# Memoryless Property of Exponential Distribution

Let $Y \sim \text{Exp}(\lambda)$

$$P(Y > a) = e^{-\lambda a}, \quad a > 0$$

$$P(\underbrace{Y > a+b}_{A} \mid \underbrace{Y > b}_{B}) = \frac{P(Y > a+b)}{P(Y > b)}$$

$$= \frac{e^{-\lambda(a+b)}}{e^{-\lambda b}} = e^{-\lambda a} = P(Y > a).$$

M.G.F. : $M_Y(t) = E(e^{tY})$

$$= \int_0^{\infty} e^{ty} \lambda e^{-\lambda y} \, dy = \frac{\lambda}{\lambda - t}, \quad 0 < t < \lambda$$

Example: The time to failure (in months) $X$ of light bulbs produced at two plants A & B obeys exponential dist$^n$ with means 5 and 2 respectively. A company buys bulbs from both plants but 3 times from B as compared to from A. What is the prob that a randomly selected bulb will have a life at least 5 months?

Sol$^n$ $P(X>5) = P(X>5|A)\,P(A) + P(X>5|B)\,P(B)$

$$= e^{-1}\cdot\frac{1}{4} + e^{-5/2}\cdot\frac{3}{4}$$

$$\approx 0.1534$$

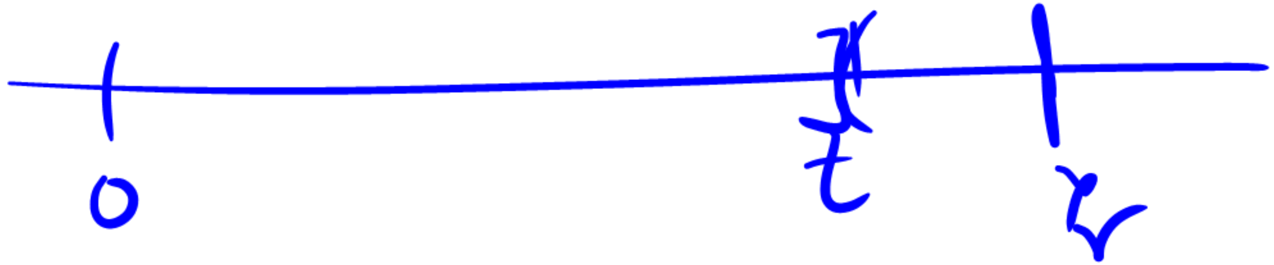
$$f_{X|A}(x) = \frac{1}{5}e^{-x/5},\quad x>0$$

$$f_{X|B}(x) = \frac{1}{2}e^{-x/2},\quad x>0$$

Further consider a Poisson process $X(t)$ with rate $\lambda$. Let $Y_r$ denote the time for the $r^{th}$ occurrence. $(r \geq 1)$

We want dist$^n$ of $Y_r$ !!

$\underline{\underline{t > 0}}$




$$P(Y_r > t) = P(X(t) \leq r-1)$$

$$= \sum_{j=0}^{r-1} P(X(t) = j)$$

$$= \sum_{j=0}^{r-1} \frac{e^{-\lambda t} (\lambda t)^j}{j!}$$

So $F_{Y_r}(t) = \begin{cases} 0, & t \le 0 \\ 1 - \sum_{j=0}^{r-1} \dfrac{e^{-\lambda t}(\lambda t)^j}{j!}, & t > 0 \end{cases}$

So the pdf of $Y_r$ is given by

$$f_{Y_r}(t) = -\frac{d}{dt}\left[ e^{-\lambda t} + \underbrace{\lambda t\, e^{-\lambda t}} + \frac{(\lambda t)^2}{2!} e^{-\lambda t} + \cdots + \frac{(\lambda t)^{r-1}}{(r-1)!} e^{-\lambda t} \right]$$

$$= - \left[ \cancel{-\lambda e^{-\lambda t}} + \cancel{\lambda e^{-\lambda t}} - \cancel{\lambda^2 t e^{-\lambda t}} + \cancel{\lambda^2 t e^{-\lambda t}} - \cdots \right.$$

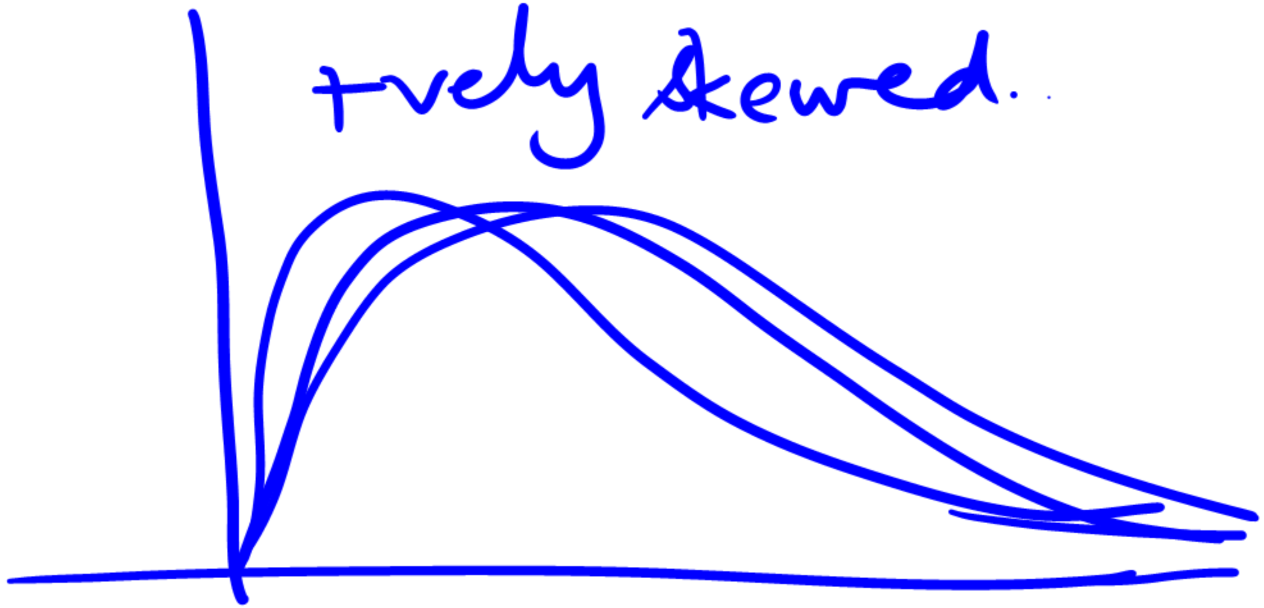
$$- \frac{\lambda^{r} t^{r-1}}{(r-1)!} e^{-\lambda t} \Big]$$

$$= \frac{1}{(r-1)!} \lambda^{r} t^{r-1} e^{-\lambda t} , \quad t > 0$$

So the pdf of $Y_r$ is

$$f_{Y_r}(t) = \begin{cases} \frac{1}{\Gamma r} \lambda^{r} t^{r-1} e^{-\lambda t}, & t > 0 \\ 0, & t \le 0 \end{cases} \quad , \lambda > 0, \; r > 0$$

This is Gamma or Erlang Distn.




$$\mu_k' = E(Y_r^k) = \int_0^\infty t^k \cdot \frac{1}{\Gamma r} e^{-\lambda t} t^{r-1} \lambda^r \, dt$$

$$= \frac{\lambda^r}{\Gamma r} \int_0^\infty t^{k+r-1} e^{-\lambda t} \, dt = \frac{\lambda^r}{\Gamma r} \cdot \frac{\Gamma k+r}{\lambda^{k+r}}$$

$$= \frac{\Gamma(k+r)}{\Gamma r} \cdot \frac{1}{\lambda^k}, \quad k = 1, 2, \cdots$$

$$\mu_1' = E(Y_r) = \frac{r}{\lambda}, \quad \mu_2' = \frac{r(r+1)}{\lambda^2}$$

$$\mu_2 = Var(Y_r) = \frac{r}{\lambda^2}$$

<u>Example</u>: The CPU time requirement on a system is a r.v. X having gamma distⁿ with mean 40 s & s.d. 20 s. Any job less than 20 s is called a

short job. What is the prob that out of 5 randomly selected jobs at least 2 are short jobs?

<u>Sol<sup></sup></u>

Solⁿ. $\frac{r}{\lambda} = 40$, $\frac{r}{\lambda^2} = 400$

So $r = 4$, $\lambda = 1/10$.

$P(\text{short job}) = P(Y_4 < 20)$

$$f_{Y_r}(t) = \left(\frac{1}{10}\right)^4 \cdot \frac{1}{\Gamma 4} e^{-t/10} \cdot t^3, \quad t > 0$$

$$= 1 - P(Y_4 \geq 20)$$

$$= 1 - \int_{20}^{\infty} \frac{1}{(10)^4} \cdot \frac{1}{6} e^{-t/10} t^3 \, dt$$

$$\boxed{\frac{t}{10} = y \qquad \frac{1}{10} dt = dy}$$

$$= 1 - \int_{2}^{\infty} \frac{1}{6} e^{-y} y^3 \, dy = 1 - \frac{19}{3} e^{-2} \approx 0.1429$$

$Z \rightarrow$ no. of shortjobs out of 5

$Z \sim \text{Bin}(5, p)$ $\quad p = 0.1429$

$$P(Z \geq 2) = \sum_{k=2}^{5} \binom{5}{k} (0.1429)^k (1 - .1429)^{5-k}$$

$$= 1 - P(Z=0) - P(Z=1)$$

$$= 1 - (1 - .1429)^5 - 5(1 - .1429)^4 (0.1429)$$

$$\approx 0.1519$$

<u>MGF</u> $M_{Y_r}(t) = E\left(e^{tY_r}\right)$

$$= \int_0^\infty e^{ty} \cdot \frac{1}{\Gamma r} \lambda^r y^{r-1} e^{-\lambda y} \, dy$$

$$= \frac{1}{\Gamma r} \lambda^r \int_0^\infty y^{r-1} e^{-(\lambda-t)y} \, dy$$

$$= \left(\frac{\lambda}{\lambda - t}\right)^r, \quad t < \lambda$$

<u>Weibull Distribution</u> : A continuous r.v. $X$ is said to have a Weibull distⁿ with parameters $\alpha$ and $\beta$ $(>0)$ if it has pdf given by

$$f_X(x) = \begin{cases} \alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}}, & x>0, \quad \alpha>0, \ \beta>0 \\ 0, & \text{otherwise} \end{cases}$$

$$F_X(x) = \begin{cases} 1 - e^{-\alpha x^{\beta}}, & x>0, \quad \alpha>0, \beta>0 \\ 0 & x \le 0 \end{cases}$$

$$\mu_k' = E(X^k) = \int_0^\infty \alpha\beta x^{\beta+k-1} e^{-\alpha x^{\beta}} dx$$

$$y = x^{\beta}$$

$$dy = \beta x^{\beta-1}$$

$$= \int_0^\infty \alpha y^{k/\beta} e^{-\alpha y} dy$$

$$= \frac{\Gamma\left(\frac{k+\beta}{\beta}\right)}{\alpha^{k/\beta}}$$

$$\mu_1' = \alpha^{-1/\beta} \cdot \Gamma\left(\frac{\beta+1}{\beta}\right)$$

$\mu_2' = ??$, $\mu_2 = ??$

Let $X$ denote the life of a system/component

We define

$R(t) = P(X > t)$ as the reliability of the system at time $t$

$= P(\text{system is working at time } t)$