MA 20104 Probability and Statistics
Hints/Solutions to Assignment No. 6

1. $P(1 < X + Y < 2)$

$$= \int_0^1 \int_{1-y}^{2-y} e^{-(x+y)} dxdy + \int_1^2 \int_0^{2-y} e^{-(x+y)} dxdy)$$

$$= (e^{-1} - e^{-2}) + (e^{-1} - 2e^{-2}) = (2e^{-1} - 3e^{-2}).$$

$P(X < Y|X < 2Y) = \frac{P(X<Y)}{P(X<2Y)}$. Now

$$P(X < Y) = \int_0^\infty \int_x^\infty e^{-(x+y)} dydx = \frac{1}{2}$$

$$P(X < 2Y) = \int_0^\infty \int_{\frac{x}{2}}^\infty e^{-(x+y)} dydx = \frac{2}{3}$$

So the required probability is 0.75.

Clearly X and Y are independent. So

$P(0 < X < 1|Y = 2) = P(0 < X < 1) = 1 - e^{-1}$.

$P(X + Y < m) = \frac{1}{2}$ is equivalent to $2(m + 1)e^{-m} - 1 = 0$. This is a nonlinear equation and can be solved numerically. Elementary numerical methods such as bisection gives $m \approx 1.68$.

2. The marginal densities of X and Y are

$f_X(x) = x + \frac{1}{2}$, $0 < x < 1$ and $f_Y(y) = y + \frac{1}{2}$, $0 < y < 1$.

Clearly X and Y are not independent.

$$E(X) = \frac{7}{12}, E(X^2) = \frac{5}{12}, V(X) = \frac{11}{144}$$

$$E(Y) = \frac{7}{12}, E(Y^2) = \frac{5}{12}, V(Y) = \frac{11}{144}$$

$$E(XY) = \frac{1}{3}, Cov(X + Y) = -\frac{1}{144}$$

$$Var(X + Y) = V(X) + V(Y) + 2Cov(X, Y) = \frac{5}{36}$$

$$Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = -\frac{1}{11}$$

The conditional pdf of $X|Y = y$ is given by

$$f_{X|Y=y}(x|y) = \frac{2(x+y)}{2y+1} \quad ,0 < x < 1, 0 < y < 1\ .$$

$$E(X|Y = y) = \frac{2+3y}{3(2y+1)}, E(X^2|Y = y) = \frac{3+4y}{6(2y+1)},$$

$$V(X|Y = y) = \frac{6y^2 + 6y + 1}{18(2y+1)^2}$$

3. Similar to calculations in Q.1 and Q. 2.
4. $(X,Y) \sim BVN(24, 28, 36, 49, 0.8)$. Then $X \sim N(24, 36)$. So
   $P(X > 30) = \Phi(-1) = 0.1587$.
   The conditional distribution of X given $Y = 35$ is $N(28.8, 12.96)$.
   So $Var(X|Y = 35) = 12.96$.
   $$P(X > 30|Y = 35) = P\left(Z > \frac{30 - 28.8}{\sqrt{12.96}}\right) = \Phi(-0.33) = 0.3707.$$
   The conditional distribution of Y given $X = 22$ is $N(26.13, 17.64)$.
   $E(Y|X = 22) = 26.13$.
5. Similar to Q. 4.
6. The marginal density of Y is $f_Y(y) = e^{-y}, y > 0$.
   The conditional density of X given $Y = y$ is
   $f_{X|y=y}(x|y) = \frac{1}{y}\ e^{-\frac{x}{y}}\ , x > 0$.
   $E(Y) = 1$, $E(X) = EE(X|Y) = E(Y) = 1$. $V(Y) = 1$.
   $V(X) = VE(X|Y) + EV(X|Y) = V(Y) + E(Y^2) = 1 + 2 = 3$.
   $E(XY) = E\big(Y\ E(X|Y)\big) = E(Y^2) = 2$. $Cov(X,Y) = 1$.
   $Corr(X,Y) = \frac{1}{\sqrt{3}}$.
7. Similar to Q. 4.
8. Similar to Q. 4.

9. In order that $f(x,y)$ is a valid density, $-1 < \alpha < 1$.
The marginal densities of X and Y are
$f_X(x) = 1, 0 < x < 1$ and $f_Y(y) = 1, 0 < y < 1$.

$$E(X) = \frac{1}{2}\ ,\ \ E(Y) = \frac{1}{2}\ , V(X) = \frac{1}{12}\ , V(Y) = \frac{1}{12}$$

$$Cov(X,Y) = E\left(X - \frac{1}{2}\right)\left(Y - \frac{1}{2}\right) = -\frac{\alpha}{36}\ , Corr(X,Y) = -\frac{\alpha}{3}$$

Clearly X and Y are independent if and only if $\alpha = 0$.

Q. 10- Q. 17 can be solved as earlier questions.