__Theorem__: Let $\underline{X} = (X_1, \ldots, X_n)$ be a continuous random vector with joint pdf $f_{\underline{X}}(\underline{x})$ and let $\underline{u}: \mathbb{R}^n \to \mathbb{R}^n$, $\underline{u} = (u_1, \ldots, u_n)$, $u_i = g_i(\underline{x})$, $i = 1, \ldots, n$. Suppose that for each $\underline{u}$ the transformation $\underline{g} = (g_1, \ldots, g_n)$ has a finite number $k = k(\underline{u})$ inverses. Suppose that $\mathbb{R}^n$ can be partitioned into k disjoint sets $A_1, \ldots, A_k$ such that transformation $\underline{g}$ from A_i into $\mathbb{R}^n$ is one-to-one with inverse transformation

$$x_1 = h_{1i}(\underline{u}), \quad \ldots \quad x_n = h_{ni}(\underline{u}), \quad i = 1 \ldots k.$$

Suppose that first order partial derivatives are continuous and each Jacobian

$$J_i = \begin{vmatrix} \frac{\partial h_{1i}}{\partial u_1} & \cdots & \frac{\partial h_{1i}}{\partial u_n} \\ \vdots & & \\ \frac{\partial h_{ni}}{\partial u_1} & \cdots & \frac{\partial h_{ni}}{\partial u_n} \end{vmatrix} \neq 0$$

in the range of the transformation. Then the joint pdf of $\underline{U} = (U_1, \ldots U_n)$ is given by

$$f_{\underline{U}}(\underline{u}) = \sum_{i=1}^{k} f_{\underline{X}}\left(h_{1i}(\underline{u}), \ldots, h_{ni}(\underline{u})\right)|J_i|$$

Example: Distribution of Order Statistics

Let $X_1, \ldots, X_n$ be i.i.d. with cdf $F(x)$ (Continuous) and pdf $f(x)$.

$X_{(1)} = \min\{X_1, \ldots, X_n\}$

$X_{(2)} =$ second smallest of $\{X_1, \ldots, X_n\}$

$\vdots$

$$X_{(n)} = \max\{X_1, \ldots, X_n\}.$$

$Y_i = X_{(i)} \rightarrow i^{th}$ order statistics

$i = 1, \ldots, n$

Distributions of Order Statistics

$$F_{Y_n}(y) = P(Y_n \leq y) = P(X_{(n)} \leq y)$$

$$= P(X_1 \leq y, X_2 \leq y, \ldots, X_n \leq y)$$

$$= \prod_{i=1}^{n} P(X_i \leq y) = [F(y)]^n$$

The pdf of Y_n is

$$f_{Y_n}(y) = n\left[F(y)\right]^{n-1} f(y)$$

$$Y_1 \rightarrow X_{(1)}$$

$$F_{Y_1}(y) = P(Y_1 \leq y) = 1 - P(X_{(1)} > y)$$

$$= 1 - P(X_1 > y, \ldots, X_n > y)$$

$$= 1 - \prod_{i=1}^{n} P(X_i > y)$$

$$= 1 - \left[1 - F(y)\right]^n$$

The pdf of Y_1 is then

$$f_{Y_1}(y) = n\left[1 - F(y)\right]^{n-1} f(y)$$

<u>Examples</u>: Let $X_1, \ldots, X_n \overset{i.i.d.}{\sim} U(0, \theta)$, $\theta > 0$

$$f(x) = \begin{cases} \frac{1}{\theta}, & 0 < x < \theta \\ 0, & \text{otherwise} \end{cases}$$

$$F(x) = \begin{cases} 0, & x \leq 0 \\ x/\theta, & 0 < x < \theta \\ 1, & x \geq \theta \end{cases}$$

Distribution of max $\rightarrow X_{(n)} = Y_n$

$$f_{Y_n}(y) = n\left(\frac{y}{\theta}\right)^{n-1} \cdot \frac{1}{\theta}, \quad 0 < y < \theta$$

$$= \begin{cases} \dfrac{n\, y^{n-1}}{\theta^n}, & 0 < y < \theta \\ 0, & \text{otherwise} \end{cases}$$

2. Let $X_1, \ldots, X_n \overset{i.i.d.}{\sim} N(0,1)$

$Y_n = X_{(n)}$ $\qquad F(x) = \Phi(x), f(x) = \phi(x)$

$$f_{Y_n}(y) = n\left[\Phi(y)\right]^{n-1}\phi(y), \quad -\infty < y < \infty$$

3. Let $X_1, \ldots, X_n \sim Exp(\lambda)$

$Y_1 = X_{(1)}$ $\qquad F(x) = \begin{cases} 1-e^{-\lambda x}, & x>0 \\ 0, & x \leq 0 \end{cases}$

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x>0 \\ 0, & x \leq 0 \end{cases}$$

$$f_{Y_1}(y) = n\left(e^{-\lambda x}\right)^{n-1} \cdot \lambda e^{-\lambda x}$$

$$= \begin{cases} n\lambda e^{-n\lambda x}, & x > 0 \\ x, & x \leq 0 \end{cases}$$

Which is again $\text{Exp}(n\lambda)$

Let us consider joint pdf of $\underline{Y} = (Y_1, \ldots, Y_n)$

$Y_1 = X_{(1)}$

$Y_2 = X_{(2)}$

$\vdots$

$Y_n = X_{(n)}$

$\underline{Y} : \mathbb{R}^n \to \mathbb{R}^n$ is $n!$ to $\underline{1}$ transformation

There are $n!$ inverse images:

(1) $x_1 = y_1$
$x_2 = y_2$
$\vdots$
$x_n = y_n$

$$J_1 = \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 1 \end{vmatrix} = 1$$

(2) $x_1 = y_2$
$x_2 = y_1$
$x_3 = y_3$
$\vdots$
$x_n = y_n$

$$J_2 = \begin{vmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix} = -1$$

$|J_2| = 1$

③

$x_1 = y_3$

$x_2 = y_2$

$x_3 = y_1$

$\vdots \quad \vdots$

$x_n = y_n$

$|J_3| = +1$

$\vdots$

$n!$

$x_1 = y_n$

$x_2 = y_{n-1}$

$\vdots$

$x_n = y_1$

$|J_{n!}| = 1$

The joint pdf of $\underset{\sim}{X} = (X_1, \cdots X_n)$ is

$$f_{\underset{\sim}{X}}(\underset{\sim}{x}) = \prod_{i=1}^{n} f(x_i) \quad , \quad x_i \in R,\ i=1 \cdots n$$

So the joint pdf of $\underset{\sim}{Y} = (Y_1, \cdots, Y_n)$

$$f_{\underset{\sim}{Y}}(\underset{\sim}{y}) = \begin{cases} n! \prod_{i=1}^{n} f(y_i) & , \quad -\infty < y_1 < y_2 < \cdots < y_n < \infty \\ 0, & \text{otherwise} \end{cases}$$

The pdf of Y_r is

$$f_{Y_r}(y) = \frac{n!}{(r-1)!\,(n-r)!} \left[F(y)\right]^{r-1} \left[1-F(y)\right]^{n-r} f(y), \quad -\infty < y < \infty$$

Example: Let $X_1, \dots X_n \overset{i.i.d.}{\sim} U(0,1)$

$$F(y) = \begin{cases} 0 & y \le 0 \\ y, & 0 < y < 1 \\ 1, & y \ge 1 \end{cases}$$

Then

$$f_{Y_r}(y) = \frac{n!}{(r-1)!\,(n-r)!}\, y^{r-1} (1-y)^{n-r}, \quad 0<y<1$$

which is Beta $(r, n-r+1)$ dist^n

Random Sampling, Population, Statistic, Sample

A statistical population is a collection of qualitative or quantitative measurements on a topic of interest.

<u>Sample</u>: $\rightarrow$ a subset of population

Random Sampling $\rightarrow$ here each unit of the population has the same probability of getting selected into the sample.

Let $X_1, \ldots, X_n$ be a random sample from a population with distribution $F(x)$ and pdf/pmf $f(x)$.

Then the joint pmf/ pdf of $X_1 \dots X_n$ is

$$f_{\underline{X}}(\underline{x}) = f(x_1) f(x_2) \cdots f(x_n)$$

A function $T = T(X_1, \dots X_n)$ is called a statistic. For example

$\overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$, $X_M \rightarrow \text{Median}(X_1 \dots X_n)$

$\rightarrow$ sample mean $\qquad \rightarrow$ sample median

$X_{(n)} \to \max\{X_1, \ldots X_n\}$, $X_{(1)} = \min\{X_1, \ldots, X_n\}$

$$S^2 = \frac{1}{(n-1)} \sum_{i=1}^{n} (X_i - \bar{X})^2 \to \text{sample variance}$$

The prob. distⁿ of a statistic is termed as a sampling distribution

Asymptotic Distribution (as $n \to \infty$) of the Sample Mean

Central Limit Theorem : Let $X_1, X_2, \cdots$ be a sequence of i.i.d. random variables with a mean μ and variance σ^2 ($<\infty$). Let

$$\boxed{\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i}$$

. Then the limiting distribution of $\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma}$ is $N(0,1)$ as $n \to \infty$.

Remark: It has been observed in practice that $n \geq 30$ is large

$$S_n = \sum_{i=1}^{n} X_i$$

$$\frac{S_n - n\mu}{\sigma\sqrt{n}} \longrightarrow N(0,1) \text{ as } n \to \infty .$$