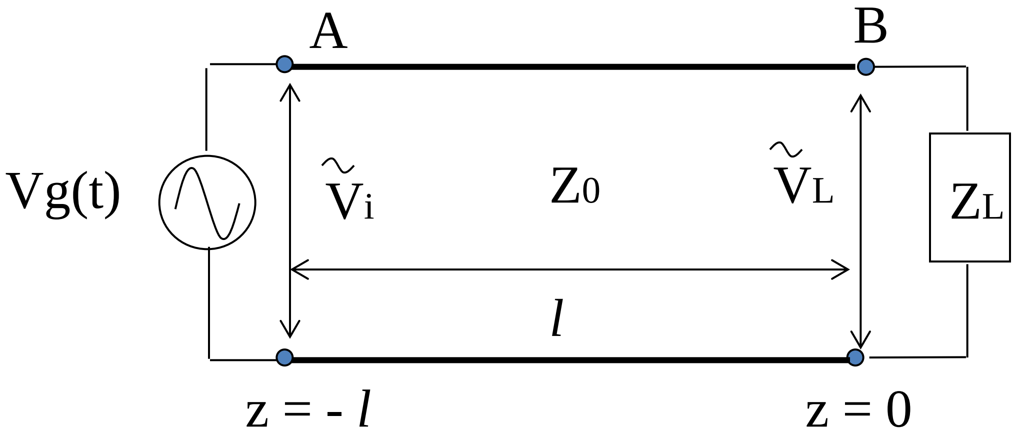
- Transmission lines

1. Reflection coefficient
2. Standing wave
3. Input impedence
4. Short-circuit Tx line
5. Open-circuit Tx Line
6. Matched Tx line
7. Quarter Wavelength transformer
8. Power Flow on transmission lines
9. Transients in Transmission Line

- <u>Voltage reflection coefficient :</u>



$$\begin{cases} \tilde{V}(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z} \\ \tilde{i}(z) = \dfrac{V_0^+}{Z_0} e^{-j\beta z} - \dfrac{V_0^-}{Z_0} e^{j\beta z} \end{cases}$$

$$\tilde{V}_L = \left.\tilde{V}(z)\right|_{z=0} = V_0^+ + V_0^-$$

$$\tilde{i}_L = \left.\tilde{i}(z)\right|_{z=0} = \frac{V_0^+}{Z_0} - \frac{V_0^-}{Z_0}$$

$$Z_L = \frac{\tilde{V}_L}{\tilde{i}_L} = \frac{V_0^+ + V_0^-}{\dfrac{V_0^+}{Z_0} - \dfrac{V_0^-}{Z_0}} \quad \Rightarrow \quad \frac{V_0^+}{V_0^-} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

- <u>Voltage reflection coefficient :</u>

$$\Gamma \equiv \frac{V_0^-}{V_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0}$$

- <u>Current reflection coefficient :</u>

$$\Gamma_i \equiv \frac{i_0^-}{i_0^+} = -\frac{V_0^-}{V_0^+} = -\Gamma$$

- <u>Note :</u>

1. $|\Gamma| \leq 1$
2. If $Z_L = Z_0$, $\Gamma = 0$. Impedance match, no reflection from the load Z_L.

- Standing wave

$$\begin{cases} \tilde{V}(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z} \\ \tilde{i}(z) = \dfrac{V_0^+}{Z_0} e^{-j\beta z} - \dfrac{V_0^-}{Z_0} e^{j\beta z} \end{cases} \qquad \text{with } \Gamma = \frac{V_0^-}{V_0^+}$$

$$\begin{cases} \tilde{V}(z) = V_0^+ (e^{-j\beta z} + \Gamma e^{j\beta z}) \\ \tilde{i}(z) = \dfrac{V_0^+}{Z_0} (e^{-j\beta z} - \Gamma e^{j\beta z}) \end{cases}$$

$$|\tilde{V}(z)| = |V_0^+|\,|\,e^{-j\beta z} + |\Gamma| e^{j\theta_r} e^{j\beta z}|$$

$$= |V_0^+|\,[1 + |\Gamma|^2 + 2|\Gamma|\cos(2\beta z + \theta_r)]^{1/2}$$

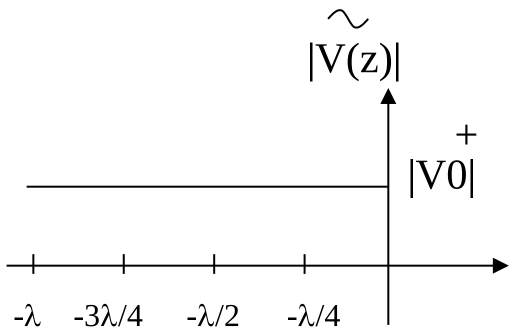
$$|\tilde{i}(z)| = |V_0^+|/|Z0|| \qquad |$$

$$= |V_0^+|/|Z_0|\,[1 + |\Gamma|^2 - 2|\Gamma|\cos(2\beta z + \theta_r)]^{1/2}$$

Special cases

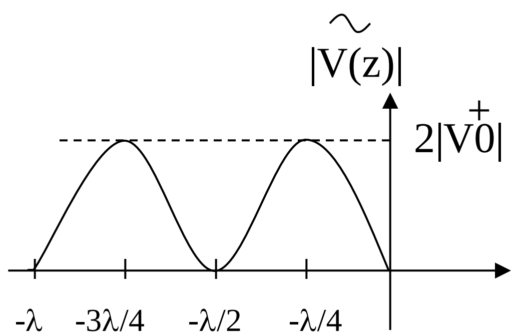
1. $Z_L = Z_0, \Gamma = 0$

$$|\tilde{V}(z)| = |V_0^+|$$



2. $Z_L = 0$, short circuit, $\Gamma = -1$

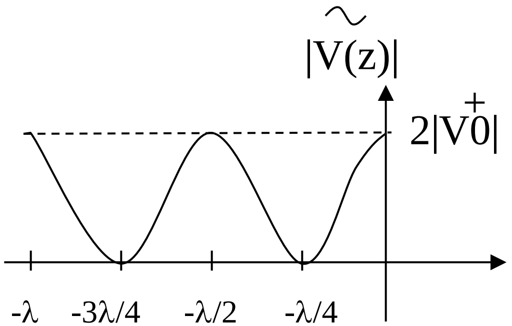
$$|\tilde{V}(z)| = |V_0^+| \left[2 + 2\cos(2\beta z + \pi)\right]^{1/2}$$



Special cases

3. $Z_L = \infty$, open circuit, $\Gamma = 1$

$$|\tilde{V}(z)| = |V_0^+| \, [2 + 2\cos(2\beta z)]^{1/2}$$



- <u>Voltage maximum</u>

$$|\tilde{V}(z)| = |V_0^+| \left[1 + |\Gamma|^2 + 2|\Gamma|\cos(2\beta z + \theta_r)\right]^{1/2}$$

$$|\tilde{V}(z)|_{max} = |V_0^+| \left[1 + |\Gamma|\right], \quad \text{when } 2\beta z + \theta_r = 2n\pi.$$

$$\boxed{\begin{array}{l} -z = \lambda\theta_r/4\pi + n\lambda/2 \\ n = 1, 2, 3, \ldots, \text{ if } \theta_r < 0 \\ n = 0, 1, 2, 3, \ldots, \text{ if } \theta_r >= 0 \end{array}}$$

- <u>Voltage minimum</u>

$$|\tilde{V}(z)| = |V_0^+| \, [1 + |\Gamma|^2 + 2|\Gamma|\cos(2\beta z + \theta_r)]^{1/2}$$

$$|\tilde{V}(z)|_{min} = |V_0^+| \, [1 - |\Gamma|], \quad \text{when } 2\beta z + \theta_r = (2n+1)\pi.$$

$$-z = \lambda\theta_r/4\pi + n\lambda/2 + \lambda/4$$

<u>Note:</u>

voltage minimums occur $\lambda/4$ away from voltage maximum, because of the $2\beta z$, the **spatial frequency doubled**. _

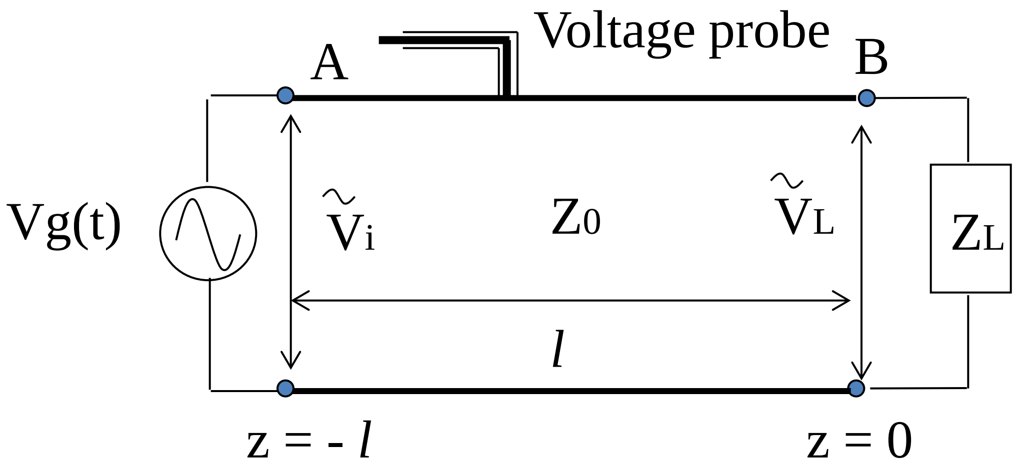
- <u>Voltage standing-wave ratio VSWR or SWR</u>

$$S \equiv \frac{|\tilde{V}(z)|_{max}}{|\tilde{V}(z)|_{min}} = \frac{1 + |\Gamma|}{1 - |\Gamma|}$$

$S = 1$, when $\Gamma = 0$,

$S = \infty$, when $|\Gamma| = 1$,

- Practical example of measurement of unknown load impedance



$S = 3$, $Z_0 = 50\Omega$, $\Delta l_{min} = 30cm$, $l_{min} = 12cm$, $Z_L = ?$

Solution:

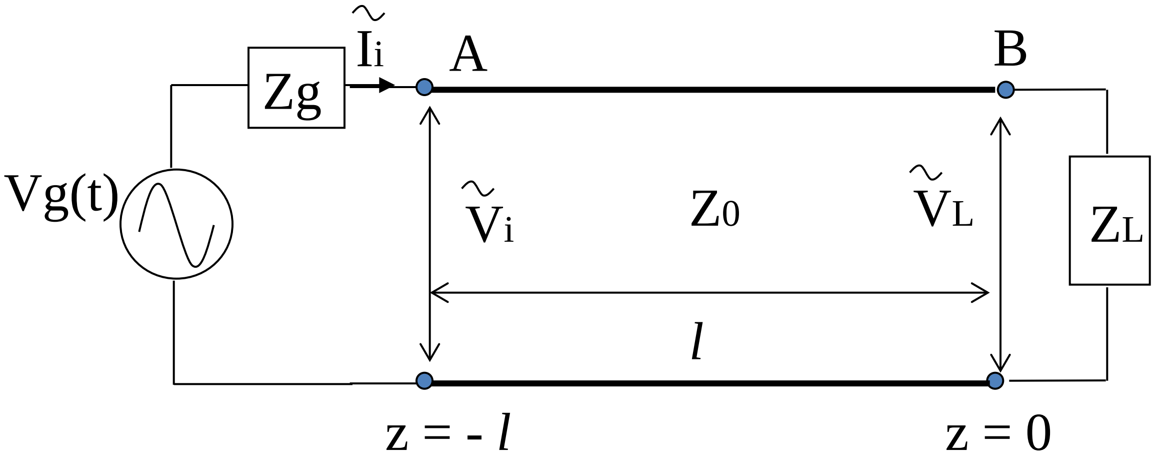
$\Delta l_{min} = 30cm, \Rightarrow \lambda = 0.6\text{m},$

$S = 3, \Rightarrow |\Gamma| = 0.5,$

$-2\beta l_{min} + \theta_r = -\pi, \Rightarrow \theta_r = -36^o,$

$\Rightarrow \Gamma$, and Z_L.

- Input impedance



$$Z_{in}(z) = \frac{\tilde{V}(z)}{\tilde{I}(z)}$$

$$= \frac{V_0^+(e^{-j\beta z} + \Gamma e^{j\beta z})}{V_0^+(e^{-j\beta z} - \Gamma e^{j\beta z})} Z_0 = \frac{(1 + \Gamma e^{j2\beta z})}{(1 - \Gamma e^{j2\beta z})} Z_0$$

$$Z_{in}(-l) = \frac{(1 + \Gamma e^{-j2\beta l})}{(1 - \Gamma e^{-j2\beta l})} Z_0$$

An example

A 1.05-GHz generator circuit with series impedance Zg = 10-Ω and voltage source given by Vg(t) = 10 sin(ωt +30º) is connected to a load ZL = 100 +j5-Ω through a 50-Ω, 67-cm long lossless transmission line. The phase velocity is 0.7c. Find V(z,t) and *i*(z,t) on the line.

Solution:

Since, $V_p = f\lambda$, $\lambda = V_p/f = 0.7c/1.05\text{GHz} = 0.2\text{m}$.

$\beta = 2\pi/\lambda$, $\beta = 10\pi$.

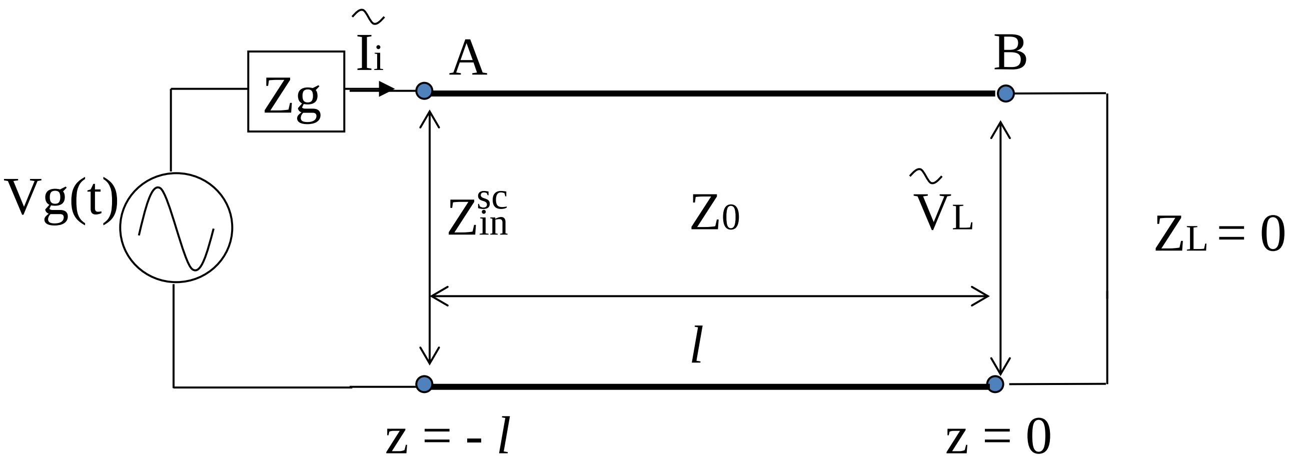
$\Gamma = (Z_L - Z_0)/(Z_L + Z_0)$, $\Gamma = 0.45\exp(j26.6^\circ)$

$$Z_{in}(-l) = \frac{(1+\Gamma e^{-j2\beta l})}{(1-\Gamma e^{-j2\beta l})} Z_0 = 21.9 + j17.4\ \Omega$$

$$V_0^+[\exp(-j\beta l) + \Gamma\exp(j\beta l)] = \frac{Z_{in}(-l)}{Z_{in}(-l) + Z_g}\tilde{V}_g$$

To find out the forward wave

short circuit line



$Z_L = 0, \Gamma = -1, S = \infty$

$$\begin{cases} \tilde{V}(z) = V_0(e^{-j\beta z} - e^{j\beta z}) = -2jV_0^+\sin(\beta z) \\ \tilde{i}(z) = \dfrac{V_0^+}{Z_0}(e^{-j\beta z} + e^{j\beta z}) = 2V_0^+\cos(\beta z)/Z_0 \end{cases}$$

$$Z_{in} = \frac{\tilde{V}(-l)}{\tilde{i}(-l)} = jZ_0\tan(\beta l)$$

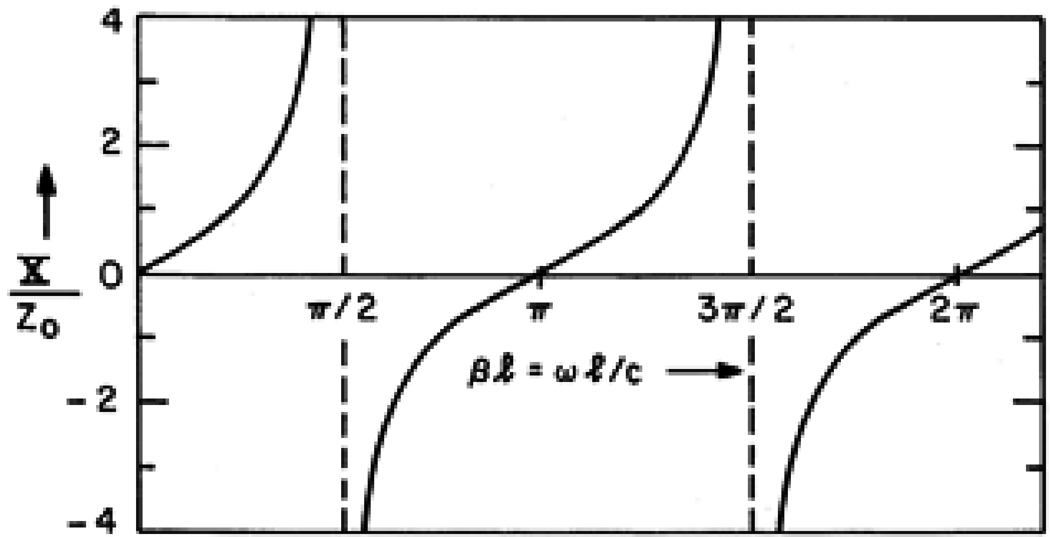
short circuit line

$$Z_{in} = \frac{\tilde{V}(-l)}{\tilde{i}(-l)} = jZ_0\tan(\beta l)$$

- If $\tan(\beta l) >= 0$, the line appears inductive, $j\omega L_{eq} = jZ_0\tan(\beta l)$,
- If $\tan(\beta l) <= 0$, the line appears capacitive, $1/j\omega C_{eq} = jZ_0\tan(\beta l)$,
- The minimum length results in transmission line as a capacitor:

$$l = 1/\beta[\pi - \tan^{-1}(1/\omega C_{eq} Z_0)],$$

Reactance as a function of normalized frequency for a shorted line



An example:

Choose the length of a shorted 50-Ω lossless line such that its input impedance at 2.25 GHz is equivalent to the reactance of a capacitor with capacitance Ceq = 4pF. The wave phase velocity on the line is 0.75c.

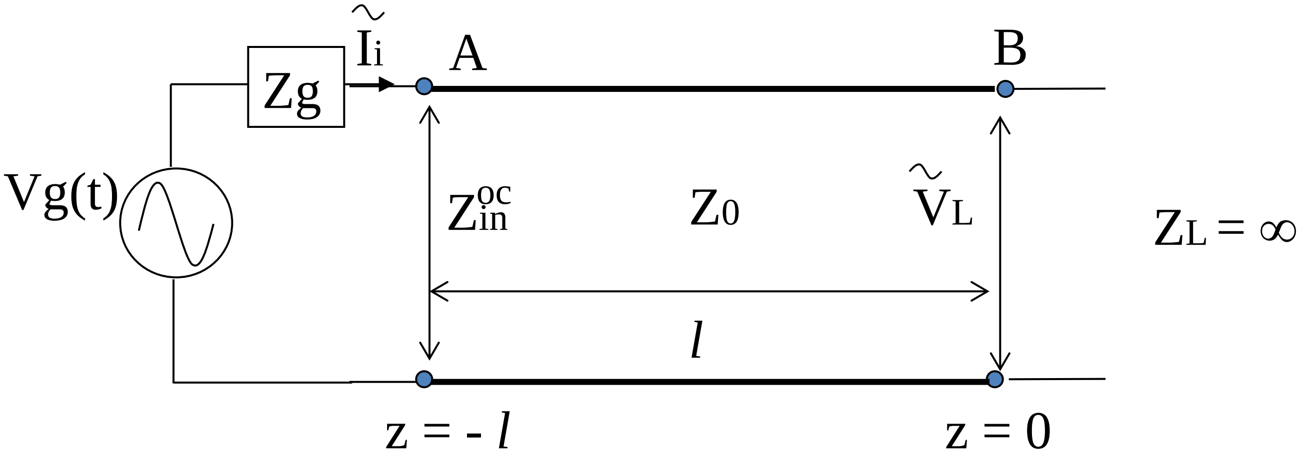
Solution:

$$V_p = \lambda f, \Rightarrow \beta = 2\pi/\lambda = 2\pi f/V_p = 62.8 \ \text{(rad/m)}$$

$$\tan(\beta l) = -1/\omega C_{eq} Z_0 = -0.354,$$

$$\beta l = \tan^{-1}(-0.354) + n\pi,$$
$$= -0.34 + n\pi,$$

open circuit line



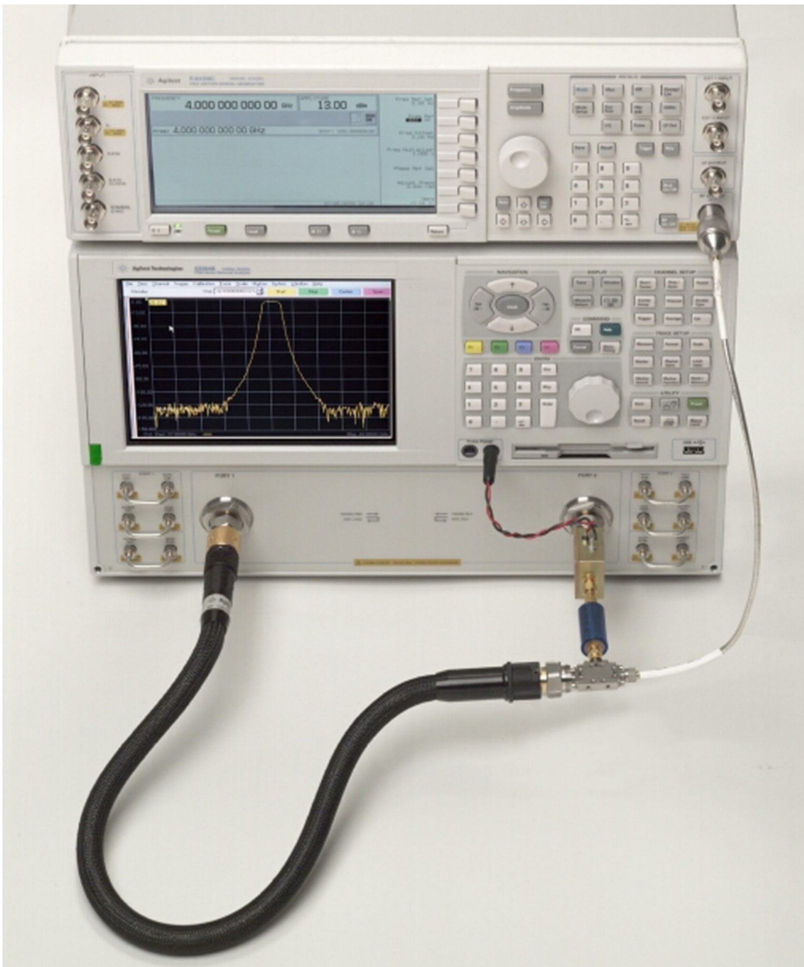
$Z_L = 0, \Gamma = 1, S = \infty$

$$\begin{cases} \tilde{V}(z) = V_0(e^{-j\beta z} + e^{j\beta z}) = 2V_0^+\cos(\beta z) \\ \tilde{i}(z) = \dfrac{V_0^+}{Z_0}(e^{-j\beta z} - e^{j\beta z}) = 2jV_0^+\sin(\beta z)/Z_0 \end{cases}$$

$$Z_{in}^{oc} = \frac{\tilde{V}(-l)}{\tilde{i}(-l)} = -jZ_0\cot(\beta l)$$

Application for short-circuit and open-circuit

- Network analyzer



- Measure Tx line parameters
- Measure Z_{in}^{sc} and Z_{in}^{oc}
- Calculate Z_0

$$Z_{in}^{sc} = jZ_0 \tan(\beta l)$$
$$Z_{in}^{oc} = -jZ_0 \cot(\beta l)$$

$$Z_0 = \sqrt{Z_{in}^{sc} Z_{in}^{oc}}$$

- Calculate βl using

$$\tan(\beta l) = -j\sqrt{\frac{Z_{in}^{sc}}{Z_{in}^{oc}}}$$

Line of length $l = n\lambda/2$

$$\tan(\beta l) = \tan((2\pi/\lambda)(n\lambda/2)) = 0,$$

$$Z_{in}(-l) = \frac{(1+\Gamma e^{-j2\beta l})}{(1-\Gamma e^{-j2\beta l})} Z_0 = Z_L$$

Any multiple of half-wavelength line doesn't modify the load impedance.

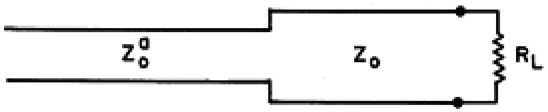
Quarter-wave transformer $l = \lambda/4 + n\lambda/2$

$$\beta l = (2\pi/\lambda)(\lambda/4 + n\lambda/2) = \pi/2 ,$$

$$Z_{in}(-l) = \frac{(1+\Gamma e^{-j2\beta l})}{(1-\Gamma e^{-j2\beta l})} Z_0 = \frac{(1+\Gamma e^{-j\pi})}{(1-\Gamma e^{-j\pi})} Z_0 = \frac{(1-\Gamma)}{(1+\Gamma)} Z_0$$

$$= Z_0^2/Z_L$$

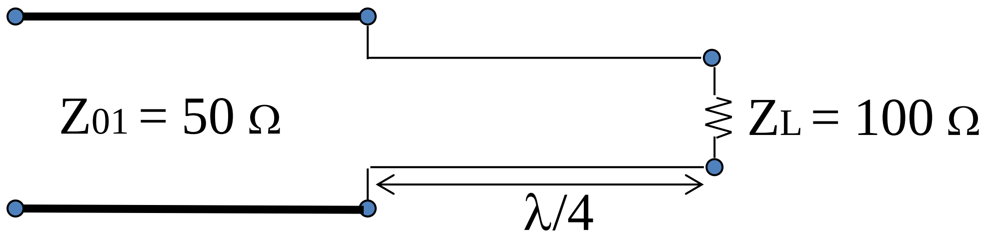
Quarter-Wave Impedance Matching Section



$$Z_o = \sqrt{Z_o^o R_L}$$

An example:

A 50-Ω lossless tarnsmission is to be matched to a resistive load impedance with Z_L = 100 Ω via a quarter-wave section, thereby eliminating reflections along the feed line. Find the characteristic impedance of the quarter-wave tarnsformer.



$$Z_{in} = Z_0^2/Z_L = 50\ \Omega$$

$$Z_0 = (Z_{in}Z_L)^{1/2} = (50*100)^{1/2}$$

Matched transmission line:

1. $Z_L = Z_0$
2. $\Gamma = 0$
3. All incident power is delivered to the load.

- Instantaneous power
- Time-average power

$$\begin{cases} \tilde{V}(z) = V_0^+ \left(e^{-j\beta z} + \Gamma e^{j\beta z}\right) \\ \tilde{i}(z) = \dfrac{V_0^+}{Z_0} \left(e^{-j\beta z} - \Gamma e^{j\beta z}\right) \end{cases}$$

At load z = 0, the incident and reflected voltages and currents:

$$\tilde{V}^i = V_0^+ \qquad \tilde{i}^i = \frac{V_0^+}{Z_0}$$

$$\tilde{V}^r = V_0^- \qquad \tilde{i}^r = \frac{V_0^-}{Z_0}$$

- Instantaneous power

$$P^i(t) = v(t)\, i(t) = \mathrm{Re}[\tilde{V}^i \exp(j\omega t)]\, \mathrm{Re}[\tilde{i}^i \exp(j\omega t)]$$

$$= \mathrm{Re}[|V_0^+|\exp(j\phi^+)\exp(j\omega t)]\, \mathrm{Re}[|V_0^+|/Z_0 \exp(j\phi^+)\exp(j\omega t)]$$

$$= (|V_0^+|^2/Z_0) \cos^2(\omega t + \phi^+)$$

$$P^r(t) = v(t)\, i(t) = \mathrm{Re}[\tilde{V}^r \exp(j\omega t)]\, \mathrm{Re}[\tilde{i}^r \exp(j\omega t)]$$

$$= \mathrm{Re}[|V_0^-|\exp(j\phi^+)\exp(j\omega t)]\, \mathrm{Re}[|V_0^-|/Z_0 \exp(j\phi^+)\exp(j\omega t)]$$

$$= -\, |\Gamma|^2(|V_0^+|^2/Z_0) \cos^2(\omega t + \phi^+ + \phi_r)$$

- Time-average

<u>Time-domain approach:</u>

$$P_{av}^{i} = \frac{1}{T}\int_0^T P^i(t)dt = \frac{\omega}{2\pi}\int_0^T (|V_0^+|^2/Z_0)\cos^2(\omega t + \phi^+)dt$$

$$= (|V_0^+|^2/2Z_0)$$

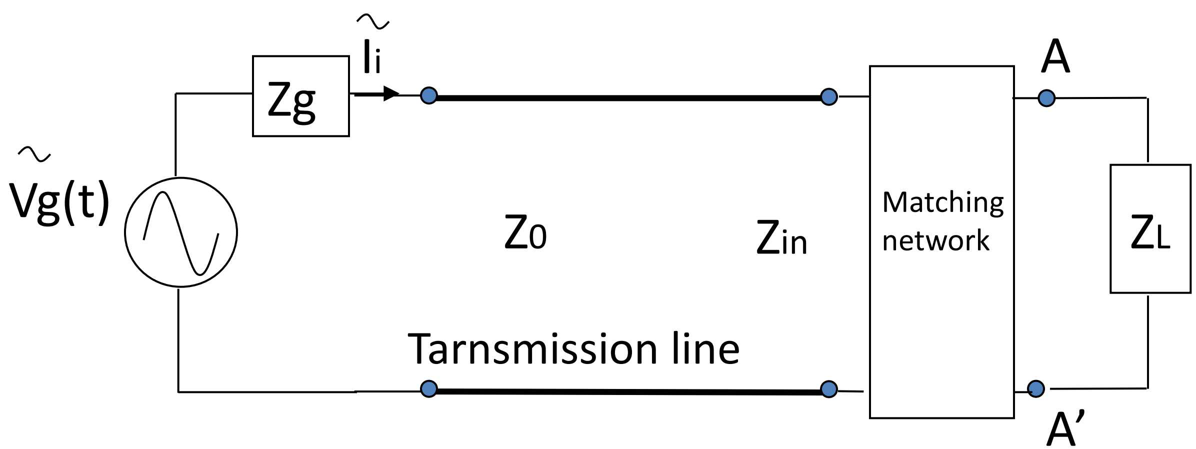
$$P_{av}^{r} = -|\Gamma|^2 (|V_0^+|^2/2Z_0)$$

<u>Net average power:</u>

$$P_{av} = P_{av}^{i} + P_{av}^{r}$$

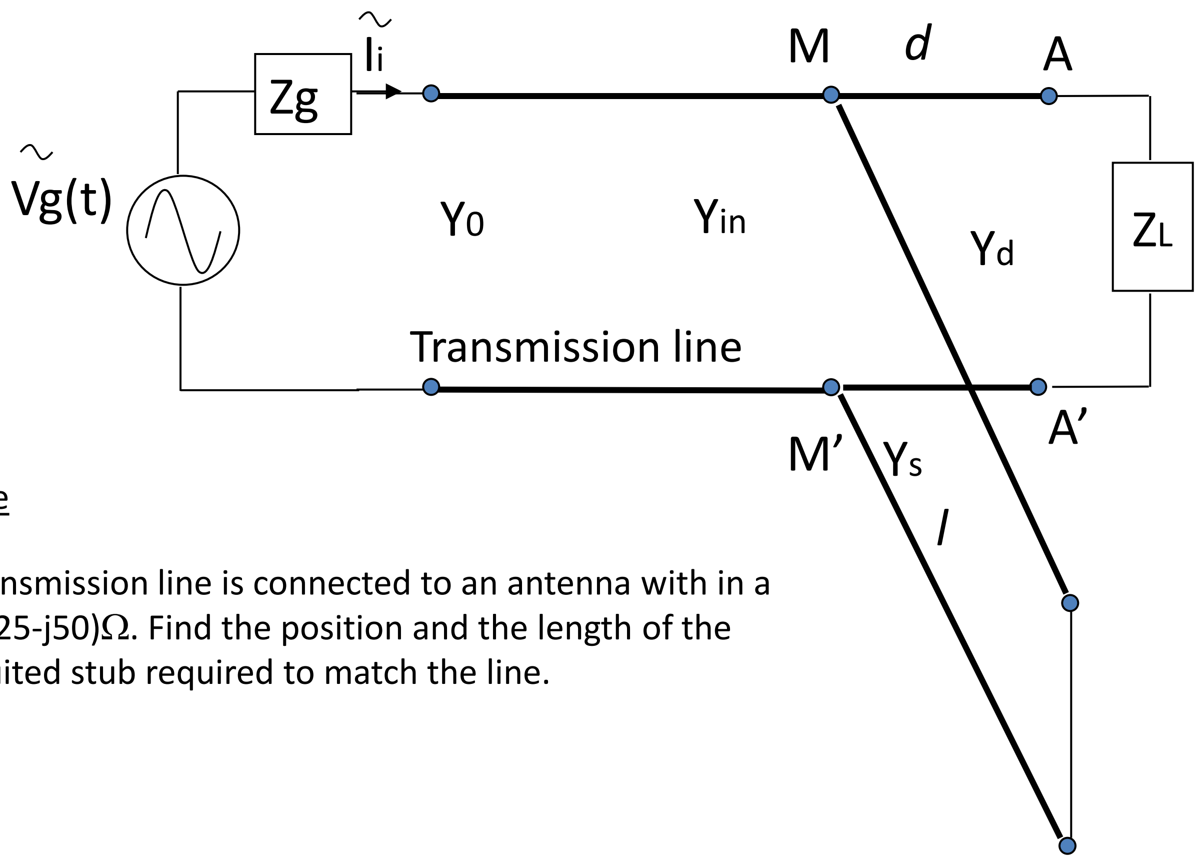
$$= (1-|\Gamma|^2)(|V_0^+|^2/2Z_0)$$

- impedance matching



$Z_{in} = Z_0$

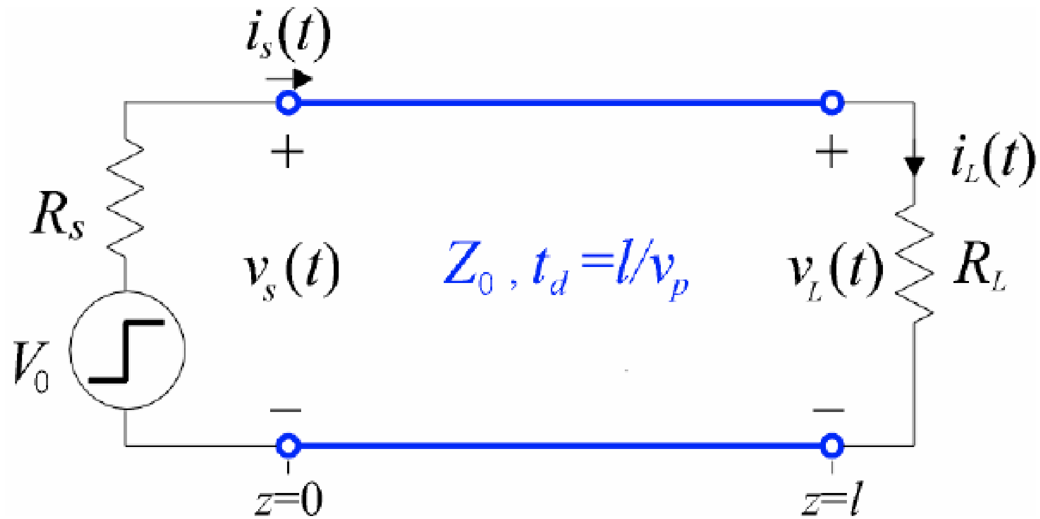
- single-stub impedance matching network



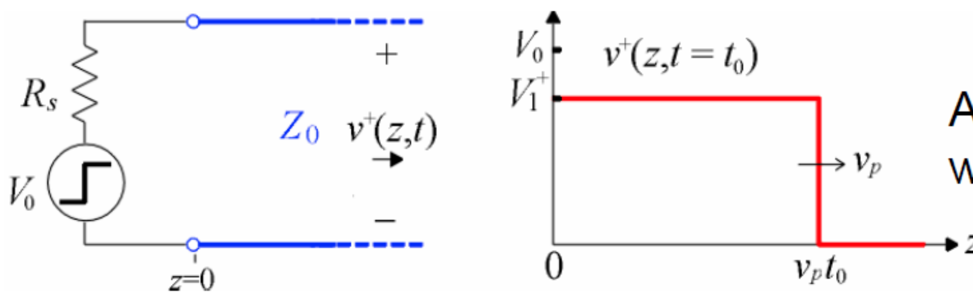
<u>An example</u>

A 50-Ω transmission line is connected to an antenna with in a load ZL = (25-j50)Ω. Find the position and the length of the short-circuited stub required to match the line.

TRANSIENTS in Tx Lines



LAUNCH of Forward Wave in Tx Lines



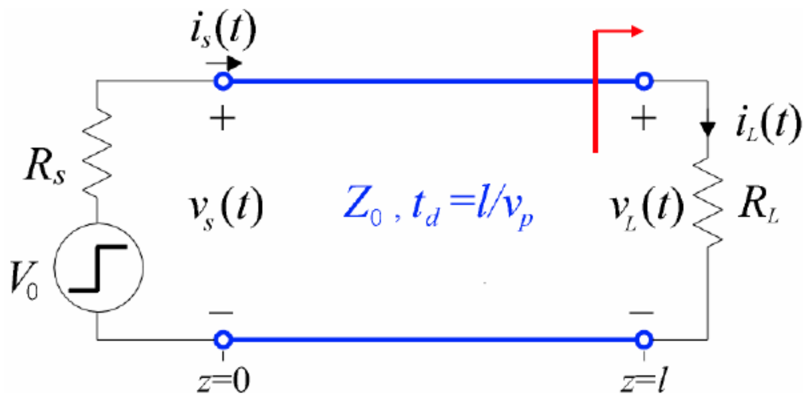
As if there were no load.

$$V_1^+ = \frac{Z_0}{Z_0 + R_S} V_0$$

LAUNCH of Backward Wave from the load end

$$\Gamma_L \equiv \frac{v_1^-(l,t)}{v_1^+(l,t)} = \frac{R_L - Z_0}{R_L + Z_0}$$

Load voltage reflection coefficient

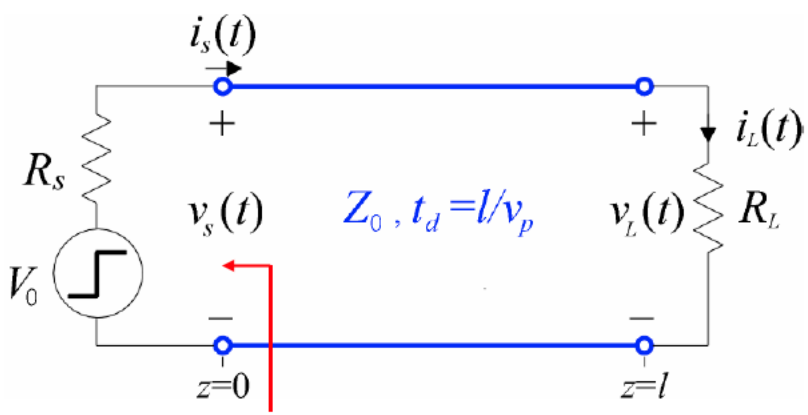


Γ_L can be used to calculate V_1^- given V_1^+ is known.

LAUNCH of 2nd Forward Wave from the source end

$$\text{At } t = 2t_d$$

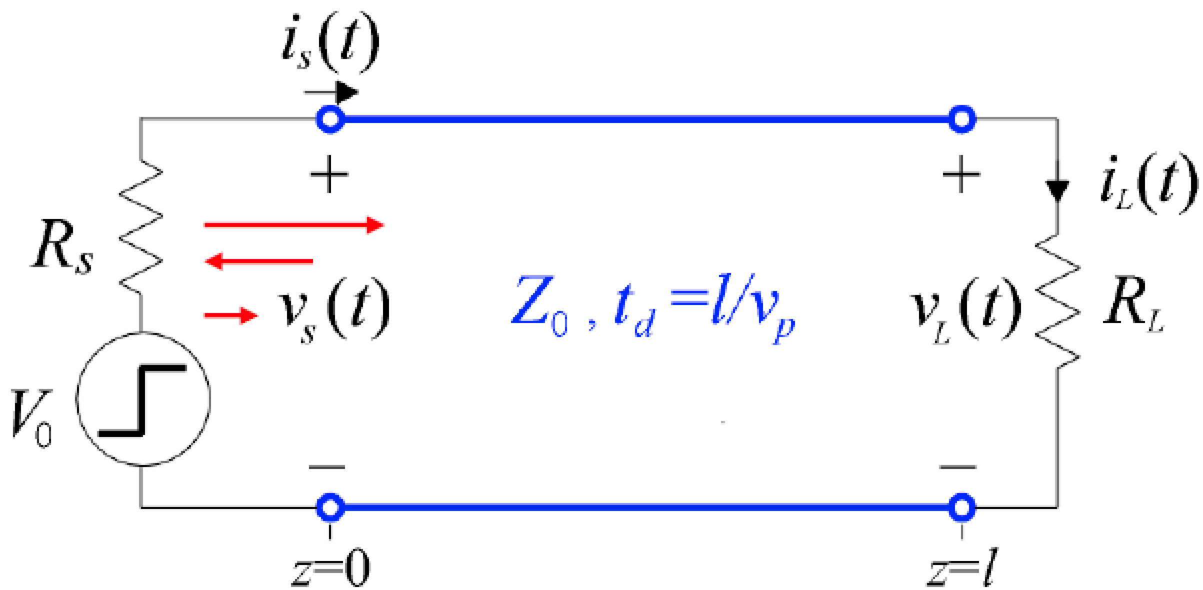
$$\Gamma_S \equiv \frac{v_2^+(l,t)}{v_1^-(l,t)} = \frac{R_S - Z_0}{R_S + Z_0}$$ Source voltage reflection coefficient



Γ_S can be used to calculate V_2^+ given V_1^- is known.

At $t = 2t_d$

$$v_S(2t_d) = v_1^+(0,2t_d) + v_1^-(0,2t_d) + v_2^+(0,2t_d) = v_1^+(0,2t_d)\left(1 + \Gamma_L + \Gamma_L \Gamma_S\right)$$



STEADY STATE PICTURE

At $t \rightarrow \infty$

$$\lim_{t\to\infty} v_S(t) = V_1^+ \cdot \left[\left(1+\Gamma_L\right)+\Gamma_L\Gamma_S\left(1+\Gamma_L\right)+\Gamma_L^2\Gamma_S^2\left(1+\Gamma_L\right)+\ldots\right]$$

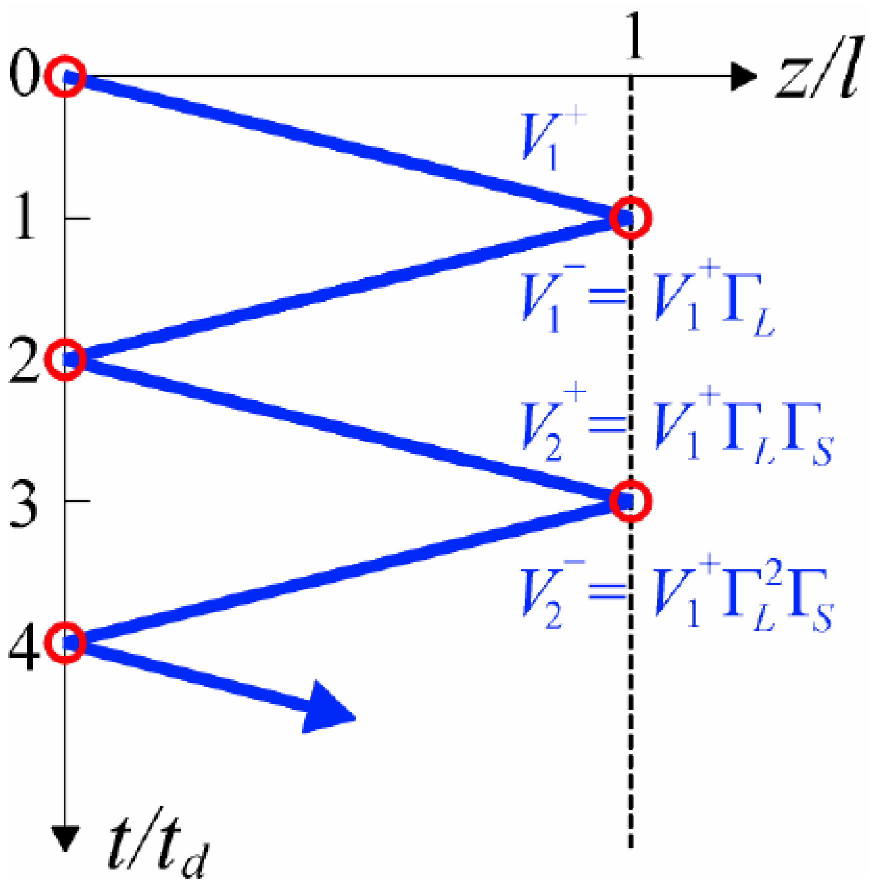
$$= \frac{R_L}{R_S + R_L} V_0$$

Steady state response is as if there were no line.

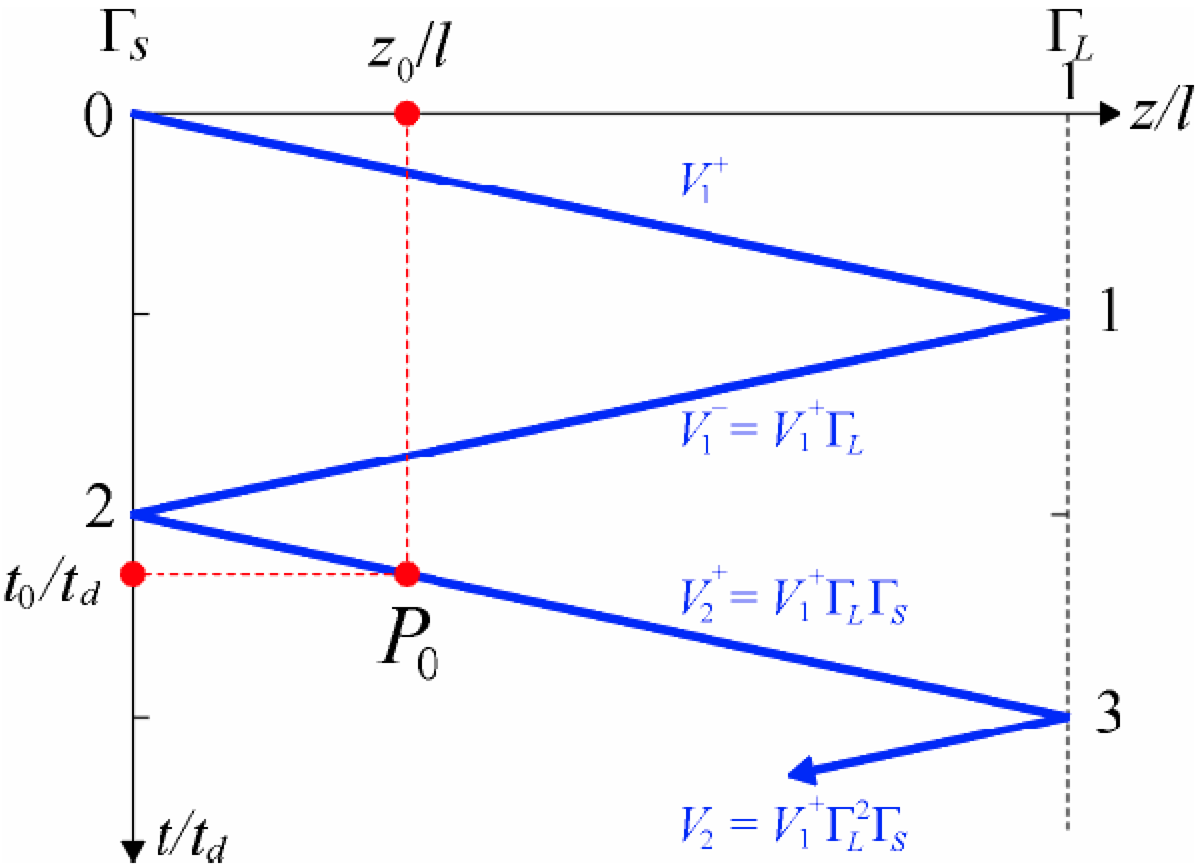
BOUNCE DIAGRAM

$$\Gamma_S = \frac{R_S - Z_0}{R_S + Z_0}$$

$$\Gamma_L = \frac{R_L - Z_0}{R_L + Z_0}$$



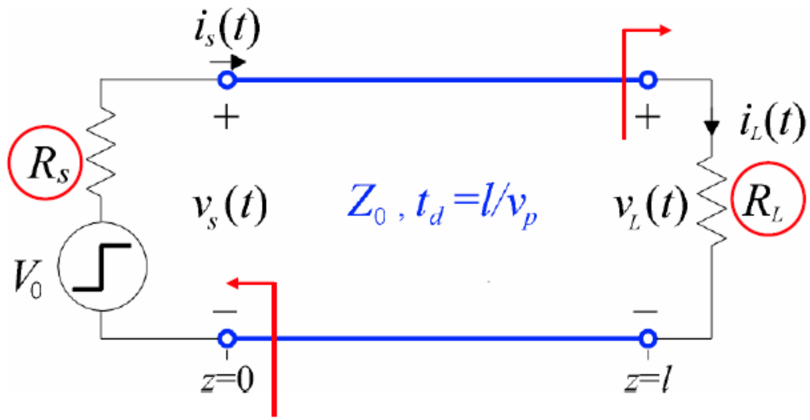
SPATIO-TEMPORAL Voltage Distribution at location z_0 time t_0



Transients in Open-ended Tx line

$$\Gamma_L = \frac{R_L - Z_0}{R_L + Z_0} = 1$$

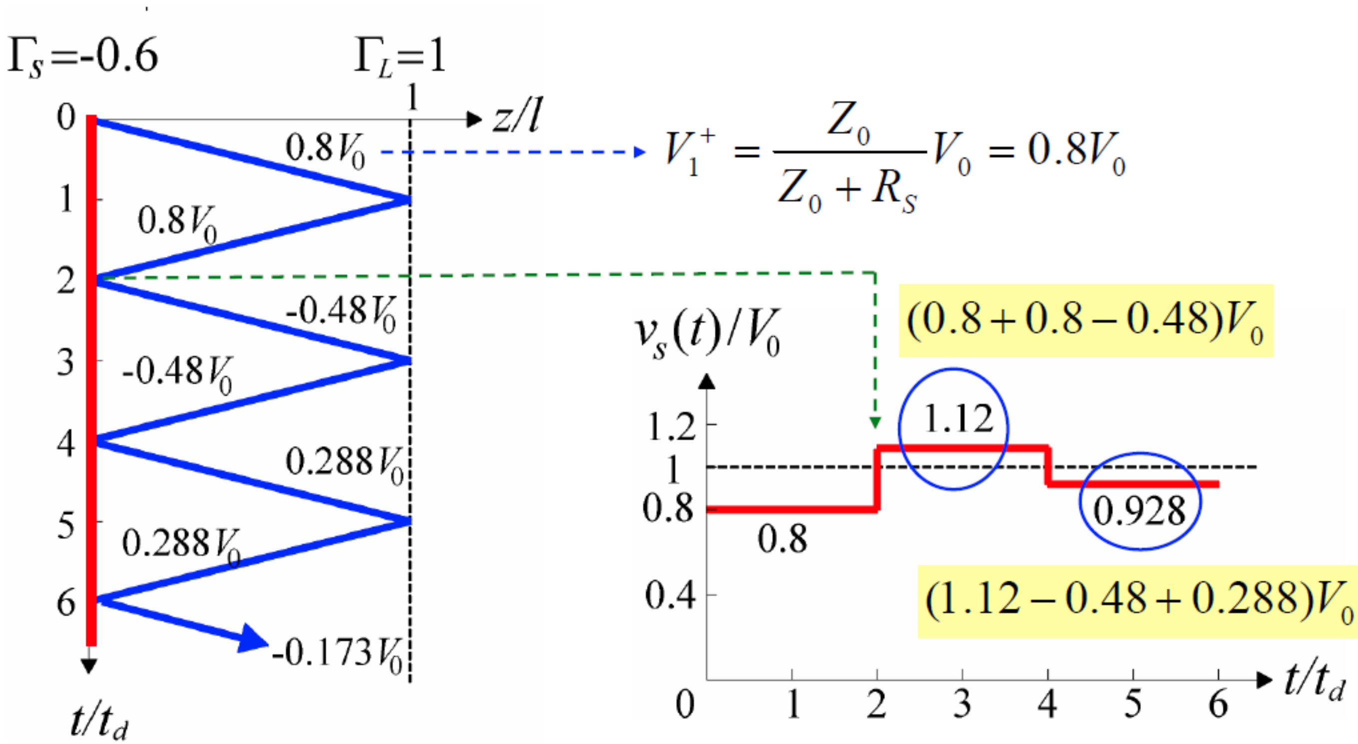
$R_s = 0.25Z_0$



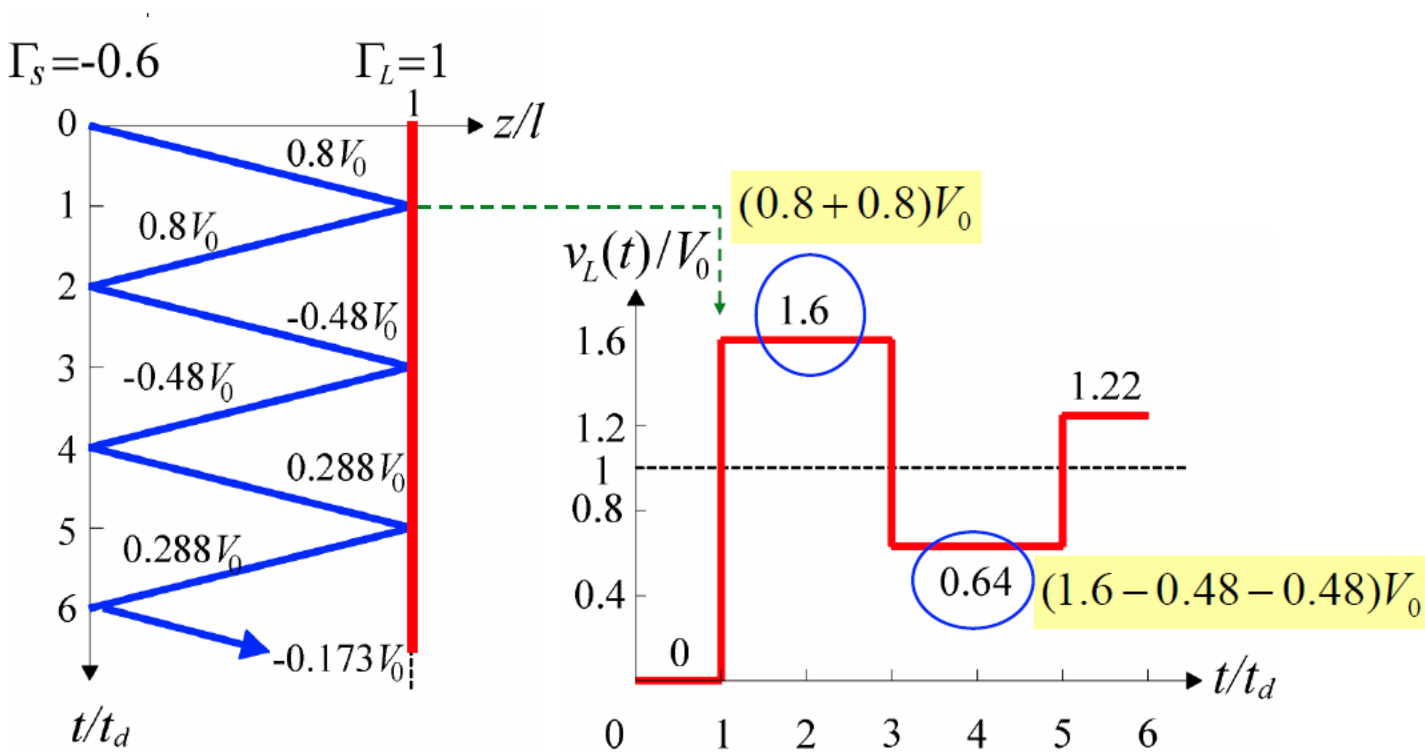
$R_L = \infty$

$$\Gamma_S = \frac{R_S - Z_0}{R_S + Z_0} = -0.6$$

Transient at SOURCE end

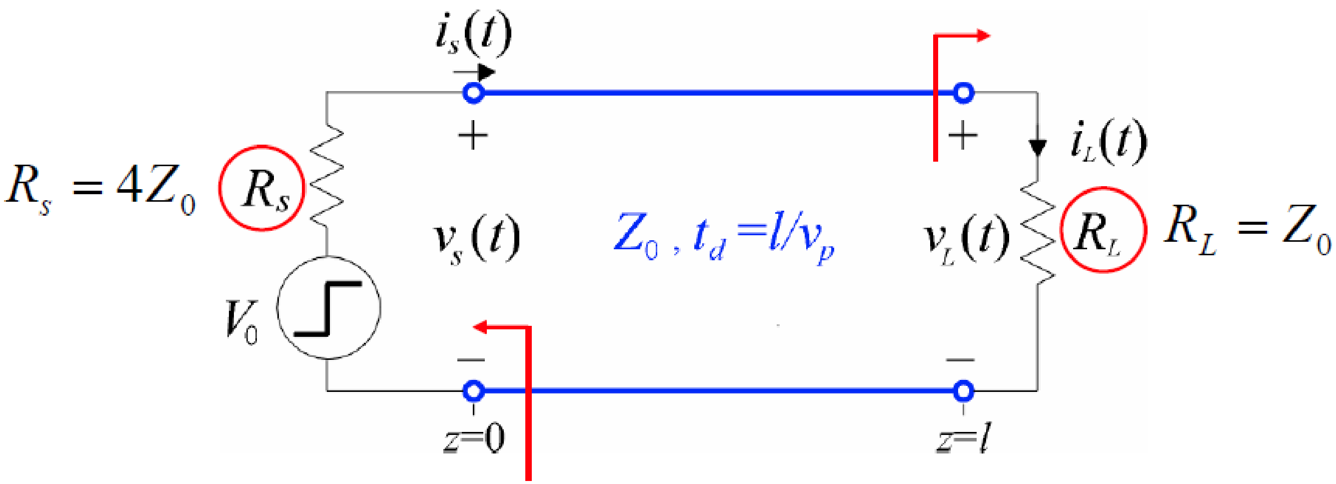


Transient at LOAD end



Transients in MATCHED Tx line

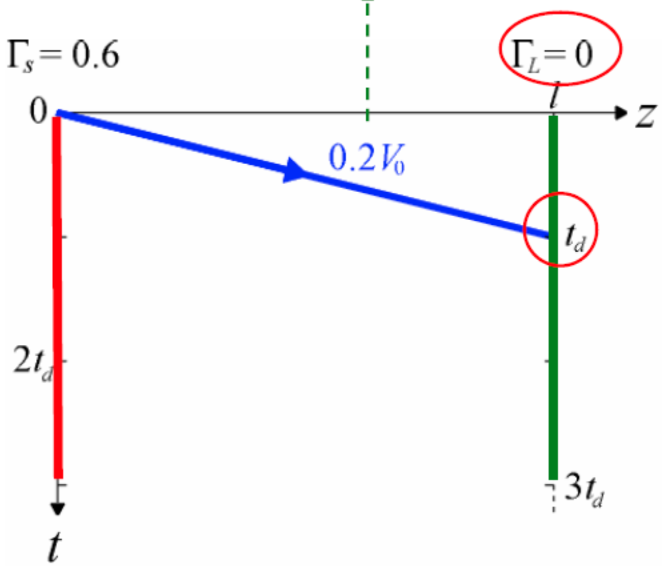
$$\Gamma_L = \frac{R_L - Z_0}{R_L + Z_0} = 0$$



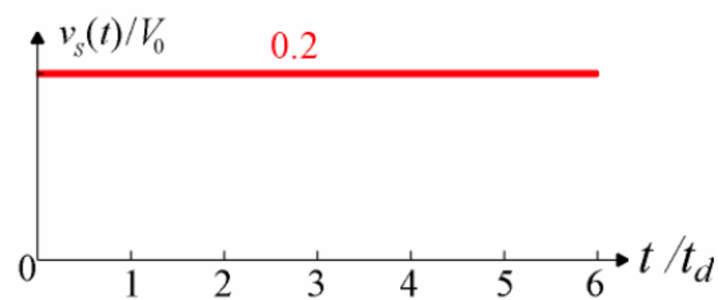
$$\Gamma_S = \frac{R_S - Z_0}{R_S + Z_0} = 0.6$$

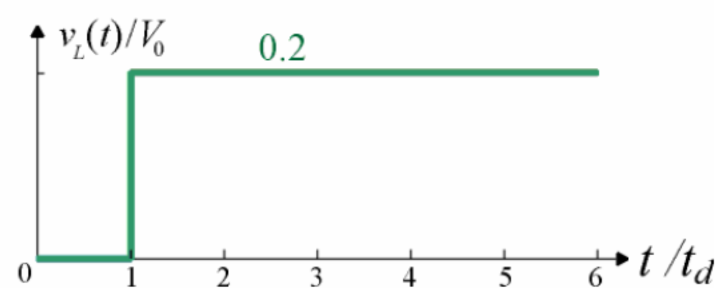
Transients at SOURCE & LOAD end

$$V_1^+ = \frac{Z_0}{Z_0 + R_S} V_0 = 0.2V_0$$



No overshooting & ringing (good!)

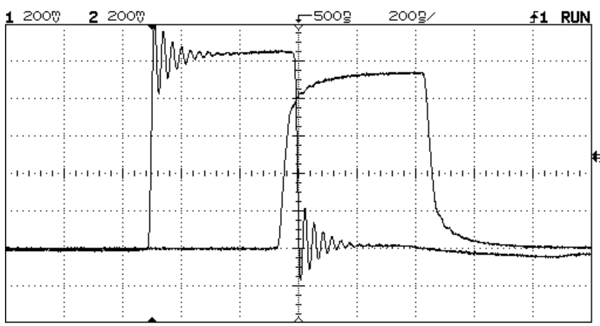


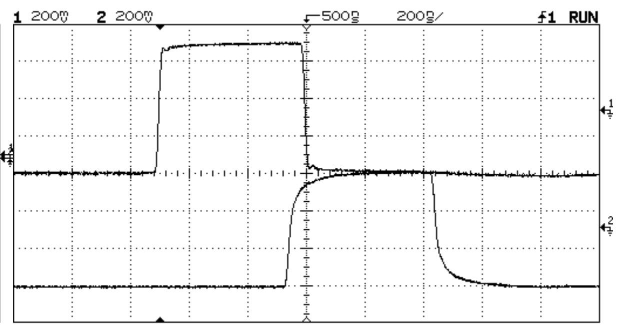


Pulses Measured with the Reels of RG58/U Cable

50 Ohm source

50 Ohm line terminated in 50 Ohms load





Improperly terminated cable connecting input to scope

Properly terminated cable connecting input to scope

Step input Measured for open ended line