Test 4: MA20104 Probability and Statistics

Time: 45 minutes **Marks: 40**

Instructions: There are ten questions. You have to answer all. They carry equal marks.

Q. 1: Let X follow an exponential distribution with mean 2. Find the variance of $Y = 1 - e^{-0.5X}$. (answer should be correct up to three decimal places)

ANS. 1: 0.083 (error range: 0.005)

Q. 2: Let $X \sim N(0,4)$ and $Y \sim N(0,9)$ be independently distributed random variables. Find the probability that (X, Y) will lie in the interior of the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$. (answer should be correct up to three decimal places)

ANS. 2: 0.393 (error range: 0.005)

Q. 3: Let (X, Y) be continuous with the joint pdf

$$f_{X,Y}(x,y) = \begin{cases} x + \frac{3y^2}{2} & 0 < x < 1, 0 < y < 1 \\ 0 & elsewhere \end{cases}.$$

Find $P(Y < 0.5 | X = 0.25)$. (answer should be correct up to three decimal places)

ANS. 3:0.250 (error range: 0.005)

Q. 4: Let X and Y be continuous with the joint pdf

$$f_{X,Y}(x,y) = \begin{cases} 6xy & 0 \le y \le \sqrt{x}\,, 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}.$$

Find $Var(X | Y = 0.5)$. (answer should be correct up to three decimal places)

ANS. 4: 0.0413 (error range: 0.005)

Q. 5: Let X and Y be continuous with the conditional density of Y given $X = x$ given by

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{x} & 0 < y < x \\ 0 & \text{elsewhere} \end{cases}.$$

Let the marginal density function of X be given by

$$f_X(x) := \begin{cases} 2x & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}.$$

Find the conditional expectation $E\left(X|Y = \frac{1}{3}\right)$. (answer should be correct up to three decimal places)

ANS. 5: 0.667 (error range: 0.005)

Q. 6: Let X and Y have the joint p.m.f. given by

$$p(x,y) := \begin{cases} \frac{1}{8} & (x,y) = (0,0),(1,0),(0,1),(1,1),(2,1),(1,2),(2,2),(3,3) \\ 0 & \text{elsewhere} \end{cases}.$$

Find the correlation coefficient of X and Y. (answer should be correct up to three decimal places)

ANS 6: 0.733 (error range: 0.005)

Q. 7: Supposed a fair coin is tossed. If head is observed, one fair dice is rolled, and if tail is observed, two fair dice are rolled independently. Suppose Y is the sum of outcome(s) on the dice. Find $P(5 \leq Y \leq 8)$. (answer should be correct up to three decimal places)

ANS. 7: 0.444 **(**error range: 0.005)

Q. 8: Let X denote the value on the upper face on one roll of a fair dice. The conditional distribution of $Y|X = x$ is a $Bin\left(x, \frac{1}{2}\right)$. Find $P(Y = 0)$. (answer should be correct up to three decimal places)

ANS. 8: 0.164 (error range: 0.005)

Q. 9: Suppose $X_1 \sim Gamma(3,2)$ and $X_2 \sim Gamma(7,2)$ are independently distributed. Find the correlation between $(X_1 + X_2)$ and $\frac{X_1}{X_1 + X_2}$. (answer should be correct up to three decimal places)

ANS. 9: 0 (error range: 0.005)

Q. 10: Suppose (X, Y) denote the marks on two consecutive class tests and follow bivariate normal $BVN(20, 24, 16, 25, 0.8)$ distribution. Find $P(\ 25 < Y \leq 34\ |X = 24)$. (answer should be correct up to three decimal places)
(Given $\Phi(0.5)= 0.6915$, $\Phi(1) = 0.8413$, $\Phi(1.96) = 0.975$, $\Phi(2) = 0.9772$,
$\Phi(2.32) = 0.99$, $\Phi(1.65) = 0.95$, $\Phi(1.28) = 0.9$, $\Phi(3) = 0.9987$)

ANS. 10: 0.819 (error range 0.005)

Q. 11: Suppose (X, Y) denote the average longevity (in years) of married couple (X denotes longevity of men and Y denote longevity of women) and follow bivariate normal $BVN(72, 75, 9, 16, 0.8)$ distribution. Find $\frac{P(\ 69<X\leq 75)}{P(73<Y\leq 77)}$ **.** (answer should be correct up to three decimal places)
(Given $\Phi(0.5)= 0.6915$, $\Phi(1) = 0.8413$, $\Phi(1.96) = 0.975$, $\Phi(2) = 0.9772$,
$\Phi(2.32) = 0.99$, $\Phi(1.65) = 0.95$, $\Phi(1.28) = 0.9$, $\Phi(3) = 0.9987$)

ANS. 11: 1.782 (error range 0.005)

Q. 12: Consider the random circle $\mathbb{C}_{X,Y}$ centered at the origin and passing through the point (X, Y). Let (X, Y) follow bivariate normal $BVN(0,0, 1, 1, 0.5)$. Find the expected value of the area of the circle $\mathbb{C}_{X,Y}$. (answer should be correct up to three decimal places)

ANS. 12: 6.283 (error range: 0.005)

Q. 13: Let X_1, X_2, X_3, X_4 be i.i.d. $Uniform(0,1)$ random variables. Find $P(0.25 < max\{X_1, \mathrm{X}_2, X_3\ , \mathrm{X}_4\} < 0.75)$. (answer should be correct up to three decimal places)

ANS. 13: 0.313 (error range: 0.005)

Q. 14: Let the joint pdf of X and Y be given by

$$f(x, y) := \begin{cases} \frac{1}{2} & 0 < x < y < 2 \\ 0 & \text{elsewhere} \end{cases}.$$

Find $Cov(X, Y)$. (answer should be correct up to three decimal places)

ANS. 14: 0.111 (error range: 0.005)

Q. 15: Suppose X and Y are independent random variables with respective density functions given by

$$f(x) = \begin{cases} 3x^2 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad f(y) = \begin{cases} 4y^3 & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}.$$

Find $P(X + Y < 1)$. (answer should be correct up to three decimal places)

ANS. 15: 0.029 (error range: 0.005)

Q. 16: Let $X \sim U\left(0, \frac{\pi}{2}\right)$. Find $Cov(\sin X, \cos X)$. (answer should be correct up to three decimal places)

ANS. 16: -0.087 (error range: 0.005)

Q. 17: Let (X, Y) have a uniform distribution on the unit circle. Find $P\left(0 < X < \frac{1}{\sqrt{2}}\right)$. (answer should be correct up to three decimal places)

ANS. 17:0.409 (error range: 0.005)

Q. 18: Let (X, Y) be continuous with the joint pdf

$$f_{X,Y}(x, y) = \begin{cases} 4x\, e^{-(x^2+2y)} & x > 0, y > 0 \\ 0 & elsewhere \end{cases}$$

Let $U = XY, V = \frac{X}{Y}$. The value of the joint pdf of (U, V) at the $(u, v) = (1, 1)$. (answer should be correct up to three decimal places)

ANS. 18: 0.1 (error range: 0.005)