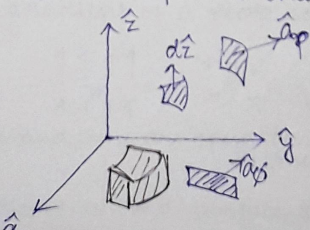
Incremental directions

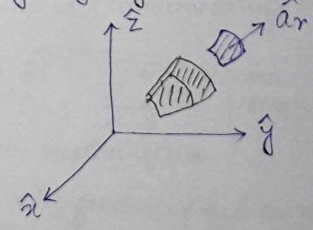
$d\vec{r} = dx\,\hat{a}_x + dy\,\hat{a}_y + dz\,\hat{a}_z \qquad d\vec{r} = d\rho\,\hat{a}_\rho + \rho\, d\phi\,\hat{a}_\phi + dz\,\hat{a}_z$

$d\vec{r} = dr\,\hat{a}_r + r\sin\theta\, d\phi\,\hat{a}_\phi + r\,d\theta\,\hat{a}_\theta$.

Incremental surfaces and volumes → characterized by normal unit vector while two other components changing slightly.




a volume → enclosed by all unit surfaces

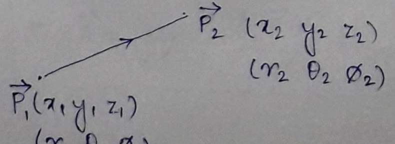
Projection Transformation

$\hat{a}_\phi = \sin\phi(-\hat{a}_x) + \cos\phi(\hat{a}_y)$

$\hat{a}_r = \sin\theta\cos\phi\,\hat{a}_x + \sin\theta\sin\phi\,\hat{a}_y + \cos\theta\,\hat{a}_z$

~~use~~ $\hat{a}_\theta =$ ~~a~~ $-\sin\theta\,\hat{a}_z + \cos\theta\cos\phi\,\hat{a}_x + \cos\theta\sin\phi\,\hat{a}_y$

→ Take any two points $\vec{P}_1$ $\vec{P}_2$ and express $\vec{P}_2 - \vec{P}_1$ vectorially in each system




conversion to cartesian then approach

# VECTOR CALCULUS

11th Jan '20

## Scalar fields

consider a function of $f(x,y,z)$ say for some point in cartesian space:-

* A gradient is defined on a scalar field only.

$f(x,y,z) = f_1 \qquad f(x,y,z) = f_1 + \Delta f$

↳ defines an equicontour

$$df(x,y,z) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = \left(\frac{\partial f}{\partial x}\hat{a}_x + \frac{\partial f}{\partial y}\hat{a}_y + \frac{\partial f}{\partial z}\hat{a}_z\right)\cdot(dx\,\hat{a}_x + dy\,\hat{a}_y + dz\,\hat{a}_z)$$

$$= (\vec{\nabla} f)\cdot d\vec{l}$$

$$\Rightarrow \boxed{df = (\vec{\nabla} f)\cdot d\vec{l}} \qquad \vec{\nabla} f \rightarrow \text{gradient}$$


now, $(df)|_{max}$ occurs ~~at~~ when $\vec{\nabla} f$ is along $d\vec{l}$

⇒ $\vec{\nabla} f$ denotes shortest path of increment (direction of maximum change)

⇒ Scalar field:- a function of $f(x,y,z)$ whose magnitude varies with $(x,y,z)$ over the entire ~~s~~ space

* Equipotential surface obtained by all paths $d\vec{l}$ such that $df = 0 \Rightarrow \boxed{\vec{\nabla} f \cdot d\vec{l} = 0}$

## Distance field

distance $(r) = \sqrt{x^2+y^2+z^2}$ as a scalar field

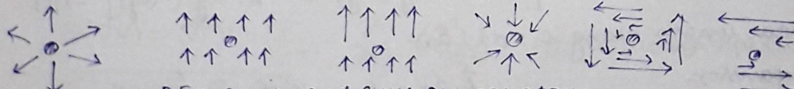
$$\Rightarrow dr = \Sigma \frac{\partial r}{\partial x}\hat{a}_x = \frac{x\,\hat{a}_x + y\,\hat{a}_y + z\,\hat{a}_z}{r} = \frac{\vec{r}}{r}$$

In spherical, $\vec{r} = r\,\hat{a}_r \Rightarrow \vec{\nabla} r = \hat{a}_r \Rightarrow$ gradient in radially outwards direction

## Vector fields

A vector function $\vec{F}(x,y,z)$ defined over Real space such that it maps a vector to each coordinate $(x,y,z)$, which are defined by functions of ~~f~~ $x,y,z$ at that point.

→ Obtained by gradient of a ~~vecto~~ scalar field [conservative]



→ some Vector fields

→ Divergence and curl are operators defined over vector fields

* **Divergence**

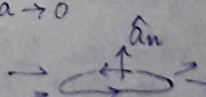
Consider a small volume, with a surface, then divergence defined as the net field flux at the point when volume → 0

$$\Rightarrow \boxed{\nabla\cdot\vec{A} = \lim_{\Delta V \to 0} \frac{\oint \vec{A}\cdot d\vec{s}}{\Delta V}}$$

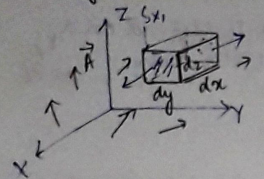

* **Curl** (circulation)

The net rotation of the vector field around the particular point along a closed contour, with contour ~~length~~ area → 0

$$\boxed{\nabla\times\vec{A} = \lim_{\Delta S \to 0} \frac{\oint_L \vec{A}\cdot d\vec{l}}{\Delta S}\,\hat{a}_n}$$



Defining $\nabla \cdot$ in cartesian system




an infinitesimal element cuboid of volume dx, dy, dz

surface given by x = constant

$$Sx_1 = \hat{a}_x\, dy\, dz \sim A(x + \tfrac{dx}{2}, y, z)$$

$$\simeq A(x,y,z) + \frac{dx}{2}\frac{\partial A_x}{\partial x}(x,y,z) \quad \leftarrow \text{Taylor expansion}$$

$$Sx_2 = -\hat{a}_x\, dy\, dz \sim A(x - \tfrac{dx}{2}, y, z)$$

$$\simeq A(x,y,z) - \frac{dx}{2}\frac{\partial A_x}{\partial x}$$

so, $\iint_{Sx} \vec{A}\cdot d\vec{S} = Sx_1 \cdot d\vec{S} + Sx_2 \cdot d\vec{S}$

$= \left(dx \frac{\partial A_x}{\partial x}\right) dy\, dz$ → only $A_x$ component contributes

also:- similarly:-

$\iint_{Sy} \vec{A}\cdot d\vec{S} = \left(dy \frac{\partial A_y}{\partial y}\right) dx\, dz$ $\qquad \iint_{Sz} \vec{A}\cdot d\vec{S} = \left(dz \frac{\partial A_z}{\partial z}\right) dx\, dy$

Thus by above:-

$$\oiint \vec{A}\cdot d\vec{S} = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) dx\, dy\, dz$$

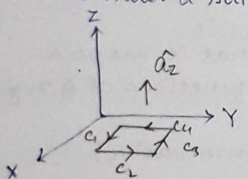
by defination,

$$\lim_{\Delta V \to 0} \frac{\oiint \vec{A}\cdot d\vec{S}}{\Delta V} = \frac{\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) \cancel{dx\, dy\, dz}}{\cancel{dx\, dy\, dz}}$$

$$\boxed{\vec{\nabla}\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}}$$ → defined in cartesian coordinate system only

Curl:-

consider a surface made by dx, dy, so that normal to plane denoted by $\hat{a}_z$




$$\oint_L \vec{A}\cdot d\vec{l} = \left(\int_{C_1} \vec{A}\cdot d\vec{l} + \int_{C_3} \vec{A}\cdot d\vec{l}\right) + \left(\int_{C_2} \vec{A}\cdot d\vec{l} + \int_{C_4} \vec{A}\cdot d\vec{l}\right)$$

$$= \left[\int \vec{A}\cdot dx\,(\hat{a}_x) + \int \vec{A}\, dx\cdot(-\hat{a}_x)\right]$$

$$\left[+ \int \vec{A}\, dy\,(\hat{a}_y) + \int \vec{A}\cdot dy\,(-\hat{a}_y)\right] \to (1)$$

incremental length along $c_1$: $dx\,(\hat{a}_x)$

From Taylor expansion, $\qquad \vec{A}\cdot\hat{a}_x = A_x \qquad \vec{A}\cdot\hat{a}_y = A_y$

~~Ax(x,y,z)~~

$$A_x(x, y \pm \tfrac{dy}{2}, z) = A_x(x,y,z) \pm \frac{dy}{2}\frac{\partial A_x}{\partial y}$$

so, $\int_{C_1 \& C_3} \vec{A}\cdot dx\,(\hat{a}_x) = (-)dy\, dx\, \cancel{\frac{\partial A_y}{\partial x}}\, \frac{\partial A_x}{\partial y}$

similarly,

$\int_{C_2 \& C_4} \vec{A}\cdot dy\,(\hat{a}_y) = dy\, dx\, \cancel{\frac{\partial A_x}{\partial y}}\, \frac{\partial A_y}{\partial x}$

combining:- $\oint \vec{A}\cdot d\vec{l} = \cancel{\frac{\partial A_x}{\partial y}}\ dx\, dy \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$

From defination:-

$$\vec{\nabla}\times\vec{A}\Big|_{\hat{a}_z} = \lim_{\Delta S \to 0} \frac{\oint_L \vec{A}\cdot d\vec{l}}{\Delta S} = \lim_{\Delta S \to 0} \frac{\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\cdot \cancel{dx\, dy}}{\cancel{dx\, dy}} = \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{a}_z$$

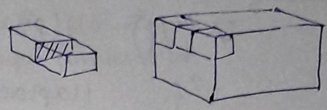
we can write

$$\boxed{\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}}$$ at $\vec{A}(x,y,z)$

## Stokes' Theorem

consider a large volume V made from ~~in~~ smaller elements $\Delta V$. By def, in each small $\Delta V$ element :-

$$\nabla \cdot \vec{A} = \lim_{\Delta V \to 0} \frac{\oint\!\!\oint \vec{A} \cdot d\vec{s}}{\Delta V}$$



now, in each subsequent shared surface by $\Delta V$'s, the $\int_{S\,(\text{shared})} \vec{A} \cdot d\vec{s}$ components cancel out and only boundary $\int \vec{A} \cdot d\vec{s}$ remain.

$$\Rightarrow \boxed{\iiint_V (\nabla \cdot \vec{A})\, dV = \oint\!\!\oint_S \vec{A} \cdot d\vec{s}}$$

## Divergence Theorem

similar argument for a large surface S and small elements $\Delta S$.

$$\boxed{\iint_S (\vec{\nabla} \times \vec{A}) \cdot d\vec{s} = \oint_L \vec{A} \cdot d\vec{l}}$$

## Helmholtz's Theorem

any vector field $\vec{A}$ can be expressed as gradient & curl:-

$$\boxed{\vec{A} = -\nabla V + \nabla \times \vec{F}}$$

and, $\boxed{\vec{\nabla} \cdot \vec{A} = \rho_v}$ (source density) $\boxed{\nabla \times \vec{A} = \vec{\rho_s}}$ (circulation density)

provided that $\rho_v$ and $\vec{\rho_s}$ vanish at infinity ($\infty$) :-

$$V \equiv \frac{1}{4\pi} \iiint \frac{\rho_v(r')}{|r - r'|} dv'$$

$r' \to$ position of source

$$\vec{F} = \frac{1}{4\pi} \iiint \frac{\vec{\rho_s}(r')}{|r - r'|} dv'$$

* ~~a vector is sino~~

* a vector field $\vec{A}$ is — → solenoidal for $\boxed{\nabla \cdot \vec{A} = 0}$ → Irrotational for $\boxed{\nabla \times \vec{A} = 0}$

$$\nabla \cdot (\nabla V) = \nabla^2 V = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) V \rightarrow \text{Laplacian operator}$$

$$\boxed{\nabla \cdot (\vec{\nabla} \times \vec{F}) = 0}$$

$$\nabla \cdot \vec{A} = \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r \sin\theta} \frac{\partial A_\phi}{\partial \phi}$$

15th Jan '20

Scalar Laplacian.

$\nabla\cdot(\nabla f)$ (gradient's divergence)

$= \nabla^2 f$ (Laplacian).

$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial y^2}\right) f$ (in cartesian)

find in spherical, cylindrical, etc.

* Curl of gradient:-

$$\boxed{\nabla\times(\nabla f) = 0}$$

* Gradient of curl:-

$$\boxed{\nabla\cdot(\nabla\times\vec{A}) = 0}$$

Curl of curl:-

$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \hat{a}_x\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{a}_y\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{a}_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \rightarrow (1)$$

From (1),

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} & \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} & \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \end{vmatrix} \quad \pm \frac{\partial^2 A_x}{\partial x^2}$$

$$= \hat{a}_x\left\{\left(\frac{\partial^2 A_y}{\partial y\,\partial x} - \frac{\partial^2 A_x}{\partial y^2}\right) - \left(\frac{\partial^2 A_x}{\partial z^2} - \frac{\partial^2 A_z}{\partial x\,\partial z}\right)\right\} + \hat{a}_y(\;) + \hat{a}_z(\;)$$

$$= \hat{a}_x\left[\underbrace{\left(\frac{\partial^2 A_y}{\partial y\,\partial x} + \frac{\partial^2 A_z}{\partial x\,\partial z} + \frac{\partial^2 A_x}{\partial x^2}\right)}_{\nabla(\nabla\cdot\vec{A})} - \underbrace{\left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2}\right)}_{\nabla^2 A_x}\right]$$

$$\Rightarrow \boxed{\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A}}$$

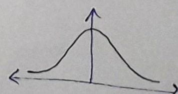
$\nabla^2\vec{A}$ → vector Laplacian.

It's valid only for cartesian form/system

Impulse function

$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-x^2/2\sigma^2}$ a Gaussian curve $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\delta(x) = \lim_{\sigma\to 0} f(x)$ a particular impulse



(Q) $\vec{A} = \frac{1}{r^2}\hat{a}_r$, find $\nabla\cdot\vec{A}$

$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\times\frac{1}{r^2}\right) = 0 \quad (r\neq 0)$

$d\vec{s} = \hat{a}_r\,(r^2\sin\theta\,d\theta\,d\phi)$

$\oint_S \vec{A}\cdot d\vec{s} = \iiint(\nabla\cdot\vec{A})\,dv = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\sin\theta\,d\theta\,d\phi = 4\pi$ → Impulse in 3D

at taken $r = 0$,

$\nabla\cdot\vec{A} = 4\pi\,\delta(r) \Rightarrow \boxed{\iiint_V \delta(r)\,dV = 1}$ → $\delta(r)$ → Spatial impulse funcn

# ELECTROSTATICS

Deals with charges which have no change of electric green function with position or time

$\Rightarrow -\frac{\partial \vec{B}}{\partial t} = 0$ so $\nabla \times \vec{E} = 0 \rightarrow (1)$

and, $\nabla \cdot \vec{D} = \rho_v$ $\quad \frac{\partial \vec{D}}{\partial t} = 0$ $\quad \nabla \times \vec{H} \rightarrow$ not considered.

$\Rightarrow$ Linear conductor follows relation:- $\boxed{\vec{J_c} = \sigma \vec{E}}$ where $\boxed{\sigma = en\mu e}$

at extreme cases, non-ohmic variables:- $\sigma(\vec{E})$. also $\mu(\vec{H})$.

also a relation:- $\boxed{\vec{D} = \epsilon \vec{E}}$ we'd taken $\epsilon \rightarrow$ a scalar constant

Ohm's law (a circuital application) is directly derived from the linear relation

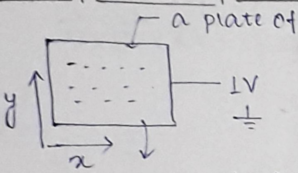
as $\nabla \times \vec{E} = 0$ for a field $\vec{E}$, we can write $\boxed{\vec{E} = -\nabla V}$ for a scalar potential V.

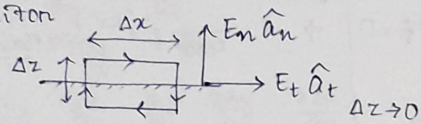
so, $\nabla \cdot \vec{D} = \epsilon \nabla \cdot \vec{E} = \epsilon(\nabla \cdot (-\nabla V)) = -\epsilon \nabla^2 V$. from Gauss' law

so, we get PDE:- (scalar $\epsilon$)

$$\boxed{\nabla^2 V + \frac{\rho_v}{\epsilon} = 0}$$ with boundary conditions

The capacitance problem




charge $q(x,y)$

net charge $Q = \iint q(x,y)\,dx\,dy$

* Boundary conditions:-
  - → tangential $\vec{E}$-field is continous at boundary.
  - → Ideal metal: no $\vec{E}$ inside metal, otherwise infinite current.

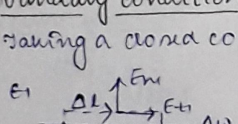
17 Jan '20

* Reading-

Ch-3: Electrostatic fields. Ch-4: Fields in material space Ch-5: BVP

Boundary condition

Taking a closed contour around boundary



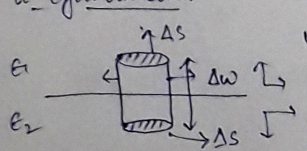

$\nabla \times \vec{E} = 0 \Rightarrow \oint_C \vec{E} \cdot \vec{dl} = 0$

$\oint_L \vec{E} \cdot \vec{dl} = E_{t_1} \Delta L - E_{n_1} \frac{\Delta w}{2} + E_{n_1} \frac{\Delta w}{2} - E_{t_2} \Delta L - E_{n_2} \frac{\Delta w}{2} + E_{n_2} \frac{\Delta w}{2} = 0$

so, $\boxed{E_{t_1} = E_{t_2}}$ $\Rightarrow$ isotropic medium:- $\boxed{\frac{D_{t_1}}{\epsilon_1} = \frac{D_{t_2}}{\epsilon_2}}$

(continous)

a cylindrical:-




using $\nabla \cdot \vec{D} = \rho_v \Rightarrow \oint \vec{D} \cdot \vec{dS} = q$

$\Rightarrow D_{n_1} \Delta S + D_{n_2} \Delta S = q \Rightarrow \boxed{\hat{n} \cdot (\vec{D_1} - \vec{D_2}) = q/\Delta S}$

* in horizontal direction $\vec{E} \cdot \vec{dS}$ cancel at diametrically opposite ends

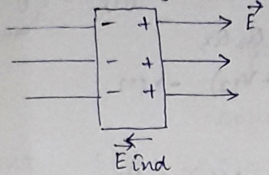
as $\Delta S \rightarrow 0$, $q/\Delta S \rightarrow \rho_s$ (surface charge density)

$\Rightarrow \lim_{\Delta S \rightarrow 0} \frac{q}{\Delta S} = \rho_s \Rightarrow \boxed{\hat{n} \cdot (\vec{D_1} - \vec{D_2}) = \rho_s}$

a metal plate is brought to a $\vec{E}$-field in the conductor

→ charge separation occurs, so that they produce an opposite field such that net $\vec{E}$-field inside is zero.

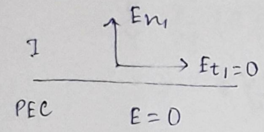
A surface charge $\rho_s$ accumulates.




* If no surface charge present, $\boxed{\vec{D}_{n1} = \vec{D}_{n2}}$, normal $\vec{D}$ component is continous.

Perfect Electric conductor

In a PEC, no electric-field is present ⇒ $E_{n2}=0$ $E_{t2}=0$




so, $E_{t1} = E_{t2} = 0$ →(1)

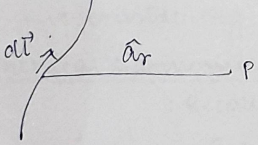
also, $\hat{n}\cdot(\vec{D_1} - \vec{D_2}) = 0$ ⇒ $\vec{D}_2 = 0$ gives $\boxed{E_{n1} = \frac{\rho_s}{\epsilon_1}}$

* ~~Et field~~

* Electric field emerge perpendicular to the boundary surface
* surface charge distribution: highest at the boundary

Electric field due to continous distribution

Eg:- A long charged body is producing some $\vec{E}$ field at P.



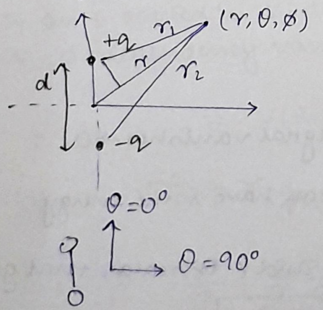

$$\vec{E} = \int \frac{\rho_l\, d\vec{l}}{4\pi\epsilon_0 R^2}\,\hat{a}_r\,(d\vec{l})$$

integral is difficult, so we measure potential V

$$V = \int \frac{\rho_l\, dl}{4\pi\epsilon_0 R}$$ a scalar and $\boxed{\vec{E} = -\nabla V}$

Electric dipole problem

two opposite and equal charges q separated by a distance d.




→ assume $d \ll r$, $d \sim Å$.

$r_1 \simeq r - \frac{d}{2}\cos\theta$

$r_2 \simeq r + \frac{d}{2}\sin\theta$

$$\Rightarrow V(r) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r - d/2\cos\theta} - \frac{1}{r + d/2\sin\theta}\right)$$

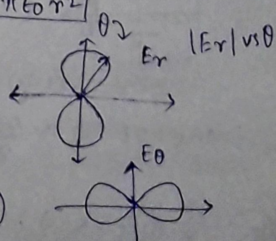
$$= \frac{q}{4\pi\epsilon_0 r}\left(\frac{1}{\left(1 - \frac{d/2\cos\theta}{r}\right)} - \frac{1}{\left(1 + \frac{d/2\cos\theta}{r}\right)}\right)$$

use $(1-x)^{-1} = 1 + x \ldots$

$$\Rightarrow \boxed{V(r) = \frac{q}{4\pi\epsilon_0}\frac{d\cos\theta}{r^2}}$$

we define dipole moment : $\boxed{\vec{p} = qd\,\hat{a}_z}$ so, $\boxed{V(r) = \frac{\vec{p}\cdot\hat{a}_r}{4\pi\epsilon_0 r^2}}$ field plots →

and to get, $\vec{E} = -\nabla V = -\frac{\partial V}{\partial r}$ ⇒ ~~$\vec{E} = \frac{+2p}{4\pi\epsilon_0 r^3}\hat{a}_r$~~




* $\theta$ variation is not uniform
⇒ $\phi$ variation is uniform (cone) at constant $\theta$)

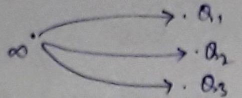
$$\vec{E} = -\nabla V = \frac{|\vec{p}|}{4\pi\epsilon_0 r^3}\left(2\cos\theta\,\hat{a}_r + \sin\theta\,\hat{a}_\theta\right)$$

→ $E_r$ max at $\theta = 0^\circ$ (axial line)

~~Energy density of distribution of charges~~

## ELECTROSTATIC ENERGY

Suppose 3 charges at infinity, brought at a location one by one, so some work gets done that contributes to gain of electrostatic energy of system



work done in bringing $Q_1, Q_2 Q_3$

$$W = 0 + Q_2 V_{21} + Q_3 (V_{31} + V_{32}) \rightarrow (1)$$

and bringing $Q_3, Q_2, Q_1$

$$W = 0 + Q_2 V_{23} + Q_1 (V_{13} + V_{12}) \rightarrow (2)$$

From (1) & (2)

$$2W = Q_2(V_2 - V_1) + Q_3(V_3 - V_1 + V_3 - V_2) +$$

$$\Rightarrow \boxed{W = \frac{1}{2} \sum_{i=1}^{n} Q_i (V_i)}$$

* For infinite charges brought, we integrate:-

$$W = \frac{1}{2} \int \rho_l (r)\, V(r)\, dl \quad \leftarrow \text{line distribution}$$

$$W = \frac{1}{2} \iiint \rho_v (r) V(r)\, dv \quad \leftarrow \text{volume distribution}$$

From Gauss's law:-

$$= \frac{1}{2} \iiint (\nabla \cdot \vec{D})\, V(r)\, dv \quad -(1)$$

* Vector identity used:-

$$\nabla \cdot (f\vec{A}) = \vec{A} \cdot \nabla f + f(\nabla \cdot \vec{A})$$

$$(\nabla \cdot \vec{A}) f = \nabla \cdot (f\vec{A}) - \vec{A} \cdot \nabla f$$

so, $(\nabla \cdot \vec{D})\, V(r) = \nabla \cdot (\vec{D} V(r)) - \vec{D} \cdot \nabla V$

contd.:-

$$\Rightarrow W = \frac{1}{2} \iiint \big( \underbrace{\nabla \cdot (V\vec{D})}_{\text{II}} - \underbrace{\vec{D} \cdot (\nabla V)}_{\vec{D} \cdot \vec{E}} \big)\, dv \quad \Rightarrow \quad \boxed{W = \frac{1}{2} \iiint \vec{D} \cdot \vec{E}\, dv}$$

so, energy density, (J/m³) $\boxed{u = \frac{1}{2} \vec{D} \cdot \vec{E}}$

The integral II, $\iiint \nabla \cdot (V\vec{D}) = \oint V\vec{D} \cdot d\vec{s}$

but as $V \sim \frac{1}{r}$, $\vec{D} \sim \frac{1}{r^2}$, the integral vanishes $\rightarrow 0$.

* Energy density is related to capacitance, so we can have some energy stored in an capacitor.

For an isotropic and regular medium $\vec{D} = \epsilon \vec{E}$ and $\epsilon$ is scalar, that gives:-

$$u = \frac{1}{2} \epsilon (\vec{E} \cdot \vec{E}) \Rightarrow \boxed{u = \frac{1}{2} \epsilon |\vec{E}|^2}$$

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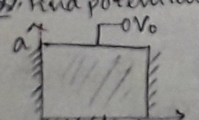
**Q.** Find potential distribution:-




boundary conditions:-

$$\begin{cases} V(x, a) = V_0 \\ V(x, 0) = V(0, y) = V(b, y) = 0 \end{cases}$$

In electrostats, no charge $\rho_v = 0$

$\nabla \cdot \vec{D} = 0$ & $\nabla \cdot \vec{E} = 0$ & $(\nabla \cdot (-\nabla V)) = 0$

$$\Rightarrow \boxed{\nabla^2 V = 0}$$

$$\Rightarrow \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \rightarrow \text{Read PDE}$$

assume $V(x,y) = X(x)\, Y(y)$ [separation of variables]

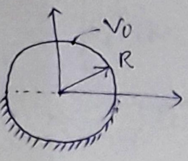
From PDE $\Rightarrow Y \frac{\partial^2 X}{\partial x^2} + X \frac{\partial^2 Y}{\partial y^2} = 0 \Rightarrow \underbrace{\frac{1}{X} \frac{\partial^2 X}{\partial x^2}}_{f(x)} + \underbrace{\frac{1}{Y} \frac{\partial^2 Y}{\partial y^2}}_{f(y)} = 0$

that imply:- $\frac{1}{X} \frac{\partial^2 X}{\partial x^2} = -\frac{1}{Y} \frac{\partial^2 Y}{\partial y^2} = \frac{k^2}{\uparrow}$ (say) $\overline{k > 0}$

$$\frac{1}{X} \frac{\partial^2 X}{\partial x^2} = k \Rightarrow X(x) = c_1 e^{kx} + c_2 e^{-kx}$$

$$Y(y) = c_3 e^{jkx} + c_4 e^{-jkx}$$

## Laplace equation in cylindrical system




→ independent of z direction:-

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2}=0$$

separation of variables:- $V(\rho,\phi)=Y(\rho)\,S(\phi)$

gives:- $\frac{\rho}{Y}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial Y}{\partial\rho}\right)=-\frac{1}{S}\frac{\partial^2 S}{\partial\phi^2}=K$

solution: $S=\sin(\sqrt{K}\phi),\ \cos(\sqrt{K}\phi)$

but, $S(\phi+2n\pi)=S(\phi)$ → inbuilt boundary condn.

⇒ $\sqrt{K}=n\in\mathbb{N}$ only ~~or~~ not 0 as $S(\phi)=1$ ?

* boundary conditions:-

if $n\neq 0$ $K=n^2$

$$\frac{\rho}{Y}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial Y}{\partial\rho}\right)=n^2$$

⇒ ~~$\rho\frac{\partial Y}{\partial\rho}$~~ ⇒ $Y(\rho)=\rho^n$ or $\rho^{-n}$

$\frac{\rho}{Y}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial Y}{\partial\rho}\right)=0$

⇒ $\frac{\partial Y}{\partial\rho}=\frac{c}{\rho}$ ⇒ Y = cmp or K const.

possible solutions:-

$$V(\rho,\phi)=\begin{cases}1,\ \text{cmp}\\ \rho^n(\cos n\phi,\ \sin n\phi)\\ \rho^{-n}(\cos n\phi,\ \sin n\phi)\end{cases}$$

→ cylindrical harmonics at z-symmetry.

* $\ln\rho$, $\rho^{-n}$ eliminated as $\rho\to 0$, the $V\to\infty$ which is not possible (when considering $\rho<R$)

but considering $\rho>R$, only $\rho^n$ is eliminated. ⇒ BCs depend on region of interest.

## Spherical harmonics

→ Recall hydrogen-atom (HELL LOT OF MATHS)

$\nabla^2 V=0$ gives:-

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}=0$$

→ If an $\vec{E}$ applied in z-direction, then it's symmetric about $\phi$, so $\phi$ dependency vanishes → $\frac{\partial^2 V}{\partial\phi^2}=0$ ⇒ $\boxed{V=e^{-im\phi}}$

⇒ $V(R,\theta)=R(r)\,Y(\theta)$ gives:-

$$\nabla^2 V=Y\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{R}{r^2\sin^2\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{dY}{\partial\theta}\right)=0$$

$$\Rightarrow\ \underbrace{\frac{r^2}{R}\left(\frac{2}{r}\frac{\partial R}{\partial r}+\frac{\partial^2 R}{\partial r^2}\right)}_{K}+\underbrace{\frac{1}{Y\sin\theta}\left(\cos\theta\frac{\partial Y}{\partial\theta}+\frac{\partial^2 Y}{\partial\theta^2}\sin\theta\right)}_{-K}=0$$

solving individually:-

$$\frac{r^2}{R}\left(\frac{2}{r}\frac{\partial R}{\partial r}+\frac{\partial^2 R}{\partial r^2}\right)=K$$

$$\Rightarrow \frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)=K$$

$$\Rightarrow r^{\cancel{2}}\frac{\partial R}{\partial r}=KR\cancel{r}$$

$$\frac{1}{Y\sin\theta}\left(\cos\theta\frac{\partial Y}{\partial\theta}+\frac{\partial^2 Y}{\partial\theta^2}\sin\theta\right)=\cancel{0}-K$$

take $\mu=\cos\theta$ $\frac{d\mu}{d\theta}=-\sin\theta$

$$\Rightarrow \frac{d}{d\mu}\left[(1-\mu^2)\frac{dY}{d\mu}\right]+KY=0$$ → Legendre DE

For $K=n(n+1)$ solution is given by

Legendre fourier → as $P_n(\mu)$ → a poly in $\cos\theta$ of order n.

$$Y(\mu)=a_0+a_1\mu$$

* all ~~ce~~ $P_n(\mu)$ are orthogonal to each other.

Orthogonality of Legendre polynomial →

$$\int_{-1}^{+1} P_m(\mu) P_n(\mu)\, d\mu = \begin{cases} 0, & m \neq n \\ \frac{2}{2n+1} & m = n \end{cases} = \frac{2}{2n+1} \delta_{mn}$$

Legendre recursion relation:-

$$\boxed{(2n+1)\mu P_n(\mu) = (n+1) P_{n+1}(\mu) + n P_{n-1}(\mu)}$$

now from $K = n(n+1)$ and radial dependence →

$$\frac{1}{R} \frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) = n(n+1)$$

let $R(r) = r^l$

$$\Rightarrow r^2 l(l-1) r^{l-2} + 2r(l) r^{l-1} - n(n+1) r^l = 0$$

$\Rightarrow \boxed{l(l+1) = n(n+1)}$ always hold.

$\Rightarrow l = n$ or $l = -(n+1)$.

so, $R(r) = \begin{cases} r^n \\ \frac{1}{r^{n+1}} \end{cases} \Rightarrow$ spherical harmonics:- $\begin{cases} r^n P_n(\cos\theta) \\ \frac{1}{r^{n+1}} P_n(\cos\theta) \end{cases}$

* Boundary conditions:-

grounded sphere → $V = 0 \;(r = R)$ $\quad \theta, \phi \;$ -(1) $\quad \vec{E} = E_0 \hat{a}_z$ -(2)

$V = -E_0 z \quad (r \gg R)$

$V(r) = \sum A_n r^n P_n(\cos\theta)$ → Linear comb. of all solutions

$+ \sum B_n \frac{1}{r^{n+1}} P_n(\cos\theta) = \sum \left(A_n r^n + \frac{B_n}{r^{n+1}}\right) P_n(\cos\theta)$

from (1),

$$A_l R^l + \frac{B_l}{R^{l+1}} = 0 \Rightarrow B_l = -A_l (R^{2l+1})$$

$$\Rightarrow \boxed{V(r) = \sum \left(A_n r^n - A_n \frac{R^{2n+1}}{r^{n+1}}\right) P_n(\cos\theta)}$$

from (2):-

$$V(r,\theta)\;(r \gg R) = -E_0 r\cos\theta$$

In general, $V(\tilde{R}, \theta) = \sum_{n=0}^{\infty} A_n \tilde{R}^n P_n(\cos\theta)$

$$\Rightarrow \langle V(\tilde{R},\theta), P_m(\cos\theta)\rangle = \sum_{n=0}^{\infty} A_n \tilde{R}^n \underline{\langle P_n(\cos\theta)\, P_m(\cos\theta)\rangle}$$

$$\Rightarrow \int_{-1}^{+1} V(\tilde{R},\theta) \cdot P_m(\cos\theta)\, d(\cos\theta) = \frac{2}{2m+1} A_m \tilde{R}^m$$

Generalized $\phi$-dependency:-

$$\frac{1}{S} \frac{\partial^2 S}{\partial \phi^2} = -m^2 \Rightarrow S = e^{im\phi} \quad m \in \mathbb{N} \text{ as } S(\phi + 2n\pi) = S(\phi)$$

due to $m^2$ term some complicacy arises-

$V(r,\theta,\phi) = R(r)\, T(\theta)\, S(\phi)$

Previously $m = 0$

$$\frac{1}{\sin\theta} \frac{d}{d\theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right) + \left(l(l+1) - \frac{m^2}{\sin^2\theta}\right) T(\theta) = 0$$

$$\Rightarrow \mu = \cos\theta \quad \frac{d}{d\mu}\left[(1-\mu^2) \frac{dT}{d\mu}\right] + \left(l(l+1) - \frac{m^2}{1-\mu^2}\right) T = 0$$ → associated Legendre

$\Rightarrow$ solution $\boxed{T_l = P_l^m(\mu)}$ → $m$th derivative of $l$th order polynomial in $\cos\theta$

taking inner product ($P_l^m$ are orthogonal):-

$$\left[P_l^m(\mu) = (-1)^m (1-\mu^2)^{m/2} \frac{d^m}{d\mu^m} P_l(x)\right] \quad m>0$$

$$\Rightarrow \int_{-1}^{1} P_l^m(\mu) P_{l'}^n(\mu)\, d\mu = \begin{cases} 0 & n\neq m \\ \frac{2}{2l+1}\frac{(l+m)!}{(l-m)!} & n=m,\ l=l' \end{cases}$$

Generalised solution:-

$$\boxed{V(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\left(A_{lm} r^l + \frac{B_{lm}}{r^{l+1}}\right) \times P_l^m(\cos\theta) \times e^{jm\phi}}$$

Generalized cylindrical Harmonics

30 Jan'20

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial\rho^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad -(1)$$ assume $V(\rho,\phi,z) = R(\rho)\,P(\phi)\,Z(z)$.

Separation of variable -

$$\frac{1}{P(\phi)}\frac{\partial^2 P}{\partial\phi^2} = -m^2 \qquad P(\phi) = A\sin(m\phi) + B\cos(m\phi)$$

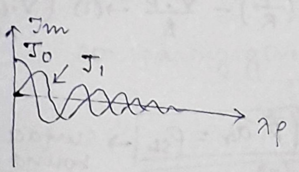
$$\frac{1}{Z(z)}\frac{\partial^2 z}{\partial z^2} = \lambda^2 \qquad \Rightarrow Z(z) = \tilde{A}\sinh(\lambda z) + \tilde{B}\cosh(\lambda z)$$

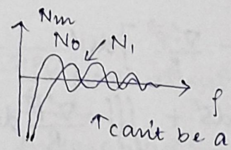
From (1), remove z - dependency:-

$$\frac{\rho}{R(\rho)}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right) + \lambda^2\rho^2 = 0 \Rightarrow \left[\rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right) + (\lambda^2\rho^2 - m^2)R = 0\right] \rightarrow \text{BESSEL function}$$

solution are:- $R(\rho) = \begin{cases} J_m(\lambda\rho) \rightarrow \text{Bessel fn. of 1st kind (more used)} \\ N_m(\lambda\rho) \rightarrow \text{Bessel fn. of 2nd kind} \end{cases}$

all $J_m$, $N_m$ are orthogonal.




↑ can't be a soln. as $\rho \to 0$, $N_m \to \infty$ & impractical

& MATLAB command "BESSEL"

→ from orthogonality:- (advanced orthogonality?)

$\langle \phi_m, \phi_n \rangle = \delta_{ij}$ and weighting fn. $= w(x) = \rho$.

$$\Rightarrow \int_0^{\rho} \underline{\underline{\rho}}\, J_m(\lambda\rho)\, J_n(\lambda'\rho)\, \partial\rho = \begin{cases} \frac{1}{\lambda}\delta(\lambda-\lambda') & , m=n \\ 0 & , m\neq n \end{cases}$$

↑ yeh aagye weighting fn!

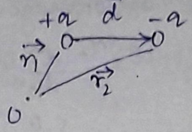
## LAPLACIAN EQUATION ($\nabla^2 V = 0$)

| cartesian | cylindrical | spherical |
|---|---|---|
| $x\begin{cases}\sin\left(\frac{n\pi}{b}x\right) \rightarrow k_x \\ \cos\left(\frac{n\pi}{b}x\right)\end{cases}$ | $P\begin{cases}\sin(m\phi) \\ \cos(m\phi)\end{cases}$ | $P\begin{cases}\sin m\phi \\ \cos m\phi\end{cases}$ |
| $y\left\{\sin\left(\underset{=k_y}{\frac{m\pi}{a}}y\right) \cos\left(\frac{m\pi}{a}y\right)\right.$ | $Z\begin{cases}\sinh(\lambda z) \\ \cosh(\lambda z)\end{cases}$ | $R\left\{ r^l, \frac{1}{r^{l+1}}\right.$ |
| $z\left\{\sinh(k_z z) \quad \cosh(k_z z)\right.$ | $R\begin{cases}J_m(\lambda\rho) \\ N_m(\lambda\rho)\end{cases}$ Bessel. | $\theta\left\{P_l^m(\cos\theta)\right.$ $m = 1, 2, \ldots l$ |

31 Jan'20

# POLARISATION

dipole moment :- $\vec{P} = Q\vec{d}$




potential due to $\vec{P}$:

$$\boxed{V(r) = \frac{\vec{P}\cdot\hat{a}_r}{4\pi\epsilon_0 r^2}}$$

$\nabla\cdot(\vec{AB})$
$= (\nabla\cdot A)\, B + (\nabla\cdot B)A$
$d(xy) = y\,dx + x\,dy$

in a dielectric material, all molecules get aligned due to external $\vec{E}$-field

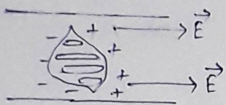
$$\text{polarisation}(\vec{P}) = \lim_{\Delta v\to 0}\frac{\sum P_k}{\Delta v_k}$$

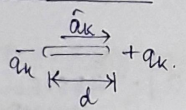
defn :- the per unit volume dipole moment in a medium. when $\vec{E}$-applied

total potential :-

$$\left[\int dV = \int_v \frac{\vec{P}\cdot\hat{a}_r\, dv'}{4\pi\epsilon_0 R^2}\right] \qquad \text{since } \nabla'\left(\frac{1}{R}\right) = \frac{\hat{a}_r}{R^2}$$

$$\int_v dV = \frac{1}{4\pi\epsilon_0}\int \vec{P}\cdot\nabla'\left(\frac{1}{R}\right)dv'$$




we define polarisation :-

$$\left[\vec{P} = \lim_{\Delta v_k\to 0}\frac{\overbrace{\sum q_k d_k\, \hat{a}_k}^{\vec{P}}}{\Delta v_k}\right] \quad C/m^2$$

for a potential :-

$$V = \int dV = \frac{1}{4\pi\epsilon_0}\iiint_{v'}\frac{\vec{P}\cdot\hat{a}_k}{R^2}\,dv'$$ ← volume integral

as $\nabla'\left(\frac{1}{R}\right) = \frac{\hat{a}_R}{R^2}$ $\quad \frac{\vec{P}\cdot\hat{a}_R}{R^2} = \vec{P}\cdot\nabla'\left(\frac{1}{R}\right)$

$$= \nabla'\left(\frac{\vec{P}}{R}\right) - \frac{\nabla'\cdot\vec{P}}{R} \rightarrow (1) \quad (\nabla\cdot(AB) \text{ rule})$$

so, $V = \frac{1}{4\pi\epsilon_0}\iiint_{v'}\left[\nabla'\left(\frac{\vec{P}}{R}\right) - \frac{\nabla'\cdot\vec{P}}{R}\right]dv'$

$$= \frac{1}{4\pi\epsilon_0}\left[\underbrace{\oint_{s'}\frac{\vec{P}\cdot\hat{a}_r}{R}\,ds'}_{\text{contribution of surface charge}} + \underbrace{\iiint_{v'}\frac{-\nabla'\cdot\vec{P}}{R}\,dv'}_{\text{contribution of volumetric charge}}\right]$$

$\boxed{\vec{P}\cdot\hat{a}_r = \rho_{sb}}$ → surface bound charge

$\boxed{-\nabla'\cdot\vec{P} = \rho_{vb}}$ → volume bound charge

from conservation of charge :-

$Q_{bs} = \oint \vec{P}\cdot d\vec{S}$ ↑ surface bound charge $\qquad Q_{bv} = \iiint(-\nabla\cdot\vec{P})\,dv'$ ↑ volume bound charge

$= \nabla\cdot(\epsilon_0\vec{E})$

so we'll have :- $\boxed{Q_{bs} + Q_{bv} = 0}$ (in absence of external charge)

* In presence of external charge:

a free charge inserted in ~~the~~ dielectric material as $\rho_v$.

$\Rightarrow \rho_t = \rho_v + \rho_{bv}$ (← bound volume) (from charge conservation)

$\Rightarrow \rho_v = \nabla\cdot(\epsilon_0\vec{E}) - (-\nabla\cdot\vec{P})$ from above

$\rho_v = \nabla\cdot(\epsilon_0\vec{E} + \vec{P})$

compare $\rho_v = \nabla\cdot\vec{D}$, $\quad \boxed{\vec{D} = \epsilon_0\vec{E} + \vec{P}}$ in the material.

→ (1)

most linear dielectric materials satisfy:-

$\vec{P} \propto \vec{E}$ and direction same

In artificial metamaterials, $\vec{P}$ opposite of $\vec{E}$ field.

at high $\vec{E}$, dielectric breakdown may occur

we write $\boxed{\vec{P} = \chi_e \epsilon_0 \vec{E}}$ in all natural cases $\chi_e > 0$

$\chi_e$ → susceptibility

so from (1)

$$\vec{D} = \epsilon_0\vec{E} + \vec{P} = \epsilon_0\vec{E} + \chi_e \epsilon_0 \vec{E} \Rightarrow \boxed{\vec{D} = \epsilon_0 \vec{E}(1+\chi_e)}$$

we define effective ~~susceptibility~~ permittivity, $\boxed{\epsilon = \epsilon_0(1+\chi_e)}$ then $\boxed{\vec{D} = \epsilon\vec{E}}$ in a polarised medium.

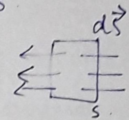
* results in steady state only. Transients are fast, decay fast and give rise to radiation

since relative permittivity, $\boxed{\epsilon_r = \frac{\epsilon}{\epsilon_0} = (1+\chi_e)}$

## CONTINUITY EQUATION

say some charges are flowing through a cross-section $d\vec{S}$

so, a current is flowing defined by:-




$$I_{out} = \oint_S \vec{J}\cdot d\vec{S} = -\frac{dQ_{in}}{dt}$$

negative as total charge enclosed in closed surface

as $dQ = \rho_v dv \Rightarrow Q = \iiint_{v'} \rho_v dv'$.

$$\Rightarrow \oint_S \vec{J}\cdot d\vec{S} = -\frac{d}{dt}\iiint_v \rho_v dv'$$

from some vector calculus, $\oint_S \vec{J}\cdot d\vec{S} = \iiint_{v'} (\nabla\cdot\vec{J})\, dv'$

comparing gives:-

$$\boxed{\nabla\cdot\vec{J} = -\frac{\partial \rho_v}{\partial t}} \rightarrow \text{continuity eqn.}$$

* this can also be derived from Ampere's law:-

$$\nabla\times\vec{H} = \vec{J}_c + \frac{\partial \vec{D}}{\partial t}$$

$\frac{\partial \vec{D}}{\partial t}$ → displacement current

take divergence both sides:-

$$\underbrace{\nabla\cdot(\nabla\times\vec{H})}_{0} = \nabla\cdot\vec{J}_c + \frac{\partial}{\partial t}(\nabla\cdot\vec{D}) \quad \text{as } \nabla\cdot\vec{D} = \rho_v \text{ from Gauss' law.}$$

$$\Rightarrow \boxed{\nabla\cdot\vec{J}_c + \frac{\partial \rho_v}{\partial t} = 0}$$

Transient time

using $\vec{J} = \sigma\vec{E}$ and $\nabla\cdot\vec{E} = \frac{\rho_v}{\epsilon}$ $\Rightarrow \vec{\nabla}\cdot\left(\frac{1}{\sigma}\vec{J}\right) = \frac{\rho_v}{\epsilon} \Rightarrow \frac{1}{\sigma}\vec{\nabla}\cdot\vec{J} = \frac{\rho_v}{\epsilon}$.

$\cancel{\nabla\times(\partial \vec{E}) + \frac{1}{\sigma}\frac{\partial}{\partial t}(\epsilon\nabla\cdot\vec{E})}$ $\quad \Rightarrow \cancel{\frac{1}{\sigma}\frac{\partial \rho_v}{\partial t} \; \rho_v}$

from continuity:-

$$\left[\frac{\partial \rho_v}{\partial t} + \left(\frac{\sigma}{\epsilon}\right)\rho_v = 0\right] \rightarrow \text{linear first order equation}$$

on solving,

$$\boxed{\rho_v(t) = \rho_{v0}\exp\left(-\frac{t}{T_r}\right)}$$

where $T_r$ : Transient time $\left[T_r = \frac{\epsilon}{\sigma}\right]$

* Typically, for most materials, $T_r \ll 1$, as in Cu: $T_r \sim 10^{-19}$ s.

# MAGNETOSTATICS

05 Feb'20

Deals with:-
(1) Steady current $\vec{J}$
(2) Biot - Savart law
(3) Ampere's law
} are correspondent

$\boxed{\nabla\cdot\vec{J}=0}$

Magnetic field ($\vec{H}$): A/m
Magnetic flux ($\phi$): Wb
Magnetic flux density ($\vec{B}$): T, Wb/m²

magnetostatics laws:- $\boxed{\begin{array}{l}\nabla\cdot\vec{B}=0\\ \nabla\times\vec{H}=J_c\end{array}}$  $\vec{B}=\mu\vec{H}$ ← permeability.

Ampere's laws: $\left[\oint_C \vec{H}\cdot d\vec{l} = \iint_S (\nabla\times\vec{H})\,d\vec{s} = \iint_S \vec{J_c}\cdot ds = I_{enc}\right]$

* current flows through whole volume

magneto-static sources →
- $I$ → current
- $I/m$ → (A/m) $\vec{J_s}$ → surface current density
- $I/m^2$ → (A/m²) $\vec{J}$ → volume current density

no such thing as $I/m^3$ as current flows through entire volume

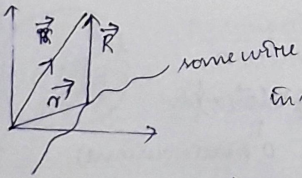
Boundary conditions:-
$H_{t_1} - H_{t_2} = J\Delta w \longrightarrow J_s \Rightarrow \boxed{(\vec{H_1}-\vec{H_2})\times\hat{n} = \vec{J_s}}$

Magnetostatics laws →
→ Oersted $d\vec{f}\propto\vec{I}$ ← Deflected force  Ampere → $\vec{H}\propto\vec{I}$ ($\nabla\times\vec{H}=\vec{J}$)
→ Biot - Savart law:-

$$\boxed{d\vec{H}(\vec{r}) = \frac{I'(\vec{r})\,d\vec{l'}\times\vec{R'}}{4\pi R^3}}$$

Henry → electromagnet
Faraday → time varying $\vec{H}$ ⇒ ~~$\nabla\times\vec{H}$~~ $\nabla\times\vec{E} = -\mu\frac{\partial\vec{H}}{\partial t}$.
Maxwell → unification into electromagnetism.

## Concept of magnetic (vector) potential $\vec{A}$



$$H(\vec{r}) = \int\frac{I'(r')\,d\vec{l'}\times\vec{R}}{4\pi R^3}\quad(=\vec{r}-\vec{r'})$$

In magnetostats, $\nabla\cdot\vec{J}=0$.
steady state current → continuity eqn. $\nabla\cdot\vec{J} = -\frac{\partial\rho_v}{\partial t} = 0$.
$\rho_v$ = const

use $\frac{d\vec{l'}\times\vec{R}}{R^3} = \nabla\left(\frac{1}{R}\right)\cdot d\vec{l'}$

use of:- $\nabla\times(V\vec{A}) = V(\nabla\times\vec{A}) + \vec{A}\times(\nabla V)$ chain  over source coord. system

compare, $\nabla\left(\frac{1}{R}\right)\cdot d\vec{l'} = -\frac{1}{R}\underbrace{(\nabla\times d\vec{l'})}_{0\text{ due to}} + \nabla\times\left(\frac{d\vec{l'}}{R}\right)$

independence of observations pt $\nabla$ & $d\vec{l'}$

$$\left[\nabla\left(\frac{1}{R}\right)\cdot d\vec{l} = +\nabla\times\left(\frac{d\vec{l'}}{R}\right)\right]$$

so, $\vec{H} = \int\frac{I'(r')}{4\pi} + \nabla\times\left(\frac{d\vec{l'}}{R}\right) = \nabla\times\underbrace{\left(\mu\int\frac{I(\vec{r'})\,d\vec{l'}}{4\pi R}\right)}_{\vec{A}(\vec{r})} = \vec{B}$

we define $\boxed{\vec{A}(\vec{r}) = \mu\int\frac{I(\vec{r'})\cdot d\vec{l'}}{4\pi R}}$ → vector potential

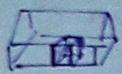
so that $\boxed{\vec{B} = \nabla\times\vec{A}}$

Energy in magnetic field 06 Feb '20

Electrostatic energy density: $\boxed{u_E = \frac{1}{2}\vec{D}\cdot\vec{E}}$

Magnetostatic energy density: $\boxed{u_B = \frac{1}{2}\vec{B}\cdot\vec{H}}$

Energy in a material



emf:- $V = -\frac{d\phi_B}{dt}$

so, $\frac{\partial w}{\partial t} = VI = I\,\frac{-\partial\phi_B}{\partial t}$

$\Rightarrow \boxed{\partial w = -I\,\partial\phi_B}$

now,

$w = \int \partial w = I\int_s \partial\vec{\phi}_B = J\Delta\sigma\int_s (\delta\vec{B})\, d\vec{a}\cdot\hat{n}$

$= \vec{J}\Delta\sigma\int_s (\nabla\times\delta\vec{A})\cdot\hat{n}\, da$

$\boxed{w = J\Delta\sigma\oint_c \delta\vec{A}\cdot d\vec{l}} = \iiint_v \delta\vec{A}\cdot\vec{J}\, dv$

now, $\iiint_v \delta\vec{A}\cdot\vec{J}\,dv = \iiint_v \delta\vec{A}\cdot(\nabla\times\vec{H})\,dv$ using $\vec{J} = \nabla\times\vec{H}$

$= \iiint_v [\vec{H}\cdot(\nabla\times\delta\vec{A}) + \underbrace{\nabla\cdot(\vec{H}\times\delta\vec{A})}_{\text{goes to 0 as } \vec{A} \,\&\, \vec{H} \text{ both } \nabla r}]\,dv$

$w = \iiint_v \vec{H}\cdot\delta\vec{B}\,dv$

$= \frac{1}{2}\iiint_v \delta(\vec{H}\cdot\vec{B})\,dv \Rightarrow \frac{\Delta w}{\Delta v} = \boxed{u_B = \frac{1}{2}\vec{H}\cdot\vec{B}}$

Energy stored in magnetic field: $\left[u_m = \frac{1}{2}\frac{\mu J^2}{4\pi^2 r^2}\right]$

**INDUCTANCE (L)**

$\psi$: Flux $\lambda$: flux linkage] are different in many cases

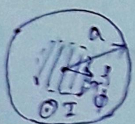
some skin effect at high frequencies

Inductance = $\frac{\text{flux linkage}}{\text{current}}$

take a cross-section uniform wire

wire of cross-section radius = $\rho$ $\Rightarrow \boxed{L = \frac{\lambda}{I}}$

carrying current =



assumptions:-

(i) DC steady state, I uniformly distributed $\vec{J} = \frac{I}{\pi a^2}\hat{n}$.

(ii) No fringing or skin effect

now, $I_{enc} = \int_{enc} \vec{J}\cdot d\vec{s} = \frac{I}{\pi a^2}\pi\rho^2 = \frac{I\rho^2}{a^2}$

* $\vec{H}, \vec{B}$ along $\phi$ direction only

so, $\oint \vec{H}\cdot d\vec{l} = I_{enc} \Rightarrow \oint_{\phi=0}^{2\pi} H_\phi\, \rho\, d\phi = \frac{I\rho^2}{a^2} \Rightarrow \boxed{H_\phi = \frac{I\rho}{2\pi a^2}} \Rightarrow \boxed{B_\phi = \frac{\mu I\rho}{2\pi a^2}\hat{a}_\phi}$

$d\vec{l} = \rho\, d\phi$

now flux ~~linkage~~ → at the whole surface

$\psi_B = \oint_s \vec{B}\cdot d\vec{s} = \iint B_\phi\hat{a}_\phi\cdot(dz\, d\rho\,\hat{a}_\phi) \Rightarrow \boxed{d\psi_B = \frac{\mu I\rho\, d\rho\, dz}{2\pi a^2}}$

and flux linkage →

$\boxed{d\lambda = d\psi\left(\frac{I_{enclosed}}{I}\right)} \Rightarrow \left[d\lambda = \frac{\mu I_{enc}\,\rho\, d\rho\, dz}{2\pi a^2}\right]$

$\Rightarrow \lambda = \iint_s d\lambda = \frac{\mu I l}{8\pi}$

## Inductance / length

assuming $z: 0 \to L$ gives flux linkage $\boxed{\lambda = \frac{\mu I l}{8\pi}}$

This gives self-inductance of wire: $\boxed{L = \frac{\mu l}{8\pi}}$

* so long transmission lines have huge inductive impedance

and inductance / length $\boxed{\frac{L}{l} = \frac{\mu}{8\pi}}$ → only material dependent value ($\mu$)

**From magnetic energy –**

we obtained, $\vec{H} = H_\phi \hat{a}_\phi$ $\quad \vec{B} = \mu H_\phi \hat{a}_\phi$. → (1)

so, magnetic energy of inductor:-

$$U_B = \frac{1}{2}LI^2 = \iiint \cancel{\frac{1}{2}} \vec{H}\cdot\vec{B}\, dv = \frac{1}{2}\iiint \vec{H}\cdot\vec{B}\, dv.$$

$$\Rightarrow LI^2 = \iiint \cancel{\frac{1}{2}} \mu (H_\phi)^2 dv \text{ from (1)}$$

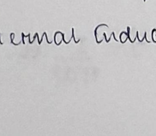
so, ~~L = ...~~

self-inductance of wire:-

$$\cancel{\Rightarrow L = \frac{\mu}{2}\left(\frac{H_\phi}{I}\right)^2 = \frac{\mu}{2}\left(\frac{\rho}{2\pi a}\right)^2}$$

$$LI^2 = \int_0^{l}\int_0^{2\pi}\int_{\rho=0}^{a} \mu \left(\frac{I\rho}{2\pi a}\right)^2 dv = \frac{\mu l I^2}{8\pi} \Rightarrow \boxed{L_{self} = \frac{\mu l}{8\pi}.}$$

Wire outside, grounded wire:-




$$\oint \vec{H}\cdot d\vec{l} = I_{enc} = I \text{ itself}$$

$$\Rightarrow \int_{\phi=0}^{2\pi} H_\phi\, \rho\, d\phi = I \Rightarrow \boxed{H_\phi = \frac{I}{2\pi\rho}} \qquad \boxed{B_\phi = \frac{\mu I}{2\pi\rho}}$$

now external inductance,

$$\frac{1}{2}LI^2 = \int_v \frac{1}{2}\vec{B}\cdot\vec{H}\, dv \Rightarrow LI^2 = \int_0^{l}\int_0^{2\pi}\int_{\rho=a}^{b} \mu \frac{I^2}{(2\pi\rho)^2}\, \rho\, d\rho\, d\phi\, dz$$

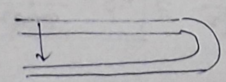
* external inductance →

$$\Rightarrow \boxed{L = \frac{\mu l}{2\pi}\ln\left(\frac{b}{a}\right)}$$ ($\rho: 0 \to a$, same $H_\phi$, $a \to b$ same $\phi$.

* external inductance calculated by ranging $\rho: a \to b$

~~as inductance at ρ=b → self due to internal + self due to external~~

when we have a flux linkage externally



→ ext. flux linkage of self-conductor.

# MUTUAL INDUCTANCE

13th Feb'2020

Energy stored between two current carrying conductors due to mutual inductance.

Contribution of wire ① to all other conductors, from all $\vec{A}$ potentials

→ Energy stored in magnetic field, $\boxed{W = \iiint_v \vec{J}\cdot\vec{A}\, dv}$

$$\sum \frac{1}{2} L I^2 = \quad W_{net} \rightarrow = \sum_{i=j}\sum_{j} \iiint \vec{J}(x_i) \cdot \frac{\mu \vec{J}(x_j)}{4\pi |x_i' - x_j'|}\, dv + \Big(\sum\sum_{i \neq j}\Big)$$

this due to Mutual Ind.

$$\boxed{W_{mut} = \frac{1}{2} \sum_i\sum_j M_{ij} I_i I_j}$$

The energy term comes as: $\sum\sum \iiint_v \frac{\mu \vec{J}(x_i)\, \vec{J}(x_j)}{4\pi |x_i' - x_j'|}$ → due to mutual interaction

So Neumann's formula → volume enclosed by closed paths $C_1\, C_2$

$$\boxed{M_{ij} = \frac{\mu}{4\pi} \oint_{C_i}\oint_{C_j} \frac{d\vec{l_i}(x_i) \cdot d\vec{l_j}(x_j)}{|x_i - x_j|}} \rightarrow \text{dot}$$

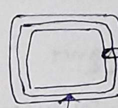
$$\oint_{C_i}\oint_{C_j} \frac{d\vec{l_i}\cdot d\vec{l_j}}{|x_i - x_j|}$$

$C_i$ → closed path for cross section

$C_j$ → closed loop of current path

⇓

both contribute to volume



→ $\oint C_i$ (gives current density dependency)

$\oint C_j$.

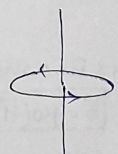
* ACITIC software (EM simulation)

Mutual Inductance of Coaxial solenoids

short solenoid carrying current I

turns/length = $n_1 / n_2$

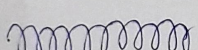
* $\vec{B}$ at axis of current loop



$\vec{J_\phi} = J_\phi \hat{a}_\phi$

14th Feb'20

SOLENOID



→ $\vec{H}$ inside is $= nI\, \hat{a}_z$

$\vec{H}$ outside is 0.

flux linkage $= N\phi$

$= (nl)(nI)(\pi a^2)\mu$

$= n^2 I a^2 l \pi$ ⇒ self inductance of solenoid,

$\Rightarrow \boxed{L = \mu n^2 (\pi a^2)\, l}$ $\qquad \mu n^2 l \underset{\text{Area}}{A}$

magnetic moment: $\boxed{\vec{M} = I\vec{A}}$ → acts as a magnetic dipole

mutual inductance of solenoid: $\boxed{M_{12} = \mu n_1 n_2\, l\, (\pi a^2)}$

Thus for two coils with self inductances $L_1, L_2$, we have:-

$\boxed{M_{12} = K\sqrt{L_1 L_2}}$ $\qquad$ K: coeff. of coupling

good inductors $K > 0.9$

# MAGNETISATION

Dipole moment per unit volume (magnetic)

$$\boxed{\vec{M}\,(A/m) = \lim_{\Delta v \to 0} \frac{\Sigma \vec{m}_k}{\Delta v}} \qquad d\vec{m} = \vec{M}\,dv'.$$

now, magnetic potential:-

$$\vec{A} = \int d\vec{A} = \frac{\mu_0}{4\pi} \iiint_V \vec{M}\,dv' \times \left(\nabla'\left(\frac{1}{R}\right)\right).$$

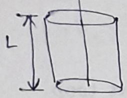
$$\vec{A} = \frac{\mu_0}{4\pi}\left[\iiint_V \frac{(\nabla' \times \vec{M})}{R}\,dv' - \iiint_V \nabla' \times \left(\frac{\vec{M}}{R}\right) dv'\right]$$

$$\vec{A} = \frac{\mu_0}{4\pi}\left[\iiint_V \frac{\vec{J}_b\,dv'}{R} - \oint_S \frac{\vec{K}_b \times \hat{a}_n}{R}\,dv'\right].$$

Bound volume ~~charge~~ current density: $\boxed{\vec{J}_b = \nabla' \times \vec{M}}$.

Bound surface ~~charge~~ current density: $\boxed{\vec{K}_b = \vec{M} \times \hat{a}_n}$

Eg:- cylindrical ~~plot~~ magnet:-




axial magnetisation: $\boxed{\vec{M} = M_0\,\hat{a}_z}$

$\boxed{\vec{J}_b = \nabla \times \vec{M} = 0}$

$\boxed{\vec{K}_b = \vec{M} \times \hat{a}_n = M_0\,\hat{a}_\phi}$

## Magnetic materials

magnetic field in a material given by:-

$\boxed{\vec{B} = \mu_0(\vec{H} + \vec{M})}$ $\vec{M}$ .. equivalent to polarisation.

for linear parts, $\vec{M} \propto \vec{H} \Rightarrow \boxed{\vec{M} = \chi_m \vec{H}}$

$\boxed{\vec{B} = \mu_0(1+\chi_m)\vec{H}}$ so, $\boxed{\mu_r = 1 + \chi_m}$ $[\mu = \mu_0 \mu_r]$

Total current: $\vec{J}_f$ (free volume current density) + $\vec{J}_b$ (Bounded vol. current density)

$$\boxed{\begin{aligned} \vec{J}_f &= \nabla \times \vec{H} \\ \vec{J}_b &= \nabla \times \vec{M} \end{aligned}}$$

Total current density:- $\boxed{\vec{J} = \nabla \times \vec{H} + \nabla \times \vec{M}}$