MGF of Normal Distribution

$$M_X(t) = E\left(e^{tX}\right)$$

$$= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$z = \frac{x-\mu}{\sigma}$$

$$dz = \frac{1}{\sigma} dx$$

$$x = \sigma z + \mu$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{t(\sigma z+\mu) - z^2/2} dz$$

$$= e^{\mu t + \frac{1}{2}\sigma^2 t^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-\sigma t)^2} dz$$

Within the integral we have pdf of a normal distribution with mean σt and variance 1. So the value of the integral is 1. So we get

$$M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$$

<u>Linearity Property</u> of a <u>Normal Dist^n</u>:

Theorem: Let $X \sim N(\mu, \sigma^2)$ and $Y = aX + b$, $a \neq 0$, $b \in \mathbb{R}$. Then

$$Y \sim N(a\mu + b, a^2\sigma^2).$$

Proof: Consider the mgf of Y,

$$M_Y(t) = E(e^{tY}) = E\{e^{t(aX+b)}\}$$

$$= e^{bt} E\{e^{(at)X}\} = e^{bt} M_X(at)$$

$$= e^{bt} e^{\mu(at) + \frac{1}{2}\sigma^2(at)^2}$$

$$= e^{(a\mu+b)t + \frac{1}{2}(a^2\sigma^2)t^2}$$

This is MGF of a $N(a\mu+b, a^2\sigma^2)$.

By the uniqueness property of mgf we conclude that $Y \sim N(a\mu+b, a^2\sigma^2)$.

Using this we can conclude that if $X \sim N(\mu, \sigma^2)$, then

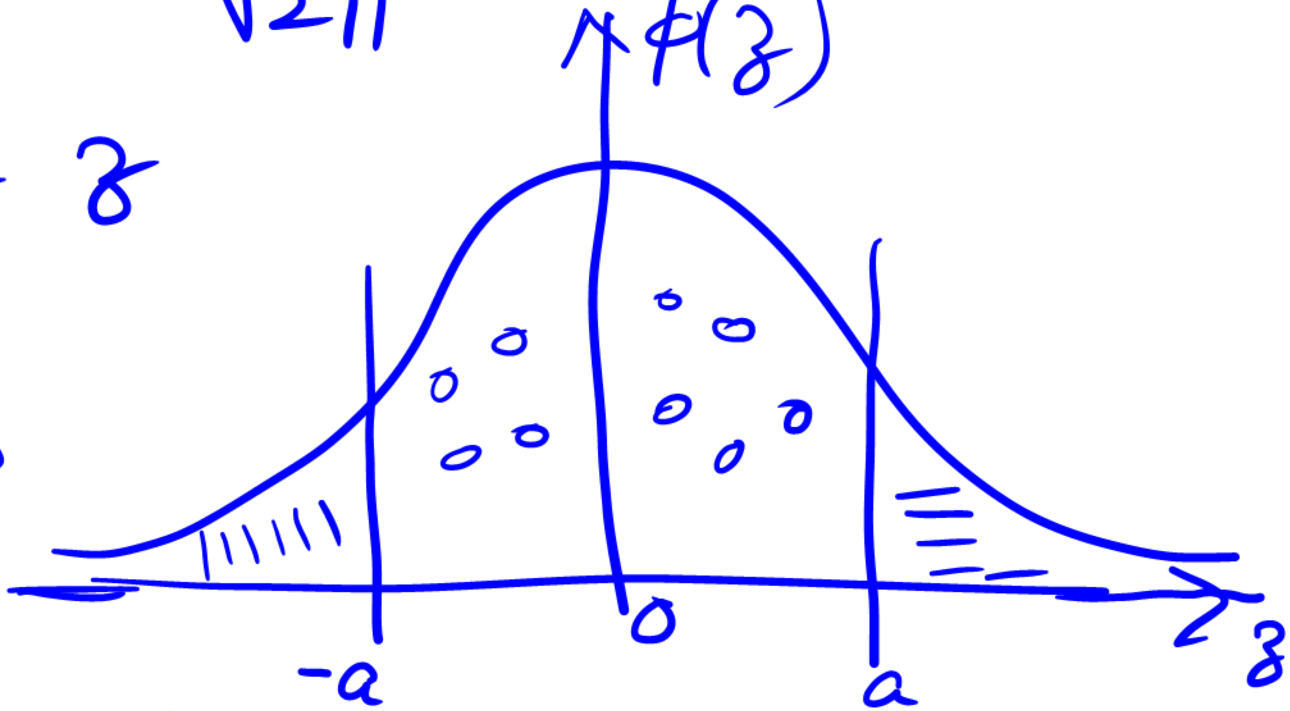
$$Z = \frac{X-\mu}{\sigma} \sim N(0, 1)$$

This is called standard normal distn.

The pdf of Z is

$$f_Z(z) = \phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \quad z \in \mathbb{R}$$

$$\phi(-z) = \phi(z) \quad \forall z$$



The cdf of Z is

$$\Phi(z) = \int_{-\infty}^{z} \phi(t)\,dt$$

Due to symmetry of pdf about 0, we

get $\Phi(-a) = 1 - \Phi(a) \quad \forall\, a$

$$\Rightarrow \Phi(a) + \Phi(-a) = 1$$

$$\Phi(0) = \frac{1}{2}.$$

We can use this transformation of any normal r.v. to a standard normal r.v. for evaluating probabilities related to any normal distn. Let $X \sim N(\mu, \sigma^2)$

$$P(a < X \leq b) = P(X \leq b) - P(X \leq a)$$

$$= P\left(\frac{X-\mu}{\sigma} \le \frac{b-\mu}{\sigma}\right) - P\left(\frac{X-\mu}{\sigma} \le \frac{a-\mu}{\sigma}\right)$$

$$= P\left(Z \le \frac{b-\mu}{\sigma}\right) - P\left(Z \le \frac{a-\mu}{\sigma}\right)$$

$$= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

Let $Z \sim N(0,1)$.

$\Phi(0) = \frac{1}{2}$, $\Phi(1) = 0.8413$, $\Phi(2) = 0.9773$

$\Phi(3) = 0.9987$

$$P(-1 \le Z \le 1) = \Phi(1) - \Phi(-1)$$

$$= 2\Phi(1) - 1 = 0.6826$$

$$P(-2 \le Z \le 2) = 2\Phi(2) - 1 = 0.9546$$

$$P(-3 \le Z \le 3) = 2\Phi(3) - 1 = 0.9974$$

So if $X \sim N(\mu, \sigma^2)$, then

$$\left.\begin{aligned} P(\mu - \sigma \le X \le \mu + \sigma) &= 0.6826 \\ P(\mu - 2\sigma \le X \le \mu + 2\sigma) &= 0.9546 \\ P(\mu - 3\sigma \le X \le \mu + 3\sigma) &= 0.9974 \end{aligned}\right\}$$

Examples. 1. A distance runner completes a one mile race in time $X \sim N(\mu, \sigma^2)$, where $\mu = 241$ sec. & $\sigma = 2$ sec. What is the prob. that this runner will take less than 4 mins? Or more than 3 mins, 55 sec?

Sol.^{n}

$$P(X < 240) = P\left(Z < \frac{240-241}{2}\right)$$

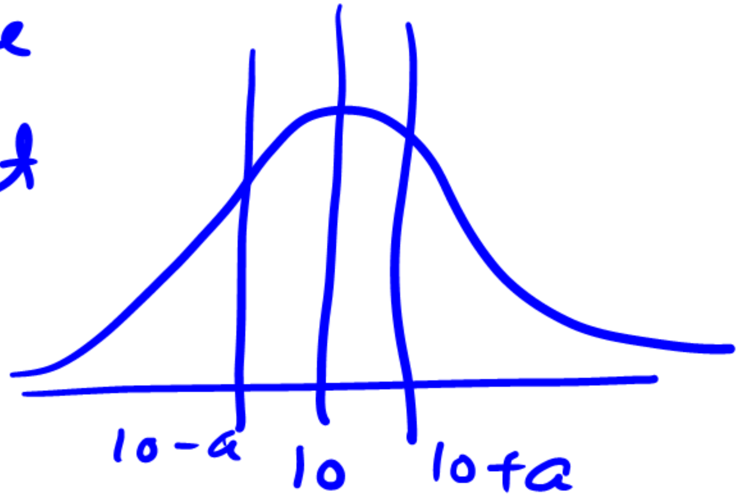
$$= P(Z < -0.5) = \Phi(-0.5) = 0.3085$$

$$P(X > 235) = P\left(Z > \frac{235-241}{2}\right) = P(Z > -3)$$

$$= P(Z < 3) = \Phi(3) = 0.9987.$$

2. Suppose the NAV (net asset value) of a share is a normal r.v. with $\mu = 10$, $\sigma = 0.25$. What is the shortest interval that has prob. 0.95 of including NAV?

Sol^n Due to the nature of the normal distribution the shortest interval will be symmetric about the mean. Let us take the interval $(10-a, 10+a)$.



So $P(10-a \leq X \leq 10+a) = 0.95$

$$\Rightarrow P\left(\frac{10-a-\mu}{\sigma} \leq \frac{X-\mu}{\sigma} \leq \frac{10+a-\mu}{\sigma}\right) = 0.95$$

$$\Rightarrow \quad P(-4a \le Z \le 4a) = 0.95$$

$$\Rightarrow \quad 2\Phi(4a) - 1 = 0.95$$

$$\Rightarrow \quad \Phi(4a) = 0.975 \Rightarrow 4a = 1.96$$

$$\Rightarrow a = 0.49$$

So the shortest interval is $(9.51, 10.49)$

The normal distribution arises as an approximation of a Binomial / Poisson distn. It also arises has a limiting distⁿ of the sample mean/ sample sum of observations from any distribution under certain conditions.

Poisson Approximation to Normal:

(De-Moivre-Laplace Limit Theorem)

Let $X \sim P(\lambda)$,

As $\lambda \to \infty$ the distribution of

$$Z = \frac{X-\lambda}{\sqrt{\lambda}} \to N(0,1)$$

Consider the mgf of Z.

$$M_Z(t) = E\left(e^{tZ}\right) = E\left\{e^{t\left(\frac{X-\lambda}{\sqrt{\lambda}}\right)}\right\}$$

$$= e^{-t\sqrt{\lambda}} E\left(e^{\frac{t}{\sqrt{\lambda}}X}\right) = e^{-t\sqrt{\lambda}} M_X\left(\frac{t}{\sqrt{\lambda}}\right)$$

$$= e^{-t\sqrt{\lambda}} e^{\lambda\left(e^{t/\sqrt{\lambda}}-1\right)}$$

$$= e^{-t\sqrt{\lambda}} e^{\lambda\left(1+\frac{t}{\sqrt{\lambda}}+\frac{t^2}{2\lambda}+\frac{t^3}{3!\,\lambda^{3/2}}+\cdots-1\right)}$$

$$= e^{\frac{t^2}{2} + \frac{1}{\lambda^{1/2}}(\cdots\cdots)} \longrightarrow e^{t^2/2} \text{ as } \lambda \to \infty$$

which is MGF of a $N(0,1)$ distn.

Let $X \sim Bin(n, p)$

$Z = \dfrac{X - np}{\sqrt{npq}}$. As $n \to \infty$ the distn of $Z \longrightarrow N(0,1)$.

<u>Proof</u>: $M_Z(t) = E(e^{tZ})$

$$= E\left[e^{t(x-np)/\sqrt{npq}}\right]$$

$$= e^{-\frac{npt}{\sqrt{npq}}} \cdot E\left[e^{\left(\frac{t}{\sqrt{npq}}x\right)}\right]$$

$$= e^{-\frac{npt}{\sqrt{npq}}} M_x\left(\frac{t}{\sqrt{npq}}\right)$$

$$= e^{-\frac{npt}{\sqrt{npq}}} \cdot \left(q+p\,e^{t/\sqrt{npq}}\right)^n$$

$$\log M_Z(t) = -\frac{npt}{\sqrt{npq}} + n\log\left(1+p\left(e^{t/\sqrt{npq}}-1\right)\right)$$

$$= -\frac{npt}{\sqrt{npq}} + n\log\left[1+p\left\{1+\frac{t}{\sqrt{npq}}+\frac{t^2}{2npq}+\frac{t^3}{3!(npq)^{3/2}}+\cdots-1\right\}\right]$$

$$= -\frac{npt}{\sqrt{npq}} + n\log\left[1+p\left\{\frac{t}{\sqrt{npq}}+\frac{t^2}{2npq}+\cdots\right\}\right]$$

For large n, we can expand using the expansion formula of $\log(1+y)$:

$$= -\frac{npt}{\cancel{\sqrt{npq}}} + n\left[p\left\{\frac{t}{\cancel{\sqrt{npq}}} + \frac{t^2}{2npq} + \frac{t^3}{3!(npq)^{3/2}} + \cdots\right\}\right.$$

$$\left. - \frac{p^2}{2}\left\{\frac{t}{\sqrt{npq}} + \frac{t^2}{2npq} + \cdots\right\}^2 + \cdots\right]$$

$$= \left(\frac{t^2}{2q} - \frac{pt^2}{2q}\right) + \frac{1}{n^{1/2}}(\cdots)$$

$$\to \frac{t^2}{2} \text{ as } n \to \infty.$$

So $M_z(t) \to e^{t^2/2}$ as $n \to \infty$

So the distⁿ of $\frac{X-np}{\sqrt{npq}} \to N(0,1)$

as $n \to \infty$.

<u>Examples</u>. 1. The probability that a patient recovers from a rare blood disease is 0.4. If 100 persons are treated what is the prob

that less than 30 survive ?

$$\sum_{j=0}^{29} \binom{100}{j} (0.4)^j (.6)^{100-j}$$

Soln. $X \to$ survivors.

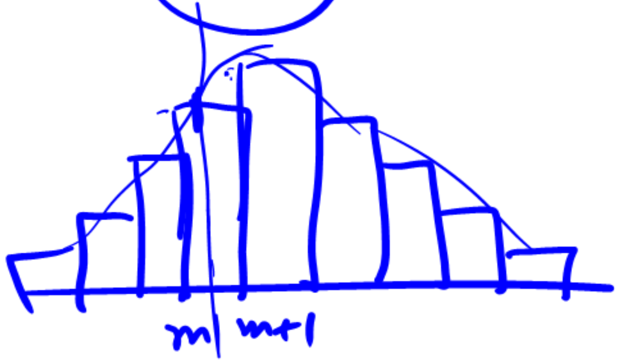
Then $X \sim Bin(100, 0.4)$.

$np = 40$, $npq = 24$, $\sqrt{npq} = 4.899$

$$Z = \frac{X-40}{4.899} \approx N(0,1)$$

continuity correction

$m + 1/2$



$$P(\underbrace{X < 30}_{X \le 29}) \simeq P(X \le 29.5)$$

$$= P\left(Z \le \frac{29.5-40}{4.899}\right) = P(Z \le -2.14)$$
$$= 0.0162$$

2. Suppose thefts occur in hostels like Poisson process with $\lambda = 1/2$ per day. What is the prob of not more than 10 thefts in a month? Not less than 17 in a month?

$$\sum_{j=0}^{10} \frac{e^{-15}(15)^j}{j!}, \quad \sum_{j=17}^{\infty} \cdots$$

for a month $\underline{\underline{X = 15}}$ $\quad \dfrac{X-15}{\sqrt{15}} \to N(0,1)$

$$P(X \le 10) \cong P(X \le 10.5)$$

$$= P\left(\frac{X-15}{\sqrt{15}} \le \frac{10.5-15}{\sqrt{15}}\right) \approx \Phi(-1.16)$$

$$= 0.123$$

$$P(X \geq 17) = 1 - P(X \le 16) \cong 1 - P(X \le 16.5)$$

$$= 1 - P\left(Z \le \frac{16.5-15}{\sqrt{15}}\right) = 1 - \Phi(0.39)$$

$$= 0.3483$$

Lognormal Distribution : Let $Y \sim N(\mu, \sigma^2)$.

Then $X = e^Y$ is said to have a lognormal distⁿ.

$$f_X(x) = \frac{1}{\sigma x \sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(\log x - \mu)^2}$$

$x > 0$

$\mu \in \mathbb{R}, \sigma > 0$

$$E(X) = E(e^Y) = M_Y(1) = e^{\mu + \sigma^2/2}$$

$$E(X^2) = E(e^{2Y}) = M_Y(2) = e^{2\mu + 2\sigma^2}$$

$$V(X) = e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right).$$

$$\mu_k' = E(X^k) = E\left(e^{kY}\right) = M_Y(k)$$

$$= e^{\mu k + \frac{1}{2}\sigma^2 k^2}$$

Example: The demand X of a certain item follows a log-normal dist^n with mean 7.43 and variance 0.56. Find $P(X > 8)$

Sol^n: $\mu_1' = e^{\mu+\sigma^2/2} = 7.43$

$$\mu + \frac{\sigma^2}{2} = 2.0055 \quad \cdots (1)$$

$$\mu_2 = \mu_2' - \mu_1'^2 \Rightarrow \mu_2' = 0.56 + (7.43)^2$$

$$=$$

$$\Rightarrow \quad 2\mu + 2\sigma^2 = 4.0211 \quad \cdots (2)$$

$$\log_e X \sim N(2, (0.1)^2)$$

$$\Rightarrow \mu \cong 2. \quad \sigma \cong 0.1$$

$$P(X > 8) = P(\log_e X > \ln_e 8)$$

$$= P\left(\frac{\ln_e X - 2}{0.1} > \frac{\ln_e 8 - 2}{0.1} \right)$$

$$= P(Z > 0.79) = \Phi(-0.79) = 0.2148$$