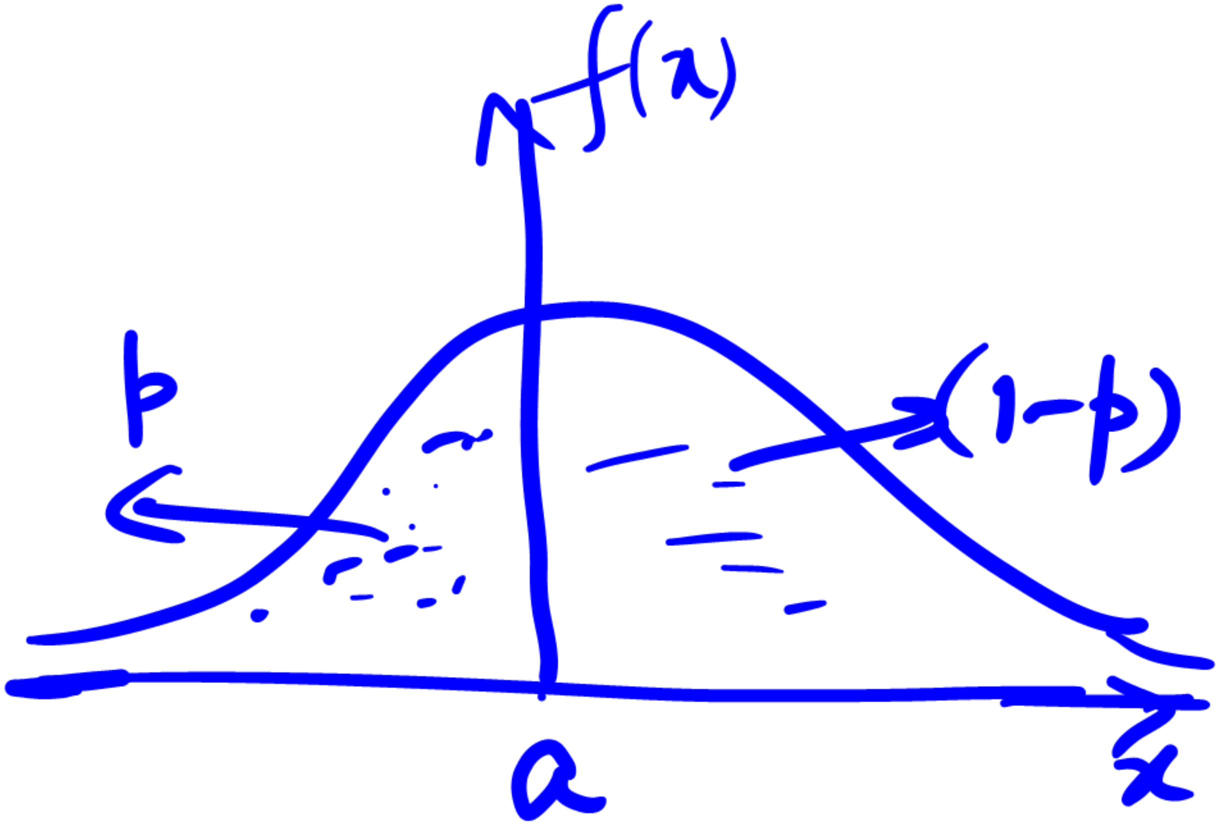
Quantiles :

$P(X \leq a) = p$



A real number Q_p satisfying

$P(X \leq Q_p) \geq p$ &

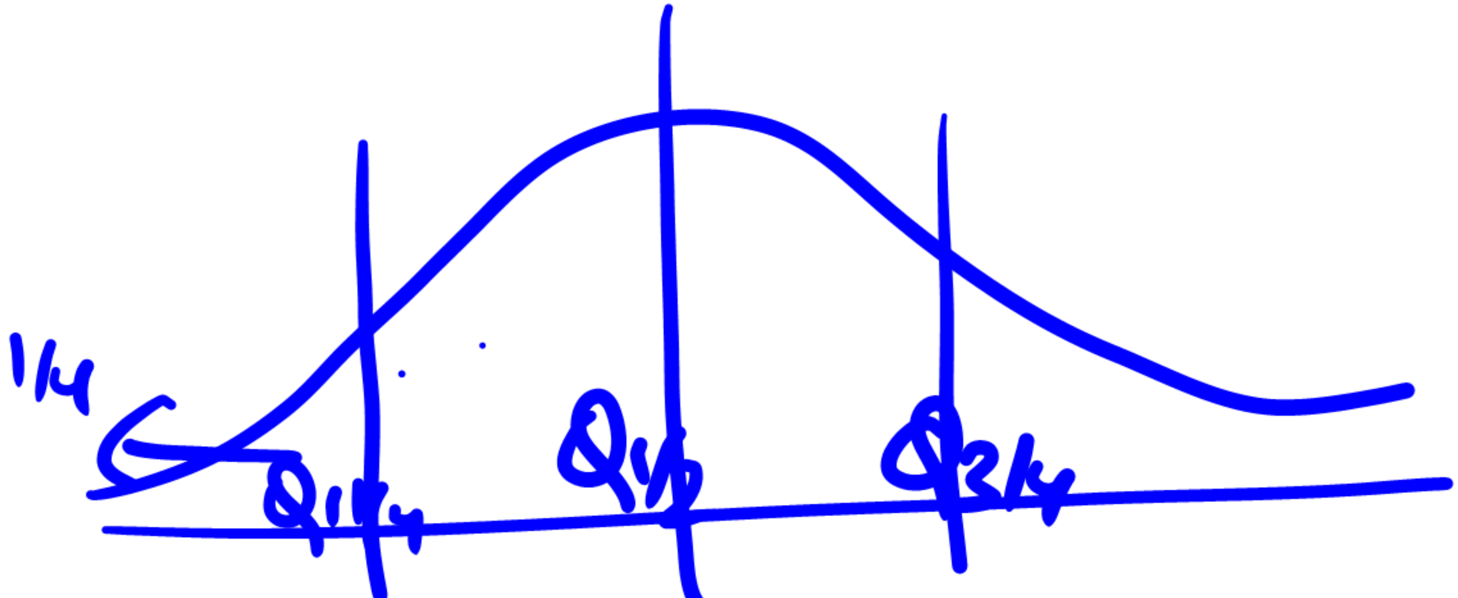
$$P(X \geq Q_p) \geq 1-p, \quad 0 < p < 1$$

is called p^{th} quantile (or quantile of order p) of distribution of X.

If F is absolutely continuous then $F(Q_p) = p$ ie there is a unique quantile.

For $p = \frac{1}{2}$, $Q_{1/2} = M$ is called the median.

$Q_{1/4}$, $Q_{1/2}$, $Q_{3/4}$ are called quartiles



$Q_{1/10}, Q_{2/10}, \ldots, Q_{9/10}$ Deciles

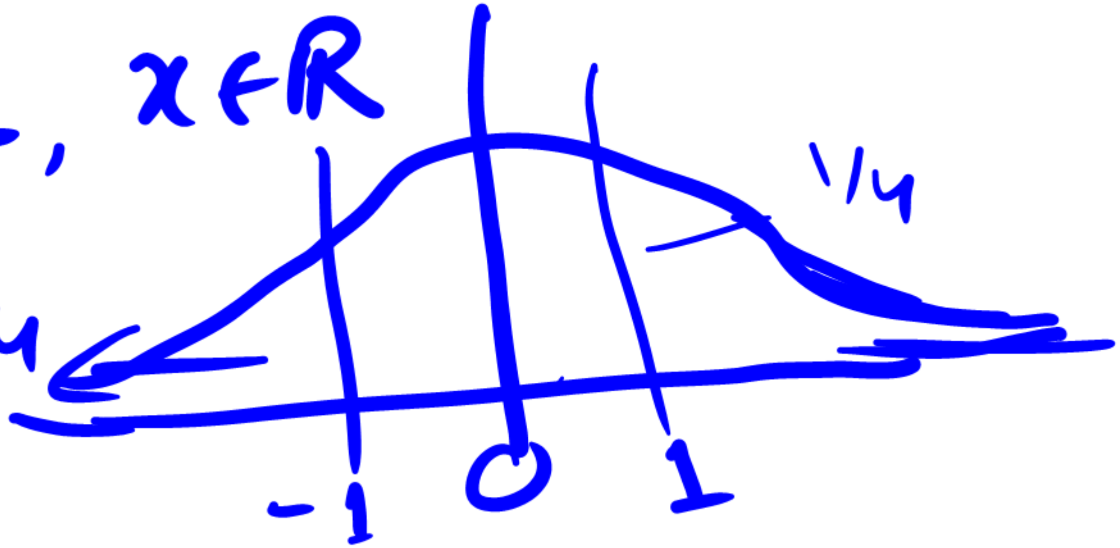
$Q_{1/100}, Q_{2/100}, \ldots, Q_{99/100}$

$\longrightarrow$ percentiles

Examples: $f_X(x) = \dfrac{1}{\pi(1+x^2)}$,

Cauchy dist^n $\quad -\infty < x < \infty$

$F_X(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1} x, \quad x \in \mathbb{R}$



$F_X(0) = \frac{1}{2}$

so $M = 0$ is the median

$$E(X) = \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{x}{(1+x^2)}\,dx$$

does not exist.

$$Q_{1/4} = -1 \quad \& \quad Q_{3/4} = 1$$

as

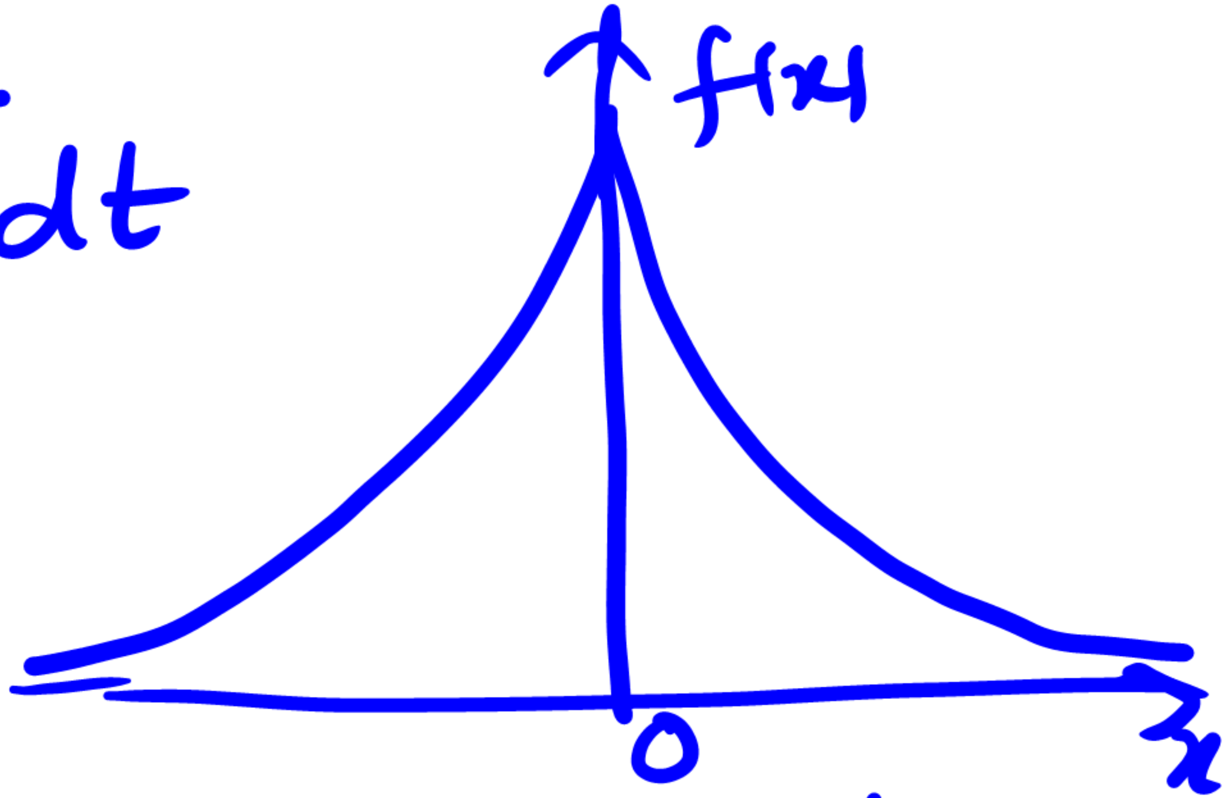
$$F(-1) = 1/4 \quad , \quad F(1) = 3/4 .$$

2: $f_X(x) = \frac{1}{2} e^{-|x|}, \quad -\infty < x < \infty$

double exponential or Laplace distⁿ

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{2} e^{t}\, dt$$

$$= \frac{1}{2} e^{x}, \quad x<0$$



$$= \frac{1}{2} + \int_{0}^{x} \frac{1}{2} e^{-t}\, dt = \frac{1}{2} - \frac{1}{2} e^{-t}\Big|_{0}^{x}$$

$$= 1 - \frac{1}{2} e^{-x}, \quad x>0$$

$$F_X(x) = \begin{cases} \frac{1}{2} e^{x}, & x < 0 \\ 1 - \frac{1}{2} e^{-x}, & x \geq 0 \end{cases}$$

$F_X(0) = \frac{1}{2}$. So Median $= 0$

$$F_X(x) = \frac{1}{4} \Rightarrow \frac{1}{\cancel{2}} e^{x} = \frac{1}{\cancel{4}\ 2}$$

$$x = -\log_e 2 \rightarrow Q_{1/4}$$

$$F_X(x) = 3/4 \;,\; 1 - \frac{1}{2} e^{-x} = \frac{3}{4}$$

$$\frac{1}{4} = \frac{1}{2} e^{-x} \Rightarrow x = \ln 2 \rightarrow Q_{3/4}$$

<u>Example</u>: $P(X=-2) = P(X=0) = \frac{1}{4}$

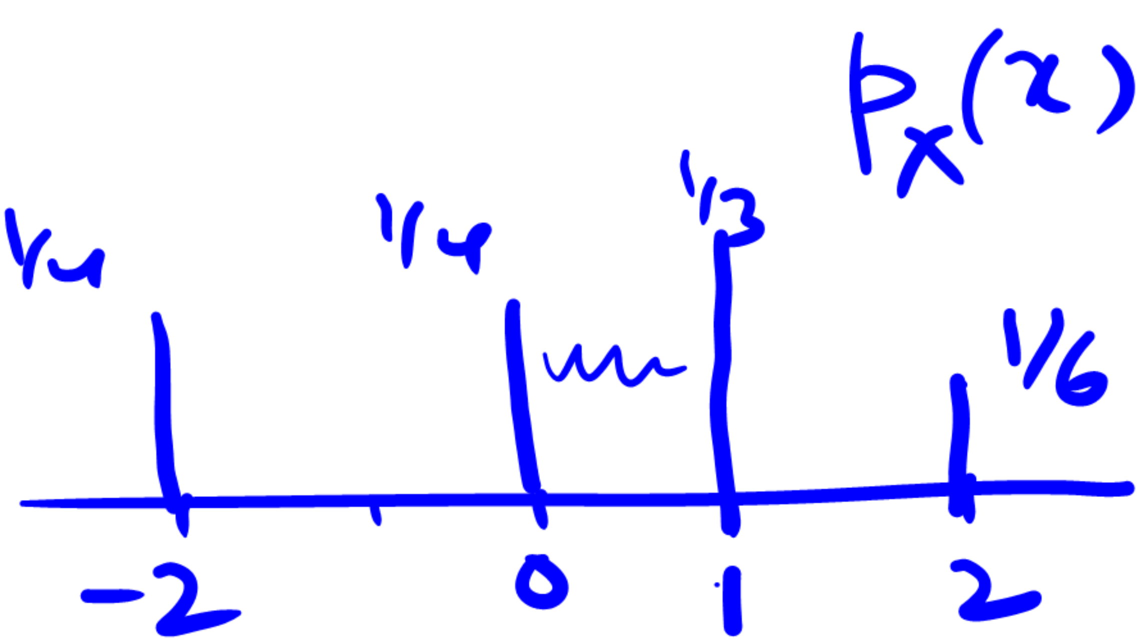
$P(X=1) = 1/3$, $P(X=2) = 1/6$

$0 \le M \le 1$

median

first quartile

$-2 \le Q_{1/4} \le 0$



Moment Generating Function

$$M_X(t) = E(e^{tX}), \quad t \in \mathbb{R}$$

is called mgf of X if it exists

<u>Example</u>: $f_X(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{ew} \end{cases}$

$$M_X(t) = E(e^{tX}) = \int_0^1 e^{tx} f_X(x)\,dx$$

$$= \int_0^1 2x e^{tx}\,dx$$

$$= 2\left[\frac{x e^{tx}}{t}\Big|_0^1 - \frac{e^{tx}}{t^2}\Big|_0^1 \right]$$

$$= 2\left[\frac{e^t}{t} - \frac{e^t}{t^2} + \frac{1}{t^2} \right]$$

2. $p_X(0) = \frac{1}{2}, \; p_X(1) = \frac{1}{2}$

$$M_X(t) = \frac{1}{2} e^{0 \cdot t} + \frac{1}{2} e^{1 \cdot t} = \left(\frac{1+e^t}{2} \right)$$

$$M_X(t) = E\left(e^{tX}\right)$$

$$= E\left[1 + \frac{tX}{1!} + \frac{t^2X^2}{2!} + \cdots\right]$$

$$= 1 + \frac{t}{1!}\mu_1' + \frac{t^2}{2!}\mu_2' + \cdots,$$

Coefficient of $t^k/k!$ in the

expansion of mgf is μ_k'.

$$\frac{d}{dt} M_X(t)\Big|_{t=0} = \mu_1'$$

$$\frac{d^2}{dt^2} M_X(t)\Big|_{t=0} = \mu_2'$$

$$\frac{d^k}{dt^k} M_X(t)\Big|_{t=0} = \mu_k'.$$

<u>Theorem</u>: If the moment of order t (>0) exists then the moment of order s $(0 < s < t)$ exists for a given r.v. X. If the moment of order $s(>0)$ does not

exist, then the moment of order t ($t > s$) does not exist.

<u>Theorem</u>: The mgf uniquely determines a cdf.

<u>Chebyshev's Inequality</u>: Let X be a r.v. with mean μ and variance σ^2. Then for any $k > 0$,

$$P(|X-\mu| \geq k) \leq \frac{\sigma^2}{k^2}.$$

Pf. Let X be continuous r.v.

with pdf $f_X(x)$.

$$\sigma^2 = Var(X) = E(X-\mu)^2$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)\,dx$$

$$|x-\mu| \le k \Rightarrow -k \le x-\mu \le k$$

$$\Rightarrow \mu-k \le x \le \mu+k$$

$$\sigma^2 \geq \int_{|x-\mu| \geq k} (x-\mu)^2 f_X(x)\, dx$$

$$\geq k^2 \int_{|x-\mu| \geq k} f_X(x)\, dx$$

$$= k^2 \, P(|X-\mu| \geq k)$$

or $$P(|X-\mu| \geq k) \leq \frac{\sigma^2}{k^2}$$

$$P(|X-\mu| < k) \geq 1-\frac{\sigma^2}{k^2}$$

Other forms

$$P(|X-\mu| < k\sigma) \geq 1-\frac{1}{k^2}$$

$$P(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}$$

Let us take $k = 1$

$$P(|X-\mu| < \sigma) \geq 0$$

trivial statement

$\underline{\underline{k=2}}$

$$P(|X-\mu| < 2\sigma) \geq 3/4$$

$$P(\mu - 2\sigma < X < \mu + 2\sigma) \geq 0.75$$

Example: 1. Let X be no. of costly purchases from a jewellery store in a day. Suppose $\mu = 18$, $\sigma = 2.5$.

With what prob. can we assert that there will be between 8 to 28 costly purchases.

$$P(8 \leq X \leq 28) = P(-10 \leq X - 18 \leq 10)$$

$$= P(|X - 18| \leq 10) \geq 1 - \frac{\sigma^2}{100} = \frac{15}{16}$$

2. Independent observations are available from a popⁿ with mean μ and variance 1. How many observations are needed in order that prob is at least 0.9 that the mean of observations differs from μ by not more than 1?

Suppose observations are $X_1, X_2 \cdots X_n$

$$E(X_i) = \mu, \quad V(X_i) = 1$$

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \quad , \quad E(\bar{X}) = \frac{1}{n}\sum_{i=1}^{n} E(X_i)$$

$$= \frac{n\mu}{n} = \mu$$

$$V(\bar{X}) = \frac{1}{n^2}\sum V(X_i) = \frac{n}{n^2} = \frac{1}{n}.$$

$$P(|\bar{X}-\mu| < 1) \geq 1 - \frac{1}{n} > 0.9$$

$$\Rightarrow n > 10$$

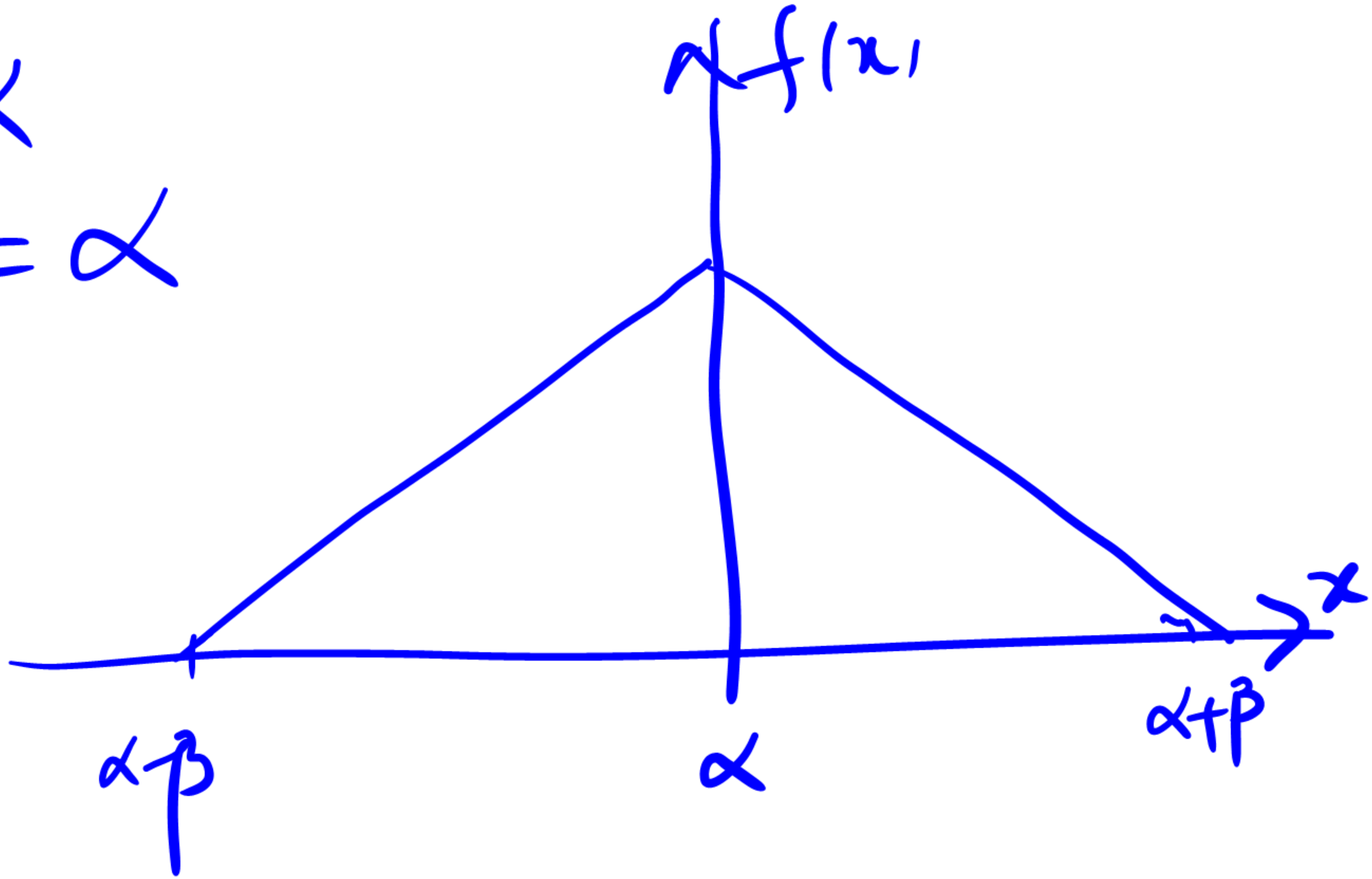
Exercise: Let X be a conti. r.v

$$f_X(x) = \frac{1}{\beta}\left\{1 - \left|\frac{x-\alpha}{\beta}\right|\right\}, \quad \alpha - \beta < x < \alpha + \beta$$

Find $E(X)$, $V(X)$, $Med(X)$, $Q_{1/4}$, $Q_{3/4}$, Measures of skewness & kurtosis

$E(X) = \alpha$

$\mathrm{Med}(X) = \alpha$



$V(X) = E(X-\mu)^2$

$$= \int_{\alpha-\beta}^{\alpha+\beta} (x-\alpha)^2 \frac{1}{\beta}\left(1-\left|\frac{x-\alpha}{\beta}\right|\right) dx$$

$$\frac{x-\alpha}{\beta} = y$$

$$= \beta^2 \int_{-1}^{1} y^2 (1-|y|)\, dy$$

$$\frac{1}{\beta} dx = dy$$

$$= \beta^2/6$$

$$\mu_4 = E(X-\alpha)^4 = \beta^4 \int_{-1}^{1} y^4 (1-|y|)\, dy$$

$Q_1 = \alpha + \beta\left(-\frac{1}{\sqrt{2}}\right)$ } Complete these calculations

Some Special Discrete Distributions

1. Discrete Uniform Distribution

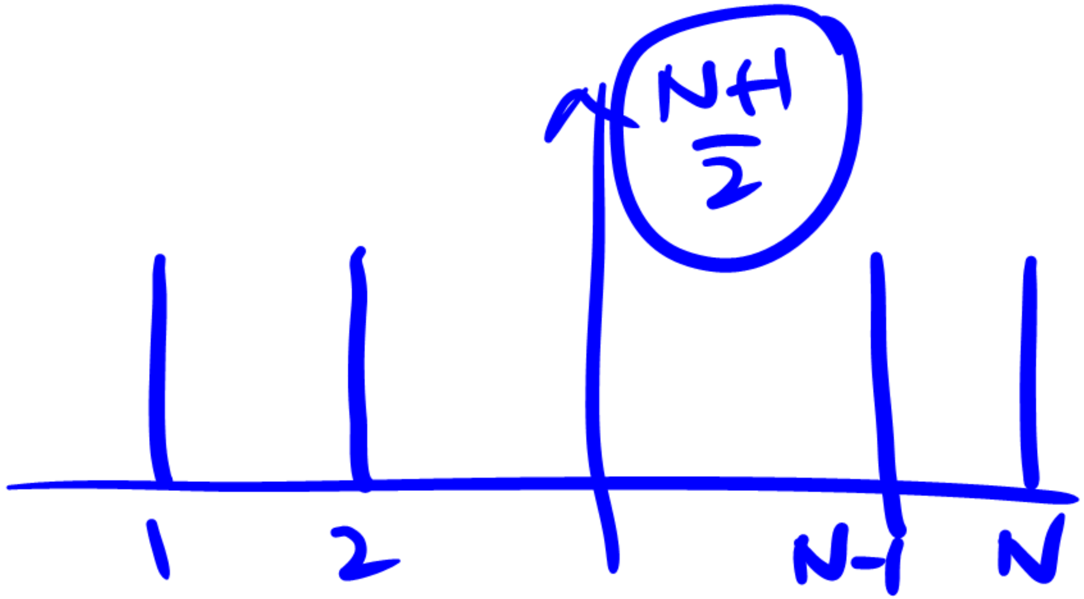
$X \to 1, 2, \ldots, N$

$P(X = i) = \frac{1}{N}, \quad i = 1, \ldots, N$

$$\mu_1' = E(X) = \sum_{i=1}^{N} \frac{i}{N} = \frac{N+1}{2}$$

$$\mu_2' = E(X^2) = \sum_{i=1}^{N} \frac{i^2}{N}$$

$$= \frac{(N+1)(2N+1)}{6}$$



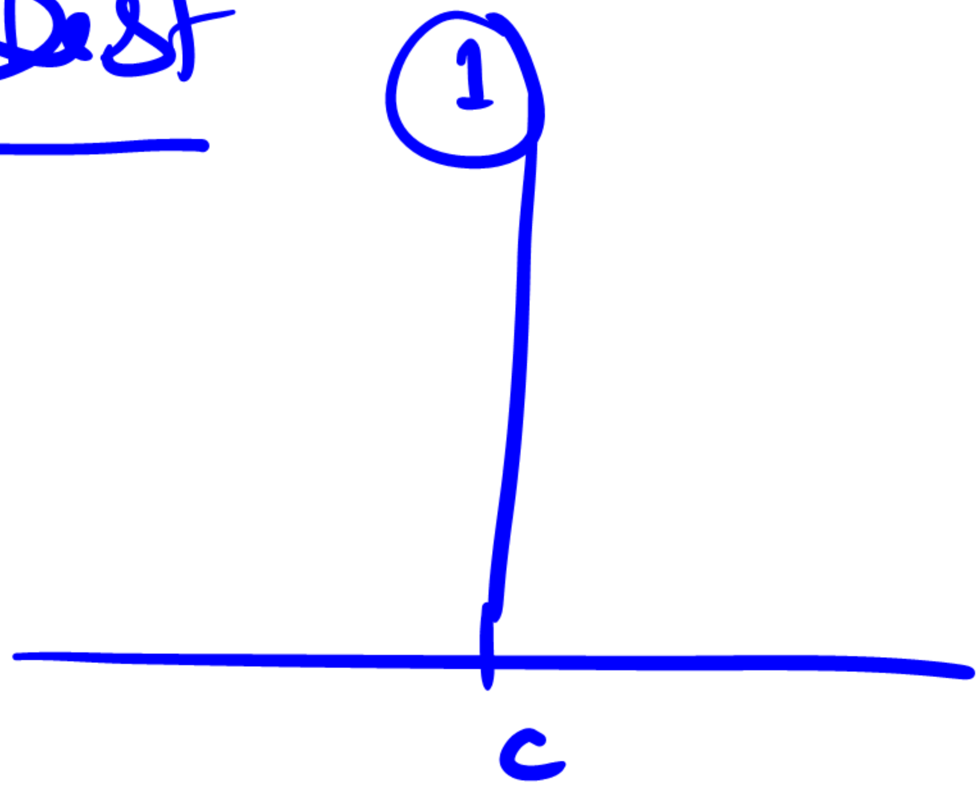
$$V(X) = \mu_2' - {\mu_1'}^2 = \frac{N^2-1}{12}$$

Moments of all orders exist.

MGF. $M_X(t) = E(e^{tX})$

$$= \sum_{i=1}^{N} \frac{e^{it}}{N} = \begin{cases} \dfrac{e^t(e^{Nt}-1)}{N(e^t-1)}, & t \neq 0 \\ 1, & t = 0 \end{cases}$$

2. <u>Degenerate Dist^n</u>



$$P(X=c)=1$$

$$\mu_k' = c^k$$

$$\mu_1' = c \,, \quad \mu_2' = c^2, \; \ldots$$

3. <u>Bernoulli Trials</u>: In a random expt if we have two possible

outcomes we associate them with success (s) and failure (f)

$$\Omega = \{ s, f \}$$

$X \rightarrow$ no of success

Bernoulli Distn

$X(s) = 1$, $X(f) = 0$

$P(X=1) = p$, $P(X=0) = 1-p$ $(= q)$

$$0 < p < 1$$

$\mu_1' = p$, $\mu_2' = p$, $\ldots$ $\mu_k' = p$

$$\mu_2 = \mu_2' - \mu_1'^2 = p - p^2 = p(1-p) = pq$$

$$M_X(t) = (1-p)\,e^{0.t} + p\,e^{1.t} = (q+pe^t)$$

4. <u>Binomial Dist^n</u>: Suppose independent Bernoulli trials are conducted under identical conditions with prob of success p. Let $X \to$ no of successes in n trials

$$X \to 0, 1, 2, \ldots, n$$

$$p_X(k) = P(X=k) = \binom{n}{k} p^k q^{n-k}, \quad k=0,1,\ldots,n, \quad 0<p<1$$

$$\sum_{k=0}^{n} p_X(k) = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k}$$

$$= (q+p)^n = 1.$$

$$\mu_1' = E(X) = \sum_{k=0}^{n} k \binom{n}{k} p^k q^{n-k}$$

$$= \sum_{k=1}^{n} \frac{n!}{(k-1)!\,(n-k)!} p^{k} q^{n-k}$$

$$= np \sum_{l=0}^{n-1} \binom{n-1}{l} p^{l} q^{n-1-l}$$

$l = k-1$

$$= np\,(q+p)^{n-1} = np.$$

$$E(X(X-1)) = \sum_{k=0}^{n} k(k-1) \binom{n}{k} p^{k} q^{n-k}$$

$$= \sum_{k=2}^{n} \frac{n!}{(k-2)!\,(n-k)!} p^{k} q^{n-k} \qquad k-2 = l$$

$$n(n-1)\boxed{p^{2} \sum_{l=0}^{n-2} \binom{n-2}{l} p^{l-2} q^{n-2-l}}$$

$$= n(n-1)p^{2}$$

$$E(X^{2}) = E\,X(X-1) + E(X)$$

$$= n(n-1)p^{2} + np$$

$$\mu_2 = V(X) = \mu_2' - \mu_1'^2$$

$$= n(n-1)p^2 + np - n^2p^2$$

$$= np(1-p) = npq.$$

$$\mu_3 = np(1-p)(1-2p),$$

$$\mu_4 = 3(npq)^2 + npq(1-6pq)$$

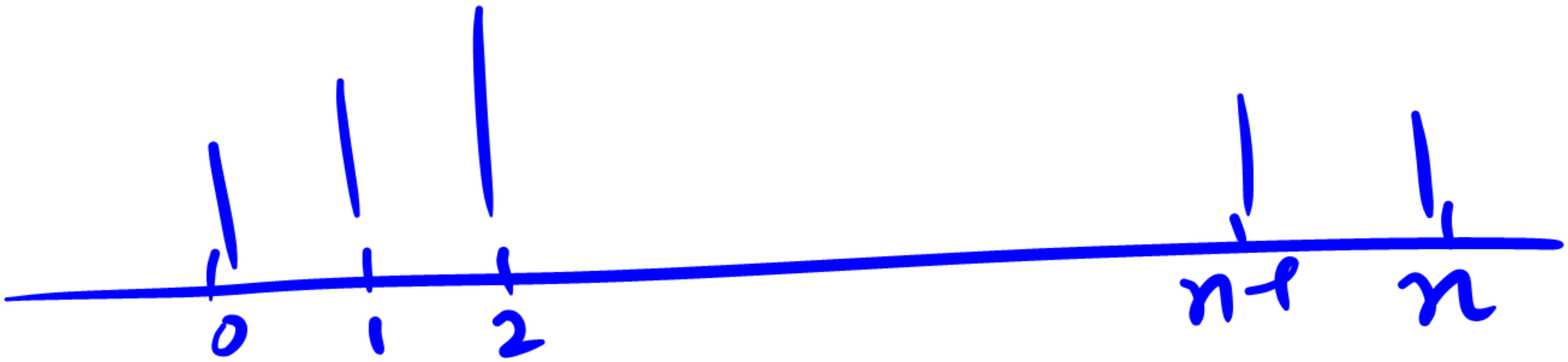
$*$ Ex

Measures of Skewness & Kurtosis

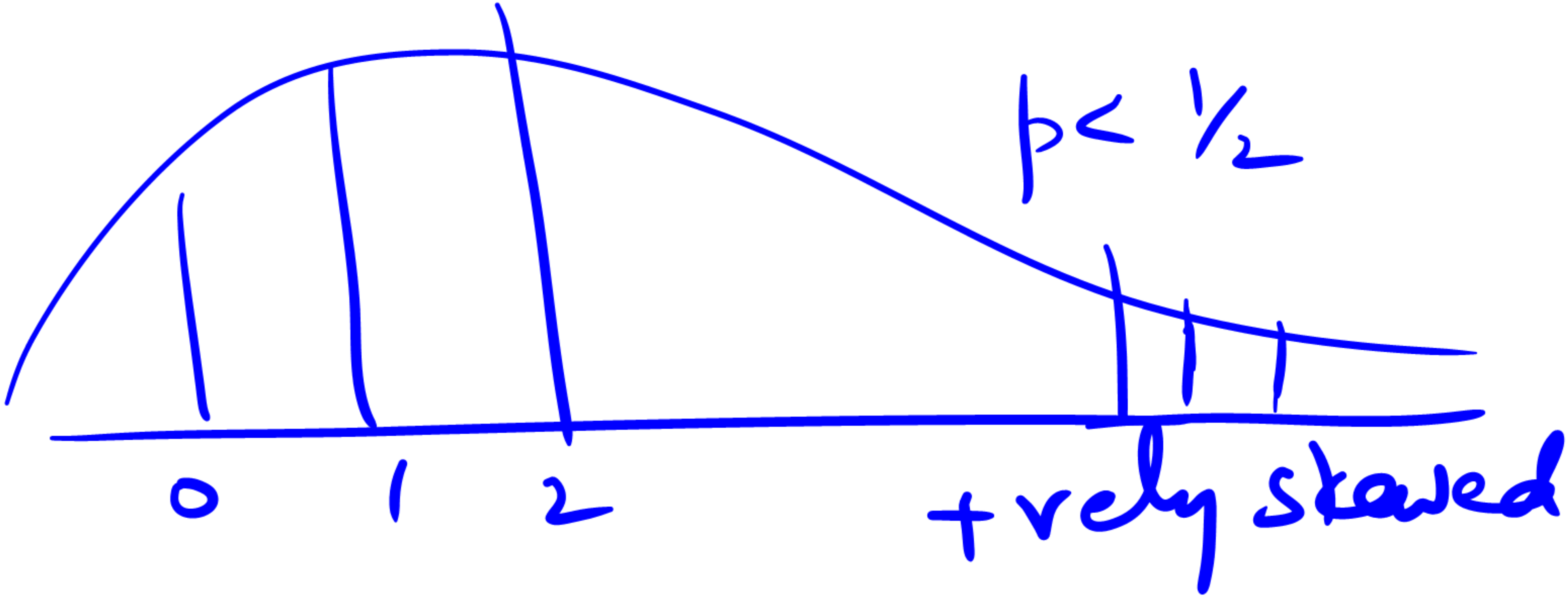
$$\beta_1 = \frac{\mu_3}{\sigma^3} = \frac{1-2p}{(npq)^{1/2}} = 0, \quad p = 1/2$$

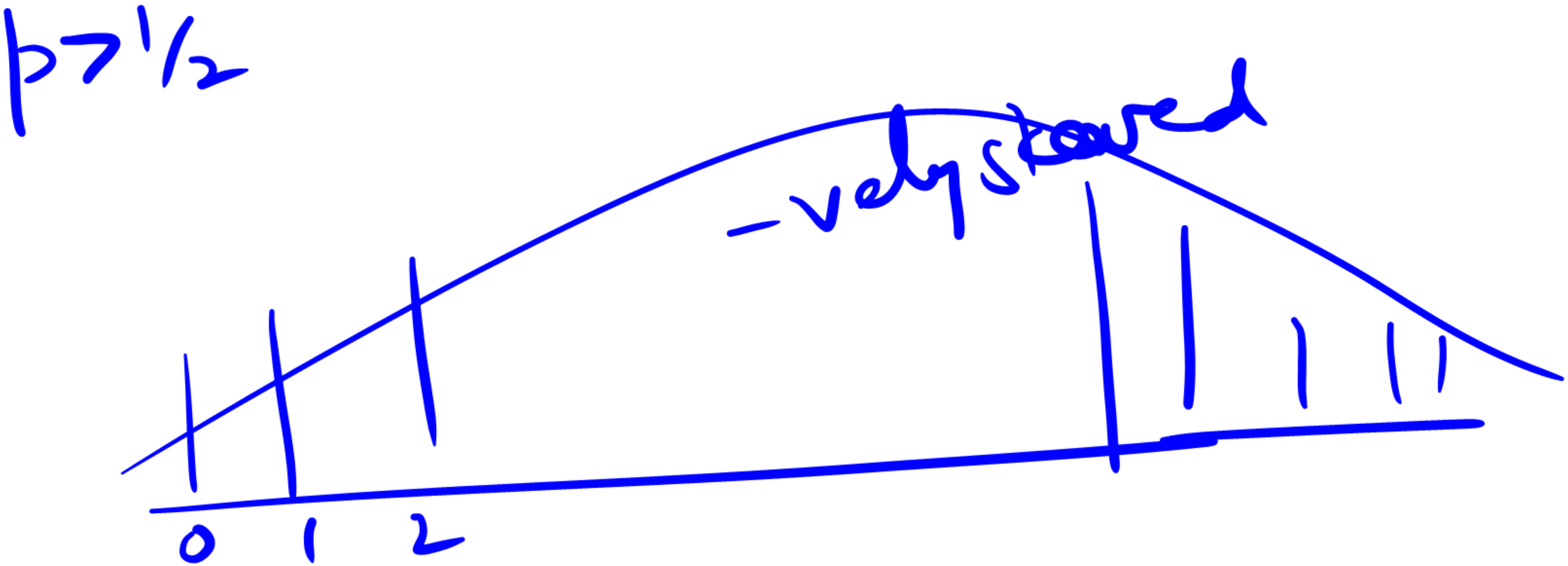
perfectly symmetric

$p = 1/2$



$\beta_1 > 0$ for $p < 1/2$


p < 1/2
0
1
2
+vely skewed


p > 1/2
-vely skewed
0
1
2

$$\beta_2 = \frac{\mu_4}{\mu_2^2} - 3 = \frac{1-6pq}{npq}$$

$= 0, \quad pq = 1/6$

$> 0 \quad pq < 1/6$

$< 0 \quad pq > 1/6$

(*) Interpret it by takinp specific values of p