INSTRUCTOR'S SOLUTIONS MANUAL FOR

PRINCIPLES OF ELECTROMAGNETICS

ASIAN EDITION

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CHAPTER 1

P. E. 1.1

(a) $\boldsymbol{A}+\boldsymbol{B}=(1,0,3)+(5,2,-6)=(6,2,-3)$

$$|\boldsymbol{A}+\boldsymbol{B}|=\sqrt{36+4+9}=\underline{\underline{7}}$$

(b) $5\boldsymbol{A}-\boldsymbol{B}=(5,0,15)-(5,2,-6)=\underline{\underline{(0,-2,21)}}$

(c) The component of $\boldsymbol{A}$ along $\mathbf{a}_y$ is $A_y=\underline{\underline{0}}$

(d) $3\boldsymbol{A}+\boldsymbol{B}=(3,0,9)+(5,2,-6)=(8,2,3)$

A unit vector parallel to this vector is

$$\boldsymbol{a}_{11}=\frac{(8,2,3)}{\sqrt{64+4+9}}$$

$$=\underline{\underline{\pm(0.9117\boldsymbol{a}_x+0.2279\boldsymbol{a}_y+0.3419\boldsymbol{a}_z)}}$$

P. E. 1.2 (a) $\boldsymbol{r}_p=\underline{\underline{\boldsymbol{a}_x-3\boldsymbol{a}_y+5\boldsymbol{a}_z}}$

$$\boldsymbol{r}_R=\underline{\underline{3\boldsymbol{a}_y+8\boldsymbol{a}_z}}$$

(b) The distance vector is

$$\boldsymbol{r}_{QR}=\boldsymbol{r}_R-\boldsymbol{r}_Q=(0,3,8)-(2,4,6)=\underline{\underline{-2\boldsymbol{a}_x-\boldsymbol{a}_y+2\boldsymbol{a}_z}}$$

(c) The distance between Q and R is

$$|r_{QR}|=\sqrt{4+1+4}=\underline{\underline{3}}$$

P. E. 1.3 Consider the figure shown on the next page:

$$\boldsymbol{u}_Z=\boldsymbol{u}_P+\boldsymbol{u}_W=-350\boldsymbol{a}_x+\frac{40}{\sqrt{2}}(-\boldsymbol{a}_x+\boldsymbol{a}_y)$$

$$=-378.28\boldsymbol{a}_x+28.28a_y \quad \text{km/hr}$$

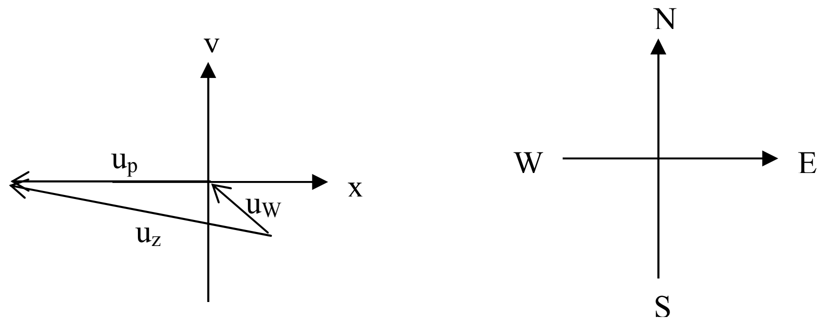
or

$$\boldsymbol{u}_z=379.3\angle 175.72^\circ \quad \text{km/hr}$$

Where $\mathbf{u}_p$ = velocity of the airplane in the absence of wind

$\mathbf{u}_w$ = wind velocity

$\mathbf{u}_z$ = observed velocity



P. E. 1.4

Using the dot product,

$$\cos\theta_{AB} = \frac{\mathbf{A}\bullet\mathbf{B}}{AB} = \frac{-13}{\sqrt{10}\sqrt{65}} = -\sqrt{\frac{13}{50}}$$

$$\underline{\underline{\theta_{AB} = 120.66^\circ}}$$

P. E. 1.5

(a) $$\boldsymbol{E}_F = (\boldsymbol{E}\cdot\boldsymbol{a}_F)\boldsymbol{a}_F = \frac{(\boldsymbol{E}\cdot\boldsymbol{F})\boldsymbol{F}}{|\boldsymbol{F}|^2} = \frac{-10(4,-10,5)}{141}$$

$$= \underline{\underline{-0.2837\boldsymbol{a}_x + 0.7092\boldsymbol{a}_y - 0.3546\boldsymbol{a}_z}}$$

(b) $$\boldsymbol{E}\times\boldsymbol{F} = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ 0 & 3 & 4 \\ 4 & -10 & 5 \end{vmatrix} = (55,16,-12)$$

$$\boldsymbol{a}_{E\times F} = \underline{\underline{\pm(0.9398, 0.2734, -0.205)}}$$

P. E. 1.6 $\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}=0$ showing that $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ form the sides of a triangle.

$\boldsymbol{a}\cdot\boldsymbol{b} = 0$,

hence it is a right angle triangle.

$$\text{Area} = \frac{1}{2}|\boldsymbol{a}\times\boldsymbol{b}| = \frac{1}{2}|\boldsymbol{b}\times\boldsymbol{c}| = \frac{1}{2}|\boldsymbol{c}\times\boldsymbol{a}|$$

$$\frac{1}{2}|\boldsymbol{a}\times\boldsymbol{b}| = \frac{1}{2}\begin{vmatrix} 4 & 0 & -1 \\ 1 & 3 & 4 \end{vmatrix} = \frac{1}{2}|(3,-17,12)|$$

$$\text{Area} = \frac{1}{2}\sqrt{9+289+144} = \underline{\underline{10.51}}$$

P. E. 1.7

(a) $P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

$= \sqrt{25+4+64} = \underline{\underline{9.644}}$

(b) $\boldsymbol{r_P} = \boldsymbol{r}_{P_1} + \lambda(\boldsymbol{r}_{P_2} - \boldsymbol{r}_{P_1})$

$= (1,2,-3) + \lambda(-5,-2,8)$

$= \underline{\underline{(1-5\lambda, 2-2\lambda, -3+8\lambda)}}.$

(c) The shortest distance is

$$\boldsymbol{d} = \boldsymbol{P_1P_3} \sin\theta = \left|\boldsymbol{P_1P_3} \times \boldsymbol{a}_{P_1P_2}\right|$$

$$= \frac{1}{\sqrt{93}}\begin{vmatrix} 6 & -3 & 5 \\ -5 & -2 & 8 \end{vmatrix}$$

$$= \frac{1}{\sqrt{93}}\left|(-14,-73,-27)\right| = \underline{\underline{8.2}}$$

P.E. 1.8

1.3 a) $A - 3B$

$= 4a_x - 2a_y + 6a_z - 3(12a_x + 18a_y - 8a_z)$

$= -32a_x - 56a_y + 30a_z$

b) $(2A + 5B)/|B|$

$$= \frac{[2(4a_x - 2a_y + 6a_z) + 5(12a_x + 18a_y - 8a_z)]}{(12^2 + 18^2 + 8^2)^{1/2}}$$

$$= \frac{68a_x + 86a_y - 28a_z}{23.06}$$

$= 2.94a_x + 3.72a_y - 1.214a_z$

c) $a_x \times A$

$= a_x \times (4a_x - 2a_y + 6a_z)$

$= 4(a_x \times a_x) - 2(a_x \times a_y) + 6(a_x \times a_z)$

$= 0 - 2a_z - 6a_y = -6a_y - 2a_z$

d) $(B \times a_x) \cdot a_y$

$((12a_x + 18a_y - 8a_z) \times a_x) \cdot a_y$

$(12(a_x \times a_x) + 18(a_y \times a_x) - 8(a_z \times a_x)) \cdot a_y$

$(0 - 18a_z - 8a_y) \cdot a_y = 0 - 8 = -8$

P.E. 1.9

$$\boldsymbol{P}\bullet\boldsymbol{Q} = (2,-6,5)\bullet(0,3,1) = 0-18+5 = \underline{-13}$$

$$\boldsymbol{P}\times\boldsymbol{Q} = \begin{vmatrix} 2 & -6 & 5 \\ 0 & 3 & 1 \end{vmatrix} = \underline{\underline{-21\boldsymbol{a}_x - 2\boldsymbol{a}_y + 6\boldsymbol{a}_z}}$$

$$\cos\theta_{PQ} = \frac{\boldsymbol{P}\cdot\boldsymbol{Q}}{PQ} = \frac{-13}{\sqrt{10}\sqrt{65}} = -0.51 \quad \longrightarrow \quad \theta_{PQ} = \underline{\underline{120.66^o}}$$

P.E. 1.10

$$\text{Area} = \frac{1}{2}|\boldsymbol{D}\times\boldsymbol{E}| = \frac{1}{2}\begin{vmatrix} 4 & 1 & -5 \\ -1 & 2 & 3 \end{vmatrix} = \frac{1}{2}|(3+10)\boldsymbol{a}_x + (5-12)\boldsymbol{a}_y + (8+1)\boldsymbol{a}_z|$$

$$= \frac{1}{2}|(13,-7,9)| = \frac{1}{2}\sqrt{169+49+81} = \underline{\underline{8.646}}$$

P.E. 1.11

$$\boldsymbol{A}\bullet\boldsymbol{B} = (4,-6,1)\bullet(2,0,5) = 8-0+5 = 13$$

(a) $|\boldsymbol{B}|^2 = 2^2 + 5^2 = 29$

$$\boldsymbol{A}\bullet\boldsymbol{B} + 2|\boldsymbol{B}|^2 = 13 + 2\times 29 = \underline{\underline{71}}$$

(b)

$$\boldsymbol{a}_\perp = \pm\frac{\boldsymbol{A}\times\boldsymbol{B}}{|\boldsymbol{A}\times\boldsymbol{B}|}$$

Let $\boldsymbol{C} = \boldsymbol{A}\times\boldsymbol{B} = \begin{vmatrix} 4 & -6 & 1 \\ 2 & 0 & 5 \end{vmatrix} = (-30,-18,12)$

$$\boldsymbol{a}_\perp = \pm\frac{\boldsymbol{C}}{|\boldsymbol{C}|} = \pm\frac{(-30,-18,12)}{\sqrt{30^2+18^2+12^2}} = \underline{\underline{\pm(-0.8111\boldsymbol{a}_x - 0.4867\boldsymbol{a}_y + 0.3244\boldsymbol{a}_z)}}$$

Prob.1.1

$$\boldsymbol{r}_{OP} = 4\boldsymbol{a}_x - 5\boldsymbol{a}_y + \boldsymbol{a}_z$$

$$\boldsymbol{a}_{r_{OP}} = \frac{\boldsymbol{r}_{OP}}{|\boldsymbol{r}_{OP}|} = \frac{(4,-5,1)}{\sqrt{(16+25+1)}} = \underline{\underline{0.6172\boldsymbol{a}_x - 0.7715\boldsymbol{a}_y + 0.1543\boldsymbol{a}_z}}$$

Prob. 1.2

$$\boldsymbol{r} = (-3,2,2) - (2,4,4) = (-5,-2,-2)$$

$$\boldsymbol{a}_r = \frac{\boldsymbol{r}}{|\boldsymbol{r}|} = \frac{(-5,-2,-2)}{\sqrt{25+4+4}} = \underline{\underline{-0.8704\boldsymbol{a}_x - 0.3482\boldsymbol{a}_y - 0.3482\boldsymbol{a}_z}}$$

Prob. 1.3

(a) $$\boldsymbol{A}-2\boldsymbol{B}=(4,-6,3)-2(-1,8,5)=(4,-6,3)-(-2,16,10)$$
$$=\underline{\underline{(6,-22,-7)}}$$

(b) $\boldsymbol{A}\bullet\boldsymbol{B}=(4,-6,3)\bullet(-1,8,5)=-4-48+15=\underline{\underline{-37}}$

(c) $$\boldsymbol{A}\times\boldsymbol{B}=\begin{vmatrix}4 & -6 & 3\\ -1 & 8 & 5\end{vmatrix}=(-30-24)\boldsymbol{a}_x+(-3-20)\boldsymbol{a}_y+(32-6)\boldsymbol{a}_z$$
$$=\underline{\underline{-54\boldsymbol{a}_x}}\,\underline{\underline{-23\boldsymbol{a}_y+26\boldsymbol{a}_z}}$$

Prob. 1.4

$$\boldsymbol{B}\times\boldsymbol{C}=\begin{vmatrix}3 & 5 & 1\\ 0 & 1 & -7\end{vmatrix}=(-35-1)\boldsymbol{a}_x+(0+21)\boldsymbol{a}_y+(3-0)\boldsymbol{a}_z$$
$$=-36\boldsymbol{a}_x+21\boldsymbol{a}_y+3\boldsymbol{a}_z$$
$$\boldsymbol{A}\bullet(\boldsymbol{B}\times\boldsymbol{C})=(4,2,1)\bullet(-36,21,3)=-144+42+3=\underline{\underline{-99}}$$

Prob. 1.5

(a) $$\boldsymbol{B}\times\boldsymbol{C}=\begin{vmatrix}1 & 1 & 1\\ 0 & 1 & 2\end{vmatrix}=\boldsymbol{a}_x-2\boldsymbol{a}_y+\boldsymbol{a}_z$$
$$\boldsymbol{A}\bullet(\boldsymbol{B}\times\boldsymbol{C})=(1,0,-1)\bullet(1,-2,1)=1+0-1=\underline{\underline{0}}$$

(b) $$\boldsymbol{A}\times\boldsymbol{B}=\begin{vmatrix}1 & 0 & -1\\ 1 & 1 & 1\end{vmatrix}=\boldsymbol{a}_x-2\boldsymbol{a}_y+\boldsymbol{a}_z$$
$$(\boldsymbol{A}\times\boldsymbol{B})\bullet\boldsymbol{C}=(1,-2,1)\bullet(0,1,2)=0-2+2=\underline{\underline{0}}$$

(c) $$\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C})=\begin{vmatrix}1 & 0 & -1\\ 1 & -2 & 1\end{vmatrix}=\underline{\underline{-2\boldsymbol{a}_x-2\boldsymbol{a}_y-2\boldsymbol{a}_z}}$$

(d) $$(\boldsymbol{A}\times\boldsymbol{B})\times\boldsymbol{C}=\begin{vmatrix}1 & -2 & 1\\ 0 & 1 & 2\end{vmatrix}=\underline{\underline{-5\boldsymbol{a}_x-2\boldsymbol{a}_y+\boldsymbol{a}_z}}$$

Prob. 1.6

(a) T = $\underline{(3, -2, 1)}$ and S = $\underline{(4, 6, 2)}$

(b) $r_{TS} = r_s - r_t = (4, 6, 2) - (3, -2, 1) = \underline{\underline{a_x + 8a_y + a_z}}$

(c) distance = $|r_{TS}| = \sqrt{1+64+1} = \underline{\underline{8.124}}$ m

Prob. 1.7

(a) If **A** and **B** are parallel, **A**=k**B**, where k is a constant.

$$(\alpha,3,-2)=k(4,\beta,8)$$

Equating coefficients gives

$$-2=8k \longrightarrow k=-\frac{1}{4}$$

$$\alpha=4k=\underline{\underline{-1}}$$

$$3=\beta k \longrightarrow \beta=3/k=\underline{\underline{-12}}$$

This can also be solved using **A** X **B** = 0.

(b) If A and B are perpendicular to each other,

$$\boldsymbol{A}\bullet\boldsymbol{B}=0 \longrightarrow \underline{\underline{4\alpha+3\beta-16=0}}$$

Prob. 1.8

(a) $A\cdot B=\mathrm{AB}\cos\theta_{AB}$

$A\times B=\mathrm{AB}\sin\theta_{AB}a_n$

$\left(A\cdot B\right)^2+\left|A\times B\right|^2=\left(AB\right)^2\left(\cos^2\theta_{AB}+\sin^2\theta_{AB}\right)=\left(\mathrm{AB}\right)^2$

(b) $\boldsymbol{a}_x\cdot\left(\boldsymbol{a}_y\times\boldsymbol{a}_z\right)=\boldsymbol{a}_x\cdot\boldsymbol{a}_x=1$. Hence,

$$\frac{\boldsymbol{a}_y\times\boldsymbol{a}_z}{\boldsymbol{a}_x\cdot\boldsymbol{a}_y\times\boldsymbol{a}_z}=\frac{\boldsymbol{a}_x}{1}=\boldsymbol{a}_x$$

$$\frac{\boldsymbol{a}_z\times\boldsymbol{a}_x}{\boldsymbol{a}_x\cdot\boldsymbol{a}_y\times\boldsymbol{a}_z}=\frac{\boldsymbol{a}_y}{1}=\boldsymbol{a}_y$$

$$\frac{\boldsymbol{a}_x\times\boldsymbol{a}_y}{\boldsymbol{a}_x\cdot\boldsymbol{a}_y\times\boldsymbol{a}_z}=\frac{\boldsymbol{a}_z}{1}=\boldsymbol{a}_z$$

Prob. 1.9

(a) $\boldsymbol{P}+\boldsymbol{Q}=\left(6,2,0\right),\boldsymbol{P}+\boldsymbol{Q}-\boldsymbol{R}=\left(7,1,-2\right)$

$\left|\boldsymbol{P}+\boldsymbol{Q}-\boldsymbol{R}\right|=\sqrt{49+1+4}=\sqrt{54}=\underline{\underline{7.3485}}$

(b) $\boldsymbol{P.Q}\times\boldsymbol{R} = \begin{vmatrix} 2 & -1 & -2 \\ 4 & 3 & 2 \\ -1 & 1 & 2 \end{vmatrix} = 2(6-2)+(8+2)-2(4+3) = 8+10-14 = \underline{\underline{4}}$

$$\boldsymbol{Q}\times\boldsymbol{R} = \begin{vmatrix} 4 & 3 & 2 \\ -1 & 1 & 2 \end{vmatrix} = (4,-10,7)$$

$$\boldsymbol{P.Q}\times\boldsymbol{R} = (2,-1,-2)\cdot(4,-10,7) = 8+10-14 = \underline{\underline{4}}$$

(c) $\boldsymbol{Q}\times\boldsymbol{P} = \begin{vmatrix} 4 & 3 & 2 \\ 2 & -1 & -2 \end{vmatrix} = (-4,12,-10)$

$$\boldsymbol{Q}\times\boldsymbol{P}\cdot\boldsymbol{R} = (-4,12,-10)\cdot(-1,1,2) = 4+12-20 = \underline{\underline{-4}}$$

or $\boldsymbol{Q}\times\boldsymbol{P}\cdot\boldsymbol{R} = \boldsymbol{R}\cdot\boldsymbol{Q}\times\boldsymbol{P} = \begin{vmatrix} -1 & 1 & 2 \\ 4 & 3 & 2 \\ 2 & -1 & -2 \end{vmatrix} = -(-6+2)-(-8-4)+2(-4-6) = \underline{\underline{-4}}$

(d) $(\boldsymbol{P}\times\boldsymbol{Q})\cdot(\boldsymbol{Q}\times\boldsymbol{R}) = (4,-12,10)\cdot(4,-10,7) = 16+120+70 = \underline{\underline{206}}$

(e) $(\boldsymbol{P}\times\boldsymbol{Q})\times(\boldsymbol{Q}\times\boldsymbol{R}) = \begin{vmatrix} 4 & -12 & 10 \\ 4 & -10 & 7 \end{vmatrix} = \underline{\underline{16a_x + 12a_y + 8a_z}}$

(f) $\cos\theta_{PR} = \dfrac{\boldsymbol{P}\cdot\boldsymbol{R}}{|\boldsymbol{P}||\boldsymbol{R}|} = \dfrac{(-2-1-4)}{\sqrt{4+1+4}\sqrt{1+1+4}} = \dfrac{-7}{3\sqrt{6}} = -0.9526$

$$\underline{\underline{\theta_{PR} = 162.3^\circ}}$$

(g) $\sin\theta_{PQ} = \dfrac{|P\times Q|}{|P||Q|} = \dfrac{\sqrt{16+144+100}}{3\sqrt{16+9+4}} = \dfrac{\sqrt{260}}{3\sqrt{29}} = 0.998$

$$\theta_{PQ} = \underline{\underline{86.45^\circ}}$$

Prob. 1.10

If **A** and **B** are parallel, then **B** = k**A** and **A** x **B** = **0.** It is evident that k = -2 and that

$$\boldsymbol{A}\times\boldsymbol{B} = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ 1 & -2 & 3 \\ -2 & 4 & -6 \end{vmatrix} = \boldsymbol{0}$$

as expected.

Prob. 1.11

(a) Using the fact that

$$(\boldsymbol{A}\times\boldsymbol{B})\times\boldsymbol{C} = (\boldsymbol{A}\cdot\boldsymbol{C})\boldsymbol{B} - (\boldsymbol{B}\cdot\boldsymbol{C})\boldsymbol{A},$$

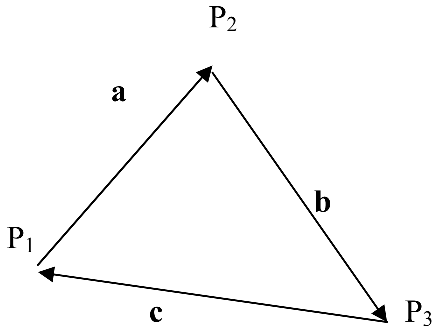
we get

$$\boldsymbol{A}\times(\boldsymbol{A}\times\boldsymbol{B}) = -(\boldsymbol{A}\times\boldsymbol{B})\times\boldsymbol{A} = \underline{\underline{(\boldsymbol{B}\cdot\boldsymbol{A})\boldsymbol{A} - (\boldsymbol{A}\cdot\boldsymbol{A})\boldsymbol{B}}}$$

(b) $$\boldsymbol{A}\times(\boldsymbol{A}\times(\boldsymbol{A}\times\boldsymbol{B})) = \boldsymbol{A}\times\left[(\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{A} - (\boldsymbol{A}\cdot\boldsymbol{A})\boldsymbol{B}\right]$$
$$= (\boldsymbol{A}\cdot\boldsymbol{B}) - (\boldsymbol{A}\times\boldsymbol{A}) - (\boldsymbol{A}\cdot\boldsymbol{A}) - (\boldsymbol{A}\times\boldsymbol{B})$$
$$= -A^2(\boldsymbol{A}\times\boldsymbol{B})$$

since **A**x**A** = **0**

Prob. 1.12



$$\boldsymbol{a} = \boldsymbol{r}_{p2} - \boldsymbol{r}_{p1} = (1,-2,4) - (5,-3,1) = (-4,1,3)$$

(a) $$\boldsymbol{b} = \boldsymbol{r}_{p3} - \boldsymbol{r}_{p2} = (3,3,5) - (1,-2,4) = (2,5,1)$$

$$\boldsymbol{c} = \boldsymbol{r}_{p1} - \boldsymbol{r}_{p3} = (5,-3,1) - (3,3,5) = (2,-6,-4)$$

Note that **a** + **b** + **c** = **0**

$\boldsymbol{a}\cdot\boldsymbol{b} = -8+5+3 = 0 \longrightarrow$ perpendicular

$\boldsymbol{b}\cdot\boldsymbol{c} = 4-30-4 \neq 0$

$\boldsymbol{c}\cdot\boldsymbol{a} = -8-6-12 \neq 0$

Hence $\underline{\underline{P_2}}$ is a right angle.

(b) $$\text{Area} = \frac{1}{2}|\boldsymbol{a}\times\boldsymbol{b}| = \frac{1}{2}\begin{vmatrix} -4 & 1 & 3 \\ 2 & 5 & 1 \end{vmatrix} = \frac{1}{2}|(1-15)\boldsymbol{a}_x + (6+4)\boldsymbol{a}_y + (-20-2)\boldsymbol{a}_z|$$

$$= \frac{1}{2}|(-14,10,-22)| = \frac{1}{2}\sqrt{196+100+484} = \underline{\underline{13.96}}$$

Prob. 1.13

Given $\boldsymbol{r}_P = (-1,4,8), \quad \boldsymbol{r}_Q = (2,-1,3), \quad \boldsymbol{r}_R = (-1,2,3)$

(a) $|PQ| = \sqrt{9+25+25} = \underline{\underline{7.6811}}$

(b) $\boldsymbol{PR} = \underline{\underline{-2\boldsymbol{a}_y - 5\boldsymbol{a}_z}}$

(c) $\angle PQR = \cos^{-1}\left(\dfrac{\boldsymbol{QP}\bullet\boldsymbol{QR}}{|\boldsymbol{QP}||\boldsymbol{QR}|}\right) = \underline{\underline{42.57^o}}$

(d) Area of triangle PQR = $\underline{11.023}$

(e) Perimeter = $\underline{17.31}$

Prob. 1.14

Let R be the midpoint of PQ.

$$\boldsymbol{r}_R = \frac{1}{2}\{(2,4,-1)+(12,16,9)\} = (7,10,4)$$

$$OR = \sqrt{49+100+16} = \sqrt{165} = 12.845$$

$$t = \frac{OR}{v} = \frac{12.845}{300} = \underline{\underline{42.82 \text{ ms}}}$$

Prob. 1.15

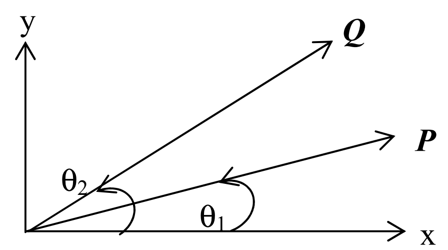
$$\boxed{\boldsymbol{A}\cdot(\boldsymbol{A}\times\boldsymbol{B}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}, \ (\boldsymbol{A}\times\boldsymbol{B})\cdot\boldsymbol{C} = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}}$$

Hence, $\boxed{\boldsymbol{A}\cdot(\boldsymbol{A}\times\boldsymbol{B}) = (\boldsymbol{A}\times\boldsymbol{B})\cdot\boldsymbol{C}}$

Also, each equals the volume of the parallelopiped formed by the three vectors as sides.

Prob. 1.16

(a) Let **P** and **Q** be as shown below:



$$|\boldsymbol{P}| = \cos^2\theta_1 + \sin^2\theta_1 = 1, |\boldsymbol{Q}| = \cos^2\theta_2 + \sin^2\theta_2 = 1,$$

Hence **P** and **Q** are unit vectors.

(b) $\boldsymbol{P}\cdot\boldsymbol{Q} = (1)(1)\cos(\theta_2 - \theta_1)$

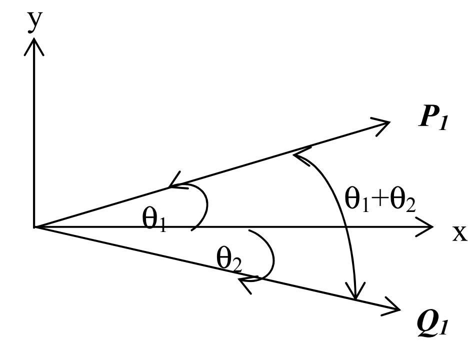
But $\boldsymbol{P}\cdot\boldsymbol{Q} = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2$. Thus,

$$\underline{\underline{\cos(\theta_2 - \theta_1) = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2}}$$

Let $\boldsymbol{P}_1 = \boldsymbol{P} = \cos\theta_1\boldsymbol{a}_x + \sin\theta_1\boldsymbol{a}_y$ and

$\boldsymbol{Q}_1 = \cos\theta_2\boldsymbol{a}_x - \sin\theta_2\boldsymbol{a}_y$.

$\boldsymbol{P}_1$ and $\boldsymbol{Q}_1$ are unit vectors as shown below:



$\boldsymbol{P}_1\cdot\boldsymbol{Q}_1 = (1)(1)\cos(\theta_1 + \theta_2)$

But $\boldsymbol{P}_1\cdot\boldsymbol{Q}_1 = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$,

$$\underline{\underline{\cos(\theta_2 + \theta_1) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2}}$$

Alternatively, we can obtain this formula from the previous one by replacing θ_2 by $-\theta_2$ in $\boldsymbol{Q}$.

(c)

$$\frac{1}{2}|\boldsymbol{P}-\boldsymbol{Q}| = \frac{1}{2}|(\cos\theta_1 - \cos\theta_2)a_x + (\sin\theta_1 - \sin\theta_2)a_y$$

$$= \frac{1}{2}\sqrt{\cos^2\theta_1 + \sin^2\theta_1 + \cos^2\theta_2 + \sin^2\theta_2 - 2\cos\theta_1\cos\theta_2 - 2\sin\theta_1\sin\theta_2}$$

$$= \frac{1}{2}\sqrt{2 - 2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)} = \frac{1}{2}\sqrt{2 - 2\cos(\theta_2 - \theta_1)}$$

Let $\theta_2 - \theta_1 = \theta$, the angle between **P** and **Q.**

$$\frac{1}{2}|\boldsymbol{P}-\boldsymbol{Q}| = \frac{1}{2}\sqrt{2 - 2\cos\theta}$$

But $\cos 2A = 1 - 2\sin^2 A$.

$$\frac{1}{2}|\boldsymbol{P}-\boldsymbol{Q}| = \frac{1}{2}\sqrt{2-2+4\sin^2\theta/2} = \sin\theta/2$$

Thus,

$$\underline{\frac{1}{2}|\boldsymbol{P}-\boldsymbol{Q}| = |\sin\frac{\theta_2-\theta_1}{2}|}$$

Prob. 1.17

$$\boldsymbol{w} = \frac{w(1,-2,2)}{3} = (1,-2,2), \quad \boldsymbol{r} = \boldsymbol{r}_p - \boldsymbol{r}_o = (1,3,4)-(2,-3,1) = (-1,6,3)$$

$$\boldsymbol{u} = \boldsymbol{w}\times\boldsymbol{r} = \begin{vmatrix} 1 & -2 & 2 \\ -1 & 6 & 3 \end{vmatrix} = (-18,-5,4)$$

$$\underline{\underline{\boldsymbol{u} = -18\boldsymbol{a}_x - 5\boldsymbol{a}_y + 4\boldsymbol{a}_z}}$$

Prob. 1.18

$$\boldsymbol{r}_1 = (1,1,1), \qquad \boldsymbol{r}_2 = (1,0,1)-(0,1,0) = (1,-1,1)$$

$$\cos\theta = \frac{\boldsymbol{r}_1\cdot\boldsymbol{r}_2}{r_1 r_2} = \frac{(1-1+1)}{\sqrt{3}\sqrt{3}} = \frac{1}{3} \longrightarrow \underline{\underline{\theta = 70.53^o}}$$

Prob. 1.19

(a) $$T_s = T\cdot a_s = \frac{T\cdot S}{|S|} = \frac{(2,-6,3)\cdot(1,2,1)}{\sqrt{6}} = \frac{-7}{\sqrt{6}} = \underline{\underline{-2.8577}}$$

(b) $$\boldsymbol{S}_T = (\boldsymbol{S}\cdot\boldsymbol{a}_T)\boldsymbol{a}_T = \frac{(\boldsymbol{S}\cdot\boldsymbol{T})\boldsymbol{T}}{\boldsymbol{T}^2} = \frac{-7(2,-6,3)}{7^2}$$

$$= \underline{\underline{-0.2857\boldsymbol{a}_x + 0.8571\boldsymbol{a}_y - 0.4286\boldsymbol{a}_z}}$$

(c) $$\sin\theta_{TS} = \frac{|\boldsymbol{T}\times\boldsymbol{S}|}{|\boldsymbol{T}||\boldsymbol{S}|} = \begin{vmatrix} 2 & -6 & 3 \\ 1 & 2 & 1 \end{vmatrix} = \frac{|(-12,1,10)|}{7\sqrt{6}} = \frac{\sqrt{245}}{7\sqrt{6}} = 0.9129$$

$$\Rightarrow \theta_{TS} = \underline{\underline{65.91^\circ}}$$

Prob. 1.20

Let $\boldsymbol{A} = \boldsymbol{A}_{B\|} + \boldsymbol{A}_{B\perp}$

$$\boldsymbol{A}_{B\|} = (\boldsymbol{A}\cdot\boldsymbol{a}_B)\boldsymbol{a}_B = \frac{\boldsymbol{A}\cdot\boldsymbol{B}}{\boldsymbol{B}\cdot\boldsymbol{B}}\boldsymbol{B}$$

Hence,

$$\boldsymbol{A}_{B\perp} = \boldsymbol{A} - \boldsymbol{A}_{B\|} = \boldsymbol{A} - \frac{\boldsymbol{A}\cdot\boldsymbol{B}}{\boldsymbol{B}\cdot\boldsymbol{B}}\boldsymbol{B}$$

Prob. 1.21

(a) $\boldsymbol{H}(1,3,-2) = 6\boldsymbol{a}_x + \boldsymbol{a}_y + 4\boldsymbol{a}_z$

$$\boldsymbol{a}_H = \frac{(6,1,4)}{\sqrt{36+1+16}} = \underline{\underline{0.8242\boldsymbol{a}_x + 0.1374\boldsymbol{a}_y + 0.5494\boldsymbol{a}_z}}$$

(b) $|\,\boldsymbol{H}\,| = 10 = \sqrt{4x^2y^2 + (x+z)^2 + z^4}$

or

$$\underline{\underline{100 = 4x^2y^2 + x^2 + 2xz + z^2 + z^4}}$$

Prob. 1.22

$$\boldsymbol{C} = 5\boldsymbol{a}_x + \boldsymbol{a}_z$$

(a) $\boldsymbol{B}\times\boldsymbol{C} = \begin{vmatrix} 1 & 1 & 0 \\ 5 & 0 & 1 \end{vmatrix} = \boldsymbol{a}_x - \boldsymbol{a}_y - 5\boldsymbol{a}_z$

$$\boldsymbol{A}\bullet(\boldsymbol{B}\times\boldsymbol{C}) = (4,-1,1)\bullet(1,-1,-5) = 4+1-5 = \underline{\underline{0}}$$

(b) $\boldsymbol{A}_B = (\boldsymbol{A}\bullet\boldsymbol{a}_B)\boldsymbol{a}_B = \dfrac{(\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{B}}{|\,\boldsymbol{B}\,|^2} = \dfrac{(4-1)(1,1,0)}{1+1} = \underline{\underline{1.5\boldsymbol{a}_x + 1.5\boldsymbol{a}_y}}$

Prob. 1.23

(a) At (1, -2, 3), x = 1, y = -2, z = 3.

$\boldsymbol{G} = \boldsymbol{a}_x + 2\boldsymbol{a}_y + 6\boldsymbol{a}_z, \qquad \boldsymbol{H} = -6\boldsymbol{a}_x + 3\boldsymbol{a}_y - 3\boldsymbol{a}_z$

$$G = \sqrt{1+4+36} = \underline{\underline{6.403}}$$

$$H = \sqrt{36+9+9} = \underline{\underline{7.348}}$$

(b) $\boldsymbol{G}\bullet\boldsymbol{H} = -6+6-18 = \underline{\underline{-18}}$

(c)

$$\cos\theta_{GH} = \frac{\boldsymbol{G}\bullet\boldsymbol{H}}{GH} = \frac{-18}{6.403\times 7.348} = -0.3826$$

$$\underline{\underline{\theta_{GH} = 112.5^o}}$$

Prob. 1.24

$$\boldsymbol{r}_{PQ} = \boldsymbol{r}_Q - \boldsymbol{r}_P = (-2,1,4)-(1,0,3) = (-3,1,1)$$

At P, $\boldsymbol{H} = 0\boldsymbol{a}_x - 1\boldsymbol{a}_z = -\boldsymbol{a}_z$

The scalar component of $\boldsymbol{H}$ along $\boldsymbol{r}_{PQ}$ is

$$D = \boldsymbol{H}\bullet a_{r_{PQ}} = \frac{\boldsymbol{H}\bullet\boldsymbol{r}_{PQ}}{|\,\boldsymbol{r}_{PQ}\,|} = \frac{-1}{\sqrt{9+1+1}} = \underline{\underline{-0.3015}}$$

Prob. 1.25

(a) At P, x = -1, y = 2, z = 4

$$\boldsymbol{D} = 8\boldsymbol{a}_x - 4\boldsymbol{a}_y - 2\boldsymbol{a}_z, \qquad \boldsymbol{E} = -10\boldsymbol{a}_x + 24\boldsymbol{a}_y + 128\boldsymbol{a}_z$$

$$\boldsymbol{C} = \boldsymbol{D} + \boldsymbol{E} = -2\boldsymbol{a}_x + 20\boldsymbol{a}_y + 126\boldsymbol{a}_z$$

(b)

$$\boldsymbol{C}\bullet\boldsymbol{a}_x = C\cos\theta_x \longrightarrow \cos\theta_x = \frac{\boldsymbol{C}\bullet\boldsymbol{a}_x}{C} = \frac{-2}{\sqrt{2^2 + 20^2 + 126^2}} = -0.01575$$

$$\underline{\underline{\theta_x = 90.9^o}}$$

Prob. 1.26

(a) At (1,2,3), $\boldsymbol{E}$ = (2,1,6)

$$|\boldsymbol{E}| = \sqrt{4+1+36} = \sqrt{41} = \underline{\underline{6.403}}$$

(b) At (1,2,3), $\boldsymbol{F}$ = (2,-4,6)

$$\boldsymbol{E}_F = (\boldsymbol{E}\cdot\boldsymbol{a}_F)\boldsymbol{a}_F = \frac{(\boldsymbol{E}\cdot\boldsymbol{F})\boldsymbol{F}}{|\boldsymbol{F}|^2} = \frac{36}{56}(2,-4,6)$$

$$= \underline{\underline{1.286\boldsymbol{a}_x - 2.571\boldsymbol{a}_y + 3.857\boldsymbol{a}_z}}$$

(c) At (0,1,–3), $\boldsymbol{E}$ = (0,1,–3), $\boldsymbol{F}$ = (0,–1,0)

$$\boldsymbol{E}\times\boldsymbol{F} = \begin{vmatrix} 0 & 1 & -3 \\ 0 & -1 & 0 \end{vmatrix} = (-3,0,0)$$

$$\boldsymbol{a}_{E\times F} = \pm\frac{\boldsymbol{E}\times\boldsymbol{F}}{|\boldsymbol{E}\times\boldsymbol{F}|} = \underline{\underline{\pm\,\boldsymbol{a}_x}}$$

CHAPTER 2

P. E. 2.1

(a) At P(1,3,5), x = 1, y = 3, z =5,

$$\rho = \sqrt{x^2 + y^2} = \sqrt{10}, \quad z = 5, \quad \phi = \tan^{-1} y/x = \tan^{-1} 3 = 71.6^o$$

$$\text{P}(\rho, \phi, z) = P(\sqrt{10}, \tan^{-1} 3, 5) = \underline{\underline{P(3.162, 71.6^o, 5)}}$$

Spherical system:

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{35} = 5.916$$

$$\theta = \tan^{-1} \sqrt{x^2 + y^2} \big/ z = \tan^{-1} \sqrt{10} \big/ 5 = \tan^{-1} 0.6325 = 32.31°$$

$$P(r, \theta, \varphi) = \underline{\underline{P(5.916, 32.31°, 71.57°)}}$$

At T(0,-4,3), x = 0 y =-4, z =3;

$$\rho = \sqrt{x^2 + y^2} = 4, z = 3, \varphi = \tan^{-1} y/x = \tan^{-1} - 4/0 = 270°$$

$$\underline{\underline{T(\rho, \varphi, z) = T(4, 270°, 3).}}$$

Spherical system:

$$r = \sqrt{x^2 + y^2 + z^2} = 5, \theta = \tan^{-1} \rho/z = \tan^{-1} 4/3 = 53.13°.$$

$$\underline{\underline{T(r, \theta, \varphi) = T(5, 53.13°, 270°).}}$$

At S(-3-4-10), x =-3, y =-4, z =-10;

$$\rho = \sqrt{x^2 + y^2} = 5, \phi = \tan^{-1}\left(\frac{-4}{-3}\right) = 233.1°$$

$$\underline{\underline{S(\rho, \phi, z) = S(5, 233.1, -10).}}$$

Spherical system:

$$r = \sqrt{x^2 + y^2 + z^2} = 5\sqrt{5} = 11.18.$$

$$\theta = \tan^{-1} \rho\big/z = \tan^{-1} \frac{5}{-10} = 153.43°;$$

$$\underline{\underline{S(r, \theta, \phi) = S(11.18, 153.43°, 233.1°).}}$$

(b) In Cylindrical system, $\rho = \sqrt{x^2 + y^2}$; $yz = z\rho \sin\phi$,

$$Q_x = \frac{\rho}{\sqrt{\rho^2 + z^2}}; \quad Q_y = 0; \quad Q_z = -\frac{z\rho \sin\phi}{\sqrt{\rho^2 + z^2}};$$

$$\begin{bmatrix} Q_\rho \\ Q_\phi \\ Q_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Q_x \\ 0 \\ Q_z \end{bmatrix};$$

$$Q_\rho = Q_x\cos\phi = \frac{\rho\cos\phi}{\sqrt{\rho^2+z^2}}, \qquad Q_\phi = -Q_x\sin\phi = \frac{-\rho\sin\phi}{\sqrt{\rho^2+z^2}}$$

Hence,

$$\boldsymbol{Q} = \frac{\rho}{\sqrt{\rho^2+z^2}}(\cos\phi\boldsymbol{a}_\rho - \sin\phi\,\boldsymbol{a}_\phi - z\sin\phi\,\boldsymbol{a}_z).$$

In Spherical coordinates:

$$Q_x = \frac{r\sin\theta}{r} = \sin\theta;$$

$$Q_z = -r\sin\phi\sin\theta\, r\cos\theta\,\frac{1}{r} = -r\sin\theta\cos\theta\sin\phi.$$

$$\begin{bmatrix} Q_r \\ Q_\theta \\ Q_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} Q_x \\ 0 \\ Q_z \end{bmatrix};$$

$$Q_r = Q_x\sin\theta\cos\phi + Q_z\cos\theta = \sin^2\theta\cos\phi - r\sin\theta\cos^2\theta\sin\phi.$$

$$Q_\theta = Q_x\cos\theta\cos\phi - Q_z\sin\theta = \sin\theta\cos\theta\cos\phi + r\sin^2\theta\cos\theta\sin\phi.$$

$$Q_\phi = -Q_x\sin\phi = -\sin\theta\sin\phi.$$

$$\therefore\ \boldsymbol{Q} = \sin\theta\left(\sin\theta\cos\phi - r\cos^2\theta\sin\phi\right)\boldsymbol{a}_r + \sin\theta\cos\theta(\cos\phi + r\sin\theta\sin\phi)\boldsymbol{a}_\theta - \sin\theta\sin\phi\,\boldsymbol{a}_\phi\ .$$

At T :

$$\boldsymbol{Q}(x,y,z) = \frac{4}{5}\boldsymbol{a}_x + \frac{12}{5}\boldsymbol{a}_z = 0.8\boldsymbol{a}_x + 2.4\boldsymbol{a}_z;$$

$$\boldsymbol{Q}(\rho,\phi,z) = \frac{4}{5}(\cos 270^\circ\,\boldsymbol{a}_\rho - \sin 270^\circ\,\boldsymbol{a}_\phi - 3\sin 270^\circ\boldsymbol{a}_z$$
$$= 0.8\boldsymbol{a}_\phi + 2.4\,\boldsymbol{a}_z;$$

$$\boldsymbol{Q}(r,\theta,\phi) = \frac{4}{5}(0 - \frac{45}{25}(-1))\boldsymbol{a}_r + \frac{4}{5}(\frac{3}{5})(0 + \frac{20}{5}(-1))\boldsymbol{a}_\theta - \frac{4}{5}(-1)\boldsymbol{a}_\phi$$
$$= \frac{36}{25}\boldsymbol{a}_r - \frac{48}{25}\boldsymbol{a}_\theta + \frac{4}{5}\boldsymbol{a}_\phi = \underline{1.44\boldsymbol{a}_r - 1.92\boldsymbol{a}_\theta + 0.8\,\boldsymbol{a}_\phi};$$

Note, that the magnitude of vector Q = 2.53 in all 3 cases above.

P.E. 2.2 (a)

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho z\sin\phi \\ 3\rho\cos\phi \\ \rho\cos\phi\sin\phi \end{bmatrix}$$

$$\boldsymbol{A} = (\rho z\cos\phi\sin\phi - 3\rho\cos\phi\sin\phi)\boldsymbol{a}_x + (\rho z\sin^2\phi + 3\rho\cos^2\phi)\boldsymbol{a}_y + \rho\cos\phi\sin\phi\,\boldsymbol{a}_z.$$

But $\rho = \sqrt{x^2+y^2}$, $\tan\phi = \dfrac{y}{x}$, $\cos\phi = \dfrac{x}{\sqrt{x^2+y^2}}$, $\sin\phi = \dfrac{y}{\sqrt{x^2+y^2}}$;

Substituting all this yields:

$$\boldsymbol{A} = \frac{1}{\sqrt{x^2+y^2}}[(xyz - 3xy)\boldsymbol{a}_x + (zy^2 + 3x^2)\boldsymbol{a}_y + xy\,\boldsymbol{a}_z].$$

$$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} r^2 \\ 0 \\ \sin\theta \end{bmatrix}$$

Since $r = \sqrt{x^2+y^2+z^2}$, $\tan\theta = \dfrac{\sqrt{x^2+y^2}}{z}$, $\tan\phi = \dfrac{y}{z}$;

and $\sin\theta = \dfrac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$, $\cos\theta = \dfrac{z}{\sqrt{x^2+y^2+z^2}}$;

and $\sin\phi = \dfrac{y}{\sqrt{x^2+y^2}}$, $\cos\phi = \dfrac{x}{\sqrt{x^2+y^2}}$;

$$B_x = r^2\sin\theta\cos\phi - \sin\theta\sin\phi = rx - \frac{y}{r} = \frac{1}{r}(r^2x - y).$$

$$B_y = r^2\sin\theta\sin\phi + \sin\theta\cos\phi = ry + \frac{x}{r} = \frac{1}{r}(r^2y + x).$$

$$B_z = r^2\cos\theta = r\,z = \frac{1}{r}(r^2z).$$

Hence,

$$\boldsymbol{B} = \frac{1}{\sqrt{x^2+y^2+z^2}}[\{x(x^2+y^2+z^2) - y\}\boldsymbol{a}_x + \{y(x^2+y^2+z^2) + x\}\boldsymbol{a}_y + z(x^2+y^2+z^2)\boldsymbol{a}_z].$$

P.E.2.3 (a) At:

$(1, \pi/3, 0), \quad \boldsymbol{H} = (0, 0.06767, 1)$

$$\boldsymbol{a}_x = \cos\phi\,\boldsymbol{a}_\rho - \sin\phi\,\boldsymbol{a}_\phi = \frac{1}{2}(\boldsymbol{a}_\rho - \sqrt{3}\,\boldsymbol{a}_\phi)$$

$\boldsymbol{H} \bullet \boldsymbol{a}_x = \underline{\underline{-0.0586.}}$

(b) At:

$(1, \pi/3, 0), \quad \boldsymbol{a}_\theta = \cos\theta\boldsymbol{a}_\rho - \sin\theta\,\boldsymbol{a}_z = -\boldsymbol{a}_z.$

$$H \times \boldsymbol{a}_z = \begin{vmatrix} \boldsymbol{a}_\rho & a_\phi & a_z \\ 0 & 0.06767 & 1 \\ 0 & 0 & 1 \end{vmatrix} = \underline{\underline{-0.06767\,a_\rho.}}$$

(c) $(\boldsymbol{H} \bullet \boldsymbol{a}_\rho)\boldsymbol{a}_\rho = \underline{\underline{0\,\boldsymbol{a}_\rho}}.$

(d)

$$\boldsymbol{H} \times \boldsymbol{a}_z = \begin{vmatrix} \boldsymbol{a}_\rho & \boldsymbol{a}_\phi & \boldsymbol{a}_z \\ 0 & 0.06767 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 0.06767\,\boldsymbol{a}_\rho.$$

$$|\boldsymbol{H} \times \boldsymbol{a}_z| = \underline{\underline{0.06767}}$$

P.E. 2.4

(a)

$\boldsymbol{A} \bullet \boldsymbol{B} = (3, 2, -6) \bullet (4, 0, 3) = \underline{\underline{-6.}}$

(b)

$$|\boldsymbol{A} \times B| = \begin{vmatrix} 3 & 2 & -6 \\ 4 & 0 & 3 \end{vmatrix} = |6\,\boldsymbol{a}_r - 33\boldsymbol{a}_\theta - 8\boldsymbol{a}_\phi|.$$

Thus the magnitude of $\boldsymbol{A} \times \boldsymbol{B} = \underline{\underline{34.48.}}$

(c)

$At\ (1, \pi/3, 5\pi/4), \quad \theta = \pi/3,$

$$\boldsymbol{a}_z = \cos\theta\,\boldsymbol{a}_r - \sin\theta\,\boldsymbol{a}_\theta = \frac{1}{2}\boldsymbol{a}_r - \frac{\sqrt{3}}{2}\boldsymbol{a}_\theta.$$

$$(A \bullet \boldsymbol{a}_z)\boldsymbol{a}_z = \left(\frac{3}{2} - \sqrt{3}\right)\left(\frac{1}{2}\boldsymbol{a}_r - \frac{\sqrt{3}}{2}\boldsymbol{a}_\theta\right) = \underline{\underline{-0.116\boldsymbol{a}_r + 0.201\boldsymbol{a}_\theta}}$$

P.E. 2.5

In spherical coordinates, the distance between two points is given by eq. 2.33:

$$d^2 = r_2^2 + r_1^2 - (2r_1 r_2 \cos\theta_2 \cos\theta_1) - (2r_1 r_2 \sin\theta_2 \sin\theta_1 \cos(\emptyset_2 - \emptyset_1))$$

$$7^2 = 10^2 + 5^2 - (2 \times 5 \times 10 \times \cos(60) \times \cos(30)) - (2 \times 5 \times 10 \times \sin(60) \times \sin(30) \times \cos(\emptyset - 50))$$

By solving the above equation, we obtain $\emptyset = 91^{\circ}$.

P.E. 2.6

In Cartesian system the dot product of two vectors $x_1\mathbf{a}_x + y_1\mathbf{a}_y + z_1\mathbf{a}_z$ and $x_2\mathbf{a}_x + y_2\mathbf{a}_y + z_2\mathbf{a}_z$ is given by

$$x_1x_2 + y_1y_2 + z_1z_2 \qquad (1)$$

Now using the eq. 2.21 to represent spherical coordinates in Cartesian system and the dot product is determined by using Eq. (1)

$$\mathbf{r}_A \cdot \mathbf{r}_B = r_1 \sin\theta_1 \cos\emptyset_1 \, r_2 \sin\theta_2 \cos\emptyset_2 + r_1 \sin\theta_1 \sin\emptyset_1 \, r_2 \sin\theta_2 \sin\emptyset_2 + r_1 \cos\theta_1 \, r_2 \cos\theta_2$$

$$= r_1r_2(\sin\theta_1 \sin\theta_2 (\cos\emptyset_1 \cos\emptyset_2 + \sin\emptyset_1 \sin\emptyset_2) + \cos\theta_1 \cos\theta_2)$$

$$= r_1r_2(\sin\theta_1 \sin\theta_2 (\cos(\emptyset_1 - \emptyset_2)) + \cos\theta_1 \cos\theta_2)$$

Dot product of the given vectors is 3

Using the above equation

$$3 = 5.099r(\sin(78.69) \times \sin(158.19) \times \cos(53.13 - 90) + \cos(78.69) \times \cos(158.19))$$

$$3 = 5.099r(0.1093)$$

$$r = 5.381$$

Hence by using the above derived equation we can directly calculate dot product of vectors in spherical system

P.E. 2.7

At $P(0,2,-5), \qquad \phi = 90°;$

$$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} B_\rho \\ B_\phi \\ B_z \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -5 \\ 1 \\ -3 \end{bmatrix}$$

$$\boldsymbol{B} = -\boldsymbol{a}_x - 5\boldsymbol{a}_y - 3\boldsymbol{a}_z$$

$(a)\ \ \boldsymbol{A}+\boldsymbol{B} = (2,4,10)+(-1,-5,-3)$

$$= \underline{\underline{\boldsymbol{a}_x - \boldsymbol{a}_y + 7\boldsymbol{a}_z}}.$$

$$(b)\ \ \cos\theta_{AB} = \frac{\boldsymbol{A}\bullet\boldsymbol{B}}{AB} = \frac{-52}{\sqrt{4200}}$$

$$\theta_{AB} = \cos^{-1}\left(\frac{-52}{\sqrt{4200}}\right) = \underline{\underline{143.36^\circ}}.$$

$$(c)\ \ A_B = \boldsymbol{A}\bullet\boldsymbol{a}_B = \frac{\boldsymbol{A}\bullet\boldsymbol{B}}{B} = -\frac{52}{\sqrt{35}} = \underline{\underline{-8.789}}.$$

Prob. 2.1

$$\rho = \sqrt{x^2+y^2} = \sqrt{4+36} = 6.324$$

(a) $\phi = \tan^{-1}\dfrac{y}{x} = \tan^{-1}\dfrac{6}{2} = 71.56^o$

P is $\underline{\underline{(6.324, 71.56^o, -4)}}$

$$r = \sqrt{x^2+y^2+z^2} = \sqrt{4+36+16} = 7.485$$

(b) $\theta = \tan^{-1}\dfrac{\sqrt{x^2+y^2}}{z} = \tan^{-1}\dfrac{6.324}{-4} = 90^o + \tan^{-1}\dfrac{6.324}{4} = 147.69^o$

P is $\underline{\underline{(7.483, 147.69^o, 71.56^o)}}$

Prob.2.2

(a) Given P(1,-4,-3), convert to cylindrical and spherical values;

$$\rho = \sqrt{x^2+y^2} = \sqrt{1^2+(-4)^2} = \sqrt{17} = 4.123.$$

$$\phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{-4}{1} = 284.04^\circ.$$

$$\therefore P(\rho,\phi,z) = \underline{\underline{(4.123,\ 284.04^\circ,\ -3)}}$$

Spherical:

$$r = \sqrt{x^2+y^2+z^2} = \sqrt{1+16+9} = 5.099.$$

$$\theta = \tan^{-1}\frac{\rho}{z} = \tan^{-1}\frac{4.123}{-3} = 126.04^\circ.$$

$$P(r,\theta,\phi) = \underline{\underline{P(5.099, 126.04^\circ,\ 284.04^\circ)}}.$$

(b) $\rho = 3, \quad \phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{0}{3} = 0^o$

$$Q(\rho,\phi,z) = \underline{\underline{Q(3,0^o,5)}}$$

$$r = \sqrt{9+0+25} = 5.831, \quad \theta = \tan^{-1}\frac{\rho}{z} = \tan^{-1}\frac{3}{5} = 30.96^o$$

$$Q(r,\theta,\phi) = \underline{\underline{Q(5.831, 30.96^o, 0^o)}}$$

(c) $\rho = \sqrt{4+36} = 6.325, \quad \phi = \tan^{-1}\frac{6}{-2} = 108.4^o$

$$R(\rho,\phi,z) = \underline{\underline{R(6.325, 108.4^o, 0)}}$$

$$r = \rho = 6.325, \quad \theta = \tan^{-1}\frac{\rho}{z} = \tan^{-1}\frac{6.325}{0} = 90^o$$

$$R(r,\theta,\phi) = \underline{\underline{R(6.325, 90^o, 108.4^o)}}$$

Prob. 2.3

(a)

$$x = \rho\cos\phi = 2\cos 30^\circ = 1.732;$$
$$y = \rho\sin\phi = 2\sin 30^\circ = 1;$$
$$z = 5;$$
$$P_1(x,y,z) = \underline{\underline{P_1(1.732, 1, 5).}}$$

(b)

$$x = 1\cos 90^\circ = 0; \quad y = 1\sin 90^\circ = 1; \quad z = -3.$$
$$P_2(x,y,z) = \underline{\underline{P_2(0, 1, -3).}}$$

(c)

$$x = r\sin\theta\cos\phi = 10\sin(\pi/4)\cos(\pi/3) = 3.535;$$
$$y = r\sin\theta\sin\phi = 10\sin(\pi/4)\sin(\pi/3) = 6.124;$$
$$z = r\cos\theta = 10\cos(\pi/4) = 7.0711$$
$$P_3(x,y,z) = \underline{\underline{P_3(3.535, 6.124, 7.0711).}}$$

(d)

$$x = 4\sin 30^\circ\cos 60^\circ = 1$$
$$y = 4\sin 30^\circ\sin 60^\circ = 1.7321$$
$$z = r\cos\theta = 4\cos 30^\circ = 3.464$$
$$P_4(x,y,z) = \underline{\underline{P_4(1, 1.7321, 3.464).}}$$

Prob. 2.4

(a)
$$x = \rho\cos\phi = 5\cos 120^o = -2.5$$
$$y = \rho\sin\phi = 5\sin 120^o = 4.33$$
$$z = 1$$
Hence Q $= \underline{\underline{(-2.5, 4.33, 1)}}$

(b)
$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{\rho^2 + z^2} = \sqrt{25+1} = 5.099$$
$$\theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z} = \tan^{-1}\frac{\rho}{z} = \tan^{-1}\frac{5}{1} = 78.69^o$$
$$\phi = 120^o$$
Hence Q $= \underline{\underline{(5.099, 78.69^o, 120^o)}}$

Prob. 2.5

$$T(r,\theta,\phi) \quad \longrightarrow \quad r = 10, \theta = 60^o, \phi = 30^o$$
$$x = r\sin\theta\cos\phi = 10\sin 60^o\cos 30^o = 7.5$$
$$y = r\sin\theta\sin\phi = 10\sin 60^o\sin 30^o = 4.33$$
$$z = r\cos\theta = 10\cos 60^o = 5$$
$$T(x,y,z) = \underline{\underline{(7.5, 4.33, 5)}}$$
$$\rho = r\sin\theta = 10\sin 60^o = 8.66$$
$$T(\rho,\phi,z) = \underline{\underline{(8.66, 30^o, 5)}}$$

Prob. 2.6

(a)
$$x = \rho\cos\phi, \qquad y = \rho\sin\phi,$$
$$V = \underline{\underline{\rho z\cos\phi - \rho^2\sin\phi\cos\phi + \rho z\sin\phi}}$$

(b)
$$\begin{aligned} U &= x^2 + y^2 + z^2 + y^2 + 2z^2 \\ &= r^2 + r^2\sin^2\theta\sin^2\phi + 2r^2\cos^2\theta \\ &= \underline{\underline{r^2[1 + \sin^2\theta\sin^2\phi + 2\cos^2\theta]}} \end{aligned}$$

Prob. 2.7
(a)

$$\begin{bmatrix} F_\rho \\ F_\phi \\ F_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dfrac{x}{\sqrt{\rho^2+z^2}} \\ \dfrac{y}{\sqrt{\rho^2+z^2}} \\ \dfrac{4}{\sqrt{\rho^2+z^2}} \end{bmatrix}$$

$$F_\rho = \frac{1}{\sqrt{\rho^2+z^2}}[\rho\cos^2\phi + \rho\sin^2\phi] = \frac{\rho}{\sqrt{\rho^2+z^2}};$$

$$F_\phi = \frac{1}{\sqrt{\rho^2+z^2}}[-\rho\cos\phi\sin\phi + \rho\cos\phi\sin\phi] = 0;$$

$$F_z = \frac{4}{\sqrt{\rho^2+z^2}};$$

$$\underline{\underline{\bar{F} = \frac{1}{\sqrt{\rho^2+z^2}}(\rho\boldsymbol{a}_\rho + 4\boldsymbol{a}_z)}}$$

In Spherical:

$$\begin{bmatrix} F_r \\ F_\theta \\ F_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \dfrac{x}{r} \\ \dfrac{y}{r} \\ \dfrac{4}{r} \end{bmatrix}$$

$$F_r = \frac{r}{r}\sin^2\theta\cos^2\phi + \frac{r}{r}\sin^2\theta\sin^2\phi + \frac{4}{r}\cos\theta = \sin^2\theta + \frac{4}{r}\cos\theta;$$

$$F_\theta = \sin\theta\cos\theta\cos^2\phi + \sin\theta\cos\theta\sin^2\phi - \frac{4}{r}\sin\theta = \sin\theta\cos\theta - \frac{4}{r}\sin\theta;$$

$$F_\phi = -\sin\theta\cos\phi\sin\phi + \sin\theta\sin\phi\cos\phi = 0;$$

$$\therefore \bar{F} = \underline{\underline{(\sin^2\theta + \frac{4}{r}\cos\theta)\boldsymbol{a}_r + \sin\theta(\cos\theta - \frac{4}{r})\boldsymbol{a}_\theta}}$$

(b)

$$\begin{bmatrix} G_\rho \\ G_\phi \\ G_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dfrac{x\rho^2}{\sqrt{\rho^2+z^2}} \\ \dfrac{y\rho^2}{\sqrt{\rho^2+z^2}} \\ \dfrac{z\rho^2}{\sqrt{\rho^2+z^2}} \end{bmatrix}$$

$$G_\rho = \frac{\rho^2}{\sqrt{\rho^2+z^2}}[\rho\cos^2\phi + \rho\sin^2\phi] = \frac{\rho^3}{\sqrt{\rho^2+z^2}};$$

$$G_\phi = 0;$$

$$G_z = \frac{z\rho^2}{\sqrt{\rho^2+z^2}};$$

$$\underline{\underline{\boldsymbol{G} = \frac{\rho^2}{\sqrt{\rho^2+z^2}}(\rho\,\boldsymbol{a}_\rho + z\,\boldsymbol{a}_z)}}$$

Spherical :

$$\boldsymbol{G} = \frac{\rho^2}{r}(x\boldsymbol{a}_x + y\boldsymbol{a}_y + z\boldsymbol{a}_z) = \frac{r^2\sin^2\theta}{r} r\boldsymbol{a}_r = \underline{\underline{r^2\sin^2\theta\boldsymbol{a}_r}}$$

Prob. 2.8

$$\boldsymbol{B} = \rho\boldsymbol{a}_x + \frac{y}{\rho}\boldsymbol{a}_y + z\boldsymbol{a}_z$$

$$\begin{bmatrix} B_\rho \\ B_\phi \\ B_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho \\ y/\rho \\ z \end{bmatrix}$$

$$B_\rho = \rho\cos\phi + \frac{y}{\rho}\sin\phi$$

$$B_\phi = -\rho\sin\phi + \frac{y}{\rho}\cos\phi$$

$$B_z = z$$

But $y = \rho\sin\phi$

$$B_\rho = \rho\cos\phi + \sin^2\phi, B_\phi = -\rho\sin\phi + \sin\phi\cos\phi$$

Hence,

$$\underline{\underline{\boldsymbol{B} = (\rho\cos\phi + \sin^2\phi)\boldsymbol{a}_\rho + \sin\phi(\cos\phi - \rho)\boldsymbol{a}_\phi + z\boldsymbol{a}_z}}$$

Prob. 2.9

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

At P, $\rho = 2, \quad \phi = \pi/2, \quad z = -1$

$$A_x = 2\cos\phi - 3\sin\phi = 2\cos 90^o - 3\sin 90^o = -3$$

$$A_y = 2\sin\phi + 3\cos\phi = 2\sin 90^o + 3\cos 90^o = 2$$

$$A_z = 4$$

Hence, $\boldsymbol{A} = \underline{\underline{-3\boldsymbol{a}_x + 2\boldsymbol{a}_y + 4\boldsymbol{a}_z}}$

Prob. 2.10

(a)

$$\begin{bmatrix} A_x \\ A_y \\ A_y \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho\sin\phi \\ \rho\cos\phi \\ -2z \end{bmatrix}$$

$$A_x = \rho\sin\phi\cos\phi - \rho\cos\phi\sin\phi = 0$$

$$A_y = \rho\sin^2\phi + \rho\cos^2\phi = \rho = \sqrt{x^2 + y^2}$$

$$A_z = -2z$$

Hence,

$$\boldsymbol{A} = \underline{\underline{\sqrt{x^2 + y^2}\,\boldsymbol{a}_y - 2z\boldsymbol{a}_z}}$$

(b)

$$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} 4r\cos\phi \\ r \\ 0 \end{bmatrix}$$

$$B_x = 4r\sin\theta\cos^2\phi + r\cos\theta\cos\phi$$

$$B_y = 4r\sin\theta\sin\phi\cos\phi + r\cos\theta\sin\phi$$

$$B_z = 4r\cos\theta\cos\phi - r\sin\theta$$

But $\quad r = \sqrt{x^2 + y^2 + z^2}, \quad \sin\theta = \dfrac{\sqrt{x^2 + y^2}}{r}, \quad \cos\theta = \dfrac{z}{r}$

$$\sin\phi = \frac{y}{\sqrt{x^2 + y^2}}, \quad \cos\phi = \frac{x}{\sqrt{x^2 + y^2}}$$

$$B_x = 4\sqrt{x^2+y^2}\,\frac{x^2}{x^2+y^2} + \frac{zx}{\sqrt{x^2+y^2}}$$

$$B_y = 4\sqrt{x^2+y^2}\,\frac{xy}{x^2+y^2} + \frac{zy}{\sqrt{x^2+y^2}}$$

$$B_z = 4z\frac{x}{\sqrt{x^2+y^2}} - \sqrt{x^2+y^2}$$

$$\boldsymbol{B} = \frac{1}{\sqrt{x^2+y^2}}\underline{\underline{\left[x(4x+z)\boldsymbol{a}_x + y(4x+z)\boldsymbol{a}_y + (4xz-x^2-y^2)\boldsymbol{a}_z\right]}}$$

Prob. 2.11

$$\begin{bmatrix} G_x \\ G_y \\ G_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \rho\sin\phi \\ -\rho\cos\phi \\ \rho \end{bmatrix}$$

$$G_x = \rho\cos\phi\sin\phi + \rho\sin\phi\cos\phi = 2\rho\sin\phi\cos\phi$$

$$G_y = \rho\sin^2\phi - \rho\cos^2\phi = \rho(1-\cos^2\phi) - \rho\cos^2\phi = \rho - 2\rho\cos^2\phi$$

$$G_z = \rho$$

$$\rho = \sqrt{x^2+y^2},\cos\phi = \frac{x}{\rho} = \frac{x}{\sqrt{x^2+y^2}},\sin\phi = \frac{y}{\rho} = \frac{y}{\sqrt{x^2+y^2}}$$

$$G_x = 2\sqrt{x^2+y^2}\,\frac{xy}{x^2+y^2} = \frac{2xy}{\sqrt{x^2+y^2}}$$

But $G_y = \sqrt{x^2+y^2} - 2\sqrt{x^2+y^2}\,\frac{x^2}{x^2+y^2} = \sqrt{x^2+y^2} - \frac{2x^2}{\sqrt{x^2+y^2}}$

$$G_z = \sqrt{x^2+y^2}$$

Thus,

$$\underline{\underline{\boldsymbol{G} = \frac{2xy}{\sqrt{x^2+y^2}}\boldsymbol{a}_x + \left(\sqrt{x^2+y^2} - \frac{2x^2}{\sqrt{x^2+y^2}}\right)\boldsymbol{a}_y + \sqrt{x^2+y^2}\,\boldsymbol{a}_z}}$$

Prob. 2.12

$$\begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}$$

$$H_x = \sin\theta\cos\theta\cos\phi + \sin\theta\cos\theta\cos\phi = 2\sin\theta\cos\theta\cos\phi$$

$$H_y = \cos\theta\sin\theta\sin\phi + \sin\theta\cos\theta\sin\phi = 2\sin\theta\cos\theta\sin\phi$$

$$H_z = \cos^2\theta - \sin^2\theta$$

But,

$$\sin\theta = \frac{\sqrt{x^2+y^2}}{r}, \qquad \cos\theta = \frac{z}{r},$$

$$\sin\phi = \frac{y}{\sqrt{x^2+y^2}}, \qquad \cos\phi = \frac{x}{\sqrt{x^2+y^2}}$$

$$H_x = \frac{2z\sqrt{x^2+y^2}}{x^2+y^2+z^2}\frac{x}{\sqrt{x^2+y^2}} = \frac{2xz}{x^2+y^2+z^2}$$

$$H_y = \frac{2z\sqrt{x^2+y^2}}{x^2+y^2+z^2}\frac{y}{\sqrt{x^2+y^2}} = \frac{2yz}{x^2+y^2+z^2}$$

$$H_z = \frac{z^2-x^2-y^2}{x^2+y^2+z^2}$$

$$\boldsymbol{H} = \underline{\underline{\frac{1}{x^2+y^2+z^2}\left(2xz\boldsymbol{a}_x + 2yz\boldsymbol{a}_y + [z^2-x^2-y^2]\boldsymbol{a}_z\right)}}$$

Prob. 2.13

(a)

$$x = \rho\cos\phi$$

$$\underline{\underline{\boldsymbol{B} = \rho\cos\phi\boldsymbol{a}_z}}$$

(b)

$$x = r\sin\theta\cos\phi$$

$$\boldsymbol{B} = r\sin\theta\cos\phi\boldsymbol{a}_z, \qquad B_x = 0 = B_y, B_z = r\sin\theta\cos\phi$$

$$\begin{bmatrix} B_r \\ B_\theta \\ B_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ r\sin\theta\cos\phi \end{bmatrix}$$

$$B_r = r\sin\theta\cos\theta\cos\phi = 0.5r\sin(2\theta)\cos\phi$$

$$B_\theta = -r\sin^2\theta\cos\phi, \qquad B_\phi = 0$$

$$\underline{\underline{\boldsymbol{B} = 0.5r\sin(2\theta)\cos\phi\boldsymbol{a}_r - r\sin^2\theta\cos\phi\boldsymbol{a}_\theta}}$$

Prob. 2.14

(a)

$$\boldsymbol{a}_x \bullet \boldsymbol{a}_\rho = (\cos\phi\, \boldsymbol{a}_\rho - \sin\phi\, \boldsymbol{a}_\phi) \bullet \boldsymbol{a}_\rho = \cos\phi$$

$$\boldsymbol{a}_x \bullet \boldsymbol{a}_\phi = (\cos\phi\, \boldsymbol{a}_\rho - \sin\phi\, \boldsymbol{a}_\phi) \bullet \boldsymbol{a}_\phi = -\sin\phi$$

$$\boldsymbol{a}_y \bullet \boldsymbol{a}_\rho = (\sin\phi\, \boldsymbol{a}_\rho + \cos\phi\, \boldsymbol{a}_\phi) \bullet \boldsymbol{a}_\rho = \sin\phi$$

$$\bar{a}_y \bullet \bar{a}_\phi = (\sin\phi\, \boldsymbol{a}_\rho + \sin\phi\, \boldsymbol{a}_\phi) \bullet \boldsymbol{a}_\phi = \cos\phi$$

(b) and (c)

In spherical system:

$$\boldsymbol{a}_x = \sin\theta\cos\phi\, \boldsymbol{a}_r + \cos\theta\cos\phi\, \boldsymbol{a}_\theta - \sin\phi\, \boldsymbol{a}_\phi.$$

$$\boldsymbol{a}_y = \sin\theta\sin\phi\, \boldsymbol{a}_r + \cos\theta\sin\phi\, \boldsymbol{a}_\theta - \cos\phi\, \boldsymbol{a}_\phi.$$

$$\boldsymbol{a}_z = \cos\theta\, \boldsymbol{a}_x - \sin\theta\, \boldsymbol{a}_\theta.$$

Hence,

$$\boldsymbol{a}_x \bullet \boldsymbol{a}_r = \sin\theta\cos\phi;$$

$$\boldsymbol{a}_x \bullet \boldsymbol{a}_\theta = \cos\theta\cos\phi;$$

$$\boldsymbol{a}_y \bullet \boldsymbol{a}_r = \sin\theta\sin\phi;$$

$$\boldsymbol{a}_y \bullet \boldsymbol{a}_\theta = \cos\theta\sin\phi;$$

$$\bar{a}_z \bullet \bar{a}_r = \cos\theta;$$

$$\bar{a}_z \bullet \bar{a}_\theta = -\sin\theta;$$

Prob. 2.15

(a)

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{\rho^2 + z^2}.$$

$$\theta = \tan^{-1}\frac{\rho}{z}; \qquad \phi = \phi.$$

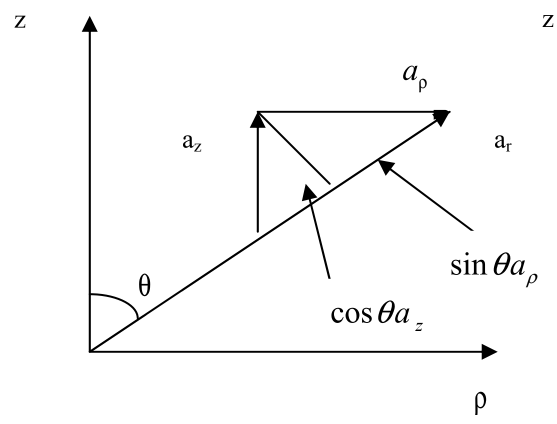
or

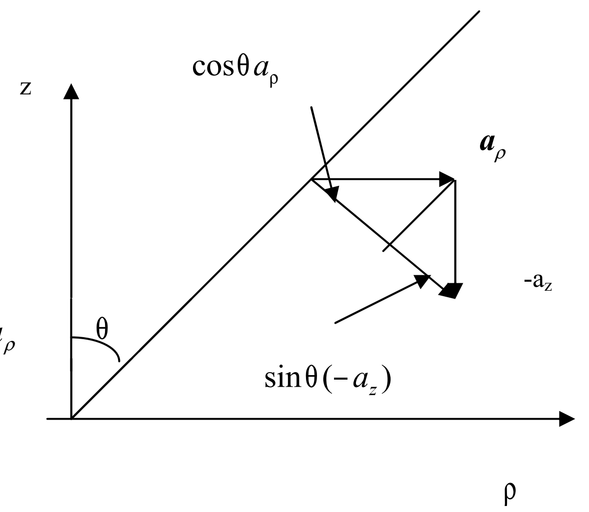
$$\rho = \sqrt{x^2 + y^2} = \sqrt{r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi}.$$

$$= r\sin\theta;$$

$$z = r\cos\theta; \qquad \phi = \phi.$$

(b) From the figures below,





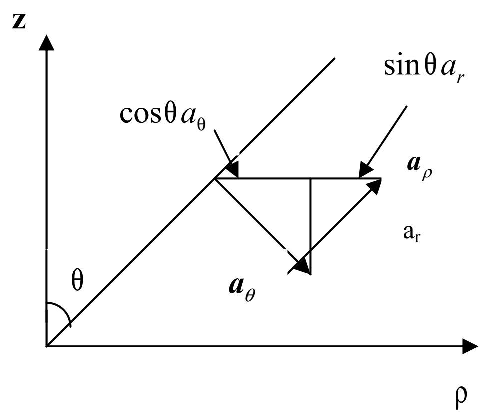
$\boldsymbol{a}_r = \sin\theta\,\boldsymbol{a}_\rho + \cos\theta\,\boldsymbol{a}_z$; $\quad \boldsymbol{a}_\theta = \cos\theta\,\boldsymbol{a}_\rho - \sin\theta\,\boldsymbol{a}_z$; $\quad \boldsymbol{a}_\phi = \boldsymbol{a}_\phi$.

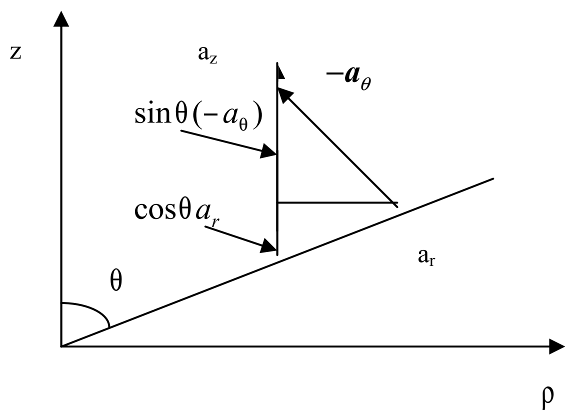
Hence,

$$\begin{bmatrix} \boldsymbol{a}_r \\ \boldsymbol{a}_\theta \\ \boldsymbol{a}_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{a}_\rho \\ \boldsymbol{a}_\phi \\ \boldsymbol{a}_z \end{bmatrix}$$

From the figures below,

$\boldsymbol{a}_\rho = \cos\theta\boldsymbol{a}_\theta + \sin\theta\,\boldsymbol{a}_r; \boldsymbol{a}_z = \cos\theta\boldsymbol{a}_r - \sin\theta\boldsymbol{a}_\theta; \boldsymbol{a}_\phi = \boldsymbol{a}_\phi.$





$$\begin{bmatrix} \boldsymbol{a}_\rho \\ \boldsymbol{a}_\phi \\ \boldsymbol{a}_z \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{a}_r \\ \boldsymbol{a}_\theta \\ \boldsymbol{a}_z \end{bmatrix}$$

Prob. 2.16

If **A** and **B** are perpendicular to each other, **A•B** = 0

$$\begin{aligned} \boldsymbol{A}\bullet\boldsymbol{B} &= \rho^2\sin^2\phi + \rho^2\cos^2\phi - \rho^2 \\ &= \rho^2(\sin^2\phi + \cos^2\phi) - \rho^2 \\ &= \rho^2 - \rho^2 \\ &= 0 \end{aligned}$$

As expected.

Prob. 2.17

(a) $\boldsymbol{A}+\boldsymbol{B} = \underline{\underline{8\boldsymbol{a}_\rho + 2\boldsymbol{a}_\phi - 7\boldsymbol{a}_z}}$

(b) $\boldsymbol{A}\bullet\boldsymbol{B} = \boldsymbol{15+0-8} = \underline{\underline{7}}$

(c) $\boldsymbol{A}\times\boldsymbol{B} = \begin{vmatrix} 3 & 2 & 1 \\ 5 & 0 & -8 \end{vmatrix}$

$$= -16\mathbf{a}_\rho + (5+24)\boldsymbol{a}_\phi - 10\boldsymbol{a}_z$$

$$= \underline{\underline{-16\mathbf{a}_\rho + 29\boldsymbol{a}_\phi - 10\boldsymbol{a}_z}}$$

(d) $\cos\theta_{AB} = \dfrac{\boldsymbol{A}\cdot\boldsymbol{B}}{AB} = \dfrac{7}{\sqrt{9+4+1}\sqrt{25+64}} = \dfrac{7}{\sqrt{14}\sqrt{89}}$

$= 0.19831$

$\underline{\underline{\theta_{\text{AB}} = 78.56^o}}$

Prob. 2.18 (a)

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho\cos\phi \\ 0 \\ \rho z^2\sin\phi \end{bmatrix}$$

$$A_x = \rho\cos^2\phi = \sqrt{x^2+y^2}\,\frac{x^2}{x^2+y^2} = \frac{x^2}{\sqrt{x^2+y^2}}$$

$$A_y = \rho\sin\phi\cos\phi = \sqrt{x^2+y^2}\,\frac{xy}{x^2+y^2} = \frac{xy}{\sqrt{x^2+y^2}}$$

$$A_z = \rho z^2\sin\phi = \rho z^2\,\frac{y}{\rho} = yz^2$$

$$\underline{\underline{\boldsymbol{A} = \frac{1}{\sqrt{x^2+y^2}}[x^2\,\boldsymbol{a}_x + xy\boldsymbol{a}_y + yz^2\sqrt{x^2+y^2}\,\boldsymbol{a}_z]}}$$

At (3,-4,0) x=3, y=-4, z=0;

$$\boldsymbol{A} = \frac{1}{5}[9\boldsymbol{a}_x - 12\,\boldsymbol{a}_y]$$

$$\underline{\underline{|\boldsymbol{A}| = 3}}$$

(b)
$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ -\cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \dfrac{x^2}{\rho} \\ \dfrac{xy}{\rho} \\ yz^2 \end{bmatrix}$$

$$x = r\sin\theta\cos\phi,\quad y = r\sin\theta\sin\phi,\quad z = r\cos\theta,\quad \rho = r\sin\theta.$$

$$A_r = \frac{r^2\sin^2\theta\cos^2\phi}{r\sin\theta}\sin\theta\cos\phi + \frac{r^2\sin^2\theta\cos\phi\sin\phi}{r\sin\theta}\sin\theta\sin\phi + r^3\sin\theta\cos^2\theta\sin\phi\cos\theta$$

$$= r\sin^2\theta\cos^3\phi + r\sin^2\theta\sin^2\phi\cos\phi + r^3\sin\theta\sin\phi\cos^3\theta$$

$$A_\theta = r\sin\theta\cos^2\phi\cos\theta\cos\phi + r\sin\theta\cos\phi\sin\phi\cos\theta\sin\phi - r^2\cos^2\theta\sin\phi\sin\theta$$

$$= r\sin\theta\cos\theta\cos\phi - r^2\sin\theta\cos^2\sin\phi$$

$$= r\sin\theta\cos\theta[\cos\phi - r\cos\theta\sin\phi]$$

$$A_\phi = -r\sin\theta\cos^2\phi\sin\phi + r\sin\theta\cos\phi\sin\phi\cos\phi = 0.$$

$\therefore$

$$\boldsymbol{A} = r\sin\theta[\cos\phi\sin\theta + \sin\theta\cos\phi\sin^2\phi + r^2\cos^3\theta\sin\phi]\,\boldsymbol{a}_r$$
$$\underline{\underline{+ r\sin\theta\cos\theta[\cos\phi - r^2\cos\theta\sin\theta\sin\phi]\,\boldsymbol{a}_\theta}}$$

$$At\ (3-4,0),\ r=5,\ \theta=\pi/2,\quad \phi=306.83$$
$$\cos\phi=3/5,\qquad \sin\phi=-4/5.$$

$$\boldsymbol{A} = 5[1^2*\frac{3}{5}+5(0)(-4/5)]\boldsymbol{a}_r + 5(1)(0)\boldsymbol{a}_\theta$$
$$= 3\boldsymbol{a}_r$$
$$\underline{\underline{|\boldsymbol{A}| = 3.}}$$

Prob. 2.19

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

$$\underline{\underline{= \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & -\frac{y}{\sqrt{x^2+y^2}} & 0 \\ \frac{y}{\sqrt{x^2+y^2}} & \frac{x}{\sqrt{x^2+y^2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}}}$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

$$\underline{\underline{= \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2+z^2}} & \frac{xz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} & \frac{-y}{\sqrt{x^2+y^2}} \\ \frac{y}{\sqrt{x^2+y^2+z^2}} & \frac{yz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} & \frac{x}{\sqrt{x^2+y^2}} \\ \frac{z}{\sqrt{x^2+y^2+z^2}} & -\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}} & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\phi \\ A_\phi \end{bmatrix}}}$$

Prob. 2.20 (a) Using the results in Prob.2.14,

$$A_\rho = \rho z \sin\phi = r^2 \sin\theta\cos\theta\sin\phi$$
$$A_\phi = 3\rho\cos\phi = 3r\sin\theta\cos\phi$$
$$A_z = \rho\cos\phi\sin\phi = r\sin\theta\cos\phi\sin\phi$$

Hence,

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} r^2\sin\theta\cos\theta\sin\phi \\ 3r\sin\theta\cos\phi \\ r\sin\theta\cos\phi\sin\phi \end{bmatrix}$$

$$\boldsymbol{A}(r,\theta,\phi) = \underline{\underline{r\sin\theta\left[\sin\phi\cos\theta\left(r\sin\theta+\cos\phi\right)\boldsymbol{a}_r + \sin\phi\left(r\cos^2\theta - \sin\theta\cos\phi\right)\boldsymbol{a}_\theta + 3\cos\phi\boldsymbol{a}_\phi\right]}}$$

At $(10,\pi/2,3\pi/4)$, $\quad r=10, \theta=\pi/2, \phi=3\pi/4$

$$\boldsymbol{A} = 10(0\boldsymbol{a}_r + 0.5\boldsymbol{a}_\theta - \frac{3}{\sqrt{2}}\boldsymbol{a}_\phi) = \underline{\underline{5\boldsymbol{a}_\theta - 21.21\boldsymbol{a}_\phi}}$$

(b) $\quad B_r = r^2 = (\rho^2 + z^2), \quad B_\theta = 0, \quad B_\phi = \sin\theta = \dfrac{\rho}{\sqrt{\rho^2+z^2}}$

$$\begin{bmatrix} B_\rho \\ B_\phi \\ B_z \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} B_r \\ B_\theta \\ B_\phi \end{bmatrix}$$

$$\boldsymbol{B}(\rho,\phi,z) = \underline{\underline{\sqrt{\rho^2+z^2}\left(\rho\boldsymbol{a}_\rho + \frac{\rho}{\rho^2+z^2}\boldsymbol{a}_\phi + z\boldsymbol{a}_z\right)}}$$

At $(2,\pi/6,1)$, $\quad \rho = 2, \phi = \pi/6, z = 1$

$$\boldsymbol{B} = \sqrt{5}(2\boldsymbol{a}_\rho + 0.4\boldsymbol{a}_\phi + \boldsymbol{a}_z) = \underline{\underline{4.472\boldsymbol{a}_\rho + 0.8944\boldsymbol{a}_\phi + 2.236\boldsymbol{a}_z}}$$

Prob. 2.21

(a) $d = \sqrt{(6-2)^2 + (-1-1)^2 + (2-5)^2} = \sqrt{29} = \underline{\underline{5.385}}$

(b)
$$d^2 = 3^2 + 5^2 - 2(3)(5)\cos\pi + (-1-5)^2 = 100$$
$$d = \sqrt{100} = \underline{\underline{10}}$$

(c)

$$d^2 = 10^2 + 5^2 - 2(10)(5)\cos\frac{\pi}{4}\cos\frac{\pi}{6} - 2(10)(5)\sin\frac{\pi}{4}\sin\frac{\pi}{6}\cos(7\frac{\pi}{4} - \frac{3\pi}{4})$$

$$= 125 - 100(\cos\frac{\pi}{4}\cos\frac{\pi}{6} - \sin\frac{\pi}{4}\sin\frac{\pi}{6}) = 125 - 100\cos 75^o = 99.12$$

$$d = \sqrt{99.12} = \underline{\underline{9.956.}}$$

Prob. 2.22

We can convert Q to cylindrical system and then use equation 2.32

At Q, $r = 4 \quad \theta = \frac{\pi}{2} \quad \phi = \frac{\pi}{2}$

$$\rho = r\sin\theta = 4\sin 90^o = 4$$

$$\phi = \frac{\pi}{2}$$

$$z = r\cos\theta = 4\cos 90^o = 0$$

Q is $(4, \pi/2, 0)$.

$$d^2 = \rho_2^2 + \rho_1^2 - 2\rho_1\rho_2\cos(\phi_2 - \phi_1) + (z_2 - z_1)^2$$

$$= 10^2 + 4^2 - 2(10)(4)\cos(\pi/4 - \pi/2) + 0 = 59.431$$

$$\underline{\underline{d = 7.709}}$$

Prob. 2.23

(a) An infinite line parallel to the z-axis.

(b) Point (2,-1,10).

(c) A circle of radius $r\sin\theta = 5$, i.e. the intersection of a cone and a sphere.

(d) An infinite line parallel to the z-axis.

(e) A semi-infinite line parallel to the x-y plane.

(f) A semi-circle of radius 5 in the y-z plane.

Prob. 2.24

At $(1, 60^o, -1)$, $\rho = 1, \phi = 60^o, z = -1$,

(a) $\boldsymbol{A} = (-2 - \sin 60^o)\boldsymbol{a}_\rho + (4 + 2\cos 60^o)\boldsymbol{a}_\phi - 3(1)(-1)\boldsymbol{a}_z$

$= -2.866\boldsymbol{a}_\rho + 5\boldsymbol{a}_\phi + 3\boldsymbol{a}_z$

$\boldsymbol{B} = 1\cos 60^o \boldsymbol{a}_\rho + \sin 60^o \boldsymbol{a}_\phi + \boldsymbol{a}_z = 0.5\boldsymbol{a}_\phi + 0.866\boldsymbol{a}_\phi + \boldsymbol{a}_z$

$\boldsymbol{A} \bullet \boldsymbol{B} = -1.433 + 4.33 + 3 = 5.897$

$AB = \sqrt{2.866^2 + 26 + 9}\sqrt{0.25 + 1 + 0.866^2} = 9.1885$

$$\cos\theta_{AB} = \frac{\boldsymbol{A}\bullet\boldsymbol{B}}{AB} = \frac{5.897}{9.1885} = 0.6419 \quad \longrightarrow \quad \theta_{AB} = \underline{\underline{50.07^o}}$$

Let **D** = **A** × **B**. At $(1, 90^o, 0)$, $\rho = 1, \phi = 90^o, z = 0$

(b) $\boldsymbol{A} = -\sin 90^o \boldsymbol{a}_\rho + 4\boldsymbol{a}_\phi = -\boldsymbol{a}_\rho + 4\boldsymbol{a}_\phi$

$\boldsymbol{B} = 1\cos 90^o \boldsymbol{a}_\rho + \sin 90^o \boldsymbol{a}_\phi + \boldsymbol{a}_z = \boldsymbol{a}_\phi + \boldsymbol{a}_z$

$$\boldsymbol{D} = \boldsymbol{A}\times\boldsymbol{B} = \begin{vmatrix} \boldsymbol{a}_\rho & \boldsymbol{a}_\phi & \boldsymbol{a}_z \\ -1 & 4 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 4\boldsymbol{a}_\rho + \boldsymbol{a}_\phi - \boldsymbol{a}_z$$

$$\boldsymbol{a}_D = \frac{\boldsymbol{D}}{D} = \frac{(4,1,-1)}{\sqrt{16+1+1}} = \underline{\underline{0.9428\boldsymbol{a}_\rho + 0.2357\boldsymbol{a}_\phi - 0.2357\boldsymbol{a}_z}}$$

Prob. 2.25

At $T(2,3,-4)$

$$\theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z} = \tan^{-1}\frac{\sqrt{13}}{-4} = 137.97$$

$$\cos\theta = \frac{-4}{\sqrt{29}} = -0.7428, \sin\theta = \frac{\sqrt{13}}{\sqrt{29}} = 0.6695$$

$$\phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{3}{2} = 56.31$$

$$\cos\phi = \frac{2}{\sqrt{13},} \quad \sin\phi = \frac{3}{\sqrt{13}}$$

$$\boldsymbol{a}_z = \cos\theta\boldsymbol{a}_r - \sin\theta\,\boldsymbol{a}_\theta = \underline{\underline{-0.7428\boldsymbol{a}_r - 0.6695\boldsymbol{a}_\theta.}}$$

$$\boldsymbol{a}_r = \sin\theta\cos\phi\ \boldsymbol{a}_x + \sin\theta\sin\phi\,\boldsymbol{a}_y + \cos\theta\,\boldsymbol{a}_z.$$

$$= \underline{\underline{0.3714\boldsymbol{a}_x + 0.5571\boldsymbol{a}_y - 0.7428\boldsymbol{a}_z.}}$$

Prob. 2.26

$$\boldsymbol{G}\bullet\boldsymbol{a}_y = G_y = G_r \sin\theta\sin\phi + G_\theta \cos\theta\sin\phi + 0$$
$$= 6r^2 \sin\theta\sin^2\phi + r^2\cos\theta\sin\phi$$

At $(2,-3,1), x=2, y=-3, z=1$

$$r^2 = x^2+y^2+z^2 = 4+9+1=14$$

$$\sin\theta = \frac{\rho}{r} = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}} = \sqrt{\frac{13}{14}}$$

$$\cos\theta = \frac{z}{r} = \frac{1}{\sqrt{14}}, \qquad \sin\phi = \frac{y}{\rho} = \frac{-3}{\sqrt{13}}$$

$$G_y = 6(14)\sqrt{\frac{13}{14}}\left(\frac{9}{13}\right) + 14\frac{1}{\sqrt{14}}\left(\frac{-3}{\sqrt{13}}\right) = 56.04 - 3.1132 = \underline{\underline{52.925}}$$

Prob. 2.27

$$\bar{G} = \cos^2\phi\boldsymbol{a}_x + \frac{2r\cos\theta\sin\phi}{r\sin\theta}\bar{a}_y + (1-\cos^2\phi)\bar{a}_z$$

$$= \cos^2\phi\,\bar{a}_x + 2\cot\theta\sin\phi\,\bar{a}_y + \sin^2\phi\,\bar{a}_z$$

$$\begin{pmatrix} G_r \\ G_\theta \\ G_\phi \end{pmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \cos^2\phi \\ 2\cot\theta\sin\phi \\ \sin^2\phi \end{bmatrix}$$

$$G_r = \sin\theta\cos^3\phi + 2\cos\theta\sin^2\phi + \cos\theta\sin^2\phi$$
$$= \sin\theta\cos^3\phi + 3\cos\theta\sin^2\phi$$

$$G_\theta = \cos\theta\cos^3\phi + 2\cot\theta\cos\theta\sin^2\phi - \sin\theta\sin^2\phi$$

$$G_\phi = -\sin\phi\cos^2\phi + 2\cot\theta\sin\phi\cos\phi$$

$$\bar{G} = [\sin\theta\cos^3\phi + 3\cos\theta\sin^2\phi]\boldsymbol{a}_r$$
$$+[\cos\theta\cos^3\phi + 2\cot\theta\cos\theta\sin^2\phi - \sin\theta\sin^2\phi]\boldsymbol{a}_\theta$$
$$\underline{\underline{+\sin\phi\cos\phi(2\cot\theta - \cos\phi)\,\boldsymbol{a}_\phi}}$$

Prob. 2.28

(a) $\boldsymbol{J}_z = (\boldsymbol{J} \bullet \boldsymbol{a}_z)\boldsymbol{a}_z.$

$At\,(2,\ \ \pi/2,\ 3\pi/2),\ \boldsymbol{a}_z = \cos\theta \boldsymbol{a}_r - \sin\theta \boldsymbol{a}_\theta = -\boldsymbol{a}_\theta.$

$\boldsymbol{J}_z = -\cos 2\theta \sin\phi \boldsymbol{a}_\theta = -\cos\pi \sin(3\pi/2)\boldsymbol{a}_\theta = -\boldsymbol{a}_\theta.$

$(b)\boldsymbol{J}_\phi = \tan\dfrac{\theta}{2}\ln r\,\boldsymbol{a}_\phi = \tan\dfrac{\pi}{4}\ln 2\,\boldsymbol{a}_\phi = \ln 2\boldsymbol{a}_\phi = 0.6931\boldsymbol{a}_\phi.$

$(c)\boldsymbol{J}_t = \boldsymbol{J} - \boldsymbol{J}_n = \boldsymbol{J} - \boldsymbol{J}_r = -\boldsymbol{a}_\theta + \ln 2\,\boldsymbol{a}_\phi = \underline{\underline{-\boldsymbol{a}_\theta + 0.6931\boldsymbol{a}_\phi}}.$

$(d)\quad \bar{J}_P = (\bar{J} \bullet \bar{a}_x)\bar{a}_x$

$\bar{a}_x = \sin\theta\cos\phi\,\bar{a}_r + \cos\theta\cos\phi\,\bar{a}_\theta - \sin\phi\,\bar{a}_\phi = \bar{a}_\phi.$

At $(2,\ \pi/2,\ 3\pi/2),$

$\bar{J}_P = \underline{\underline{\ln 2\,\bar{a}_\phi.}}$

Prob. 2.29

$$\boldsymbol{H} \bullet \boldsymbol{a}_x = H_x$$

$$\begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho^2\cos\phi \\ -\rho\sin\phi \\ 0 \end{bmatrix}$$

$H_x = \rho^2\cos^2\phi + \rho\sin^2\phi$

At P, $\rho = 2, \phi = 60^o, z = -1$

$H_x = 4(1/4) + 2(3/4) = 1 + 1.5 = \underline{\underline{2.5}}$

Prob. 2.30

(a) $5 = \boldsymbol{r}\cdot\boldsymbol{a}_x + \boldsymbol{r}\cdot\boldsymbol{a}_y = x + y$ $\quad$ $\underline{\underline{\text{a plane}}}$

(b) $10 = |\boldsymbol{r} x \boldsymbol{a}_z| = \begin{vmatrix} x & y & z \\ 0 & 0 & 1 \end{vmatrix} = |\,y\boldsymbol{a}_x - x\boldsymbol{a}_y\,| = \sqrt{x^2 + y^2} = \rho$

$\underline{\underline{\text{a cylinder of infinite length}}}$

CHAPTER 3

P. E. 3.1

(a) $DH = \int_{\phi=45^\circ}^{\phi=60^\circ} r\sin\theta d\phi \Big|_{r=3,90^o} = 3(1)[\frac{\pi}{3}-\frac{\pi}{4}] = \frac{\pi}{4} = \underline{\underline{0.7854.}}$

(b) $FG = \int_{\theta=60^\circ}^{\theta=90^\circ} r d\theta \Big|_{r=5} = 5(\frac{\pi}{2}-\frac{\pi}{3}) = \frac{5\pi}{6} = \underline{\underline{2.618}}.$

(c)

$$AEHD = \int_{\theta=60^\circ}^{\theta=90^\circ}\int_{\phi=45^\circ}^{\phi=60^\circ} r^2 \sin\theta\, d\theta\, d\phi \Big|_{r=3} = 9(-\cos\theta)\Big|_{\theta=60^\circ}^{\theta=90^\circ} \phi\Big|_{\phi=45^\circ}^{\phi=60^\circ}$$

$$= 9(\frac{1}{2})(\frac{\pi}{12}) = \frac{3\pi}{8} = \underline{\underline{1.178}}.$$

(d)

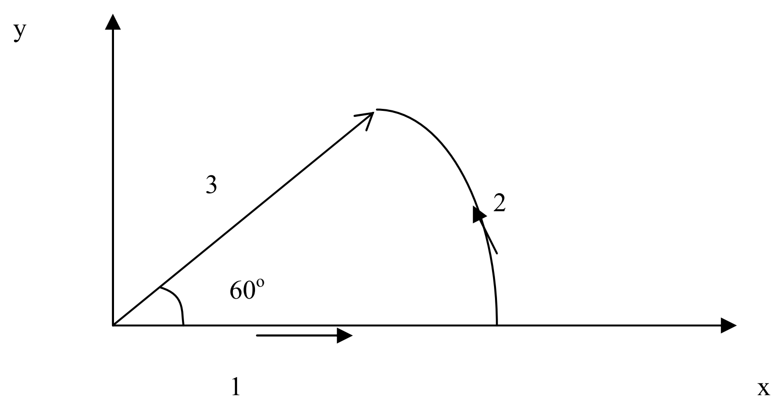
$$ABCD = \int_{r=3}^{r=5}\int_{\theta=60}^{\theta=90} r d\theta\, dr = \frac{r^2}{2}\Big|_{r=3}^{r=5}(\frac{\pi}{2}-\frac{\pi}{3}) = \frac{4\pi}{3} = \underline{\underline{4.189}}.$$

(e)

$$\text{Volume} = \int_{r=3}^{r=5}\int_{\phi=45^\circ}^{\phi=60^\circ}\int_{\theta=60}^{\theta=90} r^2\sin\theta dr\, d\theta\, d\phi = \frac{r^3}{3}\Big|_{r=3}^{r=5} (-\cos\theta)\Big|_{\theta=60^\circ}^{\theta=90^\circ} \phi\Big|_{\phi=45^\circ}^{\phi=60^\circ} = \frac{1}{3}(98)(\frac{1}{2})\frac{\pi}{12}$$

$$= \underline{\underline{\frac{49\pi}{36} = 4.276}}.$$

P.E. 3.2



$$\oint_L \boldsymbol{A}\bullet d\boldsymbol{l} = \left(\int_1 + \int_2 + \int_3\right) \boldsymbol{A}\bullet d\boldsymbol{l} = C_1 + C_2 + C_3$$

Along (1), $C_1 = \int \boldsymbol{A}\bullet d\boldsymbol{l} = \int_0^2 \rho\cos\phi\, d\rho\,|_{\phi=0} = \frac{\rho^2}{2}\Big|_0^2 = 2.$

Along (2), $d\boldsymbol{l} = \rho d\phi\, \boldsymbol{a}_\phi,\ \boldsymbol{A}\bullet d\boldsymbol{l} = 0,\quad C_2 = 0$

Along (3), $C_3 = \int_2^0 \rho\cos\phi\, d\rho_{\phi=60^\circ} = -\frac{\rho^2}{2}\Big|_0^2 \left(\frac{1}{2}\right) = -1$

$$\oint_l \boldsymbol{A}\bullet d\boldsymbol{l} = C_1 + C_2 + C_3 = 2 + 0 - 1 = \underline{\underline{1}}$$

P.E. 3.3

(a) $$\nabla U = \frac{\partial U}{\partial x}\boldsymbol{a}_x + \frac{\partial U}{\partial y}\boldsymbol{a}_y + \frac{\partial U}{\partial z}\boldsymbol{a}_z$$

$$= \underline{\underline{y(2x+z)\boldsymbol{a}_x + x(x+z)\boldsymbol{a}_y + xy\,\boldsymbol{a}_z}}$$

(b) $$\nabla V = \frac{\partial V}{\partial \rho}\boldsymbol{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\boldsymbol{a}_\phi + \frac{\partial V}{\partial z}\boldsymbol{a}_z$$

$$= \underline{\underline{(z\sin\phi + 2\rho)\boldsymbol{a}_\rho + \left(z\cos\phi - \frac{z^2}{\rho}\sin 2\phi\right)\boldsymbol{a}_\phi + (\rho\sin\phi + 2z\cos^2\phi)\boldsymbol{a}_z}}$$

(c)

$$\nabla f = \frac{\partial f}{\partial r}\boldsymbol{a}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\boldsymbol{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\boldsymbol{a}_\phi$$

$$= \left(\frac{\cos\theta\sin\phi}{r} + 2r\phi\right)\boldsymbol{a}_r - \frac{\sin\theta\sin\phi\ln r}{r}\boldsymbol{a}_\theta + \frac{(\cos\theta\cos\phi\ln r + r^2)}{r\sin\theta}\boldsymbol{a}_\phi$$

$$= \underline{\underline{\left(\frac{\cos\theta\sin\phi}{r} + 2r\phi\right)\boldsymbol{a}_r - \frac{\sin\theta\sin\phi\ln r}{r}\boldsymbol{a}_\theta + \left(\frac{\cot\theta\cos\phi\ln r}{r} + r\,\mathrm{cosec}\,\theta\right)\boldsymbol{a}_\phi}}$$

P.E. 3.4

$$\nabla\Phi = (y+z)\boldsymbol{a}_x + (x+z)\boldsymbol{a}_y + (x+y)\boldsymbol{a}_z$$

At $(1,2,3)$, $\nabla\Phi = \underline{\underline{(5,4,3)}}$

$$\nabla\Phi \bullet \boldsymbol{a}_l = (5,4,3)\bullet\frac{(2,2,1)}{3} = \frac{21}{3} = \underline{\underline{7}},$$

where $(2,2,1) = (3,4,4) - (1,2,3)$

P.E. 3.5

Let $f = x^2 y + z - 3, \quad g = x \log z - y^2 + 4,$

$$\nabla f = 2xy\,\boldsymbol{a}_x + x^2\,\boldsymbol{a}_y + \boldsymbol{a}_z,$$

$$\nabla g = \log z\,\boldsymbol{a}_x - 2y\,\boldsymbol{a}_y + \frac{x}{z}\boldsymbol{a}_z$$

At $P(-1, 2, 1)$,

$$\boldsymbol{n}_f = \pm\frac{\nabla f}{|\nabla f|} = \pm\frac{(-4\,\boldsymbol{a}_x + \boldsymbol{a}_y + \boldsymbol{a}_z)}{\sqrt{18}}, \qquad \boldsymbol{n}_g = \pm\frac{\nabla g}{|\nabla g|} = \pm\frac{(-4\,\boldsymbol{a}_y - \boldsymbol{a}_z)}{\sqrt{17}}$$

$$\cos\theta = \boldsymbol{n}_f \,.\, \boldsymbol{n}_g = \pm\frac{(-5)}{\sqrt{18\times 17}}$$

Take positive value to get acute angle.

$$\theta = \cos^{-1}\frac{5}{17.493} = \underline{\underline{73.39°}}$$

P.E. 3.6

$$(a)\quad \nabla\bullet\boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 0 + 4x + 0 = \underline{\underline{4x}}.$$

At $(1,-2,3)$, $\nabla\bullet\boldsymbol{A} = \underline{\underline{4}}.$

(b)

$$\nabla\bullet\boldsymbol{B} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho B_\rho) + \frac{1}{\rho}\frac{\partial B_\phi}{\partial\phi} + \frac{\partial B_z}{\partial\rho}$$

$$= \frac{1}{\rho}2\rho z\sin\phi - \frac{1}{\rho}3\rho z^2\sin\phi = 2z\sin\phi - 3z^2\sin\phi$$

$$= \underline{\underline{(2-3z)z\sin\phi}}.$$

$$At\,(5,\frac{\pi}{2},1), \quad \nabla\bullet\boldsymbol{B} = (2-3)(1) = \underline{\underline{-1}}.$$

(c)

$$\nabla\bullet\boldsymbol{C} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 C_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(C_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial C_\phi}{\partial\phi}$$

$$= \frac{1}{r^2}6r^2\cos\theta\cos\phi$$

$$= \underline{\underline{6\cos\theta\cos\phi}}$$

$$At\,(1,\frac{\pi}{6},\frac{\pi}{3}), \quad \nabla\bullet\boldsymbol{C} = 6\cos\frac{\pi}{6}\cos\frac{\pi}{3} = \underline{\underline{2.598.}}$$

P.E. 3.7 This is similar to Example 3.7.

$$\Psi = \oint_S \boldsymbol{D} \bullet d\boldsymbol{S} = \Psi_t + \Psi_b + \Psi_c$$

$\Psi_t = 0 = \Psi_b$ since $\boldsymbol{D}$ has no z-component

$$\Psi_c = \iint \rho^2 \cos^2\phi \rho d\phi dz = \rho^3 \int_{\phi=0}^{\phi=2\pi} \cos^2\phi d\phi \int_{z=0}^{z=1} dz \Big|_{\rho = 4}$$

$$= (4)^3 \pi (1) = 64\pi$$

$$\Psi = 0 + 0 + 64\pi = \underline{\underline{64\pi}}$$

By the divergence theorem,

$$\oint_S \boldsymbol{D} \bullet d\boldsymbol{S} = \oint_V \nabla \bullet \boldsymbol{D} dv$$

$$\nabla \bullet \boldsymbol{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho^3 \cos^2\phi) + \frac{1}{\rho}\frac{\partial}{\partial \phi} z \sin\phi + \frac{\partial A_z}{dz}$$

$$= 3\rho \cos^2\phi + \frac{z}{\rho}\cos\phi.$$

$$\Psi = \int_V \nabla \bullet \boldsymbol{D} dv = \int_V (3\rho \cos^2\phi + \frac{z}{\rho}\cos\phi)\rho d\phi dz d\rho$$

$$= 3\int_0^4 \rho^2 d\rho \int_0^{2\pi} \cos^2\phi\, d\phi \int_0^1 dz + \int_0^4 d\rho \int_0^{2\pi} \cos\phi d\phi \int_0^1 z dz$$

$$= 3(\frac{4^3}{3})\pi(1) = \underline{\underline{64\pi.}}$$

P.E. 3.8

(a)

$$\nabla \times \boldsymbol{A} = \boldsymbol{a}_x(1-0) + \boldsymbol{a}_y(y-0) + \boldsymbol{a}_z(4y-z)$$

$$= \underline{\underline{\boldsymbol{a}_x + y\boldsymbol{a}_y + (4y-z)\boldsymbol{a}_z}}$$

At $(1,-2,3), \nabla \times \boldsymbol{A} = \underline{\underline{\boldsymbol{a}_x - 2\boldsymbol{a}_y - 11\boldsymbol{a}_z}}$

(b)

$$\nabla\times\boldsymbol{B} = \boldsymbol{a}_\rho(0-6\rho z\cos\phi)+\boldsymbol{a}_\phi(\rho\sin\phi-0)+\boldsymbol{a}_z\frac{1}{\rho}(6\rho z^2\cos\phi-\rho z\cos\phi)$$

$$= \underline{\underline{-6\rho z\cos\phi\,\boldsymbol{a}_\rho+\rho\sin\phi\,\boldsymbol{a}_\phi+(6z-1)z\cos\phi\,\boldsymbol{a}_z}}$$

At $(5,\frac{\pi}{2},-1), \nabla\times\boldsymbol{B}=\underline{\underline{5\boldsymbol{a}_\phi}}$

(c)

$$\nabla\times\boldsymbol{C} = \boldsymbol{a}_r\frac{1}{r\sin\theta}(r^{-1/2}\cos\theta-0) + \frac{\boldsymbol{a}_\theta}{r}(-\frac{2r\cos\theta\sin\phi}{\sin\theta}-\frac{3}{2}r^{1/2})+\frac{\boldsymbol{a}_\phi}{r}(0+2r\sin\theta\cos\phi)$$

$$= \underline{\underline{r^{-1/2}\cot\theta\,\boldsymbol{a}_r-(2\cot\theta\sin\phi+\frac{3}{2}r^{-1/2})\boldsymbol{a}_\theta+2\sin\theta\cos\phi\,\boldsymbol{a}_\phi}}$$

At $(1,\frac{\pi}{6},\frac{\pi}{3})$, $\nabla\times\boldsymbol{C}=\underline{\underline{1.732\boldsymbol{a}_r-4.5\boldsymbol{a}_\theta+0.5\boldsymbol{a}_\phi}}$

P.E. 3.9

$$\oint_L \boldsymbol{A}\bullet d\boldsymbol{l}=\int_S(\nabla\times\boldsymbol{A})\bullet d\boldsymbol{S}$$

But $(\nabla\times\boldsymbol{A})=\sin\phi\,\boldsymbol{a}_z+\frac{z\cos\phi}{\rho}\boldsymbol{a}_\rho$ and $dS=\rho d\phi d\rho\,\boldsymbol{a}_z$

$$\int_S(\nabla\times\boldsymbol{A})\bullet d\boldsymbol{S}=\iint\rho\sin\phi\;d\phi\,d\rho$$

$$= \frac{\rho^2}{2}\Big|_0^2(-\cos\phi)\Big|_0^{60^\circ}$$

$$= 2(-\frac{1}{2}+1) = \underline{\underline{1.}}$$

P.E. 3.10

$$\nabla \times \nabla V = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ \dfrac{\partial}{\partial \boldsymbol{x}} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial V}{\partial \boldsymbol{x}} & \dfrac{\partial V}{\partial y} & \dfrac{\partial V}{\partial z} \end{vmatrix} =$$

$$= \left(\frac{\partial^2 V}{\partial y \partial z} - \frac{\partial^2 V}{\partial y \partial z}\right)\boldsymbol{a}_x + \left(\frac{\partial^2 V}{\partial x \partial z} - \frac{\partial^2 V}{\partial z \partial x}\right)\boldsymbol{a}_y + \left(\frac{\partial^2 V}{\partial x \partial y} - \frac{\partial^2 V}{\partial y \partial x}\right)\boldsymbol{a}_z = 0$$

P.E. 3.11

(a)

$$\nabla^2 U = \frac{\partial}{\partial x}(2xy + yz) + \frac{\partial}{\partial y}(x^2 + xz) + \frac{\partial}{\partial z}(xy)$$

$$= \underline{\underline{2y}}.$$

(b)

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\rho(z\sin\phi + 2\rho) + \frac{1}{\rho^2}(-\rho z\sin\phi - 2z^2\frac{\partial}{\partial \rho}\sin\phi\cos\phi) + \frac{\partial}{\partial z}(\rho\sin\phi + 2z\cos^2\phi)$$

$$= \frac{1}{\rho}(z\sin\phi + 4\rho) - \frac{1}{\rho^2}(z\rho\sin\phi + 2z^2\cos 2\phi) + 2\cos^2\phi.$$

$$= \underline{\underline{4 + 2\cos^2\phi - \frac{2z^2}{\rho^2}\cos 2\phi}}.$$

(c)

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}[r^2\frac{1}{r}\cos\theta\sin\phi + 2r^3\phi] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}[-\sin^2\theta\sin\phi\ln r]$$

$$+ \frac{1}{r^2\sin^2\theta}[-\cos\theta\sin\phi\ln r]$$

$$= \underline{\underline{\frac{1}{r^2}\cos\theta\sin\phi(1 - 2\ln r - \csc^2\theta\ln r) + 6\phi}}$$

P.E. 3.12

If $\boldsymbol{B}$ is conservative, $\nabla\times\boldsymbol{B}=\boldsymbol{0}$ must be satisfied.

$$\nabla\times\boldsymbol{B}=\begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ \dfrac{\partial}{\partial \boldsymbol{x}} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y+z\cos xz & x & x\cos xz \end{vmatrix}$$

$= 0\,\boldsymbol{a}_x+(\cos xz - xz\sin xz-\cos xz+xz\sin xz)\,\boldsymbol{a}_y+(1-1)\,\boldsymbol{a}_z=\boldsymbol{0}$

Hence $\mathbf{B}$ is a conservative field.

P.E. 3.13

$\psi=\int_S \boldsymbol{B}\bullet d\boldsymbol{S},\quad d\boldsymbol{S}=dxdy\boldsymbol{a}_z$

$$\psi=\int_{y=0}^{2}\int_{x=0}^{1}3x^2ydxdy=3\frac{x^3}{3}\Bigg|_0^1\frac{y^2}{2}\Bigg|_0^2=(1)(2)=\underline{\underline{2}}$$

P.E. 3.14

In Cartesian coordinates $\Rightarrow (\nabla\cdot\mathbf{A})_P=(6x)_P=6\times 3=18.$

For representing this vector field in cylindrical coordinates, we use eq. (2.13):

$$\begin{bmatrix}A_\rho\\A_\emptyset\\A_z\end{bmatrix}=\begin{bmatrix}\cos\emptyset & \sin\emptyset & 0\\ -\sin\emptyset & \cos\emptyset & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}3x^2+2y\\ x\\ 0\end{bmatrix}$$

$$\mathbf{A}=\big(\cos\emptyset\,(3x^2+2y)+\sin\emptyset\,(x)\big)\mathbf{a}_\rho+\big((3x^2+2y)(-\sin\emptyset)+(\cos\emptyset)x\big)\mathbf{a}_\emptyset$$

Using the following relationships, from eq. (2.8), $x=\rho\cos\emptyset$, $y=\sin\emptyset$, we obtain

$$=\Big(\cos\emptyset\big(3(\rho\cos\emptyset)^2+2(\rho\sin\emptyset)\big)+(\sin\emptyset)(\rho\cos\emptyset)\Big)\,\mathbf{a}_\rho$$
$$+\Big(\Big(\big(3(\rho\cos\emptyset)^2+2(\rho\sin\emptyset)\big)(-\sin\emptyset)\Big)+(\cos\emptyset)(\rho\cos\emptyset)\Big)\,\mathbf{a}_\emptyset$$

$$\mathbf{A}=(3\rho^2\cos^3\emptyset+2\rho\sin\emptyset\cos\emptyset+\rho\sin\emptyset\cos\emptyset)\mathbf{a}_\rho$$
$$+(-3\rho^2\cos^2\emptyset\sin\emptyset-2\rho\sin^2\emptyset+\rho\cos^2\emptyset)\mathbf{a}_\emptyset$$

P in cylindrical coordinates is $(5, 53.13°, 5)$

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\emptyset}{\partial \emptyset} + \frac{\partial A_z}{\partial z}$$

$$= \frac{1}{\rho}(9\rho^2\cos^3\emptyset + 6\rho\sin\emptyset\cos\emptyset)$$

$$+ \frac{1}{\rho}(6\rho^2\cos\emptyset\sin^2\emptyset - (3\rho^2\cos^3\emptyset)$$

$$- 4\rho\sin\emptyset\cos\emptyset - 2\rho\sin\emptyset\cos\emptyset)$$

$= 18.00$ (after substituting the values of ρ and $\emptyset$)

For representing the given vector field in spherical coordinates we use eq. (2.27):

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\emptyset \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\emptyset & \sin\theta\sin\emptyset & \cos\theta \\ \cos\theta\cos\emptyset & \cos\theta\sin\emptyset & -\sin\theta \\ -\sin\emptyset & \cos\emptyset & 0 \end{bmatrix} \begin{bmatrix} 3x^2 + 2y \\ x \\ 0 \end{bmatrix}$$

$$\mathbf{A} = \big((\sin\theta\cos\emptyset)(3x^2 + 2y) + (\sin\theta\sin\emptyset)x\big)\mathbf{a}_r$$
$$+ \big((\cos\theta\cos\emptyset)(3x^2 + 2y) + (\cos\theta\sin\emptyset)x\big)\mathbf{a}_\theta$$
$$+ \Big(\big((-\sin\emptyset)(3x^2 + 2y)\big) + \big((\cos\emptyset)x\big)\Big)\mathbf{a}_\emptyset$$

Using the following relationships, from eq. (2.22), $x = r\sin\theta\cos\emptyset$, $y = r\sin\theta\sin\emptyset$, we obtain

$$\mathbf{A} = \big((3r^2\sin^3\theta\cos^3\emptyset) + (3r\sin^2\theta\cos\emptyset\sin\emptyset)\big)\mathbf{a}_r$$
$$+ \big((3r^2\cos\theta\sin^2\theta\cos^3\emptyset) + (2r\sin\theta\cos\emptyset\cos\theta\sin\emptyset)$$
$$+ (r\sin\theta\cos\emptyset\cos\theta\sin\emptyset)\big)\mathbf{a}_\theta$$
$$+ \Big((-3r^2\sin^2\theta\cos^2\emptyset\sin\emptyset) + (-2r\sin\theta\sin^2\emptyset)$$
$$+ (r\sin\theta\cos^2\emptyset)\Big)\mathbf{a}_\emptyset$$

P in spherical system is $(7.071, 45^\circ, 53.13^\circ)$

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}A_\theta\sin\theta + \frac{1}{r\sin\theta}\frac{\partial A_\emptyset}{\partial\emptyset}$$

$\frac{1}{r^2}(12r^3\sin^3\theta\cos^3\emptyset + 9r^2\sin^2\theta\cos\emptyset\sin\emptyset) + \frac{1}{r\sin\theta}(-3r^2\sin^4\theta\cos^3\emptyset + 9r^2\cos^2\theta\sin^2\theta\cos^3\emptyset - 3r\sin^3\theta\cos\emptyset\sin\emptyset + 6r\sin\theta\cos^2\theta\cos\emptyset\sin\emptyset) + \frac{1}{r\sin\theta}(6r^2\sin^2\theta\sin^2\emptyset\cos\emptyset - 3r^2\sin^2\theta\cos^3\emptyset - 4r\sin\theta\cos\emptyset\sin\emptyset - 2r\sin\theta\cos\emptyset\sin\emptyset) = 18$ (after substituting the values of r, θ and $\emptyset$)

Note: From this example, it is clear that the divergence of a vector field is the same irrespective of the coordinate system used.

P.E. 3.15

$$\nabla \boldsymbol{F} = \boldsymbol{a}_x - 2\boldsymbol{a}_y + \boldsymbol{a}_z$$

$$a_n = \frac{\nabla \boldsymbol{F}}{|\nabla \boldsymbol{F}|} = \frac{\boldsymbol{a}_x - 2\boldsymbol{a}_y + \boldsymbol{a}_z}{\sqrt{1+4+1}} = \underline{\underline{0.4082\boldsymbol{a}_x - 0.8165\boldsymbol{a}_y + 0.4082\boldsymbol{a}_z}}$$

Prob. 3.1

(a)

$$dS = \rho d\phi dz$$

$$S = \int dS = \rho \iint d\phi dz = 2\int_0^5 dz \int_{\pi/3}^{\pi/2} d\phi = 2(5)[\frac{\pi}{2} - \frac{\pi}{3}] = \frac{10\pi}{6} = \underline{\underline{5.236}}$$

(b)

In cylindrical, $dS = \rho\, d\rho\, d\phi$

$$S = \int dS = \int_1^3 \rho\, d\rho \int_0^{\pi/4} d\phi = \frac{\pi}{4}(\frac{\rho^2}{2})\Big|_1^3 = \underline{\underline{3.142}}$$

(c) In spherical, $dS = r^2 \sin\theta\, d\phi\, d\theta$

$$S = \int dS = 100 \int_{\pi/4}^{2\pi/3} \sin\theta\, d\theta \int_0^{2\pi} d\phi = 100(2\pi)(-\cos\theta)\Big|_{\pi/4}^{2\pi/3} = 200\pi(0.5 + 0.7071) = \underline{\underline{758.4}}$$

(d)

$$dS = r\, dr\, d\theta$$

$$S = \int dS = \int_0^4 r dr \int_{\pi/3}^{\pi/2} d\theta = \frac{r^2}{2}\Big|_0^4 (\frac{\pi}{2} - \frac{\pi}{3}) = \frac{8\pi}{6} = \underline{\underline{4.189}}$$

Prob. 3.2

(a)

$$dl = \rho d\phi; \qquad \rho = 3$$

$$L = \int dl = 3\int_{\pi/4}^{\pi/2} d\phi = 3(\frac{\pi}{2} - \frac{\pi}{4}) = \frac{3\pi}{4} = \underline{\underline{2.356}}$$

(b)

$$dl = r\sin\theta d\phi; \qquad r = 1, \quad \theta = 30^\circ;$$

$$L = \int dl = r\sin\theta \int_0^{\pi/3} d\phi = (1)\sin 30^\circ [(\frac{\pi}{3}) - 0] = \underline{\underline{0.5236.}}$$

(c)

$$dl = r d\theta$$

$$L = \int dl = r\int_{\pi/6}^{\pi/2} d\theta = 4(\frac{\pi}{2} - \frac{\pi}{6}) = \frac{4\pi}{3} = \underline{\underline{4.189}}$$

Prob. 3.3

$$S = \int dS = \int_{z=0}^{2} \int_{\phi=\pi/4}^{\pi/2} \rho d\phi dz \Big|_{\rho=10}$$

$$= 10\int_0^2 dz \int_{\pi/4}^{\pi/2} d\phi = 10(2)(\pi/2 - \pi/4) = 5\pi = \underline{\underline{15.71}}$$

Prob. 3.4

$$L = \int_L dl = \int_{\phi=0}^{\pi/6} \rho d\phi \Big|_{\rho=4} = 4(\pi/6) = \underline{\underline{2.094}}$$

Prob. 3.5

(a) $dV = dxdydz$

$$V = \int dxdydz = \int_0^1 dx \int_1^2 dy \int_{-3}^{3} dz = (1)\ (2-1)(3--3) = \underline{\underline{6}}$$

(b) $dV = \rho d\phi d\rho dz$

$$V = \int_2^5 \rho d\rho \int_{-1}^{4} dz \int_{\pi/3}^{\pi} d\phi = \frac{\rho^2}{2}\Big|_2^5 (4--1)(\pi - \frac{\pi}{3}) = \frac{1}{2}(25-4)(5)(\frac{2\pi}{3}) = 35\pi = \underline{\underline{110}}$$

(c) $dV = r^2 \sin\theta \, dr d\theta d\phi$

$$V = \int_1^3 r^2 dr \int_{\pi/2}^{2\pi/3} \sin\theta \int_{\pi/6}^{\pi/2} d\phi = \frac{r^3}{3}\Big|_1^3 (-\cos\theta)\Big|_{\pi/2}^{\pi/3} (\frac{\pi}{2} - \frac{\pi}{6})$$

$$= \frac{1}{3}(27-1)(\frac{1}{2})(\frac{\pi}{3}) = \frac{26\pi}{18} = \underline{\underline{4.538}}$$

Prob. 3.6

$$\int_L \boldsymbol{H} \bullet d\boldsymbol{l} = \int (x^2 dx + y^2 \ dy)$$

But on L, $y = x^2 \quad dy = 2xdx$

$$\int_L \boldsymbol{H} \bullet d\boldsymbol{l} = \int_0^1 (x^2 + x^4.2x)dx = \frac{x^3}{3} + 2\frac{x^6}{6}\Big|_0^1 = \frac{1}{3} + \frac{1}{3} = \underline{\underline{0.6667}}$$

Prob. 3.7

(a)

$$\int \boldsymbol{F} \bullet d\boldsymbol{l} = \int_{y=0}^{1} (x^2 - z^2)dy\Big|_{x=0,z=0} + \int_{x=0}^{x=2} 2xydx\Big|_{y=1,z=0} + \int_{z=0}^{z=3} (-3xz^2)dz\Big|_{x=2,y=1}$$

$$= 0 + 2(1)\frac{x^2}{2}\Big|_0^2 - 3(2)\frac{z^3}{3}\Big|_0^3$$

$$= 0 + 4 - 54 = \underline{\underline{-50}}$$

(b)

Let $x = 2t. \quad y = t, \quad z = 3t$

$dx = 2dt, \quad dy = dt, \quad dz = 3dt;$

$$\int \boldsymbol{F} \bullet d\boldsymbol{l} = \int_0^1 (8t^2 - 5t^2 - 162t^3) \ dt$$

$$= (t^3 - 40.5t^4)\Big|_0^1 = \underline{\underline{-39.5}}$$

Prob. 3.8

$$W = \int_L \boldsymbol{F} \bullet d\boldsymbol{l} = \int_{\phi=0}^{\pi/4} z\rho d\phi \;\Big|_{z=0,\rho=2} + \int_{z=0}^{3} \rho\cos\phi dz \Big|_{\rho=2,\phi=\pi/4}$$

$$= 0 + 2\cos(\pi/4)(3) = 6\cos 45^o \;\underline{\underline{= 4.243\text{ J}}}$$

Prob. 3.9

$$\int \boldsymbol{H} \bullet d\boldsymbol{l} = \int_{x=1}^{0} (x-y)dx \Big|_{y=0,\, z=0} + \int_{z=0}^{1} 5yzdz \Big|_{x=0,\, y=0}$$

$$+ \int (x^2 + zy)dy + 5yzdz \Big|_{x=0,\, z=1-y/2}$$

$$= \int_1^0 xdx + \int_0^2 (y - \frac{y^2}{2})dy + \int_1^0 (10z - 10z^2)dz$$

$$= \underline{\underline{-1.5}}$$

Prob. 3.10

Method 1:

$$\oint_L \boldsymbol{B} \cdot d\boldsymbol{l} = -\int_{y=0}^{1} yzdy \Big|_{z=0} + \int_{z=0}^{1} xzdz \Big|_{x=1} + \int (-yzdy + xzdz) \Big|_{x=1}$$

But $z = y \quad \longrightarrow \quad dz = dy$ on the last segment (or integral).

$$\oint_L \boldsymbol{B} \cdot d\boldsymbol{l} = 0 + \frac{z^2}{2}\Big|_0^1 + \int_{y=1}^{0} (-y^2 + y)dy = \frac{1}{2} + (-\frac{y^3}{3} + \frac{y^2}{2})\Big|_1^0$$

$$= \frac{1}{2} + \frac{1}{3} - \frac{1}{2} = \frac{1}{3} = \underline{\underline{0.333}}$$

Method 2:

$$\oint_L \boldsymbol{B} \cdot d\boldsymbol{l} = \int_S \nabla \times \boldsymbol{B} \bullet d\boldsymbol{S}$$

$$\nabla \times \boldsymbol{B} = \begin{vmatrix} \frac{\partial}{\partial \boldsymbol{x}} & \frac{\partial}{\partial \boldsymbol{y}} & \frac{\partial}{\partial \boldsymbol{z}} \\ xy & -yz & xz \end{vmatrix} = y\boldsymbol{a}_x - z\boldsymbol{a}_y - x\boldsymbol{a}_z, \qquad d\boldsymbol{S} = dydz\boldsymbol{a}_x$$

$$\int_S \nabla \times \boldsymbol{B} \bullet d\boldsymbol{S} = \int_{y=0}^{1}\int_{z=0}^{y} ydzdy = \int_0^1 y^2 dy = \frac{y^3}{3}\Big|_0^1 = \frac{1}{3} = \underline{\underline{0.333}}$$

Prob. 3.11

$$\psi = \int_S \boldsymbol{A} \bullet d\boldsymbol{S} = \int\int zdxdz = \int_0^1 dx \int_0^2 zdz = (1)\frac{z^2}{2}\bigg|_0^2 = \underline{\underline{2}}$$

Prob. 3.12

(a) dv = dxdydz

$$\int_v xydv = \int_{z=0}^{2}\int_{y=0}^{1}\int_{z=0}^{1} xydxdydz = \int_0^1 xdx\int_0^1 ydy\int_0^z dz$$

$$= \frac{x^2}{2}\bigg|_0^1 \frac{y^2}{2}\bigg|_0^1 z\bigg|_0^2 = (1/2)(1/2)(2) = \underline{\underline{0.5}}$$

(b)

$$dv = \rho d\rho d\phi dz$$

$$\int_v \rho zdv = \int_{\phi=0}^{\pi}\int_{z=0}^{2}\int_{\rho=1}^{3} \rho z \;\rho d\rho d\phi dz = \int_1^3 \rho^2 d\rho \int_0^2 zdz \int_0^\pi d\phi$$

$$= \frac{\rho^3}{3}\bigg|_1^3 \frac{z^2}{2}\bigg|_0^2 (\pi) = (9 - \frac{1}{3})(2\pi) = \underline{\underline{54.45}}$$

Prob. 3.13

Let $I = \int_v \boldsymbol{A}dv = \int_v r\sin\phi\boldsymbol{a}_r dv$

$$\boldsymbol{a}_r = \sin\theta\cos\phi\boldsymbol{a}_x + \sin\theta\sin\phi\boldsymbol{a}_y + \cos\theta\boldsymbol{a}_z$$

$$A_r = r\sin\theta\sin\phi\cos\phi\boldsymbol{a}_x + r\sin\theta\sin^2\phi\boldsymbol{a}_y + r\cos\theta\sin\phi\boldsymbol{a}_z$$

$$dv = r^2\sin\theta d\theta d\phi dr$$

$$I = \int_v \boldsymbol{A} dv = \iiint r^3 \sin^2\theta \sin\phi \cos\phi \, d\theta d\phi dr \boldsymbol{a}_x$$
$$+ \iiint r^3 \sin^2\theta \sin^2\phi \, d\theta d\phi dr \boldsymbol{a}_y$$
$$+ \iiint r^3 \sin\theta \cos\theta \sin\phi \, d\theta d\phi dr \boldsymbol{a}_z$$
$$= \boldsymbol{a}_x \int_{r=1}^{1} r^3 dr \int_0^{\pi} \sin^2\theta d\theta \int_0^{2\pi} \sin\phi \cos\phi d\phi$$
$$+ \boldsymbol{a}_y \int_{r=1}^{1} r^3 dr \int_0^{\pi} \sin^2\theta d\theta \int_0^{2\pi} \sin^2\phi d\phi$$
$$+ \boldsymbol{a}_z \int_{r=1}^{1} r^3 dr \int_0^{\pi} \sin\theta \cos\theta d\theta \int_0^{2\pi} \sin\phi d\phi$$
$$= 0\boldsymbol{a}_x + 0\boldsymbol{a}_z + \boldsymbol{a}_y \left.\frac{r^4}{4}\right|_0^1 \int_0^{\pi} \frac{1}{2}(1-\cos 2\theta) d\theta \int_0^{2\pi} \frac{1}{2}(1-\cos 2\phi) d\phi$$
$$= \frac{\boldsymbol{a}_y}{4}(\pi/2)(\pi) = \frac{\pi^2}{8}\boldsymbol{a}_y = \underline{\underline{1.234\boldsymbol{a}_y}}$$

Prob. 3.14

(a) $$\nabla V_1 = \frac{\partial V_1}{\partial x}\boldsymbol{a}_x + \frac{\partial V_1}{\partial y}\boldsymbol{a}_y + \frac{\partial V_1}{\partial z}\boldsymbol{a}_z$$
$$= \underline{\underline{(6y-2z)\boldsymbol{a}_x + 6x\boldsymbol{a}_y + (1-2x)\boldsymbol{a}_z}}$$

(b) $$\nabla V_2 = \frac{\partial V_2}{\partial \rho}\boldsymbol{a}_\rho + \frac{1}{\rho}\frac{\partial V_2}{\partial \phi}\boldsymbol{a}_\phi + \frac{\partial V_2}{\partial z}\boldsymbol{a}_z$$
$$= \underline{\underline{(10\cos\phi - z)\boldsymbol{a}_\rho - 10\sin\phi \boldsymbol{a}_\phi - \rho \boldsymbol{a}_z}}$$

(c) $$\nabla V_3 = \frac{\partial V_3}{\partial r}\boldsymbol{a}_r + \frac{1}{r}\frac{\partial V_3}{\partial \theta}\boldsymbol{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V_3}{\partial \phi}\boldsymbol{a}_\phi$$
$$= -\frac{2}{r^2}\cos\phi \boldsymbol{a}_r + 0 + \frac{1}{r\sin\theta}\left(-\frac{2}{r}\sin\phi\right)\boldsymbol{a}_\phi$$
$$= \underline{\underline{-\frac{2}{r^2}\cos\phi \boldsymbol{a}_r - \frac{2\sin\phi}{r^2\sin\theta}\boldsymbol{a}_\phi}}$$

Prob. 3.15

(a)
$$\nabla U = \frac{\partial U}{\partial x}\boldsymbol{a}_x + \frac{\partial U}{\partial y}\boldsymbol{a}_y + \frac{\partial U}{\partial z}\boldsymbol{a}_z$$
$$= \underline{\underline{e^{x+2y}\cosh z\boldsymbol{a}_x + 2e^{x+2y}\cosh z\boldsymbol{a}_y + e^{x+2y}\sinh z\boldsymbol{a}_z}}$$

(b)
$$\nabla T = \frac{\partial T}{\partial \rho}\boldsymbol{a}_\rho + \frac{1}{\rho}\frac{\partial T}{\partial \phi}\boldsymbol{a}_\phi + \frac{\partial T}{\partial z}\boldsymbol{a}_z$$
$$= \underline{\underline{-\frac{3z}{\rho^2}\cos\phi\boldsymbol{a}_\rho - \frac{3z}{\rho^2}\sin\phi\boldsymbol{a}_\phi + \frac{3}{\rho}\cos\phi\boldsymbol{a}_z}}$$

(c)
$$\nabla W = \frac{\partial W}{\partial r}\boldsymbol{a}_r + \frac{1}{r}\frac{\partial W}{\partial \theta}\boldsymbol{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial W}{\partial \phi}\boldsymbol{a}_\phi$$
$$= \underline{\underline{\left(-\frac{5\cos\theta}{r^2} + 4r\sin\phi\right)\boldsymbol{a}_r + \left[2r^2\cos\phi - \frac{5\sin\phi}{r}\right]\frac{1}{r\sin\theta}\boldsymbol{a}_\phi}}$$

Prob. 3.16

$$r = \sqrt{x^2+y^2+z^2}, \qquad r^n = (x^2+y^2+z^2)^{n/2}$$

Method 1:
$$\nabla r^n = \frac{\partial r^n}{\partial x}\boldsymbol{a}_x + \frac{\partial r^n}{\partial y}\boldsymbol{a}_y + \frac{\partial r^n}{\partial z}\boldsymbol{a}_z = \frac{n}{2}(x^2+y^2+z^2)^{n/2-1}(2x)\boldsymbol{a}_x + \cdots$$
$$= n(x^2+y^2+z^2)^{n-2/2}(x\boldsymbol{a}_x + y\boldsymbol{a}_y + z\boldsymbol{a}_z) = \underline{\underline{nr^{n-2}\boldsymbol{r}}}$$

Method 2:
$$\nabla r^n = \frac{\partial r^n}{\partial r}\boldsymbol{a}_r = nr^{n-1}\frac{\boldsymbol{r}}{r} = nr^{n-2}\boldsymbol{r}$$

Prob. 3.17

$$\nabla T = 2x\,\boldsymbol{a}_x + 2y\,\boldsymbol{a}_y - \boldsymbol{a}_z$$

At $(1,1,2), \nabla T = (2,2,-1)$. The mosquito should move in the direction of

$$\underline{\underline{2\,\boldsymbol{a}_x + 2\,\boldsymbol{a}_y - \boldsymbol{a}_z}}$$

Prob. 3.18

Method 1:

$$\nabla T = \frac{\partial T}{\partial r}\boldsymbol{a}_r + \frac{1}{r}\frac{\partial T}{\partial \theta}\boldsymbol{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\boldsymbol{a}_\phi$$

$$= \sin\theta\cos\phi\boldsymbol{a}_r + \cos\theta\cos\phi\boldsymbol{a}_\theta - \sin\phi\boldsymbol{a}_\phi$$

At P, $r = 2, \theta = 60^o, \phi = 30^o$

$$\nabla T = \sin 60^o \cos 30^o \boldsymbol{a}_r + \cos 60^o \cos 30^o \boldsymbol{a}_\theta - \sin 30^o \boldsymbol{a}_\phi$$

$$= 0.75\boldsymbol{a}_r - 0.433\boldsymbol{a}_\theta - 0.5\boldsymbol{a}_\phi$$

$$|\nabla T| = \sqrt{0.75^2 + 0.433^2 + 0.5^2} = 1$$

The magnitude of T is 1 and its direction is along ∇T.

Method 2:

$$T = r\sin\theta\cos\phi = x$$

$$\nabla T = \boldsymbol{a}_x$$

$$|\nabla T| = 1$$

Prob. 3.19

$$\nabla f = \frac{\partial f}{\partial x}\boldsymbol{a}_x + \frac{\partial f}{\partial y}\boldsymbol{a}_y + \frac{\partial f}{\partial z}\boldsymbol{a}_z = (2xy - 2y^2)\boldsymbol{a}_x + (x^2 - 4xy)\boldsymbol{a}_y + 3z^2\boldsymbol{a}_z$$

At point (2,4,-3), $x = 2, y = 4, z = -3$

$$\nabla f = (16-32)\boldsymbol{a}_x + (4-32)\boldsymbol{a}_y + 27\boldsymbol{a}_z = -16\boldsymbol{a}_x - 28\boldsymbol{a}_y + 27\boldsymbol{a}_z$$

$$\boldsymbol{a} = \frac{\boldsymbol{a}_x + 2\boldsymbol{a}_y - \boldsymbol{a}_z}{\sqrt{1+4+1}} = \frac{1}{\sqrt{6}}(1,2,-1)$$

The directional derivative is

$$\nabla f \bullet \boldsymbol{a} = (-16,-28,27)\bullet\frac{1}{\sqrt{6}}(1,2,-1) = -\frac{99}{\sqrt{6}} = \underline{\underline{-40.42}}$$

Prob. 3.20

(a) Let f = ax + by + cz – d = 0

$$\nabla f = a\boldsymbol{a}_x + b\boldsymbol{a}_y + c\boldsymbol{a}_z$$

$$\boldsymbol{a}_{n1} = \frac{\nabla f}{|\nabla f|} = \frac{a\boldsymbol{a}_x + b\boldsymbol{a}_y + c\boldsymbol{a}_z}{\sqrt{a^2+b^2+c^2}}$$

Let $g = \alpha x + \beta y + \gamma z - \delta$

$$\boldsymbol{a}_{n2} = \frac{\nabla g}{|\nabla g|} = \frac{\alpha\boldsymbol{a}_x + \beta\boldsymbol{a}_y + \gamma\boldsymbol{a}_z}{\sqrt{\alpha^2+\beta^2+\gamma^2}}$$

$$\cos\theta = \boldsymbol{a}_{n1}\bullet\boldsymbol{a}_{n2} = \frac{a\alpha + b\beta + c\gamma}{\sqrt{a^2+b^2+c^2}\sqrt{\alpha^2+\beta^2+\gamma^2}}$$

$$\theta = \cos^{-1}\frac{a\alpha + b\beta + c\gamma}{\sqrt{(a^2+b^2+c^2)(\alpha^2+\beta^2+\gamma^2)}}$$

(b) $a = 1, b = 2, c = 3$

$\alpha = 1, \beta = 1, \gamma = 0$

$$\theta = \cos^{-1}\frac{1+2+0}{\sqrt{(1^2+2^2+3^2)(1^2+1^2+0^2)}} = \cos^{-1}\frac{3}{\sqrt{28}} = \cos^{-1} 0.5669 = \underline{\underline{55.46^o}}$$

Prob. 3.21

(a) $\nabla\cdot\boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \underline{\underline{3y - x}}$

(b)
$$\nabla\cdot\boldsymbol{B} = \frac{1}{\rho}2\rho z^2 + \frac{1}{\rho}\rho 2\sin\phi\cos\phi + 2\rho\sin^2\phi$$
$$= \underline{\underline{2z^2 + \sin 2\phi + 2\rho\sin^2\phi}}$$

(c) $\nabla\cdot\boldsymbol{C} = \frac{1}{r^2}3r^2 + 0 = \underline{\underline{3}}$

Prob. 3.22

$$\nabla\bullet\boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 2xy + 0 + 2y = 2y(1+x)$$

(a) At $(-3, 4, 2), x = -3, y = 4$

$$\nabla\bullet\boldsymbol{A} = 2(4)(1-3) = \underline{\underline{-16}}$$

$$\nabla\bullet \boldsymbol{B} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho B_\rho\right) + \frac{1}{\rho}\frac{\partial B_\phi}{\partial\phi} + \frac{\partial B_z}{\partial z} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(3\rho^2\sin\phi\right) + 0 + 8z\cos^2\phi$$

(b) $= 6\sin\phi + 8z\cos^2\phi$

At $(5, 30^o, 1), z = 1, \phi = 30^o$

$\nabla\bullet \boldsymbol{B} = 6\sin 30^o + 8(1)\cos^2 30^o = 3 + 6 = \underline{\underline{9}}$

$$\nabla\bullet \boldsymbol{C} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 C_r) + 0 + \frac{1}{r\sin\theta}\frac{\partial C_\phi}{\partial\phi} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^4\cos\theta) + 0 = 4r\cos\theta$$

(c) At $(2, \pi/3, \pi/2), r = 2, \theta = \pi/3$

$\nabla\bullet \boldsymbol{C} = 4(2)\cos(\pi/3) = \underline{\underline{4}}$

Prob. 3.23

$$\nabla\bullet \boldsymbol{H} = k\,\nabla\bullet\nabla T = k\,\nabla^2 T$$

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 50\sin\frac{\pi x}{2}\cosh\frac{\pi y}{2}\left(-\frac{\pi^2}{4} + \frac{\pi^2}{4}\right) = 0$$

Hence, $\nabla\bullet \boldsymbol{H} = 0$

Prob. 3.24

We convert A to cylindrical coordinates; only the ρ-component is needed.

$$A_\rho = A_x\cos\phi + A_y\sin\phi = 2x\cos\phi - z^2\sin\phi$$

But $x = \rho\cos\phi$,

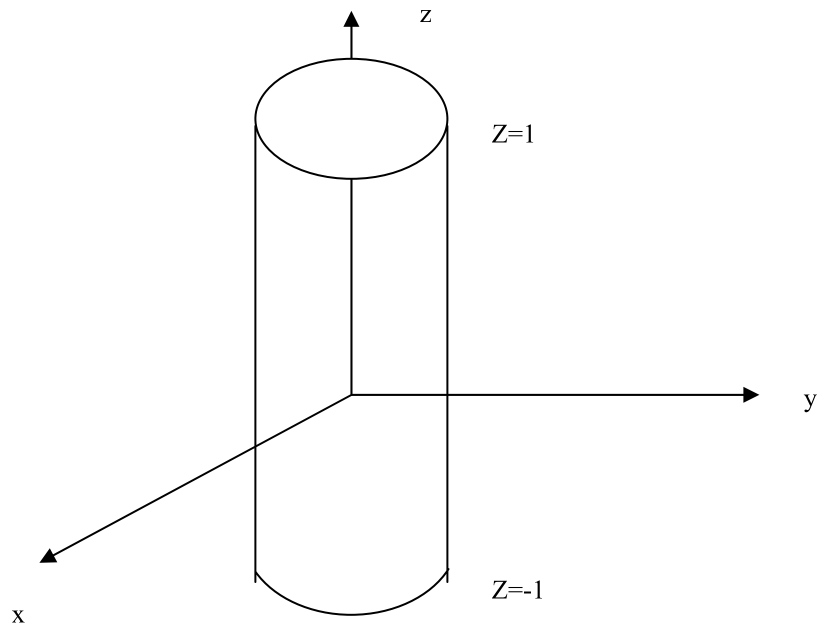
$$A_\rho = 2\rho\cos^2\phi - z^2\sin\phi$$

$$\Psi = \int_S \boldsymbol{A}\cdot d\boldsymbol{S} = \iint A_\rho \rho\, d\phi\, dz = \iint\left[2\rho^2\cos^2\phi - \rho z^2\sin\phi\right]d\phi\, dz$$

$$= 2(2)^2\int_0^{\pi/2}\frac{1}{2}(1+\cos 2\phi)\,d\phi\int_0^1 dz - 2\int_0^1 z^2\,dz\int_0^{\pi/2}\sin\phi\, d\phi$$

$$= 4\left(\phi + \frac{1}{2}\sin 2\phi\right)\Big|_0^{\pi/2} - 2\frac{z^3}{3}\Big|_0^1(-\cos\phi)\Big|_0^{\pi/2} = 2\pi - 2/3 = \underline{\underline{5.6165}}$$

Prob. 3.25



(a)

$$\oint \boldsymbol{D} \bullet d\boldsymbol{S} = [\iint_{z=-1} + \iint_{z=1} + \iint_{\rho=5}]\boldsymbol{D} \bullet d\boldsymbol{S}$$

$$= -\iint \rho^2 \cos^2 \phi d\phi d\rho + \iint \rho^2 \cos^2 \phi d\phi d\rho + \iint 2\rho^2 z^2 d\phi dz\Big|_{\rho=5}$$

$$= 2(5)^2 \int_0^{2\pi} d\phi \int_{-1}^{1} z^2 dz = +50(2\pi)(\frac{z^3}{3}\Big|_{-1}^{1})$$

$$= \frac{200\pi}{3} = \underline{\underline{209.44}}$$

(b) $\nabla \bullet \boldsymbol{D} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(2\rho^2 z^2) = 4z^2$

$$\int \nabla \bullet \boldsymbol{D} dv = \iiint 4z^2 \rho d\rho d\phi dz = 4\int_{-1}^{1} z^2 dz \int_0^5 \rho d\rho \int_0^{2\pi} d\phi$$

$$= 4x\frac{z^3}{3}\Big|_{-1}^{1} \quad \frac{\rho^2}{2}\Big|_0^5 (2\pi) = \frac{200\pi}{3} = \underline{\underline{209.44}}$$

Prob. 3.26

$$\int_S \boldsymbol{H} \bullet d\boldsymbol{S} = \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} 10\cos\theta r^2 \sin\theta d\theta d\phi \Big|_{r=1}$$

$$= 10(1)^2 \int_0^{2\pi} d\phi \int_0^{\pi/2} \sin\theta\cos\theta d\theta = 10(2\pi)\int_0^{\pi/2} \sin\theta d(\sin\theta)$$

$$= 20\pi\left(\frac{\sin^2\theta}{2}\right)\Big|_0^{\pi/2} = 10\pi = \underline{\underline{31.416}}$$

Prob. 3.27

$$\oint_S \boldsymbol{H}\cdot d\boldsymbol{S} = \int_v \nabla\cdot\boldsymbol{H}dv$$

$$\oint_S \boldsymbol{H}\cdot d\boldsymbol{S} = -\iint 2xydydz\Big|_{x=0} + \iint 2xydydz\Big|_{x=1} - \iint (x^2+z^2)dxdz\Big|_{y=1}$$

$$+\iint (x^2+z^2)dxdz\Big|_{y=2} - \iint 2yzdxdy\Big|_{z=-1} + \iint 2yzdxdy\Big|_{z=3}$$

$$= 0 + 2\int_1^2 ydy\int_{-1}^3 dz + 2\int_0^1 dx\int_1^2 ydy + 6\int_0^1 dx\int_1^2 ydy$$

$$= 12 + 3 + 9 = \underline{\underline{24}}$$

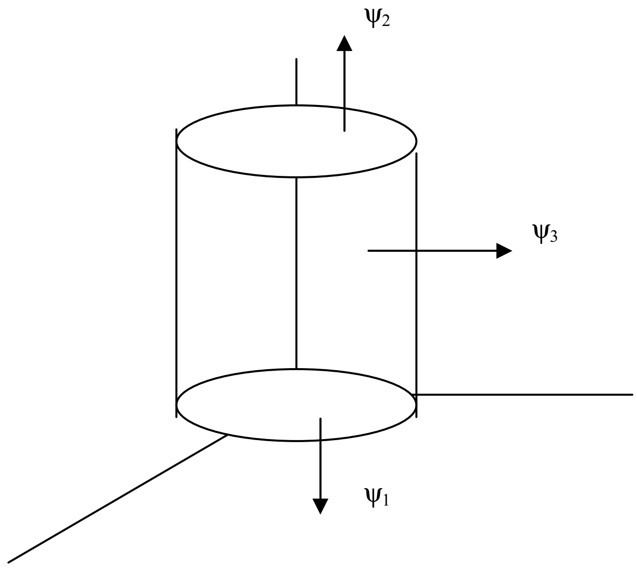
$$\nabla\cdot\boldsymbol{H} = \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 2y + 0 + 2y = 4y$$

$$\int_V \nabla\cdot\boldsymbol{H}dv = \iiint 4ydxdydz = 4\int_0^1 dx\int_1^2 ydy\int_{-1}^3 dz$$

$$= 4(1)\frac{y^2}{2}\Big|_1^2 (3+1) = \underline{\underline{24}}$$

Prob. 3.28

<u>Side 1:</u>



$$\psi = \oint_S \boldsymbol{D}\bullet d\boldsymbol{S} = \psi_1 + \psi_2 + \psi_3$$

$$= 0 + \int_{\phi=0}^{2\pi}\int_{\rho=0}^{3} 10z \times \rho d\phi d\rho \bigg|_{z=4} + \int_{z=0}^{4}\int_{\phi=0}^{2\pi} 5\rho \times \rho d\phi dz \bigg|_{\rho=3}$$

$$= 10(4)(2\pi)\left(\frac{\rho^2}{2}\right)\bigg|_0^3 + 5(9)(2\pi)(4) = 360\pi + 360\pi = \underline{\underline{2261.95}}$$

<u>Side 2:</u>

$$\psi = \int_v \nabla\bullet\boldsymbol{B}dv, \qquad \nabla\bullet\boldsymbol{B} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(5\rho^2) + 0 + 10 = 10 + 10 = 20$$

$$\psi = \iiint 20dv = 20\int_{\phi=0}^{2\pi}\int_{z=0}^{4}\int_{\rho=0}^{3} \rho d\phi d\rho dz = 20(2\pi)(4)\left(\frac{\rho^2}{2}\bigg|_0^3\right)$$

$$= 720\pi = \underline{\underline{2261.95}}$$

Prob. 3.29

Let $\psi = \oint_S \mathbf{A} \bullet d\mathbf{S} = \int_v \nabla \bullet \boldsymbol{A} dv$

$$\nabla \bullet \boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 2(x+y+z)$$

$$\nabla \bullet \boldsymbol{A} = 2(\rho\cos\phi + \rho\sin\phi + z), \qquad dv = \rho d\rho d\phi dz$$

$$\psi = \iiint 2(\rho\cos\phi + \rho\sin\phi + z)\rho d\rho d\phi dz$$

$$= 0 + 0 + 2\int_0^1 \rho d\rho \int_2^4 z dz \int_0^{2\pi} d\phi = \frac{2\rho^2}{2}\Bigg|_0^1 \frac{z^2}{2}\Bigg|_2^4 (2\pi) = \frac{1}{2}(16-4)(2\pi)$$

$$= 12\pi = \underline{\underline{37.7}}$$

Prob. 3.30

$$\nabla \bullet \boldsymbol{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^4) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(r\sin^2\theta\cos\phi)$$

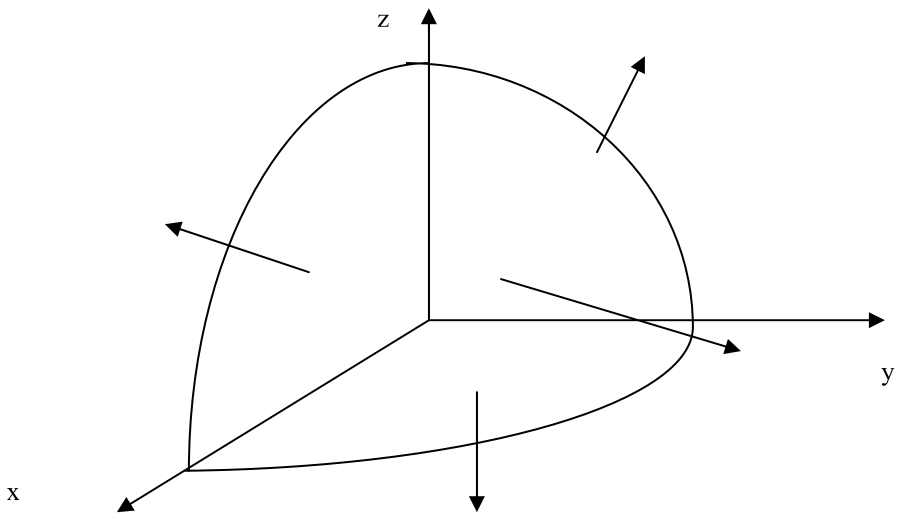
$$= 4\ r\ + 2\cos\theta\ \cos\phi$$

$$\int \nabla \bullet \boldsymbol{A} dv = \iiint 4r^3 \sin\theta d\theta d\phi dr + \iiint 2r^2 \sin\theta\cos\theta\cos\phi d\theta d\phi dr$$

$$= 4\frac{r^4}{4}\Big|_0^3 (-\cos\theta)\Big|_0^{\pi/2}(\frac{\pi}{2}) + \frac{2r^3}{3}\Big|_0^3 (-\frac{\cos^2\theta}{2})\Big|_0^{\pi/2} \sin\phi\Big|_0^{\frac{\pi}{2}}$$

$$= 81(1)(\frac{\pi}{2}) + 18(0+\frac{1}{2})(1-0)$$

$$= \frac{81\pi}{2} + 9 = \underline{\underline{136.23}}$$



$$\int \boldsymbol{A} \bullet d\boldsymbol{S} = [\iint_{\phi=0} + \iint_{\phi=\pi/2} + \iint_{r=3} + \iint_{\theta=\pi/2}]\boldsymbol{A} \bullet d\boldsymbol{S}$$

Since $\boldsymbol{A}$ has no $\phi-$component, the first two integrals on the right hand side vanish.

$$\int \boldsymbol{A}\bullet d\boldsymbol{S} = \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{\pi/2} r^4 \sin\theta\, d\theta\, d\phi\Big|_{r=3} + \int_{r=0}^{3}\int_{\phi=0}^{\pi/2} r^2 \sin^2\theta \cos\phi dr d\phi\Big|_{\theta=\pi/2}$$

$$= 81\,(\frac{\pi}{2})\,(-\cos\theta)\Big|_0^{\pi/2} + 9(1)\sin\phi\Big|_0^{\pi/2}$$

$$= \frac{81\pi}{2} + 9 \quad = \underline{\underline{136.23}}$$

Prob. 3.31

Let $\psi = \oint \boldsymbol{F}\bullet d\boldsymbol{S} = \psi_t + \psi_b + \psi_o + \psi_i$

where $\psi_t, \psi_b, \psi_o, \psi_i$ are the fluxes through the top surface, bottom surface, outer surface $(\rho = 3)$, and inner surface respectively.

For the top surface, $d\boldsymbol{S} = \rho\, d\phi\, d\rho\, \boldsymbol{a}_z, \qquad z = 5;$

$$\boldsymbol{F}\bullet d\boldsymbol{S} = \rho^2\ z\ d\phi\ dz. \text{ Hence:}$$

$$\psi_t = \int_{\rho=2}^{3}\int_{\phi=0}^{2\pi} \rho^2\ z\ d\phi\ dz\Big|_{z=5} = \frac{190\ \pi}{3} = 198.97$$

For the bottom surface, $\quad z = 0,\ d\boldsymbol{S} = \rho\, d\phi\, d\rho(-\boldsymbol{a}_z)$

$$\boldsymbol{F}\bullet d\boldsymbol{S} = -\rho^2 z\, d\phi\ d\rho = 0. \quad \text{Hence, } \psi_b = 0.$$

For the outer curved surface, $\rho = 3, \quad d\boldsymbol{S} = \rho d\phi dz\, \mathbf{a}_\rho$

$$\boldsymbol{F}\bullet d\boldsymbol{S} = \rho^2 \sin\phi\ d\phi\ dz. \text{ Hence,}$$

$$\psi_a = \int_{z=0}^{5} dz\ \rho^3 \int_{\phi=0}^{2\pi} \sin\phi\, d\phi\Big|_{\rho=3} = 0$$

For the inner curved surface, $\rho = 2, \quad d\,\boldsymbol{S} = \rho\, d\phi\, dz\,(-\boldsymbol{a}_\rho)$

$$\boldsymbol{F}\bullet d\boldsymbol{S} = -\rho^3 \sin\phi\ d\phi\ dz. \quad \text{Hence,}$$

$$\psi_a = -\int_{z=0}^{5} dz\ \rho^3 \int_{\phi=0}^{2\pi} \sin\phi\, d\phi\Big|_{\rho=2} = 0$$

$$\psi = \frac{190\pi}{3} + 0 + 0 + 0 = \frac{190\ \pi}{3} = \underline{\underline{198.97}}$$

$$\psi = \oint \boldsymbol{F} \bullet d\boldsymbol{S} = \int \nabla \bullet \boldsymbol{F} dV$$

$$\nabla \bullet \boldsymbol{F} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho^3 \sin\phi) + \frac{1}{\rho}\frac{\partial}{\partial\phi}(z\cos\phi) + \rho$$

$$= 3\rho\sin\phi - \frac{z}{\rho}\sin\phi + \rho$$

$$\int_V \nabla \bullet \boldsymbol{F} dv = \iiint (3\rho\sin\phi - \frac{z}{\rho}\sin\phi + \rho)\ \rho\, d\phi\, d\rho\, dz$$

$$= 0 + 0 + \int_0^5 dz \int_0^{2\pi} d\phi \int_2^3 \rho^2 d\rho$$

$$= \frac{190\ \pi}{3} = 198.97$$

Prob. 3.32

(a) $$\nabla x\boldsymbol{A} = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ \frac{\partial}{\partial \boldsymbol{x}} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & y^2 & -xz \end{vmatrix} = z\boldsymbol{a}_y - x\boldsymbol{a}_z$$

(b)
$$\nabla x\boldsymbol{B} = \left(\frac{1}{\rho}2\rho z 2\sin\phi\cos\phi - 0\right)\boldsymbol{a}_\rho + (2\rho z - 2z\sin^2\phi)\boldsymbol{a}_\phi + \frac{1}{\rho}\left(2\rho\sin^2\phi - 0\right)\boldsymbol{a}$$

$$= 4z\sin\phi\cos\phi\boldsymbol{a}_\rho + 2(\rho z - z\sin^2\phi)\boldsymbol{a}_\phi + 2\sin^2\phi\boldsymbol{a}_z$$

$$= 2z\sin 2\phi\boldsymbol{a}_\rho + 2z(\rho - \sin^2\phi)\boldsymbol{a}_\phi + 2\sin^2\phi\boldsymbol{a}_z$$

(c)
$$\nabla x\boldsymbol{C} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(r\cos^2\theta\sin\theta\right]\boldsymbol{a}_r - \frac{1}{r}\left[\frac{\partial}{\partial r}(r^2\cos^2\theta\right]\boldsymbol{a}_\theta$$

$$= \frac{r}{r\sin\theta}\left[(2\cos\theta)(-\sin\theta)\sin\theta + \cos\theta(\cos^2\theta)\right]\boldsymbol{a}_r - \frac{\cos^2\theta}{r}(2r)\boldsymbol{a}_\theta$$

$$= \frac{(\cos^3\theta - 2\sin^2\theta\cos\theta)}{\sin\theta}\boldsymbol{a}_r - 2\cos^2\theta\boldsymbol{a}_\theta$$

Prob. 3.33

(a)

$$\nabla\times \boldsymbol{A}=\begin{vmatrix}\boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z\\ \dfrac{\partial}{\partial \boldsymbol{x}} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ x^2y & y^2z & -2xz\end{vmatrix}= \underline{\underline{-y^2\,\boldsymbol{a}_x+2z\,\boldsymbol{a}_y-x^2\,\boldsymbol{a}_z}}$$

$$\nabla\bullet\nabla\times\boldsymbol{A}=\underline{\underline{0}}$$

(b)

$$\nabla\times\boldsymbol{A}=\left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi}-\frac{\partial A_\phi}{\partial z}\right)\boldsymbol{a}_\rho+\left(\frac{\partial A_\rho}{\partial z}-\frac{\partial A_z}{\partial\rho}\right)\boldsymbol{a}_\phi+\frac{1}{\rho}\left(\frac{\partial(\rho A_\rho)}{\partial\rho}-\frac{\partial A_\rho}{\partial\phi}\right)\boldsymbol{a}_z$$

$$=(0-0)\,\boldsymbol{a}_\rho+(\rho^2-3z^2)\,\boldsymbol{a}_\phi+\frac{1}{\rho}(4\rho^3-0)\,\boldsymbol{a}_z$$

$$=\underline{\underline{(\rho^2-3z^2)\,\boldsymbol{a}_\phi+4\rho^2\,\boldsymbol{a}_z}}$$

$$\nabla\bullet\nabla\times\boldsymbol{A}=\underline{\underline{0}}$$

(c)

$$\nabla\times\boldsymbol{A}=\frac{1}{r\sin\theta}\left[0-\frac{\sin\phi}{r^2}\right]\boldsymbol{a}_r+\frac{1}{r}\left[\frac{-1}{\sin\theta}\frac{\cos\phi}{r^2}-0\right]\boldsymbol{a}_\theta+\frac{1}{r}\left[\frac{\partial}{\partial r}\left(\frac{\cos\phi}{r}\right)-0\right]\boldsymbol{a}_\phi$$

$$\underline{\underline{=-\frac{\sin\phi}{r^3\sin\theta}\boldsymbol{a}_r+\frac{\cos\phi}{r^3\sin\theta}\boldsymbol{a}_\theta+\frac{\cos\phi}{r^3}\boldsymbol{a}_\phi}}$$

$$\nabla\bullet\nabla\times\boldsymbol{A}=\frac{-\sin\phi}{r^4\sin\theta}+0+\frac{\sin\phi}{r^4\sin\theta}=0$$

$$\nabla\bullet\nabla\times\boldsymbol{A}=\underline{\underline{0}}$$

Prob. 3.34

$$\nabla\times\boldsymbol{H}=0\boldsymbol{a}_\rho+1\boldsymbol{a}_\phi+\frac{1}{\rho}(2\rho\cos\phi-\rho\cos\phi)\boldsymbol{a}_z=\underline{\underline{\boldsymbol{a}_\phi+\cos\phi\boldsymbol{a}_z}}$$

$$\nabla\times\nabla\times\boldsymbol{H}=\left(-\frac{1}{\rho}\sin\phi-0\right)\boldsymbol{a}_\rho+0\boldsymbol{a}_\phi+\frac{1}{\rho}(1-0)\boldsymbol{a}_z=\underline{\underline{-\frac{1}{\rho}\sin\phi\boldsymbol{a}_\rho+\frac{1}{\rho}\boldsymbol{a}_z}}$$

Prob. 3.35

Method 1: We can express **A** in spherical coordinates.

$$\boldsymbol{A} = \frac{r}{r^3}\boldsymbol{a}_r = \frac{\boldsymbol{a}_r}{r^2},$$

$$\nabla \times \boldsymbol{A} = \nabla \times \left(\frac{\boldsymbol{a}_r}{r^2}\right) = \nabla\left(\frac{1}{r^2}\right) \times \boldsymbol{a}_r = \frac{-2}{r^3}\boldsymbol{a}_r \times \boldsymbol{a}_r = \mathbf{0}$$

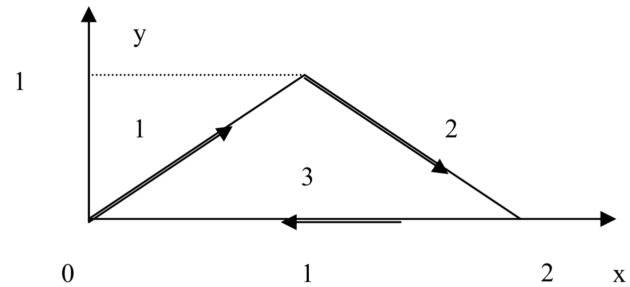
Method 2:

$$\boldsymbol{A} = \frac{x}{r^3}\boldsymbol{a}_x + \frac{y}{r^3}\boldsymbol{a}_y + \frac{z}{r^3}\boldsymbol{a}_z$$

$$\nabla \times \boldsymbol{A} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{r^3} & \frac{y}{r^3} & \frac{z}{r^3} \end{vmatrix} = \left\{-\frac{3}{2}z(x^2+y^2+z^2)^{-5/2}(2y) - -\frac{3}{2}y(x^2+y^2+z^2)^{-5/2}(2z)\right\}\boldsymbol{a}_x + \ldots$$

$$= \boldsymbol{0}$$

Prob. 3.36



(a)

$$\oint_L \boldsymbol{F} \bullet d\,\boldsymbol{l} = \left(\int_1 + \int_2 + \int_3\right) \boldsymbol{F} \bullet d\,\boldsymbol{l}$$

For 1, $y = x \quad dy = dx, d\bar{l} = dx\,\bar{a}_x + dy\,\bar{a}_y,$

$$\int_1 \boldsymbol{F} \bullet d\,\boldsymbol{l} = \int_0^1 x^3 dx - x dx = -\frac{1}{4}$$

For 2, $y = -x + 2, dy = -dx, d\bar{l} = dx\,\bar{a}_x + dy\,\bar{a}_y,$

$$\int_2 \boldsymbol{F} \bullet d\,\boldsymbol{l} = \int_1^2 (-x^3 + 2x^2 - x + 2)dx = \frac{17}{12}$$

For 3,

$$\int_3 \boldsymbol{F} \bullet d\,\boldsymbol{l} = \int_2^0 x^2 y dx\Big|_{y=0} = 0$$

$$\oint_L \boldsymbol{F} \bullet d\,\boldsymbol{l} = -\frac{1}{4} + \frac{17}{12} + 0 = \underline{\underline{\frac{7}{6}}}$$

(b)

$$\nabla \times \boldsymbol{F} = -x^2\,\boldsymbol{a}_z\,; \qquad d\boldsymbol{S} = dxdy(-\boldsymbol{a}_z)$$

$$\int (\nabla \times \boldsymbol{F}) \bullet d\,\boldsymbol{S} = -\iint (-x^2) dxdy = \int_0^1 \int_0^x x^2 dydx + \int_1^2 \int_{y=0}^{-x+2} x^2 dydx$$

$$= \int_0^1 x^2 y\Big|_0^x dx + \int_1^2 x^2 y\Big|_0^{-x+2} dx = \frac{x}{4}\Big|_0^1 + \int_1^2 x^2(-x+2)dx = \underline{\underline{\frac{7}{6}}}$$

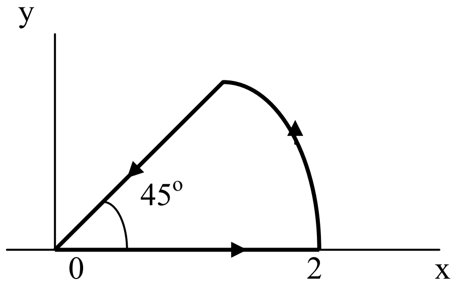
(c) <u>Yes</u>

Prob. 3.37

$$\oint \boldsymbol{A} \bullet d\boldsymbol{l} = \int_{\rho=2}^{1} \rho \sin\phi d\rho\Big|_{\phi=0} + \int_{\phi=0}^{\pi/2} \rho^2 \rho d\phi\Big|_{\rho=1} + \int_{\rho=1}^{2} \rho \sin\phi d\rho\Big|_{\phi=90^o} + \int_{\phi=\pi/2}^{0} \rho^3 d\phi\Big|_{\rho=2}$$

$$= \frac{\pi}{2} + \frac{1}{2}(4-1) + 8(-\frac{\pi}{2}) = \underline{\underline{-9.4956}}$$

Prob. 3.38



$$\oint \boldsymbol{F} \bullet dl = \int_0^2 2\rho z d\rho \Big|_{z=1} + \int_0^{\pi/4} 3z \sin\phi \, \rho d\phi \Big|_{\rho=2, z=1} + \int_2^0 2\rho z d\rho \Big|_{z=1}$$

$$= \rho^2 \Big|_0^2 + (-6\cos\phi) \Big|_0^{\pi/4} + \rho^2 \Big|_2^0 = (4-0) + 6(-\cos\pi/4 + 1) + (0-4) = \underline{\underline{1.757}}$$

$$\nabla x \boldsymbol{F} = \frac{1}{\rho}\left[3z\sin\phi - 0\right]\boldsymbol{a}_z + ...$$

$$\int (\nabla x \boldsymbol{F}) \cdot d\boldsymbol{S} = \int_{\rho=0}^{2} \int_{\phi=0}^{\pi/4} \frac{3z}{\rho} \sin\phi \, \rho d\phi d\rho \Big|_{z=} \qquad = 3(2)(-\cos\phi)\Big|_0^{\pi/4}$$

$$= 6(-\cos\pi + 1) = \underline{\underline{1.757}}$$

Prob. 3.39

$$\nabla \cdot A = 8xe^{-y} + 8xe^{-y} = 16xe^{-y}$$
$$\nabla(\nabla \cdot \boldsymbol{A}) = 16e^{-y}\boldsymbol{a}_x - 16xe^{-y}\boldsymbol{a}_y$$

$$\nabla \times \nabla(\nabla \cdot \boldsymbol{A}) = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ \frac{\partial}{\partial \boldsymbol{x}} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 16e^{-y} & -16xe^{-y} & 0 \end{vmatrix} = (-16e^{-y} + 16e^{-y})\boldsymbol{a}_z = \underline{\underline{0}}$$

Should be expected since $\nabla \times \nabla V = 0$.

Prob. 3.40

(a) $$\nabla V = \underline{\underline{-\frac{\sin\theta\cos\phi}{r^2}\boldsymbol{a}_r + \frac{\cos\theta\cos\phi}{r^2}\boldsymbol{a}_\theta - \frac{\sin\phi}{r^2}\boldsymbol{a}_\phi}}$$

(b) $\nabla x \nabla V = \underline{\underline{0}}$

(c)

$$\nabla\cdot\nabla V=\nabla^2 V=\frac{1}{r^2}\frac{\partial}{\partial r}(-\sin\theta\cos\phi)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\cos\theta\cos\phi}{r})+\frac{1}{r^2\sin^2\theta}(-\frac{\sin\theta\cos\phi}{r})$$

$$=0+\frac{\cos\phi}{r^3\sin\theta}(1-2\sin^2\theta)-\frac{\cos\phi}{r^3\sin\theta}$$

$$=\underline{\underline{-\frac{2\sin\theta\cos\phi}{r^3}}}$$

Prob. 3.41

$$\boldsymbol{Q}=\frac{r}{r\sin\theta}r\sin\theta[(\cos\phi-\sin\phi)\boldsymbol{a}_x+(\cos\phi+\sin\phi)\boldsymbol{a}_y]$$

$$=r(\cos\phi-\sin\phi)\boldsymbol{a}_x+r(\cos\phi+\sin\phi)\boldsymbol{a}_y$$

$$\begin{bmatrix}Q_r\\Q_\theta\\Q_\phi\end{bmatrix}=\begin{bmatrix}\sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta\\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta\\ -\sin\phi & \cos\phi & 0\end{bmatrix}\begin{bmatrix}Q_x\\Q_y\\Q_z\end{bmatrix}$$

$$\boldsymbol{Q}=r\sin\theta\,\boldsymbol{a}_r+r\cos\theta\,\boldsymbol{a}_\theta+r\,\boldsymbol{a}_\phi$$

(a)

$$d\boldsymbol{l}=\rho\,d\phi\,\boldsymbol{a}_\phi,\quad \rho=r\sin 30^\circ=2(\frac{1}{2})=1$$

$$z=r\cos 30^\circ=\sqrt{3}$$

$$Q_\phi=r=\sqrt{\rho^2+z^2}$$

$$\oint\boldsymbol{Q}\bullet d\boldsymbol{l}=\int_0^{2\pi}\sqrt{\rho^2+z^2}\,\rho\,d\phi=2(1)(2\pi)=\underline{\underline{4\pi}}$$

(b)

$$\nabla\times\boldsymbol{Q}=\cot\theta\,\boldsymbol{a}_r-2\,\boldsymbol{a}_\theta+\cos\theta\,\boldsymbol{a}_\phi$$

For S_1, $d\boldsymbol{S}=r^2\sin\theta\,d\theta\,d\phi\,\boldsymbol{a}_r$

$$\int_{S_1}(\nabla\times\boldsymbol{Q})\bullet d\boldsymbol{S}=\int r^2\sin\theta\cot\theta\,d\theta\,d\phi\Big|_{r=2}$$

$$=4\int_0^{2\pi}d\phi\int_0^{30^\circ}\cos\theta\,d\theta=\underline{\underline{4\pi}}$$

(c)

For S_2, $\quad d\boldsymbol{S} = r\sin\theta d\theta dr\, \boldsymbol{a}_\theta$

$$\int_{S_2} (\nabla\times\boldsymbol{Q})\bullet d\boldsymbol{S} = -2\int r\sin\theta\, d\phi\, dr\Big|_{\theta=30^\circ}$$

$$= -2\sin 30\int_0^2 r dr \int_0^{2\pi} d\phi$$

$$= \underline{\underline{-\ 4\pi}}$$

(d)

For S_1, $\quad d\boldsymbol{S} = r^2 \sin\theta\, d\phi\, d\theta\, \boldsymbol{a}_r$

$$\int_{S_1} \boldsymbol{Q}\bullet d\boldsymbol{S} = r^3 \int \sin^2\theta\, d\theta\, d\phi\Big|_{r=2}$$

$$= 8\int_0^{2\pi} d\phi \int_0^{30^\circ} \sin^2\theta\, d\theta$$

$$= 4\pi[\frac{\pi}{3} - \frac{\sqrt{3}}{2}] = \underline{\underline{2.2767}}$$

(e)

For S_2, $\quad d\boldsymbol{S} = r\sin\theta\, d\phi\, dr\, \boldsymbol{a}_\theta$

$$\int_{S_2} \boldsymbol{Q}\bullet d\boldsymbol{S} = \int r^2 \sin\theta\cos\theta\, d\phi\, dr\Big|_{\theta=30^\circ}$$

$$= \frac{4\pi\sqrt{3}}{3} = \underline{\underline{7.2552}}$$

(f)

$$\nabla\bullet\boldsymbol{Q} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^3\sin\theta) + \frac{r}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\cos\theta) + 0$$

$$= 2\sin\theta + \cos\theta\cot\theta$$

$$\int \nabla\bullet\boldsymbol{Q} dv = \int (2\sin\theta + \cos\theta\cot\theta) r^2 \sin\theta\, d\theta\, d\phi\, dr$$

$$= \frac{r^3}{3}\Big|_0^2 (2\pi)\int_0^{30} (1+\sin^2\theta) d\theta$$

$$= \frac{4\pi}{3}(\pi - \frac{\sqrt{3}}{2}) = \underline{\underline{9.532}}$$

$$Check: \int \nabla \bullet \boldsymbol{Q} dv = (\int_{S_1} + \int_{S_2} \boldsymbol{Q} \bullet d\boldsymbol{S}$$

$$= 4\pi[\frac{\pi}{3} - \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{3}]$$

$$= \frac{4\pi}{3}[\pi - \frac{\sqrt{3}}{2}] \quad \text{(It checks!)}$$

Prob. 3.42

Since $\boldsymbol{u} = \boldsymbol{\omega} \times \boldsymbol{r}$, $\nabla \times \boldsymbol{u} = \nabla \times (\boldsymbol{\omega} \times \boldsymbol{r})$. From Appendix A.10,

$$\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{A}(\nabla \bullet \boldsymbol{B}) - \boldsymbol{B}(\nabla \bullet \boldsymbol{A}) + (\boldsymbol{B} \bullet \nabla)\boldsymbol{A} - (\boldsymbol{A} \bullet \nabla)\boldsymbol{B}$$

$$\nabla \times \boldsymbol{u} = \nabla \times (\boldsymbol{\omega} \times \boldsymbol{r})$$

$$\nabla \times (\boldsymbol{\omega} \times \boldsymbol{r}) = \boldsymbol{\omega}(\nabla \bullet \boldsymbol{r}) - \boldsymbol{r}(\nabla \bullet \boldsymbol{\omega}) + (\boldsymbol{r} \bullet \nabla)\boldsymbol{\omega} - (\bar{\boldsymbol{\omega}} \bullet \nabla)\boldsymbol{r}$$

$$= \boldsymbol{\omega}(3) - \boldsymbol{\omega} = 2\boldsymbol{\omega}$$

or $\boldsymbol{\omega} = \frac{1}{2} \nabla \times \boldsymbol{u}$.

Alternatively, let $x = r\cos\omega t$, $y = r\sin\omega t$

$$\boldsymbol{u} = \frac{\partial x}{\partial t}\boldsymbol{a}_x + \frac{\partial y}{\partial t}\boldsymbol{a}_y$$

$$= -\omega r \sin\omega t\, \boldsymbol{a}_x + \omega r \cos\omega t\, \boldsymbol{a}_y$$

$$= -\omega y \boldsymbol{a}_x + \omega x \boldsymbol{a}_y$$

$$\nabla \times \boldsymbol{u} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix} = 2\omega \boldsymbol{a}_z = 2\boldsymbol{\omega}$$

i.e., $\boldsymbol{\omega} = \frac{1}{2} \nabla \times \boldsymbol{u}$

Note that we have used the fact that $\nabla \bullet \boldsymbol{\omega} = 0$, $(\boldsymbol{r} \bullet \nabla)\boldsymbol{\omega} = 0$, $(\boldsymbol{\omega} \bullet \nabla)\boldsymbol{r} = \boldsymbol{\omega}$

Prob. 3.43

$$\nabla \bullet \boldsymbol{B} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho B_\rho) + \frac{1}{\rho}\frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_z}{\partial z} = \frac{1}{\rho}2\rho\cos\phi + \rho\cos\phi - 4$$

$$= \underline{\underline{(2+\rho)\cos\phi - 4}}$$

(a)

$$\nabla \times \boldsymbol{B} = \left[\frac{1}{\rho}\frac{\partial B_z}{\partial \phi} - \frac{\partial B_\phi}{\partial z}\right]\boldsymbol{a}_\rho + \left[\frac{\partial B_\rho}{\partial z} - \frac{\partial B_z}{\partial \rho}\right]\boldsymbol{a}_\phi + \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho B_\phi) - \frac{\partial B_\rho}{\partial \phi}\right]\boldsymbol{a}_z$$

$$= 0\boldsymbol{a}_\rho + 0\boldsymbol{a}_\phi + \frac{1}{\rho}\left[3\rho^2\sin\phi + \rho\sin\phi\right]\boldsymbol{a}_z$$

$$= \underline{\underline{(3\rho+1)\sin\phi\,\boldsymbol{a}_z}}$$

(b)

$$\nabla \bullet \boldsymbol{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) + 0 + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^4\sin\theta) + \frac{1}{r\sin\theta}(-2r\sin\phi)$$

$$= \underline{\underline{4r\sin\theta - \frac{2\sin\phi}{\sin\theta}}}$$

$$\nabla \times \boldsymbol{F} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(F_\phi\sin\theta) - \frac{\partial F_\theta}{\partial \phi}\right]\boldsymbol{a}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial F_r}{\partial \phi} - \frac{\partial}{\partial r}(rF_\phi)\right]\boldsymbol{a}_\theta$$

$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(rF_\theta) - \frac{\partial F_r}{\partial \theta}\right]\boldsymbol{a}_\phi$$

$$= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(2r\sin\theta\cos\phi) - 0\right]\boldsymbol{a}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}r^2\cos\phi - 4r\cos\phi\right]\boldsymbol{a}_\theta + 0\boldsymbol{a}_\phi$$

$$= \underline{\underline{2\cot\theta\cos\phi\,\boldsymbol{a}_r + \left(\frac{r\cos\phi}{\sin\theta} - 4\cos\phi\right)\boldsymbol{a}_\theta}}$$

Prob. 3.44

(a)

$$\nabla \bullet (\mathrm{V}\nabla\mathrm{V}) = \nabla \bullet \left(\mathrm{V}\frac{\partial V}{\partial x}\boldsymbol{a}_x + \mathrm{V}\frac{\partial V}{\partial y}\boldsymbol{a}_y + \mathrm{V}\frac{\partial V}{\partial z}\boldsymbol{a}_z\right)$$

$$= \frac{\partial}{\partial x}\left(\mathrm{V}\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mathrm{V}\frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(\mathrm{V}\frac{\partial V}{\partial z}\right)$$

$$= V\frac{\partial^2 V}{\partial x^2} + V\frac{\partial^2 V}{\partial y^2} + V\frac{\partial^2 V}{\partial z^2} + \left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2$$

$$= V\nabla^2 V + |\nabla V|^2$$

$$\nabla\times V\boldsymbol{A}=\begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ VA_x & VA_y & VA_z\end{vmatrix}$$

$$=\left[\frac{\partial}{\partial y}(VA_z)-\frac{\partial}{\partial z}(VA_y)\right]\boldsymbol{a}_x+\left[\frac{\partial}{\partial z}(VA_x)-\frac{\partial}{\partial x}(VA_z)\right]\boldsymbol{a}_y+\left[\frac{\partial}{\partial x}(VA_y)-\frac{\partial}{\partial y}(VA_x)\right]\boldsymbol{a}_z$$

(b)

$$=\left[A_z\frac{\partial V}{\partial y}+V\frac{\partial A_z}{\partial y}-A_y\frac{\partial V}{\partial z}-V\frac{\partial A_y}{\partial z}\right]\boldsymbol{a}_x$$

$$+\left[A_x\frac{\partial V}{\partial z}+V\frac{\partial A_x}{\partial z}-A_z\frac{\partial V}{\partial x}-V\frac{\partial A_z}{\partial x}\right]\boldsymbol{a}_y$$

$$+\left[A_y\frac{\partial V}{\partial x}+V\frac{\partial A_y}{\partial x}-A_x\frac{\partial V}{\partial y}-V\frac{\partial A_x}{\partial y}\right]\boldsymbol{a}_z$$

$$\nabla\times V\boldsymbol{A}=V\left[\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)\boldsymbol{a}_x+\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)\boldsymbol{a}_y+\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)\boldsymbol{a}_z\right]$$

$$+\left(A_z\frac{\partial V}{\partial y}-A_y\frac{\partial V}{\partial z}\right)\boldsymbol{a}_x+\left(A_x\frac{\partial V}{\partial z}-A_z\frac{\partial V}{\partial x}\right)\boldsymbol{a}_y+\left(A_y\frac{\partial V}{\partial x}-A_x\frac{\partial V}{\partial y}\right)\boldsymbol{a}_z$$

$$=V\nabla\times\boldsymbol{A}+\nabla V\times\boldsymbol{A}$$

Prob. 3.45

(a)

$$\nabla\bullet\boldsymbol{B}=\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}=2xy+1+1=\underline{\underline{2+2xy}}$$

(b)

$$\nabla\times\boldsymbol{B}=\begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x^2y & (2x^2+y) & (z-y)\end{vmatrix}=(-1+0)\boldsymbol{a}_x+(0-0)\boldsymbol{a}_y+(4x+x^2)\boldsymbol{a}_z$$

$$=\underline{\underline{-\boldsymbol{a}_x+x(4-x)\boldsymbol{a}_z}}$$

(c)

$$\nabla(\nabla\bullet\boldsymbol{B})=\nabla(2+2xy)=\underline{\underline{2y\boldsymbol{a}_x+2x\boldsymbol{a}_y}}$$

(d)

$$\nabla\times\nabla\times\boldsymbol{B} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -1 & 0 & (4x-x^2) \end{vmatrix} = 0\boldsymbol{a}_x - (4-2x)\boldsymbol{a}_y + 0\boldsymbol{a}_z$$

$$= \underline{\underline{2(x-2)\boldsymbol{a}_y}}$$

Prob. 3.46

(a)

$$V_1 = x^3 + y^3 + z^3$$

$$\nabla^2 V_1 = \frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_1}{\partial y^2} + \frac{\partial^2 V_1}{\partial z^2}$$

$$= \frac{\partial}{\partial x}(3x^2) + \frac{\partial}{\partial y}\left(3y^2\right) + \frac{\partial}{\partial x}(3z^2)$$

$$= 6x + 6y + 6z = \underline{\underline{6(x+y+z)}}$$

(b)

$$V_2 = \rho z^2 \sin 2\phi$$

$$\nabla^2 V_2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho z^2 \sin 2\phi) - \frac{4z^2}{\rho}\sin 2\phi + \frac{\partial}{\partial z}(2\rho z \sin 2\phi)$$

$$= \frac{z^2}{\rho}\sin 2\phi - \frac{4z^2}{\rho}\sin 2\phi + 2\rho\sin 2\phi$$

$$= \underline{\underline{\left(\frac{-3z^2}{\rho} + 2\rho\right)\sin 2\phi}}$$

(c)

$$V_3 = r^2(1+\cos\theta\sin\phi)$$

$$\nabla^2 V_3 = \frac{1}{r^2}\frac{\partial}{\partial r}[2r^3(1+\cos\theta\sin\phi)]$$

$$+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(-\sin^2\theta\sin\phi)r^2 + \frac{1}{r^2\sin^2\theta}r^2(-\cos\theta\sin\phi)$$

$$= 6(1+\cos\theta\sin\phi) - \frac{2\sin\theta}{\sin\theta}\cos\theta\sin\phi - \frac{\cos\theta\sin\phi}{\sin^2\theta}$$

$$= \underline{\underline{6 + 4\cos\theta\sin\phi - \frac{\cos\theta\sin\phi}{\sin^2\theta}}}$$

Prob. 3.47

(a)

$$U = x^3 y^2 e^{xz}$$

$$\nabla^2 U = \frac{\partial}{\partial x}(3x^2 y^2 e^{xz} + x^3 y^2 z e^{xz}) + \frac{\partial}{\partial y}(2x^3 y e^{xz}) + \frac{\partial}{\partial z}(x^4 y^2 e^{xz})$$

$$= 6xy^2 e^{xz} + 3x^2 yze^{xz} + 3x^2 y^2 ze^{xz} + x^3 y^2 z^2 e^{xz} + 2x^3 e^{xz} + x^5 y^2 e^{xz}$$

$$\underline{\underline{= e^{xz}(6xy^2 + 3x^2 y^2 z + 3x^2 y^2 z + x^3 y^2 z^2 + 2x^3 + x^5 y^2)}}$$

At $(1,-1,1)$,

$$\nabla^2 U = e^1 \; (6+3+3+1+2+1) = 16e \underline{\underline{= 43.493}}$$

(b)

$$V = \rho^2 z(\cos\phi + \sin\phi)$$

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}[2\rho^2 z(\cos\phi + \sin\phi)] - z(\cos\phi + \sin\phi) + 0$$

$$= 4z(\cos\phi + \sin\phi) - z(\cos\phi + \sin\phi)$$

$$= \underline{\underline{3z(\cos\phi + \sin\phi)}}$$

At $(5, \frac{\pi}{6}, -2)$, $\nabla^2 V = -6(0.866 + 0.5) \underline{\underline{= -8.196}}$

(c)

$$W = e^{-r}\sin\theta\cos\phi$$

$$\nabla^2 W = \frac{1}{r^2}\frac{\partial}{\partial r}\;(-r^2 e^{-r}\sin\theta\cos\phi) + \frac{e^{-r}}{r^2\sin\theta}\cos\phi\frac{\partial}{\partial\theta}(\sin\theta\cos\theta)$$

$$-\frac{e^{-r}\sin\theta\cos\phi}{r^2\sin^2\theta}$$

$$= \frac{1}{r^2}(-2re^{-r}\sin\theta\cos\phi) + e^{-r}\sin\theta\cos\phi$$

$$+\frac{e^{-r}\cos\phi}{r^2\sin\theta}(1 - 2\sin^2\theta) - \frac{e^{-r}\cos\phi}{r^2\sin\theta}$$

$$\underline{\underline{\nabla^2 W = e^{-r}\sin\theta\cos\phi(1 - \frac{2}{r} - \frac{2}{r^2})}}$$

At $(1, 60°, 30°)$,

$$\nabla^2 W = e^{-1}\sin 60\cos 30(1-2-2) = -2.25e^{-1} \;\; = \underline{\underline{-0.8277}}$$

Prob. 3.48

(*a*) Let $V = 1nr = 1n\sqrt{x^2+y^2+z^2}$

$$\frac{\partial V}{\partial x} = \frac{1}{r}\ \frac{1}{2}\left(2x\right)\left(x^2+y^2+z^2\right)^{-1/2} = \frac{x}{r^2}$$

$$\nabla V = \frac{\partial V}{ox}\boldsymbol{a}_x + \frac{\partial V}{oy}\boldsymbol{a}_y + \frac{\partial V}{oz}\boldsymbol{a}_z = \frac{x\,\boldsymbol{a}_x + y\boldsymbol{a}_y + z\,\boldsymbol{a}_z}{r^2} = \underline{\underline{\frac{\boldsymbol{r}}{r^2}}}$$

(*b*) Let $\nabla V = \boldsymbol{A} = \dfrac{\boldsymbol{r}}{r^2} = \dfrac{1}{r}\boldsymbol{a}_r$ in spherical coordinates.

$$\nabla^2(1nr) = \nabla \bullet \nabla\ (1nr) = \nabla\ \bullet \boldsymbol{A} = \frac{1}{r^2}\ \ \frac{\partial}{\partial r}(r^2 A_r) = \frac{1}{r^2}\frac{\partial}{\partial r}(r)$$

$$= \underline{\underline{\frac{1}{r^2}}}$$

Prob. 3.49

$$\nabla V = \frac{\partial V}{\partial x}\boldsymbol{a}_x + \frac{\partial V}{\partial y}\boldsymbol{a}_y + \frac{\partial V}{\partial z}\boldsymbol{a}_z$$

$$= y^2 z^3 \boldsymbol{a}_x + 2xyz^3\boldsymbol{a}_y + 3xy^2z^2\boldsymbol{a}_z$$

At P(1,2,3,) $x = 1, y = 2, z = 3$

$$\nabla V = 4(27)\boldsymbol{a}_x + 2(2)(27)\boldsymbol{a}_y + 3(4)(9)\boldsymbol{a}_z$$

$$\underline{\underline{= 108(\boldsymbol{a}_x + \boldsymbol{a}_y + \boldsymbol{a}_z)}}$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

$$= \frac{\partial}{\partial x}\left(y^2z^3\right) + \frac{\partial}{\partial y}\left(2xyz^3\right) + \frac{\partial}{\partial z}\left(3xy^2z^2\right)$$

$$= 0 + 2\,xz^3 + 6xy^2 z$$

$$= 2xz\left(z^2 + 3y^2\right)$$

At P(1,2,3,) $x = 1, y = 2, z = 3.$

$$\nabla^2 V = 2(1)(3)(9 + 3\times 4) = 6(9+12)$$

$$\underline{\underline{= 126}}$$

Prob. 3.50

$$\nabla V = \frac{\partial V}{\partial \rho}\boldsymbol{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\boldsymbol{a}_\phi + \frac{\partial V}{\partial z}\boldsymbol{a}_z = \underline{\underline{2\rho z\cos\phi\boldsymbol{a}_\rho - \rho z\sin\phi\boldsymbol{a}_\phi + \rho^2\cos\phi\boldsymbol{a}_z}}$$

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(2\rho^2 z\cos\phi\right) - \frac{1}{\rho^2}\rho^2 z\cos\phi + 0$$

$$= (4-1)z\cos\phi = \underline{\underline{3z\cos\phi}}$$

Prob. 3.51

(a)

$$\nabla V = \frac{\partial V}{\partial r}\boldsymbol{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\boldsymbol{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\boldsymbol{a}_\phi$$

$$= \underline{\underline{-\frac{10}{r^3}\cos\phi\boldsymbol{a}_r - \frac{5\sin\phi}{r^3\sin\theta}\boldsymbol{a}_\phi}}$$

(b)

$$\nabla\bullet\nabla V = \nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial V}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\frac{\partial V}{\partial \theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}r^2(-\frac{10\cos\phi}{r^3}) + 0 + \frac{1}{r^2\sin^2\theta}(-\frac{5\cos\phi}{r^2})$$

$$\underline{\underline{\nabla\bullet\nabla V = \frac{10\cos\phi}{r^4} - \frac{5\cos\phi}{r^4\sin^2\theta}}}$$

(c) $\nabla\times\nabla V = 0$, see Example 3.10.

Prob. 3.52

$$\nabla U = \frac{\partial U}{\partial x}\boldsymbol{a}_x + \frac{\partial U}{\partial y}\boldsymbol{a}_y + \frac{\partial U}{\partial z}\boldsymbol{a}_z = 4yz^2\boldsymbol{a}_x + (4xz^2 + 10z)\boldsymbol{a}_y + (8xyz + 10y)\boldsymbol{a}_z$$

$$\nabla\bullet\nabla U = \frac{\partial}{\partial x}(\nabla U_x) + \frac{\partial}{\partial y}(\nabla U_y) + \frac{\partial}{\partial z}(\nabla U_z) = 0 + 0 + 8xy = 8xy$$

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0 + 0 + 8xy = 8xy$$

Hence, $\nabla^2 U = \nabla\bullet\nabla U$

Prob. 3.53

Method 1

$$\nabla^2 \boldsymbol{G}\Big|_\rho = \nabla^2 G_\rho - \frac{2}{\rho^2}\frac{\partial G_\phi}{\partial \phi} - \frac{G_\rho}{\rho^2}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}(2\rho\sin\phi) - \frac{2\rho\sin\phi}{\rho^2} + 0 + \frac{8\rho\sin\phi}{\rho^2} - \frac{2\rho\sin\phi}{\rho^2}$$

$$= \frac{2\sin\phi}{\rho} - \frac{2\sin\phi}{\rho} + \frac{8\sin\phi}{\rho} - \frac{2\sin\phi}{\rho} \qquad = \frac{6\sin\phi}{\rho}$$

$$\nabla^2 \boldsymbol{G}\Big|_\phi = \nabla^2 G_\phi + \frac{2}{\rho^2}\frac{\partial G_\rho}{\partial \phi} - \frac{G_\phi}{\rho^2}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}(4\rho\cos\phi) - \frac{1}{\rho}4\rho\cos\phi + 0 + \frac{4\rho\cos\phi}{\rho^2} - \frac{4\rho\cos\phi}{\rho^2}$$

$$= \frac{4\cos\phi}{\rho} - \frac{4\cos\phi}{\rho} + \frac{4\cos\phi}{\rho} - \frac{4\cos\phi}{\rho} = 0$$

$$\nabla^2 \boldsymbol{G}\Big|_z = \nabla^2 G_z = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left[\rho(z^2+1)\right] + 0 + \frac{\partial}{\partial z}(2z\rho)$$

$$= \frac{1}{\rho}(z^2+1) + 2\rho$$

Adding the components together gives

$$\underline{\underline{\nabla^2 \boldsymbol{G} = \frac{6\sin\phi}{\rho}\boldsymbol{a}_\rho + \left[2\rho + \frac{1}{\rho}(z^2+1)\right]\boldsymbol{a}_z}}$$

Method 2:

$$\nabla^2 \boldsymbol{G} = \nabla(\nabla \bullet \boldsymbol{G}) - \nabla \times (\nabla \times \boldsymbol{G})$$

Let $V = \nabla \bullet \boldsymbol{G} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(2\rho^2\sin\phi) + \frac{1}{\rho}(-4\rho\sin\phi) + 2z\rho = 2z\rho$

$$\nabla(\nabla \bullet \boldsymbol{G}) = \nabla V = 2z\boldsymbol{a}_\rho + 2\rho\boldsymbol{a}_z$$

Let $\quad \boldsymbol{A} = \nabla \times \boldsymbol{G} = \left[\frac{1}{\rho}0 - 0\right]\boldsymbol{a}_\rho + \left[0 - (z^2+1)\right]\boldsymbol{a}_\phi + \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(4\rho^2\cos\phi) - 2\rho\cos\phi\right]\boldsymbol{a}_z$

$$= -(z^2+1)\boldsymbol{a}_\phi + 6\cos\phi\boldsymbol{a}_z$$

$$\nabla \times \nabla \times \boldsymbol{G} = \nabla \times \boldsymbol{A} = \left[-\frac{6}{\rho}\sin\phi + 2z\right]\boldsymbol{a}_\rho + (0-0)\boldsymbol{a}_\phi + \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(-\rho(z^2+1)) - 0\right]\boldsymbol{a}_z$$

$$= \left[2z - \frac{6}{\rho}\sin\phi\right]\boldsymbol{a}_\rho - \frac{1}{\rho}(z^2+1)\boldsymbol{a}_z$$

$$\nabla^2\boldsymbol{G} = \nabla V - \nabla \times \boldsymbol{A}$$

$$= 2z\boldsymbol{a}_\rho + 2\rho\boldsymbol{a}_z - \left[2z - \frac{6}{\rho}\sin\phi\right]\boldsymbol{a}_\rho + \frac{1}{\rho}(z^2+1)\boldsymbol{a}_z$$

$$= \underline{\underline{\frac{6}{\rho}\sin\phi\boldsymbol{a}_\rho + \left[2\rho + \frac{1}{\rho}(z^2+1)\right]\boldsymbol{a}_z}}$$

Prob. 3.54

$$\nabla \bullet \boldsymbol{A} = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(z^2) + \frac{\partial}{\partial z}(yz) = z + y$$

$$\nabla(\nabla \bullet \boldsymbol{A}) = \boldsymbol{a}_y + \boldsymbol{a}_z$$

$$\nabla^2\boldsymbol{A} = \nabla^2 A_x\boldsymbol{a}_x + \nabla^2 A_y\boldsymbol{a}_y + \nabla^2 A_z\boldsymbol{a}_z = 0 + 2\boldsymbol{a}_y + 0 = 2\boldsymbol{a}_y$$

$$\nabla(\nabla \bullet \boldsymbol{A}) - \nabla^2\boldsymbol{A} = -\boldsymbol{a}_y + \boldsymbol{a}_z \qquad (1)$$

$$\nabla \times \boldsymbol{A} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & z^2 & yz \end{vmatrix} = -z\boldsymbol{a}_x + x\boldsymbol{a}_y$$

$$\nabla \times \nabla \times \boldsymbol{A} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -z & x & 0 \end{vmatrix} = -\boldsymbol{a}_y + \boldsymbol{a}_z \qquad (2)$$

From (1) and (2),

$$\nabla \times \nabla \times \boldsymbol{A} = \nabla(\nabla \bullet \boldsymbol{A}) - \nabla^2\boldsymbol{A}$$

Prob. 3.55

$$\nabla \bullet \boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 1+1+1 = 3 \neq 0$$

$$\nabla \times \boldsymbol{A} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \mathbf{0}$$

$$\nabla \bullet \boldsymbol{B} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho B_\rho) + \frac{1}{\rho}\frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_z}{\partial z} = 4\cos\phi - 4\cos\phi = 0$$

$$\nabla \times \boldsymbol{B} = \left[\frac{1}{\rho}\frac{\partial B_z}{\partial \phi} - \frac{\partial B_\phi}{\partial z}\right]\boldsymbol{a}_\rho + \left[\frac{\partial B_\rho}{\partial z} - \frac{\partial B_z}{\partial \rho}\right]\boldsymbol{a}_\phi + \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho B_\rho) - \frac{\partial B_\rho}{\partial \phi}\right]\boldsymbol{a}_z$$

$$= 0\boldsymbol{a}_\rho + 0\boldsymbol{a}_\phi + \frac{1}{\rho}\left[-8\rho\sin\phi + 2\rho\sin\phi\right]\boldsymbol{a}_z = -6\sin\phi\,\boldsymbol{a}_z \neq \mathbf{0}$$

$$\nabla \bullet \boldsymbol{C} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\sin\theta) + 0 + 0 = \frac{2\sin\theta}{r} \neq 0$$

$$\nabla \times \boldsymbol{C} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(r\sin^2\theta) - 0\right]\boldsymbol{a}_r + \frac{1}{r}\left[0 - \frac{\partial}{\partial r}(r^2\sin\theta)\right]\boldsymbol{a}_\theta$$

$$+\frac{1}{r}\left[0 - \cos\theta\right]\boldsymbol{a}_\phi$$

$$= 2\cos\theta\,\boldsymbol{a}_r - 2\sin\theta\,\boldsymbol{a}_\theta - \frac{\cos\theta}{r}\boldsymbol{a}_\phi \neq \mathbf{0}$$

(a) **B** is solenoidal.
(b) **A** is irrotational.

Prob. 3.56 (a)

$$\nabla \times \boldsymbol{G} = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ \frac{\partial}{\partial \boldsymbol{x}} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 16xy - z & 8x^2 & -x \end{vmatrix}$$

$$= 0\,\boldsymbol{a}_x + (-1+1)\,\boldsymbol{a}_y + (16x - 16x)\,\boldsymbol{a}_z = 0$$

Thus, $\boldsymbol{G}$ is irrotational.

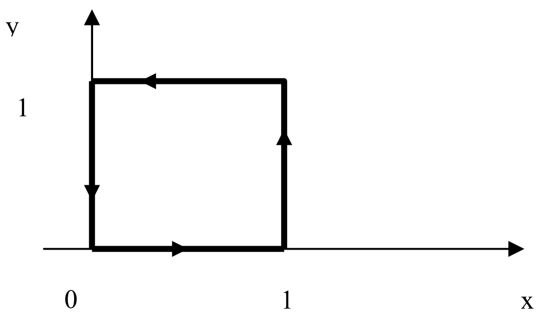
(b) Assume that ψ represents the net flux.

$$\psi = \oint \boldsymbol{G} \bullet d\boldsymbol{S} = \int \nabla \bullet \boldsymbol{G} dv$$

$$\nabla \bullet \boldsymbol{G} = 16y + 0 + 0 = 16y$$

$$\psi = \iiint 16y\,dxdydz = 16\int_0^1 dx\int_0^1 dz\int_0^1 ydy = 16(1)(1)\left(\frac{y^2}{2}\bigg|_0^1\right) = \underline{\underline{8}}$$

(c)



$$\oint_L \boldsymbol{G} \bullet d\boldsymbol{l} = \int_{x=0}^{x=1} (16xy - z)dx\Big|_{\substack{y=0\\z=0}} + \int_{y=0}^{y=1} 8x^2 dy\Big|_{\substack{x=1\\z=0}} + \int_{x=1}^{x=0} (16xy - z)dx\Big|_{\substack{y=1\\z=0}} + \int_{y=1}^{y=0} 8x^2 dy\Big|_{\substack{x=0\\z=0}}$$

$$= 0 + 8(1)y\Big|_0^1 + 16(1)\frac{x^2}{2}\Big|_1^0 + 0$$

$$= 8 - 8 = \underline{\underline{0}}$$

This is expected since $\boldsymbol{G}$ is irrotational, i.e.

$$\oint \boldsymbol{G} \bullet d\boldsymbol{l} = \int (\nabla \times \boldsymbol{G}) \bullet d\boldsymbol{S} = 0$$

$$\nabla \bullet \boldsymbol{T} = -6 + 0 = \underline{\underline{-6}}$$

Prob. 3.57

$$\nabla \cdot \boldsymbol{F} = 0$$

$$\nabla x \boldsymbol{F} = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ \dfrac{\partial}{\partial \boldsymbol{x}} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = (x-x)\boldsymbol{a}_x + (y-y)\boldsymbol{a}_y + (z-z)\boldsymbol{a}_z = \boldsymbol{0}$$

Hence **F** is both solenoidal and conservative.

Prob. 3.58

$$\nabla \times \boldsymbol{H} = \boldsymbol{0}$$

$$\oint_L H \bullet dl = \int_S (\nabla \times \boldsymbol{H}) \bullet d\boldsymbol{S} = 0$$

Prob. 3.59

From Appendix A.10,

$$\nabla \bullet (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \bullet (\nabla \times \boldsymbol{A}) - \boldsymbol{A} \bullet (\nabla \times \boldsymbol{B})$$

If **A** and **B** are irrotational,

$$\nabla \times \boldsymbol{A} = \boldsymbol{0} = \nabla \times \boldsymbol{B}$$

i.e. $\nabla \bullet (\boldsymbol{A} \times \boldsymbol{B}) = 0$

which implies that $\boldsymbol{A} \times \boldsymbol{B}$ is solenoidal.

CHAPTER 4

P. E. 4.1

$$(a)\quad \boldsymbol{F} = \frac{1\times10^{-9}}{4\pi\left(\frac{10^{-9}}{36\pi}\right)}\left[\frac{5\times10^{-9}[(1,-3,7)-(2,0,4)]}{[(1,-3,7)-(2,0,4)]^3} + \frac{(-2\times10^{-9})[(1,-3,7)-(-3,0,5)]}{[(1,-3,7)-(-3,0,5)]^3}\right]$$

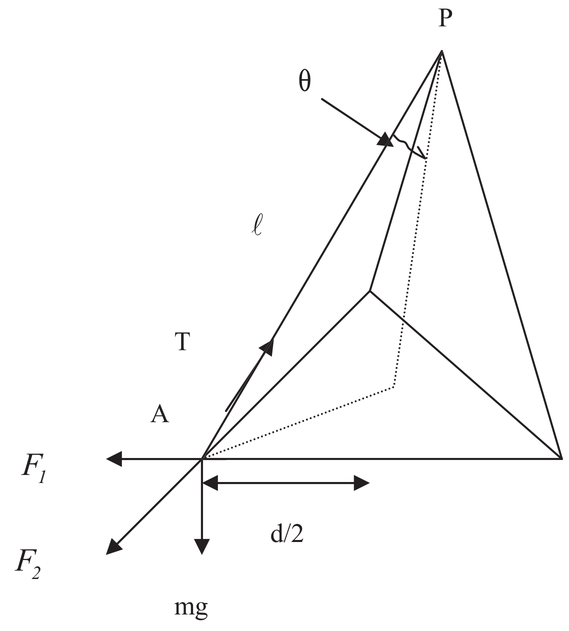
$$= [\frac{45(-1,-3,3)}{19^{3/2}} - \frac{18(4,-3,2)}{29^{3/2}}] \quad \text{nN}$$

$$= \underline{\underline{-1.004\boldsymbol{a}_x - 1.284\boldsymbol{a}_y + 1.4\boldsymbol{a}_z \ \text{nN}}}$$

$$(b)\quad \boldsymbol{E} = \frac{\boldsymbol{F}}{Q} = \underline{\underline{-1.004\boldsymbol{a}_x - 1.284\boldsymbol{a}_y + 1.4\boldsymbol{a}_z \ \text{V/m}}}$$

P. E. 4.2

Let q be the charge on each sphere, i.e. q=Q/3. The free body diagram below helps us to establish the relationship between various forces.



At point A,

$$T\sin\theta\cos 30° = F_1 + F_2\cos 60°$$

$$= \frac{q^2}{4\pi\varepsilon_0 d^2} + \frac{q^2}{4\pi\varepsilon_0 d^2}(\frac{1}{2})$$

$$= \frac{3q^2}{8\pi\varepsilon_0 d^2}$$

$$T\cos\theta = mg$$

Hence, $$\tan\theta\cos 30° = \frac{3q^2}{8\pi\varepsilon_0 d^2 mg}$$

But $$\sin\theta = \frac{h}{l} = \frac{d}{\sqrt{3}\, l} \tan\theta = \frac{\frac{d}{\sqrt{3}}}{\sqrt{l^2 - \frac{d^2}{3}}}$$

Thus, $$\frac{\frac{d}{\sqrt{3}}(\frac{\sqrt{3}}{2})}{\sqrt{l^2 - \frac{d^2}{3}}} = \frac{3q^2}{8\pi\varepsilon_0 d^2 mg}$$

or $$q^2 = \frac{4\pi\varepsilon_0 d^3 mg}{3\sqrt{l^2 - \frac{d^2}{3}}}$$

but $q = \dfrac{Q}{3} \longrightarrow q^2 = \dfrac{Q}{9}$. Hence,

$$Q^2 = \frac{12\pi\varepsilon_0 d^3 mg}{\sqrt{l^2 - \frac{d^2}{3}}}$$

P.E. 4.3

$$e\bar{E} = m\frac{d^2\bar{l}}{dt^2}$$

$$eE_0(-2\bar{a}_x + \bar{a}_y) = m\left(\frac{d^2x}{dt^2}\bar{a}_x + \frac{d^2y}{dt^2}\bar{a}_y + \frac{d^2z}{dt^2}\bar{a}_z\right)$$

where $E_0 = 200$ kV / m

$$\frac{d^2z}{dt^2} = 0 \longrightarrow z = ct + c_2$$

$$m\frac{d^2x}{dt^2} = -2eE_0 \longrightarrow x = \frac{-2eE_0t^2}{2m} + c_3t + c_4$$

$$m\frac{d^2y}{dt^2} = eE_0 \longrightarrow y = \frac{eE_0t^2}{2m} + c_5t + c_6$$

At $t = 0$, $(x, y, z) = (0, 0, 0)$ $\quad c_1 = 0 = c_4 = c_6$

Also, $\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = (0, 0, 0)$

At $t = 0 \longrightarrow \quad c_1 = 0 = c_3 = c_5$

Hence, $\quad (x, y) = \dfrac{eE_0t^2}{2m} \quad (-2, 1)$

i.e. $2\,|y| = |x|$

Thus the largest value of is

$$80 \text{ cm} = \underline{\underline{0.8 \text{ m}}}$$

P.E. 4.4

(a)

Consider an element of area dS of the disk.

The contribution due to $dS = \rho d\phi d\rho$ is

$$dE = \frac{\rho_s dS}{4\pi\varepsilon_0 r^2} = \frac{\rho_s dS}{4\pi\varepsilon_0(\rho^2 + h^2)}$$

The sum of the contribution along ρ gives zero.

$$E_z = \frac{\rho_s}{4\pi\,\varepsilon_0}\int_{\rho=0}^{a}\int_{\phi=0}^{2\pi}\frac{h\rho\,d\rho\,d\phi}{(\rho^2 + h^2)^{3/2}} = \frac{h\,\rho_s}{2\,\varepsilon_0}\int_{\rho-0}^{a}\frac{\rho\,d\rho}{(\rho^2 + h^2)^{3/2}}$$

$$= \frac{h\,\rho_s}{4\,\varepsilon_0}\int_0^a(\rho^2 + h^2)^{-3/2}d(\rho^2) = \frac{h\rho_s}{4\varepsilon_0}(-2(\rho^2 + h^2)^{-1/2}\Big|_0^a$$

$$= \frac{\rho_s}{2\varepsilon_0}\left[1 - \frac{h}{(h^2 + a^2)^{1/2}}\right]$$

(b)

As $a \longrightarrow \infty$,

$$\boldsymbol{E} = \frac{\rho_s}{2\varepsilon_0}\boldsymbol{a}_z$$

(c) Let us recall that if *a/h* <<1 then $(1+a/h)^n$ can be approximated by *(1+na/h)*. Thus the expression for E_z from (a) can be modified for $a<<h$ as follows.

$$E_z = \frac{\rho_s}{2\varepsilon_o}\left[1-\frac{1}{\sqrt{1+\frac{a^2}{h^2}}}\right] = \frac{\rho_s}{2\varepsilon_o}\left[1-\left(1+\frac{a^2}{h^2}\right)^{-\frac{1}{2}}\right] \xrightarrow[a\to 0,\ but\ \rho_s\pi a^2 = Q]{} \frac{\rho_s}{2\varepsilon_o}\left[\frac{a^2}{2h^2}\right]$$

$$= \frac{\rho_s}{2\varepsilon_o}\left[\frac{\pi a^2}{2\pi h^2}\right] = \frac{Q}{4\pi\varepsilon_o h^2}$$

This is in keeping with original Coulomb's law.

P. E. 4.5

$$Q_S = \int \rho_S dS = \int_{-2}^{2}\int_{-2}^{2} 12|y|\, dx\, dy$$

$$= 12(4)\int_0^2 2y\, dy = \underline{\underline{192\ \text{mC}}}$$

$$\boldsymbol{E} = \int \frac{\rho_s dS}{4\pi\varepsilon r^2}\boldsymbol{a}_r = \int \frac{\rho_s dS\,|\boldsymbol{r}-\boldsymbol{r'}|}{4\pi\varepsilon_o\,|\boldsymbol{r}-\boldsymbol{r'}|^3}$$

where $\boldsymbol{r}-\boldsymbol{r'} = (0,0,10)-(x,y,z) = (-x,-y,10)$.

$$\boldsymbol{E} = \int_{x=-2}^{2} \int_{y=-2}^{2} \frac{12\,|\,y\,|10^{-3}(-x,-y,10)}{4\pi(\frac{10^{-9}}{36\pi})(x^2+y^2+100)^{3/2}}$$

$$= 108(10^6)[\int_{-2}^{2}|\,y\,|\int_{-2}^{2}\frac{-xdx\,dy\ \boldsymbol{a}_x}{(x^2+y^2+100)^{3/2}} + \int_{-2}^{2}\int_{-2}^{2}\frac{-y|\,y\,|\ dy\,dx\,\boldsymbol{a}_y}{(x^2+y^2+100)^{3/2}}$$

$$+10\,\boldsymbol{a}_z\int_{-2}^{2}\int_{-2}^{2}\frac{|\,y\,|\,dx\,dy}{(x^2+y^2+100)^{3/2}}]$$

$$\boldsymbol{E} = 108(10^7)\,\boldsymbol{a}_z\int_{-2}^{2}\ [2\int_{0}^{2}\frac{\frac{1}{2}d(y^2)}{(x^2+y^2+100)^{3/2}}]dx$$

$$= -216(10^7)\,\boldsymbol{a}_z\int_{-2}^{2}\ [\frac{1}{(x^2+104)^{1/2}} - \frac{1}{(x^2+100)^{1/2}}]dx$$

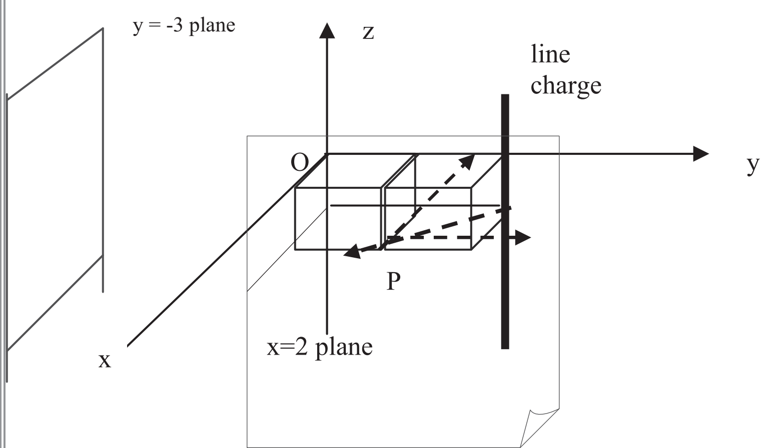
$$= -216(10^7)\,\boldsymbol{a}_z\ \ln|\frac{x+\sqrt{x^2+104}}{x+\sqrt{x^2+100}}|\,\Big|_{-2}^{2}$$

$$= -216(10^7)\boldsymbol{a}_z(\ln(\frac{2+\sqrt{108}}{2+\sqrt{104}}) - \ln(\frac{-2+\sqrt{108}}{-2+\sqrt{104}}))$$

$$= -216(10^7)\boldsymbol{a}_z(-7.6202\ (10^{-3}))$$

$$\boldsymbol{E} = \underline{\underline{16.46\ \boldsymbol{a}_z\ \text{MV/m}}}$$

P.E. 4.6



$\boldsymbol{E_1}$ and $\mathbf{E}_2$ remain the same as in Example 4.6.

$$\boldsymbol{E_3} = \frac{\rho_L}{2\pi\varepsilon_0\rho}\boldsymbol{a}_\rho$$

This expression, which represents the field due to a line charge,

is modified as follows. To get $\boldsymbol{a}_\rho$, consider the $z=-1$ plane. $\rho = \sqrt{2}$

$$\boldsymbol{a}_\rho = \boldsymbol{a}_x \cos 45° - \boldsymbol{a}_y \sin 45°$$

$$= \frac{1}{\sqrt{2}}(\boldsymbol{a}_x - \boldsymbol{a}_y)$$

$$\boldsymbol{E}_3 = \frac{10(10^{-9})}{2\pi(\frac{10^{-9}}{36\pi})}\frac{1}{2}(\boldsymbol{a}_x - \boldsymbol{a}_y)$$

$$= 90\pi(\boldsymbol{a}_x - \boldsymbol{a}_y). \quad \text{Hence,}$$

$$\boldsymbol{E} = \boldsymbol{E}_1 + \boldsymbol{E}_2 + \boldsymbol{E}_3$$

$$= -180\pi\boldsymbol{a}_x + 270\pi\boldsymbol{a}_y + 90\pi\boldsymbol{a}_x - 90\pi\boldsymbol{a}_y$$

$$= \underline{\underline{-282.7\,\boldsymbol{a}_x + 565.5\boldsymbol{a}_y \ \text{V/m}}}$$

P.E. 4.7

$$\boldsymbol{D} = \boldsymbol{D}_Q + \boldsymbol{D}_\rho = \frac{Q}{4\pi r^2}\boldsymbol{a}_r + \frac{\rho_s}{2}\boldsymbol{a}_n$$

$$= \frac{30\times10^{-9}}{4\pi(5)^2}\ \frac{[(0,4,3)-(0,0,0)]}{5} + \frac{10\times10^{-9}}{2}\boldsymbol{a}_y$$

$$= \frac{30}{500\pi}(0,4,3) + 5\boldsymbol{a}_y \ \text{nC/m}^2$$

$$= \underline{\underline{5.076\boldsymbol{a}_y + 0.0573\boldsymbol{a}_z \ \text{nC/m}^2}}$$

P.E. 4.8

(*a*) $\rho_v = \nabla \bullet \boldsymbol{D} = 4x$

$\rho_v(-1,0,3) = \underline{\underline{-4\ \text{C/m}^3}}$

(*b*) $\Psi = Q = \int \rho_v dv = \int_0^1\int_0^1\int_0^1 4x\,dx\,dy\,dz$

$= 4(1)(1)(1/2) = \underline{\underline{2\ \text{C}}}$

(*c*) $Q = \Psi = \underline{\underline{2\ \text{C}}}$

P.E. 4.9

$$Q = \int \rho v dv = \psi = \oint \boldsymbol{D} \bullet d\boldsymbol{S}$$

For $0 \le r \le 10$,

$$D_r (4\pi r^2) = \iiint 2r \ (r^2) \sin\theta \, d\theta \, dr \, d\phi$$

$$D_r (4\pi r^2) = 4\pi (\frac{2r^4}{4}\Big|_0^r) = 2\pi r^4$$

$$D_r = \frac{r^2}{2} \qquad \boldsymbol{E} = \frac{r^2}{2\varepsilon_0}\boldsymbol{a}_r \ \text{nV/m}$$

$$\boldsymbol{E}(r=2) = \frac{4(10^{-9})}{2(\frac{10^{-9}}{36\pi})}\boldsymbol{a}_r = 72\pi\boldsymbol{a}_r = \underline{\underline{226\ \boldsymbol{a}_r \ \text{V/m}}}$$

For $r \ge 10$,

$$D_r (4\pi r^2) = 2\pi r_0^{\ 4}, \qquad r_0 = 10\text{m}$$

$$D_r = \frac{r_0^{\ 4}}{2r^2} \longrightarrow \boldsymbol{E} = \frac{r_0^{\ 4}}{2\varepsilon_0 r^2}\boldsymbol{a}_r \ \text{nV/m}$$

$$\boldsymbol{E}(r=12) = \frac{10^4(10^{-9})}{2(\frac{10^{-9}}{36\pi})(144)}\boldsymbol{a}_r = 1250\pi\boldsymbol{a}_r$$

$$= \underline{\underline{3.927\boldsymbol{a}_r \ \text{kV/m}}}$$

P. E. 4.10

$$V(\boldsymbol{r}) = \sum_{k=1}^{3} \frac{Q_k}{4\pi\varepsilon_0 |\boldsymbol{r} - \boldsymbol{r}_k|} + C$$

$$At \ \ V(\infty) = 0, \quad C = 0$$

$$|\boldsymbol{r} - \boldsymbol{r}_1| = |(-1,5,2) - (2,-1,3)| = \sqrt{46}$$

$$|\boldsymbol{r} - \boldsymbol{r}_2| = |(-1,5,2) - (0,4,-2)| = \sqrt{18}$$

$$|\boldsymbol{r} - \boldsymbol{r}_3| = |(-1,5,2) - (0,0,0)| = \sqrt{30}$$

$$V(-1,5,2) = \frac{10^{-6}}{4\pi(\frac{10^{-9}}{36\pi})}[\frac{-4}{\sqrt{46}} + \frac{5}{\sqrt{18}} + \frac{3}{\sqrt{30}}]$$

$$= \underline{\underline{10.23\ k\text{V}}}$$

P.E. 4.11

$$V = \frac{Q}{4\pi\varepsilon_0 r} + C$$

If $V(0,6,-8) = V(r=10) = 2;$

$$2 = \frac{5(10^{-9})}{4\pi(\frac{10^{-9}}{36\pi})(10)} + C \qquad \longrightarrow \qquad C = -2.5$$

(a)

$$V_A = \frac{5(10^{-9})}{4\pi(\frac{10^{-9}}{36\pi})|(-3,2,6)-(0,0,0)|} - 2.5$$

$$= \underline{\underline{3.929\ V}}$$

(b)

$$V_B = \frac{45}{\sqrt{7^2+1^2+5^2}} - 2.5 = \underline{\underline{2.696\,V}}$$

(c) $V_{AB} = V_B - V_A = 2.696 - 3.929 = \underline{\underline{-1.233\,V}}$

P.E. 4.12

(a)

$$\frac{-W}{Q} = \int \boldsymbol{E} \bullet d\boldsymbol{l} = \int (3x^2 + y)dx + xdy$$

$$= \int_0^2 (3x^2+y)dx\,\Big|_{y=5} + \int_5^{-1} x\,dy\,\Big|_{x=2}$$

$$= 18 - 12 = 6 \text{ kV}$$

$$W = -6Q = \underline{\underline{12 \text{ mJ}}}$$

(b)

$$dy = -3\,dx$$

$$-\frac{W}{Q} = \int \boldsymbol{E} \bullet d\boldsymbol{l} = \int_0^2 (3x^2 + 5 - 3x)dx + x(-3)dx$$

$$= \int_0^2 (3\text{x}^2 - 6x + 5)dx = 8 - 12 + 10 = 6$$

$$W = \underline{\underline{12 \text{ mJ}}}$$

P.E. 4.13

(a)

$$(0,0,10) \longrightarrow (r=10, \theta=0, \phi=0)$$

$$V = \frac{100\cos 0}{4\pi\varepsilon_0 (10^2)}(10^{-12}) = \frac{10^{-12}}{4\pi(\frac{10^{-9}}{36\pi})} = \underline{\underline{9 \text{ mV}}}$$

$$\boldsymbol{E} = \frac{100(10^{-12})}{4\pi(\frac{10^{-9}}{36\pi})10^3}[2\cos 0\, \boldsymbol{a}_r + \sin 0\, \boldsymbol{a}_\theta]$$

$$= \underline{\underline{1.8\ \boldsymbol{a}_r \text{ mV/m}}}$$

(b)

$$At\ (1, \frac{\pi}{3}, \frac{\pi}{2}),$$

$$V = \frac{100\cos\frac{\pi}{3}(10^{-12})}{4\pi(\frac{10^{-9}}{36\pi})(1)^2} = \underline{\underline{0.45\ V}}$$

$$\boldsymbol{E} = \frac{100(10^{-12})}{4\pi(\frac{10^{-9}}{36\pi})(1)^2}(2\cos\frac{\pi}{3}\boldsymbol{a}_r + \sin\frac{\pi}{3}\boldsymbol{a}_\theta)$$

$$= \underline{\underline{0.9\,\boldsymbol{a}_r + 0.7794\boldsymbol{a}_\theta \text{ V/m}}}$$

P.E. 4.14

After Q_1, $W_1 = 0$

After Q_2, $W_2 = Q_2 V_{21} = \dfrac{Q_2\ Q_1}{4\pi\varepsilon_0|(1,0,0)-(0,0,0)|}$

$$= \frac{1(-2)(10^{-18})}{4\pi(10^{-9})\frac{1}{36\pi}} = \underline{\underline{-18 \text{ nJ}}}$$

After Q_3,

$$W_3 = Q_3(V_{31} + V_{32}) + Q_2 V_{21}$$

$$= 3(9)(10^{-9})\left\{\frac{1}{|(0,0,-1)-(0,0,0)|} + \frac{-2}{|(0,0,-1)-(1,0,0)|}\right\} - 18 \text{ nJ}$$

$$= 27(1 - \frac{2}{\sqrt{2}}) - 18$$

$$= \underline{\underline{-29.18 \text{ nJ}}}$$

After Q_4,

$$W_4 = Q_4(V_{41}+V_{42}+V_{43}) + Q_3(V_{31}+V_{32}) + Q_2V_{21}$$

$$= -4(9)(10^{-9})\left\{\frac{1}{|(0,0,1)-(0,0,0)|} + \frac{-2}{|(0,0,1)-(1,0,0)|} + \frac{3}{|(0,0,1)-(0,0,-1)|}\right\} + W_3$$

$$= -36\left(1-\frac{2}{\sqrt{2}}+\frac{3}{2}\right) + W_3$$

$$= -39.09 - 29.18\ nJ = \underline{\underline{-68.27}}\ \text{nJ}$$

P.E. 4.15

$$\boldsymbol{E} = -\nabla V = -(y+1)\boldsymbol{a}_x + (1-x)\boldsymbol{a}_y - 2\boldsymbol{a}_z$$

At (1,2,3), $\underline{\underline{\boldsymbol{E} = -3\boldsymbol{a}_x - 2\boldsymbol{a}_z \text{ V/m}}}$

$$W = \frac{1}{2}\varepsilon_o\int \boldsymbol{E}\bullet\boldsymbol{E}dv = \frac{1}{2}\varepsilon_o\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}(x^2+y^2-2x+2y+6)dxdydz$$

$$= \frac{1}{2}\varepsilon_o\left[\int_{-1}^{1}x^2dx\iint dydz + \int_{-1}^{1}y^2dy\iint dxdz - 2\int_{-1}^{1}xdx\iint dydz + 2\int_{-1}^{1}ydy\iint dxdz + 6(2)(2)(2)\right]$$

$$= \frac{1}{2}\varepsilon_o\left[2\frac{x^3}{3}\Bigg|_{-1}^{1}(2)(2)+0+0+6(8)\right] = \frac{80\varepsilon_o}{3}$$

$$= \underline{\underline{0.2358 \text{ nJ}}}$$

P.E. 4.16 (a)

$$\boldsymbol{E}(5,0,6) = \frac{Q_1}{4\pi\varepsilon_0}\frac{[(5,0,6)-(4,0,-3)]}{|(5,0,6)-(4,0,-3)|^3} + \frac{Q_2}{4\pi\varepsilon_0}\frac{[(5,0,6)-(2,0,1)]}{|(5,0,6)-(2,0,1)|^3}$$

$$= \frac{Q_1}{4\pi\varepsilon_0}\frac{(1,0,9)}{(\sqrt{82})^3} + \frac{Q_2}{4\pi\varepsilon_0}\frac{(3,0,5)}{(34)^{3/2}}$$

If $E_z = 0$, then

$$\frac{9Q_1}{4\pi\varepsilon_0}\frac{1}{(82)^{3/2}} + \frac{5Q_2}{4\pi\varepsilon_0}\frac{1}{(34)^{3/2}} = 0$$

$$Q_1 = -\frac{5}{9}Q_2\left(\frac{82}{34}\right)^{3/2} = -\frac{5}{9}4\left(\frac{82}{34}\right)^{3/2}\ \text{nC}$$

$$= \underline{\underline{-8.3232\ \text{nC}}}$$

(b)

$$\boldsymbol{F}(5,0,6) = q\boldsymbol{E}(5,0,6)$$

If $F_x = 0$, then

$$\frac{qQ_1}{4\pi\varepsilon_0(82)^{3/2}} + \frac{3qQ_2}{4\pi\varepsilon_0(34)^{3/2}} = 0$$

$$Q_1 = -3Q_2\left(\frac{82}{34}\right)^{3/2} = -12\left(\frac{82}{34}\right)^{3/2}\ nC$$

$$Q_1 = \underline{\underline{-44.945\ nC}}$$

P.E. 4.17

$$\boldsymbol{F}_e = \frac{e^2}{4\pi\varepsilon_o r^2}\boldsymbol{a}_r$$

$$\frac{F_e}{F_g} = \frac{e^2}{4\pi\varepsilon_o}\frac{1}{Gm^2} = \frac{1}{4\pi\times 10^{-9}/36\pi \times 6.67\times 10^{-11}}\left(\frac{1.6\times 10^{-19}}{9.1\times 10^{-31}}\right)^2$$

$$= \underline{\underline{4.17\times 10^{42}}}$$

P.E. 4.18

(a)

$$\boldsymbol{E} = \frac{Q\boldsymbol{R}_1}{4\pi\varepsilon_o R_1^3} - \frac{Q\boldsymbol{R}_2}{4\pi\varepsilon_o R_2^3}$$

A point on the x-axis is (x,0,0).

$$\boldsymbol{R}_1 = (x,0,0)-(0,0,d) = (x,0,-d)$$

$$R_1^3 = (x^2+d^2)^{3/2}$$

$$\boldsymbol{R}_2 = (x,0,0)-(0,0,-d) = (x,0,d)$$

$$R_2^3 = (x^2+d^2)^{3/2}$$

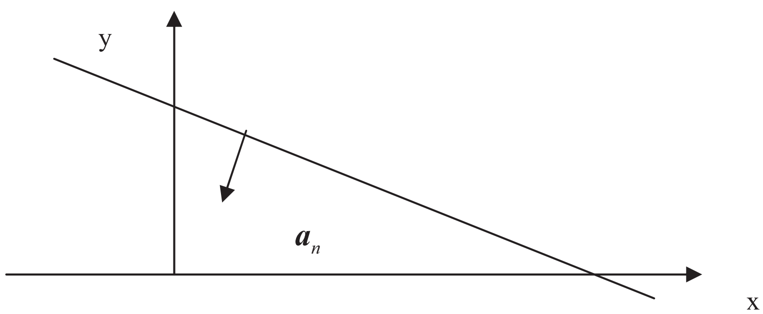
$$\boldsymbol{E} = \frac{Q}{4\pi\varepsilon_o(x^2+d^2)^{3/2}}\left[(x,0,-d)-(x,0,d)\right] = \underline{\underline{\frac{-2Qd\boldsymbol{a}_z}{4\pi\varepsilon_o(x^2+d^2)^{3/2}}}}$$

(b) A point along the z-axis is (0, 0, z).

$$\boldsymbol{R}_1 = (0,0,z) - (0,0,d) = (z-d)\boldsymbol{a}_z$$
$$R_1^3 = (z-d)^3$$

$$\boldsymbol{R}_2 = (0,0,z) - (0,0,-d) = (z+d)\boldsymbol{a}_z$$
$$R_2^3 = (z+d)^3$$
$$\boldsymbol{E} = \frac{Q(z-d)\boldsymbol{a}_z}{4\pi\varepsilon_o (z-d)^3} - \frac{Q(z+d)\boldsymbol{a}_z}{4\pi\varepsilon_o (z+d)^3} = \frac{Q\boldsymbol{a}_z}{4\pi\varepsilon_o}\left[\frac{1}{(z-d)^2} - \frac{1}{(z+d)^2}\right]$$
$$= \underline{\underline{\frac{Qdz\boldsymbol{a}_z}{\pi\varepsilon_o (z^2-d^2)^2}}}$$

P.E. 4.19



Let $f(x,y) = x + 2y - 5$; $\nabla f = \boldsymbol{a}_x + 2\boldsymbol{a}_y$

$$\boldsymbol{a}_n = \pm\frac{\nabla f}{|\nabla f|} = \pm\frac{(\boldsymbol{a}_x + 2\boldsymbol{a}_y)}{\sqrt{5}}$$

Since point $(-1,0,1)$ is below the plane,

$$\boldsymbol{a}_n = -\frac{(\boldsymbol{a}_x + 2\boldsymbol{a}_y)}{\sqrt{5}}.$$

$$\boldsymbol{E} = \frac{\rho_s}{2\varepsilon_0}\boldsymbol{a}_n = \frac{6(10^{-9})}{2(10^{-9}/36\pi)}\left(-\frac{(\boldsymbol{a}_x + 2\boldsymbol{a}_y)}{\sqrt{5}}\right)$$
$$= \underline{\underline{-151.7\boldsymbol{a}_x - 303.5\boldsymbol{a}_y \text{ V/m}}}$$

Prob. 4.1

$$\boldsymbol{F}_{Q_1} = \frac{Q_1 Q_2 (\boldsymbol{r}_{Q_1} - \boldsymbol{r}_{Q_2})}{4\pi\varepsilon \left| \boldsymbol{r}_{Q_1} - \boldsymbol{r}_{Q_2} \right|^3} = \frac{-20(10^{-12})[(3,2,1)-(-4,0,6)]}{4\pi \frac{10^{-9}}{36\pi} \left|(3,2,1)-(-4,0,6)\right|^3} = -180\frac{(7,2,-5)}{688.88}\times 10^{-3}$$

$$= \underline{\underline{-1.8291\boldsymbol{a}_x - 0.5226\boldsymbol{a}_y + 1.3065\boldsymbol{a}_z \text{ mN}}}$$

Prob. 4.2

(a)

$$\boldsymbol{E} = \sum_{k=1}^{2} \frac{Q(\boldsymbol{r}-\boldsymbol{r}_k^{'})}{4\pi\varepsilon_o \,|\, \boldsymbol{r}-\boldsymbol{r}_k^{'} \,|^3} = \frac{Q\left[(0,0,0)-(a,0,0)\right]}{4\pi\varepsilon_o \,|\,(0,0,0)-(a,0,0)\,|^3} - \frac{Q\left[(0,0,0)-(-a,0,0)\right]}{4\pi\varepsilon_o \,|\,(0,0,0)-(-a,0,0)\,|^3}$$

$$= \frac{Q(-a,0,0)}{4\pi\varepsilon_o a^3} - \frac{Q(a,0,0)}{4\pi\varepsilon_o a^3} = \underline{\underline{\frac{Q}{2\pi\varepsilon_o a^2}}}\boldsymbol{a}_x$$

(b)

$$\boldsymbol{E} = \frac{Q\left[(0,a,0)-(a,0,0)\right]}{4\pi\varepsilon_o \,|\,(0,a,0)-(a,0,0)\,|^3} - \frac{Q\left[(0,a,0)-(-a,0,0)\right]}{4\pi\varepsilon_o \,|\,(0,a,0)-(-a,0,0)\,|^3}$$

$$= \frac{Q(-a,a,0)}{4\pi\varepsilon_o (2a^2)^{3/2}} - \frac{Q(a,a,0)}{4\pi\varepsilon_o (2a^2)^{3/2}} = \underline{\underline{\frac{-Q}{4\sqrt{2}\pi\varepsilon_o a^2}\boldsymbol{a}_x}}$$

(c)

$$\boldsymbol{E} = \frac{Q\left[(a,0,a)-(a,0,0)\right]}{4\pi\varepsilon_o \,|\,(a,0,a)-(a,0,0)\,|^3} - \frac{Q\left[(a,0,a)-(-a,0,0)\right]}{4\pi\varepsilon_o \,|\,(a,0,a)-(-a,0,0)\,|^3}$$

$$= \frac{Q(0,0,a)}{4\pi\varepsilon_o a^3} - \frac{Q(2a,0,a)}{4\pi\varepsilon_o (5a^2)^{3/2}} = \underline{\underline{\frac{-Q}{10\sqrt{5}\pi\varepsilon_o a^2}\boldsymbol{a}_x + \frac{Q}{4\pi\varepsilon_o a^2}\left[1 - \frac{1}{5\sqrt{5}}\right]\boldsymbol{a}_z}}$$

Prob. 4.3

$$F = qE = mg \quad \longrightarrow \quad E = \frac{mg}{q} = \frac{2\times 9.8}{4\times 10^{-3}} = \underline{\underline{4.9 \text{ kV/m}}}$$

Prob. 4.4

(a) $$Q = \int \rho_L dl = \int_0^5 12x^2 dx \; = \; 4x^3 \Big|_0^5 mC = \underline{\underline{0.5\,C}}$$

(b)

$$Q=\int \rho_S dS = \int_{z=0}^{4}\int_{\phi=0}^{2\pi} \rho z^2 \rho\, d\phi\, dz \Big|_{\rho=3} = 9(2\pi)\frac{z^3}{3}\Big|_0^4 \ nC$$

$$= \underline{\underline{1.206 \ \mu C}}$$

(c)

$$Q=\int \rho_V dV = \iiint \frac{10}{r\sin\theta}\ r^2 \sin\theta\, d\theta\, d\phi\, dr$$

$$= 10\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \int_0^4 r dr = 10(2\pi)(\pi)\frac{4^2}{2}$$

$$= \underline{\underline{1579.1 \ C}}$$

Prob. 4.5

$$Q=\int_v \rho_v dv = \int_0^a\int_0^a\int_9^a \frac{\rho_o x}{a} dxdydz = (a)(a)\rho_o\left(\frac{x^2}{2a}\Big|_0^a\right) = \underline{\underline{\frac{a^3\rho_o}{2}}}$$

Prob. 4.6

$$Q=\int_v \rho_v dv = \int_{\rho=0}^{2}\int_{z=0}^{1}\int_{\phi=\pi/6}^{\pi/2} 5\rho^2 z \rho d\phi d\rho dz \ \text{mC}$$

$$=5\frac{\rho^4}{4}\Big|_0^2 \ \frac{z^2}{2}\Big|_0^1 \ \phi\Big|_{\pi/6}^{\pi/2} = \frac{5}{8}(16)(1)(\pi/2-\pi/6) = \frac{10\pi}{3}$$

$$Q=\underline{\underline{10.472 \ \text{mC}}}$$

Prob. 4.7

$$Q=\int \rho_s dS = \int \ 6xydxdy$$

$$=\int_{x=0}^{2}\int_{y=0}^{x} 6xydxdy + \int_{x=2}^{4}\int_{y=2}^{-x+4} 6xydxdy$$

$$=6\int_{x=0}^{2} x\frac{y^2}{2}\Big|_0^x dx + \int_{x=2}^{4} 6x\frac{y^2}{2}\Big|_0^{-x+4} dx$$

$$=6\int_{x=0}^{2} x(\frac{x^2}{2}-0)dx + 3\int_{x=2}^{4} x\left[(4-x)^2-0\right]dx$$

$$=6\int_0^2 \frac{x^3}{2}dx + 3\int_2^4 (16x-8x^2+x^3)dx$$

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$$= 3\frac{x^4}{4}\Bigg|_0^2 + 6(8x^2 - \frac{8x^3}{3} + \frac{x^4}{4})\Bigg|_2^4$$

$$= 12 + 3(128 - 32 - \frac{512}{3} + \frac{64}{3} + 64 - 4)$$

$$= 12 + 3(96 - \frac{448}{3} + 60)$$

$$Q \quad \underline{\underline{= 32 \text{ C}}}$$

Prob. 4.8

$$Q = \int_v \rho_v dv$$

$$= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} 6x^2y^2 dxdydz$$

$$= 6\int_{-1}^{1} dz \int_{-1}^{1} x^2 dx \int_{-1}^{1} y^2 dy$$

$$= 6(2)\frac{x^3}{3}\Bigg|_{-1}^{1} \frac{y^3}{3}\Bigg|_{-1}^{1}$$

$$= \frac{12}{9}(1 - -1)(1 - -1) = \frac{48}{9} \quad \text{nC}$$

$$\underline{\underline{= 5.33 \text{ nC}}}$$

Prob. 4.9

$$Q = \int_v \rho_v dv = \iint\int 4\rho^2 z \cos\phi \rho d\rho d\phi dz \qquad \text{nC}$$

$$= 4\int_0^2 \rho^3 d\rho \int_0^1 zdz \int_0^{\pi/4} \cos\phi d\phi = \rho^4 \Bigg|_0^2 \frac{z^2}{2}\Bigg|_0^1 (\sin\phi)\Bigg|_0^{\pi/4}$$

$$= (16)(0.5)(\sin\pi/4) = \underline{\underline{5.657 \text{ nC}}}$$

Prob. 4.10

For $2 < r < 4\text{cm}$

$$Q = \int_v \rho_v dv = \rho_v \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=2\text{cm}}^{4\text{cm}} 5r^2 \sin\theta dr d\theta d\phi$$

$$= 5(4\pi)\rho_v \int_{2\text{cm}}^{4\text{cm}} \mathrm{r}^2 dr$$

$$= 20\pi\rho_v \left. \frac{r^3}{3} \right|_{2\text{cm}}^{4\text{cm}}$$

$$= 20\pi \times 5 \times \frac{1}{3}(4^3 - 2^3) \times 10^{-6} \,\text{mC}$$

$$= \frac{100\pi}{3}(64-8)\text{nC}$$

$$= 5864.3 \text{ nC}$$

$$Q = \underline{\underline{5.864\mu\text{C}}}$$

Prob. 4.11

$$\boldsymbol{E} = \int_L \frac{\rho_L dl}{4\pi\varepsilon_o R^2} \boldsymbol{a}_R$$

$$\boldsymbol{R} = -a\boldsymbol{a}_\rho + h\boldsymbol{a}_z, R = |\boldsymbol{R}| = \sqrt{a^2+h^2}, dl = ad\phi$$

$$\boldsymbol{E} = \frac{\rho_L}{4\pi\varepsilon_o} \int \frac{(-a\boldsymbol{a}_\rho + h\boldsymbol{a}_z)}{(a^2+h^2)^{3/2}} ad\phi$$

Due to symmetry, the ρ-component cancels

$$\boldsymbol{E} = \frac{\rho_L}{4\pi\varepsilon_o} \int_0^{2\pi} \frac{ha\boldsymbol{a}_z d\phi}{(a^2+h^2)^{3/2}} = \frac{\rho_L ha\boldsymbol{a}_z}{4\pi\varepsilon_o (a^2+h^2)^{3/2}} (2\pi)$$

out.

$$\boldsymbol{F} = Q\boldsymbol{E} = \frac{4\times10^{-3} \times 12\times10^{-6} \times 4\times 3\boldsymbol{a}_z}{4\pi \times \frac{10^{-9}}{36\pi} \times 5^3}(2\pi) = \underline{\underline{260.58\boldsymbol{a}_z \text{ N}}}$$

Prob. 4.12

(a) At P(5,-1,4),

$$\boldsymbol{E} = \sum_{k=1}^{3} \frac{\rho_{sk}}{2\varepsilon_o}\boldsymbol{a}_{nk} = \frac{10\times10^{-6}}{2\times\frac{10^{-9}}{36\pi}}(\boldsymbol{a}_x) + \frac{-20\times10^{-6}}{2\times\frac{10^{-9}}{36\pi}}(\boldsymbol{a}_y) + \frac{30\times10^{-6}}{2\times\frac{10^{-9}}{36\pi}}(-\boldsymbol{a}_z)$$

$$= 36\pi(5,-10,-15)\times10^3 = \underline{\underline{565.5\boldsymbol{a}_x - 1131\boldsymbol{a}_y - 1696.5\boldsymbol{a}_z \text{ kV/m}}}$$

(b) At R(0,-2,1)

$$\boldsymbol{E} = 36\pi\left[5(-\boldsymbol{a}_x) - 10(\boldsymbol{a}_y) + 15(-\boldsymbol{a}_z)\right]\times10^3 = \underline{\underline{-565.5\boldsymbol{a}_x - 1131\boldsymbol{a}_y - 1696.5\boldsymbol{a}_z \text{ kV/m}}}$$

(c) At Q(3,-4,10),

$$\boldsymbol{E} = 36\pi\left[5\boldsymbol{a}_x - 10(-\boldsymbol{a}_y) + 15\boldsymbol{a}_z\right]\times10^3 = \underline{\underline{565.5\boldsymbol{a}_x + 1131\boldsymbol{a}_y + 1696.5\boldsymbol{a}_z \text{ kV/m}}}$$

Prob. 4.13

We apply $\boldsymbol{E} = \dfrac{\rho_s}{2\varepsilon_o}\boldsymbol{a}_n$

For z<0,

$$\boldsymbol{E} = \frac{1}{2\varepsilon_o}\left[-10(-\boldsymbol{a}_z) + 5(-\boldsymbol{a}_z) + 20(-\boldsymbol{a}_z\right]\times10^{-9} = \frac{1}{2\times\frac{10^{-9}}{36\pi}}(-10+5+20)(-\boldsymbol{a}_z)10^{-9}$$

$$= 18\pi\times15(-\boldsymbol{a}_z) = -848.23\boldsymbol{a}_z \text{ V/m}$$

For 0<z<1,

$$\boldsymbol{E} = \frac{1}{2\varepsilon_o}\left[-10(\boldsymbol{a}_z) + 5(-\boldsymbol{a}_z) + 20(-\boldsymbol{a}_z)\right]\times10^{-9} = \frac{1}{2\times\frac{10^{-9}}{36\pi}}(-10-5-20)(\boldsymbol{a}_z)10^{-9}$$

$$= -18\pi\times35(\boldsymbol{a}_z) = -1.979\boldsymbol{a}_z \text{ kV/m}$$

For 1 < z < 2,

$$\boldsymbol{E} = \frac{1}{2\varepsilon_o}\left[-10(\boldsymbol{a}_z) + 5(\boldsymbol{a}_z) + 20(-\boldsymbol{a}_z\right]\times 10^{-9} = \frac{1}{2\times\frac{10^{-9}}{36\pi}}(-10+5-20)(\boldsymbol{a}_z)10^{-9} = -18\pi(25)\boldsymbol{a}_z$$

$$= -1.4137\boldsymbol{a}_z \text{ kV/m}$$

For $z > 2$,

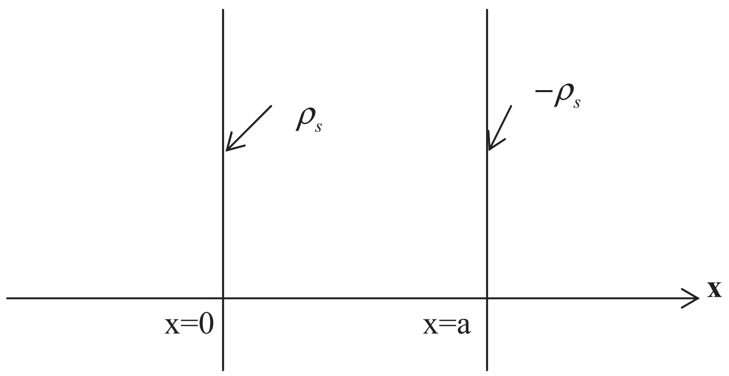
$$\boldsymbol{E} = \frac{1}{2\varepsilon_o}\left[-10(\boldsymbol{a}_z) + 5(\boldsymbol{a}_z) + 20\boldsymbol{a}_z\right]\times 10^{-9} = 18\pi(-10+5+20)(\boldsymbol{a}_z)$$

$$= 848.23\boldsymbol{a}_z \text{ V/m}$$

Thus,

$$\boldsymbol{E} = \begin{cases} -848.23\boldsymbol{a}_z \text{ V/m}, & z<0 \\ -1.979\boldsymbol{a}_z \text{ kV/m}, & 0<z<1 \\ -1.4137\boldsymbol{a}_z \text{ kV/m}, & 1<z<2 \\ 848.23\boldsymbol{a}_z \text{ V/m}, & z>2 \end{cases}$$

Prob. 4.14



(a) For $x < 0$,

$$\boldsymbol{E} = \boldsymbol{E}_1 + \boldsymbol{E}_2 = \frac{\rho_s}{2\varepsilon_o}(-\boldsymbol{a}_x) + \frac{(-\rho_s)}{2\varepsilon_o}(-\boldsymbol{a}_x) = \underline{\underline{\boldsymbol{0}}}$$

(b) For $0 < x < a$,.

$$\boldsymbol{E} = \frac{\rho_s}{2\varepsilon_o}\boldsymbol{a}_x + \frac{(-\rho_s)}{2\varepsilon_o}(-\boldsymbol{a}_x) = \underline{\underline{\frac{\rho_s}{\varepsilon_o}\boldsymbol{a}_x}}$$

(c) For $x > a$,

$$\boldsymbol{E} = \frac{\rho_s}{2\varepsilon_o}\boldsymbol{a}_x + \frac{(-\rho_s)}{2\varepsilon_o}(\boldsymbol{a}_x) = \underline{\underline{\boldsymbol{0}}}$$

Prob. 4.15

$$\boldsymbol{E} = \int_S \frac{\rho_s dS}{4\pi\varepsilon_o R^3}\boldsymbol{R}, \quad \boldsymbol{R} = \rho(-\boldsymbol{a}_\rho) + h\boldsymbol{a}_z, dS = \rho d\phi d\rho$$

$$\boldsymbol{E} = \frac{\rho_s}{4\pi\varepsilon_o}\int_S \frac{\rho d\phi d\rho}{(\rho^2+h^2)^{3/2}}(-\rho\boldsymbol{a}_\rho + h\boldsymbol{a}_z)$$

Due to symmetry, the ρ-component vanishes.

$$\boldsymbol{E} = \frac{\rho_s h\boldsymbol{a}_z}{4\pi\varepsilon_o}\int_a^b \rho(\rho^2+h^2)^{-3/2} d\rho \int_0^{2\pi} d\phi$$

Let $u=\rho^2+h^2, du = 2\rho d\rho$

$$\boldsymbol{E} = \frac{\rho_s h\boldsymbol{a}_z}{4\pi\varepsilon_o}(2\pi)\int \frac{1}{2}u^{-3/2}du = \frac{\rho_s h\boldsymbol{a}_z}{2\varepsilon_o}\frac{1/2u^{-1/2}}{-1/2} = \frac{\rho_s h\boldsymbol{a}_z}{2\varepsilon_o}\left(-\frac{1}{\sqrt{\rho^2+h^2}}\Bigg|_a^b\right)$$

$$\underline{\underline{\boldsymbol{E} = \frac{\rho_s h}{4\pi\varepsilon_o}\left[\frac{1}{\sqrt{a^2+h^2}} - \frac{1}{\sqrt{b^2+h^2}}\right]\boldsymbol{a}_z}}$$

Prob. 4.16

Let Q_1 be located at the origin. At the spherical surface of radius r,

$$Q_1 = \oint \boldsymbol{D}\bullet d\boldsymbol{S} = \varepsilon E_r(4\pi r^2)$$

Or

$$\boldsymbol{E} = \frac{Q_1}{4\pi\varepsilon r^2}\boldsymbol{a}_r \quad \text{by Gauss's law}$$

If a second charge Q_2 is placed on the spherical surface, Q_2 experiences a force

$$\boldsymbol{F} = Q_2\boldsymbol{E} = \frac{Q_1Q_2}{4\pi\varepsilon r^2}\boldsymbol{a}_r$$

which is Coulomb's law.

Prob. 4.17

For a point charge, $\boldsymbol{D} = \dfrac{Q}{4\pi R^3}\boldsymbol{R}$

For the given three point charges,

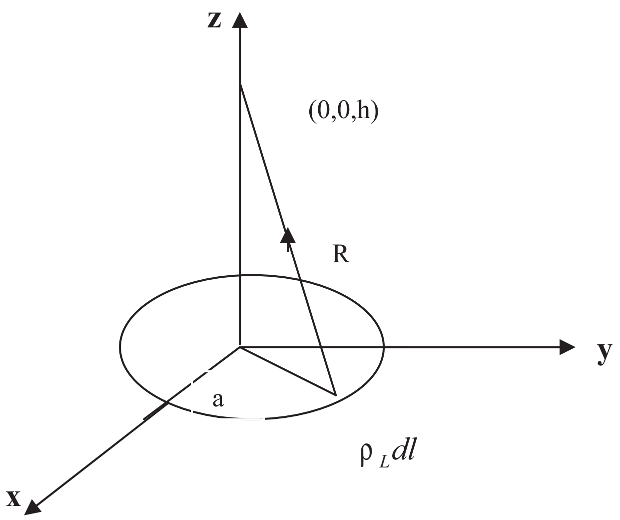
$$\boldsymbol{D} = \frac{1}{4\pi}\left[\frac{Q\boldsymbol{R_1}}{R_1^3} + \frac{Q\boldsymbol{R_2}}{R_2^3} - \frac{2Q\boldsymbol{R_3}}{R_3^3}\right]$$

$\boldsymbol{R}_1 = (0,0)-(-1,0) = (1,0), R_1 = 1$
$\boldsymbol{R}_2 = (0,0)-(1,0) = (-1,0), R_2 = 1$
$\boldsymbol{R}_3 = (0,0)-(0,1) = (0,-1), R_3 = 1$

$$\boldsymbol{D} = \frac{Q}{4\pi}\left[(1,0)+(-1,0)-2(0,-1)\right] = \frac{Q}{4\pi}(0,2) = \underline{\underline{\frac{Q}{2\pi}\boldsymbol{a}_y}}$$

Prob. 4.18

(a) Assume for now that the ring is placed on the z=0 plane.



$$\boldsymbol{D} = \int \frac{\rho_L dl\,\boldsymbol{R}}{4\pi R^3}, \boldsymbol{R} = -a\,\boldsymbol{a}_\rho + h\,\boldsymbol{a}_z$$

$$\boldsymbol{D} = \frac{\rho_L}{4\pi}\int_{\phi=0}^{\phi=2\pi} \frac{a d\phi(-a\,\boldsymbol{a}_\rho + h\,\boldsymbol{a}_z)}{(a^2+h^2)^{3/2}}$$

Due to symmetry, the ρ component vanishes.

$$\boldsymbol{D} = \frac{\rho_L\, a(2\pi h)\boldsymbol{a}_z}{4\pi(a^2+h^2)^{3/2}} = \frac{\rho_L\; a h\,\boldsymbol{a}_z}{2(a^2+h^2)^{3/2}}$$

$a = 2, \quad h = 3, \quad \rho_L = 5\,\mu\text{C/m}$

Since the ring is actually placed in $x = 0$, $\boldsymbol{a}_z$ becomes $\boldsymbol{a}_x$.

$$\boldsymbol{D} = \frac{(6)(5)\,\boldsymbol{a}_x}{2(4+9)^{3/2}} = \underline{\underline{0.32\,\boldsymbol{a}_x\ \mu\text{C/m}^2}}$$

(b)

$$\boldsymbol{D}_Q = \frac{Q}{4\pi}\frac{[(3,0,0)-(0,-3,0)]}{|(3,0,0)-(0,-3,0)|^3} + \frac{Q}{4\pi}\frac{[(3,0,0)-(0,3,0)]}{|(3,0,0)-(0,3,0)|^3}$$

$$= \frac{Q(3,3,0)}{4\pi(18)^{3/2}} + \frac{Q(3,-3,0)}{4\pi(18)^{3/2}} = \frac{6Q(1,0,0)}{4\pi(18)^{3/2}}$$

$$\boldsymbol{D} = \boldsymbol{D}_R + \boldsymbol{D}_Q = 0$$

$$0.32(10^{-6}) + \frac{6Q}{4\pi(18)^{3/2}} = 0$$

$$\therefore\ Q = -0.32(4\pi)(18^{3/2})10^{-6}\frac{1}{6} = \underline{\underline{-51.182\mu C}}$$

Prob. 4.19

(a) $$\rho_v = \nabla\bullet\boldsymbol{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \underline{\underline{8y\ \text{C/m}^2}}$$

(b) $$\rho_v = \nabla\bullet\boldsymbol{D} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial\phi} + \frac{\partial D_z}{\partial z}$$

$$= 8\sin\phi - 2\sin\phi + 4z$$

$$= \underline{\underline{6\sin\phi + 4z\ \ \text{C/m}^3}}$$

(c) $$\rho_v = \nabla\bullet\boldsymbol{D} = \frac{1}{r^2}\frac{\partial}{\partial \mathrm{r}}(\mathrm{r}^2\mathrm{D}_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\mathrm{D}_\theta\sin\theta) + 0$$

$$= -\frac{2}{\mathrm{r}^4}\cos\theta + \frac{2\cos\theta}{\mathrm{r}^4}$$

$$= \underline{\underline{0}}$$

Prob. 4.20

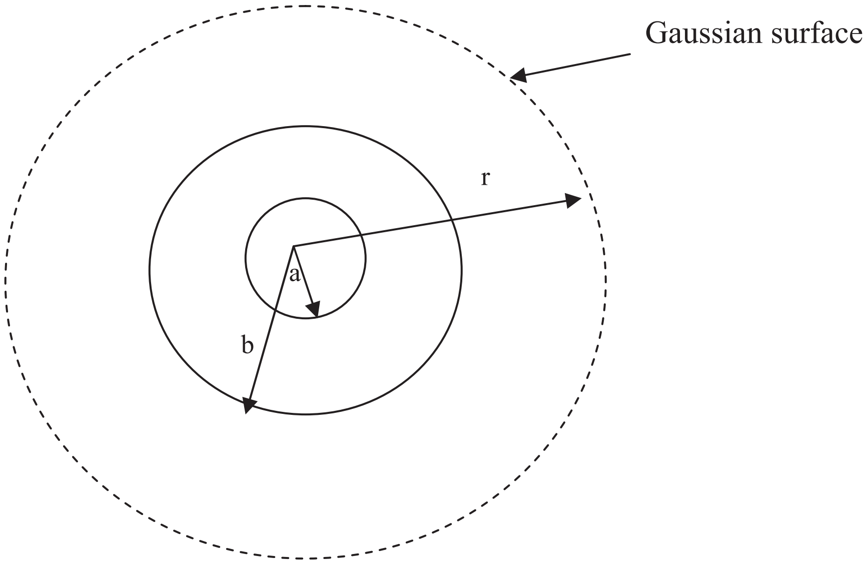
(a) $$\rho_v = \nabla\bullet\boldsymbol{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = 2(1+z^2) + 0 + 2x^2$$

$$= \underline{\underline{2(1+x^2+z^2)\ \ \text{nC/m}^3}}$$

(b) $$\psi = \int_S D\bullet dS = \int_{y=0}^{3}\int_{x=0}^{2} 2x^2 z\,dx\,dy\Big|_{z=1} = 2(1)\int_0^2 x^2 dx\int_0^3 dy$$

$$= 2\frac{x^3}{3}\Big|_0^2(3) = \underline{\underline{16\ \text{nC}}}$$

Prob. 4.21



Apply Gauss's law,

$$\oint \boldsymbol{D} \bullet d\boldsymbol{S} = Q_{enc}$$

$$D_r 4\pi r^2 = \rho_{s1} 4\pi a^2 + \rho_{s2} 4\pi b^2 = 8\times10^{-9} \times 4\pi(1)^2 + (-6\times10^{-3})\times 4\pi(2)^2 = -0.3016$$

$$D_r = \frac{-0.3016}{4\pi r^2} = \frac{=0.3016}{4\pi(3)^2} = -0.0027$$

$$\underline{\underline{\boldsymbol{D} = -2.7\boldsymbol{a}_r \quad \text{mC/m}^2}}$$

Prob. 4.22

For r <a.

$$\oint_S \boldsymbol{D} \bullet d\boldsymbol{S} = Q_{enc} = \int_v \rho_v dv$$

$$D_r(4\pi r^2) = \iiint 5r^{1/2} r^2 \sin\theta d\theta d\phi dr = 5\int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi \int_0^{r} r^{5/2} dr$$

$$= 5(2)(2\pi)\frac{r^{7/2}}{7/2}\bigg|_0^r = \frac{40\pi r^{7/2}}{7}$$

$$D_r = \varepsilon_o E_r = \frac{\frac{40\pi r^{7/2}}{7}}{4\pi r^2} = \frac{10}{7} r^{3/2}$$

$$E_r = \frac{10}{7\varepsilon_o} r^{3/2}, 0 < r < a$$

For r > a,

$$D_r(4\pi r^2) = \frac{40}{7}\pi a^{7/2}$$

$$D_r = \varepsilon_o E_r = \frac{\frac{40}{7}\pi a^{7/2}}{4\pi r^2}$$

$$E_r = \frac{10a^{7/2}}{7\varepsilon_o r^2}, r > a$$

Thus,

$$\boldsymbol{E} = \begin{cases} \dfrac{10}{7\varepsilon_o} r^{3/2}\boldsymbol{a}_r, 0 < r < a \\ \dfrac{10a^{7/2}}{7\varepsilon_o r^2}\boldsymbol{a}_r, r > a \end{cases}$$

Prob. 4.23

(a) $\rho_v = \nabla \bullet \boldsymbol{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \underline{\underline{2y \text{ C/m}^3}}$

(b) $\Psi = \int \boldsymbol{D} \cdot d\boldsymbol{S} = \iint x^2 dxdz \bigg|_{y=1} = \int_0^1 x^2\, dx \int_0^1 dz = \underline{\underline{\frac{1}{3}}} \text{ C}$

(c) $Q = \int_v \rho_v dv = \iiint 2y dxdydz = 2\int_0^1 dx \int_0^1 ydy \int_0^1 dz = \underline{\underline{1 \text{ C}}}$

Prob. 4.24

(a)

$$\rho_V = \nabla \bullet \boldsymbol{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z}$$

$$\rho_V = 4(z+1)\cos\phi - (z+1)\cos\phi + 0$$

$$\rho_V = \underline{\underline{3(z+1)\cos\phi \quad \mu\text{C/m}^3}}$$

(b)

$$Q_{enc} = \int \rho_V dv = \iiint 3(z+1)\cos\phi\ \rho d\phi\ d\rho dz$$

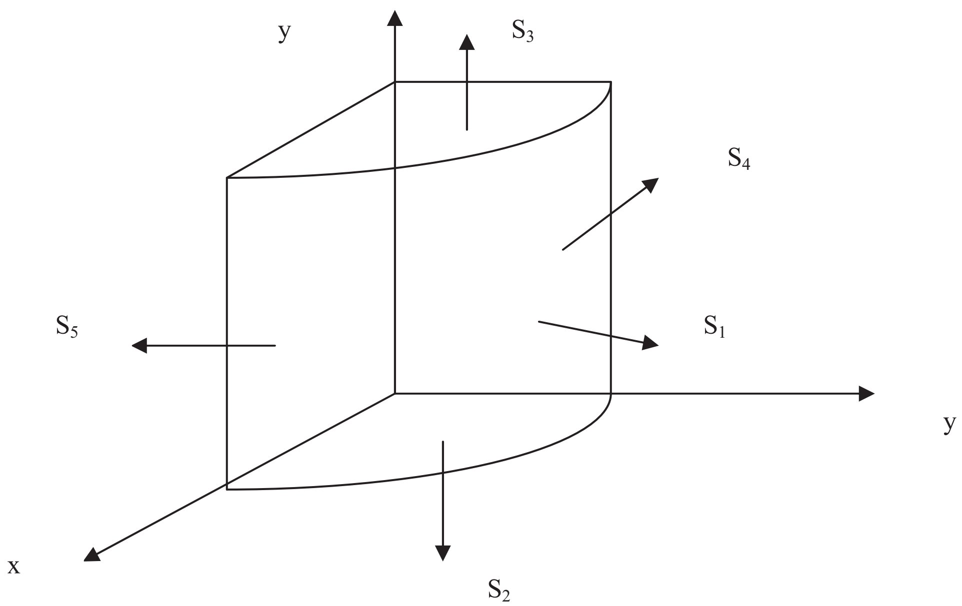
$$= 3\int_0^2 \rho d\rho \int_0^4 (z+1) \int_0^{\pi/2} \cos\phi d\phi = 3(2)(\frac{z^2}{2} + z)\Big|_0^4 (\sin\phi\Big|_0^{\pi/2})$$

$$= 6(8+4)(1-0) = \underline{\underline{72\mu\text{C}}}$$

(c)

Let $\psi = \psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5 = \oint \boldsymbol{D} \bullet d\boldsymbol{S}$

where $\psi_1,\ \psi_2,\ \psi_3,\ \psi_4,\ \psi_5$ respectively correspond witn surfaces $\mathrm{S}_1, \mathrm{S}_2, \mathrm{S}_3, \mathrm{S}_4, \mathrm{S}_4$ (in the figure below) respectively.



For S_1 $\rho = 2,\ d\boldsymbol{S} = \rho d\phi dz \boldsymbol{a}_\rho$

$$\psi_1 = \iint 2\rho(z+1)\cos\phi dS\Big|_{\rho=2} = 2(2)^2 \int_0^4 (z+1)dz \int_0^{\pi/2} \cos\phi d\phi$$

$= 8(12)(1) = 96$

For $S_2, z = 0, d\boldsymbol{S} = \rho d\phi d\rho(-\boldsymbol{a}_z)$

$$\psi_2 = -\iint \rho^2 \cos\phi \ \ \rho d\phi \ d\rho = -\int_0^2 \rho^3 d\rho \int_0^{\pi/2} \cos\phi d\phi$$

$$= -\frac{\rho^4}{4}\Big|_0^2 (1) = -4$$

For $S_{3,}\ z = 4, d\boldsymbol{S} = \rho d\phi d\rho \boldsymbol{a}_z,\ \ \psi_3 = +4$

For $\mathrm{S}_4, \phi = \pi/2, d\boldsymbol{S} = d\rho dz \boldsymbol{a}_\phi$

$$\psi_4 = -\iint \rho(z+1)\sin\phi \ d\rho dz\Big|_{d=\pi/2} = (11\int_0^2 \rho d\rho \int_0^4 (z+1)dz$$

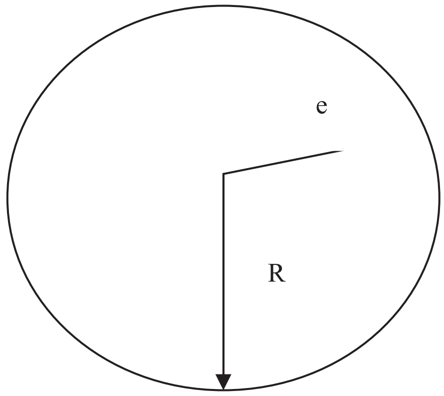
$$= -\frac{\rho^2}{2}\Big|_0^2 (12) = -(2)(12) = -24$$

For $S_5,\ \phi = 0, d\boldsymbol{S} = d\rho dz(-\boldsymbol{a}_\phi), \psi_5 = \iint \rho(z+1)\sin\phi \ \ d\rho dz\Big|_{\phi=0} = 0$

$$\psi \ = 96 - 4 + 4 - 24 + 0 = \underline{\underline{72\mu\mathrm{C}}}$$

This is exactly the answer obtained in part (b).

Prob. 4.25



$$F = eE$$

$$\rho_0 = \frac{e}{4\pi \frac{R^3}{3}} = \frac{3e}{4\pi R^3}$$

$$\rho_V = \begin{cases} \rho_{0,} & 0 < r < R \\ 0, & \text{elsewhere} \end{cases}$$

$$\oint \mathbf{D} \bullet d\mathbf{S} = Q_{enc} = \int \rho_V dV = \frac{3e}{4\pi R^3} \frac{4\pi r^3}{3} = D_r(4\pi r^2)$$

$$E_r = \frac{3e\,r}{12\pi\varepsilon_0 R^3}$$

$$F = eE = \underline{\underline{\frac{e^2\,r}{4\pi\,\varepsilon_0\,R^3}}}$$

Prob. 4.26

(a)

$$\psi = Q_{enc} \quad \text{at} \quad r = 2$$

$$Q_{enc} = \int \rho_V\, dV = \iiint \frac{10}{r^2} r^2 \sin\theta\, d\theta\, dr\, d\phi$$

$$= 10 \int_{r=1}^{2} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \sin\theta\, d\theta dr d\phi$$

$$= 10(1)\,(2\pi)\,(2) = (40\,\pi) \;\; \text{mC}$$

Thus, $\psi = \underline{\underline{125.7 \;\; \text{mC}}}$

At $r = 6$;

$$Q_{enc.} = 10 \int_{r=1}^{4} dr \int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\pi} \sin\theta\, d\theta$$

$$= 10\,(3)(2\pi)\,(2) = 120\pi \;\; \text{mC}$$

$$\psi = \underline{\underline{377 \;\; \text{mC}}}$$

(b)

$$\psi = Q_{enc}$$

But $\psi = \oint \boldsymbol{D} \bullet d\boldsymbol{S} = D_r \oint dS = D_r (4\pi r^2)$

At $r = 1$,

$$Q_{enc} = 0 \longrightarrow \underline{\underline{\boldsymbol{D} = 0}}$$

At $r = 5$, $Q_{enc} = 120\,\pi$

$$D_r = \frac{Q_{enc}}{4\pi r^2} = \frac{120\pi}{4\pi (5)^2} = 1.2$$

$$\boldsymbol{D} = \underline{\underline{1.2\boldsymbol{a}_r \text{ mC/m}^2}}$$

Prob. 4.27

$$\rho_v = \frac{Q}{\text{volume}} = \frac{Q}{4\pi a^3/3} = \frac{3Q}{4\pi a^3}$$

For r < a, $\oint \boldsymbol{D} \bullet d\boldsymbol{S} = Q_{enc} = \int \rho_v dv$

$$D_r 4\pi r^2 = \frac{3Q}{4\pi a^3}\frac{4\pi r^3}{3} = \frac{Qr^3}{a^3} \longrightarrow D_r = \frac{Qr}{4\pi a^3}$$

For r > a, $\oint \boldsymbol{D} \bullet d\boldsymbol{S} = Q$

$$D_r 4\pi r^2 = Q \longrightarrow D_r = \frac{Q}{4\pi r^2}$$

Hence,

$$\underline{\underline{\boldsymbol{D} = \begin{cases} \dfrac{Qr}{4\pi a^3}\boldsymbol{a}_r, & r < a \\ \dfrac{Q}{4\pi r^2}\boldsymbol{a}_r, & r > a \end{cases}}}$$

Prob. 4.28

$$V_P = \frac{Q_1}{4\pi\varepsilon_o r_1} + \frac{Q_2}{4\pi\varepsilon_o r_2} = \frac{10^{-9}}{4\pi \times \dfrac{10^{-9}}{36\pi}}\left[\frac{2}{|(1,-2,3)-(1,0,3)|} - \frac{4}{|(1,-2,3)-(-2,1,5)|}\right]$$

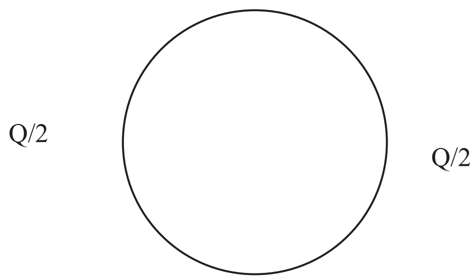
$$= 9\left[\frac{2}{2} - \frac{4}{\sqrt{9+9+4}}\right] = \underline{\underline{1.325 \text{ V}}}$$

Prob. 4.29

$$V = 4\frac{Q}{4\pi\varepsilon_o r}, \qquad r = \sqrt{a^2 + a^2 + h^2} = \sqrt{2^2 + 2^2 + 3^2} = \sqrt{17} \text{ cm}$$

$$V = \frac{4\times 8\times 10^{-9}}{4\pi\times\frac{10^{-9}}{36\pi}\times\sqrt{17}\times 10^{-2}} = \underline{\underline{6.985 \text{ kV}}}$$

Prob. 4.30 (a)



$$V = \frac{2\frac{Q}{2}}{4\pi\varepsilon_0 r} = \frac{Q}{4\pi\varepsilon_0 r}$$

$$= \frac{60(10^{-6})}{4\pi\times\frac{10^{-9}}{36\pi}x4} = \underline{\underline{135 \; k\text{V}}}$$

(b)

$$V = \frac{3\,(\frac{Q}{3})}{4\pi\varepsilon_0 r} = \underline{\underline{135 \; k\text{V}}}$$

(c)

$$V = \int\frac{\rho_L dl}{4\pi\varepsilon_0 r} = \frac{\frac{Q}{8\pi}2\pi(4)}{4\pi\varepsilon_0 r} \quad \frac{Q}{4\pi\varepsilon_0 r} \; = \; \underline{\underline{135 \; k\text{V}}}$$

Prob. 4.31

(a)

$$V_P = \sum \frac{Q_k}{4\pi | \boldsymbol{r}_p - \boldsymbol{r}_k |}$$

$$4\pi \varepsilon_o V_p = \frac{10^{-3}}{|(-1,1,2)-(0,0,4)|} + \frac{-2(10^{-3})}{|(-1,1,2)-(-2,5,1)|} + \frac{3(10^{-3})}{|(-1,1,2)-(3,-4,6)|}$$

$$4\pi \varepsilon_0 (10^3)\, V_p = \frac{1}{|(-1,1,-2)|} - \frac{2}{|(1,-4,1)} + \frac{3}{|(-4,5,-4)|} = \frac{1}{\sqrt{6}} - \frac{2}{\sqrt{18}} + \frac{3}{\sqrt{57}}$$

$$4\pi \frac{10^{-9}}{36\pi}(10^3)\, V_p = 0.3542$$

$$\therefore V_p = \underline{\underline{3.008\times 10^6 \text{ V}}}$$

(b)

$$V_Q = \sum \frac{Q_k}{4\pi\varepsilon_o | \boldsymbol{r}_p - \boldsymbol{r}_k |}$$

$$4\pi \varepsilon_o V_Q = \frac{10^{-3}}{|(1,2,3)-(0,0,4)|} + \frac{-2(10^{-3})}{|(1,2,3)-(-2,5,1)|} + \frac{3(10^{-3})}{|(1,2,3)-(3,-4,6)|}$$

$$4\pi \varepsilon_0 (10^3)\, V_p = \frac{1}{|(1,2,-1)|} - \frac{2}{|(3,-3,2)} + \frac{3}{|(-2,6,-3)|} = \frac{1}{\sqrt{6}} - \frac{2}{\sqrt{22}} + \frac{3}{7}$$

$$4\pi \frac{10^{-9}}{36\pi}(10^3)\, V_p = 0.410$$

$$V_Q = \underline{\underline{3.694\ (10^6)\text{V}}}$$

$$\therefore V_{PQ} = V_Q - V_P = 0.686(10^6) = \underline{\underline{686\ k\text{V}}}$$

Prob. 4.32

$$V = \int_S \frac{\rho_S dS}{4\pi\varepsilon_0 r}; \qquad \rho_S = \frac{1}{\rho}; \quad dS = \rho\, d\phi\, d\rho; \quad r = \sqrt{\rho^2 + h^2}$$

$$V = \frac{1}{4\pi\varepsilon_0} \iint \frac{\frac{1}{\rho}(\rho d\phi d\rho)}{(\rho^2 + h^2)^{1/2}} = \frac{1}{4\pi\varepsilon_0} \int_0^{2\pi} d\phi \int_{\rho=0}^{a} \frac{d\rho}{(\rho^2 + h^2)}$$

$$= \frac{2\pi}{4\pi\varepsilon_0} \ln(\rho + \sqrt{\rho^2 + h^2})\Big|_{\rho=0}^{a} = \frac{1}{2\varepsilon_0}[\ln(a + \sqrt{a^2 + h^2}) - \ln h]$$

$$= \underline{\underline{\frac{1}{2\varepsilon_0} \ln \frac{a + \sqrt{a^2 + h^2}}{h}}}$$

Prob. 4.33

(a)

$$\boldsymbol{E} = -\left(\frac{\partial V}{\partial x}\boldsymbol{a}_x + \frac{\partial V}{\partial y}\boldsymbol{a}_y + \frac{\partial V}{\partial z}\boldsymbol{a}_z\right)$$

$$= -2xy(z+3)\boldsymbol{a}_x - x^2(z+3)\boldsymbol{a}_y - x^2 y\,\boldsymbol{a}_z$$

At $(3,4,-6)$, $x=3,\ y=4,\ z=-6,$

$$\boldsymbol{E} = -2(3)(4)(-3)\boldsymbol{a}_x - 9(-3)\boldsymbol{a}_y - 9(4)\boldsymbol{a}_z$$

$$= \underline{\underline{72\boldsymbol{a}_x + 27\boldsymbol{a}_y - 36\boldsymbol{a}_z \text{ V/m}}}$$

(b)

$$\rho_V = \nabla \bullet \boldsymbol{D} = \varepsilon_0 \nabla \bullet \boldsymbol{E} = -\varepsilon_0 (2y)(z+3)$$

$$\psi = Q_{enc} = \int \rho_V dV = -2\varepsilon_0 \iiint y(z+3)dx\,dy\,dz$$

$$= -2\varepsilon_0 \int_0^1 dx \int_0^1 y\,dy \int_0^1 (z+3)dz = -2\varepsilon_0(1)(1/2)\left(\frac{z^2}{2}+3z\right)\Big|_0^1$$

$$= -\varepsilon_0\left(\frac{1}{2}+3\right) = \frac{-7}{2}\left(\frac{10^{-9}}{36\pi}\right)$$

$$Q_{enc} = \underline{\underline{-30.95 \ \ pC}}$$

Prob. 4.34

(a)

$$Q = \int_v \rho_v dv = \iiint \rho_o \left(1 - \frac{r^2}{a^2}\right) r^2 \sin\theta d\theta d\phi dr$$

$$= \rho_o \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \int_0^a \left(r^2 - \frac{r^4}{a^2}\right) dr = \rho_o (2\pi)(2)\left(\frac{a^3}{3} - \frac{a^3}{5}\right) = \underline{\underline{\frac{8\pi}{15}a^3\rho_o}}$$

(b) Outside the nucleus, r >a,

$$\oint_S \boldsymbol{D}\bullet d\boldsymbol{S} = Q_{enc} \longrightarrow \boldsymbol{E} = \frac{Q_{enc}}{4\pi\varepsilon_o r^2}\boldsymbol{a}_r$$

$$\boldsymbol{E} = \frac{\frac{8\pi a^3 \rho_o}{15}}{4\pi\varepsilon_o r^2}\boldsymbol{a}_r = \underline{\underline{\frac{2a^3\rho_o}{15\varepsilon_o r^2}\boldsymbol{a}_r}}$$

$$V = -\int \boldsymbol{E}\bullet d\boldsymbol{l} = -\int E_r dr = \frac{2a^3\rho_o}{15\varepsilon_o r} + C_1$$

Since $V(\infty) = 0, C_1 = 0.$

$$V = \underline{\underline{\frac{2a^3\rho_o}{15\varepsilon_o r}}}$$

(c) Inside the nucleus, r <a

$$Q_{enc} = 4\pi\rho_o\left(\frac{r^3}{3} - \frac{r^5}{5a^2}\right)$$

$$\boldsymbol{E} = \frac{Q_{enc}}{4\pi\varepsilon_o r^2}\boldsymbol{a}_r = \underline{\underline{\frac{\rho_o}{\varepsilon_o}\left(\frac{r}{3} - \frac{r^3}{5a^2}\right)\boldsymbol{a}_r}}$$

$$V = -\int E_r dr = -\frac{\rho_o}{\varepsilon_o}\left(\frac{r^2}{6} - \frac{r^4}{20a^2}\right) + C_2$$

$$V(r=a) = \frac{2a^2\rho_o}{15\varepsilon_o} = \frac{\rho_o}{\varepsilon_o}\left(\frac{a^2}{20} - \frac{a^2}{6}\right) + C_2$$

$$C_2 = \frac{2a^2\rho_o}{15\varepsilon_o} + \frac{7a^2\rho_o}{60\varepsilon_o} = \frac{a^2\rho_o}{4\varepsilon_o}$$

$$V = \underline{\underline{\frac{\rho_o}{\varepsilon_o}\left(\frac{r^4}{20a^2} - \frac{r^2}{6}\right) + \frac{a^2\rho_o}{4\varepsilon_o}}}$$

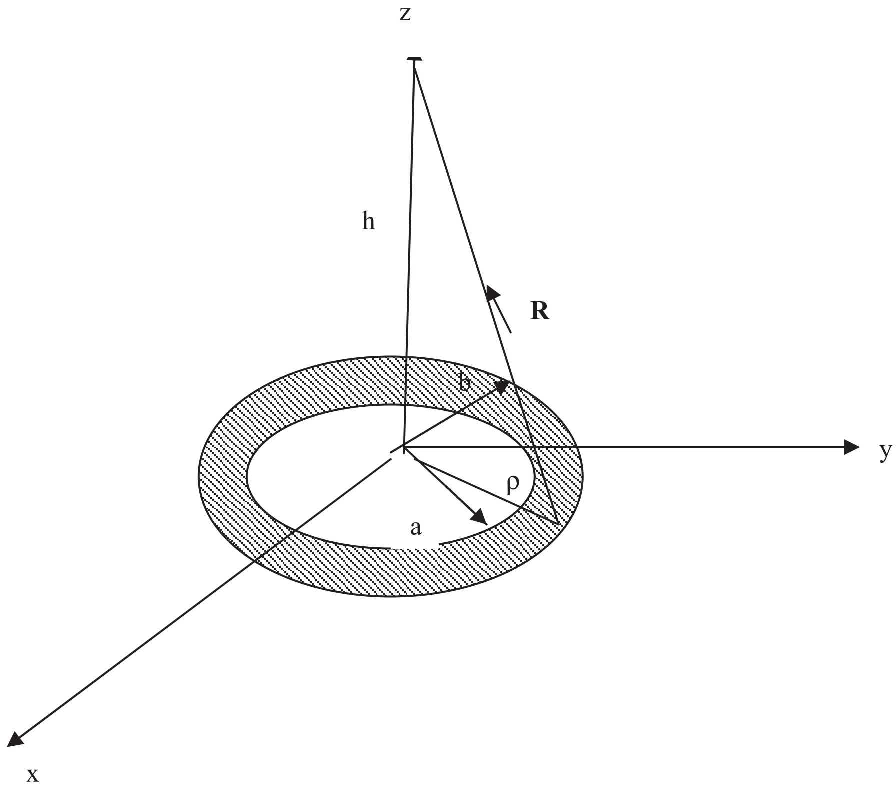
(d) E is maximum when

$$\frac{dE}{dr} = 0 = \frac{\rho_o}{\varepsilon_o}\left(\frac{1}{3} - \frac{3r^2}{5a^2}\right) \longrightarrow 9r^2 = 5a^2$$

$$r = \frac{\sqrt{5}}{3}a = \underline{\underline{0.7454a}}$$

We are able to say maximum because $\frac{d^2E}{dr^2} = -\frac{6r}{5a^2} < 0.$

Prob. 4.35



$$\mathbf{D} = \int \frac{\rho_s dS\mathbf{R}}{4\pi R^3}, \qquad \mathbf{R} = -\rho \mathbf{a}_\rho + h\mathbf{a}_z, \qquad R = |\mathbf{R}| = \sqrt{\rho^2 + h^2}, dS = \rho d\phi d\rho$$

$$\rho_s = \frac{Q}{S} = \frac{Q}{\pi(b^2 - a^2)}$$

$$\mathbf{D} = \frac{\rho_s}{4\pi} \int \frac{\rho d\phi d\rho(-\rho \mathbf{a}_\rho + h\mathbf{a}_z)}{(\rho^2 + h^2)^{3/2}}$$

Due to symmetry, the component along $\mathbf{a}_\rho$ vanishes.

$$D_z = \frac{\rho_s h}{4\pi} \int_{\rho=a}^{b} \int_{\phi=0}^{2\pi} \frac{\rho d\phi d\rho}{(\rho^2 + h^2)^{3/2}} = \frac{\rho_s h}{4\pi}(2\pi) \int_a^b (\rho^2 + h^2)^{-3/2} \rho d\rho$$

$$= \frac{\rho_s h}{2} \left[\frac{-1}{\sqrt{\rho^2 + h^2}} \right]_a^b = \frac{\rho_s h}{2} \left[\frac{1}{\sqrt{a^2 + h^2}} - \frac{1}{\sqrt{b^2 + h^2}} \right]$$

$$\underline{\underline{\mathbf{D} = \frac{Qh}{2\pi(b^2 - a^2)} \left[\frac{1}{\sqrt{a^2 + h^2}} - \frac{1}{\sqrt{b^2 + h^2}} \right] \mathbf{a}_z}}$$

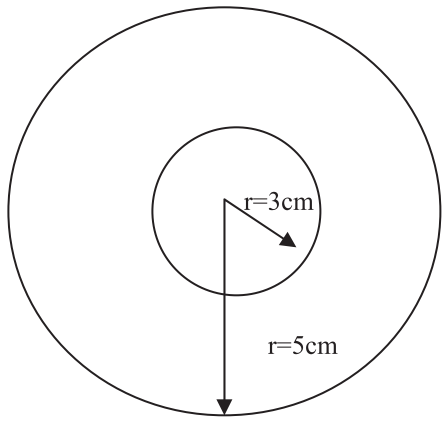
Prob. 4.36

$$\rho_v = \nabla \bullet D = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = 4 - 20y + 2z$$

At P(1,2,3), x=1, y= 2, z=3

$$\rho_v = 4 - 20(2) + 2(3) = \underline{\underline{-30 \text{ C/m}^3}}$$

Prob. 4.37



For r < 3cm, $Q_{enc} = 0 \longrightarrow \boldsymbol{D} = 0$

For 3 < r < 5cm,

$$\oint \boldsymbol{D} \bullet d\boldsymbol{S} = Q_{enc} = 10 \text{ nC}$$

$$D_r 4\pi r^2 = 10 \text{ nC} \longrightarrow D_r = \frac{10}{4\pi r^2} \text{ nC/m}^2$$

For r > 5 cm,

$$\oint \boldsymbol{D} \bullet d\boldsymbol{S} = Q_{enc} = 10 \text{ -5 = 5nC} \longrightarrow D_r = \frac{5}{4\pi r^2} \text{ nC/m}^2$$

Thus,

$$\boldsymbol{D} = \begin{cases} 0, & r < 3 \text{ cm} \\ \dfrac{10}{4\pi r^2}\boldsymbol{a}_r \text{ nC/m}^2, & 3 < r < 5 \text{ cm} \\ \dfrac{5}{4\pi r^2}\boldsymbol{a}_r \text{ nC/m}^2, & r > 5 \text{ cm} \end{cases}$$

Prob. 4.38

$$\rho_v = \nabla \bullet \boldsymbol{D} = \nabla \bullet \varepsilon_o \boldsymbol{E} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\frac{\varepsilon_o E_o \rho^2}{a}\right) \quad = \underline{\underline{\frac{2\varepsilon_o E_o}{a}}}, 0 < \rho < a$$

Prob. 4.39

Let us choose the following path of two segments.
$(2,1,-1) \rightarrow (5,1,-1) \rightarrow (5,1,2)$

$$W = -q\int \boldsymbol{E} \bullet d\boldsymbol{l}$$

$$-\frac{W}{q} = \int \boldsymbol{E} \bullet d\boldsymbol{l} = \int_{x=2}^{5} 2xyzdx \Bigg|_{z=-1,\, y=1} + \int_{z=-1}^{2} x^2 ydz \Bigg|_{x=5,\, y=1}$$

$$= 2(1)(-1)\frac{x^2}{2}\Bigg|_2^5 + (5)^2(1)z\Bigg|_{-1}^{2} = -21 + 75 = 54$$

$$W = -54q = \underline{\underline{-108\ \mu\text{J}}}$$

Prob. 4.40

(a)

From A to B, $d\boldsymbol{l} = rd\theta\, \boldsymbol{a}_\theta$,

$$W_{AB} = -Q\int_{\theta=30^\circ}^{90^\circ} 10r\cos\theta\, r\, d\theta \Big|_{r=5} = \underline{\underline{-1250\ \text{nJ}}}$$

(b)

From A to C, $d\boldsymbol{l} = dr\, \boldsymbol{a}_r$,

$$W_{AC} = -Q\int_{r=5}^{10} 20r\sin\theta\, dr \Big|_{\theta=30^\circ} = \underline{\underline{-3750\ \text{nJ}}}$$

(c)

From A to D, $d\boldsymbol{l} = r\sin\theta\, d\phi\, \boldsymbol{a}_\phi$,

$$W_{AD} = -Q\int 0(r\sin\theta)\, d\phi = \underline{\underline{0\ \text{J}}}$$

(d)

$W_{AE} = W_{AD} + W_{DF} + W_{FE}$

where F is $(10, 30^o, 60^o)$. Hence,

$$W_{AE} = -Q\left\{\int_{r=5}^{10} 20r\sin\theta\, dr \Big|_{\theta=30^\circ} + \int_{\theta=30^o}^{90^o} 10r\cos\theta\, r\, d\theta \Big|_{r=10}\right\}$$

$$= -100[\frac{75}{2} + \frac{100}{2}]\ \text{nJ} = \underline{\underline{-8750\ \text{nJ}}}$$

Prob. 4.41

$$V_{AB} = -\int_A^B \boldsymbol{E} \bullet d\boldsymbol{l} = -\int_1^5 \frac{10}{r^2} dr$$

$$= \frac{10}{r}\Big|_1^5 = 10(\frac{1}{5} - 1) = \underline{\underline{-8V}}$$

Prob. 4.42

$$A = (2, 3, -1)$$

$$\downarrow d\boldsymbol{l} = dx\boldsymbol{a}_x$$

$$A'(8, 3, -1) \xrightarrow{d\boldsymbol{l} = dy\boldsymbol{a}_y} B(8, 0, -1)$$

$$W = -Q\int \boldsymbol{E} \bullet d\boldsymbol{l}$$

$$-\frac{W}{Q} = \int \boldsymbol{E} \bullet d\boldsymbol{l} = \int_{x=2}^{8} 2xy^2 dx\Big|_{y=3} + \int_{y=3}^{0} 2y^2(x^2+1)dy\Big|_{x=8}$$

$$= 2(3)^2 \frac{x^2}{2}\Big|_2^8 + 2(8^2+1)\frac{y^3}{3}\Big|_3^0$$

$$= 9(64-4) + \frac{2}{3}65(0-81)$$

$$= 540 - 3510 = -2970$$

$$W = 2970Q = 5940 \text{ nJ}$$

$$\underline{\underline{= 5.94\ \mu J}}$$

Prob. 4.43

Method 1:

$$W = -Q\int_L \mathbf{E}\bullet d\mathbf{l}, \qquad d\mathbf{l} = \rho d\phi \mathbf{a}_\phi$$

$$\begin{bmatrix} E_\rho \\ E_\phi \\ E_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

$$E_\phi = -E_x \sin\phi + E_y \cos\phi = -20x\sin\phi + 40y\cos\phi$$

$$x = \rho\cos\phi, y = \rho\sin\phi$$

$$E_\phi = -20\rho\cos\phi\sin\phi + 40\rho\sin\phi\cos\phi = 20\rho\cos\phi\sin\phi$$

$$W = -Q\int_L \mathbf{E}\bullet d\mathbf{l} = -2\times10^{-3}\int 20\rho\cos\phi\sin\phi\rho d\phi \bigg|_{\rho=2}$$

$$= -2(20)(2)^2\int_0^\pi \sin\phi d(\sin\phi) \text{ mJ} = 160\frac{\sin^2\phi}{2}\bigg|_0^{\pi/2} = \underline{\underline{-80 \text{ mJ}}}$$

Method 2:

$$-\frac{W}{Q} = \int_L \mathbf{E}\bullet d\mathbf{l} = \int 20\text{xdx} + 40\text{ydy}$$

$$y = 2 - x, dy = -dx$$

$$-\frac{W}{Q} = \int 20\text{xdx} + 40(2-\text{x})(-\text{dx}) = \int_{\text{x}=2}^{0} (60\text{x} - 80)\text{dx}$$

$$= \frac{60x^2}{2} - 80x\bigg|_2^0 = 40$$

$$W = -40Q = \underline{\underline{-80 \text{ mJ}}}$$

Method 3:

$$\nabla\times\mathbf{E} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 20x & 40y & -10z \end{vmatrix} = 0$$

$$V = -\int_L \mathbf{E}\bullet d\mathbf{l} = -10\text{x}^2 - 20y^2 + 5z^2 + C$$

$$W = Q(V_2 - V_1) = Q(-20\times4 + 10\times4) = -40Q$$

$$W = -40Q = \underline{\underline{-80 \text{ mJ}}}$$

Prob. 4.44

$$W = -Q\int_L \boldsymbol{E}\bullet d\boldsymbol{l}$$

$$\boldsymbol{E} = \frac{\rho_s}{2\varepsilon_o}\boldsymbol{a}_n = \frac{\rho_s}{2\varepsilon_o}\boldsymbol{a}_x, \qquad d\boldsymbol{l} = dx\boldsymbol{a}_x$$

$$W = -\frac{Q\rho_s}{2\varepsilon_o}\int_3^1 dx = -\frac{Q\rho_s}{2\varepsilon_o}(-2) = \frac{Q\rho_s}{\varepsilon_o}$$

$$= \frac{10\times10^{-6}\times40\times10^{-9}}{\dfrac{10^{-9}}{36\pi}} = 400\times36\pi\times10^{-6} = \underline{\underline{45.24 \text{ mJ}}}$$

Prob. 4.45

(a) $$\boldsymbol{E} = -\nabla V = -\frac{\partial V}{\partial x}\boldsymbol{a}_x - \frac{\partial V}{\partial y}\boldsymbol{a}_y - \frac{\partial V}{\partial z}\boldsymbol{a}_y = \underline{\underline{-4x\boldsymbol{a}_x - 8y\boldsymbol{a}_y}}$$

$$\rho_v = \nabla\bullet D = \varepsilon_o\nabla\bullet E = \varepsilon_o(-4-8) = -12\varepsilon_o = \underline{\underline{-106.25 \text{ pC/m}^3}}$$

(b) $$\boldsymbol{E} = -\nabla V = -\frac{\partial V}{\partial \rho}\boldsymbol{a}_\rho - \frac{1}{\rho}\frac{\partial V}{\partial \phi}\boldsymbol{a}_\phi - \frac{\partial V}{\partial z}\boldsymbol{a}_z$$

$$= \underline{\underline{-(20\rho\sin\phi + 6z)\boldsymbol{a}_\rho - 10\rho\cos\phi\boldsymbol{a}_\phi - 6\rho\boldsymbol{a}_z}}$$

$$\rho_v = \nabla\bullet D = \varepsilon_o\nabla\bullet E = \varepsilon_o\left(-\frac{1}{\rho}\left[40\rho\sin\phi + 6z\right] + 10\sin\phi\right)$$

$$= \underline{\underline{-\left(30\sin\phi + \frac{6z}{\rho}\right)\varepsilon_o \text{ C/m}^3}}$$

(c) $$\boldsymbol{E} = -\nabla V = -\frac{\partial V}{\partial r}\boldsymbol{a}_r - \frac{1}{r}\frac{\partial V}{\partial \theta}\boldsymbol{a}_\theta - \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\boldsymbol{a}_\phi$$

$$= \underline{\underline{-10r\cos\theta\sin\phi\boldsymbol{a}_r + 5r\sin\theta\sin\phi\boldsymbol{a}_\theta - 5r\cot\theta\cos\phi\boldsymbol{a}_\phi}}$$

$$\rho_v = \nabla\bullet D = \varepsilon_o\nabla\bullet E$$

$$\frac{\rho_v}{\varepsilon_o} = \frac{1}{r^2}(-30r^2\cos\theta\sin\phi) + \frac{5r\sin\phi}{r\sin\theta}2\sin\theta\cos\theta + \frac{5r\cot\theta\sin\phi}{r\sin\theta}$$

$$\rho_v = \underline{\underline{\varepsilon_o(5\sin\phi\csc^2\theta\cos\theta - 20\cos\theta\sin\phi) \text{ C/m}^3}}$$

Prob. 4.46

$$V = -r^{-3}\sin\theta\cos\phi$$

$$-\boldsymbol{E} = \nabla V = \frac{\partial V}{\partial r}\boldsymbol{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\boldsymbol{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\boldsymbol{a}_\phi$$

$$= -3r^{-4}\sin\theta\cos\phi\boldsymbol{a}_r + r^{-4}\cos\theta\cos\phi\boldsymbol{a}_\theta + \frac{r^{-4}\sin\theta}{\sin\theta}(-\sin\phi)\boldsymbol{a}_\phi$$

$$\boldsymbol{E} = \frac{3}{r^4}\sin\theta\cos\phi\boldsymbol{a}_r - \frac{1}{r^4}\cos\theta\cos\phi\boldsymbol{a}_\theta + \frac{\sin\phi}{r^4}\boldsymbol{a}_\phi$$

At $(1, 30^o, 60^o), r = 1, \theta = 30^o, \phi = 60^o$

$$\boldsymbol{E} = 3\sin 30^o\cos 60^o\,\boldsymbol{a}_r - \cos 30^o\cos 60^o\,\boldsymbol{a}_\theta + \sin 60^o\,\boldsymbol{a}_\phi$$

$$= 0.75\boldsymbol{a}_r - 0.433\boldsymbol{a}_\theta + 0.866\boldsymbol{a}_\phi$$

$$\boldsymbol{D} = \varepsilon_o\boldsymbol{E} = \frac{10^{-9}}{36\pi}(0.75\boldsymbol{a}_r - 0.433\boldsymbol{a}_\theta + 0.866\boldsymbol{a}_\phi)$$

$$= 6.635\boldsymbol{a}_r - 3.829\boldsymbol{a}_\theta + 7.657\boldsymbol{a}_\phi \text{ pC/m}^2$$

Prob. 4.47

For a < r < b, we apply Gauss's law.

$$\oint_S D \bullet dS = Q_{enc} = Q$$

$$D_r(4\pi r^2) = Q \quad \longrightarrow \quad E_r = \frac{D_r}{\varepsilon_o} = \frac{Q}{4\pi\varepsilon_o r^2}$$

$$V_{ab} = -\int_a^b \boldsymbol{E}\bullet d\boldsymbol{l} = -\frac{Q}{4\pi\varepsilon_o}\int_a^b \frac{1}{r^2}dr = \frac{Q}{4\pi\varepsilon_o}\frac{1}{r}\Bigg|_a^b = -\frac{Q}{4\pi\varepsilon_o}\left(\frac{1}{a} - \frac{1}{b}\right)$$

Prob. 4.48

$$V = \int_S \frac{\rho_s dS}{4\pi\varepsilon_o r} = \frac{\rho_s}{4\pi\varepsilon_o}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2}\frac{r^2\sin\theta d\theta d\phi}{r} = \frac{\rho_s}{4\pi\varepsilon_o r}(2\pi)\int_0^{\pi/2}\sin\theta d\theta$$

$$= \frac{\rho_s}{2\varepsilon_o r}\left(-\cos\theta\Bigg|_0^{\pi/2}\right)\Bigg|_{r=a}$$

$$V = \frac{\rho_s}{2\varepsilon_o a}$$

Prob. 4.49

$$\nabla\times\boldsymbol{E}=0 \quad\longrightarrow\quad \nabla\times\boldsymbol{D}=0$$

$$\nabla\times\boldsymbol{D}=\left[\frac{1}{\rho}\frac{\partial D_z}{\partial\phi}-\frac{\partial D_\phi}{\partial z}\right]\boldsymbol{a}_\rho+\left[\frac{\partial D_\rho}{\partial z}-\frac{\partial D_z}{\rho}\right]\boldsymbol{a}_\phi+\frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho D_\phi)-\frac{\partial D_\rho}{\partial\phi}\right]\boldsymbol{a}_z$$

$$=0\,\boldsymbol{a}_\rho-0\boldsymbol{a}_\phi-\frac{1}{\rho}2\rho\cos\phi\boldsymbol{a}_z\neq 0$$

Hence D is not a genuine EM field.

$$\psi=\int_S \boldsymbol{D}\bullet d\boldsymbol{S}=\int_{\phi=0}^{\pi/4}\int_{z=0}^{1}2\rho\sin\phi\rho d\phi dz=2\int_0^{\pi/4}\sin\phi d\phi\int_0^1 dz\rho^2\Big|_{\rho=1}$$

$$=-2\cos\phi\Big|_0^{\pi/4}(1)(1)^2=-2(\cos\pi/4-1)=\underline{\underline{0.5858\text{ C}}}$$

Prob. 4.50

(a)

$m\dfrac{d^2y}{dt^2}=eE$; divide by m, and integrate once, one obtains:

$$u=\frac{dy}{dt}=\frac{eEt}{m}+c_0$$

$$y=\frac{eEt^2}{2m}+c_0t+c_1 \qquad (1)$$

"From rest" implies $c_1=0=c_0$

At $t=t_0$, $y=d$, $E=\dfrac{V}{d}$ or $V=Ed$.

Substituting this in (1) yields:

$$t^2=\frac{2md}{eE}$$

Hence:

$$u=\frac{eE}{m}\sqrt{\frac{2md}{eE}}=\sqrt{\frac{2eEd}{m}}=\sqrt{\frac{2eV}{m}}$$

that is, $u\ \alpha\ \sqrt{V}$

or $u=k\sqrt{V}$

(b)

$$k=\sqrt{\frac{2e}{m}}=\sqrt{\frac{2\,(1.603)\,10^{-19}}{9.1066\ (10^{-31})}}$$

$$=\underline{\underline{5.933\times10^5}}$$

(c)

$$V = \frac{u^2 \ m}{2e} = \frac{9(10^{16})\ \dfrac{1}{100}}{2(1.76)\ (10^{11})} = \underline{\underline{2.557 \ k \ \text{V}}}$$

Prob. 4.51

(a)

This is similar to Example 4.3.

$$u_y = \frac{eEt}{m}, \quad u_x = u_0$$

$$y = \frac{eEt^2}{2m}, \quad x = u_0 t$$

$$t = \frac{x}{u_0} = \frac{10(10^{-2})}{10^7} = 10 \text{ ns}$$

Since $x = 10\,\text{cm}$ when $y = 1\text{cm}$,

$$E = \frac{2my}{et^2} = \frac{2(10^{-2})}{1.76(10^{11})(10^{-16})} = 1.136 \ k\text{V/m}$$

$$\boldsymbol{E} = \underline{\underline{-\,1.136\ \boldsymbol{a}_y \ \text{kV/m}}}$$

(b)

$$u_x = u_0 = 10^7,$$

$$u_y = \frac{2000}{1.76}(1.76)10^{11}(10^{-8}) = 2(10^6)$$

$$\boldsymbol{u} = \underline{\underline{(\boldsymbol{a}_x + 0.2\boldsymbol{a}_y)(10^7) \ \text{m/s}}}$$

Prob. 4.52

$$\nabla \bullet \boldsymbol{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{20\cos\theta}{r}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{10\sin^2\theta}{r^3}\right) + 0$$

$$= -\frac{20\cos\theta}{r^4} + \frac{20\cos\theta}{r^4} = 0$$

$$\nabla \times \boldsymbol{E} = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(\frac{10\sin\theta}{r^2}\right) + \frac{20\sin\theta}{r^3}\right]\boldsymbol{a}_\phi$$

$$= \frac{1}{r}\left[-\frac{20\sin\theta}{r^3} + \frac{20\sin\theta}{r^3}\right]\boldsymbol{a}_\phi = \mathbf{0}$$

At r = 0, $\nabla \bullet \boldsymbol{E}$ and $\nabla \times \boldsymbol{E}$ are not defined. So they are zero every where except at the origin.

Prob. 4.53

The dipole is oriented along $y-$axis.

$$V=\frac{\boldsymbol{p}\bullet\boldsymbol{r}}{4\pi\varepsilon_0 r^2};\ \boldsymbol{p}\bullet\boldsymbol{r}=Q\,d\ \boldsymbol{a}_y\bullet\boldsymbol{a}_r=Qd\sin\theta\sin\phi$$

$$V=\frac{Qd\sin\theta\sin\phi}{4\pi\varepsilon_0 r^2}$$

$$\boldsymbol{E}=-\nabla V=-\frac{\partial V}{\partial r}\boldsymbol{a}_r-\frac{1}{r}\frac{\partial V}{\partial\theta}\boldsymbol{a}_\theta-\frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}\boldsymbol{a}_\phi$$

$$=\frac{Q\,d}{4\pi\varepsilon_0}\left\{\frac{2\sin\theta\sin\phi}{r^3}\boldsymbol{a}_r-\frac{\cos\theta\sin\phi}{r^3}\boldsymbol{a}_\theta-\frac{\cos\phi}{r^3}\boldsymbol{a}_\phi\right\}$$

$$\boldsymbol{E}=\underline{\underline{\frac{Q\,d}{4\pi\varepsilon_0 r^3}(2\sin\theta\sin\phi\,\boldsymbol{a}_r-\cos\theta\sin\phi\,\boldsymbol{a}_\theta-\cos\phi\,\boldsymbol{a}_\phi)}}$$

Prob. 4.54

$$V=\frac{p\cos\theta}{4\pi\varepsilon_0 r^2}=\frac{k\cos\theta}{r^2}$$

At (0,1 nm), $\theta=0$, $r=1$ nm, $V=9$;

that is, $9=\dfrac{k(1)}{1(10^{-18})}$, $\therefore\ k=9(10^{-18})$

$$V=9(10^{-18})\frac{\cos\theta}{r^2}$$

At (1,1) nm, $r=\sqrt{2}$ nm, $\theta=45°$,

$$V=\frac{9(10^{-18})\cos 45°}{10^{-18}(\sqrt{2})^2}=\frac{9}{2\sqrt{2}}=\underline{\underline{3.182\ \text{V}}}$$

Prob. 4.55

$$W=W_1+W_2=0+Q_2V_{21}=Q_2\frac{Q_1}{4\pi\varepsilon_o\,|(2,0,0)-(0,0,1)|}$$

$$=\frac{40\times10^{-9}\times(-50)\times10^{-9}}{4\pi\times\frac{10^{-9}}{36\pi}|(2,0,-1)|}=\frac{40\times9\times(-50)\times10^{-9}}{\sqrt{4+1}}$$

$$=\underline{\underline{-8.05\ \mu\text{J}}}$$

Prob. 4.56

$$\boldsymbol{E} = -\nabla V = -\frac{\partial V}{\partial x}\boldsymbol{a}_x - \frac{\partial V}{\partial y}\boldsymbol{a}_y = -4x\boldsymbol{a}_x - 12y\boldsymbol{a}_y \quad \text{V/m}$$

$$W = \frac{1}{2}\varepsilon_o \int_v |\boldsymbol{E}|^2\, dv = \frac{1}{2}\varepsilon_o \int_{z=-1}^{1}\int_{y=-1}^{1}\int_{x=-1}^{1} (16x^2 + 144y^2)dxdydz$$

$$= \frac{1}{2}\varepsilon_o \left[16(4)\frac{x^3}{3}\Bigg|_{-1}^{1} + 144(4)\frac{y^3}{3}\Bigg|_{-1}^{1}\right] = \frac{1}{2}\frac{10^{-9}}{36\pi}(160)(4)\frac{1}{3}(1+1)$$

$$= \underline{\underline{1.886 \text{ nJ}}}$$

Prob. 4.57

Given that $\boldsymbol{E} = 2r\sin\theta\cos\phi\boldsymbol{a}_r + r\cos\theta\cos\phi\boldsymbol{a}_\theta - r\sin\phi\boldsymbol{a}_\phi$

$$E^2 = 4r^2\sin^2\theta\cos^2\phi + r^2\cos^2\theta\cos^2\phi + r^2\sin^2\phi$$

$$= r^2\cos^2\phi\left(4\sin^2\theta + \cos^2\theta\right) + r^2\sin^2\phi$$

$$= \quad r^2\cos^2\phi + 3r^2\cos^2\phi\sin^2\theta + r^2\sin^2\phi$$

$$= r^2(1 + 3\cos^2\phi\sin^2\theta)$$

$$W = \frac{\varepsilon}{2}\iiint E^2 r^2 \sin\theta drd\theta d\phi$$

$$= \frac{\varepsilon}{2}\int_0^2 r^4 dr \int_{\theta=0}^{\pi}\int_{\phi}^{\pi}(1 + 3\cos^2\phi\sin^2\theta)\sin\theta d\theta d\phi$$

$$= \frac{16\varepsilon}{5}\int_0^{\pi}(\pi\sin\theta + \frac{3\pi}{2}\sin^2\theta)d\theta$$

$$= \frac{16}{5}x\frac{10^{-9}}{36\pi}(4\pi) = \frac{16}{45} \text{ nJ} = \underline{\underline{0.36 \text{ nJ}}}$$

Prob. 4.58

Method 1:

$$W = \frac{1}{2}\int_S \rho_s V dS = \frac{V}{2}\int_S \rho_s dS = \frac{1}{2}QV$$

But $\quad V = \dfrac{Q}{4\pi\varepsilon_o a}$

$$W = \underline{\underline{\frac{Q^2}{8\pi\varepsilon_o a}}}$$

Method 2:

$$W = \frac{1}{2}\int_v \boldsymbol{D}\bullet\boldsymbol{E}dv = \frac{1}{2}\varepsilon_o\int_v E^2 dv$$

$$= \frac{1}{2}\varepsilon_o\iiint\left(\frac{Q}{4\pi\varepsilon_o r^2}\right)^2 r^2\sin\theta d\theta dr d\phi$$

$$= \frac{1}{2}\varepsilon_o\int_0^{2\pi} d\phi\int_0^{\pi}\sin\theta d\theta\int_{r=a}^{\infty}\frac{Q^2}{16\pi^2\varepsilon_o^2 r^2}dr = \frac{\varepsilon_o}{2}(2\pi)(2)\frac{Q^2}{16\pi^2\varepsilon_o^2 a}$$

$$W = \underline{\underline{\frac{Q^2}{8\pi\varepsilon_o a}}}$$

Prob. 4.59

$$W = \frac{1}{2}\int_v \varepsilon_o\,|\,\boldsymbol{E}\,|^2\,dv$$

$$= \frac{\varepsilon_o}{2}\int_{z=0}^{3}\int_{y=0}^{2}\int_{x=0}^{1}(9x^2 + 25z^2)dxdydz$$

$$= \frac{1}{2}\frac{10^{-9}}{36\pi}\left[9\int_0^3 dz\int_0^2 dy\int_0^1 x^2 dx + 25\int_0^3 z^2 dz\int_0^2 dy\int_0^1 dx\right]$$

$$= \frac{1}{72\pi}\left[9(3)(2)\frac{x^3}{3}\bigg|_0^1 + 25(1)(2)\frac{z^3}{3}\bigg|_0^3\right]\text{ nJ} = \frac{1}{72\pi}(18 + 50\times 9) = \frac{468}{72\pi}\text{ nJ}$$

W=$\underline{\underline{2.069\text{ nJ}}}$

CHAPTER 5

P. E. 5.1 $\quad d\mathbf{S} = \rho d\phi dz \boldsymbol{a}_\rho$

$$I = \int J \bullet dS = \int_{\phi=0}^{2\pi}\int_{z=1}^{5} 10z\sin^2\phi\rho dz d\phi\,|_{\rho=2} = 10(2)\frac{z^2}{2}\Big|_1^5 \int_0^{2\pi}\frac{1}{2}(1-\cos 2\phi)d\phi = 240\pi$$

$\underline{\underline{I = 754\text{ A}}}$

P. E. 5.2

$$I = \rho_s wu = 0.5\times10^{-6}\times0.1\times10 = 0.5\mu\text{ A}$$

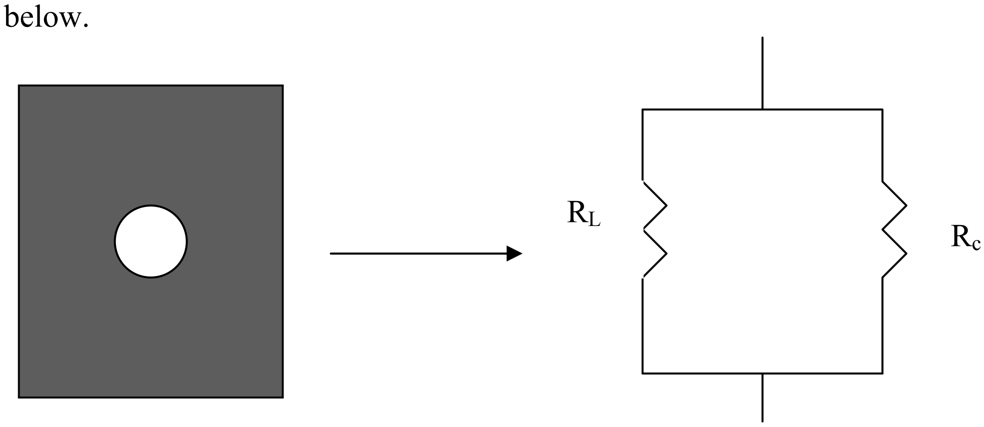
$$V = IR = 10^{14}\times0.5\times10^{-6} = \underline{\underline{50\text{ MV}}}$$

P. E. 5.3 $\quad \sigma = 5.8\times10^7$ S/m

$$J = \sigma E \quad \longrightarrow \quad E = \frac{J}{\sigma} = \frac{8\times10^6}{5.8\times10^7} = \underline{\underline{0.138\text{ V/m}}}$$

$$J = \rho_v u \quad \longrightarrow \quad u = \frac{J}{\rho_v} = \frac{8\times10^6}{1.81\times10^{10}} = \underline{\underline{4.42\times10^{-4}\text{ m/s}}}$$

P. E. 5.4 The composite bar can be modeled as a parallel combination of resistors as shown below.



For the lead, $\quad R_L = \dfrac{l}{\sigma_L S_L}, \quad S_L = d^2 - \pi r^2 = 9 - \dfrac{\pi}{4}\text{ cm}^2$

$R_L = 0.974\text{ m}\Omega$

For copper, $\quad R_c = \dfrac{l}{\sigma_c S_c}, \quad S_c = \pi r^2 = \dfrac{\pi}{4}\text{ cm}^2$

$$R_c = \frac{4}{5.8\times10^7 \times \frac{\pi}{4}\times10^{-4}} = 0.8781 \ \text{m}\,\Omega$$

$$R = \frac{R_L R_c}{R_L + R_c} = \frac{0.974x0.8781}{0.974+0.8781} = \underline{\underline{461.7\mu\Omega}}$$

P. E. 5.5 $\rho_{Ps} = \boldsymbol{P} \bullet \boldsymbol{a}_x = ax^2 + b$

$$\rho_{ps}\Big|_{x=0} = \boldsymbol{P}\bullet(-\boldsymbol{a}_x)\Big|_{x=0} = \underline{\underline{-b}}$$

$$\rho_{ps}\Big|_{x=L} = \boldsymbol{P}\bullet\boldsymbol{a}_x \ \Big|_{x=L} = \underline{\underline{aL^2 + b}}$$

$$Q_s = \int \rho_{ps} dS = -bA + (aL^2 + b)A = AaL^2$$

$$\rho_{pv} = -\nabla\bullet\boldsymbol{P} = -\frac{d}{dx}(ax^2 + b) = -2ax$$

$$\rho_{pv}\Big|_{x=0} = \underline{\underline{0}}, \qquad \rho_{pv}\Big|_{x=L} = \underline{\underline{-2aL}}$$

$$Q_v = \int \rho_{pv} dv = \int_0^L (-2ax)Adx = -AaL^2$$

Hence,

$$Q_T = Q_v + Q_s = -AaL^2 + AaL^2 = 0$$

P. E. 5.6

$$\boldsymbol{E} = \frac{V}{d}\boldsymbol{a}_x = \frac{10^3}{2x10^{-3}}\boldsymbol{a}_x = 500\boldsymbol{a}_x \ \ \text{kV/m}$$

$$\boldsymbol{P} = \chi_e\varepsilon_o\boldsymbol{E} = (2.55-1)x\frac{10^{-9}}{36\pi}x0.5x10^6\boldsymbol{a}_x = \underline{\underline{6.853\boldsymbol{a}_x\mu C/\text{m}^2}}$$

$$\rho_{ps} = \boldsymbol{P}\bullet\boldsymbol{a}_x = \underline{\underline{6.853\mu\text{C/m}^2}}$$

P. E. 5.7 (a) Since $P = \varepsilon_o\chi_e E$, $P_x = \varepsilon_o\chi_e E_x$

$$\chi_e = \frac{P_x}{\varepsilon_o E_x} = \frac{3x10^{-9}}{10\pi}\frac{1}{5}x36\pi x10^9 = \underline{\underline{2.16}}$$

(b) $E = \dfrac{P}{\chi_e \varepsilon_o} = \dfrac{36\pi \times 10^9}{2.16} \dfrac{1}{10\pi}(3,-1,4)10^{-9} = \underline{\underline{5a_x - 1.67a_y + 6.67a_z}}$ V/m

(c)

$D = \varepsilon_o \varepsilon_r E = \dfrac{\varepsilon_r P}{\chi_e} = \dfrac{3.16}{2.16}\left(\dfrac{1}{10\pi}\right)(3,-1,4)$ nC/m^2 $= \underline{\underline{139.7a_x - 46.6a_y + 186.3a_z}}$ pC/m^2

P. E. 5.8 From Example 5.8,

$$F = \frac{\rho_s^2 S}{2\varepsilon_o} \longrightarrow \rho_s^2 = \frac{2\varepsilon_o F}{S}$$

But $\rho_s = \varepsilon_o E = \varepsilon_o \dfrac{V_d}{d}$. Hence

$$\rho_s^2 = \frac{2\varepsilon_o F}{S} = \frac{\varepsilon_o^2 V_d^2}{d^2} \longrightarrow V_d^2 = \frac{2Fd^2}{\varepsilon_o S}$$

i.e.

$$V_d = V_1 - V_2 = \sqrt{\frac{2Fd^2}{\varepsilon_o S}}$$

as required.

P. E. 5.9 (a) Since $a_n = a_x$,

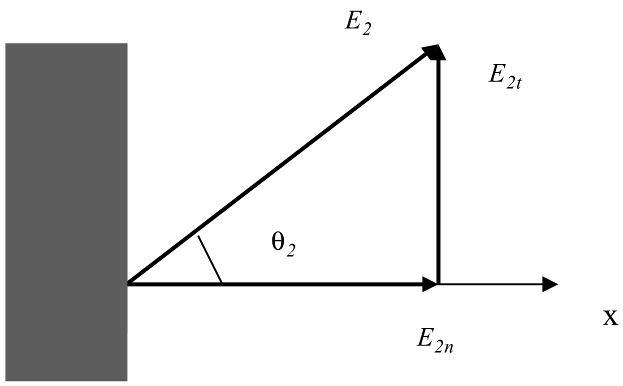
$D_{1n} = 12a_x, \quad D_{1t} = -10a_x + 4a_z, \quad D_{2n} = D_{1n} = 12a_x$

$$E_{2t} = E_{1t} \longrightarrow D_{2t} = \frac{\varepsilon_2 D_{1t}}{\varepsilon_1} = \frac{1}{2.5}(-10a_y + 4a_z) = -4a_y + 1.6a_z$$

$D_2 = D_{2n} + D_{2t} = \underline{\underline{12a_x - 4a_y + 1.6a_z}}$ nC/m^2.

$$\tan\theta_2 = \frac{D_{2t}}{D_{2n}} = \frac{\sqrt{(-4)^2 + (1.6)^2}}{12} = 0.359 \longrightarrow \underline{\underline{\theta_2 = 19.75^o}}$$

(b) $E_{1t} = E_{2t} = E_2 \sin\theta_2 = 12\sin 60^o = 10.392$



$$E_{1n} = \frac{\varepsilon_{r2}}{\varepsilon_{r1}} E_{2n} = \frac{1}{2.5} 12\cos 60^o = 2.4$$

$$E_1 = \sqrt{E_{1t}^{\ 2} + E_{1n}^{\ 2}} = \underline{\underline{10.67}}$$

$$\tan\theta_1 = \frac{\varepsilon_{r1}}{\varepsilon_{r2}} \tan\theta_2 = \frac{2.5}{1} \tan 60^o = 4.33 \quad \longrightarrow \quad \underline{\underline{\theta_1 = 77^o}}$$

Note that $\theta_1 > \theta_2$.

P. E. 5.10

$$D = \varepsilon_o E = \frac{10^{-9}}{36\pi}(60,20,-30)x10^{-3} = \underline{\underline{0.531a_x + 0.177a_y - 0.265a_z}} \ \text{pC/m}^2$$

$$\rho_s = D_n = |D| = \frac{10^{-9}}{36\pi}(10)\sqrt{36+4+9}(10^{-3}) = \underline{\underline{0.619}} \ \ \text{pC/m}^2$$

Prob. 5.1

$$I = \int \boldsymbol{J} \bullet d\boldsymbol{S}, \qquad d\boldsymbol{S} = dydz\boldsymbol{a}_x$$

$$I = \iint e^{-x}\cos(4y)dydz \Big|_{x=2} = e^{-2}\int_0^{\pi/3}\cos(4y)dy\int_0^4 dz$$

$$= 4e^{-2}\left(\frac{\sin 4y}{4}\Big|_0^{\pi/3}\right) = e^{-2}\left(\sin(\frac{4\pi}{3}) - 0\right) = \underline{\underline{-0.1172 \text{ A}}}$$

Prob. 5.2

Method 1:

$$I = \int \boldsymbol{J} \bullet d\boldsymbol{S} = \iint \frac{10}{r} e^{-10^3 t} r^2 \sin d\theta d\phi \Big|_{t=2ms, r=4m}$$

$$= 10(4)e^{-10^3\times 2\times 10^{-3}} \int_{\theta=0}^{\pi} \sin\theta d\theta \int_{\phi=0}^{2\pi} d\phi = 40e^{-2}(2)(2\pi) = 160\pi e^{-2}$$

$$= \underline{\underline{68.03 \text{ A}}}$$

Method 2:

$$I=\int \boldsymbol{J}\bullet d\boldsymbol{S}=\iint \frac{10}{r}e^{-10^3 t}dS=\frac{10}{r}e^{-10^3 t}(4\pi r^2)$$

since r is constant on the surface.

$$\text{I}=40\pi r e^{-2}=160\pi e^{-2} \quad =\underline{\underline{68.03 \text{ A}}}$$

Prob. 5.3

$$I=-\frac{dQ}{dt}=3x10^{-4}e^{-3t}$$

I(t=0) = $\underline{0.3 \text{ mA}}$, I(t=2.5) = 0.3 $e^{-7.5}$ = $\underline{166 \text{ nA}}$

Prob. 5.4

$$I=\int \boldsymbol{J}\bullet d\boldsymbol{S} \quad =5\int_{\rho=0}^{a}\int_{\phi=0}^{2\pi} e^{-10\rho}\rho d\phi d\rho=5\int_0^{2\pi}d\phi\int_{\rho=0}^{a}\rho e^{-10\rho}d\rho$$

$$=5(2\pi)\left(\frac{e^{-10\rho}}{100}(-10\rho-1)\right)\Bigg|_0^a=\frac{10\pi}{100}\left[e^{-10a}(-10a-1)-1(-0-1)\right]$$

$$=\frac{\pi}{10}\left[e^{-0.04}(-0.04-1)+1\right]=\frac{\pi}{10}(0.00078)=\underline{\underline{244.7\ \mu\text{A}}}$$

Prob. 5.5

$$I=\int \boldsymbol{J}\bullet d\boldsymbol{S}=\iint \frac{10}{\rho}\sin\phi \rho d\phi dz=10\int_0^5 dz\int_0^{\pi}\sin\phi d\phi$$

$$=10(5)(-\cos\phi)\Big|_0^{\pi}=\underline{\underline{100 \text{ A}}}$$

Prob. 5.6

$$R=\frac{l}{\sigma S} \quad \longrightarrow \quad \sigma=\frac{l}{RS}=\frac{2\times10^{-2}}{10^6(\pi)(4\times10^{-3})^2}=\underline{\underline{3.978\times10^{-4}\text{ S/m}}}$$

Prob. 5.7 (a) $R=\dfrac{l}{\sigma S}=\dfrac{8\times10^{-2}}{3\times10^4\pi(25)10^{-6}}=\dfrac{8}{75\pi}=\underline{\underline{33.95\text{m}\Omega}}$

(b) $I=V/R=9\times\dfrac{75\pi}{8}=\underline{\underline{265.1}}$ A

(c) P = IV = 2.386 kW

Prob. 5.8

If R and S are the same,

$$R_1 = \frac{\ell_1}{\sigma_1 S} = R_2 = \frac{\ell_2}{\sigma_2 S} \longrightarrow \ell_1 = \ell_2 \frac{\sigma_1}{\sigma_2}$$

If 1 corresponds to copper and 2 to silver,

$$\sigma_1 = 5.8\times10^7 \text{ S/m}, \quad \sigma_2 = 6.1\times10^7 \text{ S/m}$$

$$\ell_1 = \ell_2 \frac{5.8}{6.1} = 0.951\ell_2$$

That is, the copper wire is shorter than silver wire or the silver wire is longer.

Prob. 5.9 (a) $S_i = \pi r_i^2 = \pi(1.5)^2 x10^{-4} = 7.068x10^{-4}$

$$S_o = \pi(r_o^2 - r_i^2) = \pi(4-2.25)\times10^{-4} = 5.498\times10^{-4}$$

$$R_I = \frac{\rho_I l}{S_I} = \frac{11.8\times10^{-8}\times10}{7.068X10^{-4}} = 16.69\times10^{-4}$$

$$R_o = \frac{\rho_o l}{S_o} = \frac{1.77\times10^{-8}\times10}{5.498\times10^{-4}} = 3.219\times10^{-4}$$

$$R = R_i // R_o = \frac{R_i R_o}{R_i + R_o} = \frac{16.69\times3.219\times10^{-4}}{16.69+3.219} = \underline{\underline{0.27\text{m}\Omega}}$$

(b) $V = I_i R_i = I_o R_o \longrightarrow \frac{I_i}{I_o} = \frac{R_o}{R_i} = \frac{0.3219}{1.669} = 0.1929$

$$I_i + I_o = 1.1929 I_o = 60 \text{ A}$$

I_o = $\underline{\underline{50.3\text{ A}}}$ (copper), I_i = $\underline{\underline{9.7\text{ A}}}$ (steel)

Alternatively, using the principle of current division,

$$I_o = 60\frac{R_i}{R_i + R_o} = 50.3 \text{ A}$$

$$I_i = 60\frac{R_o}{R_i + R_o} = 9.7 \text{ A}$$

(c) $R = \frac{10\times1.77\times10^{-8}}{1.75\pi\times10^{-4}} = \underline{\underline{0.322\text{m}\Omega}}$

Prob. 5.10

From eq. (5.16),

$$R_1 = \frac{\rho_1 \ell}{S_1} = \frac{\rho_1 \ell}{ab}, \qquad R_2 = \frac{\rho_2 \ell}{S_2} = \frac{\rho_2 \ell}{ac}$$

$$R = R_1 \,||\, R_2 = \frac{R_1 R_2}{R_1 + R_2} = \frac{\frac{\rho_1 \ell}{ab}\frac{\rho_2 \ell}{ac}}{\frac{\rho_1 \ell}{ab} + \frac{\rho_2 \ell}{ac}} = \frac{\rho_1 \rho_2 \ell^2}{ac\rho_1 \ell + ab\rho_2 \ell}$$

$$R = \underline{\underline{\frac{\rho_1 \rho_2 \ell}{a(c\rho_1 + b\rho_2)}}}$$

Prob. 5.11

$$|\mathbf{P}| = n\,|\mathbf{p}| = nQd = 2ned = \chi_e \varepsilon_o E \qquad (Q = 2e)$$

$$\chi_e = \frac{2ned}{\varepsilon_o E} = \frac{2 \times 5 \times 10^{25} \times 1.602 \times 10^{-19} \times 10^{-18}}{\frac{10^{-9}}{36\pi} \times 10^4} = 0.000182$$

$$\varepsilon_r = 1 + \chi_e = \underline{\underline{1.000182}}$$

Prob. 5.12

$$\mathbf{P} = \frac{\sum_{i=1}^{N} q_i \mathbf{d}_i}{v} = \frac{\sum_{i=1}^{N} \mathbf{p}_i}{v}$$

$$|\mathbf{P}| = \frac{N}{v}\,|\mathbf{p}| = 2 \times 10^{19} \times 1.8 \times 10^{-27} = 3.6 \times 10^{-8}$$

$$\mathbf{P} = |\mathbf{P}|\,\mathbf{a}_x = \underline{\underline{3.6 \times 10^{-8} \mathbf{a}_x \text{ C/m}^2}}$$

But $\quad P = \chi_e \varepsilon_o E \quad$ or $\quad \chi_e = \dfrac{P}{\varepsilon_o E} = \dfrac{3.6 \times 36\pi \times 10^9 \times 10^{-18}}{10^5} = 0.0407$

$$\varepsilon_r = 1 + \chi_e = \underline{\underline{1.0407}}$$

Prob. 5.13

$$\boldsymbol{E} = \frac{Q}{4\pi\varepsilon_o\varepsilon_r r^2}\boldsymbol{a}_r$$

$$\boldsymbol{P} = \chi_e\varepsilon_o\boldsymbol{E} = \frac{\chi_e Q}{4\pi\varepsilon_r r^2}\boldsymbol{a}_r = \frac{3(10)10^{-3}}{4\pi(4)1^2}\boldsymbol{a}_r = \underline{\underline{596.8\boldsymbol{a}_r \quad \mu\text{C/m}^2}}$$

Prob. 5.14

$$\boldsymbol{P} = \chi_e\varepsilon_o\boldsymbol{E} \quad \longrightarrow \quad \boldsymbol{E} = \frac{\boldsymbol{P}}{\chi_e\varepsilon_o} = \frac{100\times10^{-9}}{2.5\frac{10^{-9}}{36\pi}(2)}\boldsymbol{a}_\rho = \underline{\underline{2.261\boldsymbol{a}_\rho \text{ kV/m}}}$$

$$\boldsymbol{D} = \varepsilon_o\varepsilon_r\boldsymbol{E} = 3.5\times\frac{10^{-9}}{36\pi}2.261\times10^3\boldsymbol{a}_\rho = \underline{\underline{70\boldsymbol{a}_\rho \text{ nC/m}^2}}$$

Prob. 5.15

(a)

$$Q_{s1} = \int_S \boldsymbol{P}\bullet d\boldsymbol{S}, d\boldsymbol{S} = r^2 \sin d\theta d\phi(-\boldsymbol{a}_r)$$

$$= -\iint 4r\, r^2 \sin\theta d\theta d\phi \Big|_{r=1.2cm}$$

$$= -4(1.2)^3 (10^{-6})\int_0^{2\pi} d\phi \int \sin\theta d\theta (10^{-12})$$

$$= -6.912(2\pi)(2)\times10^{-18}$$

$$= \underline{\underline{-86.86\times10^{-18} \text{ C}}}$$

(b)

$$Q_{s2} = \int_S \boldsymbol{P}\bullet d\boldsymbol{S}, d\boldsymbol{S} = r^2 \sin d\theta d\phi(-\boldsymbol{a}_r)$$

$$= -\iint 4r\, r^2 \sin\theta d\theta d\phi \Big|_{r=2.6cm}$$

$$= -4(2.6)^3 (10^{-6})\int_0^{2\pi} d\phi \int \sin\theta d\theta (10^{-12})$$

$$= -4(2.6)^3(2\pi)(2)\times10^{-18} = \underline{\underline{883.5\times10^{-18} \text{ C}}}$$

(c)

$$\rho_{pv} = -\nabla \bullet \boldsymbol{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}(4r^3) \ \text{pC/m}^3 = -12\text{pC/m}^3$$

$$Q_v = \int_v \rho_{pv} dv = -12\iiint dv = -12\int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi \int_{1.2}^{2.6} r^2 dr (10^{-18})$$

$$= -12(2)(2\pi)\frac{r^3}{3}\bigg|_{1.2}^{2.6}(10^{-18}) = -16\pi(2.6^3 - 1.2^3)(10^{-18})$$

$$= \underline{\underline{-796.61\times 10^{-18} \ \text{C}}}$$

Prob. 5.16

$$\boldsymbol{D} = \varepsilon_o\varepsilon_r\boldsymbol{E} = 2.1x\frac{10^{-9}}{36\pi}(6,12,-20) = \underline{\underline{0.1114\boldsymbol{a}_x + 0.2228\boldsymbol{a}_y - 0.3714\boldsymbol{a}_z \ \text{nC/m}^2}}$$

$$\boldsymbol{P} = \chi_e\varepsilon_o\boldsymbol{E} = 1.1x\frac{10^{-9}}{36\pi}(6,12,-20) = \underline{\underline{0.0584\boldsymbol{a}_x + 0.1167\boldsymbol{a}_y - 0.1945\boldsymbol{a}_z \ \text{nC/m}^2}}$$

Prob. 5.17

$$\boldsymbol{E} = -\nabla V = -\left(\frac{\partial V}{\partial \rho}\boldsymbol{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\boldsymbol{a}_\phi + \frac{\partial V}{\partial z}\boldsymbol{a}_z\right)$$

$$= \left(10\text{z}\sin\phi\boldsymbol{a}_\rho + 10z\cos\phi\boldsymbol{a}_\phi + 10\rho\sin\phi\boldsymbol{a}_z\right)$$

$$\boldsymbol{D} = \varepsilon\boldsymbol{E} = 5\varepsilon_o\boldsymbol{E}$$

$$= \underline{\underline{-50\varepsilon_0\left(\text{z}\sin\phi\boldsymbol{a}_\rho + z\cos\phi\boldsymbol{a}_\phi + \rho\sin\phi\boldsymbol{a}_z\right) \ \text{C/m}^2}}$$

Prob. 5.18

(a) $\boldsymbol{E} = -\nabla V = -\left(\frac{\partial V}{\partial x}\boldsymbol{a}_x + \frac{\partial V}{\partial y}\boldsymbol{a}_y + \frac{\partial V}{\partial z}\boldsymbol{a}_z\right) = \underline{\underline{-20xyz\boldsymbol{a}_x - 10x^2z\boldsymbol{a}_y - 10(x^2y - z)\boldsymbol{a}_z \text{V/m}}}$

(b) $\boldsymbol{D} = \varepsilon\boldsymbol{E} = 5\varepsilon_o\boldsymbol{E} = \underline{\underline{-0.8842xyz\boldsymbol{a}_x - 0.4421x^2z\boldsymbol{a}_y - 0.4421(x^2y - z)\boldsymbol{a}_z \ \text{nC/m}^2}}$

(c) $\boldsymbol{P} = \chi_e\varepsilon_o\boldsymbol{E} = 4\varepsilon_o\boldsymbol{E} = \underline{\underline{-0.7073xyz\boldsymbol{a}_x - 0.3537x^2z\boldsymbol{a}_y - 0.3537(x^2y - z)\boldsymbol{a}_z \ \text{nC/m}^2}}$

(d) $\rho_v = -\varepsilon\nabla^2 V$

$$\nabla^2 V = \frac{\partial}{\partial x}(20xyz) + \frac{\partial}{\partial y}(10x^2z) + \frac{\partial}{\partial z}(10x^2y - 10z) = 20yz - 10$$

$$\rho_v = -5\varepsilon_o 10(2yz - 1) = \underline{\underline{-0.8854yz + 0.4427 \ \text{nC/m}^3}}$$

Prob. 5.19

$$\boldsymbol{P} = \chi_e \varepsilon_o \boldsymbol{E} = \chi_e \varepsilon_o \frac{\boldsymbol{D}}{\varepsilon_o \varepsilon_r} = \frac{(\varepsilon_r - 1)\boldsymbol{D}}{\varepsilon_r} = \frac{1.4}{2.4} \times 450\boldsymbol{a}_x \text{ nC/m}^2$$

$$\underline{\underline{\boldsymbol{P} \quad = 262.5\boldsymbol{a}_x \text{ nC/m}^2}}$$

Prob. 5.20 (a) Applying Coulomb's law,

$$E_r = \begin{cases} \dfrac{D_r}{\varepsilon_o} = \dfrac{Q}{4\pi\varepsilon_o r^2}, & r > b \\ \dfrac{D_r}{\varepsilon} = \dfrac{Q}{4\pi\varepsilon r^2}, & a < r < b \end{cases}$$

$$P = \frac{\varepsilon_r - 1}{\varepsilon_r} D \qquad (= D - \varepsilon_O E)$$

Hence

$$P_r = \underline{\underline{\frac{\varepsilon_r - 1}{\varepsilon_r} . \frac{Q}{4\pi r^2}}}, \qquad a < r < b$$

(b) $$\rho_{pv} = -\nabla \bullet \boldsymbol{P} = -\frac{1}{r^2}\frac{d}{dr}(r^2 P_r) = \underline{\underline{0}}$$

(c)

$$\rho_{ps} = \boldsymbol{P} \bullet (-\boldsymbol{a}_r) = \underline{\underline{-\frac{Q}{4\pi a^2}(\frac{\varepsilon_r - 1}{\varepsilon_r})}}, \qquad r = a$$

$$\rho_{ps} = \boldsymbol{P} \bullet (\boldsymbol{a}_r) = \underline{\underline{-\frac{Q}{4\pi b^2}(\frac{\varepsilon_r - 1}{\varepsilon_r})}}, \qquad r = b$$

Prob. 5.21

$$F_1 = \frac{Q_1 Q_2}{4\pi\varepsilon_o d^2} = 2.6 \text{ nN}, \qquad F_2 = \frac{Q_1 Q_2}{4\pi\varepsilon_o \varepsilon_r d^2} = 1.5 \text{ nN}$$

$$\frac{F_1}{F_2} = \frac{2.6}{1.5} = \varepsilon_r = \underline{\underline{1.733}}$$

Prob. 5.22

(a) By Gauss's law,

$$\oint_S \mathbf{D}\bullet d\mathbf{S} = Q_{enc} \longrightarrow D_r = \frac{Q}{4\pi r^2}$$

$$E_r = \frac{D_r}{\varepsilon} = \frac{Q}{4\pi\varepsilon r^2}$$

$$W = \int_v \frac{1}{2}\varepsilon |\mathbf{E}|^2\, dv, \qquad dv = r^2 \sin\theta dr d\phi d\theta$$

$$W = \frac{1}{2}\varepsilon \int_{\phi=0}^{2\pi}\int_{\theta}^{\pi}\int_{r=a}^{\infty} \frac{Q^2}{16\pi^2\varepsilon^2 r^4} r^2 \sin\theta dr d\phi d\theta = \underline{\underline{\frac{Q^2}{8\pi\varepsilon a}}}$$

(b) D_r remains the same but

$$E_r = \frac{D_r}{\varepsilon} = \frac{Q}{4\pi r^2 \varepsilon_o \left(1+\frac{a}{r}\right)^2} = \frac{Q}{4\pi\varepsilon_o (r+a)^2}$$

$$W = \int_v \frac{1}{2}\varepsilon |E|^2\, dv = \frac{1}{2}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=a}^{\infty} \frac{Q^2 r^2 \sin\theta dr d\theta d\phi}{16\pi^2\varepsilon^2 (r+a)^4}\varepsilon_o\left(\frac{r+a}{r}\right)^2$$

$$= \frac{Q^2}{32\pi^2\varepsilon_o}(4\pi)\int_a^{\infty}\frac{dr}{(r+a)^2} = \frac{Q^2}{8\pi\varepsilon_o}\left(-\frac{1}{r+a}\bigg|_a^{\infty}\right) = \frac{Q^2}{8\pi\varepsilon_o}\frac{1}{2a}$$

$$W = \underline{\underline{\frac{Q^2}{16 a\pi\varepsilon_o}}}$$

Prob. 5.23

(a)

$$\rho_v = \begin{cases} \rho_o, & 0 < r < a \\ 0, & r > a \end{cases}$$

For $r < a$, $\varepsilon E_r (4\pi r^2) = \rho_o \dfrac{4\pi r^3}{3} \longrightarrow E_r = \dfrac{\rho_o r}{3\varepsilon}$

$$V = -\int \mathbf{E}\bullet d\mathbf{l} = -\frac{\rho_o r^2}{6\varepsilon} + c_1$$

For $r > a$, $\varepsilon_o E_r (4\pi r^2) = \rho_o \dfrac{4\pi a^3}{3} \longrightarrow E_r = \dfrac{\rho_o a^3}{3\varepsilon_o r^2}$

$$V = -\int E\bullet dl = \frac{\rho_o a^3}{3\varepsilon_o r} + c_2$$

As $r \longrightarrow \infty$, $V = 0$ and $c_2 = 0$

At $r = a$, $V(a^+) = V(a^-)$

$$-\frac{\rho_o a^2}{6\varepsilon_o \varepsilon_r} + c_1 = \frac{\rho_o a^2}{3\varepsilon_o} \longrightarrow c_1 = \frac{\rho_o a^2}{6\varepsilon_o \varepsilon_r}(2\varepsilon_r + 1)$$

$$\text{V(r=0)} = c_1 = \underline{\underline{\frac{\rho_o a^2 (2\varepsilon_r + 1)}{6\varepsilon_o \varepsilon_r}}}$$

(b) $V(r = a) = \underline{\underline{\dfrac{\rho_o a^2}{3\varepsilon_o}}}$

Prob. 5.24

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \varepsilon_o E_o \begin{bmatrix} 4 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$D_x = \varepsilon_o E_o (4 + 1 - 1) = 4\varepsilon_o E_o$$
$$D_y = \varepsilon_o E_o (1 + 3 - 1) = 3\varepsilon_o E_o$$
$$D_z = \varepsilon_o E_o (1 + 1 - 2) = 0$$
$$\boldsymbol{D} = \underline{\underline{\varepsilon_o E_o (4\boldsymbol{a}_x + 3\boldsymbol{a}_y) \text{ C/m}^2}}$$

Prob. 5.25

Since $\dfrac{\partial \rho_v}{\partial t} = 0$, $\nabla \bullet \boldsymbol{J} = 0$ must hold.

(a) $\nabla \bullet \boldsymbol{J} = 6x^2 y + 0 - 6x^2 y = 0 \longrightarrow$ This is <u>possible</u>.

(b) $\nabla \bullet \boldsymbol{J} = y + (z + 1) \neq 0 \longrightarrow$ This is <u>not possible</u>.

(c) $\nabla \bullet \boldsymbol{J} = \dfrac{1}{\rho}\dfrac{\partial}{\partial \rho}(z^2) + \cos\phi \neq 0 \longrightarrow$ This is <u>not possible.</u>

(d) $\nabla \bullet \boldsymbol{J} = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(\sin\theta) = 0 \longrightarrow$ This is <u>possible</u>.

Prob. 5.26

$$\nabla \bullet J = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} = 2e^{-2y}\cos 2x - 2e^{-2y}\cos 2x + 1 = 1 = -\frac{\partial \rho_v}{\partial t}$$

Hence, $\dfrac{\partial \rho_v}{\partial t} = \underline{\underline{-1 \text{ C/m}^3 s}}$

Prob. 5.27

(a) $\nabla \bullet \boldsymbol{J} = \dfrac{1}{\rho}\dfrac{\partial}{\partial \rho}\left(\dfrac{100}{\rho}\right) = -\dfrac{100}{\rho^3}$

$$-\frac{\partial \rho_v}{\partial t} = \nabla \bullet \boldsymbol{J} = -\frac{100}{\rho^3} \quad \longrightarrow \quad \frac{\partial \rho_v}{\partial t} = \underline{\underline{\frac{100}{\rho^3} \text{ C/m}^3.s}}$$

(b) $I = \int \boldsymbol{J} \bullet d\boldsymbol{S} = \iint \dfrac{100}{\rho^2}\rho\, d\phi\, dz\Big|_{\rho=2} = \dfrac{100}{2}\int_0^{2\pi} d\phi \int_0^1 dz = 100\pi = \underline{\underline{314.16 \text{ A}}}$

Prob. 5.28

$$T_r = \frac{\varepsilon}{\sigma} = \frac{2.5\times 10^{-9}}{5\times 10^{-6}\times 36\pi} = 4.42\ \mu\text{s}$$

$$\rho_{vo} = \frac{Q}{V} = \frac{1}{\frac{4\pi}{3}\times 10^{-6}\times 8} = \underline{\underline{29.84 \text{ k}C/\text{m}^3}}$$

$$\rho_v = \rho_{vo}e^{-t/T_r} = 29.84e^{-2/4.42} = \underline{\underline{18.98 \text{ kC/m}^3}}$$

Prob. 5.29

$$-\frac{\partial \rho_v}{\partial t} = \nabla \bullet \boldsymbol{J} = \frac{\partial J_x}{\partial x} = 0.5\pi\cos \pi x$$

At P(2,4,-3), x= 2

$$\frac{\partial \rho_v}{\partial t} = -0.5\pi\cos(2\pi) = -0.5\pi = \underline{\underline{-1.571 \text{ C/m}^2 s}}$$

Prob. 5.30

(a)

$$\frac{\varepsilon}{\sigma} = \frac{3.1\times\frac{10^{-9}}{36\pi}}{10^{-15}} = \underline{\underline{2.741\times 10^4 \text{ s}}}$$

(b) $\dfrac{\varepsilon}{\sigma} = \dfrac{6\times\frac{10^{-9}}{36\pi}}{10^{-15}} = \underline{\underline{5.305\times 10^4 \text{ s}}}$

(c) $$\frac{\varepsilon}{\sigma} = \frac{80\times\frac{10^{-9}}{36\pi}}{10^{-4}} = \underline{\underline{7.07\ \mu\text{s}}}$$

Prob. 5.31

$$\nabla\bullet\boldsymbol{J} = -\frac{\partial\rho_v}{\partial t} \quad\longrightarrow\quad \rho_v = -\int\nabla\bullet\boldsymbol{J}dt$$

$$\nabla\bullet\boldsymbol{J} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2J_r) = \frac{1}{r^2}\frac{\partial}{\partial r}(0.5r\exp(-10^4t)) = \frac{0.5}{r^2}\exp(-10^4t)$$

$$\rho_v = -\int\nabla\bullet\boldsymbol{J}dt = \frac{0.5}{r^2 10^4}\exp(-10^4t) + C$$

If $\rho_v \to 0$ as $t \to \infty$, C = 0.

$$\underline{\underline{\rho_v = \frac{50}{r^2}\exp(-10^4t)\ \ \mu\text{C/m}^3}}$$

Prob. 5.32

$$\boldsymbol{P}_1 = \chi_{e1}\varepsilon_o\boldsymbol{E}_1 = \chi_{e1}\varepsilon_o\frac{\boldsymbol{D}_1}{\varepsilon_o\varepsilon_{r1}} = \frac{4-1}{4}\boldsymbol{D}_1 = \frac{3}{4}\boldsymbol{D}_1$$

$$\underline{\underline{= 12\boldsymbol{a}_x + 22.5\boldsymbol{a}_y - 15a_z\ \ \text{nC/m}^2}}$$

$$\boldsymbol{D}_{2n} = \boldsymbol{D}_{1n} = -20\boldsymbol{a}_z$$

$$\boldsymbol{E}_{2t} = \boldsymbol{E}_{1t} \quad\longrightarrow\quad \frac{\boldsymbol{D}_{2t}}{\varepsilon_2} = \frac{\boldsymbol{D}_{1t}}{\varepsilon_1}$$

$$\boldsymbol{D}_{2t} = \frac{\varepsilon_2}{\varepsilon_1}\boldsymbol{D}_{1t} = \frac{6.5\varepsilon_o}{4\varepsilon_o}(16\boldsymbol{a}_x + 30\boldsymbol{a}_y) = 26\boldsymbol{a}_x + 48.75\boldsymbol{a}_y$$

$$\boldsymbol{D}_2 = \boldsymbol{D}_{2n} + \boldsymbol{D}_{2t} = \underline{\underline{26\boldsymbol{a}_x + 48.75\boldsymbol{a}_y - 20\boldsymbol{a}_z\ \ \text{nC/m}^2}}$$

Prob. 5.33

Let x > 0 be region 1 and x < 0 be region 2.

$$\boldsymbol{D}_{1n} = 50\boldsymbol{a}_x, \qquad \boldsymbol{D}_{1t} = 80\boldsymbol{a}_y - 30\boldsymbol{a}_z$$

$$\boldsymbol{D}_{2n} = \boldsymbol{D}_{1n} = 50\boldsymbol{a}_x$$

$$\boldsymbol{E}_{2t} = \boldsymbol{E}_{1t} \quad\longrightarrow\quad \frac{\boldsymbol{D}_{2t}}{\varepsilon_2} = \frac{\boldsymbol{D}_{1t}}{\varepsilon_1}$$

$$\boldsymbol{D}_{2t} = \frac{\varepsilon_2}{\varepsilon_1}\boldsymbol{D}_{1t} = \frac{7.6}{2.1}(80\boldsymbol{a}_y - 30\boldsymbol{a}_z) = 289.5\boldsymbol{a}_y - 108.6\boldsymbol{a}_z$$

$$\boldsymbol{D}_2 = \boldsymbol{D}_{2t} + \boldsymbol{D}_{2n} = \underline{\underline{50\boldsymbol{a}_x + 289.5\boldsymbol{a}_y - 108.6\boldsymbol{a}_z\ \ \text{nC/m}^2}}$$

Prob. 5.34

f(x,y)= 4x +3y –10=0

$$\nabla f = 4\boldsymbol{a}_x + 3\boldsymbol{a}_y \quad \longrightarrow \quad \boldsymbol{a}_n = -\frac{\nabla f}{|\nabla f|} = \frac{-(4\boldsymbol{a}_x + 3\boldsymbol{a}_y)}{5} = -0.8\boldsymbol{a}_x - 0.6\boldsymbol{a}_y$$

The minus sign is chosen for $\mathbf{a}_n$ because it is directed toward the origin.

$$\boldsymbol{D}_{1n} = (\boldsymbol{D}_1 \cdot \boldsymbol{a}_n)\boldsymbol{a}_n = (1.6 - 2.4)\boldsymbol{a}_n = -0.64\boldsymbol{a}_x - 0.48\boldsymbol{a}_y$$

$$\boldsymbol{D}_{1t} = \boldsymbol{D}_1 - \boldsymbol{D}_{1n} = 2.64\boldsymbol{a}_x - 3.52\boldsymbol{a}_y + 6.5\boldsymbol{a}_z$$

$$\boldsymbol{D}_{2n} = \boldsymbol{D}_{1n} = -0.64\boldsymbol{a}_x - 0.48\boldsymbol{a}_y$$

$$\boldsymbol{E}_{2t} = \boldsymbol{E}_{1t} \quad \longrightarrow \quad \frac{\boldsymbol{D}_{2t}}{\varepsilon_2} = \frac{\boldsymbol{D}_{1t}}{\varepsilon_2}$$

$$\boldsymbol{D}_{2t} = \frac{\varepsilon_2}{\varepsilon_1}\boldsymbol{D}_{1t} = \frac{2.5}{1}(2.64, -3.52, 6.5) = (6.6, -8.8, 16.25)$$

$$\boldsymbol{D}_2 = \boldsymbol{D}_{2n} + \boldsymbol{D}_{2t} = \underline{\underline{5.96\boldsymbol{a}_x - 9.28\boldsymbol{a}_y + 16.25\boldsymbol{a}_z \ \text{nC/m}^2}}$$

$$\theta_2 = \cos^{-1}\frac{\boldsymbol{D}_2 \bullet \boldsymbol{a}_n}{|\boldsymbol{D}_2|} = \underline{\underline{87.66^o}}$$

Prob. 5.35

(a) $$\boldsymbol{P}_1 = \varepsilon_o \chi_{e1} \boldsymbol{E}_1 = 2 \times \frac{10^{-9}}{36\pi}(10, -6, 12) = \underline{\underline{0.1768\boldsymbol{a}_x - 0.1061\boldsymbol{a}_y + 0.2122\boldsymbol{a}_z \ \text{nC/m}^2}}$$

(b) $$\boldsymbol{E}_{1n} = -6\boldsymbol{a}_y, \qquad \boldsymbol{E}_{2t} = \boldsymbol{E}_{1t} = 10\boldsymbol{a}_x + 12\boldsymbol{a}_z$$

$$\boldsymbol{D}_{2n} = \boldsymbol{D}_{1n} \quad \longrightarrow \quad \varepsilon_2 \boldsymbol{E}_{2n} = \varepsilon_1 \boldsymbol{E}_{1n}$$

or $$\boldsymbol{E}_{2n} = \frac{\varepsilon_1}{\varepsilon_2}\boldsymbol{E}_{1n} = \frac{3\varepsilon_o}{4.5\varepsilon_o}(-6\boldsymbol{a}_z) = -4\boldsymbol{a}_y$$

$$\boldsymbol{E}_2 = \underline{\underline{10\boldsymbol{a}_x - 4\boldsymbol{a}_y + 12\boldsymbol{a}_z \ \text{V/m}}}$$

$$\tan\theta_2 = \frac{E_{2t}}{E_{2n}} = \frac{\sqrt{10^2 + 12^2}}{4} = 3.905 \quad \longrightarrow \quad \underline{\underline{\theta_2 = 75.64^o}}$$

(c) $w_E = \frac{1}{2}\boldsymbol{D}\bullet\boldsymbol{E} = \frac{1}{2}\varepsilon|\boldsymbol{E}|^2$

$$w_{E1} = \frac{1}{2}\varepsilon_1|\boldsymbol{E}_1|^2 = \frac{1}{2}x3x\frac{10^{-9}}{36\pi}(10^2+6^2+12^2) = \underline{\underline{3.7136\ \text{nJ/m}^3}}$$

$$w_{E2} = \frac{1}{2}\varepsilon_2|\boldsymbol{E}_2|^2 = \frac{1}{2}x4.5x\frac{10^{-9}}{36\pi}(10^2+4^2+12^2) = \underline{\underline{5.1725\ \text{nJ/m}^3}}$$

Prob. 5.36 (a) $\boldsymbol{D}_{2n} = 12\boldsymbol{a}_\rho = \boldsymbol{D}_{1n}, \qquad \boldsymbol{D}_{2t} = -6\boldsymbol{a}_\phi + 9\boldsymbol{a}_z$

$$\boldsymbol{E}_{2t} = \boldsymbol{E}_{2t} \quad \longrightarrow \quad \frac{\boldsymbol{D}_{1t}}{\varepsilon_1} = \frac{\boldsymbol{D}_{2t}}{\varepsilon_2}$$

$$\boldsymbol{D}_{1t} = \frac{\varepsilon_1}{\varepsilon_2}\boldsymbol{D}_{2t} = \frac{3.5\varepsilon_o}{1.5\varepsilon_o}(-6\boldsymbol{a}_\phi + 9\boldsymbol{a}_z) = -14\boldsymbol{a}_\phi + 21\boldsymbol{a}_z$$

$$\underline{\underline{\boldsymbol{D}_1 = 12\boldsymbol{a}_\rho - 14\boldsymbol{a}_\phi + 21\boldsymbol{a}_z\ \ \text{nC/m}^2}}$$

$$\boldsymbol{E}_1 = \boldsymbol{D}_1/\varepsilon_1 = \frac{(12,-14,21)\times10^{-9}}{3.5\times\frac{10^{-9}}{36\pi}} = 387.8\boldsymbol{a}_\rho - 452.4\boldsymbol{a}_\phi + 678.6\boldsymbol{a}_z$$

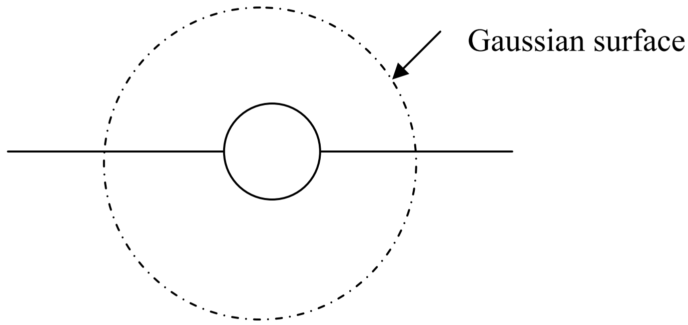
(b) $\boldsymbol{P}_2 = \varepsilon_o\chi_{e2}\boldsymbol{E}_2 = 0.5\varepsilon_o\frac{\boldsymbol{D}_2}{\varepsilon_2} = \frac{0.5\varepsilon_o}{1.5\varepsilon_o}(12,-6,9) = \underline{\underline{4\boldsymbol{a}_\rho - 2\boldsymbol{a}_\phi + 3\boldsymbol{a}_z\ \ \text{nC/m}^2}}$

$$\rho_{v2} = \nabla\bullet\boldsymbol{P}_2 = 0$$

(c) $w_{E1} = \frac{1}{2}\boldsymbol{D}_1\bullet\boldsymbol{E}_1 = \frac{1}{2}\frac{\boldsymbol{D}_1\bullet\boldsymbol{D}_1}{\varepsilon_o\varepsilon_{r1}} = \frac{1}{2}\frac{(12^2+14^2+21^2)x10^{-18}}{3.5x\frac{10^{-9}}{36\pi}} = \underline{\underline{12.62\ \ \mu\text{J/m}^2}}$

$$w_{E2} = \frac{1}{2}\frac{\boldsymbol{D}_2\bullet\boldsymbol{D}_2}{\varepsilon_o\varepsilon_{r2}} = \frac{1}{2}\frac{(12^2+6^2+9^2)x10^{-18}}{1.5x\frac{10^{-9}}{36\pi}} = \underline{\underline{9.839\ \ \mu\text{J/m}^2}}$$

Prob. 5.37

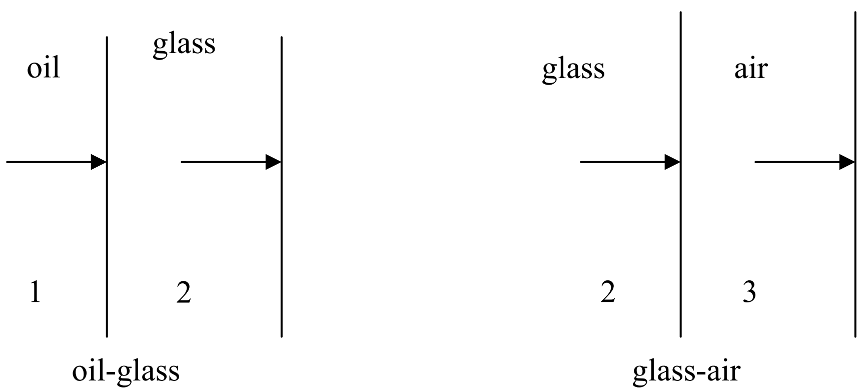


$$Q = \int \boldsymbol{D}.dS = \varepsilon_1 E_r \frac{4\pi r^2}{2} + \varepsilon_2 E_r \frac{4\pi r^2}{2} = 2\pi r^2 (\varepsilon_1 + \varepsilon_2) E_r$$

$$E_r = \begin{cases} \dfrac{Q}{2\pi(\varepsilon_1 + \varepsilon_2) r^2}, & r > a \\ 0, & r < a \end{cases}$$

Prob. 5.38

(a) The two interfaces are shown below



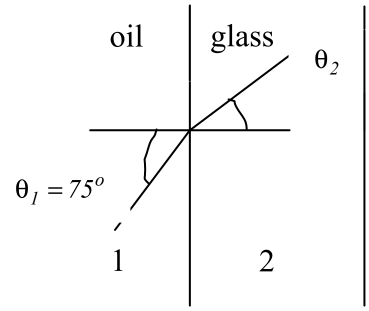
$$E_{1n} = 2000, \qquad E_{1t} = 0 = E_{2t} = E_{3t}$$

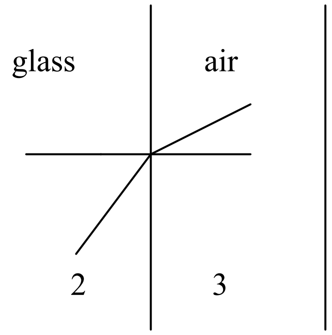
$$D_{1n} = D_{2n} = D_{3n} \qquad \longrightarrow \qquad \varepsilon_1 E_{1n} = \varepsilon_2 E_{2n} = \varepsilon_3 E_{3n}$$

$$E_{2n} = \frac{\varepsilon_1}{\varepsilon_2} E_{1n} = \frac{3.0}{8.5}(2000) = \underline{\underline{705.9 \text{ V/m}, \ \theta_2 = 0^o}}$$

$$E_{3n} = \frac{\varepsilon_1}{\varepsilon_3} E_{1n} = \frac{3.0}{1.0}(2000) = \underline{\underline{6000 \text{ V/m}, \ \theta_3 = 0^o}}$$

(b)





$$E_{1n} = 2000\cos 75^o = 517.63, \qquad E_{1t} = 2000\sin 75^o = E_{2t} = E_{3t} = 1931.85$$

$$E_{2n} = \frac{\varepsilon_1}{\varepsilon_2} E_{1n} = \frac{3}{8.5}(517.63) = 182.7, \qquad E_{3n} = \frac{\varepsilon_1}{\varepsilon_3} E_{1n} = \frac{3}{1}(517.63) = 1552.89$$

$$E_2 = \sqrt{E_{2n}{}^2 + E_{2t}{}^2} = 1940.5, \qquad \theta_2 = \tan^{-1}\frac{E_{2t}}{E_{2n}} = \underline{\underline{84.6^o}},$$

$$E_3 = \sqrt{E_{3n}{}^2 + E_{3t}{}^2} = 2478.6, \qquad \theta_3 = \tan^{-1}\frac{E_{3t}}{E_{3n}} = \underline{\underline{51.2^o}}$$

Prob. 5.39

$$\rho_s = D_n = \varepsilon_o E = \frac{10^{-9}}{36\pi}\sqrt{30^2 + 40^2 + 20^2} \times 10^{-3} = \frac{\sqrt{2900}}{36\pi} \text{ pC/m}^2$$

$$= \underline{\underline{0.476 \text{ pC/m}^2}}$$

Prob. 5.40 (a) $\rho_s = D_n = \varepsilon_o E_n = \dfrac{10^{-9}}{36\pi}\sqrt{15^2 + 8^2} = \underline{\underline{0.1503 \text{ nC/m}^2}}$

(b) $D_n = \rho_s = -20$ nC

$$D = D_n a_n = (-20 \text{ nC})(-a_y) = \underline{\underline{20 a_y \text{ nC/m}^2}}$$

Prob. 5.41

At the interface between ε_o and $2\varepsilon_o$,

$E_{1n} = E_o \cos 30^o, \qquad E_{1t} = E_o \sin 30^o$

$E_{2t} = E_{1t} = 0.5E_o$

$D_{2n} = D_{1n} \longrightarrow \varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}$

$$E_{2n} = \frac{\varepsilon_1}{\varepsilon_2} E_{1n} = \frac{\varepsilon_o}{2\varepsilon_o}(0.866E_o) = 0.433E_o$$

The angle **E** makes with the z-axis is

$$\theta_1 = \tan^{-1}\frac{E_{2t}}{E_{2n}} = \tan^{-1}\frac{0.5}{0.433} = \underline{\underline{49.11^o}}$$

At the interface between $2\varepsilon_o$ and $3\varepsilon_o$,

$E_{3t} = E_{2t} = 0.5E_o$

$$D_{3n} = D_{2n} \longrightarrow E_{3n} = \frac{\varepsilon_2}{\varepsilon_3} E_{2n} = \frac{2\varepsilon_o}{3\varepsilon_o}(0.433E_o) = 0.2887E_o$$

The angle **E** makes with the z-axis is

$$\theta_2 = \tan^{-1}\frac{E_{3t}}{E_{3n}} = \tan^{-1}\frac{0.5}{0.2887} = \underline{\underline{60^o}}$$

At the interface between $3\varepsilon_o$ and ε_o,

$E_{4t} = E_{3t} = 0.5E_o$

$$D_{4n} = D_{3n} \longrightarrow E_{4n} = \frac{\varepsilon_3}{\varepsilon_4} E_{3n} = \frac{3\varepsilon_o}{\varepsilon_o}(0.2887E_o) = 0.866E_o$$

The angle **E** makes with the z-axis is

$$\theta_3 = \tan^{-1}\frac{E_{4t}}{E_{4n}} = \tan^{-1}\frac{0.5}{0.866} = \underline{\underline{30^o}}$$

CHAPTER 6

P. E. 6.1

$$\nabla^2 V = -\frac{\rho}{\varepsilon} \longrightarrow \frac{d^2V}{dx^2} = -\frac{\rho_o x}{\varepsilon a}$$

$$V = -\frac{\rho_o x^3}{6\varepsilon a} + Ax + B$$

$$\boldsymbol{E} = -\frac{dV}{dx}\boldsymbol{a}_x = \left(\frac{\rho_o x^2}{2\varepsilon a} - A\right)\boldsymbol{a}_x$$

If **E** = 0 at x =0, then

$$0 = 0 - A \longrightarrow A = 0$$

If V= 0 at x =a, then

$$0 = -\frac{\rho_o a^3}{6\varepsilon a} + B \longrightarrow B = \frac{\rho_o a^2}{6\varepsilon}$$

Thus

$$\underline{\underline{V = \frac{\rho_o}{6\varepsilon a}(a^3 - x^3)}}, \qquad \underline{\underline{\boldsymbol{E} = \frac{\rho_o x^2}{2\varepsilon a}\boldsymbol{a}_x}}$$

P. E. 6.2 $V_1 = A_1 x + B_1, \quad V_2 = A_2 x + B_2$

$$V_1(x=d) = V_o = A_1 d + B_1 \longrightarrow B_1 = V_o - A_1 d$$

$$V_1(x=0) = 0 = 0 + B_2 \longrightarrow B_2 = 0$$

$$V_1(x=a) = V_2(x=a) \longrightarrow aA_1 + B_1 = A_2 a$$

$$D_{1n} = D_{2n} \longrightarrow \varepsilon_1 A_1 = \varepsilon_2 A_2 \longrightarrow A_2 = \frac{\varepsilon_1}{\varepsilon_2}A_1$$

$$A_1 a + V_o - A_1 d = \frac{\varepsilon_1}{\varepsilon_2}aA_1 \longrightarrow V_o = A_1\left(-a + d + \frac{\varepsilon_1}{\varepsilon_2}a\right)$$

or

$$A_1 = \frac{V_o}{d - a + \varepsilon_1 a/\varepsilon_2}, \qquad A_2 = \frac{\varepsilon_1}{\varepsilon_2}A_1 \frac{\varepsilon_1 V_o}{\varepsilon_2 d - \varepsilon_2 a + \varepsilon_1 a}$$

Hence

$$\boldsymbol{E}_1 = -A_1\boldsymbol{a}_x = \underline{\underline{\frac{-V_o a_x}{d - a + \varepsilon_1 a/\varepsilon_2}}}, \qquad \boldsymbol{E}_2 = -A_2\boldsymbol{a}_x = \underline{\underline{\frac{-V_o\boldsymbol{a}_x}{a + \varepsilon_2 d/\varepsilon_1 - \varepsilon_2 a/\varepsilon_1}}}$$

P. E. 6.3 From Example 6.3,

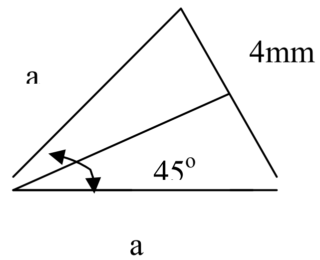
$$\boldsymbol{E} = -\frac{V_o}{\rho\phi_o}\boldsymbol{a}_\phi, \qquad D = \varepsilon_o E$$

$$\rho_s = D_n(\phi = 0) = -\frac{V_o\varepsilon}{\rho\phi_o}$$

The charge on the plate $\phi = 0$ is

$$Q = \int \rho_s dS = -\frac{V_o\varepsilon}{\phi_o}\int_{z=0}^{L}\int_{\rho=a}^{b}\frac{1}{\rho}dz\,d\rho = -\frac{V_o\varepsilon}{\phi_o}L\ln(b/a)$$

$$C = \frac{|Q|}{V_o} = \frac{\varepsilon L}{\phi_o}\ln\frac{b}{a}$$



$$a\sin\frac{45^o}{2} = 2 \quad \longrightarrow \quad a = \frac{2}{\sin 22.5^o} = 5.226 \text{ mm}$$

$$C = \frac{1.5\times\frac{10^{-9}}{36\pi}}{\frac{\pi}{4}}5\ln\frac{1000}{5.226} = 444 \text{ pF}$$

$$Q = CV_o = 444\times10^{-12}\times50 \text{ C} = \underline{\underline{22.2 \text{ nC}}}$$

P. E. 6.4 From Example 6.4,

$V_o = 50, \quad \theta_2 = 45^o, \quad \theta_1 = 90^o, \quad r = \sqrt{3^2+4^2+2^2} = \sqrt{29}, \quad \theta = \tan^{-1}\frac{\rho}{z} =$

$\tan^{-1}\frac{5}{2} \quad \longrightarrow \quad \theta = 68.2^o; \qquad \tan 45^o = 1$

$$V = \frac{50\ln(\tan 34.1^o)}{\ln(\tan 22.5^o)} = \underline{\underline{22.125 \text{ V}}},$$

$$\boldsymbol{E} = \frac{50\boldsymbol{a}_\theta}{\sqrt{29}\sin 68.2^o\ln(\tan 22.5^o)} = \underline{\underline{11.35\boldsymbol{a}_\theta \text{ V/m}}}$$

P. E. 6.5

$$\boldsymbol{E} = -\nabla V = -\frac{\partial V}{\partial x}\boldsymbol{a}_x - \frac{\partial V}{\partial y}\boldsymbol{a}_y$$

$$= -\frac{4V_o}{b}\sum_{n=\text{odd}}^{\infty}\frac{1}{\sinh n\pi a/b}\left[\cos(n\pi x/b)\sinh(n\pi y/b)\,\mathbf{a}_x + \sin(n\pi x/b)\cosh(n\pi y/b)\mathbf{a}_y\right]$$

(a) At (x,y) = (a, a/2),

$$V = \frac{400}{\pi}(0.3775 - 0.0313 + 0.00394 - 0.000585 + ...) = \underline{\underline{44.51 \text{ V}}}$$

$$\boldsymbol{E} = 0\boldsymbol{a}_x + (-115.12 + 19.127 - 3.9411 + 0.8192 - 0.1703 + 0.035 - 0.0074 + ...)\boldsymbol{a}_y$$
$$= \underline{\underline{-99.25\boldsymbol{a}_y \text{ V/m}}}$$

(b) At (x,y) = (3a/2, a/4),

$$V = \frac{400}{\pi}(0.1238 + 0.006226 - 0.00383 + 0.0000264 + ...) = \underline{\underline{16.50 \text{ V}}}$$

$$\boldsymbol{E} = (24.757 - 3.7358 - 0.3834 + 0.0369 + 0.00351 - 0.00033 + ...)\boldsymbol{a}_x$$
$$+(-66.25 - 4.518 + 0.3988 + 0.03722 - 0.00352 - 0.000333 + ...)\boldsymbol{a}_y$$
$$= \underline{\underline{20.68\boldsymbol{a}_x - 70.34\boldsymbol{a}_y \text{ V/m}}}$$

P. E. 6.6

$$V(y=a) = V_o\sin(7\pi x/b) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/b)\sinh(n\pi a/b)$$

By equating coefficients, we notice that $c_n = 0$ for $n \neq 7$. For n=7,

$$V_o\sin(7\pi x/b) = c_7\sin(7\pi x/b)\sinh(7\pi a/b) \quad \longrightarrow \quad c_7 = \frac{V_o}{\sinh(7\pi a/b)}$$

Hence

$$V(x,y) = \underline{\underline{\frac{V_o}{\sinh(7\pi a/b)}\sin(7\pi x/b)\sinh(7\pi y/b)}}$$

P. E. 6.7 Let $V(r,\theta,\phi) = R(r)F(\theta)\Phi(\phi)$.
Substituting this in Laplace's equation gives

$$\frac{\Phi F}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{R\Phi}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dF}{d\theta}\right) + \frac{RF}{r^2\sin^2\theta}\frac{d^2\Phi}{d\phi^2} = 0$$

Dividing by $RF\Phi / r^2 \sin^2\theta$ gives

$$\frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2R'\right)+\frac{\sin\theta}{F}\frac{d}{d\theta}\left(\sin\theta F'\right)=-\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}=\lambda^2$$

$$\underline{\underline{\Phi''+\lambda^2\Phi=0}}$$

$$\frac{1}{R}\frac{d}{dr}\left(r^2R'\right)+\frac{1}{F\sin\theta}\frac{d}{d\theta}\left(\sin\theta F'\right)=\lambda^2/\sin^2\theta$$

$$\frac{1}{R}\frac{d}{dr}\left(r^2R'\right)=\frac{\lambda^2}{\sin^2\theta}-\frac{1}{F\sin\theta}\frac{d}{d\theta}\left(\sin\theta F'\right)=\mu^2$$

$$2rR'+r^2R''=\mu^2R$$

or

$$\underline{\underline{R''+\frac{2}{r}R'-\frac{\mu^2}{r^2}R=0}}$$

$$\frac{\sin\theta}{F}\frac{d}{d\theta}\left(\sin\theta F'\right)-\lambda^2+\mu^2\sin^2\theta=0$$

or

$$\underline{\underline{F''+\cos\theta F'+(\mu^2\sin\theta-\lambda^2\csc\theta)F=0}}$$

P. E. 6.8 (a) This is similar to Example 6.8(a) except that here $0<\phi<2\pi$ instead of $0<\phi<\pi/2$. Hence

$$I=\frac{2\pi tV_o\sigma}{\ln(b/a)} \quad \text{and} \quad R=\frac{V_o}{I}=\underline{\underline{\frac{\ln\frac{b}{a}}{2\pi t\sigma}}}$$

(b) This to similar to Example 6.8(b) except that here $0<\phi<2\pi$. Hence

$$I=\frac{V_o\sigma}{t}\int_a^b\int_0^{2\pi}\rho\, d\rho\, d\phi=\frac{V_o\sigma\pi(b^2-a^2)}{t}$$

and $R=\dfrac{V_o}{I}=\underline{\underline{\dfrac{t}{\sigma\pi(b^2-a^2)}}}$

P. E. 6.9 From Example 6.9,

$$J_1 = \frac{\sigma_1 V_o}{\rho \ln \frac{b}{a}}, \qquad J_2 = \frac{\sigma_2 V_o}{\rho \ln \frac{b}{a}}$$

$$I = \int J \bullet dS = \int_{z=0}^{L} \left[\int_{\phi=0}^{\pi} J_1 \rho d\phi + \int_{\phi=\pi}^{2\pi} J_2 \rho d\phi \right] dz = \frac{V_o l}{\ln \frac{b}{a}} [\pi\sigma_1 + \pi\sigma_2]$$

$$R = \frac{V_o}{I} = \frac{\ln \frac{b}{a}}{\pi l [\sigma_1 + \sigma_2]}$$

P. E. 6.10 (a) $C = \dfrac{4\pi\varepsilon}{\frac{1}{a} - \frac{1}{b}}$, C_1 and C_2 are in series.

$$C_1 = 4\pi x \frac{10^{-9}}{36\pi} \left(\frac{2.5}{\frac{10^3}{2} - \frac{10^3}{3}} \right) = 5/3 \text{ pF}, \qquad C_2 = 4\pi x \frac{10^{-9}}{36\pi} \left(\frac{3.5}{\frac{10^3}{1} - \frac{10^3}{2}} \right) = 7/9 \text{ pF}$$

$$C = \frac{C_1 C_2}{C_1 + C_2} = \frac{(5/3)(7/9)}{(5/3) + (7/9)} = \underline{\underline{0.53 \text{ pF}}}$$

(b) $C = \dfrac{2\pi\varepsilon}{\frac{1}{a} - \frac{1}{b}}$, C_1 and C_2 are in parallel.

$$C_1 = 2\pi \times \frac{10^{-9}}{36\pi} \left(\frac{2.5}{\frac{10^3}{1} - \frac{10^3}{3}} \right) = 5/24 \text{ pF}, \qquad C_2 = 2\pi \times \frac{10^{-9}}{36\pi} \left(\frac{3.5}{\frac{10^3}{1} - \frac{10^3}{3}} \right) = 7/24 \text{ pF}$$

$$C = C_1 + C_2 = \underline{\underline{0.5 \text{ pF}}}$$

P. E. 6.11 As in Example 6.8, assuming $V(\rho = a) = 0, \quad V(\rho = b) = V_o,$

$$V = V_o \frac{\ln \rho / a}{\ln b / a}, \qquad E = -\nabla V = -\frac{V_o}{\rho \ln b / a} a_\rho$$

$$Q = \int \varepsilon E \bullet dS = \frac{V_o \varepsilon}{\ln b / a} \int_{z=0}^{L} \int_{\phi=0}^{2\pi} \frac{1}{\rho} dz \rho d\phi = \frac{V_o 2\pi\varepsilon L}{\ln b / a}$$

$$C = \frac{Q}{V_o} = \underline{\underline{\frac{2\pi\varepsilon L}{\ln b / a}}}$$

P. E. 6.12

(a) Let C_1 and C_2 be capacitances per unit length of each section and C_T be the total capacitance of 10m length. C_1 and C_2 are in series.

$$C_1 = \frac{2\pi\varepsilon_{r1}\varepsilon_o}{\ln b / c} = \frac{2\pi x 2.5}{\ln 3/2} \frac{10^{-9}}{36\pi} = 342.54 \text{ pF/m},$$

$$C_2 = \frac{2\pi\varepsilon_{r2}\varepsilon_o}{\ln c / a} = \frac{2\pi x 3.5}{\ln 2} \frac{10^{-9}}{36\pi} = 280.52 \text{ pF/m}$$

$$C = \frac{C_1 C_2}{C_1 + C_2} = \frac{342.54 x 280.52}{342.54 + 280.52} = 154.22 \text{ pF}$$

$$C_T = Cl = \underline{\underline{1.54}} \text{ nF}$$

(b) C_1 and C_2 are in parallel.

$$C = C_1 + C_2 = \frac{\pi\varepsilon_{r1}\varepsilon_o}{\ln b / a} + \frac{\pi\varepsilon_{r2}\varepsilon_o}{\ln b / a} = \frac{\pi(\varepsilon_{r1} + \varepsilon_{r2})\varepsilon_o}{\ln b / a} = \frac{6\pi}{\ln 3} \frac{10^{-9}}{36\pi} = 151.7 \text{ pF/m}$$

$$C_T = Cl = \underline{\underline{1.52}} \text{ nF}$$

P. E. 6.13 Instead of Eq. (6.31), we now have

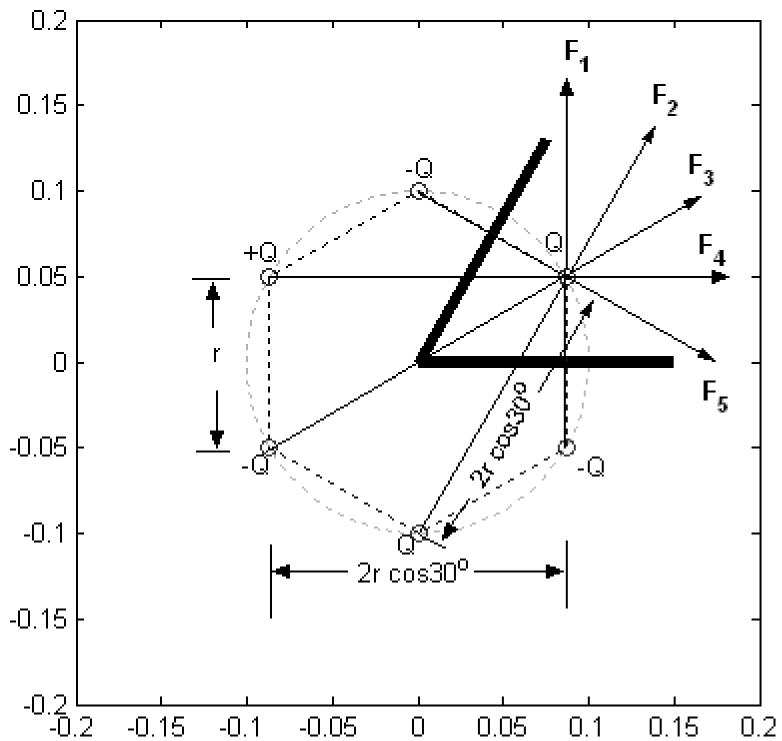
$$V = -\int_b^a \frac{Qdr}{4\pi\varepsilon r^2} = -\int_b^a \frac{Qdr}{4\pi \frac{10\varepsilon_o}{r} r^2} = -\frac{Q}{40\pi\varepsilon_o} \ln b / a$$

$$C = \frac{Q}{|V|} = \frac{40\pi}{\ln 4 / 1.5} \frac{10^{-9}}{36\pi} = \underline{\underline{1.13}} \text{ nF}$$

P. E. 6.14 Let

$$F = F_1 + F_2 + F_3 + F_4 + F_5$$

where $F_i, \quad i = 1,2,...,5$ are shown on in the figure below.



$$\boldsymbol{F} = -\frac{Q^2}{4\pi\varepsilon_o r^2}\boldsymbol{a}_y + \frac{Q^2(\boldsymbol{a}_x \sin 30^o + \boldsymbol{a}_y \cos 30^o)}{4\pi\varepsilon_o (2r\cos 30^o)^2} - \frac{Q^2(\boldsymbol{a}_x \cos 30^o + \boldsymbol{a}_y \sin 30^o)}{4\pi\varepsilon_o (2r)^2} + \frac{Q^2\boldsymbol{a}_x}{4\pi\varepsilon_o (2r\cos 30^o)^2}$$

$$- \frac{Q^2(\boldsymbol{a}_x \cos 30^o - \boldsymbol{a}_y \sin 30^o)}{4\pi\varepsilon_o r^2}$$

$$= \frac{Q^2}{4\pi\varepsilon_o r^2}\left[-\boldsymbol{a}_y + \frac{1}{3}\left(\frac{\boldsymbol{a}_x}{2} + \frac{\sqrt{3}\boldsymbol{a}_y}{2}\right) - \frac{1}{4}\left(\frac{\sqrt{3}\boldsymbol{a}_x}{2} + \frac{\boldsymbol{a}_y}{2}\right) + \frac{1}{3}a_x - \frac{\sqrt{3}\boldsymbol{a}_x}{2} + \frac{\boldsymbol{a}_y}{2}\right]$$

$$= 9x10^{-5}\left[\boldsymbol{a}_x\left(\frac{1}{2} - \frac{5\sqrt{3}}{8}\right) + \boldsymbol{a}_y\left(\frac{-5}{8} + \frac{\sqrt{3}}{6}\right)\right] = -52.4279\boldsymbol{a}_x - 30.27\boldsymbol{a}_y \quad \mu\text{N}$$

$$\underline{\underline{|\boldsymbol{F}| = 60.54\ \mu\text{N}}}$$

Note that the force tends to pull Q toward the origin.

P.E. 6.15

(a) $E_1 = 1.35 \times 10^3$ V/mm and $E_2 = 1 \times 10^3$ V/mm

(b) $E_1 = 1.72 \times 10^3$ V/mm and $E_2 = 0.59 \times 10^3$ V/mm

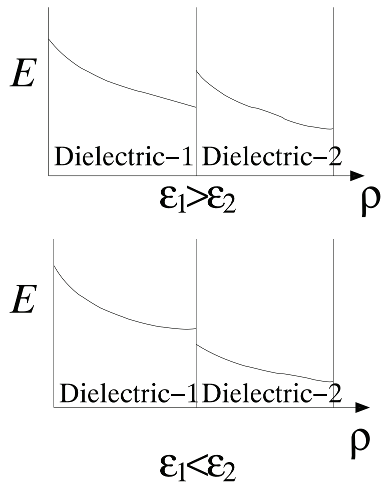


Figure. Electric field intensity distribution in a multi-dielectric coaxial capacitor

It can be noted from this practice exercise that, in a non-uniform field, the dielectric with higher permittivity should be placed in the high field region and the one with lower permittivity in the low field region (this is case-a). The difference in the average stress levels of the two dielectrics reduces. If the placement is done the other way round, the difference increases further (case-b). The electric field intensities in the two cases are plotted as a function of radius in the figure above.

P.E. 6.16

(a) $V_2 = 66.66$ kV, breakdown voltage $= 6 \times 10 = 60$ kV, breakdown occurs

(b) $V_2 = 85.71$ kV, breakdown voltage $= 9 \times 10 = 90$ kV, breakdown does not occur

Note: One has to properly size the thickness of dielectric layers with lower permittivity, as in case-b in this practice exercise.

P.E. 6.17

(a) For the parallel-plate capacitor,

$$\boldsymbol{E} = -\frac{V_o}{d}\boldsymbol{a}_x$$

From Example 6.11,

$$C = \frac{1}{V_o^2}\int \varepsilon \,|\, E\,|^2 dv = \frac{1}{V_o^2}\int \varepsilon \frac{V_o^2}{d^2} dv = \frac{\varepsilon}{d^2} Sd = \frac{\varepsilon S}{d}$$

(b) For the cylindrical capacitor,

$$\boldsymbol{E} = -\frac{V_o}{\rho \ln b/a}\boldsymbol{a}_\rho$$

From Example 6.8,

$$C = \frac{1}{V_o^2}\iiint \frac{\varepsilon V_o^2}{\left(\rho \ln b/a\right)^2}\rho d\rho d\phi dz = \frac{2\pi\varepsilon L}{\left(\ln b/a\right)^2}\int_a^b \frac{d\rho}{\rho} = \frac{2\pi\varepsilon L}{\ln b/a}$$

(c) For the spherical capacitor,

$$\boldsymbol{E} = \frac{V_o}{r^2(1/a - 1/b)}\boldsymbol{a}_r$$

From Example 6.10,

$$C = \frac{1}{V_o^2}\iiint \frac{\varepsilon V_o^2}{r^4\left(1/a - 1/b\right)^2} r^2 \sin\theta d\theta dr d\phi = \frac{\varepsilon}{\left(1/a - 1/b\right)^2} 4\pi \int_a^b \frac{dr}{r^2} = \frac{4\pi\varepsilon}{\dfrac{1}{a} - \dfrac{1}{b}}$$

Prob. 6.1

(a)

$$-\boldsymbol{E} = \nabla V = -\frac{20}{r^3}\cos\theta\sin\phi\boldsymbol{a}_r - \frac{10}{r^3}\sin\theta\sin\phi\boldsymbol{a}_\theta + \frac{1}{\mathrm{r}\sin\theta}\frac{10}{r^2}\cos\theta\cos\phi\boldsymbol{a}_\phi$$

At P$(1,60^o,30^o), r=1, \theta = 60^o, \phi = 30^o$

$$\boldsymbol{E} = \frac{20}{1^3}\cos 60^o \sin 30^o \boldsymbol{a}_r + \frac{10}{1^3}\sin 60^o \sin 30^o \boldsymbol{a}_\theta - \frac{1}{\sin 60^o}\frac{10\cos 60^o \cos 30^o}{1^3}\boldsymbol{a}_\phi$$

$$= \underline{\underline{5\boldsymbol{a}_r + 4.33\boldsymbol{a}_\theta - 5\boldsymbol{a}_\phi \text{ V/m}}}$$

(b)

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{-20\cos\theta\sin\phi}{r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{-10\sin^2\theta\sin\phi}{r^2}\right)$$

$$-\frac{1}{r^2\sin^2\theta}\frac{10\cos\theta\sin\phi}{r^2}$$

$$= \frac{20\cos\theta\sin\phi}{r^4} - \frac{20\sin\theta\cos\theta\sin\phi}{r^4\sin\theta} - \frac{10\cos\theta\sin\phi}{r^4\sin^2\theta}$$

$$= -\frac{10\cos\theta\sin\phi}{r^4\sin^2\theta}$$

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon} \longrightarrow \rho_v = -\varepsilon\nabla^2 V = \frac{10\varepsilon_o\cos\theta\sin\phi}{r^4\sin^2\theta}$$

At P, r =1, θ=60°, ϕ=30°

$$\rho_v = 10\times\frac{10^{-9}}{36\pi}\frac{\cos 60^o \sin 30^o}{1^4\sin^2 60^o} = \underline{\underline{29.47 \text{ pC/m}^3}}$$

Prob. 6.2

(a)

$$\boldsymbol{E} = -\nabla V = -\left(\frac{\partial V}{\partial x}\boldsymbol{a}_x + \frac{\partial V}{\partial z}\boldsymbol{a}_y + \frac{\partial V}{\partial z}\boldsymbol{a}_z\right)$$

$$= -(15x^2y^2z\boldsymbol{a}_x + 10x^3yz\boldsymbol{a}_y + 5x^3y^2\boldsymbol{a}_z)$$

At P, x=-3, y=1, z=2,

$$\boldsymbol{E} = -15(9)(1)(2)\boldsymbol{a}_x + 10(-27)(1)(2)\boldsymbol{a}_y - 5(-27)(1)\boldsymbol{a}_z = \underline{\underline{-270\boldsymbol{a}_x + 540\boldsymbol{a}_y + 135\boldsymbol{a}_z \text{ V/m}}}$$

(b) $\rho_v = \nabla \bullet \boldsymbol{D}$ or $\rho_v = -\varepsilon\nabla^2 V$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial}{\partial x}(15x^2y^2z) + \frac{\partial}{\partial y}(10x^3yz) + \frac{\partial}{\partial z}(5x^3y^2)$$

$$= 30xy^2z + 10x^3z$$

At P,

$$\rho_v = -\varepsilon\nabla^2 V = -2.25\times\frac{10^{-9}}{36\pi}\left[30(-3)(1)(2) + 10(-27)(2)\right] = \underline{\underline{14.324 \text{ nC/m}^3}}$$

Prob. 6.3

(a) $\boldsymbol{E} = -\nabla V = -\frac{\partial V}{\partial \rho}\boldsymbol{a}_\rho - \frac{1}{\rho}\frac{\partial V}{\partial \phi}\boldsymbol{a}_\phi = \frac{\sin 3\phi}{\rho^2}\boldsymbol{a}_\rho - \frac{3\cos 3\phi}{\rho^2}\boldsymbol{a}_\phi$

At A, ρ=1, ϕ=20°, z=4,

$$\boldsymbol{E} = \frac{\sin 60^o}{1^2}\boldsymbol{a}_\rho - \frac{3\cos 60^o}{1^2}\boldsymbol{a}_\phi = \underline{\underline{0.866\boldsymbol{a}_\rho - 1.5\boldsymbol{a}_\phi \text{ V/m}}}$$

$$\boldsymbol{P} = \chi_e\varepsilon_o\boldsymbol{E} = 1.8\times\frac{10^{-9}}{36\pi}(0.866\boldsymbol{a}_\rho - 1.5\boldsymbol{a}_\phi)$$

$$\underline{\underline{= 13.783\boldsymbol{a}_\rho - 23.87\boldsymbol{a}_\phi \text{ pC/m}^2}}$$

(b) $\rho_v = \nabla\bullet\boldsymbol{D} = \varepsilon\nabla\bullet\boldsymbol{E} = \varepsilon\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\frac{\sin 3\phi}{\rho}\right) + \frac{9\sin 3\phi}{\rho^3}\right]$

$$\frac{\rho_v}{\varepsilon} = \frac{-\sin 3\phi}{\rho^3} + \frac{9\sin 3\phi}{\rho^3}$$

$$\rho_v = \frac{8\varepsilon\sin 3\phi}{\rho^3}$$

At A, $\rho_v = 8\times 2.8\times\frac{10^{-9}}{36\pi}\frac{\sin 60^o}{1^3} = \underline{\underline{171.52 \text{ pC/m}^3}}$

Prob. 6.4

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon}$$

$$\frac{d^2V}{dy^2} = -\frac{y}{4\pi}\cdot\frac{10^{-9}}{4\varepsilon_o} = -\frac{y}{4\pi}\cdot\frac{10^{-9}}{4\frac{10^{-9}}{36\pi}} = -2.25y$$

$$\frac{dV}{dy} = -2.25\frac{y^2}{2} + B$$

$$V = -0.375y^3 + By + C$$

$$V(1) = 0 = -0.375 + B + C \qquad (1)$$

$$V(3) = 50 = -10.125 + 3B + C \qquad (2)$$

From (1) and (2), B=29.875 and C=-29.5

$$V = -0.375y^3 + 29.875y - 29.5$$

$$V(2) = \underline{\underline{27.25\text{ V}}}$$

Prob. 6.5

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon} \longrightarrow \frac{d^2V}{dz^2} = -\frac{\rho_o z}{\varepsilon d}$$

$$\frac{dV}{dz} = -\frac{\rho_o z^2}{2\varepsilon d} + A$$

$$V = -\frac{\rho_o z^3}{6\varepsilon d} + Az + B$$

$$z = 0, V = 0 \longrightarrow 0 = 0 + B, \quad \text{i.e. } B = 0$$

$$z = d, V = V_o \longrightarrow V_o = -\frac{\rho_o d^2}{6\varepsilon} + Ad$$

$$A = \frac{V_o}{d} + \frac{\rho_o d}{6\varepsilon}$$

Hence,

$$V = \underline{\underline{-\frac{\rho_o z^3}{6\varepsilon d} + \left(\frac{V_o}{d} + \frac{\rho_o d}{6\varepsilon}\right)z}}$$

Prob. 6.6

$$\nabla^2 V = \frac{d^2V}{dx^2} = -\frac{\rho_v}{\varepsilon} = -\frac{50(1-y^2)x10^{-6}}{\varepsilon} = -k(1-y^2)$$

where $k = \dfrac{50\times 10^{-6}}{3\times\dfrac{10^{-9}}{36\pi}} = 600\pi\times 10^3$

$$\frac{dV}{dy} = -k(y - y^3/3) + A$$

$$V = -k\left(\frac{y^2}{2} - \frac{y^4}{12}\right) + Ay + B = 50\pi.10^3 y^4 - 300\pi.10^3 y^2 + Ay + B$$

When y=2cm, V=30X10^3,

$$30\times 10^3 = 50\pi\times 10^3 \times 16\times 10^{-6} - 300\pi\times 10^3\times 4\times 10^{-4} + Ay + B$$

or

$$30,374.5 = 0.02A + B \qquad (1)$$

When y=-2cm, V=30x10^3,

$$30,374.5 = -0.02A + B \qquad (2)$$

From (1) and (2), A=0, B=30,374.5. Thus,

$$\underline{\underline{V = 157.08y^4 - 942.5y^2 + 30.374 \text{ kV}}}$$

Prob. 6.7

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon} = \frac{-\dfrac{10}{\rho}\times 10^{-12}}{3.6\times\dfrac{10^{-9}}{36\pi}} = -\frac{0.1\pi}{\rho}$$

Let $\alpha = 0.1\pi.$

$$\nabla^2 V = -\frac{\alpha}{\rho} = \frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dV}{d\rho}\right)$$

$$-\alpha = \frac{d}{d\rho}\left(\rho\frac{dV}{d\rho}\right)$$

$$\rho\frac{dV}{d\rho} = -\alpha\rho + A$$

$$\frac{dV}{d\rho} = -\alpha + \frac{A}{\rho}$$

$$V = -\alpha\rho + A\ln\rho + B$$

At ρ=2, V=0 $\longrightarrow$ $0 = -2\alpha + A\ln 2 + B$ (1)

At ρ=5, V=60 $\longrightarrow$ $60 = -5\alpha + A\ln 5 + B$ (2)

Subtracting (1) from (2),

$$60 = -3\alpha + A\ln 5/2 \longrightarrow A = \frac{60+3\alpha}{\ln 2.5} = 66.51$$

From (1),

$$B = 2\alpha - A\ln 2 = -45.473$$

$$\boldsymbol{E} = -\frac{dV}{d\rho}\boldsymbol{a}_\rho = (\alpha - \frac{A}{\rho})\boldsymbol{a}_\rho = \underline{\underline{(0.3142 - \frac{66.51}{\rho})\boldsymbol{a}_\rho}}$$

Prob. 6.8

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0 + 0 - 2 = -2 \neq 0$$

Does <u>not</u> satisfy Laplace's equation.

Prob. 6.9

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 6xy + 0 + 2c = 0$$

$\underline{\underline{c = -3xy}}$

Prob. 6.10

$V = V(\rho), \qquad \nabla^2 V = \frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{dV}{d\rho}\right) = 0$

$\rho \frac{dV}{d\rho} = A \qquad \longrightarrow \qquad \frac{dV}{d\rho} = \frac{A}{\rho}$

$V = A\ln\rho + B$

When ρ=4mm=a, V=0 $\longrightarrow$ 0=Alna+B or B = -Alna

When ρ=12mm=b, V=V_o $\longrightarrow$ V_o=Alnb-Alna or $A = \frac{V_o}{\ln\frac{b}{a}}$

$$\boldsymbol{E} = -\nabla V = -\frac{dV}{d\rho}\boldsymbol{a}_\rho = -\frac{A}{\rho}\boldsymbol{a}_\rho = -\frac{V_o}{\rho\ln\frac{b}{a}}\boldsymbol{a}_\rho$$

At ρ=8mm,

$$-6\boldsymbol{a}_\rho \text{ kV/m} = -\frac{V_o}{8\times10^{-3}\ln\frac{12}{4}}\boldsymbol{a}_\rho \qquad \longrightarrow \qquad V_o = 48\ln 3 = \underline{\underline{53.73}}$$

Prob. 6.11

$$\nabla^2 V = \frac{d^2V}{dz^2} = 0 \qquad \longrightarrow \qquad V = Az + B$$

When z=0, V = 0 $\longrightarrow$ B=0

When z=d, V = V_o $\longrightarrow$ V_o=Ad or A = V_o/d

Hence,

$$V = \frac{V_o z}{d}$$

$$\boldsymbol{E} = -\nabla V = -\frac{dV}{dz}\boldsymbol{a}_z = -\frac{V_o}{d}\boldsymbol{a}_z$$

$$\boldsymbol{D} = \varepsilon\boldsymbol{E} = -\varepsilon_o\varepsilon_r\frac{V_o}{d}\boldsymbol{a}_z$$

Since V_o = 50 V and d = 2mm,

<u>V = 25z kV,</u> <u>**E** = - 25$\mathbf{a}_z$ kV/m</u>

$$\boldsymbol{D} = -\frac{10^{-9}}{36\pi}(1.5)25\times10^3\boldsymbol{a}_z = \underline{\underline{-332\boldsymbol{a}_z \text{ nC/m}^2}}$$

$$\rho_s = D_n = \underline{\underline{\pm 332 \text{ nC/m}^2}}$$

The surface charge density is positive on the plate at z=d and negative on the plate at z=0.

Prob. 6.12 From Example 6.8, solving $\nabla^2 V = 0$ when $V = V(\rho)$ leads to

$$V = \frac{V_o \ln \rho / a}{\ln b / a} = V_o \frac{\ln(a/\rho)}{\ln(a/b)}$$

$$\boldsymbol{E} = -\nabla V = -\frac{V_o}{\rho \ln b / a}\boldsymbol{a}_\rho = \frac{V_o}{\rho \ln a / b}\boldsymbol{a}_\rho, \qquad \boldsymbol{D} = \varepsilon \boldsymbol{E} = -\frac{\varepsilon_o \varepsilon_r V_o}{\rho \ln b / a}\boldsymbol{a}_\rho$$

$$\rho_s = D_n = \pm \frac{\varepsilon_o \varepsilon_r V_o}{\rho \ln b / a}\bigg|_{\rho = a,b}$$

In this case, V_o=100 V, b=5mm, a=15mm, $\varepsilon_r = 2$. Hence at ρ=10mm,

$$V = \frac{100 \ln(10/15)}{\ln(5/15)} = \underline{\underline{36.91 \text{ V}}}$$

$$\boldsymbol{E} = \frac{100}{10x10^{-3} \ln 3}\boldsymbol{a}_\rho = \underline{\underline{9.102\boldsymbol{a}_\rho \text{ kV/m}}}$$

$$\boldsymbol{D} = 9.102x10^3 x \frac{10^{-9}}{36\pi} 2\boldsymbol{a}_\rho = \underline{\underline{161\boldsymbol{a}_\rho \text{ nC/m}^2}}$$

$$\rho_s(\rho = 5\text{mm}) = \frac{10^{-9}}{36\pi}(2)\frac{10^5}{5\ln 3} = \underline{\underline{322 \text{ nC/m}^2}}$$

$$\rho_s(\rho = 15\text{mm}) = -\frac{10^{-9}}{36\pi}(2)\frac{10^5}{15\ln 3} = \underline{\underline{-107.3 \text{ nC/m}^2}}$$

Prob. 6.13

(a)

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + 0$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(2c_1\rho^2 - 2c_2\rho^{-2}\right)\sin 2\phi - \frac{4}{\rho^2}(c_1\rho^2 + c_2\rho^{-2})\sin 2\phi$$

$$= (4c_1 + 4c_2\rho^{-4} - 4c_1 - 4c_2\rho^{-4})\sin 2\phi = 0$$

(b)

At $P(2, 45^o, 1), \rho = 1, \phi = 45^o$

$$50 = (c_1 + c_2)\sin 90^o = c_1 + c_2 \qquad (1)$$

$$-\boldsymbol{E} = \nabla V = \frac{\partial V}{\partial \rho}\boldsymbol{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\boldsymbol{a}_\phi + \boldsymbol{0}$$

$$= (2c_1\rho - 2c_2\rho^{-3})\sin 2\phi \boldsymbol{a}_\rho + (c_1\rho + c_2\rho^{-3})(2)\cos 2\phi \boldsymbol{a}_\phi$$

At P,

$$-\boldsymbol{E} = (2c_1 - 2c_2)(1)\boldsymbol{a}_\rho + \boldsymbol{0}$$

$$|\mathbf{E}| = 100 = 2c_1 - 2c_2 \quad \longrightarrow \quad 50 = c_1 - c_2 \qquad (2)$$

From (1) and (2), $\underline{\underline{c_1 = 50, c_2 = 0}}$

Prob. 6.14

$$\frac{1}{\rho}\frac{d^2V}{d\phi^2} = 0 \quad \longrightarrow \quad \frac{d^2V}{d\phi^2} = 0 \quad \longrightarrow \quad \frac{dV}{d\phi} = A$$

$$V = A\phi + B$$

$$0 = 0 + B \quad \longrightarrow \quad B = 0$$

$$50 = A\pi/2 \quad \longrightarrow \quad A = \frac{100}{\pi}$$

$$\boldsymbol{E} = -\nabla V = -\frac{1}{\rho}\frac{dV}{d\phi}\boldsymbol{a}_\phi = -\frac{A}{\rho}\boldsymbol{a}_\phi = \underline{\underline{-\frac{100}{\pi\rho}\boldsymbol{a}_\phi}}$$

Prob. 6.15

(a)

$$\frac{\partial V}{\partial \rho} = V_o(1 + \frac{a^2}{\rho^2})\sin\phi$$

$$\rho\frac{\partial V}{\partial \rho} = V_o(\rho + \frac{a^2}{\rho})\sin\phi$$

$$\frac{\partial}{\partial \rho}\left(\rho\frac{\partial V}{\partial \rho}\right) = V_o(1 - \frac{a^2}{\rho^2})\sin\phi$$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial V}{\partial \rho}\right) = V_o(\frac{1}{\rho} - \frac{a^2}{\rho^3})\sin\phi$$

$$\frac{\partial^2 V}{\partial \phi^2} = -V_o(\rho - \frac{a^2}{\rho})\sin\phi$$

$$\frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} = -V_o(\frac{1}{\rho} - \frac{a^2}{\rho^3})\sin\phi$$

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial V}{\partial \rho}\right) + \frac{\partial^2 V}{\partial \phi^2} = 0$$

(b)

If $\rho^2 \gg a^2$, then $a^2/\rho^2 \ll 1$ and $V \simeq V_o \rho \sin\phi$

$$\boldsymbol{E} = -\nabla V = -\frac{\partial V}{\partial \rho}\boldsymbol{a}_\rho - \frac{1}{\rho}\frac{\partial V}{\partial \phi}\boldsymbol{a}_\phi = \underline{\underline{-V_o \sin\phi \boldsymbol{a}_\rho - V_o \cos\phi \boldsymbol{a}_\phi}}$$

Prob. 6.16

$$\nabla^2 V = \frac{d^2V}{dx^2} = 0 \longrightarrow V = Ax + B$$

At $x = 20$ mm $= 0.02$ m, $V = 0$

$$0 = 0.02A + B \longrightarrow (1)$$

$$\boldsymbol{E} = -\frac{dV}{dx}\boldsymbol{a}_x \longrightarrow A = 110 \longrightarrow (2)$$

From (1) $B = -0.02A = -2.2$

Then $V = 110x - 2.2$

At $x = 0$ $\underline{\underline{V = -2.2V}}$

At x = 50 mm = 0.05 m,

$$V = 110 \times 0.05 - 2.2 = \underline{\underline{3.3V}}$$

Prob. 6.17

$$\nabla^2 V = 0 \longrightarrow V = -A/r + B$$

At r=0.5, V=-50 $\longrightarrow$ -50 = -A/0.5 + B

Or

-50 = -2A + B (1)

At r = 1, V =50 $\longrightarrow$ 50 = -A + B (2)

From (1) and (2), A = 100, B = 150, and

$$\underline{\underline{V = -\frac{100}{r} + 150}}$$

$$\boldsymbol{E} = -\nabla V = -\frac{A}{r^2}\boldsymbol{a}_r = \underline{\underline{-\frac{100}{r^2}\boldsymbol{a}_r \text{ V/m}}}$$

Prob. 6.18 From Example 6.4,

$$V = \frac{V_o \ln\left(\dfrac{\tan\theta/2}{\tan\theta_1/2}\right)}{\ln\left(\dfrac{\tan\theta_2/2}{\tan\theta_1/2}\right)}$$

$V_o = 100,\ \theta_1 = 30^o,\ \ \theta_2 = 120^o,\ \ r = \sqrt{3^2+0^2+4^2} = 5, \qquad \theta = \tan^{-1}\rho/z = \tan^{-1}3/4 = 36.87^o$

$$V = 100\frac{\ln\left(\dfrac{\tan 18.435^o}{\tan 15^o}\right)}{\ln\left(\dfrac{\tan 60^o}{\tan 15^o}\right)} = \underline{\underline{11.7\text{ V}}}$$

$$\boldsymbol{E} = \frac{-V_o\boldsymbol{a}_\theta}{r\sin\theta\ln\left(\dfrac{\tan\theta_2/2}{\tan\theta_1/2}\right)} = \frac{-100\boldsymbol{a}_\theta}{5\sin 36.87^o\ \ln 6.464} = \underline{\underline{-17.86\boldsymbol{a}_\theta\ \text{ V/m}}}$$

Prob. 6.19

(a)

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) = 0 \quad\longrightarrow\quad V = A\ln\rho + B$$

$$V(\rho=b)=0 \quad\longrightarrow\quad 0 = A\ln b + B \quad\longrightarrow\quad B = -A\ln b$$

$$V(\rho=a)=V_o \quad\longrightarrow\quad V_o = A\ln a/b \quad\longrightarrow\quad A = -\frac{V_o}{\ln b/a}$$

$$V = -\frac{V_o}{\ln b/a}\ln\rho/b = \frac{V_o\ln b/\rho}{\ln b/a}$$

$$V(\rho=15\text{mm}) = 70\frac{\ln 2}{\ln 50} = \underline{\underline{12.4\text{ V}}}$$

(b) As the electron decelerates, potential energy gained = K.E. loss

$$e[70-12.4] = \frac{1}{2}m[(10^7)^2 - u^2] \quad\longrightarrow\quad 10^{14} - u^2 = \frac{2e}{m}\times 57.6$$

$$u^2 = 10^{14} - \frac{2\times1.6\times10^{-19}}{9.1\times10^{-31}}\times 57.6 = 10^{12}(100-20.25)$$

$$\underline{\underline{u = 8.93\times10^6\ \text{ m/s}}}$$

Prob. 6.20 This is similar to case 1 of Example 6.5.

$$X = c_1 x + c_2, \quad Y = c_3 y + c_4$$

But $X(0) = 0 \longrightarrow 0 = c_2, \quad Y(0) = 0 \longrightarrow 0 = c_4$

Hence,

$$V(x,y) = XY = a_o xy, \quad a_o = c_1 c_3$$

Also, $V(xy = 4) = 20 \longrightarrow 20 = 4a_o \longrightarrow a_o = 5$

Thus,

$$V(x,y) = 5xy \text{ and } \boldsymbol{E} = -\nabla V = -5y\boldsymbol{a}_x - 5x\boldsymbol{a}_y$$

At (x,y) = (1,2),

$$\underline{\underline{V = 10 \text{ V}, \quad \boldsymbol{E} = -10\boldsymbol{a}_x - 5\boldsymbol{a}_y \text{ V/m}}}$$

Prob. 6.21

(a) As in Example 6.5, $X(x) = A\sin(n\pi x / b)$

For Y,

$$Y(y) = c_1 \cosh(n\pi y / b) + c_2 \sinh(n\pi y / b)$$

$$Y(a) = 0 \longrightarrow 0 = c_1 \cosh(n\pi a / b) + c_2 \sinh(n\pi a / b) \longrightarrow c_1 = -c_2 \tanh(n\pi a / b)$$

$$V = \sum_{n=1}^{\infty} a_n \sin(n\pi x / b)\left[\sinh(n\pi y / b) - \tanh(n\pi a / b)\cosh(n\pi y / b)\right]$$

$$V(x, y = 0) = V_o = -\sum_{n=1}^{\infty} a_n \tanh(n\pi a / b)\sin(n\pi x / b)$$

$$-a_n \tanh(n\pi a / b) = \frac{2}{b}\int_0^b V_o \sin(n\pi x / b)dx = \begin{cases} \dfrac{4V_o}{n\pi}, & n = \text{odd} \\ 0, & n = \text{even} \end{cases}$$

Hence,

$$V = -\frac{4V_o}{\pi}\sum_{n=\text{odd}}^{\infty}\sin(n\pi x/b)\left[\frac{\sinh(n\pi y/b)}{n\tanh(n\pi a/b)} - \frac{\cosh(n\pi y/b)}{n}\right]$$

$$= -\frac{4V_o}{\pi}\sum_{n=\text{odd}}^{\infty}\frac{\sin(n\pi x/b)}{n\sinh(n\pi a/b)}\left[\sinh(n\pi y/b)\cosh(n\pi a/b) - \cosh(n\pi y/b)\sinh(n\pi a/b)\right]$$

$$\underline{\underline{= \frac{4V_o}{\pi}\sum_{n=\text{odd}}^{\infty}\frac{\sin(n\pi x/b)\sinh[n\pi(a-y)/b]}{n\sinh(n\pi a/b)}}}$$

Alternatively, for Y

$$Y(y) = c_1\sinh n\pi(y - c_2)/b$$

$$Y(a) = 0 \quad \longrightarrow \quad 0 = c_1\sinh[n\pi(a - c_2)/b] \quad \longrightarrow \quad c_2 = a$$

$$V = \sum_{n=1}^{\infty} b_n\sin(n\pi x/b)\sinh[n\pi(y-a)/b]$$

where

$$b_n = \begin{cases} -\dfrac{4V_o}{n\pi\sinh(n\pi a/b)}, & n = \text{odd} \\ 0, & n = \text{even} \end{cases}$$

(b) This is the same as Example 6.5 except that we exchange y and x. Hence

$$\underline{\underline{V(x,y) = \frac{4V_o}{\pi}\sum_{n=\text{odd}}^{\infty}\frac{\sin(n\pi y/a)\sinh(n\pi x/a)]}{n\sinh(n\pi b/a)}}}$$

(c) This is the same as part (a) except that we must exchange x and y. Hence

$$\underline{\underline{V(x,y) = \frac{4V_o}{\pi}\sum_{n=\text{odd}}^{\infty}\frac{\sin(n\pi y/a)\sinh[n\pi(b-x)/a]}{n\sinh(n\pi b/a)}}}$$

Prob. 6.22 (a) X(x) is the same as in Example 6.5. Hence

$$V(x,y) = \sum_{n=1}^{\infty} \sin(n\pi x / b)\left[a_n \sinh(n\pi y / b) + b_n \cosh(n\pi y / b)\right]$$

At y=0, V = V_1

$$V_1 = \sum_{n=1}^{\infty} b_n \sin(n\pi x / b) \quad \longrightarrow \quad b_n = \begin{cases} \dfrac{4V_1}{n\pi}, & n = \text{odd} \\ 0, & n = \text{even} \end{cases}$$

At y=a, V = V_2

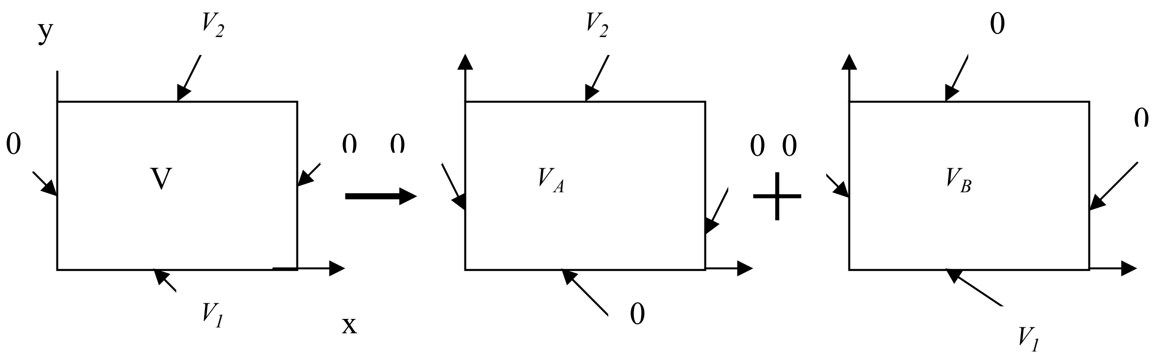
$$V_2 = \sum_{n=1}^{\infty} \sin(n\pi x / b)\left[a_n \sinh(n\pi a / b) + b_n \cosh(n\pi a / b)\right]$$

$$a_n \sinh(n\pi a / b) + b_n \cosh(n\pi a / b) = \begin{cases} \dfrac{4V_2}{n\pi}, & n = \text{odd} \\ 0, & \text{n=even} \end{cases}$$

or

$$a_n = \begin{cases} \dfrac{4V_2}{n\pi \sinh(n\pi a / b)}\left(V_2 - V_1 \cosh(n\pi a / b)\right), & n = \text{odd} \\ 0, & \text{n=even} \end{cases}$$

Alternatively, we may apply superposition principle.



i.e. $V = V_A + V_B$

V_A is exactly the same as Example 6.5 with $V_o = V_2$, while V_B is exactly the same **as** Prob. 6.19(a). Hence

$$V = \frac{4}{\pi}\sum_{n=\text{odd}}^{\infty}\frac{\sin(n\pi x/b)}{n\sinh(n\pi a/b)}\left[V_1\sinh[n\pi(a-y)/b] + V_2\sinh(n\pi y/b)\right]$$

(b)

$$V(x,y) = (a_1 e^{-\alpha x} + a_2 e^{+\alpha x})(a_3\sin\alpha y + a_4\cos\alpha y)$$

$$\lim_{x\longrightarrow\infty} V(x,y) = 0 \quad\longrightarrow\quad a_2 = 0$$

$$V(x, y=0) = 0 \quad\longrightarrow\quad a_4 = 0$$

$$V(x, y=a) = 0 \quad\longrightarrow\quad \alpha = n\pi/a, \qquad n = 1,2,3,...$$

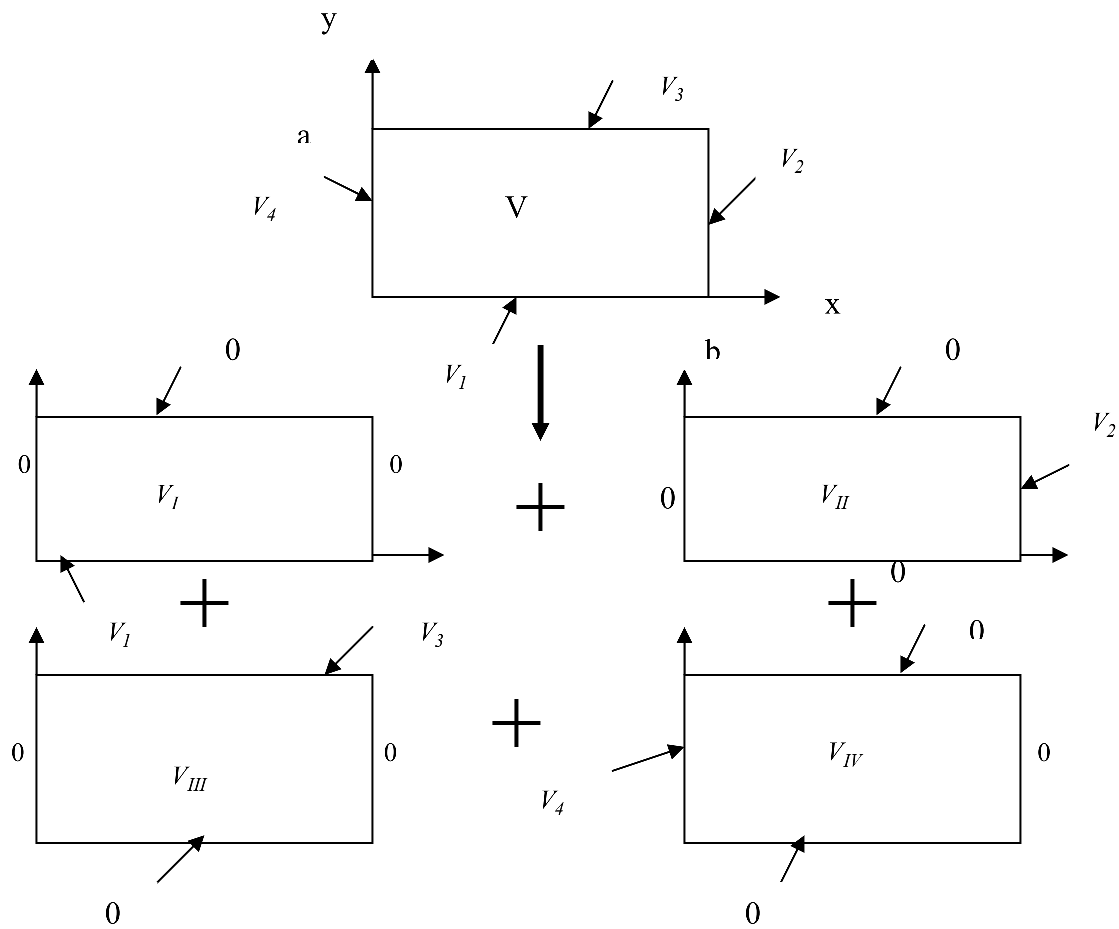
Hence,

$$V(x,y) = \sum_{n=1}^{\infty} a_n e^{-n\pi x/a}\sin(n\pi y/a)$$

$$V(x=0,y) = V_o = \sum_{n=1}^{\infty} a_n\sin(n\pi y/a) \quad\longrightarrow\quad a_n = \begin{cases}\dfrac{4V_o}{n\pi}, & n = \text{odd}\\ 0, & n = \text{even}\end{cases}$$

$$V(x,y) = \frac{4V_o}{\pi}\sum_{n=\text{odd}}^{\infty}\frac{\sin(n\pi y/a)}{n}\exp(-n\pi x/a)$$

(c) The problem is easily solved using superposition theorem, as illustrated below.



Therefore,

$$V = V_I + V_{II} + V_{III} + V_{IV}$$

$$= \frac{4}{\pi}\sum_{n=odd}^{\infty}\frac{1}{n}\left\{\begin{aligned}&\frac{\sin(n\pi x/b)}{\sinh(n\pi a/b)}\left[V_1\sinh(n\pi(a-y)/b)+V_3\sinh(n\pi y/b)\right]\\&+\frac{\sin(n\pi x/a)}{\sinh(n\pi b/a)}\left[V_2\sinh(n\pi y/a)+V_4\sinh(n\pi(b-x)/a)\right]\end{aligned}\right\}$$

where

$$V_I = \frac{4V_1}{\pi}\sum_{n=\text{odd}}^{\infty}\frac{\sin(n\pi x/b)\sinh[n\pi(a-y)/b]}{n\sinh(n\pi a/b)}$$

$$V_{II} = \frac{4V_2}{\pi}\sum_{n=\text{odd}}^{\infty}\frac{\sin(n\pi x/a)\sinh(n\pi y/a)}{n\sinh(n\pi b/a)}$$

$$V_{III} = \frac{4V_3}{\pi} \sum_{n=\text{odd}}^{\infty} \frac{\sin(n\pi x/b)\sinh(n\pi y/b)}{n\sinh(n\pi a/b)}$$

$$V_{IV} = \frac{4V_4}{\pi} \sum_{n=\text{odd}}^{\infty} \frac{\sin(n\pi y/a)\sinh[n\pi(b-x)/a]}{n\sinh(n\pi b/a)}$$

Prob. 6.23

$$\boldsymbol{E} = -\nabla V = -\frac{\partial V}{\partial x}\boldsymbol{a}_x - \frac{\partial V}{\partial y}\boldsymbol{a}_y$$

$$E_x = \frac{4V_o}{\pi} \sum_{n=odd}^{\infty} \frac{n\pi}{a} \frac{\sin(n\pi y/a)}{n} \exp(-n\pi x/a)$$

$$E_y = -\frac{4V_o}{\pi} \sum_{n=odd}^{\infty} \frac{n\pi}{a} \frac{\cos(n\pi y/a)}{n} \exp(-n\pi x/a)$$

$$\boldsymbol{E} = \underline{\underline{\frac{4V_o}{a} \sum_{n=odd}^{\infty} \exp(-n\pi x/a)\left[\sin(n\pi y/a)\boldsymbol{a}_x - \cos(n\pi y/a)\boldsymbol{a}_y\right]}}$$

Prob. 6.24

This is similar to Example 6.5 except that we must exchange x and y. Going through the same arguments, we have

$$V(x,y) = \sum c_n \sinh\left(\frac{n\pi x}{b}\right)\sin\left(\frac{n\pi y}{b}\right)$$

Applying the condition at x=a, we get

$$V_o \sin\left(\frac{\pi y}{b}\right) = \sum c_n \sinh\left(\frac{n\pi a}{b}\right)\sin\left(\frac{n\pi y}{b}\right)$$

This yields

$$c_n \sinh\left(\frac{n\pi a}{b}\right) = \begin{cases} V_o, & n=1 \\ 0, & n \neq 1 \end{cases}$$

Hence,

$$V(x,y) = \underline{\underline{V_o \frac{\sinh\left(\frac{\pi x}{b}\right)\sin\left(\frac{\pi y}{b}\right)}{\sinh\left(\frac{\pi a}{b}\right)}}}$$

Prob. 6.25

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2} = 0$$

If we let $V(\rho,\phi) = R(\rho)\Phi(\phi)$,

$$\frac{\Phi}{\rho}\frac{\partial}{\partial\rho}(\rho R') + \frac{1}{\rho^2}R\Phi'' = 0$$

or

$$\frac{\rho}{R}\frac{\partial}{\partial\rho}(\rho R') = -\frac{\Phi''}{\Phi} = \lambda$$

Hence

$$\underline{\underline{\Phi'' + \lambda\Phi = 0}}$$

and

$$\frac{\partial}{\partial\rho}(\rho R') - \frac{\lambda R}{\rho} = 0$$

or

$$\underline{\underline{R'' + \frac{R'}{\rho} - \frac{\lambda R}{\rho^2} = 0}}$$

Prob. 6.26

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial V}{\partial\theta}) = 0$$

If $V(r,\theta) = R(r)F(\theta), \quad r \neq 0,$

$$F\frac{d}{dr}(r^2R') + \frac{R}{\sin\theta}\frac{d}{d\theta}(\sin\theta F') = 0$$

Dividing through by RF gives

$$\frac{1}{R}\frac{d}{dr}(r^2R') = -\frac{1}{F\sin\theta}\frac{d}{d\theta}(\sin\theta F') = \lambda$$

Hence,

$$\sin\theta F'' + \cos\theta F' + \lambda F\sin\theta = 0$$

or

$$\underline{\underline{F''+\cot\theta F'+\lambda F=0}}$$

Also,

$$\frac{d}{dr}(r^2R')-\lambda R=0$$

or

$$\underline{\underline{R''+\frac{2R'}{r}-\frac{\lambda}{r^2}R=0}}$$

Prob. 6.27 If the centers at $\phi=0$ and $\phi=\pi/2$ are maintained at a potential difference of V_o, from Example 6.3,

$$E_\phi=\frac{2V_o}{\pi\rho}, \qquad J=\sigma E$$

Hence,

$$I=\int J\bullet dS=\frac{2V_o\sigma}{\pi}\int_{\rho=a}^{b}\int_{z=0}^{t}\frac{1}{\rho}d\rho dz=\frac{2V_o\sigma t}{\pi}\ln(b/a)$$

and

$$R=\frac{V_o}{I}=\frac{\pi}{2\sigma t\ln(b/a)}$$

Prob. 6.28 If $V(r=a)=0, \quad V(r=b)=V_o$, from Example 6.9,

$$E=\frac{V_o}{r^2(1/a-1/b)}, \qquad J=\sigma E$$

Hence,

$$I=\int J\bullet dS=\frac{V_o\sigma}{1/a-1/b}\int_{\theta=0}^{\alpha}\int_{\phi=0}^{2\pi}\frac{1}{r^2}r^2\sin\theta d\theta d\phi=\frac{2\pi V_o\sigma}{1/a-1/b}(-\cos\theta)\Big|_0^{\alpha}$$

$$R=\frac{V_o}{I}=\frac{\frac{1}{a}-\frac{1}{b}}{2\pi\sigma(1-\cos\alpha)}$$

Prob. 6.29

This is the same as Problem 6.30 except that $\alpha = \pi$. Hence,

$$R = \frac{1}{2\pi\sigma(1-\cos\pi)}\left(\frac{1}{a}-\frac{1}{b}\right) = \underline{\underline{\frac{1}{4\pi\sigma}\left(\frac{1}{a}-\frac{1}{b}\right)}}$$

Prob. 6.30

For a spherical capacitor, from Eq. (6.38),

$$R = \frac{\frac{1}{a}-\frac{1}{b}}{4\pi\sigma}$$

For the hemisphere, $R' = 2R$ since the sphere consists of two hemispheres in parallel. As

$b \longrightarrow \infty$,

$$R' = \lim_{b\longrightarrow\infty} \frac{2\left[\frac{1}{a}-\frac{1}{b}\right]}{4\pi\sigma} = \frac{1}{2\pi a\sigma}$$

$$G = 1/R' = 2\pi a\sigma$$

Alternatively, for an isolated sphere, $C = 4\pi\varepsilon a$. But

$$RC = \frac{\varepsilon}{\sigma} \quad \longrightarrow \quad R = \frac{1}{4\pi a\sigma}$$

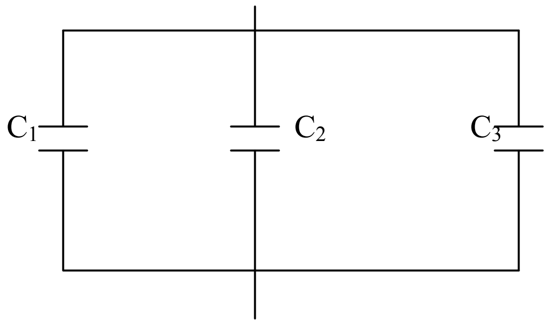
$$R' = 2R = \frac{1}{2\pi a\sigma} \quad \text{or} \quad G = 2\pi a\sigma$$

Prob. 6.31

$$C = \frac{\varepsilon S}{d} \quad \longrightarrow \quad S = \frac{Cd}{\varepsilon_o\varepsilon_r} = \frac{2\times10^{-9}\times10^{-6}}{4\times10^{-9}/36\pi}\ \text{m}^2 = \underline{\underline{0.5655\ \text{cm}^2}}$$

Prob. 6.32

This can be regarded as three capacitors in parallel**.**



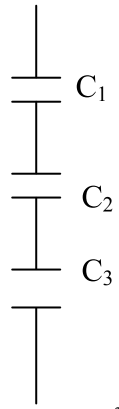
$$C = C_1 + C_2 + C_3 = \sum \frac{\varepsilon_o \varepsilon_{rk} S_k}{d_k}$$

$$= \frac{\varepsilon_o}{2\times10^{-3}}\left[3\times15\times10^{-2}\times20\times10^{-2} + 5\times15\times10^{-2}\times20\times10^{-2} + 8\times15\times10^{-2}\times20\times10^{-2}\right]$$

$$= \frac{10^{-9}}{36\pi}\times\frac{15\times10^{-2}\times20\times10^{-2}}{2\times10^{-3}}[3+5+8] = \underline{\underline{2.122 \text{ nF}}}$$

Prob. 6.33

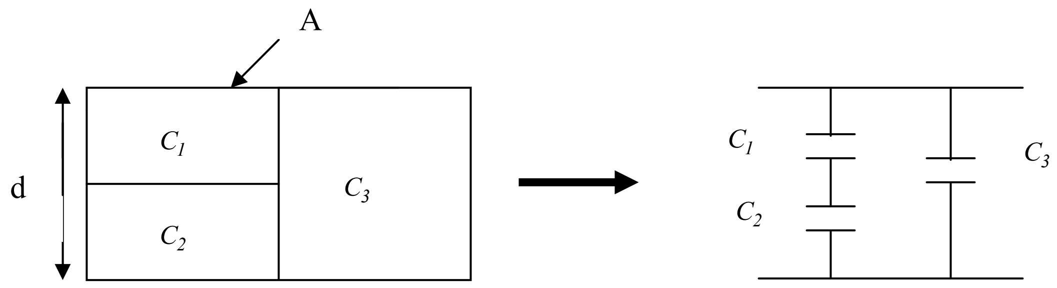
This may be regarded as three capacitors in series.



$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} = \sum_{k=1}^{3} \frac{d_k}{\varepsilon_k S_k} = \frac{1\times10^{-3}}{\frac{10^{-9}}{36\pi}\times80\times10^{-4}}\left[\frac{1}{3}+\frac{1}{5}+\frac{1}{8}\right] = \frac{36\pi}{8}\times10^{9}\times0.6583$$

$$C_{eq} = \frac{8}{36\pi x 0.6583} \text{ nF} = \underline{\underline{0.1076 \text{ nF}}}$$

Prob. 6.34



From the figure above,

$$C = \frac{C_1 C_2}{C_1 + C_2} + C_3$$

here

$$C_1 = \frac{\varepsilon_o A/2}{d/2} = \frac{\varepsilon_o A}{d}, \qquad C_2 = \frac{\varepsilon_o \varepsilon_r A}{d}, \qquad C_3 = \frac{\varepsilon_o A}{2d}$$

$$C = \frac{\varepsilon_o{}^2 \varepsilon_r A^2/d^2}{\varepsilon_o(\varepsilon_r + 1)A/d} + \frac{\varepsilon_o A}{2d} = \frac{\varepsilon_o A}{d}\left(\frac{1}{2} + \frac{\varepsilon_r}{\varepsilon_r + 1}\right) = \frac{10^{-9}}{36\pi}\frac{10\times 10^{-4}}{2\times 10^{-3}}\left(\frac{1}{2} + \frac{6}{7}\right) \cong \underline{\underline{6\text{ pF}}}$$

Prob. 6.35

$$C = \frac{\varepsilon_o S}{d} \quad \longrightarrow S = \frac{Cd}{\varepsilon_o}$$

$$S = \frac{1\times 1\times 10^{-3}}{10^{-9}/36\pi} = 36\pi \times 10^{6}$$

$$\underline{\underline{S = 1.131\times 10^{8}\text{ m}^2}}$$

Prob. 6.36

$$Fdx = dW_E \longrightarrow F = \frac{dW_E}{dx}$$

$$W_E = \int \frac{1}{2}\varepsilon |E|^2 \, dv = \frac{1}{2}\varepsilon_o \varepsilon_r E^2 xad + \frac{1}{2}\varepsilon_o E^2 da(1-x)$$

where $E = V_o / d$.

$$\frac{dW_E}{dx} = \frac{1}{2}\varepsilon_o \frac{V_o^{\,2}}{d^2}(\varepsilon_r - 1)da \longrightarrow F = \frac{\varepsilon_o(\varepsilon_r - 1)V_o^{\,2}a}{2d}$$

Alternatively, $W_E = \frac{1}{2}CV_o^{\,2}$, where

$$C = C_1 + C_2 = \frac{\varepsilon_o \varepsilon_r ax}{d} + \frac{\varepsilon_o \varepsilon_r (1-x)}{d}$$

$$\frac{dW_E}{dx} = \frac{1}{2}\varepsilon_o \frac{V_o^{\,2}a}{d}(\varepsilon_r - 1)$$

$$\underline{\underline{F = \frac{\varepsilon_o(\varepsilon_r - 1)V_o^{\,2}a}{2d}}}$$

Prob.6.37

(a)

$$C = \frac{4\pi\varepsilon}{\frac{1}{a} - \frac{1}{b}} = \frac{4\pi \times 2.25 \times \frac{10^{-9}}{36\pi}}{\frac{1}{5x10^{-2}} - \frac{1}{10x10^{-2}}} = \underline{\underline{25 \text{ pF}}}$$

(b) Q = C V_o= 25x80 pC

$$\rho_s = \frac{Q}{4\pi r^2} = \frac{25 \times 80}{4\pi \times 25 \times 10^{-4}} \text{ pC/m}^2 = \underline{\underline{63.66 \text{ nC/m}^2}}$$

Prob. 6.38

$$C_1 = \frac{\varepsilon_o \varepsilon_r S}{d}, \qquad C_2 = \frac{\varepsilon_o S}{d}$$

$$\frac{C_1}{C_2} = \varepsilon_r \longrightarrow \varepsilon_r = \frac{56\mu\text{F}}{32\mu\text{F}} = \underline{\underline{1.75}}$$

Prob. 6.39

$$\frac{d^2V}{dz^2}=0 \longrightarrow V = Az + B$$

At z=0, V=40 $\longrightarrow$ 40=B

At z=2mm=d, V=0 $\longrightarrow 0 = Ad + 40 \longrightarrow A = -40/d$

$$V = -\frac{40}{d}z + 40 = -\frac{40z}{2\times10^{-3}} + 40 = \underline{\underline{-20\times10^3 z + 40 \text{ V}}}$$

Prob. 6.40

(a) $$C = \frac{\varepsilon S}{d} = \frac{6.8\times\frac{10^{-9}}{36\pi}\times0.5}{4\times10^{-3}} = \underline{\underline{7.515 \text{ nF}}}$$

(b) $$\rho_s = \frac{Q}{S}, \quad C = \frac{Q}{V} \longrightarrow Q = CV$$

$$\rho_s = \pm\frac{CV}{S} = \pm\frac{7.515\times10^{-9}\times9}{0.5} = \underline{\underline{\pm135.27 \text{ nC/m}^2}}$$

Prob. 6.41

$$W_E = \frac{Q^2}{2C}$$

$$C = \frac{\varepsilon_o\varepsilon_r S}{d}$$

$$W_E = \frac{Q^2 d}{2\varepsilon_0\varepsilon_r S}$$

When the plate spacing is doubled

$$W_E = \frac{Q^2(2d)}{2\varepsilon_0\varepsilon_r S} = \underline{\underline{\frac{Q^2 d}{\varepsilon_0\varepsilon_r S}}}$$

When the plate spacing is halved

$$W_E = \frac{Q^2(d/2)}{2\varepsilon_0\varepsilon_r S} = \underline{\underline{\frac{Q^2 d}{4\varepsilon_0\varepsilon_r S}}}$$

Prob.6.42

This can be treated as three capacitors in series

$$C_1 = \frac{\varepsilon_1 S}{d_1} = \frac{\varepsilon_o (2.5)(0.4)}{1.5\times10^{-3}} = 666.67\varepsilon_o$$

$$C_2 = \frac{\varepsilon_2 S}{d_2} = \frac{\varepsilon_o (5.6)(0.4)}{1.5\times10^{-3}} = 1493.33\varepsilon_o$$

$$C_3 = \frac{\varepsilon_3 S}{d_3} = \frac{\varepsilon_o (8.1)(0.4)}{2\times10^{-3}} = 1620\varepsilon_o$$

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} = \frac{0.002787}{\varepsilon_0}$$

$$C = 358.81\varepsilon_o = 358.81\times\frac{10^{-9}}{36\pi}$$

$$C = \underline{\underline{3.1726 \text{ nF}}}$$

Prob. 6.43

(a)

$$C = \frac{\varepsilon_o S}{d} = \frac{10^{-9}}{36\pi}\frac{200\times10^{-4}}{3\times10^{-3}} = \underline{\underline{59 \text{ pF}}}$$

(b) $\rho_s = D_n = 10^{-6}$ nC/m^2. But

$$D_n = \varepsilon E_n = \frac{\varepsilon_o V_o}{d} = \rho_s$$

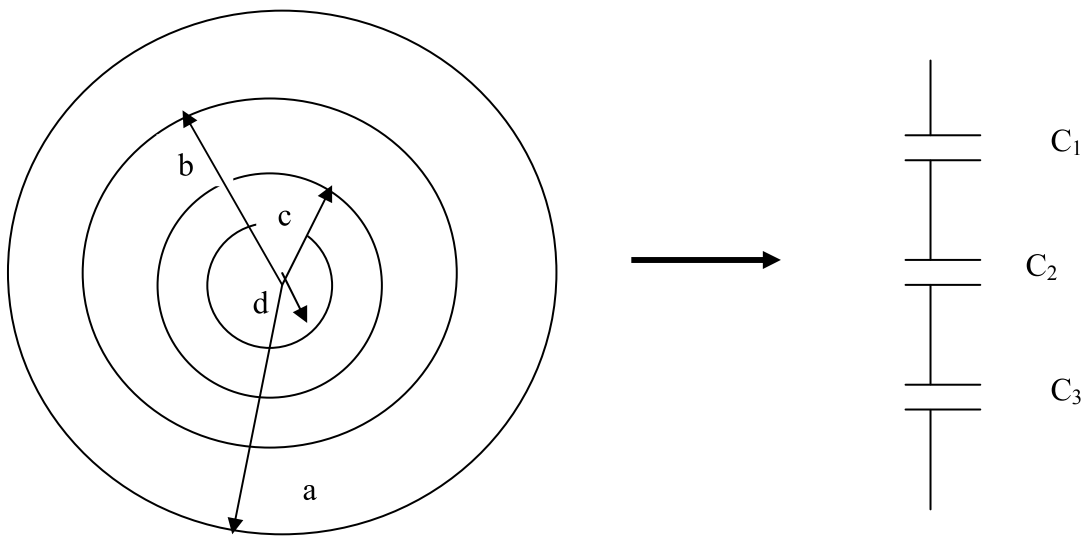
or

$$V_o = \frac{\rho_s d}{\varepsilon_o} = 10^{-6}\times3\times10^{-3}\times36\pi\times10^{9} = \underline{\underline{339.3 \text{ V}}}$$

(c)

$$F = \frac{Q^2}{2S\varepsilon_o} = \frac{\rho_s^{\,2} S}{2\varepsilon_o} = \frac{10^{-12}\times200\times10^{-4}\times36\pi\times10^{9}}{2} = \underline{\underline{1.131 \text{ mN}}}$$

Prob. 6.44



$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$$

where $C_1 = \dfrac{4\pi\varepsilon_3}{\dfrac{1}{b} - \dfrac{1}{a}}, \qquad C_2 = \dfrac{4\pi\varepsilon_2}{\dfrac{1}{c} - \dfrac{1}{b}}, \qquad C_3 = \dfrac{4\pi\varepsilon_1}{\dfrac{1}{d} - \dfrac{1}{c}},$

$$\frac{4\pi}{C} = \frac{1/b - 1/a}{\varepsilon_3} + \frac{1/c - 1/b}{\varepsilon_2} + \frac{1/d - 1/c}{\varepsilon_1}$$

$$C = \frac{4\pi}{\dfrac{\varepsilon_1}{\dfrac{1}{d} - \dfrac{1}{c}} + \dfrac{\varepsilon_2}{\dfrac{1}{c} - \dfrac{1}{b}} + \dfrac{\varepsilon_3}{\dfrac{1}{b} - \dfrac{1}{a}}}$$

Prob. 6.45
We may place a charge Q on the inner conductor. The negative charge –Q is on the outer surface of the shell. Within the shell, **E** = 0, i.e. between r=c and r=b. Otherwise,

$$\boldsymbol{E} = \frac{Q}{4\pi\varepsilon_o r^2}\boldsymbol{a}_r$$

The potential at r=a is

$$V_a = -\int_{-\infty}^{a} \boldsymbol{E}\bullet d\boldsymbol{l} = -\int_{-\infty}^{c} E_r dr - \int_{c}^{b} E_r dr - \int_{b}^{a} E_r dr$$

$$= -\frac{Q}{4\pi\varepsilon_o}\int_{-\infty}^{c}\frac{dr}{r^2} - 0 - \frac{Q}{4\pi\varepsilon_o}\int_{b}^{a}\frac{dr}{r^2} = \frac{Q}{4\pi\varepsilon_o c} + \frac{Q}{4\pi\varepsilon_o}\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$C = \frac{Q}{V_a} = \underline{\underline{\frac{1}{\dfrac{1}{4\pi\varepsilon_o c} + \dfrac{1}{4\pi\varepsilon_o}\left(\dfrac{1}{a} - \dfrac{1}{b}\right)}}}$$

Prob. 6.46

$$C = \frac{2\pi\varepsilon L}{\ln(b/a)} = \frac{2\pi \times 2.5 \times \dfrac{10^{-9}}{36\pi} \times 3 \times 10^3}{\ln(8/5)} = \underline{\underline{0.8665\ \mu\text{F}}}$$

Prob. 6.47
Let the plate at ϕ=0 be 0, i.e. V(0)=0 and let the plate at $\phi=\pi/4$ be V_o, i.e. $V(\pi/4)=V_o$.

$$\nabla^2 V = \frac{1}{\rho^2}\frac{d^2V}{d\phi^2} = 0 \quad\longrightarrow\quad \frac{dV}{d\phi} = A \quad\longrightarrow\quad V = A\phi + B$$

$$V(0) = 0 \quad\longrightarrow\quad 0 = 0 + B \quad\longrightarrow\quad B = 0$$

$$V(\pi/4) = V_o \quad\longrightarrow\quad V_o = A\pi/4 \quad\longrightarrow\quad A = \frac{4V_o}{\pi}$$

$$\boldsymbol{E} = -\nabla V = -\frac{1}{\rho}\frac{dV}{d\phi}\boldsymbol{a}_\phi = -\frac{A}{\rho}\boldsymbol{a}_\phi = -\frac{4V_o}{\pi\rho}\boldsymbol{a}_\phi$$

$$\boldsymbol{D} = \varepsilon\boldsymbol{E} = -\frac{4\varepsilon V_o}{\pi\rho}\boldsymbol{a}_\phi$$

$$\rho_s = D_n = -\frac{4\varepsilon V_o}{\pi\rho}$$

$$Q = \int \rho_s dS = -\int_{\rho=a}^{b}\int_{z=0}^{L}\frac{4\varepsilon V_o}{\pi\rho}d\rho dz = -\frac{4\varepsilon V_o}{\pi}L\ln(b/a)$$

$$C = \frac{|Q|}{V_o} = \underline{\underline{\frac{4\varepsilon L}{\pi}\ln(b/a)}}$$

Prob. 6.48

$$C = \frac{2\pi\varepsilon L}{\ln(b/a)} = \frac{2\pi \times \dfrac{10^{-9}}{36\pi} \times 4.2 \times 400 \times 10^{-3}}{\ln(3.5/1)} = \underline{\underline{74.5 \text{ pF}}}$$

Prob. 6.49

$$C = \frac{2\pi\varepsilon_o L}{\ln(b/a)} = \frac{2\pi \times \dfrac{10^{-9}}{36\pi} \times 100 \times 10^{-6}}{\ln(600/20)} = 1.633 \times 10^{-15} \text{ F}$$

$$V = Q/C = \frac{50 \times 10^{-15}}{1.633 \times 10^{-15}} = \underline{\underline{30.62 \text{ V}}}$$

Prob. 6.50

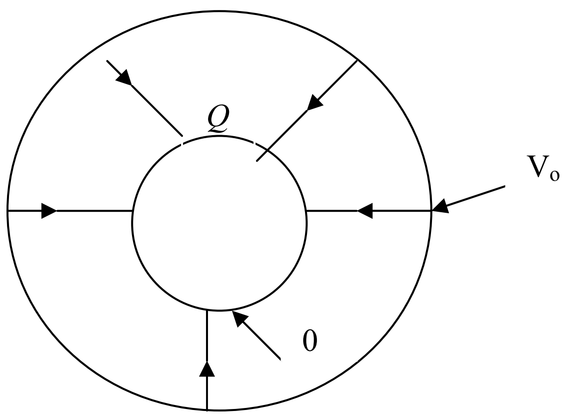
$$C_1 = \frac{2\pi\varepsilon_1}{\ln(b/a)}, \qquad C_2 = \frac{2\pi\varepsilon_2}{\ln(c/b)}$$

Since the capacitance are in series, the total capacitance per unit length is

$$C = \frac{C_1 C_2}{C_1 + C_2} = \underline{\underline{\frac{2\pi\varepsilon_1\varepsilon_2}{\varepsilon_2 \ln(b/a) + \varepsilon_1 \ln(c/b)}}}$$

Prob. 6.51

$$\boldsymbol{E} = \frac{Q}{4\pi\varepsilon r^2}\boldsymbol{a}_r$$



$$W = \frac{1}{2}\int \varepsilon |\boldsymbol{E}|^2\, dv = \iiint \frac{Q^2}{32\pi^2\varepsilon^2 r^4}\varepsilon r^2 \sin\theta d\theta d\phi dr$$

$$= \frac{Q^2}{32\pi^2\varepsilon}(2\pi)(2)\int_c^b \frac{dr}{r^2} = \frac{Q^2}{8\pi\varepsilon}\left(\frac{1}{c} - \frac{1}{b}\right)$$

$$W = \frac{Q^2(b-c)}{8\pi\varepsilon bc}$$

Prob. 6.52

(a) Method 1: $\boldsymbol{E} = \frac{\rho_s}{\varepsilon}(-\boldsymbol{a}_x)$, where ρ_s is to be determined.

$$V_o = -\int \boldsymbol{E} \bullet dl = -\int \frac{-\rho_s}{\varepsilon} dx = \rho_s \int_0^d \frac{1}{\varepsilon_o} \frac{d}{d+x} dx = \frac{\rho_s}{\varepsilon} d \ln(x+d) \Big|_0^d$$

$$V_o = \rho_s d \ln \frac{2d}{d} \longrightarrow \rho_s = \frac{V_o \varepsilon_o}{d \ln 2}$$

$$\boldsymbol{E} = -\frac{\rho_s}{\varepsilon} \boldsymbol{a}_x = \underline{\underline{-\frac{V_o}{(x+d)\ln 2} \boldsymbol{a}_x}}$$

Method 2: We solve Laplace's equation

$$\nabla \bullet (\varepsilon \nabla V) = \frac{d}{dx}(\varepsilon \frac{dV}{dx}) = 0 \longrightarrow \varepsilon \frac{dV}{dx} = A$$

$$\frac{dV}{dx} = \frac{A}{\varepsilon} = \frac{Ad}{\varepsilon_o (x+d)} = \frac{c_1}{x+d}$$

$$V = c_1 \ln(x+d) + c_2$$

$$V(x=0) = 0 \longrightarrow 0 = c_1 \ln d + c_2 \longrightarrow c_2 = -c_1 \ln d$$

$$V(x=d) = V_o \longrightarrow V_o = c_1 \ln 2d - c_1 \ln d = c_1 \ln 2$$

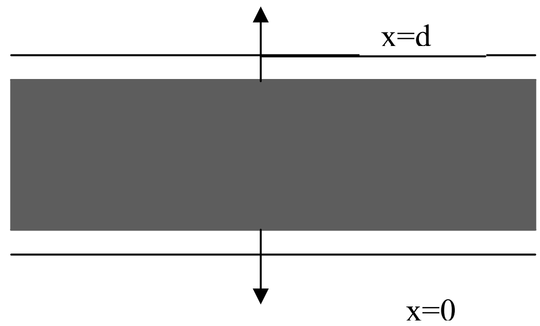
$$c_1 = \frac{V_o}{\ln 2}$$

$$V = c_1 \ln \frac{x+d}{d} = \underline{\underline{\frac{V_o}{\ln 2} \ln \frac{x+d}{d}}}$$

$$\boldsymbol{E} = -\frac{dV}{dx} \boldsymbol{a}_x = \underline{\underline{-\frac{V_o}{(x+d)\ln 2} \boldsymbol{a}_x}}$$

(b) $\boldsymbol{P} = (\varepsilon_r - 1)\varepsilon_o \boldsymbol{E} = -\left(\frac{x+d}{d} - 1\right)\frac{\varepsilon_o V_o}{(x+d)\ln 2}\boldsymbol{a}_x = \underline{\underline{-\frac{\varepsilon_o x V_o}{d(x+d)\ln 2}\boldsymbol{a}_x}}$

(c)



$$\rho_{ps}\big|_{x=0} = \boldsymbol{P} \bullet (-\boldsymbol{a}_x)\big|_{x=0} = \underline{\underline{0}}$$

$$\rho_{ps}\big|_{x=d} = \boldsymbol{P} \bullet \boldsymbol{a}_x\big|_{x=d} = \underline{\underline{-\frac{\varepsilon_o V_o}{2d\ln 2}}}$$

(d) $\boldsymbol{E} = \frac{\rho_s}{\varepsilon}\boldsymbol{a}_x = \frac{Q}{\varepsilon S}\boldsymbol{a}_x = \frac{Q}{\varepsilon_o(1+\frac{x}{d})S}\boldsymbol{a}_x$

$$V = -\int \boldsymbol{E} \bullet d\boldsymbol{l} = -\frac{Q}{\varepsilon_o S}\int_a^d \frac{dx}{(1+\frac{x}{d})} = \frac{Q}{\varepsilon_o S} d\ln 2$$

$$C = \frac{Q}{V} = \underline{\underline{\frac{\varepsilon_o S}{d\ln 2}}}$$

Prob. 6.53

We solve Laplace's equation for an inhomogeneous medium.

$$\nabla\bullet(\varepsilon\nabla V)=\frac{d}{dx}\left(\varepsilon\frac{dV}{dx}\right)=0 \quad\longrightarrow\quad \varepsilon\frac{dV}{dx}=A$$

$$\frac{dV}{dx}=\frac{A}{\varepsilon}=\frac{A}{2\varepsilon_o}\left[1+\left(\frac{x}{d}\right)^2\right]$$

$$V=\frac{A}{2\varepsilon_o}\left(x+\frac{x^3}{3d^2}\right)+B$$

When x=d, V=V_o,

$$V_o=\frac{A}{2\varepsilon_o}\left(d+\frac{d}{3}\right)+B \quad\longrightarrow\quad V_o=\frac{2Ad}{3\varepsilon_o}+B \qquad (1)$$

When x = -d, V=0,

$$0=\frac{A}{2\varepsilon_o}\left(-d-\frac{d}{3}\right)+B \quad\longrightarrow\quad 0=-\frac{2Ad}{3\varepsilon_o}+B \qquad (2)$$

Adding (1) and (2), $V_o=2B \quad\longrightarrow\quad B=V_o/2$

From (2),

$$B=\frac{2Ad}{3\varepsilon_o}=\frac{V_o}{2} \quad\longrightarrow\quad A=\frac{3\varepsilon_o V_o}{4d}$$

$$\boldsymbol{E}=-\nabla V=-\frac{dV}{dx}\boldsymbol{a}_x=-\frac{A}{\varepsilon}\boldsymbol{a}_x=-\frac{3\varepsilon_o V_o}{4d}\frac{\left[1+\left(\frac{x}{d}\right)^2\right]}{2\varepsilon_o}\boldsymbol{a}_x=-\frac{-3V_o}{8d}\left[1+\left(\frac{x}{d}\right)^2\right]\boldsymbol{a}_x$$

$$\rho_s=D\bullet a_n=\varepsilon E\bullet a_x\Big|_{x=d}=-A=-\frac{3\varepsilon_o V_o}{4d}$$

$$Q=\int_S \rho_s dS=\rho_s S=-\frac{3S\varepsilon_o V_o}{4d}$$

$$C=\frac{|Q|}{V_o}=\underline{\underline{\frac{3\varepsilon_o S}{4d}}}$$

Prob. 6.54

Method 1: Using Gauss's law,

$$Q = \int \boldsymbol{D} \bullet dS = 4\pi r^2 D_r \quad \longrightarrow \quad \boldsymbol{D} = \frac{Q}{4\pi r^2}\boldsymbol{a}_r, \quad \varepsilon = \frac{\varepsilon_o k}{r^2}$$

$$\boldsymbol{E} = \boldsymbol{D} / \varepsilon = \frac{Q}{4\pi\varepsilon_o k}\boldsymbol{a}_r$$

$$V = -\int \boldsymbol{E} \bullet d\boldsymbol{l} = -\frac{Q}{4\pi\varepsilon_o k}\int_b^a dr = -\frac{Q}{4\pi\varepsilon_o k}(b-a)$$

$$C = \frac{Q}{|V|} = \underline{\underline{\frac{4\pi\varepsilon_o k}{b-a}}}$$

Method 2: Using the inhomogeneous Laplace's equation,

$$\nabla \bullet (\varepsilon \nabla V) = 0 \quad \longrightarrow \quad \frac{1}{r^2}\frac{d}{dr}\left(\frac{\varepsilon_o k}{r^2} r^2 \frac{dV}{dr}\right) = 0$$

$$\varepsilon_o k \frac{dV}{dr} = A' \quad \longrightarrow \quad \frac{dV}{dr} = A \text{ or } V = Ar + B$$

$$V(r=a) = 0 \quad \longrightarrow \quad 0 = Aa + B \quad \longrightarrow \quad B = -Aa$$

$$V(r=b) = V_o \quad \longrightarrow \quad V_o = Ab + B = A(b-a) \quad \longrightarrow \quad A = \frac{V_o}{b-a}$$

$$\boldsymbol{E} = -\frac{dV}{dr}\boldsymbol{a}_r = -A\boldsymbol{a}_r = -\frac{V_o}{b-a}\boldsymbol{a}_r$$

$$\rho_s = D_n = -\frac{V_o}{b-a}\frac{\varepsilon_o k}{r^2}\Big|_{r=a,b}$$

$$Q = \int \rho_s dS = -\frac{V_o \varepsilon_o k}{b-a}\iint \frac{1}{r^2} r^2 \sin\theta d\theta d\phi = -\frac{V_o \varepsilon_o k}{b-a}4\pi$$

$$C = \frac{|Q|}{V_o} = \underline{\underline{\frac{4\pi\varepsilon_o k}{b-a}}}$$

Prob. 6.55

$$C = 4\pi\varepsilon_o a = 4\pi \times \frac{10^{-9}}{36\pi} \times 6.37 \times 10^6 = \underline{\underline{0.708 \text{ mF}}}$$

Prob.6.56

$$C = \frac{Q}{V}$$

$$\boldsymbol{D} = \frac{Q}{2\pi\rho L}\boldsymbol{a}_\rho$$

$$\boldsymbol{E} = \frac{D}{\varepsilon} = \frac{Q}{2\pi\rho L\varepsilon_o(3)(1+\rho)}$$

$$V = -\int \boldsymbol{E}\bullet dl = \frac{Q}{6\pi L\varepsilon_o}\int_a^b \frac{d\rho}{\rho(1+\rho)}$$

Let $\frac{1}{\rho(1+\rho)} = \frac{A}{\rho} + \frac{B}{1+\rho}$

Using partial fractions

A=1, B= -1

$$V = \frac{Q}{6\pi\varepsilon_0 L}\left[\int_a^b \frac{d\rho}{\rho} - \int_a^b \frac{d\rho}{1+\rho}\right]$$

$$= \frac{Q}{6\pi\varepsilon_0 L}\left[\ln\rho - \ln(1+\rho)\right]\Big|_a^b$$

$$= \frac{Q}{6\pi\varepsilon_0 L}\left[\ln\frac{b}{1+b} - \ln\frac{a}{1+a}\right]$$

If a=1 mm, and b=5 mm

$$C = \frac{Q}{|V|}$$

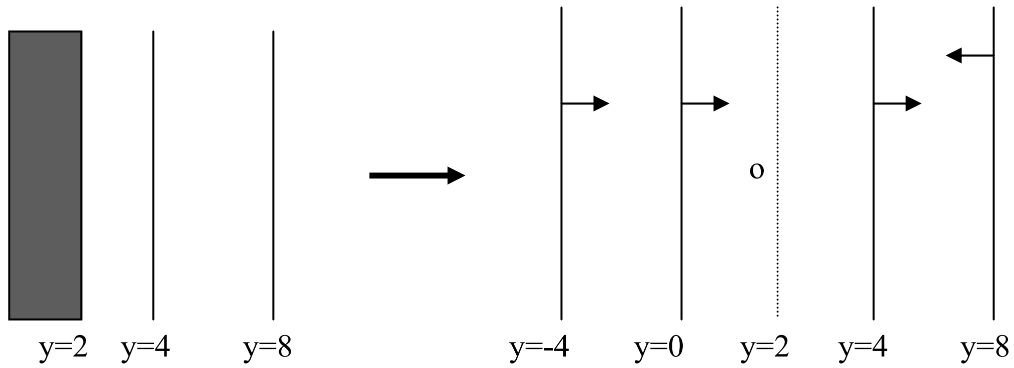
$$= \frac{6\pi\varepsilon_o}{\ln\frac{b}{1+b} - \ln\frac{a}{1+a}}$$

$$= \frac{6\pi\times\frac{10^{-9}}{36\pi}}{\ln\frac{5}{6} - \ln\frac{1}{2}}$$

$$= \frac{\frac{1}{6}\times 10^{-9}}{\ln 0.8333 - \ln 0.5} = \frac{1}{6}\times 1.9591 \text{ nF}$$

$C = \underline{\underline{0.326 \text{ nF}}}$

Prob. 6.57



At P(0,0,0), $\underline{\underline{\mathbf{E}=0}}$ since **E** does not exist for y<2.

At Q(-4,6,2), y=6 and

$$\boldsymbol{E} = \sum \frac{\rho_s}{2\varepsilon_o}\boldsymbol{a}_n = \frac{10^{-9}}{2x10^{-9}/36\pi}(-30\boldsymbol{a}_y + 20\boldsymbol{a}_y - 20\boldsymbol{a}_y - 30\boldsymbol{a}_y) = 18\pi(-60)\boldsymbol{a}_y$$
$$= \underline{\underline{-3.4\boldsymbol{a}_y \text{ kV/m}}}$$

Prob. 6.58

4nC	-3nC		3nC	-4nC
-2	-1	0	1	2
4	3		2	1

(a) Q_i = -(3nC – 4nC) = $\underline{\underline{1nC}}$

(b) The force of attraction between the charges and the plates is

$$\boldsymbol{F} = \boldsymbol{F}_{13} + \boldsymbol{F}_{14} + \boldsymbol{F}_{23} + \boldsymbol{F}_{24}$$

$$|F| = \frac{10^{-18}}{4\pi \times 10^{-9}/36\pi}\left[\frac{9}{2^2} - \frac{2(12)}{3^2} + \frac{16}{4^2}\right] = \underline{\underline{5.25 \text{ nN}}}$$

Prob. 6.59

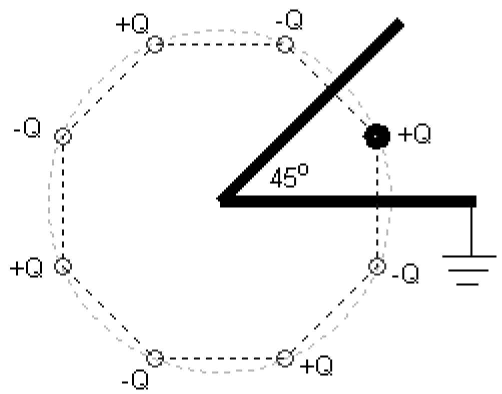
We have 7 images as follows: -Q at (-1,1,1), -Q at (1,-1,1), -Q at (1,1,-1),

-Q at (-1,-1,-1), Q at (1,-1,-1), Q at (-1,-1,1), and Q at (-1,1,-1). Hence,

$$\boldsymbol{F} = \frac{Q^2}{4\pi\varepsilon_o}\left[\begin{array}{l} -\dfrac{2}{2^3}\boldsymbol{a}_x - \dfrac{2}{2^3}\boldsymbol{a}_y - \dfrac{2}{2^3}\boldsymbol{a}_z - \dfrac{(2\boldsymbol{a}_x + 2\boldsymbol{a}_y + 2\boldsymbol{a}_z)}{12^{3/2}} + \dfrac{(2\boldsymbol{a}_y + 2\boldsymbol{a}_z)}{8^{3/2}} \\ + \dfrac{(2\boldsymbol{a}_x + 2\boldsymbol{a}_y)}{8^{3/2}} + \dfrac{(2\boldsymbol{a}_x + 2\boldsymbol{a}_z)}{8^{3/2}} \end{array}\right]$$

$$= 0.9(\boldsymbol{a}_x + \boldsymbol{a}_y + \boldsymbol{a}_z)\left(-\frac{1}{4} - \frac{1}{12\sqrt{3}} + \frac{1}{4\sqrt{2}}\right) = \underline{\underline{-0.1092(\boldsymbol{a}_x + \boldsymbol{a}_y + \boldsymbol{a}_z) \text{ N}}}$$

Prob. 6.60



$$N = \left(\frac{360^o}{45^o} - 1 \right) = \underline{\underline{7}}$$

Prob. 6.61

(a)

$$\boldsymbol{E} = \boldsymbol{E}_+ + \boldsymbol{E}_- = \frac{\rho_L}{2\pi\varepsilon_o}\left(\frac{\boldsymbol{a}_{\rho 1}}{\rho_1} - \frac{\boldsymbol{a}_{\rho 2}}{\rho_2}\right) = \frac{16\times 10^{-9}}{2\pi\times\frac{10^{-9}}{36\pi}}\left[\frac{(2,-2,3)-(3,-2,4)}{|(2,-2,3)-(3,-2,4)|^2} - \frac{(2,-2,3)-(3,-2,-4)}{|(2,-2,3)-(3,-2,-4)|^2}\right]$$

$$= 18x16\left[\frac{(-1,0,-1)}{2} - \frac{(-1,0,7)}{50}\right] = \underline{\underline{-138.2\boldsymbol{a}_x - 184.3\boldsymbol{a}_y \text{ V/m}}}$$

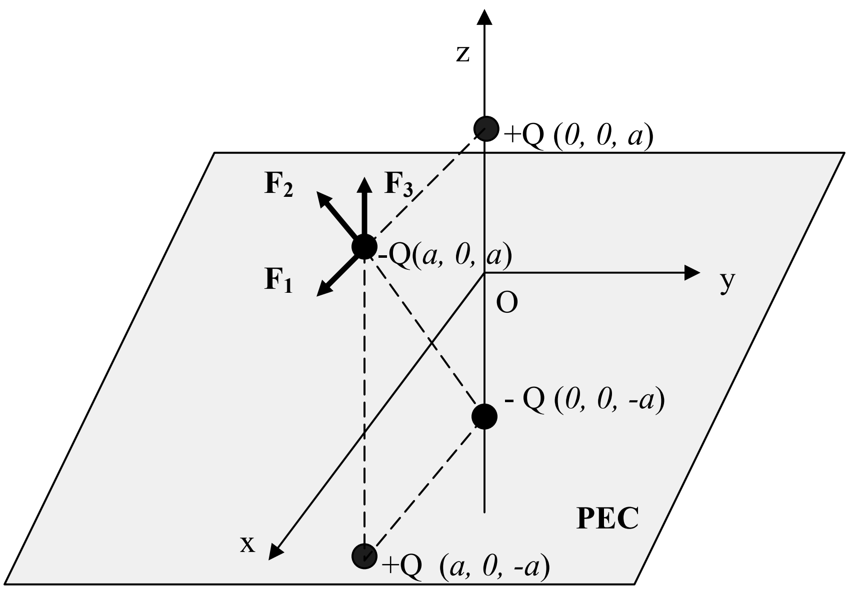
(b) $\rho_s = D_n$

$$\boldsymbol{D} = \boldsymbol{D}_+ + \boldsymbol{D}_- = \frac{\rho_L}{2\pi}\left(\frac{\boldsymbol{a}_{\rho 1}}{\rho_1} - \frac{\boldsymbol{a}_{\rho 2}}{\rho_2}\right) = \frac{16x10^{-9}}{2\pi}\left[\frac{(5,-6,0)-(3,-6,4)}{|(5,-6,0)-(3,-6,4)|^2} - \frac{(5,-6,0)-(3,-6,-4)}{|(5,-6,0)-(3,-6,-4)|^2}\right]$$

$$= \frac{8}{\pi}\left[\frac{(2,0,-4)}{20} - \frac{(2,0,4)}{20}\right] \text{ nC/m}^2 = -1.018\boldsymbol{a}_z \text{ nC/m}^2$$

$$\underline{\underline{\rho_s = -1.018 \text{ nC/m}^2}}$$

Prob. 6.62
The images are shown with proper sign at proper locations. Figure does not show the actual direction of forces but they are expressed a follows:



$$\mathbf{F}_1 = \frac{Q^2}{4\pi\varepsilon_o}\left[\frac{-\mathbf{a}_x}{a^2}\right]$$

$$\mathbf{F}_2 = \frac{Q^2}{4\pi\varepsilon_o}\left[\frac{a\mathbf{a}_x + 2a\mathbf{a}_z}{\left(\sqrt{a^2+4a^2}\right)^3}\right]$$

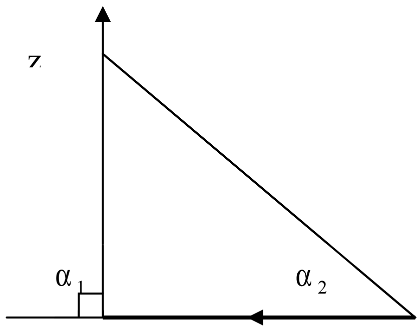
$$\mathbf{F}_3 = \frac{Q^2}{4\pi\varepsilon_o}\left[\frac{-\mathbf{a}_z}{4a^2}\right]$$

$$\mathbf{F}_{\text{total}} = \frac{Q^2}{4\pi\varepsilon_o a^2}\left[\left(\frac{1}{5\sqrt{5}}-1\right)\mathbf{a}_x + \left(\frac{2}{5\sqrt{5}}-\frac{1}{4}\right)\mathbf{a}_z\right]$$

$$= \underline{\underline{\frac{Q^2}{4\pi\varepsilon_o a^2}\left[-0.91\boldsymbol{a}_x - 0.071\boldsymbol{a}_y\right]\text{ N}}}$$

CHAPTER 7

P.E. 7.1



$\rho = 5,\ \ \cos\alpha_1 = 0,\ \ \ \cos\alpha_2 = \sqrt{\frac{2}{27}}$

$$\boldsymbol{a}_\phi = \boldsymbol{a}_l \times \boldsymbol{a}_\rho = \left(\frac{-\boldsymbol{a}_x - \boldsymbol{a}_y}{\sqrt{2}}\right) \times \boldsymbol{a}_z = \frac{-\boldsymbol{a}_x + \boldsymbol{a}_y}{\sqrt{2}}$$

$$\boldsymbol{H}_3 = \frac{10}{4\pi(5)}\left(\sqrt{\frac{2}{27}} - 0\right)\left(\frac{-\boldsymbol{a}_x + \boldsymbol{a}_y}{\sqrt{2}}\right) = \underline{\underline{-30.63\boldsymbol{a}_x + 30.63\boldsymbol{a}_y}} \ \text{ mA/m}$$

P.E. 7.2

(a) $\boldsymbol{H} = \frac{2}{4\pi(2)}\left(1 + \frac{3}{\sqrt{13}}\right)\boldsymbol{a}_z = \underline{\underline{0.1458\boldsymbol{a}_z}}$ A/m

(b) $\rho = \sqrt{3^2 + 4^2} = 5, \alpha_2 = 0, \cos\alpha_1 = -\frac{12}{13},$

$$\boldsymbol{a}_\phi = -\boldsymbol{a}_y \times \left(\frac{3\boldsymbol{a}_x - 4\boldsymbol{a}_z}{5}\right) = \frac{4\boldsymbol{a}_x + 3\boldsymbol{a}_z}{5}$$

$$\boldsymbol{H} = \frac{2}{4\pi(5)}\left(1 + \frac{12}{13}\right)\left(\frac{4\boldsymbol{a}_x + 3\boldsymbol{a}_z}{5}\right) = \frac{1}{26\pi}\left(4\boldsymbol{a}_x + 3\boldsymbol{a}_z\right)$$

$$= \underline{\underline{48.97\boldsymbol{a}_x + 36.73\boldsymbol{a}_z}} \ \text{ mA/m}$$

P.E. 7.3

(a) From Example 7.3,

$$\boldsymbol{H} = \frac{Ia^2}{2(a^2 + z^2)^{3/2}}\boldsymbol{a}_z$$

At (0,0,-1cm), z = 2cm,

$$\boldsymbol{H} = \frac{50\times10^{-3}\times25\times10^{-4}}{2(5^2 + 2^2)^{3/2}\times10^{-6}}\boldsymbol{a}_z = \underline{\underline{400.2\boldsymbol{a}_z}} \ \text{ mA/m}$$

(b) At (0,0,10cm), z = 9cm,

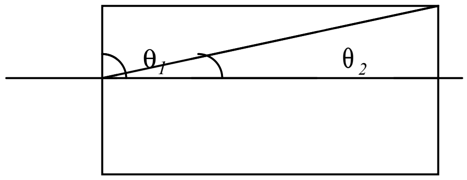
$$\boldsymbol{H} = \frac{50\times10^{-3}\times25\times10^{-4}}{2(5^2+9^2)^{3/2}\times10^{-6}}\boldsymbol{a}_z = \underline{\underline{57.3\boldsymbol{a}_z \text{ mA/m}}}$$

P.E. 7.4

$$\boldsymbol{H} = \frac{NI}{2L}\left(\cos\theta_2 - \cos\theta_1\right)\boldsymbol{a}_z = \frac{2\times10^3\times50\times10^{-3}(\cos\theta_2-\cos\theta_1)\boldsymbol{a}_z}{2\times0.75}$$

$$= \frac{100}{1.5}\left(\cos\theta_2 - \cos\theta_1\right)a_z$$

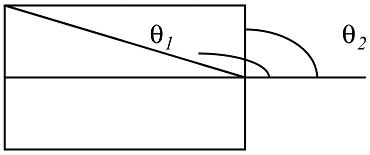
(a) At (0,0,0), $\theta = 90^o$, $\cos\theta_2 = \dfrac{0.75}{\sqrt{0.75^2+0.05^2}} = 0.9978$



$$\boldsymbol{H} = \frac{100}{1.5}\left(0.9978 - 0\right)\boldsymbol{a}_z$$

$= \underline{\underline{66.52\ \mathbf{a}_z \text{ A/m}}}$

(b) At (0,0,0.75), $\theta_2 = 90^o, \cos\theta_1 = -0.9978$

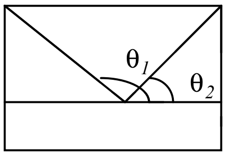


$$\boldsymbol{H} = \frac{100}{1.5}(0 + 0.9978)\boldsymbol{a}_z$$

$= \underline{\underline{66.52\mathbf{a}_z \text{ A/m}}}$

(c) At (0,0,0.5), $\cos\theta_1 = \dfrac{-0.5}{\sqrt{0.5^2+0.05^2}} = -0.995$

$$\cos\theta_1 = \frac{0.25}{\sqrt{0.25^2+0.05^2}} = 0.9806$$



$$\boldsymbol{H} = \frac{100}{1.5}(0.9806 + 0.995)\boldsymbol{a}_z$$

$= \underline{\underline{131.7\mathbf{a}_z \text{ A/m}}}$

P.E. 7.5

$$\boldsymbol{H} = \frac{1}{2}\boldsymbol{K}\times\boldsymbol{a}_n$$

(a) $\boldsymbol{H}(0,0,0) = \frac{1}{2}50\boldsymbol{a}_z\times(-\boldsymbol{a}_y) = \underline{\underline{25\boldsymbol{a}_x}}$ mA/m

(b) $\boldsymbol{H}(1,5,-3) = \frac{1}{2}50\boldsymbol{a}_z\times\boldsymbol{a}_y = \underline{\underline{-25\boldsymbol{a}_x}}$ mA/m

P.E. 7.6

$$|\boldsymbol{H}| = \begin{cases} \dfrac{NI}{2\pi\rho}, \rho - a < \rho < \rho + a, \quad 9<\rho<11 \\ 0, \qquad \text{otherwise} \end{cases}$$

(a) At $(3,-4,0), \rho = \sqrt{3^2 + 4^2} = 5\text{cm} < 9\text{cm}$

$$|\boldsymbol{H}| = \underline{\underline{0}}$$

(b) At $(6,9,0), \rho = \sqrt{6^2 + 9^2} = \sqrt{117} < 11$

$$|\boldsymbol{H}| = \frac{10^3 \times 100 \times 10^{-3}}{2\pi\sqrt{117} \times 10^2} = \underline{\underline{147.1}} \text{ A/m}$$

P.E. 7.7

(a) $B = \nabla \times A = (-4xz - 0)a_x + (0 + 4yz)a_y + (y^2 - x^2)a_z$

$$\boldsymbol{B}(-1,2,5) = \underline{\underline{20\boldsymbol{a}_x + 40\boldsymbol{a}_y + 3\boldsymbol{a}_z}} \text{ Wb/m}^2$$

(b) $\psi = \int \boldsymbol{B}.d\boldsymbol{S} = \int_{y=-1}^{4} \int_{x=0}^{1} (y^2 - x^2)dxdy = \int_{-1}^{4} y^2 dy - 5\int_{0}^{1} x^2 dx$

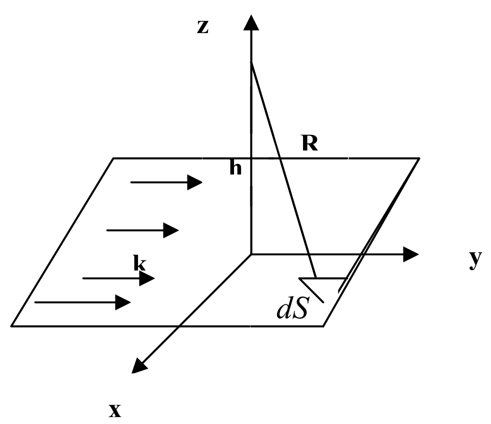
$$= \frac{1}{3}(64+1) - \frac{5}{3} = \underline{\underline{20 \text{ Wb}}}$$

Alternatively,

$$\psi = \int \boldsymbol{A}.d\boldsymbol{l} = \int_{0}^{1} x^2(-1)dx + \int_{-1}^{4} y^2(1)dy + \int_{1}^{0} x^2(4)dx + 0$$

$$= -\frac{5}{3} + \frac{65}{3} = \underline{\underline{20 \text{ Wb}}}$$

P.E. 7.8



$$\boldsymbol{H} = \int \frac{\boldsymbol{k}dS \times \boldsymbol{R}}{4\pi R^3},$$

$dS = dxdy, \boldsymbol{k} = k_y \boldsymbol{a}_{y,}$

$R = (-x, -y, h),$

$$\boldsymbol{k}\times\boldsymbol{R}=(h\boldsymbol{a}_x+x\boldsymbol{a}_z)k_y\,,$$

$$\boldsymbol{H}=\int\frac{k_y(h\boldsymbol{a}_x+x\boldsymbol{a}_z)dxdy}{4\pi(x^2+y^2+h^2)^{3/2}}$$

$$=\frac{k_yh\boldsymbol{a}_x}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dxdy}{(x^2+y^2+h^2)^{3/2}}+\frac{k_y\boldsymbol{a}_z}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{xdxdy}{(x^2+y^2+h^2)^{3/2}}$$

The integrand in the last term is zero because it is an odd function of x.

$$\boldsymbol{H}=\frac{k_yh\boldsymbol{a}_x}{4\pi}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{\infty}\frac{\rho d\phi d\rho}{(\rho^2+h^2)^{3/2}}=\frac{k_yh2\pi\boldsymbol{a}_x}{4\pi}\int_0^{\infty}(\rho^2+h^2)^{-3/2}\frac{d(\rho^2)}{2}$$

$$=\frac{k_yh}{2}\boldsymbol{a}_x\left(\frac{-1}{(\rho^2+h^2)^{1/2}}\right)\Bigg|_0^{\infty}=\frac{k_y}{2}\boldsymbol{a}_x$$

Similarly, for point (0,0,-h), $\mathbf{H}=-\frac{1}{2}k_y\boldsymbol{a}_x$

Hence,

$$\boldsymbol{H}=\begin{cases}\frac{1}{2}k_y\boldsymbol{a}_x, & z>0\\ \frac{1}{2}k_y\boldsymbol{a}_x, & z<0\end{cases}$$

P.E. 7.9

$$\boldsymbol{H}\ =\ \frac{\mathrm{I}}{2\pi\rho}\ \boldsymbol{a}_\phi$$

But $\quad \boldsymbol{H}\ =\ -\nabla V_{\mathrm{m}}\quad (\boldsymbol{J}=\ 0)$

$$\frac{\mathrm{I}}{2\pi\rho}\boldsymbol{a}_\phi\ =\ -\ \frac{1}{\rho}\ \frac{\partial V_m}{\partial\phi}\ \mathbf{a}_\varphi\ \ \rightarrow\ \ V_m\ =\ -\frac{\mathrm{I}}{2\pi}\phi\ +\mathrm{C}$$

At $(10, 60^{\circ}, 7)$, $\phi\ =\ 60^{\circ}\ =\ \frac{\pi}{3}$, $\mathrm{V_m}\ =\ 0\rightarrow\ \ 0\ =\ -\frac{\mathrm{I}}{2\pi}\cdot\frac{\pi}{3}\ +\mathrm{C}$

or $\quad \mathrm{C}\ =\ \frac{I}{6}$

$$V_m\ =\ -\frac{\mathrm{I}}{2\pi}\phi\ +\ \frac{I}{6}$$

At $(4, 30^{\circ}, -2)$, $\phi\ =\ 30^{\circ}\ =\ \frac{\pi}{6}$,

$$\mathrm{V_m}\ =\ -\frac{\mathrm{I}}{2\pi}\cdot\frac{\pi}{6}\ +\ \frac{I}{6}\ =\ \frac{\mathrm{I}}{12}\ =\ \frac{12}{12}$$

$$\underline{\underline{\mathrm{V_m}\ =\ 1\ \mathrm{A}}}$$

P.E. 7.10

$$(a)\quad \boldsymbol{B} = \nabla\times\mathbf{A} = \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x^2y+yz & xy^2-xz^3 & -6xy+2z^2y^2 \end{vmatrix}$$

$$\boldsymbol{B} = \underline{\underline{(-6xz+4x^2y+3xz^2)\boldsymbol{a}_x+\left(y+6yz\text{-}4xy^2\right)\boldsymbol{a}_y+\left(y^2-z^3-2x^2-z\right)\boldsymbol{a}_z}} \text{ Wb/m}^2$$

(b)

$$\psi = \int_{z=0}^{2}\int_{y=0}^{2}\left(-6xz+4x^2y+3xz^2\right)\,dy\,dz\,\Big|_{x=1}$$

$$= \int\int_0(-6xz)\,dy\,dz + 4\int\int_0 x^2y\,dy\,dz + 3\int\int_0 xz^2\,dy\,dz$$

$$= -6\int_0^2 dz\int_0^2 dy + 4\int_0^2 dz\int_0^2 y\,dy + 3\int_0^2 dy\int_0^2 z^2dz$$

$$= -6(2)(2) + 4(2)\left(\frac{y^2}{2}\Big|_0^2\right) + 3(2)\left(\frac{z^3}{3}\Big|_0^2\right) = -24+16+16$$

$$\psi = \underline{\underline{8\text{ Wb}}}$$

$$(c)\quad \nabla\cdot\boldsymbol{A} = \frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z} = 4xy+2xy-6xy = 0$$

$$\nabla\cdot\boldsymbol{B} = -6z+8xy+3z^3+6z-8xy+1-3z^3-1 = 0$$

As a matter of mathematical necessity,

$$\nabla\bullet\boldsymbol{B}=\nabla\bullet(\nabla\times\boldsymbol{A})=0$$

Prob. 7.1

(a) See text

(b) Let **H = H**$_y$ + **H**$_z$

For $\boldsymbol{H}_z = \dfrac{I}{2\pi\rho}\boldsymbol{a}_\phi \qquad \rho=\sqrt{(-3)^2+4^2}=5$

$$a_\phi = -a_z\times\frac{(-3a_x+4a_y)}{5} = \frac{(3a_y-4a_x)}{5}$$

$$H_z = \frac{20}{2\pi(25)}(4a_x+3a_y) = 0.5093a_x+0.382a_y$$

For $H_y = \dfrac{I}{2\pi\rho}a_\phi,\ \rho=\sqrt{(-3)^2+5^2}=\sqrt{34}$

$$\boldsymbol{a}_\phi = \boldsymbol{a}_y\times\frac{(-3\boldsymbol{a}_x+5\boldsymbol{a}_z)}{\sqrt{34}} = \frac{3\boldsymbol{a}_z+5\boldsymbol{a}_x}{\sqrt{34}}$$

$$\boldsymbol{H}_y = \frac{10}{2\pi(34)}(5\boldsymbol{a}_x + 3\boldsymbol{a}_z) = 0.234\boldsymbol{a}_x + 0.1404\boldsymbol{a}_z$$

$\mathbf{H} = \mathbf{H}_y + \mathbf{H}_z$

$= \underline{\underline{0.7433\mathbf{a}_x + 0.382\mathbf{a}_y + 0.1404\mathbf{a}_z \text{ A/m}}}$

Prob. 7.2

$$d\boldsymbol{H} = \frac{Id\boldsymbol{l} \times \boldsymbol{R}}{4\pi R^3}$$

(a) At (1,0,0), **R**=(1,0,0) - (0,0,0) = (1,0,0)

$$d\boldsymbol{H} = \frac{4\boldsymbol{a}_x \times \boldsymbol{a}_x}{4\pi(1)^3} = \underline{\underline{\boldsymbol{0}}}$$

(b) At (0, 1,0), $\mathbf{R} = \mathbf{a}_y$

$$d\boldsymbol{H} = \frac{4\boldsymbol{a}_x \times \boldsymbol{a}_y}{4\pi(1)^3} = \underline{\underline{0.3183\boldsymbol{a}_z \text{ A/m}}}$$

(c) At (0,0,1), $\mathbf{R} = \mathbf{a}_z$

$$d\boldsymbol{H} = \frac{4\boldsymbol{a}_x \times \boldsymbol{a}_z}{4\pi(1)^3} = \underline{\underline{-0.3183\boldsymbol{a}_y \text{ A/m}}}$$

(d) At (1,1,1), **R**=(1,1,1)

$$d\boldsymbol{H} = \frac{4\boldsymbol{a}_x \times (\boldsymbol{a}_x + \boldsymbol{a}_y + \boldsymbol{a}_z)}{4\pi(3)^{3/2}} = \underline{\underline{61.26(-\boldsymbol{a}_y + \boldsymbol{a}_z) \text{ mA/m}}}$$

Prob. 7.3

Let $\boldsymbol{H} = \boldsymbol{H}_1 + \boldsymbol{H}_2$

where $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ are respectively due to the lines located at (0,0) and (0,5).

$$\boldsymbol{H}_1 = \frac{I}{2\pi\rho}\boldsymbol{a}_\phi, \qquad \rho = 5, \qquad \boldsymbol{a}_\phi = \boldsymbol{a}_\ell \times \boldsymbol{a}_\rho = \boldsymbol{a}_z \times \boldsymbol{a}_x = \boldsymbol{a}_y$$

$$\boldsymbol{H}_1 = \frac{10}{2\pi(5)}\boldsymbol{a}_y = \frac{\boldsymbol{a}_y}{\pi}$$

$$\boldsymbol{H}_2 = \frac{I}{2\pi\rho}\boldsymbol{a}_\phi, \qquad \rho = 5\sqrt{2}, \qquad \boldsymbol{a}_\phi = \boldsymbol{a}_\ell \times \boldsymbol{a}_\rho, \boldsymbol{a}_\ell = -\boldsymbol{a}_z$$

$$\boldsymbol{a}_\rho = \frac{5\boldsymbol{a}_x - 5\boldsymbol{a}_y}{5\sqrt{2}} = \frac{\boldsymbol{a}_x - \boldsymbol{a}_y}{\sqrt{2}}$$

$$\boldsymbol{a}_\phi = -\boldsymbol{a}_z \times \left(\frac{\boldsymbol{a}_x - \boldsymbol{a}_y}{\sqrt{2}}\right) = \frac{-\boldsymbol{a}_x - \boldsymbol{a}_y}{\sqrt{2}}$$

$$\boldsymbol{H}_2 = \frac{10}{2\pi 5\sqrt{2}}\left(\frac{-\boldsymbol{a}_x - \boldsymbol{a}_y}{\sqrt{2}}\right) = \frac{1}{2\pi}(-\boldsymbol{a}_x - \boldsymbol{a}_y)$$

$$\boldsymbol{H} = \boldsymbol{H}_1 + \boldsymbol{H}_2 = \frac{\boldsymbol{a}_y}{\pi} + \frac{1}{2\pi}(-\boldsymbol{a}_x - \boldsymbol{a}_y) = \underline{\underline{-0.1592\boldsymbol{a}_x + 0.1592\boldsymbol{a}_y}}$$

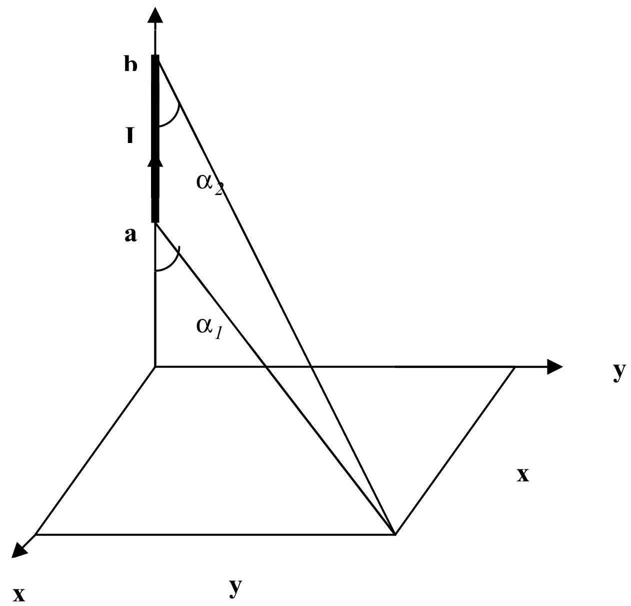
Prob. 7.4

$$\boldsymbol{H} = \frac{I}{2\pi\rho}\boldsymbol{a}_\phi, \qquad \rho = 5, I = 12$$

$$\boldsymbol{a}_\phi = \boldsymbol{a}_\ell \times \boldsymbol{a}_\rho = -\boldsymbol{a}_z \times \left(\frac{3\boldsymbol{a}_x + 4\boldsymbol{a}_y}{5}\right) = \frac{4}{5}\boldsymbol{a}_x - \frac{3}{5}\boldsymbol{a}_y$$

$$\boldsymbol{H} = \frac{12}{2\pi(5)}\left(\frac{4}{5}\boldsymbol{a}_x - \frac{3}{5}\boldsymbol{a}_y\right) = \underline{\underline{0.3056\boldsymbol{a}_x - 0.2292\boldsymbol{a}_y}}$$

Prob. 7.5



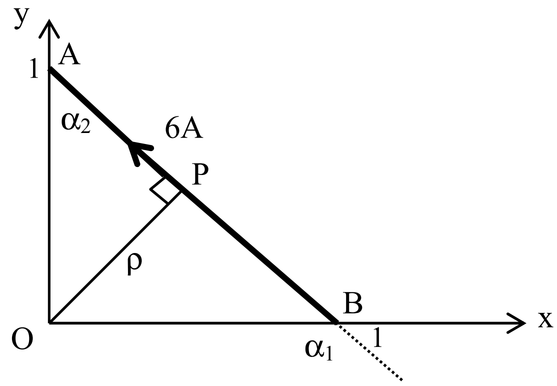
$$\boldsymbol{H} = \frac{I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1)\boldsymbol{a}_\phi$$

$$\rho = \sqrt{x^2 + y^2}, \cos\alpha_1 = \frac{a}{\sqrt{a^2+\rho^2}}, \cos\alpha_2 = \frac{b}{\sqrt{b^2+\rho^2}}$$

$\boldsymbol{a}_\phi = \boldsymbol{a}_l \times \boldsymbol{a}_\rho = \boldsymbol{a}_z \times \boldsymbol{a}_\rho = \boldsymbol{a}_\phi$. Hence,

$$\boldsymbol{H} = \frac{I}{4\pi\sqrt{x^2+y^2}}\underline{\underline{\left[\frac{b}{\sqrt{x^2+y^2+b^2}} - \frac{a}{\sqrt{x^2+y^2+a^2}}\right]}}\boldsymbol{a}_\phi$$

Prob. 7.6



$$H = \frac{I}{4\pi\rho}\left(\cos\alpha_2 - \cos\alpha_1\right)\boldsymbol{a}_\phi$$

$$\alpha_1 = 135^o, \quad \alpha_2 = 45^o, \quad \rho = \frac{1}{2}\sqrt{2} = \frac{\sqrt{2}}{2}$$

$$\boldsymbol{a}_\phi = \boldsymbol{a}_l \times \boldsymbol{a}_\rho = \left(\frac{-\boldsymbol{a}_x + \boldsymbol{a}_y}{\sqrt{2}}\right) \times \left(\frac{-\boldsymbol{a}_x - \boldsymbol{a}_y}{\sqrt{2}}\right) = \frac{1}{2}\begin{vmatrix} -1 & 1 & 0 \\ -1 & -1 & 0 \end{vmatrix} = \boldsymbol{a}_z$$

$$H = \frac{6}{4\pi\frac{\sqrt{2}}{2}}\left(\cos 45^o - \cos 135^o\right)\boldsymbol{a}_z = \frac{3}{\pi}\boldsymbol{a}_z$$

$$H\left(0,0,0\right) = \underline{\underline{0.954\boldsymbol{a}_z \text{ A/m}}}$$

Prob. 7.7

(a) At (5,0,0), $\rho = 5, \quad \boldsymbol{a}_\phi = \boldsymbol{a}_y, \quad \cos\alpha_1 = 0, \quad \cos\alpha_2 = \frac{10}{\sqrt{125}}$

$$H = \frac{2}{4\pi(5)}\left(\frac{10}{\sqrt{125}}\right)\boldsymbol{a}_y = \underline{\underline{28.471\boldsymbol{a}_y \text{ mA/m}}}$$

(b) At (5,5,0), $\rho = 5\sqrt{2}, \quad \cos\alpha_1 = 0, \quad \cos\alpha_2 = \frac{10}{\sqrt{150}}$

$$\boldsymbol{a}_\phi = \frac{-\boldsymbol{a}_x + \boldsymbol{a}_y}{\sqrt{2}}$$

$$H = \frac{2}{4\pi(5\sqrt{2})}\left(\frac{10}{\sqrt{150}}\right)\left(\frac{-\boldsymbol{a}_x + \boldsymbol{a}_y}{\sqrt{2}}\right) = \underline{\underline{13(-\boldsymbol{a}_x + \boldsymbol{a}_y) \text{ mA/m}}}$$

(c) At (5,15,0), $\rho = \sqrt{250} = 5\sqrt{10}$, $\cos\alpha_1 = 0$, $\cos\alpha_2 = \dfrac{10}{\sqrt{350}}$

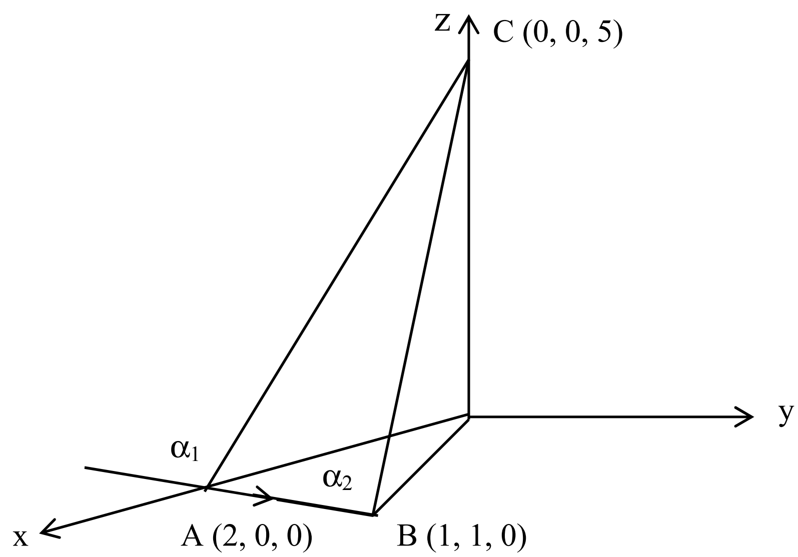
$$\boldsymbol{a}_\phi = \frac{5\boldsymbol{a}_y - 15\boldsymbol{a}_x}{5\sqrt{10}}$$

$$\boldsymbol{H} = \frac{2}{4\pi(5\sqrt{10})}\left(\frac{10}{\sqrt{350}}\right)\left(\frac{-15\boldsymbol{a}_x + 5\boldsymbol{a}_y}{5\sqrt{10}}\right) = \underline{\underline{-5.1\boldsymbol{a}_x + 1.7\boldsymbol{a}_y \text{ mA/m}}}$$

d) At (5,-15,0), by symmetry,

$$\boldsymbol{H} = \underline{\underline{5.1\boldsymbol{a}_x + 1.7\boldsymbol{a}_y \text{ mA/m}}}$$

Prob. 7.8



(*a*) Consider the figure above.

$$AB = (1, 1, 0) - (2, 0, 0) = (-1, 1, 0)$$
$$AC = (0, 0, 5) - (2, 0, 0) = (-2, 0, 5)$$

$AB \cdot AC = 2$, i.e AB and AC are not perpendicular.

$$\cos(180^o - \alpha_1) = \frac{AB \cdot AC}{|AB||AC|} = \frac{2}{\sqrt{2}\sqrt{29}} \rightarrow \cos\alpha_1 = -\sqrt{\frac{2}{29}}$$

$$BC = (0, 0, 5) - (1, 1, 0) = (-1, -1, 5)$$
$$BA = (1, -1, 0)$$

$$\cos\alpha_2 = \frac{\overline{B}C \cdot B\overline{A}}{|BC||BA|} = \frac{-1+1}{|BC||BA|} = 0$$

i.e. $BC = \boldsymbol{\rho} = (-1, -1, 5), \quad \rho = \sqrt{27}$

$$\boldsymbol{a}_\phi = \boldsymbol{a}_l \times \boldsymbol{a}_\rho = \frac{(-1, 1, 0)}{\sqrt{2}} \times \frac{(-1, -1, 5)}{\sqrt{27}} = \frac{(5, 5, 2)}{\sqrt{54}}$$

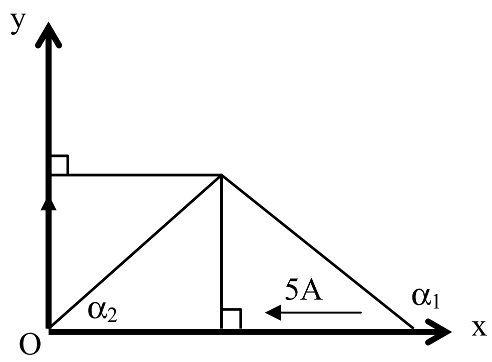
$$\boldsymbol{H}_2 = \frac{10}{4\pi\sqrt{27}}\left(0 + \sqrt{\frac{2}{29}}\right)\frac{(5, 5, 2)}{\sqrt{2}\sqrt{27}} = \frac{5}{2\pi\sqrt{29}} \cdot \frac{(5, 5, 2)}{27} \text{ A/m}$$

$$= \underline{\underline{27.37\,\boldsymbol{a}_x + 27.37\boldsymbol{a}_y + 10.95\,\boldsymbol{a}_z \text{ mA/m}}}$$

(b) $$\boldsymbol{H} = \boldsymbol{H}_1 + \boldsymbol{H}_2 + \boldsymbol{H}_3 = (0, -59.1, 0) + (27.37, 27.37, 10.95) + (-30.63, 30.63, 0)$$

$$= \underline{\underline{-3.26\,\boldsymbol{a}_x - 1.1\,\boldsymbol{a}_y + 10.95\boldsymbol{a}_z \text{ mA/m}}}$$

Prob. 7.9



(*a*) Let $\boldsymbol{H} = \boldsymbol{H}_x + \boldsymbol{H}_y = 2\boldsymbol{H}_x$

$$\boldsymbol{H}_x = \frac{I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1)\boldsymbol{a}_\phi$$

where $\boldsymbol{a}_\phi = -\boldsymbol{a}_x \times \boldsymbol{a}_y = -\boldsymbol{a}_z,\ \alpha_1 = 180^\text{o},\ \alpha_2 = 45^\text{o}$

$$\boldsymbol{H}_x = \frac{5}{4\pi(2)}\left(\cos 45^\text{o} - \cos 180^\text{o}\right)\left(-\boldsymbol{a}_z\right)$$

$$= \underline{\underline{-0.6792\ \boldsymbol{a}_z\ \text{A/m}}}$$

(*b*) $\boldsymbol{H} = \boldsymbol{H}_x + \boldsymbol{H}_y$

where $\boldsymbol{H}_x = \dfrac{5}{4\pi(2)}(1-0)\ \boldsymbol{a}_\phi,\quad \boldsymbol{a}_\phi = -\boldsymbol{a}_x \times -\boldsymbol{a}_y = \boldsymbol{a}_z$

$$= 198.9\boldsymbol{a}_z\ \text{mA/m}$$

$\boldsymbol{H}_y = 0$ since $\alpha_1 = \alpha_2 = 0$

$\boldsymbol{H} = \underline{\underline{0.1989\ \boldsymbol{a}_z\ \text{A/m}}}$

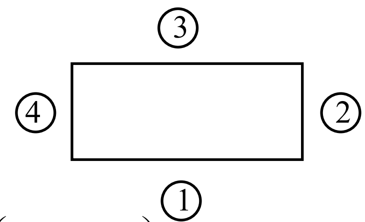
(*c*) $\boldsymbol{H} = \boldsymbol{H}_x + \boldsymbol{H}_y$

where $\boldsymbol{H}_x = \dfrac{5}{4\pi(2)}(1-0)\ \left(-\boldsymbol{a}_x \times \boldsymbol{a}_z\right) = 198.9\ \boldsymbol{a}_y\ \text{mA/m}$

$$\boldsymbol{H}_y = \frac{5}{4\pi(2)}(1-0)\ \left(\boldsymbol{a}_y \times \boldsymbol{a}_z\right) = 198.9\ \boldsymbol{a}_x\ \text{mA/m}$$

$$\boldsymbol{H} = \underline{\underline{0.1989\ \boldsymbol{a}_x + 0.1989\ \boldsymbol{a}_y\ \text{A/m.}}}$$

Prob. 7.10



Let $\boldsymbol{H} = \boldsymbol{H}_1 + \boldsymbol{H}_2 + \boldsymbol{H}_3 + \boldsymbol{H}_4$

where $\overline{H}_n$ is the contribution by side n.

(a) $\boldsymbol{H} = 2\boldsymbol{H}_1 + \boldsymbol{H}_2 + \boldsymbol{H}_4$ since $\boldsymbol{H}_1 = \boldsymbol{H}_3$

$$\boldsymbol{H}_1 = \frac{\text{I}}{4\pi\rho}\left(\cos\alpha_2 - \cos\alpha_1\right)\boldsymbol{a}_\phi = \frac{10}{4\pi(2)}\left(\frac{6}{\sqrt{40}} + \frac{1}{\sqrt{2}}\right)\boldsymbol{a}_z$$

$$\boldsymbol{H}_2 = \frac{10}{4\pi(6)}\left(2\times\frac{2}{\sqrt{40}}\right)\boldsymbol{a}_z,\quad \boldsymbol{H}_4 = \frac{10}{4\pi(2)}\left(2\cdot\frac{1}{\sqrt{2}}\right)\boldsymbol{a}_z$$

$$\boldsymbol{H} = \left[\frac{5}{2\pi}\left(\frac{3}{\sqrt{10}} + \frac{1}{\sqrt{2}}\right) + \frac{5}{6\pi\sqrt{10}} + \frac{5}{2\pi\sqrt{2}}\right]\boldsymbol{a}_z = \underline{\underline{1.964\boldsymbol{a}_z\ \text{A/m}}}$$

(*b*) At $(4, 2, 0)$, $\boldsymbol{H} = 2\left(\boldsymbol{H}_1 + \boldsymbol{H}_4\right)$

$$\boldsymbol{H}_1 = \frac{10}{4\pi(2)}\ \frac{8}{\sqrt{20}}\ \boldsymbol{a}_z,\ \boldsymbol{H}_4 = \frac{10}{4\pi(4)}\ \frac{4}{\sqrt{20}}\ \boldsymbol{a}_z$$

$$\boldsymbol{H} = \frac{2\sqrt{5}}{\pi}\left(1 + \frac{1}{4}\right)\boldsymbol{a}_z = \underline{\underline{1.78\boldsymbol{a}_z\ \text{A/m}}}$$

(c) At $(4, 8, 0)$, $\boldsymbol{H} = \boldsymbol{H}_1 + 2\boldsymbol{H}_2 + \boldsymbol{H}_3$

$$\boldsymbol{H}_1 = \frac{10}{4\pi(8)}\left(2\cdot\frac{4}{4\sqrt{5}}\right)\boldsymbol{a}_z,\quad \boldsymbol{H}_2 = \frac{10}{4\pi(4)}\left(\frac{8}{4\sqrt{5}} - \frac{1}{\sqrt{2}}\right)\boldsymbol{a}_z$$

$$\boldsymbol{H}_3 = \frac{10}{4\pi(4)}\left(\frac{2}{\sqrt{2}}\right)(-\boldsymbol{a}_z)$$

$$\boldsymbol{H} = \frac{5}{8\pi}(\boldsymbol{a}_z)\left(\frac{1}{\sqrt{5}} + \frac{4}{\sqrt{5}} - \frac{4}{\sqrt{2}}\right) = \underline{\underline{-0.1178\boldsymbol{a}_z \text{ A/m}}}$$

(d) At $(0, 0, 2)$,

$$\boldsymbol{H}_1 = \frac{10}{4\pi(2)}\left(\frac{8}{\sqrt{68}} - 0\right)(\boldsymbol{a}_x \times \boldsymbol{a}_z) = -\frac{10}{\pi\sqrt{68}}\boldsymbol{a}_y$$

$$\boldsymbol{H}_2 = \frac{10}{4\pi\sqrt{68}}\left(\frac{4}{\sqrt{84}} - 0\right)\boldsymbol{a}_y \times \left(\frac{2\boldsymbol{a}_z - 8\boldsymbol{a}_x}{\sqrt{68}}\right) = \frac{5(\boldsymbol{a}_x + 4\boldsymbol{a}_z)}{17\pi\sqrt{84}}$$

$$\boldsymbol{H}_3 = \frac{10}{4\pi\sqrt{20}}\left(-\frac{8}{\sqrt{84}} - 0\right)\boldsymbol{a}_x \times \left(\frac{2\boldsymbol{a}_x - 4\boldsymbol{a}_y}{\sqrt{20}}\right) = \frac{\boldsymbol{a}_y + 2\boldsymbol{a}_z}{\pi\sqrt{21}}$$

$$\boldsymbol{H}_4 = \frac{10}{4\pi\sqrt{2}}\left(0 + \frac{4}{\sqrt{20}}\right)(-\boldsymbol{a}_y \times \boldsymbol{a}_z) = \frac{-5\boldsymbol{a}_x}{\pi\sqrt{20}}$$

$$\boldsymbol{H} = \left(\frac{5}{34\pi\sqrt{21}} - \frac{5}{\pi\sqrt{20}}\right)\boldsymbol{a}_x + \left(\frac{1}{\pi\sqrt{21}} - \frac{10}{\pi\sqrt{68}}\right)\boldsymbol{a}_y + \left(\frac{20}{34\pi\sqrt{21}} + \frac{2}{\pi\sqrt{21}}\right)\boldsymbol{a}_z$$

$$= \underline{\underline{-0.3457\,\bar{a}_x - 0.3165\,\bar{a}_y + 0.1798\,\bar{a}_z \text{ A/m}}}$$

Prob. 7.11

For the side of the loop along y-axis,

$$\boldsymbol{H}_1 = \frac{\mathrm{I}}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1)\,\boldsymbol{a}_\phi$$

where $\boldsymbol{a}_\phi = -\boldsymbol{a}_x$, $\rho = 2\tan 30^\circ = \frac{2}{\sqrt{3}}$, $\alpha_2 = 30^\circ$, $\alpha_1 = 150^\circ$

$$\boldsymbol{H}_1 = \frac{5}{4\pi}\frac{\sqrt{3}}{2}(\cos 30^\circ - \cos 150^\circ)(-\boldsymbol{a}_x) = -\frac{15}{8\pi}\boldsymbol{a}_x$$

$$\boldsymbol{H} = 3\boldsymbol{H}_1 = -1.79\boldsymbol{a}_x \text{ A/m}$$

Prob. 7.12

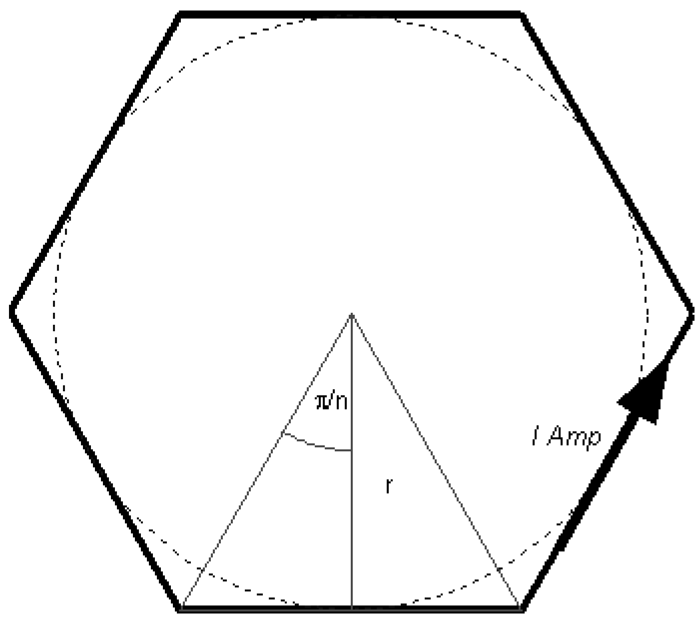
$$\boldsymbol{H} = 4\boldsymbol{H}_1 = 4\frac{I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1)\boldsymbol{a}_\phi$$

$$\rho = a = 2cm, I = mA, \alpha_2 = 45^o, \alpha_1 = 90^o + 45^o = 135^o$$

$$\boldsymbol{a}_\phi = \boldsymbol{a}_\ell \times \boldsymbol{a}_\rho = \boldsymbol{a}_y \times (-\boldsymbol{a}_x) = \boldsymbol{a}_z$$

$$\boldsymbol{H} = \frac{I}{\pi a}\left(\frac{1}{\sqrt{2}} - -\frac{1}{\sqrt{2}}\right)\boldsymbol{a}_z = \frac{\sqrt{2}I}{\pi a}\boldsymbol{a}_z = \frac{\sqrt{2}\times 5\times 10^{-3}}{\pi\times 2\times 10^{-2}}\boldsymbol{a}_z = \underline{\underline{0.1125\boldsymbol{a}_z}}$$

Prob. 7.13



(a) Consider one side of the polygon as shown. The angle subtended by the Side At the center of the circle

$$\frac{360°}{n} = \frac{2\pi}{n}$$

The filed due to this side is

$$H_1 = \frac{I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1)$$

where $\rho = r$, $\cos\alpha_2 = \cos(90 - \frac{\pi}{n}) = \sin\frac{\pi}{n}$

$$\cos\alpha_1 = -\sin\frac{\pi}{n}$$

$$H_1 = \frac{I}{4\pi r} 2\sin\frac{\pi}{n}$$

$$H = nH_1 = \frac{nI}{2\pi r}\sin\frac{\pi}{n}$$

(b) For $n = 3$, $H = \frac{3I}{2\pi r}\sin\frac{\pi}{3}$

$$r \cot 30^o = 2 \rightarrow r = \frac{2}{\sqrt{3}}$$

$$H = \frac{3\times5}{2\pi\,2/\sqrt{3}}\cdot\frac{\sqrt{3}}{2} = \frac{45}{8\pi} = 1.79 \text{ A/m.}$$

For $n = 4$, $H = \frac{4I}{2\pi r}\sin\frac{\pi}{4} = \frac{4\times5}{2\pi(2)}\cdot\frac{1}{\sqrt{2}}$

$= 1.128$ A/m.

(c) As $n \rightarrow \infty$,

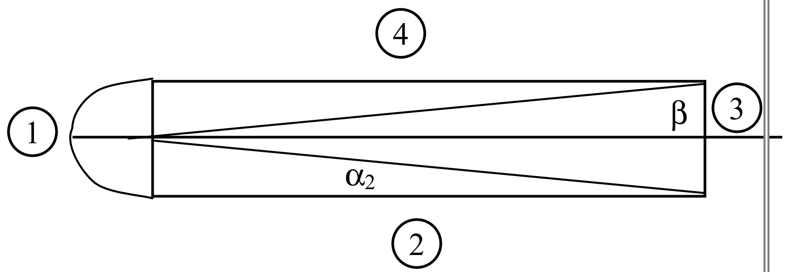
$$H = \lim_{n\rightarrow\infty}\frac{nI}{2\pi r}\sin\frac{\pi}{n} = \frac{nI}{2\pi r}\cdot\frac{\pi}{n} = \frac{I}{2r}$$

From Example 7.3, when $h = 0$,

$$H = \frac{I}{2r}$$

which agrees.

Prob. 7.14



Let $\boldsymbol{H} = \boldsymbol{H}_1 + \mathbf{H}_2 + \mathbf{H}_3 + \mathbf{H}_4$

$$\boldsymbol{H}_1 = \frac{I}{4a}\,\mathbf{a}_z = \frac{10}{4\times4\times10^{-2}}\,\mathbf{a}_z = 62.5\,\mathbf{a}_z$$

$$\boldsymbol{H}_2 = \boldsymbol{H}_4 = \frac{I}{4\pi\times4\times10^{-2}}\left(\cos\alpha_2 - \cos 90^o\right)\mathbf{a}_z, \quad \alpha_2 = \tan^{-1}\frac{4}{100} = 2.29^o$$

$$= 19.88\,\mathbf{a}_z$$

$$\boldsymbol{H}_3 = \frac{I}{4\pi(1)}\,2\cos\beta\mathbf{a}_z, \quad \beta = \tan^{-1}\frac{100}{4} = 87.7^o$$

$$= \frac{10}{4\pi}\,2\cos 87.7^o\mathbf{a}_z = 0.06361\,\mathbf{a}_z$$

$$\boldsymbol{H} = \left(62.5 + 2\times19.88 + 0.06361\right)\mathbf{a}_z$$

$$= 102.32\,\mathbf{a}_z \text{ A/m.}$$

Prob. 7.15

From Example 7.3, $\boldsymbol{H}$ due to circular loop is

$$\boldsymbol{H}_i = \frac{I\rho^2}{2\left(\rho^2 + z^2\right)}\,\mathbf{a}_z$$

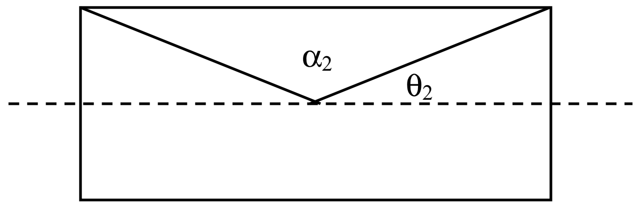
(a) $$\boldsymbol{H}(0,0,0) = \frac{5\times2^2}{2\left(2^2+0^2\right)^{3/2}}\,\mathbf{a}_z + \frac{5\times2^2}{2\left(2^2+4^2\right)^{3/2}}\mathbf{a}_z$$

$$= \underline{\underline{1.36\,\mathbf{a}_z \text{ A/m}}}$$

(b) $$\boldsymbol{H}(0,0,2) = 2\frac{5\times2^2}{2\left(2^2+2^2\right)^{3/2}}\mathbf{a}_z$$

$$= \underline{\underline{0.884\,\mathbf{a}_z \text{ A/m}}}$$

Prob. 7.16

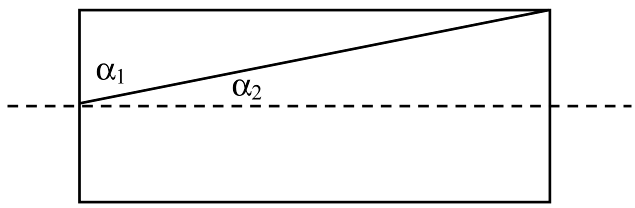


$$|\boldsymbol{H}| = \frac{nI}{2}(\cos\theta_2 - \cos\theta_1)$$

$$\cos\theta_2 = -\cos\theta_1 = \frac{\ell/2}{\left(a^2 + \ell^2/4\right)^{1/2}}$$

$$|\boldsymbol{H}| = \frac{nI\ell}{2\left(a^2 + \ell^2/4\right)^{1/2}} = \frac{0.5\times150\times2\times10^{-2}}{2\times10^{-3}\times\sqrt{4^2+10^2}} = \underline{\underline{69.63\text{ A/m}}}$$

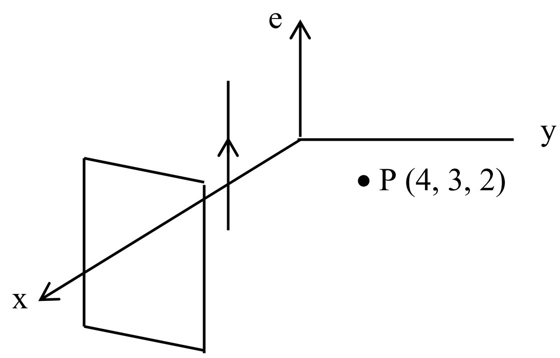
(b)



$$\alpha_1 = 90^{\circ},\quad \tan\theta_2 = \frac{a}{b} = \frac{4}{20} = 0.2 \rightarrow \quad \theta_2 = 11.31^{\circ}$$

$$|\boldsymbol{H}| = \frac{nI}{2}\cos\theta_2 = \frac{150\times0.5}{2}\cos 11.31^{\circ} = \underline{\underline{36.77\text{ A/m}}}$$

Prob. 7.17



Let $\boldsymbol{H} = \boldsymbol{H}_l + \boldsymbol{H}_p$

$$\boldsymbol{H}_l = \frac{1}{2\pi\rho}\mathbf{a}_\phi$$

$$\boldsymbol{\rho} = (4, 3, 2) - (1, -2, 2) = (3, 5, 0), \quad \rho = |\boldsymbol{\rho}| = \sqrt{34}$$

$$\mathbf{a}_\rho = \frac{3\mathbf{a}_x + 5\mathbf{a}_y}{\sqrt{34}}, \quad \mathbf{a}_l = \mathbf{a}_z$$

$$\mathbf{a}_\phi = \mathbf{a}_l \times \mathbf{a}_\rho = \mathbf{a}_z \times \left(\frac{3\mathbf{a}_x + 5\mathbf{a}_y}{\sqrt{34}}\right) = \frac{3\mathbf{a}_y - 5\mathbf{a}_x}{\sqrt{34}}$$

$$\boldsymbol{H}_l = \frac{20\pi}{2\pi}\left(\frac{-5\mathbf{a}_x + 3\mathbf{a}_y}{34}\right)\text{x}10^{-3} = (-1.47\mathbf{a}_y + 0.88\mathbf{a}_y) \text{ mA/m}$$

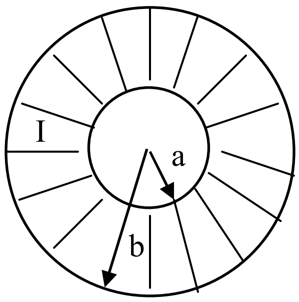
$$\boldsymbol{H}_p = \frac{1}{2}\mathbf{K} \times \mathbf{a}_n = \frac{1}{2}\left(100\times10^{-3}\right)\mathbf{a}_z \times (\text{-}\mathbf{a}_x) = -0.05\mathbf{a}_y \text{ A/m}$$

$$\boldsymbol{H} = \boldsymbol{H}_l + \mathbf{H}_p = \underline{\underline{-1.47\mathbf{a}_x - 49.12\,\mathbf{a}_y \text{ mA/m}}}$$

Prob. 7.18

(a) See text

(b)



For $\rho < \text{a}$, $\oint \boldsymbol{H} \cdot d\boldsymbol{l} = \text{I}_{\text{enc}} = 0 \rightarrow \text{H} = 0$

For $\text{a} < \rho < \text{b}$, $\text{H}_\phi \cdot 2\pi\rho = \dfrac{\text{I}\pi\left(\rho^2 - a^2\right)}{\pi\left(b^2 - a^2\right)}$

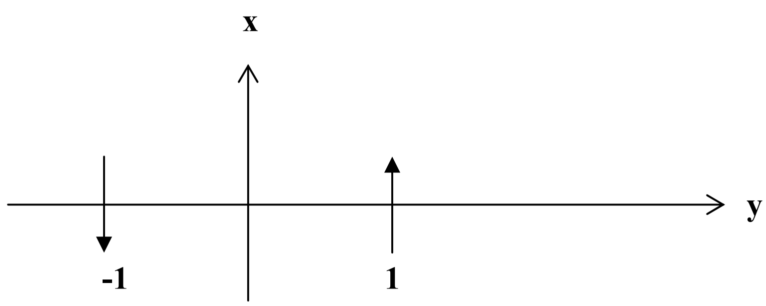
$$\text{H}_\phi = \frac{\text{I}}{2\pi\rho}\left(\frac{\rho^2 - a^2}{b^2 - a^2}\right)$$

For $\rho > \text{b}$, $\text{H}_\phi \cdot 2\pi\rho = \text{I} \rightarrow \text{H}_\phi = \dfrac{\text{I}}{2\pi\rho}$

Thus,

$$\text{H}_\phi = \begin{cases} 0, & \rho < \text{a} \\ \dfrac{\text{I}}{2\pi\rho}\left(\dfrac{\rho^2 - a^2}{b^2 - a^2}\right), & \text{a} < \rho < b \\ \dfrac{\text{I}}{2\pi\rho}, & \rho > b \end{cases}$$

Prob. 7.19



$$\boldsymbol{H} = \sum \frac{1}{2}\boldsymbol{K} \times \boldsymbol{a}_n$$
$$= \frac{1}{2}(20\boldsymbol{a}_x) \times (-\boldsymbol{a}_y) + \frac{1}{2}(-20\boldsymbol{a}_x) \times \boldsymbol{a}_y$$
$$= 10(-\boldsymbol{a}_z) - 10(\boldsymbol{a}_z)$$
$$\underline{\underline{= -20\boldsymbol{a}_z \text{ A/m}}}$$

Prob. 7.20

$$\boldsymbol{H}_P = \frac{1}{2}\boldsymbol{k} \times \boldsymbol{a}_n = \frac{1}{2}10\boldsymbol{a}_x \times \boldsymbol{a}_z = -5\boldsymbol{a}_y$$

$$\boldsymbol{H}_L = \frac{I}{2\pi\rho}\boldsymbol{a}_\phi = \frac{I}{2\pi(3)}(\boldsymbol{a}_x \times -\boldsymbol{a}_z) = \frac{I}{6\pi}\boldsymbol{a}_y$$

$$\boldsymbol{H}_P + \boldsymbol{H}_L = -5\boldsymbol{a}_y + \frac{I}{6\pi}\boldsymbol{a}_y = 0 \quad \longrightarrow \quad I = 30\pi = \underline{\underline{94.25 \text{ A}}}$$

Prob. 7.21

(a) Applying Ampere's law,

$$\mathrm{H}_\phi \cdot 2\pi\rho = \mathrm{I} \cdot \frac{\pi\rho^2}{\pi a^2} \quad \rightarrow \quad \mathrm{H}_\phi = \mathrm{I} \cdot \frac{\rho^2}{2\pi a^2}$$

i.e $$\mathbf{H} = \frac{\mathrm{I}\rho}{2\pi a^2}\boldsymbol{a}_\phi$$

(*b*) From Eq. (7.29),

$$H_\phi = \begin{cases} \dfrac{I\rho}{2\pi a^2}, & \rho < a \\ \dfrac{I}{2\pi\rho}, & \rho > a \end{cases}$$

At $(0, 1\text{ cm}, 0)$,

$$H_\phi = \frac{3\times 1\times 10^{-2}}{2\pi\times 4\times 10^{-4}} = \frac{300}{8\pi}$$

$$\boldsymbol{H} = \underline{\underline{11.94\, \boldsymbol{a}_\phi \text{ A/m}}}$$

At $(0, 4\text{ cm}, 0)$,

$$H_\phi = \frac{3}{2\pi\times 4\times 10^{-2}} = \frac{300}{8\pi}$$

$$\boldsymbol{H} = \underline{\underline{11.94\, \boldsymbol{a}_\phi \text{ A/m}}}$$

Prob. 7.22

For $0 < \rho < a$

$$\oint_L \boldsymbol{H}\bullet d\boldsymbol{l} = I_{enc} = \int \boldsymbol{J}\bullet d\boldsymbol{S}$$

$$H_\phi 2\pi\rho = \int_{\phi=0}^{2\pi}\int_{\rho=0}^{\rho} \frac{J_o}{\rho}\rho\, d\phi\, d\rho$$

$$= J_o 2\pi\rho$$

$$H_\phi = J_o$$

For $\rho > a$

$$\int \mathbf{H}\bullet d\boldsymbol{l} = \int \boldsymbol{J}\bullet d\boldsymbol{S} = \int_{\phi=0}^{2\pi}\int_{\rho=0}^{a} \frac{J_o}{\rho}\rho\, d\phi\, d\rho$$

$$H_\rho 2\pi\rho = J_o 2\pi a$$

$$H_\phi = \frac{J_o a}{\rho}$$

$$\underline{\underline{\text{Hence } H_\phi = \begin{cases} J_o, & 0<\rho<a \\ \dfrac{J_o a}{\rho}, & \rho>a \end{cases}}}$$

Prob. 7.23

(a) $$\boldsymbol{J} = \nabla \times \boldsymbol{H} = \frac{1}{\rho}\frac{d}{d\rho}(\rho H_\phi)\boldsymbol{a}_z = \frac{1}{\rho}\frac{d}{d\rho}(k_o\frac{\rho^2}{a})\boldsymbol{a}_z = \underline{\underline{\frac{2k_o}{a}\boldsymbol{a}_z}}$$

(b) For ρ>a,

$$\oint \boldsymbol{H}\cdot d\boldsymbol{l} = I_{enc} = \int \boldsymbol{J}\cdot dS = \int_{\rho=0}^{a}\int_{\phi=0}^{2\pi}\frac{2k_o}{a}\rho d\rho d\phi = \frac{2k_o}{a}(2\pi)\frac{\rho^2}{2}\bigg|_0^a$$

$$H_\phi 2\pi\rho = 2\pi k_o a \quad \longrightarrow \quad H_\phi = \frac{k_o a}{\rho}$$

$$\boldsymbol{H} = \underline{\underline{k_o\left(\frac{a}{\rho}\right)\boldsymbol{a}_\phi, \qquad \rho > a}}$$

Prob. 7.24

$$\boldsymbol{J} = \nabla \times \boldsymbol{H} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & 0 \end{vmatrix} = (2x - 2y)\boldsymbol{a}_z$$

At (1,-4,7), x =1, y = -4, z=7,

$$\boldsymbol{J} = \left[2(1) - 2(-4)\right]\boldsymbol{a}_z = \underline{\underline{10\boldsymbol{a}_z \text{ A/m}^2}}$$

Prob. 7.25

(a)

$$\boldsymbol{J} = \nabla \times \boldsymbol{H} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho H_\phi)\boldsymbol{a}_z = \frac{1}{\rho}\frac{\partial}{\partial\rho}(10^3\rho^3)\boldsymbol{a}_z$$

$$= \underline{\underline{3\rho\times 10^3\boldsymbol{a}_z \text{ A/m}^2}}$$

(b)

Method 1:

$$I = \int_S \boldsymbol{J}\bullet d\boldsymbol{S} = \iint 3\rho\ \rho d\phi d\rho 10^3 = 3\times 10^3\int_0^2 \rho^2 d\rho\int_0^{2\pi} d\phi$$

$$= 3\times 10^3(2\pi)\frac{\rho^3}{3}\bigg|_2^2 = 16\pi\times 10^3 A = \underline{\underline{50.265 \text{ kA}}}$$

Method 2:

$$I = \oint_L \boldsymbol{H}\bullet d\boldsymbol{l} = 10^3\int_0^{2\pi}\rho^2\ \rho d\phi = 10^3(8)(2\pi) = \underline{\underline{50.265 \text{ kA}}}$$

Prob. 7.26

Let $\boldsymbol{H} = \boldsymbol{H}_1 + \boldsymbol{H}_2$

where $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ are due to the wires centered at x = 0 and x = 10cm respectively.

(*a*) For $\boldsymbol{H}_1$, $\rho = 50$ cm, $\boldsymbol{a}_\phi = \boldsymbol{a}_l \times \boldsymbol{a}_\rho = \boldsymbol{a}_z \times \boldsymbol{a}_x = \boldsymbol{a}_y$

$$\boldsymbol{H}_1 = \frac{5}{2\pi\left(5\times10^{-2}\right)}\boldsymbol{a}_y = \frac{50}{\pi}\boldsymbol{a}_y$$

For $\boldsymbol{H}_2$, $\rho = 5$ cm, $\boldsymbol{a}_\phi = -\boldsymbol{a}_z \times -\boldsymbol{a}_x = \boldsymbol{a}_y$, $\boldsymbol{H}_2 = \boldsymbol{H}_1$

$$\mathbf{H} = 2\boldsymbol{H}_1 = \frac{100}{\pi}\boldsymbol{a}_y$$

$$= \underline{\underline{31.83\,\boldsymbol{a}_y \text{ A/m}}}$$

(*b*) For $\boldsymbol{H}_1$, $\boldsymbol{a}_\phi = \boldsymbol{a}_z \times \left(\frac{2\boldsymbol{a}_x + \boldsymbol{a}_y}{\sqrt{5}}\right) = \frac{2\boldsymbol{a}_y - \boldsymbol{a}_x}{\sqrt{5}}$

$$\boldsymbol{H}_1 = \frac{5}{2\pi 5\sqrt{5}\times10^{-2}}\left(\frac{-\boldsymbol{a}_x + 2\boldsymbol{a}_y}{\sqrt{5}}\right) = -3.183\boldsymbol{a}_x + 6.366\boldsymbol{a}_y$$

For $\boldsymbol{H}_2$, $\boldsymbol{a}_\rho = -\boldsymbol{a}_z \times \boldsymbol{a}_y = \boldsymbol{a}_x$

$$\boldsymbol{H}_2 = \frac{5}{2\pi(5)}\boldsymbol{a}_x = 15.915\boldsymbol{a}_x$$

$$\boldsymbol{H} = \boldsymbol{H}_1 + \boldsymbol{H}_2$$

$$= \underline{\underline{12.3\,\boldsymbol{a}_x + 6.366\boldsymbol{a}_y \text{ A/m}}}$$

Prob. 7.27

(a) $\boldsymbol{B} = \dfrac{\mu_o I}{2\pi\rho}\boldsymbol{a}_\phi$

At (-3,4,5), ρ=5.

$$B = \frac{4\pi\times10^{-7}\times2}{2\pi(5)}a_\phi = \underline{\underline{80a_\phi \text{ nW/m}^2}}$$

(b) $\Psi = \int \boldsymbol{B}\bullet dS = \dfrac{\mu_o I}{2\pi}\iint\dfrac{d\rho dz}{\rho} = \dfrac{4\pi\times10^{-7}\times2}{2\pi}\ln\rho\Big|_2^6 z\Big|_0^4$

$$= 16\times10^{-7}\ln 3 = \underline{\underline{1.756\ \mu\text{Wb}}}$$

Prob. 7.28

(a) $I = \int \boldsymbol{J} \bullet d\boldsymbol{S}$

$$= \int_{\phi=0}^{2\pi}\int_{\rho=0}^{a} J_o(1-\frac{\rho^2}{a^2})\rho d\rho d\phi \qquad = J_o \int_0^{2\pi} d\phi \int_0^a (\rho - \frac{\rho^3}{a^2}) d\rho$$

$$= 2\pi J_o \left(\frac{\rho^2}{2} - \frac{\rho^4}{4a^2}\right)\Bigg|_0^a \qquad = \frac{2\pi}{2} J_o \left(a^2 - \frac{a^2}{2}\right)$$

$$\underline{\underline{= \frac{1}{2}\pi a^2 J_o}}$$

(b) $\oint \boldsymbol{H} \bullet d\boldsymbol{l} = I_{enc} = \int \boldsymbol{J} \bullet d\boldsymbol{S}$

For ρ < a,

$$H_\phi 2\pi\rho = \int \boldsymbol{J} \bullet d\boldsymbol{S}$$

$$= 2\pi J_o \left(\frac{\rho^2}{2} - \frac{\rho^4}{4a^2}\right)$$

$$H_\rho \bullet 2\pi\rho = 2\pi J_o \frac{\rho^2}{4}\left(2 - \frac{\rho^2}{a^2}\right)$$

$$H_\rho = \frac{J_o \rho}{4}\left(2 - \frac{\rho^2}{a^2}\right)$$

For ρ > a,

$$\oint \mathbf{H} \bullet dl = \int J_o d\boldsymbol{S} = I$$

$$H_\phi \bullet 2\pi\rho = \frac{1}{2}\pi a^2 J_o$$

$$H_\phi = \frac{a^2 J_o}{4\rho}$$

$$\text{Hence} \quad \underline{\underline{H_\phi = \begin{cases} \frac{J_o \rho}{4}\left(2 - \frac{\rho^2}{a^2}\right), \rho < a \\ \frac{a J_o}{4\rho}, \quad \rho > a \end{cases}}}$$

Prob. 7.29

$$\boldsymbol{B} = \frac{\mu_0 \mathrm{I}}{2\pi\rho}\boldsymbol{a}_\phi$$

$$\psi = \int \boldsymbol{B}\cdot \mathrm{d}\boldsymbol{S} = \int_{\rho=\mathrm{d}}^{\mathrm{d+a}} \int_{\mathrm{z}=0}^{\mathrm{b}} \frac{\mu_0 \mathrm{I}}{2\pi\rho}\, \mathrm{d}\rho \mathrm{dz}$$

$$= \frac{\mu_0 \mathrm{Ib}}{2\pi} \mathrm{In} \frac{d+a}{d}$$

Prob. 7.30

For a whole circular loop of radius a, Example 7.3 gives

$$\boldsymbol{H} = \frac{Ia^2\boldsymbol{a}_z}{2\left[a^2+h^2\right]^{3/2}}$$

Let $\mathrm{h} \longrightarrow 0$

$$\boldsymbol{H} = \frac{I}{2a}\boldsymbol{a}_z$$

For a semicircular loop, H is halfed

$$\boldsymbol{H} = \frac{I}{4a}\boldsymbol{a}_z$$

$$\boldsymbol{B} = \mu_o \boldsymbol{H} = \underline{\underline{\frac{\mu_o I}{4a}\boldsymbol{a}_z}}$$

Prob. 7.31

(a) $\nabla \bullet \boldsymbol{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0$

showing that **B** satisfies Maxwell's equation.

(b) $d\boldsymbol{S} = dydz\boldsymbol{a}_x$

$$\Psi = \int \boldsymbol{B}\bullet d\boldsymbol{S} = \int_{z=1}^{4}\int_{y=0}^{1} y^2 dydz = \frac{y^3}{3}\bigg|_0^1 (z)\bigg|_1^4 = \underline{\underline{1\ \mathrm{Wb}}}$$

(c) $\nabla \times \mathbf{H} = \mathbf{J} \longrightarrow \boldsymbol{J} = \nabla \times \dfrac{\boldsymbol{B}}{\mu_o}$

$$\nabla \times \boldsymbol{B} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & z^2 & x^2 \end{vmatrix} = -2z\boldsymbol{a}_x - 2x\boldsymbol{a}_y - 2y\boldsymbol{a}_z$$

$$\underline{\underline{\boldsymbol{J} = -\frac{2}{\mu_o}(z\boldsymbol{a}_x + x\boldsymbol{a}_y + y\boldsymbol{a}_z)\ \text{A/m}^2}}$$

Prob. 7.32

On the slant side of the ring, $z = \dfrac{h}{6}(\rho - a)$

where $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ are due to the wires centered at x = 0 and x = 10cm respectively.

$$\psi = \int \boldsymbol{B}.\mathrm{d}\boldsymbol{S} = \int \frac{\mu_o I}{2\pi\rho}\,\mathrm{d}\rho\,\mathrm{dz}$$

$$= \frac{\mu_o I}{2\pi}\int_{\rho=a}^{a+b}\int_{z=0}^{\frac{h}{b}(\rho-a)} \frac{dz\ \mathrm{d}\rho}{\rho} = \frac{\mu_o Ih}{2\pi b}\int_{\rho=a}^{a+b}\left(1-\frac{a}{\rho}\right)\mathrm{d}\rho$$

$$= \underline{\underline{\frac{\mu_o Ih}{2\pi b}\left(b - a\ \ln\frac{a+b}{a}\right)}} \text{ as required.}$$

If a = 30 cm, b = 10 cm, h = 5 cm, I = 10 A,

$$\psi = \frac{4\pi\times10^{-7}\times10\times0.05}{2\pi\left(10\times10^{-2}\right)}\left(0.1 - 0.3\ \ln\frac{4}{3}\right)$$

$$= \underline{\underline{1.37 \times 10^{-8}\ \text{Wb}}}$$

Prob. 7.33

$$\psi = \int \mathbf{B}\bullet \mathrm{d}\boldsymbol{S} = \mu_o \int_{z=0}^{0.2}\int_{\phi=0}^{50^o} \frac{10^6}{\rho}\sin 2\phi\ \rho\ \mathrm{d}\phi\ \mathrm{dz}$$

$$\psi = 4\pi\times10^{-7}\times10^6(0.2)\left(-\frac{\cos\ 2\phi}{2}\right)\Bigg|_0^{50^o}$$

$$= 0.04\pi\left(1 - \cos\ 100^o\right)$$

$$= \underline{\underline{0.1475\ \text{Wb}}}$$

Prob. 7.34

$$\psi = \int_S \boldsymbol{B} \bullet d\boldsymbol{S} = \int_{\phi=0}^{\pi/4} \int_{\rho=1}^{2} \frac{20}{\rho} \sin^2\phi \; \rho d\rho d\phi = 20\int_1^2 d\rho \int_0^{\pi/4} \sin^2\phi d\phi$$

$$= 20(1)\int_0^{\pi/4} \frac{1}{2}(1-\cos 2\phi)d\phi = 10(\phi - \frac{1}{2}\sin 2\phi)\Big|_0^{\pi/4}$$

$$= 10(\frac{\pi}{4} - \frac{1}{2}) = \underline{\underline{2.854 \text{ Wb}}}$$

Prob. 7.35

$$\psi = \int_S \boldsymbol{B} \bullet d\boldsymbol{S}, \quad d\boldsymbol{S} = r^2 \sin\theta d\theta d\phi \boldsymbol{a}_r$$

$$\psi = \iint \frac{2}{r^3}\cos\theta \; r^2 \sin\theta d\theta d\phi \Big|_{r=1} = 2\int_0^{2\pi} d\phi \int_0^{\pi/3} \cos\theta \sin\theta d\theta$$

$$= 2(2\pi)\int_0^{\pi/3} \sin\theta d(\sin\theta) = 4\pi \frac{\sin^2\theta}{2}\Big|_0^{\pi/3} = 2\pi \sin^2(\pi/3)$$

$$= \underline{\underline{4.7123 \text{ Wb}}}$$

Prob. 7.36

$$\boldsymbol{B} = \mu_o \boldsymbol{H} = \frac{\mu_o}{4\pi}\int_v \frac{\boldsymbol{J}\times\boldsymbol{R}}{R^3} dv$$

Since current is the flow of charge, we can express this in terms of a charge moving with velocity **u**. **J**dv = dq**u**.

$$\boldsymbol{B} = \frac{\mu_o}{4\pi}\left[\frac{q\boldsymbol{u}\times\boldsymbol{R}}{R^3}\right]$$

In our case, $\boldsymbol{u}$ and $\boldsymbol{R}$ are perpendicular. Hence,

$$B = \frac{\mu_o}{4\pi}\frac{qu}{R^2} = \frac{4\pi\times10^{-7}}{4\pi}\times\frac{1.6\times10^{-19}\times2.2\times10^6}{(5.3\times10^{-11})^2} = \frac{1.6\times10^{-20}}{(5.3)^2\times10^{-22}}$$

$$= \underline{\underline{12.53 \text{ Wb/m}^2}}$$

Prob. 7.37

(*a*) $\nabla \bullet A = -ya \sin ax \neq 0$

$$\nabla \times A = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y \cos ax & 0 & y + e^{-x} \end{vmatrix}$$

$$= \boldsymbol{a}_x + e^{-x}\boldsymbol{a}_y - \cos ax\boldsymbol{a}_z \neq \boldsymbol{0}$$

A is neither electrostatic nor magnetostatic field

(*b*) $\nabla \cdot \boldsymbol{B} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho B_\rho\right) = \frac{1}{\rho}\frac{\partial}{\partial \rho}(20) = 0$

$\nabla \times \boldsymbol{B} = \boldsymbol{0}$

B can be **E**-field in a charge-free region.

(*c*) $\nabla \cdot \mathbf{C} = \frac{1}{r \sin\theta}\frac{\partial}{\partial \phi}(r^2 \sin\theta) = 0$

$$\nabla \times \boldsymbol{C} = \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(r^2 \sin^2 \theta\right)\boldsymbol{a}_r - \frac{1}{r}\frac{\partial}{\partial r}(r^3 \sin\theta)\boldsymbol{a}_\theta \neq \boldsymbol{0}$$

C is possibly ***H*** field.

Prob. 7.38

(*a*) $\nabla \cdot \boldsymbol{D} = 0$

$$\nabla \times \boldsymbol{D} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 z & 2(x+1)yz & -(x+1)z^2 \end{vmatrix}$$

$$= 2(x+1)y\boldsymbol{a}_x + \ldots \neq \boldsymbol{0}$$

D is possibly a magnetostatic field.

(*b*) $\nabla \cdot \mathbf{E} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left((z+1)\cos\phi\right) + \frac{\partial}{\partial z}\left(\frac{\sin\phi}{\rho}\right) = 0$

$\nabla \times \boldsymbol{E} = \frac{1}{\rho^2} \cos\theta\, \boldsymbol{a}_\rho + \ldots \neq \boldsymbol{0}$

E could be a magnetostatic field.

(*c*) $\nabla \cdot \boldsymbol{F} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(2\cos\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\frac{\sin\theta}{r^2}\right) \neq 0$

$$\nabla \times \boldsymbol{F} = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r^{-1}\sin\theta\right) + \frac{2\sin\theta}{r^2}\right]\boldsymbol{a}_\theta \neq \boldsymbol{0}$$

F can be neither electrostatic nor magnetostatic field.

Prob. 7.39

$$\boldsymbol{A} = \int \frac{\mu_o I d\boldsymbol{l}}{4\pi r} = \frac{\mu_o I L \boldsymbol{a}_z}{4\pi r}$$

This requires no integration since L << r.

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} = \frac{1}{\rho}\frac{\partial A_z}{\partial \phi}\boldsymbol{a}_\rho - \frac{\partial A_z}{\partial \rho}\boldsymbol{a}_\phi$$

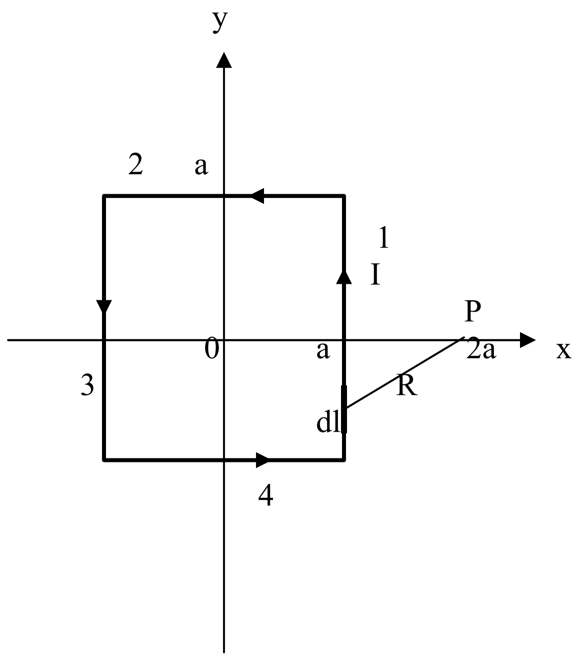
But r $= \sqrt{\rho^2 + z^2}$

$$\boldsymbol{A} = \frac{\mu_o I L \boldsymbol{a}_z}{4\pi(\rho^2 + z^2)^{1/2}}$$

$$\frac{\partial A_z}{\partial \rho} = \frac{\mu_o I L}{4\pi}\frac{\partial}{\partial \rho}(\rho^2 + z^2)^{1/2} = \frac{\mu_o I L}{4\pi}(-\frac{1}{2})(\rho^2 + z^2)^{-3/2}(2\rho)$$

$$\boldsymbol{B} = \underline{\underline{\frac{\mu_o I L \rho \mathbf{a}_\phi}{4\pi(\rho^2 + z^2)^{3/2}} = \frac{\mu_o I L \rho \mathbf{a}_\phi}{4\pi r^3}}}$$

Prob. 7.40



Divide the loop into four segments as shown above. Due to segment 1,

$$\mathbf{A}_1 = \int \frac{\mu_o I d\mathbf{l}}{4\pi R}, \qquad d\mathbf{l} = dy\mathbf{a}_y, R = \sqrt{y^2 + a^2}$$

$$\mathbf{A}_1 = \frac{\mu_o I}{4\pi}\mathbf{a}_y \int_{y=-a}^{a} \frac{dy}{\sqrt{y^2+a^2}} = \frac{\mu_o I}{4\pi}\mathbf{a}_y \left(\ln(y+\sqrt{y^2+a^2}\right)\Bigg|_{-a}^{a}$$

$$= \frac{\mu_o I}{4\pi}\mathbf{a}_y \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) = \frac{\mu_o I}{2\pi}\ln(\sqrt{2}+1)\,\mathbf{a}_y$$

By symmetry, the contributions due to sides 2 and 4 cancel. For side 3,

$$\mathbf{A}_3 = \int \frac{\mu_o I d\mathbf{l}}{4\pi R}, \qquad d\mathbf{l} = dy(-\mathbf{a}_y), R = \sqrt{y^2 + (-3a)^2}$$

$$\mathbf{A}_3 = \frac{\mu_o I}{4\pi}(\text{-}\mathbf{a}_y)\left(\ln(y+\sqrt{y^2+9a^2}\right)\Bigg|_{-a}^{a} = \frac{\mu_o I}{4\pi}(\text{-}\mathbf{a}_y)\ln\left(\frac{\sqrt{10}+1}{\sqrt{10}-1}\right)$$

$$= \frac{\mu_o I}{2\pi}\ln\left(\frac{\sqrt{10}+1}{3}\right)(\text{-}\mathbf{a}_y)$$

$$\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3 + \mathbf{A}_4 \qquad = \frac{\mu_o I}{2\pi}\ln(\sqrt{2}+1)\,\mathbf{a}_y - \qquad \frac{\mu_o I}{2\pi}\ln\left(\frac{\sqrt{10}+1}{3}\right)\mathbf{a}_y$$

$$= \qquad \underline{\underline{\frac{\mu_o I}{2\pi}\ln\left(\frac{3(\sqrt{2}+1)}{\sqrt{10}+1}\right)\mathbf{a}_y}}$$

Prob. 7.41

$$\mathbf{B} = \nabla \times \mathbf{A} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & A_z(x,y) \end{vmatrix} = \frac{\partial A_z}{\partial y}\mathbf{a}_x - \frac{\partial A_z}{\partial x}\mathbf{a}_y$$

$$\underline{\underline{= -\frac{\pi}{2}\sin\frac{\pi x}{2}\sin\frac{\pi y}{2}\mathbf{a}_x - \frac{\pi}{2}\cos\frac{\pi x}{2}\cos\frac{\pi y}{2}\mathbf{a}_y}}$$

Prob. 7.42

$$\mathbf{B} = \mu_o \mathbf{H} = \nabla \times \mathbf{A} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & A_z(x,y) \end{vmatrix} = \frac{\partial A_z}{\partial y}\mathbf{a}_x - \frac{\partial A_z}{\partial x}\mathbf{a}_z = -2\mu_o ky\mathbf{a}_x + 2\mu_o kx\mathbf{a}_y$$

$$\mathbf{H} = \underline{\underline{-2ky\mathbf{a}_x + 2kx\mathbf{a}_y}}$$

Prob. 7.43

$$\boldsymbol{B_1} = \nabla \times \boldsymbol{A_1} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & (\sin x + x\sin y) & 0 \end{vmatrix} = (\cos x + \sin y)\boldsymbol{a}_z$$

$$\boldsymbol{B_2} = \nabla \times \boldsymbol{A_2} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos y & \sin x & 0 \end{vmatrix} = (\cos x + \sin y)\boldsymbol{a}_z$$

$\boldsymbol{B_1} = \boldsymbol{B}_2 = \boldsymbol{B}$

Hence, $\mathbf{A}_2$ and $\mathbf{A}_2$ give the same **B**.

$$\nabla \bullet \boldsymbol{B} = 0$$

showing that **B** is solenoidal.

Prob. 7.44

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} = \frac{1}{\rho}\frac{\partial A_z}{\partial \phi}\boldsymbol{a}_\rho - \frac{\partial A_z}{\partial \rho}\boldsymbol{a}_\phi$$

$$= \frac{15}{\rho}\mathrm{e}^{-\rho}\cos\phi\,\boldsymbol{a}_\rho + 15\,\mathrm{e}^{-\rho}\sin\phi\,\boldsymbol{a}_\phi$$

$$\boldsymbol{B}\left(3, \frac{\pi}{4}, \text{-}10\right) = 5\,\mathrm{e}^{-3}\,\frac{1}{\sqrt{2}}\boldsymbol{a}_\rho + 15\,\mathrm{e}^{-3}\,\frac{1}{\sqrt{2}}\,\boldsymbol{a}_\phi$$

$$\mathbf{H} = \frac{\boldsymbol{B}}{\mu_o} = \frac{10^7}{4\pi}\,\frac{15}{\sqrt{2}}\,\mathrm{e}^{-3}\left(\frac{1}{3}\,\boldsymbol{a}_\rho + \boldsymbol{a}_\phi\right)$$

$$\mathbf{H} = \left(14\,\boldsymbol{a}_\rho + 42\,\boldsymbol{a}_\phi\right)\cdot 10^4 \text{ A/m}$$

$$\psi = \int \boldsymbol{B}\cdot \mathrm{d}\boldsymbol{S} = \iint \frac{15}{\rho}\,\mathrm{e}^{-\rho}\cos\phi\,\rho\,\mathrm{d}\phi\,\mathrm{dz}$$

$$= 15\ \mathrm{z}\Big|_0^{10}\ \left(\sin\phi\right)\Big|_0^{\pi/2}\ \mathrm{e}^{-5} = 150\,\mathrm{e}^{-5} \qquad \Rightarrow \qquad \psi = \underline{\underline{1.011\text{ Wb}}}$$

Prob. 7.45

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(A_\phi \sin\theta) - \frac{\partial A_\theta}{\partial\phi}\right]\boldsymbol{a}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right]\boldsymbol{a}_\theta$$

$$+\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]\boldsymbol{a}_\phi$$

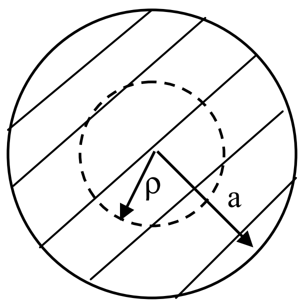
$$= \frac{1}{r\sin\theta}\frac{10}{r}2\sin\theta\cos\theta\boldsymbol{a}_r - \frac{1}{r}\frac{\partial}{\partial r}(10)\sin\theta\boldsymbol{a}_\theta + 0\boldsymbol{a}_\phi$$

$$\boldsymbol{B} = \frac{20}{r^2}\cos\theta\boldsymbol{a}_r$$

At $(4, 60^o, 30^o)$, $r = 4$, $\theta = 60^o$

$$\boldsymbol{H} = \frac{\boldsymbol{B}}{\mu_o} = \frac{1}{4\pi\times10^{-7}}\left[\frac{20}{4^2}\cos 60^o\boldsymbol{a}_r\right] = \underline{\underline{4.974\times10^5\boldsymbol{a}_r \text{ A/m}}}$$

Prob. 7.46



Applying Ampere's law gives

$$H_\phi \cdot 2\pi\rho = J_o \cdot \pi\rho^2$$

$$H_\phi = \frac{J_o}{2}\rho$$

$$B_\phi = \mu_o H_\phi = \mu_o \frac{J_o\rho}{2}$$

But $\boldsymbol{B} = \nabla\times\mathbf{A} = -\frac{\partial A_Z}{\partial\rho}\bar{a}_\varphi + \ . \ . \ .$

$$-\frac{\partial A_Z}{\partial\rho} = \frac{1}{2}\mu J_o\rho \longrightarrow A_Z = -\mu_o\frac{J_o\rho^2}{4}$$

$$\text{or} \quad \mathbf{A} = \underline{\underline{-\frac{1}{4}\mu_o J_o\rho^2 \ \boldsymbol{a}_z}}$$

Prob. 7.47

$$\boldsymbol{B} = \mu_o \boldsymbol{H} = \nabla \times \boldsymbol{A} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 10\sin \pi y & 0 & 4+\cos \pi x \end{vmatrix} = \pi \sin \pi x \boldsymbol{a}_y - 10\pi \cos \pi y \boldsymbol{a}_z$$

$$\underline{\underline{\boldsymbol{H} = \frac{\pi}{\mu_o}\left(\sin \pi x \boldsymbol{a}_y - 10\cos \pi y \boldsymbol{a}_z \right)}}$$

$$\boldsymbol{J} = \nabla \times \boldsymbol{H} = \frac{\pi}{\mu_o}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & \sin \pi x & -10\cos \pi y \end{vmatrix} = \frac{\pi}{\mu_o}\left(10\pi \sin \pi y \boldsymbol{a}_x + \pi \cos \pi x \boldsymbol{a}_z \right)$$

$$\underline{\underline{\boldsymbol{J} = \frac{\pi^2}{\mu_o}\left(10\sin \pi y \boldsymbol{a}_x + \cos \pi x \boldsymbol{a}_z \right)}}$$

Prob. 7.48

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} = \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_\phi \sin\theta)\boldsymbol{a}_r - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\boldsymbol{a}_\theta$$

$$= \frac{1}{r\sin\theta}\frac{A_o}{r^2}(2\sin\theta\cos\theta)\boldsymbol{a}_r - \frac{A_o}{r}\sin\theta(-r^{-2})\boldsymbol{a}_\theta$$

$$= \underline{\underline{\frac{A_o}{r^3}(2\cos\theta \boldsymbol{a}_r + \sin\theta \boldsymbol{a}_\theta)}}$$

Prob. 7.49

(a) $$\boldsymbol{J} = \nabla \times \boldsymbol{H} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & x^2 z & -y^2 z \end{vmatrix} = (-2yz - x^2)\boldsymbol{a}_x + (2xz - 2xy)\boldsymbol{a}_z$$

At (2,-1,3), x=2, y=-1, z=3.

$\boldsymbol{J} = \underline{\underline{2\boldsymbol{a}_x + 16\boldsymbol{a}_z \text{ A/m}^2}}$

(b) $$-\frac{\partial \rho_v}{\partial t} = \nabla \bullet \boldsymbol{J} = 0 - 2x + 2x = 0$$

At (2,-1,3),

$$\frac{\partial \rho_v}{\partial t} = \underline{\underline{0 \text{ C/m}^3\text{s}}}$$

Prob. 7.50

$$(a)\ \mathbf{B} = \nabla \times \boldsymbol{A}$$

$$= \left[\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right]\boldsymbol{a}_\rho + \left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right]\boldsymbol{a}_\phi + \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial \phi}\right]\boldsymbol{a}_z$$

$$= -\frac{\partial A_z}{\partial \rho}\boldsymbol{a}_\phi = 20\rho \boldsymbol{a}_\phi \ \mu\text{Wb/m}^2$$

$$\boldsymbol{H} = \frac{\boldsymbol{B}}{\mu_o} = \underline{\underline{\frac{-20\rho}{\mu_o}\boldsymbol{a}_\phi \ \mu\text{A/m}}}$$

$$\boldsymbol{J} = \nabla \times \boldsymbol{H} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\phi)\boldsymbol{a}_z$$

$$= \frac{1}{\mu_o \rho}(-40\rho)\boldsymbol{a}_z = \underline{\underline{\frac{-40}{\mu_o}\boldsymbol{a}_z \ \mu\text{A/m}^2}}$$

(b) $$I = \int \boldsymbol{J} \bullet d\boldsymbol{S} = \frac{-40}{\mu_o}\int_{\rho=0}^{2}\int_{\phi=0}^{2\pi} \rho\, d\phi\, d\rho, \quad d\boldsymbol{S} = \rho\, d\phi\, d\rho\, \boldsymbol{a}_z$$

$$= \frac{-40}{\mu_o}\int_0^2 \rho\, d\rho \int_0^{2\pi} d\phi = \frac{-40}{\mu_o}\frac{\rho^2}{2}\Big|_0^2 (2\pi)$$

$$= \frac{-80\pi \times 2 \times 10^{-6}}{4\pi \times 10^{-7}} = \underline{\underline{-400\ \text{A}}}$$

Prob. 7.51

$$\mathbf{H} = -\nabla V_m \quad \rightarrow \quad V_m = -\int \boldsymbol{H} \cdot d\boldsymbol{l} = -\text{mmf}$$

From Example 7.3, $\boldsymbol{H} = \dfrac{Ia^2}{2\left(z^2 + a^2\right)^{3/2}}\boldsymbol{a}_z$

$$V_m = -\frac{Ia^2}{2}\int \left(z^2 + a^2\right)^{-3/2} dz = \frac{-Iz}{2\left(z^2 + a^2\right)^{1/2}} + c$$

As $z \rightarrow \infty$, $V_m = 0$, i.e.

$$0 = -\frac{I}{2} + c \quad \rightarrow \quad c = \frac{I}{2}$$

Hence,

$$V_m = \frac{I}{2}\left[1 - \frac{z}{\sqrt{z^2 + a^2}}\right]$$

Prob. 7.52

$$\nabla'\frac{1}{R} = \left(\frac{\partial}{\partial x'}\,\boldsymbol{a}_x + \frac{\partial}{\partial y'}\,\boldsymbol{a}_y + \frac{\partial}{\partial z'}\,\boldsymbol{a}_z\right)\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-1/2}$$

$$= \left(-\frac{1}{2}\right)(-2)\,(x-x')\boldsymbol{a}_x\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-3/2} + \boldsymbol{a}_y \text{and } \boldsymbol{a}_z \text{ terms}$$

$$= \frac{\boldsymbol{R}}{R^3}$$

$$R = |\boldsymbol{r} - \boldsymbol{r}'| = \left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{1/2}$$

$$\nabla\frac{1}{R} = \left(\frac{\partial}{\partial x}\,\boldsymbol{a}_x + \frac{\partial}{\partial y}\,\boldsymbol{a}_y + \frac{\partial}{\partial z}\boldsymbol{a}_z\right)\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-1/2}$$

$$= -\frac{1}{2}\,2\,(x-x')\,\boldsymbol{a}_x\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-3/2} + \boldsymbol{a}_y \text{ and } \boldsymbol{a}_z \text{ terms}$$

$$= \frac{-\left[(x-x')\boldsymbol{a}_z+(y-y')\boldsymbol{a}_y+(z-z')\boldsymbol{a}_z\right]}{R^3} = -\frac{\bar{R}}{R^3}$$

CHAPTER 8

P.E. 8.1

(a) $\boldsymbol{F} = m\dfrac{\partial \boldsymbol{u}}{\partial t} = Q\boldsymbol{E} = \underline{\underline{6\boldsymbol{a}_z N}}$

(b) $\dfrac{\partial \boldsymbol{u}}{\partial t} = 6\boldsymbol{a}_z = \dfrac{\partial}{\partial t}(u_x, u_y, u_z) \quad \Rightarrow$

$$\frac{\partial u_x}{\partial t} = 0 \rightarrow u_x = A$$

$$\frac{\partial u_y}{\partial t} = 0 \rightarrow u_y = B$$

$$\frac{\partial u_z}{\partial t} = 6 \rightarrow u_z = 6t + C$$

Since $\boldsymbol{u}(t=0) = 0$, A = B = C = 0

$u_x = 0 = u_y, \quad u_z = 6t$

$$u_x = \frac{\partial x}{\partial t} = 0 \rightarrow x = A$$

$$u_y = \frac{\partial y}{\partial t} = 0 \rightarrow y = B$$

$$u_z = \frac{\partial z}{\partial t} = 6t \rightarrow z = 3t^2 + C_1$$

At t = 0, (x,y,z) = (0,0,0) $\rightarrow A_1 = 0 = B_1 = C_1$

Hence , (x,y,z) = $(0,0,3t^2)$,

$\boldsymbol{u} = 6t\boldsymbol{a}_z$ at any time. At P(0,0,12), $z = 12 = 3t^2 \rightarrow t = 2s$

$$\underline{\underline{t = 2s}}$$

(c) $\boldsymbol{u} = 6t\boldsymbol{a}_z = 12\boldsymbol{a}_z$ m/s .

$$\boldsymbol{a} = \frac{\partial \boldsymbol{u}}{\partial t} = \underline{\underline{6\boldsymbol{a}_z \, m/s^2}}$$

(d) $K.E = \dfrac{1}{2}m|\boldsymbol{u}|^2 = \dfrac{1}{2}(1)(144) = \underline{\underline{72J}}$

P.E. 8.2

(a) $m\boldsymbol{a} = e\boldsymbol{u} \times \boldsymbol{B} = (eB_o u_y, -eB_o u_x, 0)$

$$\frac{d^2x}{dt^2} = \frac{eB_o}{m}\frac{dy}{dt} = \omega\frac{dy}{dt} \qquad (1)$$

$$\frac{d^2y}{dt^2} = -\frac{eBo}{m}\frac{dx}{dt} = -\omega\frac{dx}{dt} \qquad (2)$$

$$\frac{d^2z}{dt^2} = 0; \Rightarrow \frac{dz}{dt} = C_1 \qquad (3)$$

From (1) and (2),

$$\frac{d^3x}{dt^3} = \omega\frac{d^2y}{dt^2} = -\omega^2\frac{dx}{dt}$$

$(D^2 + w^2 D)x = 0 \rightarrow Dx = (0, \pm j\omega)x$

$x = c_2 + c_3\cos\omega t + c_4\sin\omega t$

$$\frac{dy}{dt} = \frac{1}{\omega}\frac{d^2x}{dt^2} = -c_3\omega\cos\omega t - c_4\omega\sin\omega t$$

At t = 0, $\boldsymbol{u} = (\alpha, 0, \beta)$. Hence,

$$c_1 = \beta, c_3 = 0, c_4 = \frac{\alpha}{\omega}$$

$$\underline{\underline{\frac{dx}{dt} = \alpha\cos\omega t, \frac{dy}{dt} = -\alpha\sin\omega t, \frac{dz}{dt} = \beta}}$$

(b) Solving these yields

$$x = \frac{a}{\omega}\sin\omega t, y = \frac{\alpha}{\omega}\cos\omega t, z = \beta t$$

The starting point of the particle is $(0, \frac{\alpha}{\omega}, 0)$

(c) $x^2 + y^2 = \frac{\alpha^2}{\omega^2}$, $z=\beta t$

showing that the particles move along a helix of radius α/ω placed along the z-axis.

P.E. 8.3

(a) From Example 8.3, QuB = QE regardless of the sign of the charge.

$E = uB = 8 \times 10^6 \times 0.5 \times 10^{-3} = \underline{\underline{4 \text{ kV/m}}}$

(b) Yes, since QuB = QE holds for any Q and m.

P.E. 8.4

By Newton's 3rd law, $\boldsymbol{F}_{12} = \boldsymbol{F}_{21}$, the force on the infinitely long wire is:

$$\boldsymbol{F}_l = -\boldsymbol{F} = \frac{\mu_o I_1 I_2 b}{2\pi}\left(\frac{1}{\rho_o} - \frac{1}{\rho_o + a}\right)\boldsymbol{a}_\rho$$

$$= \frac{4\pi\times10^{-7}\times50\times3}{2\pi}\left(\frac{1}{2}-\frac{1}{3}\right)\boldsymbol{a}_\rho = \underline{\underline{5\boldsymbol{a}_\rho \ \ \mu N}}$$

P.E. 8.5

$$\boldsymbol{m} = IS\boldsymbol{a}_n = 10\times10^{-4}\times50\frac{(2,6,-3)}{7}$$

$= 7.143 \text{ x } 10^{-3}$ (2, 6, -3)

$$= \underline{\underline{(1.429\boldsymbol{a}_x + 4.286\boldsymbol{a}_y - 2.143\boldsymbol{a}_z)\times10^{-2} \text{ A-m}^2}}$$

P.E. 8.6

(a)
$$\boldsymbol{T} = \boldsymbol{m}\times\boldsymbol{B} = \frac{10\times10^{-4}\times50}{7\times10}\begin{vmatrix} 2 & 6 & -3 \\ 6 & 4 & 5 \end{vmatrix}$$

$$= \underline{\underline{0.03\boldsymbol{a}_x - 0.02\boldsymbol{a}_y - 0.02\boldsymbol{a}_z \text{ N-m}}}$$

(b) $\quad |\boldsymbol{T}| = ISB\sin\theta \rightarrow \quad |\boldsymbol{T}|_{\max} = ISB$

$$|\boldsymbol{T}|_{max} = \frac{50\times10^{-3}}{10}|6\boldsymbol{a}_x + 4\boldsymbol{a}_y + 5\boldsymbol{a}_z| = \underline{\underline{0.04387 \text{ Nm}}}$$

P.E. 8.7

(a) $\quad \mu_r = \dfrac{\mu}{\mu_o} = 4.6, \ \chi_m = \mu_r - 1 = \underline{\underline{3.6}}$

(b) $\quad \boldsymbol{H} = \dfrac{\boldsymbol{B}}{\mu} = \dfrac{10\times10^{-3}e^{-y}}{4\pi\times10^{-7}\times4.6}\boldsymbol{a}_z \ A/m \quad = \underline{\underline{1730e^{-y}\boldsymbol{a}_z}}$ A/m

(c) $\quad \boldsymbol{M} = \chi_m\boldsymbol{H} = \underline{\underline{6228e^{-y}\boldsymbol{a}_z}}$ A/m

P.E. 8.8

$$\boldsymbol{a}_n = \frac{3\boldsymbol{a}_x + 4\boldsymbol{a}_y}{5} = \frac{6\boldsymbol{a}_x + 8\boldsymbol{a}_y}{10}$$

$$\boldsymbol{B}_{1n} = (\boldsymbol{B}_1 \bullet \boldsymbol{a}_n)\boldsymbol{a}_n = \frac{(6+32)(6\boldsymbol{a}_x + 8\boldsymbol{a}_y)}{1000}$$

$$= 0.228\boldsymbol{a}_x + 0.304\boldsymbol{a}_y = \boldsymbol{B}_{2n}$$

$$\boldsymbol{B}_{1t} = \boldsymbol{B}_1 - \boldsymbol{B}_{1n} = -0.128\boldsymbol{a}_x + 0.096\boldsymbol{a}_y + 0.2\boldsymbol{a}_z$$

$$\boldsymbol{B}_{2t} = \frac{\mu_2}{\mu_1}\boldsymbol{B}_{1t} = 10\boldsymbol{B}_{1t} = -1.28\boldsymbol{a}_x + 0.96\boldsymbol{a}_y + 2\boldsymbol{a}_z$$

$$\boldsymbol{B}_2 = \boldsymbol{B}_{2n} + \boldsymbol{B}_{2t} = \underline{\underline{-1.052\boldsymbol{a}_x + 1.264\boldsymbol{a}_y + 2\boldsymbol{a}_z}} \text{ Wb/m}^2$$

P.E. 8.9

(a) $\boldsymbol{B}_{1n} = \boldsymbol{B}_{2n} \rightarrow \mu_1\boldsymbol{H}_{1n} =_z \mu_2\boldsymbol{H}_{2n}$

or $\mu_1\boldsymbol{H}_1 \bullet \boldsymbol{a}_{n21} = \mu_2\boldsymbol{H}_2 \bullet \boldsymbol{a}_{n21}$

$$\mu_o\frac{(60+2-36)}{7} = 2\mu_o\frac{(6H_{2x}-10-12)}{7}$$

$$35 = 6H_{2x}$$

$$\underline{\underline{H_{2x} = 5.833 \text{ A/m}}}$$

(b) $\boldsymbol{K} = (\boldsymbol{H}_1 - \boldsymbol{H}_2)\times\boldsymbol{a}_{n12} = \boldsymbol{a}_{n21}\times(\boldsymbol{H}_1 - \boldsymbol{H}_2)$

$$= \boldsymbol{a}_{n21}\times\left[(10,1,12) - (35/6, -5, 4)\right]$$

$$= \frac{1}{7}\begin{vmatrix} 6 & 2 & -3 \\ 25/6 & 6 & 8 \end{vmatrix}$$

$$\boldsymbol{K} = \underline{\underline{4.86\boldsymbol{a}_x - 8.64\boldsymbol{a}_y + 3.95\boldsymbol{a}_z}} \text{ A/m}$$

(c) Since $\boldsymbol{B} = \mu\boldsymbol{H}$, $\boldsymbol{B}_1$ and $\boldsymbol{H}_1$ are parallel, i.e. they make the same angle with the normal to the interface.

$$\cos\theta_1 = \frac{\boldsymbol{H}_1 \bullet \boldsymbol{a}_{n21}}{|\boldsymbol{H}_1|} = \frac{26}{7\sqrt{100+1+144}} = 0.2373$$

$$\underline{\underline{\theta_1 = 76.27^o}}$$

$$\cos\theta_2 = \frac{\boldsymbol{H}_2 \bullet \boldsymbol{a}_{n21}}{|\boldsymbol{H}_2|} = \frac{13}{7\sqrt{(5.833)^2+25+16}} = 0.2144$$

$$\underline{\underline{\theta_2 = 77.62^o}}$$

P.E. 8.10

(a) $L' = \mu_o\mu_r n^2 S = 4\pi\times10^{-7}\times1000\times16\times10^6\times4\times10^{-4}$

= <u>8.042 H/m</u>

(b) $W_m' = \frac{1}{2} L' I^2 = \frac{1}{2}(8.042)(0.5^2) = \underline{\underline{1.005}}$ J/m

P.E. 8.11 From Example 8.11**,**

$$L_{in} = \frac{\mu_o l}{8\pi}$$

$$L_{ext} = \frac{2w_m}{I^2} = \frac{1}{I^2}\iiint \frac{\mu I^2}{4\pi^2 \rho^2}\rho d\rho d\phi dz$$

$$= \frac{1}{4\pi^2}\int_0^l dz \int_0^{2\pi} d\phi \int_a^b \frac{2\mu_o}{(1+\rho)\rho} d\rho$$

$$= \frac{2\mu_o}{4\pi^2} \bullet 2\pi l \int_a^b \left[\frac{1}{\rho} - \frac{1}{(1+\rho)}\right] d\rho$$

$$= \frac{\mu_o l}{\pi}\left[\ln\frac{b}{a} - \ln\frac{1+b}{1+a}\right]$$

$$L = L_{in} + L_{ext} = \underline{\underline{\frac{\mu_o l}{8\pi} + \frac{\mu_o l}{\pi}\left[\ln\frac{b}{a} - \ln\frac{1+b}{1+a}\right]}}$$

P.E. 8.12

(a) $L'_{in} = \frac{\mu_o}{8\pi} = \frac{4\pi \times 10^{-7}}{8\pi} = \underline{0.05\ \mu\text{H/m}}$

$L'_{ext} = L' - L'_{in} = 1.2 - 0.05 = \underline{1.15\ \mu\text{H/m}}$

(b) $L' = \frac{\mu_o}{2\pi}\left[\frac{1}{4} + \ln\frac{d-a}{a}\right]$

$$\ln\frac{d-a}{a} = \frac{2\pi L'}{\mu_o} - 0.25 = \frac{2\pi \times 1.2 \times 10^{-6}}{4\pi x 10^{-7}} - 0.25$$

$$= 6 - 0.25 = 5.75$$

$$\frac{d-a}{a} = e^{5.75} = 314.19$$

$$d - a = 314.19a = 314.19 \times \frac{2.588 \times 10^{-3}}{2} = 406.6mm$$

$$d = 407.9mm = \underline{\underline{40.79cm}}$$

P.E. 8.13
This is similar to Example 8.13. In this case, however, h=0 so that

$$\mathbf{A}_1 = \frac{\mu_o I_1 a^2 b}{4b^3}\mathbf{a}_\phi$$

$$\phi_{12} = \frac{\mu_o I_1 a^2}{4b^2} \bullet 2\pi b = \frac{\mu_o \pi I_1 a^2}{2b}$$

$$m_{12} = \frac{\phi_{12}}{I_1} = \frac{\mu_o \pi a^2}{2b} = \frac{4\pi \times 10^{-7} \times \pi \times 4}{2 \times 3}$$

$= \underline{\underline{2.632\ \mu\text{H}}}$

P.E. 8.14

$$L_{in} = \frac{\mu_o}{8\pi} l = \frac{\mu_o 2\pi\rho_o}{8\pi} = \frac{4\pi \times 10^{-7} \times 10 \times 10^{-2}}{4}$$

$= \underline{\underline{31.42\ \text{nH}}}$

P.E. 8.15
(a) From Example 7.6,

$$B_{ave} = \frac{\mu_o NI}{l} = \frac{\mu_o NI}{2\pi\rho_o}$$

$$\phi = B_{ave} \bullet S = \frac{\mu_o NI}{2\pi\rho_o} \bullet \pi a^2$$

or $I = \dfrac{2\rho_o \phi}{\mu a^2 N} = \dfrac{2 \times 10 \times 10^{-2} \times 0.5 \times 10^{-3}}{4\pi \times 10^{-7} \times 10^{-4} \times 10^3}$

$= \underline{\underline{795.77\text{A}}}$

Alternatively, using circuit approach

$$R = \frac{l}{\mu S} = \frac{2\pi\rho_o}{\mu_o S} = \frac{2\pi\rho_o}{\mu_o \pi a^2}$$

$\Im = NI = \dfrac{\phi\Re}{N} = \dfrac{2\rho_o\phi}{\mu a^2 N}$, as obtained before.

$$\Re = \frac{2\rho_o}{\mu a^2} = \frac{2 \times 10 \times 10^{-2}}{4\pi \times 10^{-7} \times 10^{-4}} = 1.591 \times 10^9$$

$\Im = \phi\Re = 0.5\text{x}10^{-3}\text{x}1.591\text{x}10^{9} = 7.9577\text{x}10^{5}$

$I = \dfrac{\Im}{N} = 795.77A$ as obtained before.

(b) If $\mu=500\mu_o$,

$$I = \frac{795.77}{500} = \underline{\underline{1.592}}\text{ A}$$

P.E. 8.16

$$\Im = \frac{B^2{}_a S}{2\mu_o} = \frac{(1.5)^2 \times 10\times 10^{-4}}{2\times 4\pi \times 10^{-7}} = \frac{22500}{8\pi} = \underline{\underline{895.25\text{N}}}$$

P.E. 8.17

We may approximate the longer solenoid as infinite so that $B_1 = \frac{\mu_o N_1 I_1}{l_1}$. The flux linking the second solenoid is:

$$\psi_2 = N_2 B_1 S_1 = \frac{\mu_o N_1 I_1}{l_1} \bullet \pi r_1^2 \bullet N_2$$

$$M = \frac{\psi_2}{I_1} = \frac{\mu_o N_1 N_2}{l_1} \bullet \pi r_1^2$$

Here we assume air-core solenoids.

P.E. 8.18

$$R = \frac{\ell}{\mu S}$$

$$\ell = 2\pi\rho_o = 2\pi \times \frac{1}{2}(5+6)10^{-2} = 11\pi \times 10^{-2}$$

$$S = 1.5\times 10^{-2}(6-5)10^{-2} = 1.5\times 10^{-4}$$

$$F = NI = \psi R = \psi \frac{\ell}{\mu S} \longrightarrow \mu = \frac{\psi \ell}{NIS}$$

$$\mu = \frac{12\times 10^{-3}(11\pi \times 10^{-2})}{500(2)1.5\times 10^{-4}} = \underline{\underline{27.65\times 10^{-3}\text{ H/m}}}$$

$$B = \frac{\psi}{S} = \frac{12\times 10^{-3}}{1.5\times 10^{-4}} = \underline{\underline{80\text{ Wb}}}$$

Prob. 8.1

$$F = m\omega^2 r = 9.11\times10^{-31}\times(2\times10^{16})^2(0.4\times10^{-10}) = \underline{\underline{14.576 \text{ nN}}}$$

Prob. 8.2
(a)

$$\boldsymbol{F} = Q(\boldsymbol{u}\times\boldsymbol{B}) = 10^{-3}\begin{vmatrix} 10 & -2 & 6 \\ 0 & 0 & 25 \end{vmatrix} = 10^{-3}(-50\boldsymbol{a}_x - 250\boldsymbol{a}_y)$$

$$= \underline{\underline{-0.05\boldsymbol{a}_x - 0.25\boldsymbol{a}_y \text{ N}}}$$

(b) Constant velocity implies that acceleration **a** = **0**.

$$\boldsymbol{F} = m\boldsymbol{a} = 0 = Q(\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B})$$

$$\boldsymbol{E} = -\boldsymbol{u}\times\boldsymbol{B} = \underline{\underline{50\boldsymbol{a}_x + 250\boldsymbol{a}_y \text{ V/m}}}$$

Prob. 8.3
At P, x = 2, y = 5, z = -3

$$\boldsymbol{E} = 2(2)(5)(-3)\boldsymbol{a}_x + (2)^2(-3)\boldsymbol{a}_y + (2)^2(5)\boldsymbol{a}_z = -60\boldsymbol{a}_x - 12\boldsymbol{a}_y + 20\boldsymbol{a}_z$$

$$\boldsymbol{B} = (5)^2\boldsymbol{a}_x + (-3)^2\boldsymbol{a}_y + 2^2\boldsymbol{a}_z = 25\boldsymbol{a}_x + 9\boldsymbol{a}_y + 4\boldsymbol{a}_z$$

$$\boldsymbol{F} = Q(\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B})$$

$$\boldsymbol{u}\times\boldsymbol{B} = \begin{vmatrix} 1.4 & 3.2 & -1 \\ 25 & 9 & 4 \end{vmatrix} = 21.8\boldsymbol{a}_x - 30.6\boldsymbol{a}_y - 67.4\boldsymbol{a}_z$$

$$\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B} = (-60,-12,20) + (21.8,-30.6,-67.4) = (-38.2,-42.6,-47.4)$$

$$\boldsymbol{F} = Q(\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B}) = 4(\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B}) \text{ mN}$$

$$= \underline{\underline{-152.8\boldsymbol{a}_x - 170.4\boldsymbol{a}_y - 189.6\boldsymbol{a}_z \text{ mN}}}$$

Prob. 8.4

$$\boldsymbol{F} = q\boldsymbol{E} = m\boldsymbol{a} = m\frac{d\boldsymbol{u}}{dt}$$

$$\frac{d\boldsymbol{u}}{dt} = \frac{q\boldsymbol{E}}{m} = \frac{10\times10^{-3}}{2}(30,0,0)\times10^{3}$$

$$\frac{d}{dt}(u_x, u_y, u_z) = (150,0,0)$$

Equating components gives

$$\frac{du_x}{dt} = 150 \longrightarrow u_x = 150t + c_1$$

$$\frac{du_y}{dt} = 0 \longrightarrow u_y = c_2$$

$$\frac{du_z}{dt} = 0 \longrightarrow u_z = c_3$$

At t = 0, **u**=$(2,5,0)\times10^3$.

$$2000 = 0 + c_1 \longrightarrow c_1 = 2000$$

$$5000 = c_2$$

$$0 = c_3$$

Hence, u= (150t+2000,5000,0)

At t = 4s,

$$u = \underline{\underline{(2600, 5000, 0)\text{ m/s}}}$$

$$u_x = \frac{dx}{dt} = 150t + 2000 \longrightarrow x = 75t^2 + 2000t + c_4$$

$$u_y = \frac{dy}{dt} = 5000 \longrightarrow y = 5000t + c_5$$

$$u_z = \frac{dz}{dt} = 0 \longrightarrow z = +c_6$$

At t=0, (x,y,z)=(0,0,0) $\longrightarrow$ $c_4 = 0 = c_5 = c_6$

Hence,

$$(x, y, z) = (75t^2 + 2000t, 5000t, 0)$$

At t = 4s, x=9,200, y=20,000, z=0.

i.e. $(x, y, z) = \underline{\underline{(9200, 20000, 0)}}$

Prob. 8.5

$$m\boldsymbol{a} = Q\boldsymbol{u}\times\boldsymbol{B}$$

$$10^{-3}\boldsymbol{a} = -2\times10^{-3}\begin{vmatrix} u_x & u_y & u_z \\ 0 & 6 & 0 \end{vmatrix}$$

$$\frac{d}{dt}(u_x, u_y, u_z) = (12u_z, 0, -12u_x)$$

i.e. $$\frac{du_x}{dt} = 12u_z \qquad (1)$$

$$\frac{du_y}{dt} = 0 \rightarrow u_y = A_1 \qquad (2)$$

$$\frac{du_z}{dt} = -12u_x \qquad (3)$$

From (1) and (3),

$$\ddot{u}_x = 12\dot{u}_z = -144u_x$$

or

$$\ddot{u}_x + 144u_x = 0 \rightarrow u_x = c_1\cos 12t + c_2\sin 12t$$

From (1), $u_z = -c_1\sin 12t + c_2\cos 12t$

At t=0,

$u_x=5$, $u_y=0$, $u_z=0 \rightarrow A_1=0=c_2$, $c_1=5$

Hence,

$$\boldsymbol{u} = (5\cos 12t, 0, -5\sin 12t)$$

$$\boldsymbol{u}(t=10s) = (5\cos 120, 0, -5\sin 120) = \underline{\underline{4.071\boldsymbol{a}_x - 2.903\boldsymbol{a}_z}} \text{ m/s}$$

$$u_x = \frac{dx}{dt} = 5\cos 12t \rightarrow x = \frac{5}{12}\sin 12t + B_1$$

$$u_y = \frac{dy}{dt} = 0 \rightarrow y = B_2$$

$$u_z = \frac{dz}{dt} = -5\sin 12t \rightarrow z = \frac{5}{12}\cos 12t + B_3$$

At t=0, (x, y, z) = (0, 1, 2) $\rightarrow B_1=0$, $B_2=1$, $B_3=\frac{19}{12}$

$$(x, y, z) = \left(\frac{5}{12}\sin 12t, 1, \frac{5}{12}\cos 12t + \frac{19}{12}\right) \qquad (4)$$

At t=10s,

$$(x,y,z)=\left(\frac{5}{12}\sin 120, 1, \frac{5}{12}\cos 120+\frac{19}{12}\right)=\underline{(0.2419,\ 1,\ 1.923)}$$

By eliminating t from (4),

$x^2+(z-19/12)^2=(5/12)^2,\ y=1$ which is a circle in the y=1 plane with center at (0,1,19/12). The particle gyrates.

Prob. 8.6

(a) $m\boldsymbol{a}=-e(\boldsymbol{u}\times\boldsymbol{B})$

$$-\frac{m}{e}\frac{d}{dt}(u_x,u_y,u_z)=\begin{vmatrix}u_x & u_y & u_z\\ 0 & 0 & B_o\end{vmatrix}=u_yB_o\vec{a}_x-B_ou_x\vec{a}_y$$

$$\frac{du_z}{dt}=0\rightarrow u_z=c=0$$

$$\frac{du_x}{dt}=-u_y\frac{B_oe}{m}=-u_yw,\ \text{ where w} = \frac{B_oe}{m}$$

$$\frac{du_y}{dt}=u_xw$$

Hence,

$$\ddot{u}_x=-w\dot{u}_y=-w^2u_x$$

or $\ddot{u}_x+w^2u_x=0\rightarrow u_x=A\cos wt+B\sin wt$

$$u_y=-\frac{\dot{u}_x}{w}=A\sin wt-B\cos wt$$

At t=0, $u_x = u_o$, $u_y = 0 \rightarrow A = u_o$, B=0

Hence,

$$u_x=u_o\cos wt=\frac{dx}{dt}\rightarrow x=\frac{u_o}{w}\sin wt+c_1$$

$$u_y=u_o\sin wt=\frac{dy}{dt}\rightarrow y=-\frac{u_o}{w}\cos wt+c_2$$

At t=0, x = 0 = y $\rightarrow$ c_1=0, $c_2=\frac{u_o}{w}$. Hence,

$$x=\frac{u_o}{w}\sin wt,\ y=\frac{u_o}{w}(1-\cos wt)$$

$$\frac{u^2{}_o}{w^2}(\cos^2 wt+\sin^2 wt)=\left(\frac{u_o}{w}\right)^2=x^2+(y-\frac{u_o}{w})^2$$

showing that the electron would move in a circle centered at (0, $\frac{u_o}{w}$). But since the field does not exist throughout the circular region, the electron passes through a semi-circle and leaves the field horizontally.

(b) d = twice the radius of the semi-circle

$$= \frac{2u_o}{w} = \frac{2u_o m}{B_o e}$$

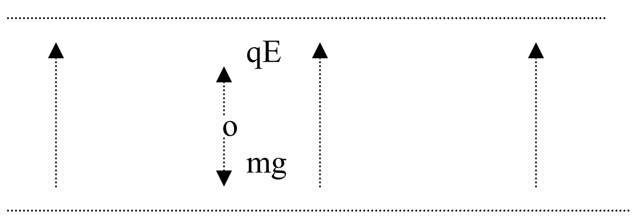
Prob. 8.7

$$\boldsymbol{F} = \int I d\boldsymbol{l} \times \boldsymbol{B} = \int_0^{0.2} 2dy(-\boldsymbol{a}_y) \times (4\boldsymbol{a}_x - 8\boldsymbol{a}_z)$$

$$(-\boldsymbol{a}_y) \times (4\boldsymbol{a}_x - 8\boldsymbol{a}_z) = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ 0 & -1 & 0 \\ 4 & 0 & -8 \end{vmatrix} = 8\boldsymbol{a}_x + 4\boldsymbol{a}_z$$

$$\boldsymbol{F} = 2(8\boldsymbol{a}_x + 4\boldsymbol{a}_z)(0.2) = \underline{\underline{3.2\boldsymbol{a}_x + 1.6\boldsymbol{a}_z \text{ N}}}$$

Prob. 8.8



$$mg = qE \quad \longrightarrow \quad q = \frac{mg}{E} = \frac{0.4 \times 10^{-3} \times 9.81}{1.5 \times 10^5} = \underline{\underline{26.67 \text{ nC}}}$$

Prob. 8.9

$$\mathfrak{F} = I\boldsymbol{L} \times \boldsymbol{B} \rightarrow \mathfrak{F} = \frac{\boldsymbol{F}}{L} = I_1 \boldsymbol{a}_l \times \boldsymbol{B}_2 = \frac{\mu_o I_1 I_2 \boldsymbol{a}_l \times \boldsymbol{a}_\phi}{2\pi\rho}$$

(a) $$\boldsymbol{F}_{21} = \frac{\boldsymbol{a}_z \times (-\boldsymbol{a}_y) 4\pi \times 10^{-7}(-100)(200)}{2\pi} = \underline{\underline{4\boldsymbol{a}_x \text{ mN/m}}} \quad \text{(repulsive)}$$

(b) $$\boldsymbol{F}_{12} = -\boldsymbol{F}_{21} = -4\boldsymbol{a}_x \text{ mN/m} \quad \text{(repulsive)}$$

(c) $$\boldsymbol{a}_l \times \boldsymbol{a}_\phi = \boldsymbol{a}_z \times \left(-\frac{4}{5}\boldsymbol{a}_x + \frac{3}{5}\boldsymbol{a}_y\right) = -\frac{3}{5}\boldsymbol{a}_x - \frac{4}{5}\boldsymbol{a}_y, \rho = 5$$

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$$F_{31} = \frac{4\pi\times10^{-7}(-3\times10^{4})}{2\pi(5)}\left(-\frac{3}{5}\boldsymbol{a}_x - \frac{4}{5}\boldsymbol{a}_y\right)$$

$= \underline{\underline{0.72\boldsymbol{a}_x + 0.96\boldsymbol{a}_y}}$ mN/m (attractive)

(d) $\boldsymbol{F}_3 = \boldsymbol{F}_{31} + \boldsymbol{F}_{32}$

$$\boldsymbol{F}_{32} = \frac{4\pi\times10^{-7}\times6\times10^{4})}{2\pi(3)}\left(\boldsymbol{a}_z\times\boldsymbol{a}_y\right) = -4\boldsymbol{a}_x \text{ mN/m(attractive)}$$

$\boldsymbol{F}_3 = \underline{\underline{-3.28\boldsymbol{a}_x + 0.96\boldsymbol{a}_y}}$ mN/m

(attractive due to L_2 and repulsive due to L_1)

Prob. 8.10

$$F = \frac{\mu_o I_1 I_2}{2\pi\rho} = \frac{4\pi\times10^{-7}(10)10}{2\pi(20\times10^{-2})} = \underline{\underline{100\ \mu\text{N}}}$$

Prob. 8.11

$$W = -\int \boldsymbol{F}\bullet d\boldsymbol{l}, \qquad \boldsymbol{F} = \int L d\boldsymbol{l}\times\boldsymbol{B} = 3(2\boldsymbol{a}_z)\times\cos\phi/3\ \boldsymbol{a}_\phi$$

$$F = 6\cos\phi/3\ \vec{a}_\phi \text{ N}$$

$$W = -\int_0^{2\pi} 6\cos\phi/3\ \rho_o d\phi = -6\rho_o\times3\sin\phi/3\Big|_0^{2\pi} \text{ J}$$

$$= -1.8\sin\frac{2\pi}{3} = \underline{\underline{-1.559\ \text{J}}}$$

Prob. 8.12

(a)
$$\boldsymbol{F}_1 = \int_{\rho=2}^{6} \frac{\mu_o I_1 I_2}{2\pi\rho} d\rho\boldsymbol{a}_\rho\times\boldsymbol{a}_\phi = \frac{4\pi\times10^{-7}}{2\pi}(2)(5)\ln 6/2\ \boldsymbol{a}_z$$

$$= 2\ln 3\boldsymbol{a}_z\ \mu\text{N} = \underline{\underline{2.197\boldsymbol{a}_z\ \mu\text{N}}}$$

(b)
$$\boldsymbol{F}_2 = \int I_2 d\boldsymbol{l}_2\times\boldsymbol{B}_1$$

$$= \frac{\mu_o I_1 I_2}{2\pi}\int\frac{1}{\rho}\left[d\rho\boldsymbol{a}_\rho + dz\boldsymbol{a}_z\right]\times\boldsymbol{a}_\phi$$

$$= \frac{\mu_o I_1 I_2}{2\pi}\int\frac{1}{\rho}\left[d\rho\boldsymbol{a}_z - dz\boldsymbol{a}_\rho\right]$$

But ρ = z+2, dz=dρ

$$\boldsymbol{F}_2 = \frac{4\pi\times10^{-7}}{2\pi}(5)(2)\int_{\rho=4}^{2}\frac{1}{\rho}\left[d\rho\boldsymbol{a}_z - dz\boldsymbol{a}_\rho\right]$$

$$2\ln \tfrac{2}{4}(\boldsymbol{a}_z - \boldsymbol{a}_\rho)\mu N = 1.386\boldsymbol{a}_\rho - 1.386\boldsymbol{a}_z\boldsymbol{a}_z\mu N$$

$$\boldsymbol{F}_3 = \frac{\mu_o I_1 I_2}{2\pi}\int \frac{1}{\rho}\left[d\rho \boldsymbol{a}_z - dz\boldsymbol{a}_\rho\right]$$

But z = -ρ + 6, dz = -dρ

$$\boldsymbol{F}_3 = \frac{4\pi\times 10^{-7}}{2\pi}(5)(2)\int_{\rho=6}^{4}\frac{1}{\rho}\left[d\rho \boldsymbol{a}_z - dz\boldsymbol{a}_\rho\right]$$

$$2\ln \tfrac{4}{6}(\boldsymbol{a}_z + \boldsymbol{a}_\rho)\mu N = -0.8109\boldsymbol{a}_\rho - 0.8109\boldsymbol{a}_z\mu N$$

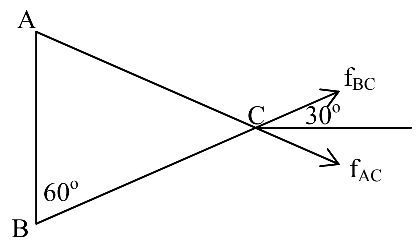
$$\boldsymbol{F} = \boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3$$

$$= \boldsymbol{a}_\rho(\ln 4 + \ln 4 - \ln 9) + \boldsymbol{a}_z(\ln 9 - \ln 4 + \ln 4 - \ln 9)$$

$$= \underline{\underline{0.575\boldsymbol{a}_\rho \ \mu\text{N}}}$$

Prob. 8.13

From Prob. 8.7,



$$\boldsymbol{f} = \frac{\mu_o I_1 I_2}{2\pi\rho}\boldsymbol{a}_\rho$$

$$\boldsymbol{f} = \boldsymbol{f}_{AC} + \boldsymbol{f}_{BC}$$

$$|\boldsymbol{f}_{AC}| = |\boldsymbol{f}_{BC}| = \frac{4\pi\times 10^{-7}\times 75\times 150}{2\pi\times 2} = 1.125\times 10^{-3}$$

$$\boldsymbol{f} = 2\times 1.125\cos 30^o\,\boldsymbol{a}_x\,\text{mN/m}$$

$$= \underline{\underline{1.949\boldsymbol{a}_x}}\ \text{mN/m}$$

Prob. 8.14

The field due to the current sheet is

$$\boldsymbol{B} = \frac{\mu}{2}\boldsymbol{K}\times\boldsymbol{a}_n = \frac{\mu_o}{2}10\boldsymbol{a}_x\times(-\boldsymbol{a}_z) = 5\mu_o\boldsymbol{a}_y$$

$$\boldsymbol{F} = I_2\int d\boldsymbol{l}_2\times\boldsymbol{B} = 2.5\int_0^L dx\boldsymbol{a}_x\times(5\mu_o\boldsymbol{a}_y) = 2.5L\times 5\mu_o(\boldsymbol{a}_z)$$

$$\frac{\boldsymbol{F}}{L} = 12.5\times 4\pi\times 10^{-7}(\boldsymbol{a}_z) = \underline{\underline{15.71\boldsymbol{a}_z \ \ \mu\text{N/m}}}$$

Prob. 8.15

$$\boldsymbol{F} = \int I d\boldsymbol{l}\times\boldsymbol{B} = I\boldsymbol{L}\times\boldsymbol{B} = 5(2\boldsymbol{a}_z)\times 40\boldsymbol{a}_x 10^{-3} = \underline{\underline{0.4\boldsymbol{a}_y\ \text{N}}}$$

Prob. 8.16

$$\boldsymbol{T} = \boldsymbol{m}\times\boldsymbol{B} = \left[0.4(0.6)(3)\boldsymbol{a}_x\right]\times(0.5\boldsymbol{a}_x + 0.8\boldsymbol{a}_y) = 0.72(0.8)\boldsymbol{a}_z$$
$$= \underline{\underline{0.576\boldsymbol{a}_z \text{ Nm}}}$$

Prob. 8.17

$$\boldsymbol{F} = \int I d\boldsymbol{l}\times\boldsymbol{B} \quad\longrightarrow\quad F = IB\ell = 520\times0.4\times10^{-3}\times30\times10^{-3}$$

$$F = \underline{\underline{6.24 \text{ mN}}}$$

Prob. 8.18

$$m = IS \quad\longrightarrow\quad I = \frac{m}{S} = \frac{m}{\pi r^2}$$
$$I = \frac{8\times10^{22}}{\pi(6370\times10^3)^2} = 6.275\times10^8 = \underline{\underline{627.5 \text{ MA}}}$$

Prob. 8.19

Let $\boldsymbol{F} = \boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3$

$$\boldsymbol{F}_1 = \int I d\boldsymbol{l}\times\boldsymbol{B} = \int_5^0 2dx\boldsymbol{a}_x\times30\boldsymbol{a}_z \text{ mN}$$
$$= -60\boldsymbol{a}_y x\Big|_5^0 = 300\boldsymbol{a}_y \text{ mN}$$
$$\boldsymbol{F}_2 = \int_0^5 2dy\boldsymbol{a}_y\times30\boldsymbol{a}_z \text{ mN}$$
$$= 60\boldsymbol{a}_x y\Big|_0^5 = 300\boldsymbol{a}_x \text{ mN}$$
$$\boldsymbol{F}_3 = \int_0^5 2(dx\boldsymbol{a}_x + dz\boldsymbol{a}_z)\times30\boldsymbol{a}_z \text{ mN}$$
$$= 60(-\boldsymbol{a}_y)x\Big|_0^5 = -300\boldsymbol{a}_y \text{ mN}$$
$$\boldsymbol{F} = \boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 = 300\boldsymbol{a}_y + 300\boldsymbol{a}_x - 300\boldsymbol{a}_y \text{ mN} = \underline{\underline{300\boldsymbol{a}_x \text{ mN}}}$$
$$\boldsymbol{T} = \boldsymbol{m}\times\boldsymbol{B} = IS\boldsymbol{a}_n\times\boldsymbol{B} = 2(\frac{1}{2})(5)(5)\boldsymbol{a}_y\times30\boldsymbol{a}_z10^{-3} = \underline{\underline{0.75\boldsymbol{a}_x \text{ N.m}}}$$

Prob. 8.20

For each turn, $\boldsymbol{T} = \boldsymbol{m} \times \boldsymbol{B}$, $\boldsymbol{m} = \mathrm{IS}\boldsymbol{a}_\mathrm{n}$

For N turns,

$$T = NISB = 50 \times 4 \times 12 \times 10^{-4} \times 100 \times 10^{-3} = \underline{\underline{24 \text{ mNm}}}$$

Prob. 8.21

$$f(x,y,z) = x + 2y - 5z - 12 = 0 \quad \longrightarrow \quad \nabla f = \boldsymbol{a}_x + 2\boldsymbol{a}_y - 5\boldsymbol{a}_z$$

$$\boldsymbol{a}_n = \frac{\nabla f}{|\nabla f|} = \frac{\boldsymbol{a}_x + 2\boldsymbol{a}_y - 5\boldsymbol{a}_z}{\sqrt{30}}$$

$$\boldsymbol{m} = NIS\boldsymbol{a}_n = 2 \times 60 \times 8 \times 10^{-4} \frac{(\boldsymbol{a}_x + 2\boldsymbol{a}_y - 5\boldsymbol{a}_z)}{\sqrt{30}} = \underline{\underline{17.53\boldsymbol{a}_x + 35.05\boldsymbol{a}_y - 87.64\boldsymbol{a}_z \text{ mAm}}}$$

Prob. 8.22

$$M = \chi_m H = \chi_m \frac{B}{\mu_o \mu_r} = \frac{\chi_m B}{\mu_o (1 + \chi_m)}$$

Prob. 8.23

(a) $\boldsymbol{M} = \chi_\mathrm{m} \boldsymbol{H} = \chi_\mathrm{m} \dfrac{\boldsymbol{B}}{\mu_\mathrm{o} \mu}$

$$\mathrm{M} = \frac{4999}{5000} \times \frac{1.5}{4\pi \times 10^{-7}} = \underline{\underline{1.193 \times 10^6 \text{ A/m}}}$$

(b) $\boldsymbol{M} = \dfrac{\sum_{\mathrm{k}=1}^{\mathrm{N}} \boldsymbol{m}_\mathrm{k}}{\Delta v}$

If we assume that all $\boldsymbol{m}_\mathrm{k}$ align with the applied $\boldsymbol{B}$ field,

$$\mathrm{M} = \frac{Nm_\mathrm{k}}{\Delta v} \quad \rightarrow \quad m_\mathrm{k} = \frac{Nm_\mathrm{k}}{N/\Delta v} = \frac{1.193 \times 10^6}{8.5 \times 10^{28}}$$

$$\mathrm{m}_\mathrm{k} = \underline{\underline{1.404 \times 10^{-23} \text{ A} \cdot \mathrm{m}^2}}$$

Prob. 8.24

$$\mu_r = \chi_m + 1 = 6.5 + 1 = \underline{\underline{7.5}}$$

$$\boldsymbol{M} = \chi_m \boldsymbol{H} \quad \longrightarrow \quad \boldsymbol{H} = \frac{\boldsymbol{M}}{\chi_m} = \frac{24y^2}{6.5}\boldsymbol{a}_z$$

At y = 2cm,

$$\boldsymbol{H} = \frac{24 \times 4 \times 10^{-4}}{6.5}\boldsymbol{a}_z = \underline{\underline{1.477\boldsymbol{a}_z \text{ mA/m}}}$$

$$\boldsymbol{J} = \nabla \times \boldsymbol{H} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & \frac{24y^2}{6.5} \end{vmatrix} = \frac{48y}{6.5}\mathbf{a}_x$$

At y=2cm,

$$\boldsymbol{J} = \frac{48 \times 2 \times 10^{-2}}{6.5}\mathbf{a}_x = \underline{\underline{0.1477\boldsymbol{a}_x \text{ A/m}^2}}$$

Prob. 8.25

(a) $\chi_m = \mu_r - 1 = \underline{\underline{3.5}}$

(b) $\boldsymbol{H} = \dfrac{\boldsymbol{B}}{\mu} = \dfrac{4y\,\boldsymbol{a}_z \times 10^{-3}}{4\pi \times 10^{-7} \times 4.5} = \underline{\underline{707.3y\,\boldsymbol{a}_z \text{ A/m}}}$

(c) $\boldsymbol{M} = \chi_m \boldsymbol{H} = \underline{\underline{2.476y\,\boldsymbol{a}_z \text{ kA/m}}}$

(d) $\boldsymbol{J}_b = \nabla \times \mathbf{M} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & M_z(y) \end{vmatrix} = \dfrac{dM_z}{dy}\boldsymbol{a}_x$

$= \underline{\underline{2.476\boldsymbol{a}_x \text{ kA/m}^2}}$

Prob. 8.26

When H = 250,

$$B = \frac{2H}{100 + H} = \frac{2(250)}{100 + 250} = 1.4286 \text{ mWb/m}^2$$

But $B = \mu_o \mu_r H$

$$\mu_r = \frac{B}{\mu_o H} = \frac{1.4286 \times 10^{-3}}{4\pi \times 10^{-7} \times 250} = \underline{\underline{4.54}}$$

Prob. 8.27

$$\oint \boldsymbol{H} \cdot d\boldsymbol{l} = I_{enc}$$

$$H_\phi \cdot 2\pi\rho = \frac{\pi\rho^2}{\pi a^2} \cdot I \quad \rightarrow \quad H_\phi = \frac{I\rho}{2\pi a^2}$$

$$\boldsymbol{M} = \chi_m \boldsymbol{H} = \underline{\underline{(\mu_r - 1)\, \frac{I\rho}{2\pi a^2}\, \boldsymbol{a}_\phi}}$$

$$\boldsymbol{J}_b = \nabla \times \boldsymbol{M} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho M_\phi\right) \boldsymbol{a}_z = \underline{\underline{(\mu_r - 1)\, \frac{I}{\pi a^2} \boldsymbol{a}_z}}$$

Prob. 8.28

(a) From $H_{1t} - H_{2t} = K$ and $M = \chi_m H$, we obtain:

$$\underline{\underline{\frac{M_{1t}}{\chi_{m1}} - \frac{M_{2t}}{\chi_{m2}} = K}}$$

Also from $B_{1n} - B_{2n} = 0$ and $B = \mu H = (\mu/\chi_m)M$, we get:

$$\underline{\underline{\frac{\mu_1 M_{1n}}{\chi_{m1}} = \frac{\mu_2 M_{2n}}{\chi_{m2}}}}$$

(b) From $B_1\cos\theta_1 = B_{1n} = B_{2n} = B_2\cos\theta_2$ (1)

and $\dfrac{B_1 \sin\theta_1}{\mu_1} = H_{1t} = K + H_{2t} = K + \dfrac{B_2 \sin\theta_2}{\mu_2}$ (2)

Dividing (2) by (1) gives

$$\frac{\tan\theta_1}{\mu_1} = \frac{k}{B_2\cos\theta_2} + \frac{\tan\theta_2}{\mu_2} = \frac{\tan\theta_2}{\mu_2}\left(1 + \frac{k\mu_2}{B_2\sin\theta_2}\right)$$

i.e. $$\underline{\underline{\frac{\tan\theta_1}{\tan\theta_2} = \frac{\mu_1}{\mu_2}\left(1 + \frac{k\mu_2}{B_2\sin\theta_2}\right)}}$$

Prob. 8.29

$\boldsymbol{B}_{2n} = \boldsymbol{B}_{1n} = 1.8\boldsymbol{a}_z$

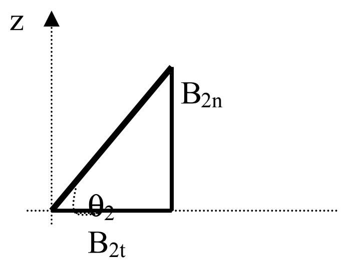
$\boldsymbol{H}_{2t} = \boldsymbol{H}_{1t} \quad \longrightarrow \quad \dfrac{\boldsymbol{B}_{2t}}{\mu_2} = \dfrac{\boldsymbol{B}_{1t}}{\mu_1}$

$\boldsymbol{B}_{2t} = \dfrac{\mu_2}{\mu_1}\boldsymbol{B}_{1t} = \dfrac{4\mu_o}{2.5\mu_o}(6\boldsymbol{a}_x - 4.2\boldsymbol{a}_y) = 9.6\boldsymbol{a}_x - 6.72\boldsymbol{a}_y$

$\boldsymbol{B}_2 = \boldsymbol{B}_{2n} + \boldsymbol{B}_{2t} = 9.6\boldsymbol{a}_x - 6.72\boldsymbol{a}_y + 1.8\boldsymbol{a}_z \text{ mWb/m}^2$

$\boldsymbol{H}_2 = \dfrac{\boldsymbol{B}_2}{\mu_2} = \dfrac{10^{-3}(9.6, -6.72, 1.8)}{4\times 4\pi\times 10^{-7}}$

$\quad = \underline{\underline{1,909.86\,\boldsymbol{a}_x - 1,336.9\boldsymbol{a}_y + 358.1\boldsymbol{a}_z \text{ A/m}}}$



$\tan\theta_2 = \dfrac{B_{2n}}{B_{2t}} = \dfrac{1.8}{\sqrt{9.6^2 + 6.72^2}} = 0.1536$

$\theta_2 = \underline{\underline{8.73^o}}$

Prob. 8.30

(a) $\quad \mathbf{B}_{1n} = \mathbf{B}_{2n} = 15\boldsymbol{a}_\phi$

$\mathbf{H}_{1t} = \mathbf{H}_{2t} \rightarrow \dfrac{\mathbf{B}_{1t}}{\mu_1} = \dfrac{\boldsymbol{B}_{2t}}{\mu_2}$

$\boldsymbol{B}_{1t} = \dfrac{\mu_1}{\mu_2}\boldsymbol{B}_{2t} = \dfrac{2}{5}\left(10\boldsymbol{a}_\rho - 20\boldsymbol{a}_z\right) = 4\boldsymbol{a}_\rho - 8\boldsymbol{a}_z$

Hence,

$\boldsymbol{B}_1 = \underline{\underline{4\boldsymbol{a}_\rho + 15\boldsymbol{a}_\phi - 8\boldsymbol{a}_z \text{ mWb/m}^2}}$

(b) $\quad w_{m1} = \dfrac{1}{2}\boldsymbol{B}_1\cdot\boldsymbol{H}_1 = \dfrac{B_1^2}{2\mu_1} = \dfrac{\left(4^2 + 15^2 + 8^2\right)\times 10^{-6}}{2\times 2\times 4\pi\times 10^{-7}}$

$w_{m1} = \underline{\underline{60.68 \;\; J/m^3}}$

$w_{m2} = \dfrac{B_2^2}{2\mu_2} = \dfrac{\left(10^2 + 15^2 + 20^2\right)\times 10^{-6}}{2\times 5\times 4\pi\times 10^{-7}} = \underline{\underline{57.7 \;\; J/m^3}}$

Prob. 8.31

$$\boldsymbol{B}_{2n} = \boldsymbol{B}_{1n} = 40\boldsymbol{a}_x$$

$$\boldsymbol{B}_{2n} = \mu_2 \boldsymbol{H}_{2n} \quad \longrightarrow \quad \boldsymbol{H}_{2n} = \frac{40\boldsymbol{a}_x}{\mu_2} = \frac{40\boldsymbol{a}_x}{50\mu_o}$$

$$\boldsymbol{H}_{2t} = \boldsymbol{H}_{1t} \quad \longrightarrow \quad \frac{\boldsymbol{B}_{2t}}{\mu_2} = \frac{\boldsymbol{B}_{1t}}{\mu_1}$$

$$\boldsymbol{B}_{2t} = \frac{\mu_2}{\mu_1}\boldsymbol{B}_{1t}$$

$$\boldsymbol{H}_{2t} = \frac{\boldsymbol{B}_{2t}}{\mu_2} = \frac{\boldsymbol{B}_{1t}}{\mu_1} = \frac{(-30\boldsymbol{a}_x + 10\boldsymbol{a}_y)}{\mu_o}$$

$$\boldsymbol{H}_2 = \boldsymbol{H}_{2n} + \boldsymbol{H}_{2t} = \frac{1}{\mu_o}(\frac{40}{50}, -30, 10)\cdot 10^{-3} = \frac{10^{-3}}{4\pi\times 10^{-7}}(0.8, -30, 10)$$

$$\boldsymbol{H}_2 = \underline{\underline{0.6366\boldsymbol{a}_x - 23.87\boldsymbol{a}_y + 7.957\boldsymbol{a}_z \quad \text{kA/m}}}$$

Prob. 8.32

$$\boldsymbol{H}_{2t} = \boldsymbol{H}_{1t} = \alpha\boldsymbol{a}_x + \delta\boldsymbol{a}_z$$

$$\boldsymbol{B}_{2n} = \boldsymbol{B}_{1n} \quad \longrightarrow \quad \mu_2 \boldsymbol{H}_{2n} = \mu_1 \boldsymbol{H}_{1n}$$

$$\boldsymbol{H}_{2n} = \frac{\mu_1}{\mu_2}\boldsymbol{H}_{1n} = \frac{\mu_{r1}}{\mu_{r2}}\beta\boldsymbol{a}_y$$

$$\boldsymbol{H} = \underline{\underline{\alpha\boldsymbol{a}_x + \frac{\mu_{r1}}{\mu_{r2}}\beta\boldsymbol{a}_y + \delta\boldsymbol{a}_z}}$$

Prob. 8.33

$$\boldsymbol{B}_{2n} = \boldsymbol{B}_{1n} = 0.6\boldsymbol{a}_y$$

$$\boldsymbol{H}_{2t} = \boldsymbol{H}_{1t} \quad \longrightarrow \quad \frac{\boldsymbol{B}_{2t}}{\mu_2} = \frac{\boldsymbol{B}_{1t}}{\mu_1}$$

$$\boldsymbol{B}_{1t} = \frac{\mu_1}{\mu_2}\boldsymbol{B}_{2t} = \frac{\mu_o}{12\mu_o}(1.4\boldsymbol{a}_x - 2\boldsymbol{a}_z) = 0.1167\boldsymbol{a}_x - 0.1667\boldsymbol{a}_z$$

$$\boldsymbol{B}_1 = \boldsymbol{B}_{1n} + \boldsymbol{B}_{1t} = \underline{\underline{0.1167\boldsymbol{a}_x + 0.6\boldsymbol{a}_y - 0.1667\boldsymbol{a}_z \quad \text{Wb/m}^2}}$$

$$\boldsymbol{H}_1 = \frac{\boldsymbol{B}_1}{\mu_1} = \frac{10^{-3}(0.1167, 0.6, -0.1667)}{4\pi\times 10^{-7}}$$

$$\underline{\underline{= (0.0929\,\boldsymbol{a}_x + 0.4775\boldsymbol{a}_y - 0.1327\boldsymbol{a}_z)\cdot 10^6 \quad \text{A/m}}}$$

Prob. 8.34

$$f(x,y,z) = x - y + 2z$$
$$\nabla f = \boldsymbol{a}_x - \boldsymbol{a}_y + 2\boldsymbol{a}_z$$
$$a_n = \frac{\nabla f}{|\nabla f|} = \frac{1}{\sqrt{6}}(\boldsymbol{a}_x - \boldsymbol{a}_y + 2\boldsymbol{a}_z)$$

(a)

$$\boldsymbol{H}_{1n} = (\boldsymbol{H}_1 \bullet \boldsymbol{a}_n)\boldsymbol{a}_n = (40-20-60)\frac{(\boldsymbol{a}_x - \boldsymbol{a}_y + 2\boldsymbol{a}_z)}{6}$$
$$= \underline{\underline{-6.667\boldsymbol{a}_x + 6.667\boldsymbol{a}_y - 13.333\boldsymbol{a}_z \text{ A/m}}}$$

(b)

$$\boldsymbol{H}_2 = \boldsymbol{H}_{2n} + \boldsymbol{H}_{2t}$$

But $\quad \boldsymbol{B}_{2n} = \boldsymbol{B}_{1n} \quad \longrightarrow \quad \mu_2 \boldsymbol{H}_{2n} = \mu_1 \boldsymbol{H}_{1n}$

$$\boldsymbol{B}_2 = \mu_2 \boldsymbol{H}_2 = \mu_2 \boldsymbol{H}_{2n} + \mu_2 \boldsymbol{H}_{2t} = \mu_1 \boldsymbol{H}_{1n} + \mu_2 \boldsymbol{H}_{2t} = \mu_o(2\boldsymbol{H}_{1n} + 5\boldsymbol{H}_{2t})$$
$$= 4\pi\times10^{-7}\left[(-13.333, 13.333, -26.667) + (233.333, 66.666, -83.333\right]$$
$$= 4\pi\times10^{-7}(220, 80, -110)$$
$$= \underline{\underline{276.5\boldsymbol{a}_x + 100.5\boldsymbol{a}_y - 138.2\boldsymbol{a}_z \quad \mu\text{Wb/m}^2}}$$

Prob. 8.35

$$\boldsymbol{a}_n = \boldsymbol{a}_\rho$$
$$\boldsymbol{B}_{2n} = \boldsymbol{B}_{1n} = 22\mu_o \boldsymbol{a}_\rho$$
$$\boldsymbol{H}_{2t} = \boldsymbol{H}_{1t} \quad \longrightarrow \quad \frac{\boldsymbol{B}_{2t}}{\mu_2} = \frac{\boldsymbol{B}_{1t}}{\mu_1}$$
$$\boldsymbol{B}_{2t} = \frac{\mu_2}{\mu_1}\boldsymbol{B}_{1t} = \frac{\mu_o}{800\mu_o}(45\mu_o \boldsymbol{a}_\phi) = 0.05625\mu_o \boldsymbol{a}_\phi$$

$$\boldsymbol{B}_2 = \underline{\underline{\mu_o(22\boldsymbol{a}_\rho + 0.05625\boldsymbol{a}_\phi) \text{ Wb/m}^2}}$$

Prob. 8.36

r = a is the interface between the two media.

$$\boldsymbol{B}_{2n} = \boldsymbol{B}_{1n} \quad \longrightarrow \quad B_{o1}(1+1.6)\cos\theta \boldsymbol{a}_r = B_{o2}\cos\theta \boldsymbol{a}_r$$
$$2.6B_{o1} = B_{o2} \qquad (1)$$

$$\boldsymbol{H}_{2t} = \boldsymbol{H}_{1t} \longrightarrow \frac{\boldsymbol{B}_{2t}}{\mu_2} = \frac{\boldsymbol{B}_{1t}}{\mu_1}$$

$$\mu_2 \boldsymbol{B}_{1t} = \mu_1 \boldsymbol{B}_{2t}$$

$$\mu_2 B_{o1}(-0.2)\sin\theta \boldsymbol{a}_\theta = \mu_o B_{o2}(-\sin\theta)\boldsymbol{a}_\theta$$

$$\mu_2 = \frac{\mu_o B_{o2}}{0.2 B_{o1}} \qquad (2)$$

Substituting (1) into (2) gives

$$\mu_2 = \frac{\mu_o}{0.2}(2.6) = \underline{\underline{13\mu_o}}$$

Prob. 8.37

(a) $$\boldsymbol{H} = \tfrac{1}{2}\boldsymbol{K}\times\boldsymbol{a}_n = \tfrac{1}{2}(30-40)\boldsymbol{a}_x \times(-\boldsymbol{a}_z) = \underline{\underline{-5\boldsymbol{a}_y}} \text{ A/m}$$

$$\boldsymbol{B} = \mu_o\boldsymbol{H} = 4\pi\times10^{-7}(-5\boldsymbol{a}_y) = \underline{\underline{-6.28\boldsymbol{a}_y}}\mu \text{ Wb/m}^2$$

(b) $$\boldsymbol{H} = \tfrac{1}{2}(-30-40)\boldsymbol{a}_y = \underline{\underline{-35\boldsymbol{a}_y}} \text{ A/m}$$

$$\boldsymbol{B} = \mu_o\mu_r\boldsymbol{H} = 4\pi\times10^{-7}(2.5)(-35\boldsymbol{a}_y) = \underline{\underline{-110\boldsymbol{a}_y \ \mu\text{Wb/m}^2}}$$

(c) $$\boldsymbol{H} = \tfrac{1}{2}(-30+40)\boldsymbol{a}_y = \underline{\underline{5\boldsymbol{a}_y}}$$

$$\boldsymbol{B} = \mu_o\boldsymbol{H} = \underline{\underline{6.283\boldsymbol{a}_y}}\mu \text{ Wb/m}^2$$

Prob. 8.38

$$\boldsymbol{H}_{1n} = -3\boldsymbol{a}_z, \qquad \boldsymbol{H}_{1t} = 10\boldsymbol{a}_x + 15\boldsymbol{a}_y$$

$$\boldsymbol{H}_{2t} = \boldsymbol{H}_{1t} = 10\boldsymbol{a}_x + 15\boldsymbol{a}_y$$

$$\boldsymbol{H}_{2n} = \frac{\mu_1}{\mu_2}\boldsymbol{H}_{1n} = \frac{1}{200}(-3\boldsymbol{a}_z) = -0.015\boldsymbol{a}_z$$

$$\boldsymbol{H}_2 = 10\boldsymbol{a}_x + 15\boldsymbol{a}_y - 0.015\boldsymbol{a}_z$$

$$\boldsymbol{B}_2 = \mu_2\boldsymbol{H}_2 = 200\times4\pi\times10^{-7}(10, 15, -0.015)$$

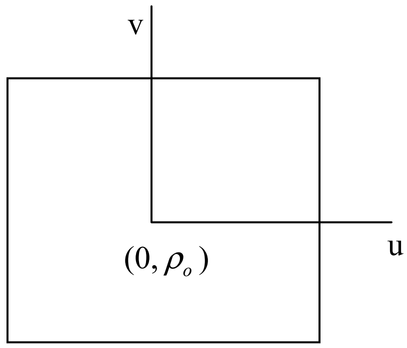
$$\boldsymbol{B}_2 = \underline{\underline{2.51\boldsymbol{a}_x + 3.77\boldsymbol{a}_y - 0.0037\boldsymbol{a}_z}} \text{ mWb/m}^2$$

$$\tan\alpha = \frac{B_{2n}}{B_{2t}}$$

or $$\alpha = \tan^{-1}\frac{0.0037}{\sqrt{2.51^2+3.77^2}} = \underline{\underline{0.047^\circ}}$$

Prob. 8.39

(a) The square cross-section of the toroid is shown below. Let (u,v) be the local coordinates and ρ_o=mean radius. Using Ampere's law around a circle passing through P, we get



$$H(2\pi)(\rho_o + v) = NI \quad \longrightarrow \quad H = \frac{NI}{2\pi(\rho_o + v)}$$

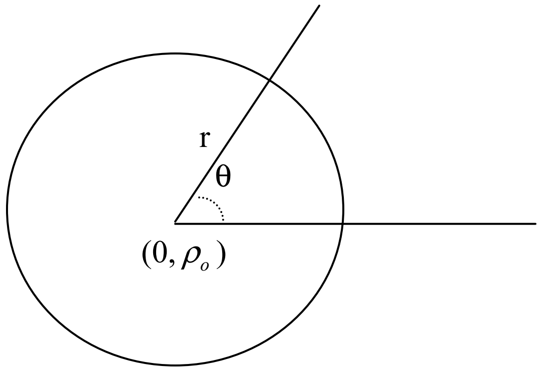
The flux per turn is

$$\Psi = \int_{u=-a/2}^{a/2} \int_{v=-a/2}^{a/2} B du dv = \frac{\mu_o N I a}{2\pi} \ln\left(\frac{\rho_o + a/2}{\rho_o - a/2}\right)$$

$$L = \frac{N\Psi}{I} = \frac{\mu_o N^2 a}{2\pi} \ln\left(\frac{2\rho_o + a}{2\rho_o - a}\right)$$

(b) The circular cross-section of the toroid is shown below. Let (r,θ) be the local coordinates. Consider a point P($r\cos\theta, \rho_o + r\sin\theta$) and apply Ampere's law around a circle that passes through P.

$$H(2\pi)(\rho_o + r\sin\theta) = NI \quad \longrightarrow \quad H = \frac{NI}{2\pi(\rho_o + r\sin\theta)} \approx \frac{NI}{2\pi\rho_o}\left(1 - \frac{r\sin\theta}{\rho_o}\right)$$



Flux per turn $\Psi = \int_{r=0}^{a}\int_{\theta}^{2\pi} \frac{\mu NI}{2\pi\rho_o}\left(1-\frac{r\sin\theta}{\rho_o}\right) r dr d\theta = \frac{\mu NI}{2\pi\rho_o}\frac{a^2}{2}(2\pi)$

$$L = \frac{N\Psi}{I} = \frac{\mu N^2 a^2}{2\rho_o}$$

Or from Example 8.10,

$$L = L'l = \frac{\mu_o N^2 lS}{l^2} = \frac{\mu_o N^2 \pi a^2}{2\pi\rho_o} = \frac{\mu_o N^2 a^2}{2\rho_o}$$

Prob. 8.40

$$\rho_o = \frac{1}{2}(3+5) = 4\text{cm}$$

$$a = 2 \text{ cm}$$

$$L = \frac{\mu_o N^2 a}{2\pi}\ln\left[\frac{2\rho_o + a}{2\rho_o - a}\right]$$

$$N^2 = \frac{2\pi L}{\mu_o a \ln\left[\frac{2\rho_o + a}{2\rho_o - a}\right]} = \frac{2\pi(45\times10^{-6})}{4\pi\times10^{-7}(2\times10^{-2})\ln\left(\frac{8+2}{8-2}\right)} = 22,023.17$$

$$\underline{\underline{N = 148.4 \text{ or } 148}}$$

Prob. 8.41

$$L = \frac{\mu_o \ell}{8\pi} \quad \longrightarrow \quad \frac{L}{\ell} = \frac{4\pi\times10^{-7}}{8\pi} = \underline{\underline{50 \text{ nH/m}}}$$

Prob. 8.42

$$L_{in} = \frac{\mu_o \ell}{8\pi}, \qquad L_{ext} = \frac{\mu_o \ell}{2\pi}\ln(b/a)$$

If $L_{in} = 2L_{ext} \longrightarrow \frac{\mu_o \ell}{8\pi} = \frac{\mu_o \ell}{\pi}\ln(b/a)$

$$\ln(b/a) = \frac{1}{8} \qquad \frac{b}{a} = e^{1/8} = 1.1331$$

$$b = 1.1331a = \underline{\underline{7.365 \text{ mm}}}$$

Prob. 8.43

$$L' = \frac{\mu}{2\pi}\left[\frac{1}{4} + \ln\frac{b}{a}\right] = \frac{4\pi\times10^{-7}}{2\pi}[0.25 + \ln(6/2.5)] = \underline{\underline{225 \text{ nH}}}$$

Prob. 8.44

$$\psi_{12} = \int \boldsymbol{B}_1 \bullet d\boldsymbol{S} = \int_{\rho=\rho_o}^{\rho_o+a}\int_{z=0}^{b} \frac{\mu_o I}{2\pi\rho} dz d\rho = \frac{\mu_o I b}{2\pi}\ln\frac{a+\rho_o}{\rho_o}$$

$$M_{12} = \frac{N\psi_{12}}{I} = \frac{N\mu_o b}{2\pi}\ln\frac{a+\rho_o}{\rho_o}$$

For N = 1,

$$M_{12} = \frac{\psi_{12}}{I_1} = \frac{\mu_o b}{2\pi}\ln\frac{a+\rho_o}{\rho_o}$$

$$= \frac{4\pi\times10^{-7}}{2\pi}(1)\ln 2 = \underline{\underline{0.1386\mu}}\text{ H}$$

Prob. 8.45

$$\boldsymbol{H} = \frac{I}{2\pi\rho}\boldsymbol{a}_\rho$$

$$w_m = \frac{1}{2}\mu|\boldsymbol{H}|^2 = \frac{1}{2}\mu\frac{I^2}{4\pi^2\rho^2}$$

$$W = \int w_m dv = \iiint \frac{1}{2}\mu\frac{I^2}{4\pi^2\rho^2}\rho d\phi d\rho dz = \frac{1}{4\pi}\mu I^2 L\ln(b/a)$$

$$= \frac{1}{4\pi}\times 4\times 4\pi\times 10^{-7}(625\times 10^{-6})3\ln(18/12) = \underline{\underline{304.1 \text{ pJ}}}$$

Alternatively,

$$W = \frac{1}{2}LI^2 = \frac{1}{2}\frac{\mu L}{2\pi}\ln\frac{b}{a}\times I^2 = \frac{\mu I^2 L}{4\pi}\ln\frac{b}{a}$$

Prob. 8.46

$$\mu_r = \chi_m + 1 = 20$$

$$w_m = \frac{1}{2}\boldsymbol{B}_1\cdot\boldsymbol{H}_1 = \frac{1}{2}\mu\mathbf{H}\cdot\mathbf{H}$$

$$= \frac{1}{2}\mu\left(25x^4y^2z^2 + 100x^2y^4z^2 + 225x^2y^2z^4\right)$$

$$W_m = \int w_m dv$$

$$= \frac{1}{2}\mu\left[25\int_0^1 x^4dx\int_0^2 y^2dy\int_{-1}^2 z^2dz + 100\int_0^1 x^2dx\int_0^2 y^4dy\int_{-1}^2 z^2dz\right.$$

$$\left. + 225\int_0^1 x^2dx\int_0^2 y^2dy\int_{-1}^2 zdz\right]$$

$$= \frac{25\mu}{2}\left[\left.\frac{x^5}{5}\right|_0^1 \left.\frac{y^3}{3}\right|_0^2 \left.\frac{z^3}{3}\right|_{-1}^2 + 4\left.\frac{x^3}{3}\right|_0^1 \left.\frac{y^5}{5}\right|_0^2 \left.\frac{z^3}{3}\right|_{-1}^2\right.$$

$$\left. + 9\left.\frac{x^3}{3}\right|_0^1 \left.\frac{y^3}{3}\right|_0^2 \left.\frac{z^5}{5}\right|_{-1}^2\right]$$

$$= \frac{25\mu}{2}\left(\frac{1}{5}\cdot\frac{8}{3}\cdot\frac{9}{3} + \frac{4}{3}\cdot\frac{32}{3}\cdot\frac{9}{3} + \frac{9}{3}\cdot\frac{8}{3}\cdot\frac{33}{5}\right)$$

$$= \frac{25}{2}\times 4\pi\times 10^{-7}\times 20\times\frac{3600}{45}$$

$$W_m = \underline{\underline{25.13 \text{ mJ}}}$$

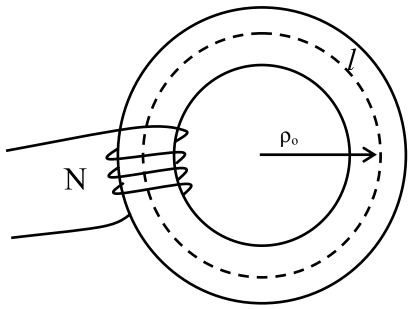
Prob. 8.47

$$W = \frac{1}{2}\int_v \mu H^2 dv = \frac{1}{2}\iiint 4.5 \times 4\pi \times 10^{-7}[200^2 + 500^2]10^{-6}\,dxdydz$$

$$= 2\pi(4.5)10^{-7}(29 \times 10^4)10^{-6}(2)(2)(2)10^{-6} = \underline{\underline{6.56 \text{ pJ}}}$$

Prob. 8.48

$$NI = Hl = \frac{Bl}{\mu}$$



$$N = \frac{Bl}{\mu_o \mu_r I} = \frac{1.5 \times 0.6\pi}{4\pi \times 10^{-7} \times 600 \times 12}$$

$$= \underline{\underline{313 \text{ turns}}}$$

Prob. 8.49

F = NI = 400 x 0.5 = 200 A.t

$$R_a = \frac{100}{4\pi} \text{ MAt/Wb}, \quad R_1 = R_2 = \frac{6}{4\pi} \text{ MAt/Wb}, \quad R_3 = \frac{1.8}{4\pi} \text{ MAt/Wb}$$

$$F_a = \frac{R_a F}{R_a + R_3 + R_1 // R_2} = \underline{\underline{190.8}} \text{ A.t}$$

$$H_a = \frac{F_a}{l_a} = \frac{190.8}{1 \times 10^{-2}} = \underline{\underline{19080}} \text{ A/m}$$

Prob. 8.50

Total F = NI = 2000 x 10 = 20,000 A.t

$$R_c = \frac{l_c}{\mu_o \mu_r S} = \frac{(24 + 20 - 0.6) \times 10^{-2}}{4\pi \times 10^{-7} \times 1500 \times 2 \times 10^{-4}} = \underline{\underline{0.115 \times 10^7 \text{ A.t/m}}}$$

$$R_a = \frac{l_a}{\mu_o \mu_r S} = \frac{0.6 \times 10^{-2}}{4\pi \times 10^{-7}(1) \times 2 \times 10^{-4}} = \underline{\underline{2.387 \times 10^7 \text{ A.t/m}}}$$

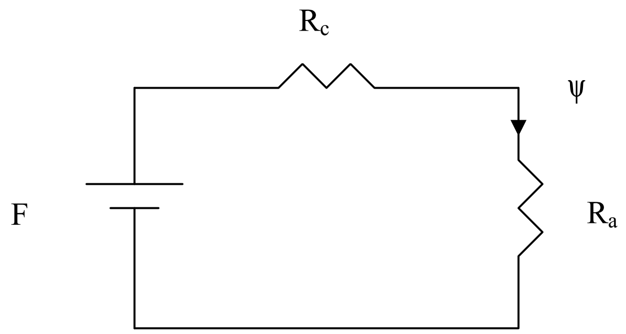
R = R_a + R_c = 2.502 x 10^7 A.t/m

$$\psi = \frac{\Im}{R} = \psi_a = \psi_c = \frac{20{,}000}{2.502 \times 10^7} = \underline{\underline{8 \times 10^{-4} \text{ Wb/m}^2}}$$

$$\Im_a = \frac{R_a}{R_a + R_c}\Im = \frac{2.387 \times 20{,}000}{2.502} = \underline{\underline{19{,}081}} \text{ A.t}$$

$$\Im_c = \frac{R_c}{R_a + R_c}\Im = \frac{0.115 \times 20{,}000}{2.502} = \underline{\underline{919}} \text{ A.t}$$

Prob. 8.51



F = NI = 500 x 0.2 = 100 A.t

$$R_c = \frac{l_c}{\mu S} = \frac{42\times10^{-2}}{4\pi\times10^{-7}\times10^{3}\times4\times10^{-4}} = \frac{42\times10^{6}}{16\pi}$$

$$R_a = \frac{l_a}{\mu_o S} = \frac{10^{-3}}{4\pi\times10^{-7}\times4\times10^{-4}} = \frac{10^{8}}{16\pi}$$

$$R_a + R_c = \frac{1.42\times10^{8}}{16\pi}$$

$$\psi = \frac{F}{R_a + R_c} = \frac{16\pi\times100}{1.42\times10^{8}} = \frac{16\pi}{1.42} \quad \mu\text{Wb}$$

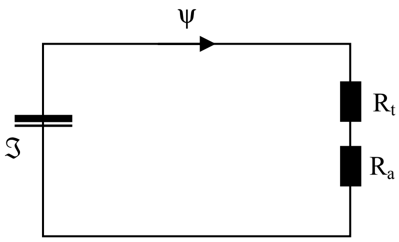
$$B_a = \frac{\psi}{S} = \frac{16\pi\times10^{-6}}{1.42\times4\times10^{-4}} = \underline{\underline{88.5 \text{ mWb/m}^2}}$$

Prob. 8.52

$$F=\frac{B^2S}{2\mu_o}=\frac{\psi^2}{2\mu_oS}=\frac{4\times10^{-6}}{2\times4\pi\times10^{-7}\times0.3\times10^{-4}}=\underline{\underline{53.05}}\ \text{kN}$$

Prob. 8.53

(a) F = NI = 200 x 10^{-3} x 750 = 150 A.t.



$$R_a=\frac{l_a}{\mu_oS}=\frac{10^{-3}}{25\times10^{-6}\mu_o}=3.183\times10^7$$

$$R_t=\frac{l_t}{\mu_o\mu_rS}=\frac{2\pi\times0.1}{\mu_o\times300\times25\times10^{-6}}=6.7\text{ x }10^7$$

$$\psi=\frac{\Im}{R_a+R_t}=\frac{150}{10^7(3.183+20/3)}=15.23\times10^{-7}$$

$$F=\frac{B^2S}{2\mu_o}=\frac{\psi^2}{2\mu_oS}=\frac{2.32\times10^{-12}}{2\times4\pi\times10^{-7}\times25\times10^{-6}}$$

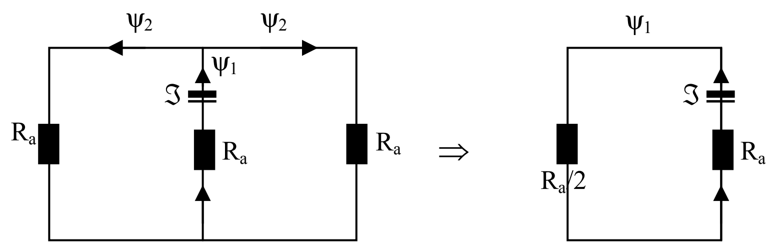
$= \underline{37\ \text{mN}}$

(b) If $\mu_t\to\infty,\ R_t=0,\ \psi=\dfrac{\Im}{R_a}=\dfrac{150}{3.183\times10^7}$

$$F_2=I_2dl_2\bullet B_1=I_2dl_2\frac{\psi_1}{S}=\frac{2\times10^{-3}\times5\times10^{-3}\times150}{3.183\times10^7\times25\times10^{-6}}$$

$F_2 = \underline{1.885\ \mu\text{N}}$

Prob. 8.54



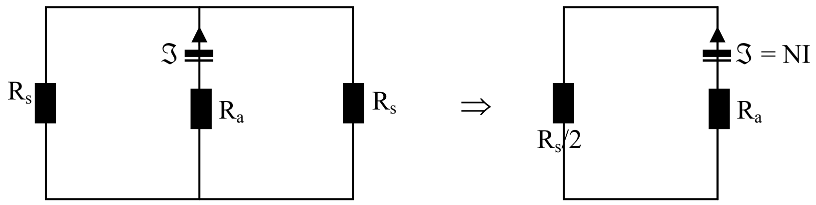
$$\psi_1=2\psi_2,\psi_1=\frac{\Im}{{}^3\!/_2R_a}=\frac{2\Im}{3R_a}\to\psi_2=\frac{\Im}{3R_a}$$

$$\Im=2\left(\frac{\psi_2{}^2}{2\mu_oS}\right)+\frac{\psi_1}{2\mu_oS}=\frac{3\psi_1{}^2}{4\mu_oS}=\frac{\Im^2}{3R_a{}^2\mu_oS}$$

$$= \frac{\mu_o S \Im^2}{3l_a^{\ 2}} = \frac{4\pi\times10^{-7}\times200\times10^{-4}\times9\times10^{6}}{3\times10^{-6}}$$

$$= 24\pi\times10^{3} = mg \rightarrow m = \frac{24\pi\times10^{3}}{9.8} = \underline{\underline{7694}}\ \text{kg}$$

Prob. 8.55



Since $\mu \rightarrow \infty$ for the core (see Figure), $R_c = 0$.

$$\Im = NI = \psi\left(R_a + \frac{R_s}{2}\right) = \frac{\psi(a/2 + x)}{\mu_o S}$$

$$= \frac{\psi(2x+a)}{2\mu_o S}$$

$$\Im = \frac{B^2 S}{2\mu_o} = \psi^2 \frac{1}{2\mu_o S} = \frac{1}{2\mu_o S} \bullet \frac{N^2 I^2 4\mu_o^2 S^2}{(a+2x)^2}$$

$$= \frac{2N^2 I^2 \mu_o S}{(a+2x)^2}$$

$\boldsymbol{F} = -F\boldsymbol{a}_x$ since the force is attractive, i.e.

$$\boldsymbol{F} = \frac{-2N^2 I^2 \mu_o S \boldsymbol{a}_x}{(a+2x)^2}$$

CHAPTER 9

P.E. 9.1

(a) $V_{emf} = \int (\boldsymbol{u} \times \boldsymbol{B}) \cdot d\boldsymbol{l} = uBl = 8(0.5)(0.1) = \underline{\underline{0.4}}$ V

(b) $I = \dfrac{V_{emf}}{R} = \dfrac{0.4}{20} = \underline{\underline{20}}$ mA

(c) $\boldsymbol{F}_m = I\boldsymbol{l} \times \boldsymbol{B} = 0.02(-0.1\boldsymbol{a}_y \times 0.5\boldsymbol{a}_z) = \underline{\underline{-\boldsymbol{a}_x}}$ mN

(d) $P = FU = I^2 R = 8$ mW

or $\quad P = \dfrac{V_{emf}}{R} = \dfrac{(0.4)^2}{20} = \underline{\underline{8}}$ mW

P.E. 9.2

(a) $V_{emf} = \int (\boldsymbol{u} \times \boldsymbol{B}) \cdot d\boldsymbol{l}$

where $\boldsymbol{B} = B_o \boldsymbol{a}_y = B_o(\sin\phi \boldsymbol{a}_\rho + \cos\phi \boldsymbol{a}_\phi)$, $B_o = 0.05$ Wb/m^2

$$(\boldsymbol{u} \times \boldsymbol{B}) \cdot d\boldsymbol{l} = -\rho\omega B_o \sin\phi dz = -0.2\pi \sin\left(\omega t + \pi/2\right) dz$$

$$V_{emf} = \int_0^{0.03} (\boldsymbol{u} \times \boldsymbol{B}) \cdot d\boldsymbol{l} = -6\pi\cos(100\pi t) \text{ mV}$$

At t = 1ms,

$$V_{emf} = -6\pi\cos 0.1\pi = \underline{\underline{-17.93}} \text{ mV}$$

$$i = \frac{V_{emf}}{R} = -60\pi\cos(100\pi.t) \text{ mA}$$

At t = 3ms, $i = -60\pi\cos 0.3\pi = \underline{\underline{-110.8 \text{ mA}}}$

(b) Method 1:

$$\Psi = \int \boldsymbol{B} \cdot d\boldsymbol{S} = \int B_o t(\cos\phi \boldsymbol{a}_\rho - \sin\phi \boldsymbol{a}_\phi) \cdot d\rho dz \boldsymbol{a}_\phi = -\int_0^{\rho_o}\int_0^{z_o} B_o t \sin\phi d\rho dz = -B_o \rho_o z_o t \sin\phi$$

where $B_o = 0.02$, $\rho_o = 0.04$, $z_o = 0.03$

$$\phi = \omega t + \pi/2$$

$$\Psi = -B_o \rho_o z_o t \cos\omega t$$

$$V_{emf} = -\frac{\partial \Psi}{\partial t} = B_o \rho_o z_o \cos\omega t - B_o \rho_o z_o t \omega \sin\omega t$$

$$=(0.02)(0.04)(0.03)[\cos\omega t-\omega t\sin\omega t]$$

$$=24[\cos\omega t-\omega t\sin\omega t]\,\mu V$$

Method 2:

$$V_{emf}=-\int\frac{\partial \boldsymbol{B}}{\partial t}\bullet d\boldsymbol{S}+\int(\boldsymbol{u}\times\boldsymbol{B}).d\boldsymbol{l}$$

$$\boldsymbol{B}=B_o t\boldsymbol{a}_x=B_o t(\cos\phi\boldsymbol{a}_\rho-\sin\phi\boldsymbol{a}_\phi),\phi=\omega t+\pi/2$$

$$\frac{\partial \boldsymbol{B}}{\partial t}=B_o(\cos\phi\boldsymbol{a}_\rho-\sin\phi\boldsymbol{a}_\phi)$$

Note that only explicit dependence of ***B*** on time is accounted for, i.e. we make φ = constant because it is transformer (stationary) emf. Thus,

$$V_{emf}=-B_o\int_0^{\rho_o}\int_0^{z_o}(\cos\phi\boldsymbol{a}_\rho-\sin\phi\boldsymbol{a}_\phi)\bullet d\rho dz\boldsymbol{a}_\phi+\int_{z_o}^{0}-\rho_o\omega B_o t\cos\phi dz$$

$$=B_o\rho_o z_o(\sin\phi-\omega t\cos\phi),\phi=\omega t+\pi/2$$

$$=B_o\rho_o z_o(\cos\omega t-\omega t\sin\omega t)$$ as obtained earlier.

At t = 1ms,

$$V_{emf}=24[\cos 18^o-100\pi\times10^{-3}\sin 18^o]\mu V$$

$$=\underline{\underline{20.5\mu V}}$$

At t = 3ms,

$$i=240[\cos 54^o-.03\pi\sin 54^o]mA$$

$$=\underline{\underline{-41.93\text{ mA}}}$$

P.E. 9.3

$$V_1=-N_1\frac{d\psi}{dt},V_2=-N_2\frac{d\psi}{dt}$$

$$\frac{V_2}{V_1}=\frac{N_2}{N_1}\to V_2=\frac{N_2}{N_1}V_1=\frac{300\times120}{500}=\underline{\underline{72V}}$$

P.E. 9.4

(a) $$\boldsymbol{J}_d=\frac{\partial\boldsymbol{D}}{\partial t}=\underline{\underline{-20\omega\varepsilon_o\sin(\omega t-50x)\boldsymbol{a}_y A/m^2}}$$

(b) $\nabla \times \boldsymbol{H} = \boldsymbol{J}_d \rightarrow -\frac{\partial H_z}{\partial x}\boldsymbol{a}_y = -20\omega\varepsilon_o \sin(\omega t - 50x)\boldsymbol{a}_y$

$$\text{or } \boldsymbol{H} = \frac{20\omega\varepsilon_o}{50}\cos(\omega t - 50x)\boldsymbol{a}_z$$

$$= \underline{\underline{0.4\omega\varepsilon_o \cos(\omega t - 50x)\boldsymbol{a}_z}} \text{ A/m}$$

(c) $\nabla \times \boldsymbol{E} = -\mu_o \frac{\partial \boldsymbol{H}}{\partial t} \rightarrow \frac{\partial E_y}{\partial x}\boldsymbol{a}_z = 0.4\mu_o\omega^2\varepsilon_o \sin(\omega t - 50x)\boldsymbol{a}_z$

$$1000 = 0.4\mu_o\varepsilon_o\omega^2 = 0.4\frac{\omega^2}{c^2}$$

or $\omega = \underline{1.5 \text{ x } 10^{10} \text{ rad/s}}$

P.E. 9.5

(a) $j^3\left(\frac{1+j}{2-j}\right)^2 = -j\left[\frac{\sqrt{2}\angle 45^o}{\sqrt{5}\angle -26.56^o}\right]^2 = -j\left(2/5\angle 143.13^o\right)$

$$= \underline{\underline{0.24 + j0.32}}$$

(b) $6\angle 30^o + j5 - 3 + e^{j45^o} = 5.196 + j3 + j5 - 3 + 0.7071(1+j)$

$$= \underline{\underline{2.903 + j8.707}}$$

P.E. 9.6

$$\boldsymbol{P} = 2\sin(10t + x - \pi/4)\boldsymbol{a}_y = 2\cos\left(10t + x - \pi/4 - \pi/2\right)\boldsymbol{a}_y, w = 10$$

$$= R_e\left(2e^{j(x-3\pi/4)}\boldsymbol{a}_y e^{jwt}\right) = R_e\left(\boldsymbol{P}_s e^{jwt}\right)$$

i.e. $\boldsymbol{P}_s = \underline{\underline{2e^{j(x-3\pi/4)}\boldsymbol{a}_y}}$

$$\boldsymbol{Q} = R_e\left(\boldsymbol{Q}_s e^{jwt}\right) = R_e\left(e^{j(x+wt)}(\boldsymbol{a}_x - \boldsymbol{a}_z)\right)\sin \pi y$$

$$= \underline{\underline{\sin \pi y \cos(wt + x)(\boldsymbol{a}_x - \boldsymbol{a}_z)}}$$

P.E. 9.7

$$-\mu\frac{\partial \boldsymbol{H}}{\partial t} = \nabla \times \boldsymbol{E} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(E_\phi \sin\theta)\boldsymbol{a}_r - \frac{1}{r}\frac{\partial}{\partial r}(rE_\phi)\boldsymbol{a}_\theta$$

$$= \frac{2\cos\theta}{r^2}\cos(\omega t-\beta r)\boldsymbol{a}_r - \frac{\beta}{r}\sin\theta\sin(\omega t-\beta r)\boldsymbol{a}_\theta$$

$$\boldsymbol{H} = -\frac{2\cos\theta}{\mu\omega r^2}\sin(\omega t-\beta r)\boldsymbol{a}_r - \frac{\beta}{\mu\omega r}\sin\theta\cos(\omega t-\beta r)\boldsymbol{a}_\theta$$

$$\beta = \frac{\omega}{c} = \frac{6\times10^7}{3\times10^8} = \underline{\underline{0.2}} \text{ rad/m}$$

$$\boldsymbol{H} = \underline{\underline{-\frac{1}{12\pi r^2}\cos\theta\sin(6\times10^7-0.2r)\boldsymbol{a}_r - \frac{1}{120\pi r}\sin\theta\cos(6\times10^7-0.2r)\boldsymbol{a}_\theta}}$$

P.E. 9.8

$$\omega = \frac{3}{\sqrt{\mu\varepsilon}} = \frac{3c}{\sqrt{\mu_r\varepsilon_r}} = \frac{9\times10^8}{\sqrt{10}} = \underline{\underline{2.846\times10^8}} \text{ rad/s}$$

$$\boldsymbol{E} = \frac{1}{\varepsilon}\int\nabla\times\boldsymbol{H}dt = -\frac{6}{\omega\varepsilon}\cos(\omega t-3y)\boldsymbol{a}_x$$

$$= \frac{-6}{\frac{9\times10^8}{\sqrt{10}}\bullet\frac{10^{-9}}{36\pi}(5)}\cos(\omega t-3y)\boldsymbol{a}_x$$

$$\boldsymbol{E} = \underline{\underline{-476.86\cos(2.846\times10^8 t-3y)\boldsymbol{a}_x \text{ V/m}}}$$

P.E. 9.9

$$V = -\frac{\partial\psi}{\partial t} = -\frac{\partial}{\partial t}\int\boldsymbol{B}\bullet d\boldsymbol{S} = -\frac{\partial\boldsymbol{B}}{\partial t}\bullet\boldsymbol{S}$$

$= 3770 \sin377t \text{ x } \pi(0.2)^2 \text{ x } 10^{-3}$

$= \underline{\underline{0.4738 \sin377t \text{ V}}}$

P.E. 9.10

$$V = \int(\boldsymbol{u}\times\boldsymbol{B})\bullet d\boldsymbol{l}$$

$$u = \rho\omega\boldsymbol{a}_\phi, \qquad B = B_o\boldsymbol{a}_z$$

$$V = \int_{\rho=0}^{\ell}\rho\omega B_o d\rho = \frac{1}{2}\omega B_o\rho^2\Big|_0^\ell = \frac{\omega B_o\ell^2}{2}$$

$$= \frac{30}{2}\times60\times10^{-3}(8\times10^{-2})^2 = \underline{\underline{5.76 \text{ mV}}}$$

Prob. 9.1

Measuring the induced emf in the clockwise direction,

$$V_{\text{emf}} = \oint (\boldsymbol{u} \times \boldsymbol{B}) \bullet d\boldsymbol{l}$$

$$= \int_0^{1.2} (5\boldsymbol{a}_x \times 0.2\boldsymbol{a}_z) \bullet dy\boldsymbol{a}_y + \int_{1.2}^{0} (15\boldsymbol{a}_x \times 0.2\boldsymbol{a}_z) dy\boldsymbol{a}_x$$

$$= -\int_0^{1.2} (1)\, dy - \int_{1.2}^{0} (3) dy$$

$$= -1.2 + 1.2 \times 3 = -1.2 + 3.6$$

$$\underline{\underline{= 2.4 \text{ V}}}$$

Prob. 9.2

$$\psi = \boldsymbol{B} \bullet \boldsymbol{S} = (0.2)^2 \pi \bullet 40 \times 10^{-3} \sin 10^4 t$$

$$V = -\frac{\partial \psi}{\partial t} = -16\pi \cos 10^4 t$$

$$i = \frac{V}{R} = \frac{16\pi}{4} \cos 10^4 t$$

$$\underline{\underline{= -12.57 \cos 10^4 t \text{ A}}}$$

Prob. 9.3

$$V_{emf} = -\frac{\partial \psi}{\partial t} = -\frac{\partial}{\partial t} \iint \boldsymbol{B} \bullet d\boldsymbol{S} = -\iint \frac{\partial \boldsymbol{B}}{\partial t} \bullet dxdy\boldsymbol{a}_z$$

$$= \int_{y=0}^{0.1} \int_{x=0}^{0.8} 30\pi \times 40 \sin(30\pi t - 3y) dxdy \text{ mV}$$

$$= 1200\pi \int_0^{0.8} dx \int_0^{0.1} \sin(30\pi t - 3y) dy$$

$$= 1200\pi (0.8) \left(-\frac{1}{-3} \cos(30\pi t - 3y) \Big|_0^{0.1} \right)$$

$$= 320\pi \left[\cos(30\pi t - 0.3) - \cos(30\pi t) \right] \text{ mV}$$

$$I = \frac{V_{emf}}{R} = \frac{V_{emf}}{10+4} = \frac{320\pi}{14} \left[-2\sin(30\pi t - 0.15) \sin(-0.15) \right]$$

$$= 143.62 \sin(30\pi t - 0.15) \sin(0.15)$$

$$I = \underline{\underline{21.46 \sin(30\pi t - 0.15) \text{ mA}}}$$

Prob. 9.4

$$V_{emf} = -\frac{\partial}{\partial t}\int \boldsymbol{B} \bullet d\boldsymbol{S} = -\int \frac{\partial \boldsymbol{B}}{\partial t} \bullet d\boldsymbol{S}$$

$$= -\iint (-4\omega)\sin\omega t\sqrt{x^2+y^2}\,dxdy = 4\omega\sin\omega t\iint \sqrt{x^2+y^2}\,dxdy$$

We change variables from Cartesian to cylindrical coordinates.

$$V_{emf} = 4\omega\sin\omega t\int_{\phi=0}^{2\pi}\int_{\rho=0}^{3} \rho\cdot\rho d\rho d\phi = 4\omega\sin\omega t(2\pi)\frac{\rho^3}{3}\Bigg|_0^3$$

$$= 72\pi\omega\sin\omega t = \underline{\underline{226.2\omega\sin\omega t \text{ V}}}$$

Prob. 9.5

$$\boldsymbol{B} = \frac{\mu_o I}{2\pi y}(-\boldsymbol{a}_x)$$

$$\psi = \int \boldsymbol{B}\bullet d\boldsymbol{S} = \frac{\mu_o I}{2\pi}\int_{z=0}^{a}\int_{y=\rho}^{\rho+a}\frac{dzdy}{y} = \frac{\mu_o I a}{2\pi}\ln\frac{\rho+a}{\rho}$$

$$V_{emf} = -\frac{\partial\psi}{\partial t} = -\frac{\partial\psi}{\partial\rho}\bullet\frac{\partial\rho}{\partial t} = -\frac{\mu_o I a}{2\pi}u_o\frac{d}{d\rho}[\ln(\rho+a)-\ln\rho]$$

$$= -\frac{\mu_o I a}{2\pi}u_o\left[\frac{1}{\rho+a}-\frac{1}{\rho}\right] = \underline{\underline{\frac{\mu_o a^2 I u_o}{2\pi\rho(\rho+a)}}}$$

where $\rho = \rho_o + u_o t$

Prob. 9.6

$$V_{emf} = \int_{\rho}^{\rho+a} 3\boldsymbol{a}_z\times\frac{\mu_o I}{2\pi\rho}\boldsymbol{a}_\phi\bullet d\rho\boldsymbol{a}_\rho = -\frac{3\mu_o I}{2\pi}\ln\frac{\rho+a}{\rho}$$

$$= -\frac{4\pi\times10^{-7}}{2\pi}\times15\times3\ln\frac{60}{20} = -9.888\mu V$$

Thus the induced emf = <u>9.888μV, point A at higher potential.</u>

Prob. 9.7

$$V_{emf} = -N\frac{\partial\psi}{\partial t} = -N\frac{\partial}{\partial t}\int \boldsymbol{B}\bullet d\boldsymbol{S} = -NB\frac{dS}{dt}$$

$$= -NB\ell\frac{d}{dt}(\rho\phi) = -NB\ell\rho\frac{d\phi}{dt} = -NB\ell\rho\omega$$

$$= -50(0.2)(30\times10^{-4})(60) = \underline{\underline{-1.8\text{V}}}$$

Prob. 9.8

<u>Method 1</u>:
We assume that the sliding rode is on $-\ell < z < \ell$
$\ell = x/\sqrt{3} = 5t/\sqrt{3}$

$$V_{emf} = \int (\boldsymbol{u}\times\boldsymbol{B})\bullet d\boldsymbol{l} = \int 5\boldsymbol{a}_x \times 0.6\boldsymbol{a}_z \bullet dy\boldsymbol{a}_y = -3x\int_{-\ell}^{\ell} dy = -3x\times 25t^{23} = \underline{\underline{-86.6025t^t}}$$

<u>Method 2</u>:
The flux linkage is given by

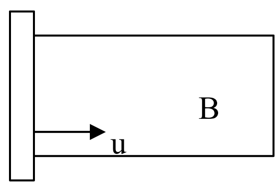
$$\psi = \int_{x=o}^{5t}\int_{y=-x/\sqrt{3}}^{x.\sqrt{3}} 0.6xdxdy = 0.6\times\frac{2}{\sqrt{3}}\times 125t^3/3 = 28,8675t^3$$

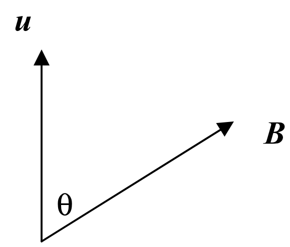
$$V_{emf} = -\frac{d\psi}{dt} = \underline{\underline{-86.602t^2}}$$

Prob. 9.9

$$V_{emf} = uB\ell = 410\times 0.4\times 10^{-6}\times 36 = \underline{\underline{5.904\text{ mV}}}$$

Prob. 10





$$V_{emf} = \int (\boldsymbol{u}\times\boldsymbol{B})\cdot d\boldsymbol{l} = uBl\cos\theta$$

$$= \left(\frac{120\times 10^3}{3600} m/s\right)\left(4.3\times 10^{-5}\right)(1.6)\,cos65^o$$

$$= 2.293cos65^o = \underline{\underline{0.97}}\text{ mV}$$

Prob. 9.11

$d\psi = 0.64 - 0.45 = 0.19$, $dt = 0.02$

$$V_{emf} = N\frac{d\psi}{dt} = 10\left(\frac{0.19}{0.02}\right) = 95V$$

$$I = \frac{V_{emf}}{R} = \left(\frac{95}{15}\right) = \underline{\underline{6.33\ \text{A}}}$$

Using Lenz's law, the direction of the induced current is <u>counterclockwise.</u>

Prob. 9.12

$$V = \int (\boldsymbol{u}\times\boldsymbol{B})\bullet d\boldsymbol{l}\text{, where } \boldsymbol{u} = \rho\omega\boldsymbol{a}_\phi, \quad \boldsymbol{B} = B_o\boldsymbol{a}_z$$

$$V = \int_{\rho_1}^{\rho_2} \rho\omega B_o d\rho = \frac{\omega B_o}{2}(\rho^2{}_2 - \rho^2{}_1)$$

$$V = \frac{60\times 15}{2}\bullet 10^{-3}(100-4)\bullet 10^{-4} = \underline{\underline{4.32\text{ mV}}}$$

Prob. 9.13

$$J_{ds} = j\omega D_s \rightarrow \left|J_{ds}\right|_{\max} = \omega\varepsilon E_s = \omega\varepsilon\frac{V_s}{d}$$

$$= \frac{10^{-9}}{36\pi}\times\frac{2\pi\times 20\times 10^6\times 50}{0.2\times 10^{-3}}$$

$$= \underline{\underline{277.8}}\ \text{A/m}^2$$

$$I_{ds} = J_{ds}\bullet S = \frac{1000}{3.6}\times 2.8\times 10^{-4} = \underline{\underline{77.78}}\ \text{mA}$$

Prob. 9.14

$$J_c = \sigma E, \qquad J_d = \frac{\partial D}{\partial t} = \varepsilon\frac{\partial E}{\partial t}$$

$$|J_c| = \sigma|E|, \qquad |J_d| = \varepsilon\omega|E|$$

If $I_c = I_d$, then $|J_c| = |J_d| \longrightarrow \sigma = \varepsilon\omega$

$$\omega = 2\pi f = \frac{\sigma}{\varepsilon}$$

$$f = \frac{\sigma}{2\pi\varepsilon} = \frac{4}{2\pi\times 9\times\frac{10^{-9}}{36\pi}} = \underline{\underline{8\text{ GHz}}}$$

Prob. 9.15 $\frac{J_c}{J_d} = \frac{\sigma E}{\omega\varepsilon E} = \frac{\sigma}{\omega\varepsilon}$

(a) $\frac{\sigma}{\omega\varepsilon} = \frac{2\times10^{-3}}{2\pi\times10^{9}\times81\times\frac{10^{-9}}{36\pi}} = \underline{\underline{0.444\times10^{-3}}}$

(b) $\frac{\sigma}{\omega\varepsilon} = \frac{25}{2\pi\times10^{9}\times81\times\frac{10^{-9}}{36\pi}} = \underline{\underline{5.555}}$

(c) $\frac{\sigma}{\omega\varepsilon} = \frac{2\times10^{-4}}{2\pi\times10^{9}\times5\times\frac{10^{-9}}{36\pi}} = \underline{\underline{7.2\times10^{-4}}}$

Prob. 9.16

$$\frac{J_d}{J} = \frac{\omega\varepsilon E}{\sigma E} = \frac{\omega\varepsilon}{\sigma} = 1 \quad \longrightarrow \quad \omega = \frac{\sigma}{\varepsilon} = \frac{10^{-4}}{3\times\frac{10^{-9}}{36\pi}} = 12\pi\times10^{5}$$

$$2\pi f = 12\pi\times10^{5} \quad \longrightarrow \quad f = \underline{\underline{600 \text{ kHz}}}$$

Prob. 9.17

$$J_c = \sigma E = 0.4\cos(2\pi\times10^{3}t)$$

$$E = \frac{0.4}{\sigma}\cos(2\pi\times10^{3}t)$$

$$J_d = \varepsilon\frac{\partial E}{\partial t} = -\frac{0.4\varepsilon}{\sigma}(2\pi\times10^{3})\sin(2\pi\times10^{3}t)$$

$$= -\frac{0.4\times4.5\times\frac{10^{-9}}{36\pi}}{10^{-4}}(2\pi\times10^{3})\sin(2\pi\times10^{3}t)$$

$$\underline{\underline{= -100\sin(2\pi\times10^{3}t) \text{ A/m}^2}}$$

Prob. 9.18

(a) $\nabla\bullet\boldsymbol{E}_s = \rho_s/\varepsilon, \nabla\bullet\boldsymbol{H}_s = 0$

$$\underline{\underline{\nabla\times\boldsymbol{E}_s = j\omega\mu\boldsymbol{H}_s, \quad \nabla\times\boldsymbol{H}_s = (\sigma - j\omega\varepsilon)\boldsymbol{E}_s}}$$

(b) $\nabla \bullet \boldsymbol{D} = \rho_v \rightarrow \dfrac{\partial D_x}{\partial x} + \dfrac{\partial D_y}{\partial y} + \dfrac{\partial D_z}{\partial z} = \rho_v$ (1)

$$\nabla \bullet \boldsymbol{B} = 0 \rightarrow \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 \quad (2)$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \rightarrow \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t} \quad (3)$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{\partial B_y}{\partial t} \quad (4)$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{\partial B_z}{\partial t} \quad (5)$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \rightarrow \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = J_x + \frac{\partial D_x}{\partial t} \quad (6)$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = J_y + \frac{\partial D_y}{\partial t} \quad (7)$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z + \frac{\partial D_z}{\partial t} \quad (8)$$

Prob. 9.19

If $\boldsymbol{J} = 0 = \rho_v$, then $\nabla \bullet \boldsymbol{B} = 0$ (1)

$$\nabla \bullet \boldsymbol{D} = \rho_v \quad (2)$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \quad (3)$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \quad (4)$$

Since $\nabla \bullet \nabla \times \boldsymbol{A} = 0$ for any vector field $\boldsymbol{A}$,

$$\nabla \bullet \nabla \times \boldsymbol{E} = -\frac{\partial}{\partial t} \nabla \bullet \boldsymbol{B} = 0$$

$$\nabla \bullet \nabla \times \boldsymbol{H} = -\frac{\partial}{\partial t} \nabla \bullet \boldsymbol{D} = 0$$

showing that (1) and (2) are incorporated in (3) and (4). Thus Maxwell's equations can be reduced to (3) and (4), i.e.

$$\underline{\underline{\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \quad \nabla \times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t}}}$$

Prob. 9.20

$$\nabla \bullet \boldsymbol{E} = 0 \longrightarrow (1)$$

$$\nabla \bullet \boldsymbol{H} = 0 \longrightarrow (2)$$

$$\nabla \times \boldsymbol{E} = -\mu \frac{\partial \boldsymbol{H}}{\partial t} \longrightarrow (3)$$

$$\nabla \times \boldsymbol{E} = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \\ 0 & E_y(x,t) & 0 \\ \end{bmatrix}$$

$$= \frac{\partial E_y}{\partial x} \boldsymbol{a}_z = -E_o \sin x \cos t \boldsymbol{a}_z$$

$$H = -\frac{1}{\mu} \int \nabla \times \boldsymbol{E} dt = \frac{E_o}{\mu_o} \sin \times \sin t \boldsymbol{a}_z$$

$$\nabla \times \boldsymbol{H} = \quad \varepsilon \frac{\partial \boldsymbol{E}}{\partial t} \longrightarrow (4)$$

$$\nabla \times \boldsymbol{H} = \begin{bmatrix} \frac{\partial}{\partial \boldsymbol{x}} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \\ 0 & 0 & H_z(x,t) \\ \end{bmatrix}$$

$$= -\frac{\partial H_z}{\partial \boldsymbol{x}} \boldsymbol{a}_y = -\frac{E_o}{\mu_o} \cos x \sin t \boldsymbol{a}_y$$

$$\boldsymbol{E} \quad = \frac{\boldsymbol{1}}{\varepsilon} \int \nabla \times \boldsymbol{H} dt = \frac{E_o}{\mu_o \varepsilon} \cos x \cos t \boldsymbol{a}_y$$

which is off the given $\boldsymbol{E}$ by a factor. Thus, Maxwell's equations (1) to (3) are satisfied, but (4) is not. The only way (4) is satisfied is for $\mu_o \varepsilon = 1$ which is not true.

Prob. 9.21

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\nabla \times \nabla \times \boldsymbol{E} = -\frac{\partial}{\partial t} \nabla \times \boldsymbol{B} = -\mu \frac{\partial}{\partial t} \nabla \times \boldsymbol{H} = -\mu \frac{\partial \boldsymbol{J}}{\partial t} - \mu\varepsilon \frac{\partial^2 \boldsymbol{E}}{\partial t^2}$$

But

$$\nabla\times\nabla\times\boldsymbol{E}=\nabla(\nabla\bullet\boldsymbol{E})-\nabla^2\boldsymbol{E}$$

$$\nabla(\nabla\bullet\boldsymbol{E})-\nabla^2\boldsymbol{E}=-\mu\frac{\partial\boldsymbol{J}}{\partial t}-\mu\varepsilon\frac{\partial^2\boldsymbol{E}}{\partial t^2},\qquad \boldsymbol{J}=\sigma\boldsymbol{E}$$

In a source-free region, $\nabla\bullet\boldsymbol{E}=\rho_v/\varepsilon=0$. Thus,

$$\underline{\underline{\nabla^2 E=\mu\sigma\frac{\partial E}{\partial t}+\mu\varepsilon\frac{\partial^2 E}{\partial t^2}}}$$

Prob. 9.22

$$\nabla\bullet J=(0+0+3z^2)\sin 10^4 t=-\frac{\partial\rho_v}{\partial t}$$

$$\rho_v=-\int\nabla\bullet J dt=-\int 3z^2\sin 10^4 t dt=\frac{3z^2}{10^4}\cos 10^4 t+C_o$$

If $\rho_v|_{z=0}=0$, then $C_o=0$ and

$$\underline{\underline{\rho_v=0.3z^2\cos 10^4 t\ \ \text{mC/m}^3}}$$

Prob. 9.23

$$\boldsymbol{J}_d=\frac{\partial\boldsymbol{D}}{\partial t}=\varepsilon_o\frac{\partial\boldsymbol{E}}{\partial t}=\frac{50\varepsilon_o}{\rho}(-10^8)\sin(10^8 t-kz)\boldsymbol{a}_\rho=\underline{\underline{-\frac{4.421\times10^{-2}}{\rho}\sin(10^8 t-kz)\boldsymbol{a}_\rho\ \text{A/m}^2}}$$

$$\nabla\times\boldsymbol{E}=-\mu_o\frac{\partial\boldsymbol{H}}{\partial t}$$

$$\nabla\times\boldsymbol{E}=\frac{\partial E_\rho}{\partial z}\boldsymbol{a}_\phi=\frac{50k}{\rho}\sin(10^8 t-kz)\boldsymbol{a}_\phi$$

$$\boldsymbol{H}=-\frac{1}{\mu_o}\int\nabla\times\boldsymbol{E}dt=\frac{1}{4\pi\times10^{-7}}\frac{50k}{10^8\rho}\cos(10^8 t-kz)\boldsymbol{a}_\phi$$

$$\underline{\underline{\boldsymbol{H}=\frac{2.5k}{2\pi\rho}\cos(10^8 t-kz)\boldsymbol{a}_\phi\ \text{A/m}}}$$

$$\nabla\times\boldsymbol{H}=-\frac{\partial H_\phi}{\partial z}\boldsymbol{a}_\rho=-\frac{2.5k^2}{2\pi\rho}\sin(10^8 t-kz)\boldsymbol{a}_\rho$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}_d \quad \longrightarrow \quad -\frac{4.421x10^{-2}}{\rho}\sin(10^8 t - kz)\boldsymbol{a}_\rho = \frac{-2.5k^2}{2\pi\rho}\sin(10^8 t - kz)\boldsymbol{a}_\rho$$

$$k^2 = \frac{2\pi}{2.5}\times 4.421\times 10^{-2} \quad \longrightarrow \quad \underline{\underline{k = 0.333}}$$

Prob. 9.24

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} = 0 + \varepsilon\frac{\partial \boldsymbol{E}}{\partial t} \quad \longrightarrow \quad \boldsymbol{E} = \frac{1}{\varepsilon}\int \nabla \times \boldsymbol{H} dt$$

$$\nabla \times \boldsymbol{H} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & 10\cos(\omega t + \beta x) \end{vmatrix} = 10\beta\sin(\omega t + \beta x)\boldsymbol{a}_y$$

$$\boldsymbol{E} = \frac{1}{\varepsilon}\int 10\beta\sin(\omega t + \beta x)dt\boldsymbol{a}_y = \frac{-10\beta}{\omega\varepsilon}\cos(\omega t + \beta x)\boldsymbol{a}_y$$

But $\nabla \times \boldsymbol{E} = -\mu\frac{\partial \boldsymbol{H}}{\partial t} \quad \longrightarrow \quad \boldsymbol{H} = -\frac{1}{\mu}\int \nabla \times \boldsymbol{E} dt$

$$\nabla \times \boldsymbol{E} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & \frac{-10\beta}{\omega\varepsilon}\cos(\omega t + \beta x) & 0 \end{vmatrix} = \frac{10\beta^2}{\omega\varepsilon}\sin(\omega t + \beta x)\boldsymbol{a}_z$$

$$\boldsymbol{H} = -\frac{1}{\mu}\int \frac{10\beta^2}{\omega\varepsilon}\sin(\omega t + \beta x)dt\boldsymbol{a}_z = \frac{10\beta^2}{\omega^2\mu\varepsilon}\cos(\omega t + \beta x)\boldsymbol{a}_z$$

Comparing this with the given **H**,

$$10 = \frac{10\beta^2}{\omega^2\mu\varepsilon} \quad \longrightarrow \quad \beta = \omega\sqrt{\mu\varepsilon} = 2\pi\times 10^9\sqrt{4\pi\times 10^{-7}\times\frac{10^{-9}}{36\pi}\times 81}$$

$$\beta = 60\pi = \underline{\underline{188.5 \text{ rad/m}}}$$

$$E = \frac{-10\beta}{\omega\varepsilon}\cos(\omega t + \beta x)\boldsymbol{a}_y = \underline{\underline{-148\cos(\omega t + \beta x)\boldsymbol{a}_y \text{ V}/m}}$$

Prob. 9.25

$$\boldsymbol{D} = \varepsilon_o \boldsymbol{E} = \underline{\underline{\varepsilon_o E_o \cos(\omega t - \beta z)\boldsymbol{a}_x}}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \quad \longrightarrow \quad \boldsymbol{B} = -\int \nabla \times \boldsymbol{E} dt$$

$$\nabla \times \boldsymbol{E} = \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_o \cos(\omega t - \beta z) & 0 & 0 \end{vmatrix} = -\beta E_o \sin(\omega t - \beta z)\boldsymbol{a}_y$$

$$\boldsymbol{B} = \underline{\underline{\frac{\beta E_o}{\omega} \cos(\omega t - \beta z)\boldsymbol{a}_y}}$$

$$\boldsymbol{H} = \frac{\boldsymbol{B}}{\mu_o} = \underline{\underline{\frac{\beta E_o}{\mu_o \omega} \cos(\omega t - \beta z)\boldsymbol{a}_y}}$$

Prob. 9.26

(a) $\boldsymbol{J}_d = \dfrac{\partial \boldsymbol{D}}{dt} \quad \longrightarrow \quad \boldsymbol{D} = \int \boldsymbol{J}_d \, dt$

$$\boldsymbol{D} = \frac{-60 \times 10^{-3}}{109} \cos(10^9 t - \beta z)\boldsymbol{a}_x = \underline{\underline{-60 \times 10^{-12} \cos(10^9 t - \beta z)\boldsymbol{a}_x \ \text{C/m}^2}}$$

$$\nabla \times \boldsymbol{E} = \mu \frac{\partial \boldsymbol{H}}{\partial t} \quad \longrightarrow \quad \nabla \times \frac{\boldsymbol{D}}{\varepsilon} = -\mu \frac{\partial \boldsymbol{H}}{\partial t}$$

$$\nabla \times \frac{\boldsymbol{D}}{\varepsilon} = \frac{1}{\varepsilon} \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ D_x & 0 & 0 \end{vmatrix} = \frac{1}{\varepsilon}(-60)(-1) \times 10^{-12} \sin(10^9 t - \beta z)\boldsymbol{a}_x$$

$$= \frac{60\beta}{\varepsilon} \times 10^{-12} \sin(10^9 t - \beta z)\boldsymbol{a}_y$$

$$\boldsymbol{H} = -\frac{1}{\mu} \int \nabla \times \frac{\boldsymbol{D}}{\varepsilon} dt = -\frac{1}{\mu}(-1)\frac{60\beta}{\varepsilon} \times \frac{10^{-12}}{10^9} \cos(10^9 t - \beta z)\boldsymbol{a}_y$$

$$= \underline{\underline{\frac{60\beta}{\mu\varepsilon} \times 10^{-21} \cos(10^9 t - \beta z)\boldsymbol{a}_y \ \text{A/m}}}$$

(b) $\nabla\times \boldsymbol{H} = \boldsymbol{J} + \boldsymbol{J}_\phi = 0 + \boldsymbol{J}_d$

$$\boldsymbol{J}_d = \nabla\times\boldsymbol{H} = \begin{vmatrix} \frac{\partial}{\partial_x} & \frac{\partial}{\partial_y} & \frac{\partial}{\partial_z} \\ 0 & H_y & 0 \end{vmatrix} = \frac{(-\beta)(-1)60\beta}{\mu\varepsilon}\times(10^{-21})\sin(10^9 t - \beta z)\boldsymbol{a}_x$$

Equating this with the given $\boldsymbol{J}_d$

$$60\times10^{-3} = \frac{60\beta^2}{\mu\varepsilon}\times10^{-21}$$

$$\beta^2 = \mu\varepsilon \bullet 10^{18} = 2\times4\pi\times10^{-7}\times10\times\frac{10^{-9}}{36\pi} = \frac{2000}{9}$$

$$\beta = \underline{\underline{14.907 \text{ rad/m}}}$$

Prob. 9.27

$$\nabla\times\boldsymbol{E} = -\mu_o\frac{\partial\boldsymbol{H}}{\partial t} \quad\longrightarrow\quad \boldsymbol{H} = -\frac{1}{\mu_o}\int\nabla\times\boldsymbol{E}dt$$

$$\nabla\times\boldsymbol{E} = \frac{1}{r}\frac{\partial}{\partial r}(rE_\theta)\boldsymbol{a}_\phi = \frac{1}{r}\frac{\partial}{\partial r}\left[10\sin\theta\cos(\omega t - \beta r)\right]\boldsymbol{a}_\phi$$

$$= \frac{10\beta}{r}\sin\theta\cos(\omega t - \beta r)\boldsymbol{a}_\phi$$

$$\boldsymbol{H} = -\frac{10\beta}{\mu r}\sin\theta\int\sin(\omega t - \beta r)dt\boldsymbol{a}_\phi$$

$$\underline{\underline{= \frac{10\beta}{\omega\mu_o r}\sin\theta\cos(\omega t - \beta r)\boldsymbol{a}_\phi}}$$

Prob. 9.28

(a) $\nabla\bullet\boldsymbol{A} = 0$

$$\nabla\times\boldsymbol{A} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & E_z(x,t) \end{vmatrix} = -\frac{\partial E_z(x,t)}{\partial x}\boldsymbol{a}_y \neq 0$$

<u>Yes</u>, A is a possible EM field.

(b) $\nabla \bullet B = 0$

$$\nabla \times \boldsymbol{B} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left[10\cos(\omega t - 2\rho)\right]\boldsymbol{a}_z \neq 0$$

Yes, B is a possible EM field.

(c) $\nabla \bullet \boldsymbol{C} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(3\rho^3 \cot\phi \sin\omega t\right) - \frac{\sin\phi \sin\omega t}{\rho^2} \neq 0$

$$\nabla \times \boldsymbol{C} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\cos\phi \sin\omega t\right)\boldsymbol{a}_z - 3\rho^2 \frac{\partial}{\partial \phi}(\cot\phi \sin\omega t)\boldsymbol{a}_z \neq 0$$

No, C cannot be an EM field.

(d) $\nabla \bullet \boldsymbol{D} = \frac{1}{r^2 \sin\theta}\sin(\omega t - 5r)\frac{\partial}{\partial \theta}(\sin^2\theta) \neq 0$

$$\nabla \times \boldsymbol{D} = -\frac{\partial D_\theta}{\partial \phi}\boldsymbol{a}_r + \frac{1}{r}\frac{\partial}{\partial r}(rD_\theta)\boldsymbol{a}_\phi = \frac{1}{r}\sin\theta(-5)\sin(\omega t - 5r)\boldsymbol{a}_\phi \neq 0$$

No, $\boldsymbol{D}$ cannot be an EM field.

Prob. 9.29

From Maxwell's equations,

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad (1)$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \qquad (2)$$

Dotting both sides of (2) with $\vec{E}$ gives:

$$\boldsymbol{E} \bullet (\nabla \times \boldsymbol{H}) = \boldsymbol{E} \bullet \boldsymbol{J} + \boldsymbol{E} \bullet \frac{\partial \boldsymbol{D}}{\partial t} \qquad (3)$$

But for any arbitrary vectors $\vec{A}$ and $\vec{B}$,

$$\nabla \bullet (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \bullet (\nabla \times \boldsymbol{A}) - \boldsymbol{A} \bullet (\nabla \times \boldsymbol{B})$$

Applying this on the left-hand side of (3) by letting $\boldsymbol{A} \equiv \boldsymbol{H}$ and $\boldsymbol{B} \equiv \boldsymbol{E}$, we get

$$\boldsymbol{H} \bullet (\nabla \times \boldsymbol{E}) + \nabla \bullet (\boldsymbol{H} \times \boldsymbol{E}) = \boldsymbol{E} \bullet \boldsymbol{J} + \tfrac{1}{2}\frac{\partial}{\partial t}(\boldsymbol{D} \bullet \boldsymbol{E}) \qquad (4)$$

From (1),

$$\boldsymbol{H} \bullet (\nabla \times \boldsymbol{E}) = \boldsymbol{H} \bullet \left(-\frac{\partial \boldsymbol{B}}{\partial t}\right) = \tfrac{1}{2}\frac{\partial}{\partial t}(\boldsymbol{B} \bullet \boldsymbol{H})$$

Substituting this in (4) gives:

$$-\tfrac{1}{2}\frac{\partial}{\partial t}(\boldsymbol{B} \bullet \boldsymbol{H}) - \nabla \bullet (\boldsymbol{E} \times \boldsymbol{H}) = \boldsymbol{J} \bullet \boldsymbol{E} + \tfrac{1}{2}\frac{\partial}{\partial t}(\boldsymbol{D} \bullet \boldsymbol{E})$$

Rearranging terms and then taking the volume integral of both sides:

$$\int_v \nabla \bullet (\boldsymbol{E}\times\boldsymbol{H})dv = -\frac{\partial}{\partial t}\tfrac{1}{2}\int_v (\boldsymbol{E}\bullet\boldsymbol{D}+\boldsymbol{H}\bullet\boldsymbol{B})dv - \int_v \boldsymbol{J}\bullet\boldsymbol{E}dv$$

Using the divergence theorem, we get

$$\oint_s (\boldsymbol{E}\times\boldsymbol{H})\bullet d\boldsymbol{S} = -\frac{\partial W}{\partial t} - \int_v \boldsymbol{J}\bullet\boldsymbol{E}dv$$

or $\dfrac{\partial W}{\partial t} = -\oint_s (\boldsymbol{E}\times\boldsymbol{H})\bullet d\boldsymbol{S} - \int_v \boldsymbol{E}\bullet\boldsymbol{J}dv$ as required.

Prob. 9.30

$$-\frac{\partial B}{\partial t} = \nabla\times\boldsymbol{E} = \beta E_o \sin(\omega t+\beta y-\beta z)(\boldsymbol{a}_y+\boldsymbol{a}_z)$$

$$-\mu\frac{\partial \boldsymbol{H}}{\partial t} = \nabla\times\boldsymbol{E} \quad\longrightarrow\quad \boldsymbol{H} = -\frac{1}{\mu}\int \nabla\times\boldsymbol{E}dt$$

$$\underline{\underline{\boldsymbol{H} = \frac{\beta E_o}{\mu\omega}\cos(\omega t+\beta y-\beta z)(\boldsymbol{a}_y+\boldsymbol{a}_z)\text{ A/m}}}$$

Prob. 9.31 Using Maxwell's equations,

$$\nabla\times\boldsymbol{H} = \sigma\boldsymbol{E}+\varepsilon\frac{\partial\boldsymbol{E}}{\partial t} \qquad (\sigma=0) \qquad\longrightarrow\qquad \boldsymbol{E} = \frac{1}{\varepsilon}\int\nabla\times\boldsymbol{H}dt$$

But

$$\nabla\times\boldsymbol{H} = -\frac{1}{r\sin\theta}\frac{\partial H_\theta}{\partial\phi}\boldsymbol{a}_r + \frac{1}{r}\frac{\partial}{\partial r}(rH_\theta)\boldsymbol{a}_\phi = \frac{12\sin\theta}{r}\beta\sin(2\pi\times10^8 t-\beta r)\boldsymbol{a}_\phi$$

$$\boldsymbol{E} = \frac{12\sin\theta}{\varepsilon_o}\beta\int \sin(2\pi\times10^8 t-\beta r)dt\boldsymbol{a}_\phi$$

$$= \underline{\underline{-\frac{12\sin\theta}{\omega\varepsilon_o r}\beta\cos(\omega t-\beta r)\boldsymbol{a}_\phi}}, \qquad \omega = 2\pi\times10^8$$

Prob. 9.32
With the given **A**, we need to prove that

$$\nabla^2 \boldsymbol{A} = \mu\varepsilon \frac{\partial^2 \boldsymbol{A}}{\partial t^2}$$

$\nabla^2 \boldsymbol{A} = \mu\varepsilon (j\omega)(j\omega)\boldsymbol{A} = -\omega^2 \mu\varepsilon \boldsymbol{A}$

Let $\beta^2 = \omega^2 \mu\varepsilon$, then $\nabla^2 \boldsymbol{A} = -\beta^2 \boldsymbol{A}$ is to be proved. We recognize that

$$\boldsymbol{A} = \frac{\mu_o}{4\pi r} e^{j\omega t} e^{-j\beta r} \boldsymbol{a}_z$$

Assume $\varphi = \dfrac{e^{-j\beta r}}{r}$, $\boldsymbol{A} = \dfrac{\mu_o}{4\pi} e^{j\omega t} \varphi \boldsymbol{a}_z$

$$\nabla^2 \varphi = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}(r^2 \sin\theta \frac{\partial \varphi}{dr})\right] = \frac{1}{r^2}\left[\frac{\partial}{\partial r}(r^2)\left(\frac{-j\beta}{r} - \frac{1}{r^2}\right)e^{-j\beta r}\right]$$

$$= \frac{1}{r^2}\left(-\beta^2 r + j\beta - j\beta\right)e^{-j\beta r} = -\beta^2 \frac{e^{-j\beta r}}{r} = -\beta^2 \varphi$$

Therefore, $\nabla^2 \boldsymbol{A} = -\beta^2 \boldsymbol{A}$

We can find V using Lorentz gauge.

$$V = \frac{-1}{\mu_o \varepsilon_o} \int \nabla \bullet \boldsymbol{A} dt = \frac{-1}{j\omega \mu_o \varepsilon_o} \nabla \bullet \boldsymbol{A}$$

$$= \frac{-1}{j\omega \mu_o \varepsilon_o} \frac{\partial}{\partial r}\left(\frac{\mu_o}{4\pi r} e^{-j\beta r} e^{j\omega t}\right) = \frac{-1}{j\omega \varepsilon_o (4\pi)}\left(\frac{-j\beta}{r} - \frac{1}{r^2}\right)e^{-j\beta r} e^{j\omega t} \cos\theta$$

$$\underline{\underline{V = \frac{\cos\theta}{j4\pi\omega\varepsilon_o r}\left(j\beta + \frac{1}{r}\right)e^{j(\omega t - \beta r)}}}$$

Prob. 9.33
Take the curl of both sides of the equation.

$$\nabla \times \boldsymbol{E} = -\nabla \times \nabla V - \frac{\partial}{\partial t} \nabla \times \boldsymbol{A}$$

But $\nabla \times \nabla V = \mathbf{0}$ and $\boldsymbol{B} = \nabla \times \boldsymbol{A}$. Hence,

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

which is Faraday's law.

Prob. 9.34

(a) $\nabla \cdot \boldsymbol{A} = \dfrac{\partial A_z}{\partial z} = \dfrac{x}{c}$, $\dfrac{\partial V}{\partial t} = -xc$, $-\mu_o \varepsilon_o \dfrac{\partial V}{\partial t} = \dfrac{x}{c^2} c = \dfrac{x}{c}$

Hence, $\nabla \cdot \boldsymbol{A} = -\mu_o \varepsilon_o \dfrac{\partial V}{\partial t}$

(b) $$\boldsymbol{E} = -\nabla V - \frac{\partial A}{\partial t} = -\left(\frac{\partial V}{\partial x}\boldsymbol{a}_x + \frac{\partial V}{\partial z}\boldsymbol{a}_z\right) + x\boldsymbol{a}_z = -(z\boldsymbol{a}_x + x\boldsymbol{a}_z) + x\boldsymbol{a}_z$$

$$\underline{\underline{\boldsymbol{E} \quad = -z\boldsymbol{a}_x}}$$

Prob. 9.35

$$\nabla \bullet \boldsymbol{A} = 0 = -\mu\varepsilon\frac{\partial V}{\partial t} \quad \longrightarrow \quad \underline{\underline{V = \text{constant}}}$$

(a) $$\boldsymbol{E} = -\nabla V - \frac{\partial \boldsymbol{A}}{\partial t} = \boldsymbol{0} - A_o\omega\cos(\omega t - \beta z)\boldsymbol{a}_x$$

$$= \underline{\underline{-A_o\omega\cos(\omega t - \beta z)\boldsymbol{a}_x}}$$

(b) Using Maxwell's equations, we can show that

$$\underline{\underline{\beta = \omega\sqrt{\mu_o\varepsilon_o}}}$$

Prob. 9.36

(a)

$$z = 4\angle 30^o - 10\angle 50^o = 3.464 + 2j - 6.427 - j7.66 = -2.963 - j5.66$$

$$= 6.389\angle -117.64^o$$

$$z^{1/2} = \underline{\underline{2.5277\angle -58.82^o}}$$

(b)

$$\frac{1 + j2}{6 - j8 - 7\angle 15^o} = \frac{2.236\angle 63.43^o}{6 - j8 - 7.761 - j1.812} = \frac{2.236\angle 63.43^o}{9.841\angle 265.57^o}$$

$$= \underline{\underline{0.2272\angle -202.1^o}}$$

(c) $$z = \frac{(5\angle 53.13^o)^2}{12 - j7 - 6 - j10} = \frac{25\angle 106.26^o}{18.028\angle -70.56^o}$$

$$= \underline{\underline{1.387\angle 176.8^o}}$$

(d)

$$\frac{1.897\angle -100^o}{(5.76\angle 90^o)(9.434\angle -122^o)} = \underline{\underline{0.0349\angle -68^o}}$$

Prob. 9.37

(a) $$\boldsymbol{A} = 5\cos(2t + \pi/3 - \pi/2)\boldsymbol{a}_x + 3\cos(2t + 30^o)\boldsymbol{a}_y = \text{Re}(\boldsymbol{A}_s e^{j\omega t}), \omega = 2$$

$$\boldsymbol{A}_s = \underline{\underline{5e^{-j30^o}\boldsymbol{a}_x + 3e^{j30^o}\boldsymbol{a}_y}}$$

(b) $\boldsymbol{B} = \dfrac{100}{\rho}\cos(\omega t - 2\pi z - 90^o)\boldsymbol{a}_\rho$

$$\boldsymbol{B}_s = \underline{\underline{\frac{100}{\rho} e^{-j(2\pi z + 90^o)}\boldsymbol{a}_\rho}}$$

(c) $\boldsymbol{C} = \dfrac{\cos\theta}{r}\cos(\omega t - 3r - 90^o)\boldsymbol{a}_\theta$

$$\boldsymbol{C}_s = \underline{\underline{\frac{\cos\theta}{r} e^{-j(3r + 90^o)}\boldsymbol{a}_z}}$$

(d) $\boldsymbol{D}_s = \underline{\underline{10\cos(k_1 x)e^{-jk_2 z}\boldsymbol{a}_y}}$

Prob. 9.39

We can use Maxwell's equations or borrow ideas from chapter 10.

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \eta_o\sqrt{\frac{1}{\varepsilon_r}} = \frac{120\pi}{9}$$

$$H_o = \frac{E_o}{\eta} = \frac{10\times 9}{120\pi} = \underline{\underline{0.2387}}$$

$$\beta = \omega\sqrt{\mu\varepsilon} = \frac{\omega}{c}\sqrt{\varepsilon_r} = \frac{2\pi\times 10^9}{3\times 10^8}\sqrt{81} = 60\pi = \underline{\underline{188.5 \text{ rad/m}}}$$

Prob. 9.40

(a)

$$\boldsymbol{H} = \text{Re}\left[40e^{j(10^9 t - \beta z)}\boldsymbol{a}_x\right], \quad \omega = 10^9$$

$$= \text{Re}\left[40e^{-j\beta z}\boldsymbol{a}_x e^{j\omega t}\right] = \text{Re}\left[\boldsymbol{H}_s e^{j\omega t}\right]$$

$$\boldsymbol{H}_s = \underline{\underline{40e^{-j\beta z}\boldsymbol{a}_x}}$$

(b)

$$\boldsymbol{J}_d = \nabla\times\boldsymbol{H} = \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 40\cos(10^9 t - \beta z) & 0 & 0 \end{vmatrix}$$

$$= \underline{\underline{40\beta}}\,\underline{\underline{\sin(10^9 t - \beta z)\boldsymbol{a}_y \text{ A/m}^2}}$$

Prob. 9.41

$(j\omega)^2 Y + 4j\omega Y + Y = 2\angle 0^o, \qquad \omega = 3$

$Y(-\omega^2 + 4j\omega + 1) = 2$

$$Y = \frac{2}{-\omega^2 + 4j\omega + 1} = \frac{2}{-9 + j12 + 1} = \frac{2}{-8 + j12} = -0.0769 - j0.1154$$

$$= 0.1387\angle -123.7^o$$

$$y(t) = \mathrm{Re}(Ye^{j\omega t}) = \underline{\underline{0.1387\cos(3t - 123.7^o)}}$$

CHAPTER 10

P. E. 10.1 (a)

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{2\times10^8} = \underline{\underline{31.42\ \text{ns}}},$$

$$\lambda = uT = 3\times10^8\times31.42\times10^{-9} = \underline{\underline{9.425\ \text{m}}}$$

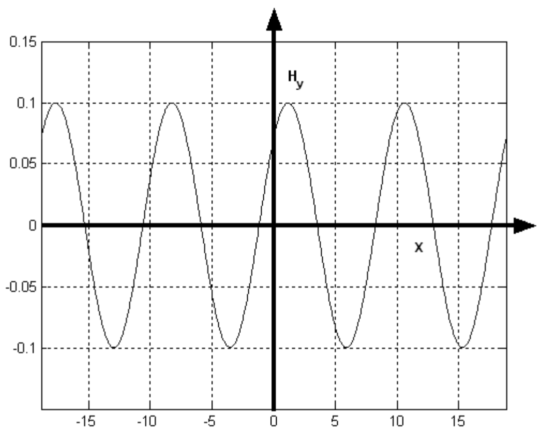
$$k = \beta = 2\pi/\lambda = \underline{\underline{0.6667\ \text{rad/m}}}$$

(b) $t_1 = T/8 = \underline{\underline{3.927\ \text{ns}}}$

(c)

$$\boldsymbol{H}(t=t_1) = 0.1\cos(2\times10^8\frac{\pi}{8\times10^8} - 2x/3)\boldsymbol{a}_y = 0.1\cos(2x/3 - \pi/4)\boldsymbol{a}_y$$

as sketched below.



P. E. 10.2 Let $x_o = \sqrt{1+(\sigma/\omega\varepsilon)^2}$, then

$$\alpha = \omega\sqrt{\frac{\mu_o\varepsilon_o}{2}\mu_r\varepsilon_r(x_o-1)} = \frac{\omega}{c}\sqrt{\frac{16}{2}}\sqrt{x_o-1}$$

or $$\sqrt{x_o-1} = \frac{\alpha c}{\omega\sqrt{8}} = \frac{1/3\times3\times10^8}{10^8\sqrt{8}} = \frac{1}{\sqrt{8}} \longrightarrow x_o = 9/8$$

$$x_o^2 = \frac{81}{64} = 1+(\sigma/\omega\varepsilon)^2 \longrightarrow \frac{\sigma}{\omega\varepsilon} = 0.5154$$

$\tan 2\theta_\eta = 0.5154 \longrightarrow \theta_\eta = 13.63^o$

$$\frac{\beta}{\alpha} = \sqrt{\frac{x_o + 1}{x_o - 1}} = \sqrt{17}$$

(a) $\beta = \alpha\sqrt{17} = \frac{\sqrt{17}}{3} = \underline{\underline{1.374 \text{ rad/m}}}$

(b) $\frac{\sigma}{\omega\varepsilon} = \underline{\underline{0.5154}}$

(c) $|\eta| = \frac{\sqrt{\mu/\varepsilon}}{\sqrt{x_o}} = \frac{120\pi\sqrt{2/8}}{\sqrt{9/8}} = 177.72$

$\eta = \underline{\underline{177.72\angle 13.63^o \, \Omega}}$

(d) $u = \frac{\omega}{\beta} = \frac{10^8}{1.374} = \underline{\underline{7.278 \times 10^7}}$ m/s

(e) $\boldsymbol{a}_H = \boldsymbol{a}_k \times \boldsymbol{a}_E \longrightarrow \boldsymbol{a}_z \times \boldsymbol{a}_x = \boldsymbol{a}_H \longrightarrow \boldsymbol{a}_H = \boldsymbol{a}_y$

$$\boldsymbol{H} = \frac{0.5}{177.5} e^{-z/3} \sin(10^8 t - \beta z - 13.63^o)\boldsymbol{a}_y = \underline{\underline{2.817 e^{-z/3} \sin(10^8 t - \beta z - 13.63^o)\boldsymbol{a}_y \text{ mA/m}}}$$

P. E. 10.3 (a) Along $\underline{\underline{\text{-z direction}}}$

(b) $\lambda = \frac{2\pi}{\beta} = 2\pi / 2 = \underline{\underline{3.142 \text{ m}}}$

$f = \frac{\omega}{2\pi} = \frac{10^8}{2\pi} = \underline{\underline{15.92 \text{ MHz}}}$

$\beta = \omega\sqrt{\mu\varepsilon} = \omega\sqrt{\mu_o\varepsilon_o}\sqrt{\mu_r\varepsilon_r} = \frac{\omega}{c}\sqrt{(1)\varepsilon_r}$

or $\sqrt{\varepsilon_r} = \beta c / \omega = \frac{3\times 10^8 \times 2}{10^8} = 6 \longrightarrow \varepsilon_r = 36$

(c) $\theta_\eta = 0, |\eta| = \sqrt{\mu/\varepsilon} = \sqrt{\mu_o/\varepsilon_o}\sqrt{1/\varepsilon_r} = \frac{120\pi}{6} = 20\pi$

$\boldsymbol{a}_k = \boldsymbol{a}_E \times \boldsymbol{a}_H \longrightarrow -\boldsymbol{a}_z = \boldsymbol{a}_y \times \boldsymbol{a}_H \longrightarrow \boldsymbol{a}_H = \boldsymbol{a}_x$

$$\boldsymbol{H} = \frac{50}{20\pi}\sin(\omega t + \beta z)\boldsymbol{a}_x = \underline{\underline{795.8\sin(10^8 t + 2z)\boldsymbol{a}_x}} \quad \text{mA/m}$$

P. E. 10.4 (a)

$$\frac{\sigma}{\omega\varepsilon} = \frac{10^{-2}}{10^9 \pi \times 4 \times \frac{10^{-9}}{36\pi}} = 0.09$$

$$\alpha \cong \omega\sqrt{\frac{\mu\varepsilon}{2}\left[1+\frac{1}{2}\left(\frac{\sigma}{\omega\varepsilon}\right)^2 - 1\right]} = \frac{\omega}{2c}\sqrt{\mu_r\varepsilon_r}\,\frac{\sigma}{\omega\varepsilon} = \frac{10^9\pi}{2\times3\times10^8}(2)(0.09) = 0.9425 \text{ Np/m}$$

$$\beta \cong \omega\sqrt{\frac{\mu\varepsilon}{2}\left[1+\frac{1}{2}\left(\frac{\sigma}{\omega\varepsilon}\right)^2 + 1\right]} = \frac{10^9\pi}{3\times10^8}\sqrt{2[2+0.5(0.09)^2]} = 20.965 \text{ rad/m}$$

$$\boldsymbol{E} = 30e^{-0.9425y}\cos(10^9\pi t - 20.96y + \pi/4)\boldsymbol{a}_z$$

At t = 2ns, y = 1m,

$$\boldsymbol{E} = 30e^{-0.9425}\cos(2\pi - 20.96 + \pi/4)\boldsymbol{a}_z = \underline{\underline{2.844\boldsymbol{a}_z}} \quad \text{V/m}$$

(b) $\beta y = 10^o = \frac{10\pi}{180}$ rad

or

$$y = \frac{\pi}{18}\frac{1}{\beta} = \frac{\pi}{18\times20.965} = \underline{\underline{8.325 \text{ mm}}}$$

(c) $30(0.6) = 30\ e^{-\alpha y}$

$$y = \frac{1}{\alpha}\ln(1/0.6) = \frac{1}{0.9425}\ln\frac{1}{0.6} = \underline{\underline{542 \text{ mm}}}$$

(d)

$$|\eta| \cong \frac{\sqrt{\mu/\varepsilon}}{[1+\frac{1}{4}(0.09)^2]} = \frac{60\pi}{1.002} = 188.11\ \Omega$$

$$2\theta_\eta = \tan^{-1} 0.09 \longrightarrow \theta_\eta = 2.571^o$$

$$\boldsymbol{a}_H = \boldsymbol{a}_k \times \boldsymbol{a}_E = \boldsymbol{a}_y \times \boldsymbol{a}_z = \boldsymbol{a}_x$$

$$\boldsymbol{H} = \frac{30}{188.11} e^{-0.9425y} \cos(10^9 \pi t - 20.96y + \pi/4 - 2.571^o)\boldsymbol{a}_x$$

At y = 2m, t = 2ns,

$$\boldsymbol{H} = (0.1595)(0.1518)\ \cos(-34.8963 rad)\boldsymbol{a}_x = \underline{\underline{-22.83\boldsymbol{a}_x}} \text{ mA/m}$$

P. E. 10.5

$$I_s = \int_0^w \int_0^\infty J_{xs} dy dz = J_{xs}(0) \int_0^w dy \int_0^\infty e^{-z(1+j)/\delta} dz = \frac{J_{xs}(0) w\delta}{1+j}$$

$$\underline{\underline{|I_s| = \frac{J_{xs}(0) w\delta}{\sqrt{2}}}}$$

P. E. 10.6 (a)

$$\frac{R_{ac}}{R_{dc}} = \frac{a}{2\delta} = \frac{a}{2}\sqrt{\pi f \mu \sigma} = \frac{1.3\times10^{-3}}{2}\sqrt{\pi \times 10^7 \times 4\pi \times 10^{-7} \times 3.5\times10^7} = \underline{24.16}$$

(b)

$$\frac{R_{ac}}{R_{dc}} = \frac{1.3\times10^{-3}}{2}\sqrt{\pi \times 2\times10^9 \times 4\pi \times 10^{-7} \times 3.5\times10^7} = \underline{\underline{341.7}}$$

P. E. 10.7

$$\begin{aligned} E &= \text{Re}[\boldsymbol{E}_s e^{j\omega t}] = \text{Re}\left[E_o e^{j\omega t} e^{-j\beta z}\boldsymbol{a}_x + E_o e^{-j\pi/2} e^{j\omega t} e^{-j\beta z}\boldsymbol{a}_y\right] \\ &= E_o \cos(\omega t - \beta z)\boldsymbol{a}_x + E_o \cos(\omega t - \beta z - \pi/2)\boldsymbol{a}_y \\ &= E_o \cos(\omega t - \beta z)\boldsymbol{a}_x + E_o \sin(\omega t - \beta z)\boldsymbol{a}_y \end{aligned}$$

At z = 0, $E_x = E_o \cos\omega t, E_y = E_o \sin \omega t$

$$\cos^2 \omega t + \sin^2 \omega t = 1 \longrightarrow \left(\frac{E_x}{E_o}\right)^2 + \left(\frac{E_y}{E_o}\right)^2 = 1$$

which describes a circle. Hence the polarization is circular.

P. E. 10.8

$$\mathcal{P}_{ave} = \frac{1}{2}\eta H_o^2 \boldsymbol{a}_x$$

(a) Let $f(x,z) = x + y - 1 = 0$

$$\boldsymbol{a}_n = \frac{\nabla f}{|\nabla f|} = \frac{\boldsymbol{a}_x + \boldsymbol{a}_y}{\sqrt{2}}, \qquad d\boldsymbol{S} = \text{dS}\boldsymbol{a}_\text{n}$$

$$P_t = \int \mathbf{P}.\text{d}\mathbf{S} = \mathbf{P}.S\mathbf{a_n} = \frac{1}{2}\eta H_o^2 \boldsymbol{a}_x.\frac{\boldsymbol{a}_x + \boldsymbol{a}_y}{\sqrt{2}}$$

$$= \frac{1}{2\sqrt{2}}(120\pi)(0.2)^2(0.1)^2 = \underline{\underline{53.31 \text{ mW}}}$$

(b) $d\boldsymbol{S} = \text{dydz}\boldsymbol{a}_\text{x}$, $P_t = \int \mathcal{P}.\text{d}\mathbf{S} = \frac{1}{2}\eta H_o^2 S$

$$P_t = \frac{1}{2}(120\pi)(0.2)^2 \pi (0.05)^2 = \underline{\underline{59.22 \text{ mW}}}$$

P. E. 10.9 $\eta_1 = \eta_o = 120\pi, \eta_2 = \sqrt{\frac{\mu}{\varepsilon}} = \frac{\eta_o}{2}$

$$\tau = \frac{2\eta_2}{\eta_2 + \eta_1} = 2/3, \Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = -1/3$$

$$E_{ro} = \Gamma E_{io} = -\frac{10}{3}$$

$$\underline{\underline{\boldsymbol{E}_{rs} = -\frac{10}{3} e^{j\beta_1 z} \boldsymbol{a}_x}} \text{ V/m}$$

where $\beta_1 = \omega / c = 100\pi / 3$.

$$E_{to} = \tau E_{io} = \frac{20}{3}$$

$$\underline{\underline{\boldsymbol{E}_{ts} = \frac{20}{3} e^{-j\beta_2 z} \boldsymbol{a}_x \text{ V/m}}}$$

where $\beta_2 = \omega\sqrt{\varepsilon_r} / c = 2\beta_1 = 200\pi / 3$.

P. E. 10.10

$$\alpha_1 = 0, \quad \beta_1 = \frac{\omega}{c}\sqrt{\mu_r \varepsilon_r} = \frac{2\omega}{c} = 5 \longrightarrow \omega = 5c/2 = 7.5\times 10^8$$

$$\frac{\sigma_2}{\omega\varepsilon_2} = \frac{0.1}{7.5\times 10^8 \times 4 \times \frac{10^{-9}}{36\pi}} = 1.2\pi$$

$$\alpha_2 = \frac{\omega}{c}\sqrt{\frac{4}{2}\left[\sqrt{1+1.44\pi^2} - 1\right]} = 6.021$$

$$\beta_2 = \frac{\omega}{c}\sqrt{\frac{4}{2}\left[\sqrt{1+1.44\pi^2} + 1\right]} = 7.826$$

$$|\eta_2| = \frac{60\pi}{\sqrt[4]{1+1.44\pi^2}} = 95.445, \eta_1 = 120\pi\sqrt{\varepsilon_{r1}} = 754$$

$$\tan 2\theta_{\eta_2} = 1.2\pi \longrightarrow \theta_{\eta_2} = 37.57^o$$

$$\eta_2 = 95.445\angle 37.57^o$$

(a)

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{95.445\angle 37.57^o - 754}{95.445\angle 37.57^o + 754} = \underline{\underline{0.8186\angle 171.08^o}}$$

$$\tau = 1 + \Gamma = \underline{\underline{0.2295\angle 33.56^o}}$$

$$s = \frac{1+|\Gamma|}{1-|\Gamma|} = \frac{1+0.8186}{1-0.8186} = \underline{\underline{10.025}}$$

(b) $\quad \boldsymbol{E}_i = 50\sin(\omega t - 5x)\boldsymbol{a}_y = \mathrm{Im}(\boldsymbol{E}_{is}e^{j\omega t})$, where $E_{is} = 50e^{-j5x}a_y$.

$$E_{ro} = \Gamma E_{io} = 0.8186e^{j171.08^o}(50) = 40.93e^{j171.08^o}$$

$$\boldsymbol{E}_{rs} = 40.93e^{j5x + j171.08^o}\boldsymbol{a}_y$$

$$\boldsymbol{E}_r = \mathrm{Im}(\boldsymbol{E}_{rs}e^{j\omega t}) = \underline{\underline{40.93\sin(\omega t + 5x + 171.1^o)\boldsymbol{a}_y}} \text{ V/m}$$

$$\boldsymbol{a}_H = \boldsymbol{a}_k \times \boldsymbol{a}_E = -\boldsymbol{a}_x \times \boldsymbol{a}_y = -\boldsymbol{a}_z$$

$$\boldsymbol{H}_r = -\frac{40.93}{754}\sin(\omega t + 5x + 171.1^o)\boldsymbol{a}_z = \underline{\underline{-0.0543\sin(\omega t + 5x + 171.1^o)\boldsymbol{a}_z}} \quad \text{A/m}$$

(c)

$$E_{to} = \tau E_{io} = 0.229e^{j33.56^o}(50) = 11.475e^{j33.56^o}$$

$$\boldsymbol{E}_{ts} = 11.475e^{-j\beta_2 x + j33.56^o}e^{-\alpha_2 x}\boldsymbol{a}_y$$

$$\boldsymbol{E}_t = \text{Im}(\boldsymbol{E}_{ts}e^{j\omega t}) = \underline{\underline{11.475e^{-6.021x}\sin(\omega t - 7.826x + 33.56^o)\boldsymbol{a}_y}} \quad \text{V/m}$$

$$\boldsymbol{a}_H = \boldsymbol{a}_k \times \boldsymbol{a}_E = \boldsymbol{a}_x \times \boldsymbol{a}_y = \boldsymbol{a}_z$$

$$\boldsymbol{H}_t = \frac{11.495}{95.445}e^{-6.021x}\sin(\omega t - 7.826x + 33.56^o - 37.57^o)\boldsymbol{a}_z$$

$$= \underline{\underline{0.1202e^{-6.021x}\sin(\omega t - 7.826x - 4.01^o)\boldsymbol{a}_z}} \quad \text{A/m}$$

(d)

$$\mathcal{P}_{1\text{ave}} = \frac{E_{io}^{\ 2}}{2\eta_1}\boldsymbol{a}_x + \frac{E_{ro}^{\ 2}}{2\eta_1}(-\boldsymbol{a}_x) = \frac{1}{2(240\pi)}[50^2\boldsymbol{a}_x - 40.93^2\boldsymbol{a}_x] = \underline{\underline{0.5469\boldsymbol{a}_x}} \quad \text{W/m}^2$$

$$\boldsymbol{P}_{2\text{ave}} = \frac{E_{to}^{\ 2}}{2|\eta_2|}e^{-2\alpha_2 x}\cos\theta_{\eta_2}\boldsymbol{a}_x = \frac{(11.475)^2}{2(95.445)}\cos 37.57^o e^{-2(6.021)x}\boldsymbol{a}_x = \underline{\underline{0.5469e^{-12.04x}\boldsymbol{a}_x}} \quad \text{W/m}^2$$

P. E. 10.11 (a)

$$\boldsymbol{k} = -2\boldsymbol{a}_y + 4\boldsymbol{a}_z \longrightarrow k = \sqrt{2^2 + 4^2} = \sqrt{20}$$

$$\omega = kc = 3\times 10^8\sqrt{20} = \underline{\underline{1.342\times 10^9 \text{ rad/s}}},$$

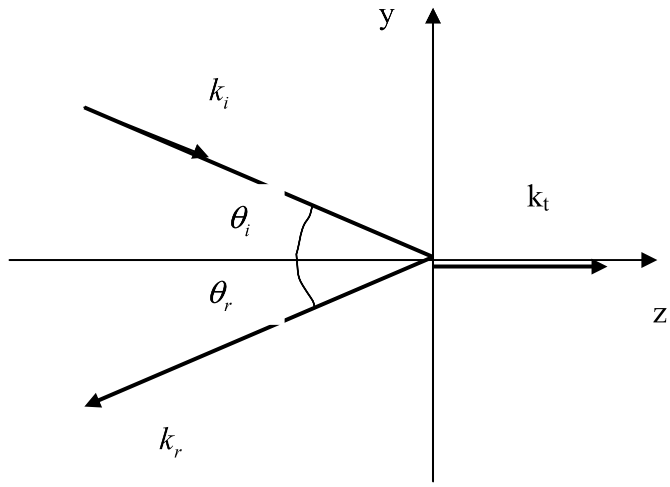
$$\lambda = 2\pi / k = \underline{\underline{1.405\text{m}}}$$

(b) $$\boldsymbol{H} = \frac{\boldsymbol{a}_k \times \boldsymbol{E}}{\eta_o} = \frac{(-2\boldsymbol{a}_y + 4\boldsymbol{a}_z)}{\sqrt{20}(120\pi)} \times (10\boldsymbol{a}_y + 5\boldsymbol{a}_z)\cos(\omega t - \boldsymbol{k}.\boldsymbol{r})$$

$$= \underline{\underline{-29.66\cos(1.342x10^9 t + 2y - 4z)\boldsymbol{a}_x}} \quad \text{mA/m}$$

(c) $\mathcal{P}_{\text{ave}} = \dfrac{|E_o|^2}{2\eta_o}\boldsymbol{a}_k = \dfrac{125}{2(120\pi)}\dfrac{(-2\boldsymbol{a}_y + 4\boldsymbol{a}_z)}{\sqrt{20}} = \underline{\underline{-74.15\boldsymbol{a}_y + 148.9\boldsymbol{a}_z}}$ mW/m^2

P. E. 10.12 (a)



$$\tan\theta_i = \frac{k_{iy}}{k_{iz}} = \frac{2}{4} \longrightarrow \underline{\underline{\theta_i = 26.56 = \theta_r}}$$

$$\sin\theta_t = \sqrt{\frac{\mu_1 \varepsilon_1}{\mu_2 \varepsilon_2}}\sin\theta_i = \frac{1}{2}\sin 26.56^o \longrightarrow \underline{\underline{\theta_t = 12.92^o}}$$

(b) $\eta_1 = \eta_o, \eta_2 = \eta_o / 2$ **E** is parallel to the plane of incidence. Since $\mu_1 = \mu_2 = \mu_o$,

we may use the result of Prob. 10.42, i.e.

$$\Gamma_{\backslash\backslash} = \frac{\tan(\theta_t - \theta_i)}{\tan(\theta_t + \theta_i)} = \frac{\tan(-13.64^o)}{\tan(39.48^o)} = \underline{\underline{-0.2946}}$$

$$\tau_{\backslash\backslash} = \frac{2\cos 26.56^o \sin 12.92^o}{\sin 39.48^o \cos(-13.64^o)} = \underline{\underline{0.6474}}$$

(c) $\boldsymbol{k}_r = -\beta_1 \sin\theta_r \boldsymbol{a}_y - \beta_1 \cos\theta_r \boldsymbol{a}_z$. Once k_r is known, E_r is chosen such that

$\boldsymbol{k}_r.\boldsymbol{E}_r = 0$ or $\nabla.\boldsymbol{E}_r = 0$. Let

$$\boldsymbol{E}_r = \pm E_{or}(-\cos\theta_r \boldsymbol{a}_y + \sin\theta_r \boldsymbol{a}_z)\cos(\omega t + \beta_1 \sin\theta_r y + \beta_1 \cos\theta_r z)$$

Only the positive sign will satisfy the boundary conditions. It is evident that

$$\boldsymbol{E}_i = E_{oi}(\cos\theta_i \boldsymbol{a}_y + \sin\theta_i \boldsymbol{a}_z)\cos(\omega t + 2y - 4z)$$

Since $\theta_r = \theta_i$,

$$E_{or}\cos\theta_r = \Gamma_{//}E_{oi}\cos\theta_i = 10\Gamma_{//} = -2.946$$

$$E_{or}\sin\theta_r = \Gamma_{//}E_{oi}\sin\theta_i = 5\Gamma_{//} = -1.473$$

$$\beta_1\sin\theta_r = 2, \beta_1\cos\theta_r = 4$$

i.e.

$$E_r = -(2.946a_y - 1.473a_z)\cos(\omega t + 2y + 4z)$$

$$\boldsymbol{E}_1 = \boldsymbol{E}_i + \boldsymbol{E}_r = \underline{\underline{(10\boldsymbol{a}_y + 5\boldsymbol{a}_z)\cos(\omega t + 2y - 4z) + (-2.946\boldsymbol{a}_y + 1.473\boldsymbol{a}_z)\cos(\omega t + 2y + 4z)}}$$

V/m

(d) $\boldsymbol{k}_t = -\beta_2\sin\theta_t \boldsymbol{a}_y + \beta_2\cos\theta_t \boldsymbol{a}_z$. Since $k_r \bullet E_r = 0$, let

$$\boldsymbol{E}_t = E_{ot}(\cos\theta_t \boldsymbol{a}_y + \sin\theta_t \boldsymbol{a}_z)\cos(\omega t + \beta_2 y\sin\theta_t - \beta_2 z\cos\theta_t)$$

$$\beta_2 = \omega\sqrt{\mu_2\varepsilon_2} = \beta_1\sqrt{\varepsilon_{r2}} = 2\sqrt{20}$$

$$\sin\theta_t = \frac{1}{2}\sin\theta_i = \frac{1}{2\sqrt{5}}, \qquad \cos\theta_t = \frac{\sqrt{9}}{\sqrt{20}}$$

$$\beta_2\cos\theta_t = 2\sqrt{20}\sqrt{\frac{19}{20}} = 8.718$$

$$E_{ot}\cos\theta_t = \tau_{//}E_{oi}\cos\theta_t = 0.6474\sqrt{125}\sqrt{\frac{19}{20}} = 7.055$$

$$E_{ot}\sin\theta_t = \tau_{//}E_{oi}\sin\theta_t = 0.6474\sqrt{125}\sqrt{\frac{1}{20}} = 1.6185$$

Hence

$$\boldsymbol{E}_2 = \boldsymbol{E}_t = \underline{\underline{(7.055\boldsymbol{a}_y + 1.6185\boldsymbol{a}_z)\cos(\omega t + 2y - 8.718z)}} \text{ V/m}$$

(d) $\tan\theta_{B//} = \sqrt{\dfrac{\varepsilon_2}{\varepsilon_1}} = 2 \longrightarrow \underline{\underline{\theta_{B//} = 63.43^o}}$

P.E. 10.13

$S_i = \dfrac{1+0.4}{1-0.4} = \dfrac{1.4}{0.6} = \underline{\underline{2.333}}$

$S_o = \dfrac{1+0.2}{1-0.2} = \dfrac{1.2}{0.8} = \underline{\underline{1.5}}$

Prob. 10.1 (a) Wave propagates $\underline{\underline{\text{along } +\mathbf{a}_x}}$.

(b)

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{2\pi \times 10^6} = \underline{\underline{1\mu s}}$$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{6} = \underline{\underline{1.047\text{m}}}$$

$$u = \frac{\omega}{\beta} = \frac{2\pi \times 10^6}{6} = \underline{\underline{1.047 \times 10^6 \text{ m/}s}}$$

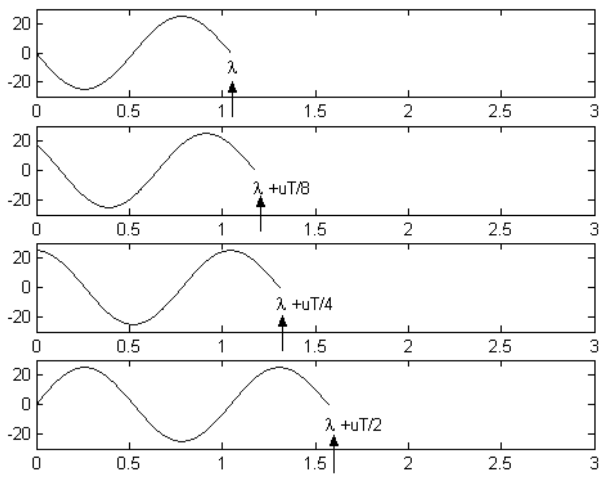
(c) At t=0, $E_z = 25\sin(-6x) = -25\sin 6x$

At t=T/8, $E_z = 25\sin(\frac{2\pi}{T}\frac{T}{8} - 6x) = 25\sin(\frac{\pi}{4} - 6x)$

At t=T/4, $E_z = 25\sin(\frac{2\pi}{T}\frac{T}{4} - 6x) = 25\sin(-6x + 90^o) = 25\cos 6x$

At t=T/2, $E_z = 25\sin(\frac{2\pi}{T}\frac{T}{2} - 6x) = 25\sin(-6x + \pi) = 25\sin 6x$

These are sketched below.



Prob. 10.2

(a)

$$\frac{\partial E}{\partial x} = -\sin(x+\omega t) - \sin(x-\omega t)$$

$$\frac{\partial^2 E}{\partial x^2} = -\cos(x+\omega t) - \cos(x-\omega t) = -E$$

$$\frac{\partial E}{\partial t} = -\omega\sin(x+\omega t) - \omega\sin(x-\omega t)$$

$$\frac{\partial^2 E}{\partial t^2} = -\omega^2\cos(x+\omega t) - \omega^2\cos(x-\omega t) = -\omega^2 E$$

$$\frac{\partial^2 E}{\partial t^2} - u^2\frac{\partial^2 E}{\partial x^2} = -\omega^2 E + u^2 E = 0$$

if $\omega^2 = u^2$ and hence, eq. (10.1) is satisfied.

(b) $\underline{\underline{u = \omega}}$

Prob. 10.3

(a) $\omega = \underline{\underline{10^8 \text{ rad/s}}}$

(b) $\beta = \dfrac{\omega}{c} = \dfrac{10^8}{3\times 10^8} = \underline{\underline{0.333 \text{ rad/m}}}$

(c) $\lambda = \dfrac{2\pi}{\beta} = 6\pi = \underline{\underline{18.85 \text{ m}}}$

(d) Along $-\mathbf{a}_y$

At y=1, t=10ms,

(e) $H = 0.5\cos(10^8 t \times 10\times 10^{-9} + \dfrac{1}{3}\times 3) = 0.5\cos(1+1)$

$= \underline{\underline{-0.1665 \text{ A/m}}}$

Prob. 10.4

(a) $\lambda = \frac{c}{f} = \frac{3\times10^8}{60} = \underline{\underline{5\times10^6\ \text{m}}}$

(b) $\lambda = \frac{3\times10^8}{2\times10^6} = \underline{\underline{150\ \text{m}}}$

(c) $\lambda = \frac{3\times10^8}{120\times10^6} = \underline{\underline{2.5\ \text{m}}}$

(d) $\lambda = \frac{3\times10^8}{2.4\times10^9} = \underline{\underline{0.125\ \text{m}}}$

Prob. 10.5 If

$\gamma^2 = j\omega\mu(\sigma + j\omega\varepsilon) = -\omega^2\mu\varepsilon + j\omega\mu\sigma$ and $\gamma = \alpha + j\beta$, then

$|\gamma^2| = \sqrt{(\alpha^2 - \beta^2) + 4\alpha^2\beta^2} = \sqrt{(\alpha^2 + \beta^2)^2} = \alpha^2 + \beta^2$

i.e.

$$\alpha^2 + \beta^2 = \omega\mu\sqrt{(\sigma^2 + \omega^2\varepsilon^2)} \qquad (1)$$

$$\text{Re}(\gamma^2) = \alpha^2 - \beta^2 = -\omega^2\mu\varepsilon$$

$$\beta^2 - \alpha^2 = \omega^2\mu\varepsilon \qquad (2)$$

Subtracting and adding (1) and (2) lead respectively to

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]}$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right]}$$

(b) From eq. (10.25), $\boldsymbol{E}_s(z) = E_o e^{-\gamma z}\boldsymbol{a}_x$.

$$\nabla \times \boldsymbol{E} = -j\omega\mu \boldsymbol{H}_s \longrightarrow \boldsymbol{H}_s = \frac{j}{\omega\mu}\nabla \times \boldsymbol{E}_s = \frac{j}{\omega\mu}(-\gamma E_o e^{-\gamma z}\boldsymbol{a}_y)$$

But $\boldsymbol{H}_s(z) = H_o e^{-\gamma z}\boldsymbol{a}_y$, hence $H_o = \dfrac{E_o}{\eta} = -\dfrac{j\gamma}{\omega\mu}E_o$

$$\eta = \frac{j\omega\mu}{\gamma}$$

(c) From (b),

$$\eta = \frac{j\omega\mu}{\sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} = \frac{\sqrt{\mu/\varepsilon}}{\sqrt{1 - j\dfrac{\sigma}{\omega\varepsilon}}}$$

$$|\eta| = \frac{\sqrt{\mu/\varepsilon}}{\sqrt[4]{1 + \left(\dfrac{\sigma}{\omega\varepsilon}\right)^2}}, \tan 2\theta_\eta = \left(\frac{\omega\varepsilon}{\sigma}\right)^{-1} = \frac{\sigma}{\omega\varepsilon}$$

Prob. 10.6 (a)

$$\frac{\sigma}{\omega\varepsilon} = \frac{8\times 10^{-2}}{2\pi \times 50\times 10^6 \times 3.6 \times \dfrac{10^{-9}}{36\pi}} = 8$$

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]} = \frac{2\pi\times 50\times 10^6}{3\times 10^8}\sqrt{\frac{2.1\times 3.6}{2}[\sqrt{65} - 1]} = 5.41$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right]} = 6.129$$

$$\gamma = \alpha + j\beta = \underline{\underline{5.41 + j6.129}} \text{ /m}$$

(b) $\lambda = \dfrac{2\pi}{\beta} = \dfrac{2\pi}{6.129} = \underline{\underline{1.025}}\text{ m}$

(c) $u = \frac{\omega}{\beta} = \frac{2\pi \times 50 \times 10^6}{6.129} = \underline{\underline{5.125 \times 10^7}}$ m/s

(d) $$|\eta| = \frac{\sqrt{\frac{\mu}{\varepsilon}}}{\sqrt[4]{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}} = \frac{120\pi\sqrt{\frac{2.1}{3.6}}}{\sqrt[4]{65}} = 101.4$$

$$\tan 2\theta_\eta = \frac{\sigma}{\omega\varepsilon} = 8 \longrightarrow \theta_\eta = 41.44^o$$

$$\eta = \underline{\underline{101.41\angle 41.44^o\,\Omega}}$$

(e) $$\boldsymbol{H}_s = \boldsymbol{a}_k \times \frac{\boldsymbol{E}_s}{\eta} = \boldsymbol{a}_x \times \frac{6}{\eta} e^{-\gamma z}\boldsymbol{a}_z = -\frac{6}{\eta} e^{-\gamma z}\boldsymbol{a}_y = \underline{\underline{-59.16 e^{-j41.44^o} e^{-\gamma z}\boldsymbol{a}_y}} \text{ mA/m}$$

Prob. 10.7

(a) $$\tan\theta = \frac{\sigma}{\omega\varepsilon} = \frac{10^{-2}}{2\pi \times 12 \times 10^6 \times 10 \times \frac{10^{-9}}{36\pi}} = \underline{\underline{1.5}}$$

(b) $$\tan\theta = \frac{10^{-4}}{2\pi \times 12 \times 10^6 \times 4 \times \frac{10^{-9}}{36\pi}} = \underline{\underline{3.75 \times 10^{-2}}}$$

(c) $$\tan\theta = \frac{4}{2\pi \times 12 \times 10^6 \times 81 \times \frac{10^{-9}}{36\pi}} = \underline{\underline{74.07}}$$

Prob. 10.8

(a)

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]} = \frac{2\pi \times 15 \times 10^9}{3 \times 10^8}\sqrt{\frac{1 \times 9.6}{2}\left[\sqrt{1 + 9 \times 10^{-8}} - 1\right]}$$

$$= 100\pi\sqrt{4.8\left(\frac{1}{2} \times 9 \times 10^{-8}\right)} = 0.146$$

$$\delta = \frac{1}{\alpha} = \underline{\underline{6.85 \text{ m}}}$$

(b) $A = \alpha\ell = 0.146 \times 5 \times 10^{-3} = \underline{\underline{0.73 \times 10^{-3} \text{ Np}}}$

Prob. 10.9

The phase difference is θ_η

$$\tan 2\theta_\eta = \frac{\sigma}{\omega\varepsilon} = \frac{8\times10^{-3}}{2\pi\times20\times10^{6}\times4\times\dfrac{10^{-9}}{36\pi}} = 1.8$$

$$2\theta_\eta = 60.9^o \quad \longrightarrow \quad \theta_\eta = \underline{\underline{30.47^o}}$$

Prob. 10.10

For silver, the loss tangent is

$$\frac{\sigma}{\omega\varepsilon} = \frac{6.1\times10^{7}}{2\pi\times10^{8}\times\dfrac{10^{-9}}{36\pi}} = 6.1\times18\times10^{8} \gg 1$$

Hence, silver is a good conductor

For rubber,

$$\frac{\sigma}{\omega\varepsilon} = \frac{10^{-15}}{2\pi\times10^{8}\times3.1\times\dfrac{10^{-9}}{36\pi}} = \frac{18}{3.1}\times10^{-14} \ll 1$$

Hence, rubber is a poor conductor or a good insulator.

Prob. 10.11

$$\frac{\sigma}{\omega\varepsilon} = \frac{4}{2\pi\times10^{5}\times80\times10^{-9}/36\pi} = 9{,}000 >> 1$$

$$\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\frac{2\pi\times10^{5}}{2}\times4\pi\times10^{-7}\times4} = 0.4\pi$$

(a) $u = \omega/\beta = \dfrac{2\pi\times10^{5}}{0.4\pi} = \underline{\underline{5\times10^{5}}}$ m/s

(b) $\lambda = 2\pi/\beta = \dfrac{2\pi}{0.4\pi} = \underline{\underline{5}}$ m

(c) $\delta = 1/\alpha = \dfrac{1}{0.4\pi} = \underline{\underline{0.796}}\,\text{m}$

(d) $\eta = |\eta| \angle\theta_\eta, \theta_\eta = 45^o$

$$|\eta| = \frac{\sqrt{\frac{\mu}{\varepsilon}}}{\sqrt[4]{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}} \cong \sqrt{\frac{\mu}{\varepsilon}\frac{\omega\varepsilon}{\sigma}} = \sqrt{\frac{4\pi\times10^{-7}\times2\pi\times10^5}{4}} = 0.4443$$

$\eta = \underline{\underline{0.4443\angle45^o \quad \Omega}}$

Prob. 10.12

$$\frac{\sigma}{\omega\varepsilon} = \frac{1}{2\pi\times10^9\times4\times\frac{10^{-9}}{36\pi}} = 4.5$$

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]}$$

$$= 2\pi\times10^9\sqrt{\frac{4\pi}{2}\times10^{-7}\times4\times9\times\frac{10^{-9}}{36\pi}\left[\sqrt{1+4.5^2} - 1\right]}$$

$$= 20\pi\sqrt{2[\sqrt{21.25} - 1]} = \underline{\underline{168.8 \text{ Np/m}}}$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} + 1\right]} = 20\pi\sqrt{2[\sqrt{21.25} + 1]} = \underline{\underline{210.5 \text{ rad/m}}}$$

$$\tan 2\theta_\eta = \frac{\sigma}{\omega\varepsilon} = 4.5 \quad \longrightarrow \quad \theta_\eta = 38.73^o$$

$$|\eta| = \frac{\sqrt{\mu/\varepsilon}}{\sqrt[4]{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}} = \frac{120\pi\sqrt{9/4}}{\sqrt[4]{1+4.5^2}} = 263.38$$

$\eta = \underline{\underline{263.38\angle38.73^o \quad \Omega}}$

$$u = \frac{\omega}{\beta} = \frac{2\pi\times10^9}{210.5} = \underline{\underline{2.985\times10^7 \text{ m/s}}}$$

Prob. 10.13

This is a lossy medium in which $\mu = \mu_o$.

Let $x = \left(\dfrac{\sigma}{\omega\varepsilon}\right)^2$

$$\eta = \frac{j\omega\mu}{\gamma} = \frac{j2\pi\times10^9\times4\pi}{100+j200} = 35.31\angle26.57^o$$

$$E_o = 0.05\times35.31 = 1.765$$

$$\boldsymbol{a}_E = \boldsymbol{a}_H \times \boldsymbol{a}_k = -\boldsymbol{a}_z$$

Thus, we obtain

$$\underline{\underline{\boldsymbol{E} = -1.765\cos(2\pi\times10^9 t - 200x + 26.57^o)\boldsymbol{a}_z \text{ V/m}}}$$

$$\sqrt{\varepsilon_r(1/3)} = \frac{\alpha c}{\omega} = \frac{100\times3\times10^8}{2\pi\times10^9} = \frac{15}{\pi}$$

$$\varepsilon_r\frac{1}{3} = 4.776 \longrightarrow \varepsilon_r = 14.32$$

$$\tan2\theta_\eta = \frac{\sigma}{\omega\varepsilon} = \frac{4}{3} \longrightarrow \theta_\eta = 26.57^o$$

$$|\eta| = \frac{\sqrt{\mu/\varepsilon}}{\sqrt[4]{1+x}} = \frac{\dfrac{377}{\sqrt{14.32}}}{\sqrt{5/3}} = 77.175$$

$$E_o = |\eta| H_o = 77.175\times50\times10^{-3} = 3.858$$

$$\boldsymbol{a}_E = -(\boldsymbol{a}_k\times\boldsymbol{a}_H) = -(\boldsymbol{a}_x\times\boldsymbol{a}_y) = -\boldsymbol{a}_z$$

$$\boldsymbol{E} = \underline{\underline{-3.858e^{-100x}\cos(2\pi\times10^9 t - 200x + 26.57^o)\boldsymbol{a}_z \text{ V/m}}}$$

Prob. 10.14 (a)

$$T = 1/f = 2\pi/\omega = \frac{2\pi}{\pi x10^8} = \underline{\underline{20 \text{ ns}}}$$

(b) Let $x = \sqrt{1+\left(\dfrac{\sigma}{\omega\varepsilon}\right)^2}$

$$\frac{\alpha}{\beta} = \left(\frac{x-1}{x+1}\right)^{1/2}$$

But $\alpha = \dfrac{\omega}{c}\sqrt{\dfrac{\mu_r\varepsilon_r}{2}}\sqrt{x-1}$

$$\sqrt{x-1} = \frac{\alpha c}{\omega\sqrt{\frac{\mu_r \varepsilon_r}{2}}} = \frac{0.1\times 3\times 10^8}{\pi\times 10^8\sqrt{2}} = 0.06752 \longrightarrow x = 1.0046$$

$$\beta = \left(\frac{x+1}{x-1}\right)^{1/2}\alpha = \left(\frac{2.0046}{0.0046}\right)^{1/2} 0.1 = 2.088$$

$$\lambda = 2\pi/\beta = \frac{2\pi}{2.088} = \underline{\underline{3}}\text{ m}$$

(c) $$|\eta| = \frac{\sqrt{\mu/\varepsilon}}{\sqrt{x}} = \frac{377}{2\sqrt{1.0046}} = 188.1$$

$$x = \sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} = 1.0046$$

$$\frac{\sigma}{\omega\varepsilon} = 0.096 = \tan 2\theta_\eta \longrightarrow \theta_\eta = 2.74^o$$

$$\eta = 188.1\angle 2.74^o \quad \Omega$$

$$E_o = \eta H_o = 12\times 188.1 = 2257.2$$

$$\boldsymbol{a}_E\times\boldsymbol{a}_H = \boldsymbol{a}_k \longrightarrow \boldsymbol{a}_E\times\boldsymbol{a}_x = \boldsymbol{a}_y \longrightarrow \boldsymbol{a}_E = \boldsymbol{a}_z$$

$$\boldsymbol{E} = \underline{\underline{2.257e^{-0.1y}\sin(\pi\times 10^8 t - 2.088y + 2.74^o)\boldsymbol{a}_z}} \quad \text{kV/m}$$

(d) The phase difference is $\underline{\underline{2.74^o}}$.

Prob. 10.15

(a) $\beta = 6.5 = \omega\sqrt{\mu_o \varepsilon_o} = \dfrac{\omega}{c}$

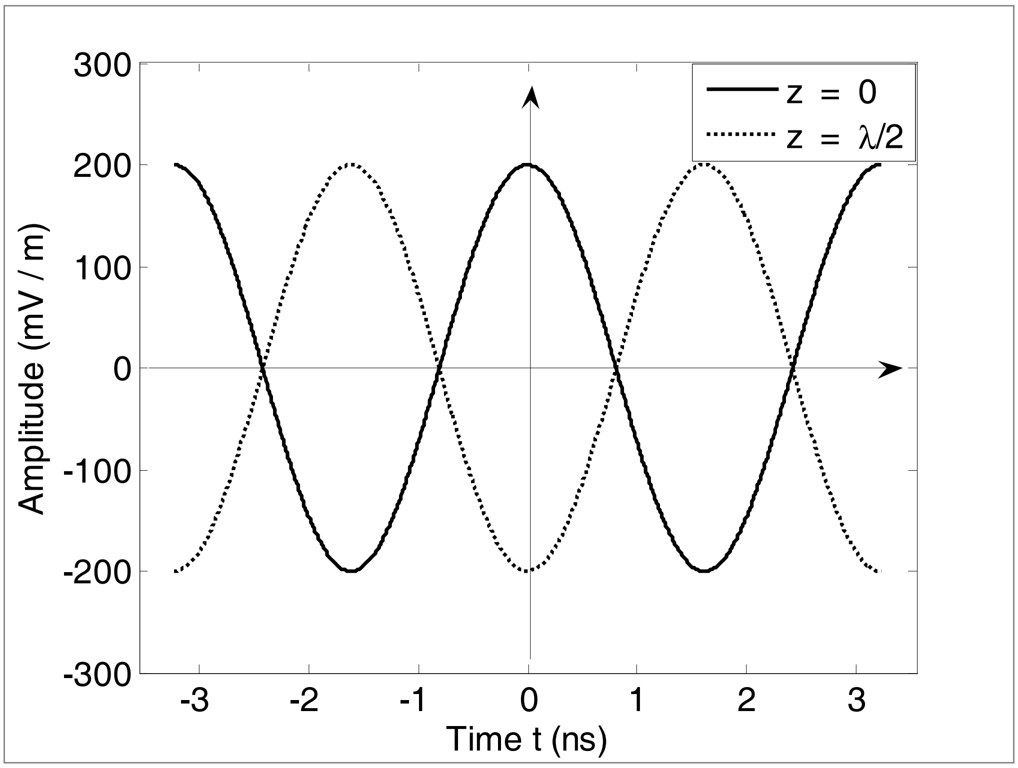
$\omega = \beta c = 6.5 \times 3 \times 10^8 = \underline{\underline{1.95 \times 10^9 \text{ rad/s}}}$

$\lambda = \dfrac{2\pi}{\beta} = \dfrac{2\pi}{6.5} = \underline{\underline{0.9666 \text{ m}}}$

(b) For z=0, $E_z = 0.2\cos\omega t$

For z=λ/2, $E_z = 0.2\cos(\omega t - \dfrac{2\pi}{\lambda}\dfrac{\lambda}{2}) = -0.2\cos\omega t$

The two waves are sketched below.



(c) $\boldsymbol{H} = H_o \cos(\omega t - 6.5z)\boldsymbol{a}_H$

$H_o = \dfrac{E_o}{\eta_o} = \dfrac{0.2}{377} = 5.305 \times 10^{-4}$

$\boldsymbol{a}_E \times \boldsymbol{a}_H = \boldsymbol{a}_k \quad \longrightarrow \quad \boldsymbol{a}_x \times \boldsymbol{a}_H = \boldsymbol{a}_z \quad \longrightarrow \quad \boldsymbol{a}_H = \boldsymbol{a}_y$

$\boldsymbol{H} = \underline{\underline{0.5305\cos(\omega t - 6.5z)\boldsymbol{a}_y \text{ mA/m}}}$

Prob. 10.16

$$u = \frac{c}{\sqrt{\varepsilon_r \mu_r}} = \frac{3\times10^8}{\sqrt{3\times4}} = \underline{\underline{8.66\times10^7 \text{ m/s}}}$$

$$\lambda = \frac{u}{f} = \frac{8.66\times10^7}{60\times10^6} = \underline{\underline{1.443 \text{ m}}}$$

$$\eta = \eta_o\sqrt{\frac{\mu_r}{\varepsilon_r}} = 377\sqrt{\frac{4}{3}} = \underline{\underline{435.32\ \Omega}}$$

Prob. 10.17 (a) Along $\underline{\underline{\text{-x direction.}}}$

(b) $\beta = 6, \quad \omega = 2\times10^8$,

$$\beta = \omega\sqrt{\mu\varepsilon} = \frac{\omega}{c}\sqrt{\mu_r\varepsilon_r}$$

$$\sqrt{\varepsilon_r} = \beta c / \omega = \frac{6\times3\times10^8}{2\times10^8} = 9 \quad \longrightarrow \quad \varepsilon_r = 81$$

$$\varepsilon = \varepsilon_o\varepsilon_r = \frac{10^{-9}}{36\pi}\times81 = \underline{\underline{7.162\times10^{-10}}} \text{ F/m}$$

(c) $\eta = \sqrt{\mu/\varepsilon} = \sqrt{\mu_o/\varepsilon_o}\sqrt{\mu_r/\varepsilon_r} = \dfrac{120\pi}{9} = 41.89\ \Omega$

$$E_o = H_o\eta = 25\times10^{-3}\times41.88 = 1.047$$

$$\boldsymbol{a}_E\times\boldsymbol{a}_H = \boldsymbol{a}_k \longrightarrow \boldsymbol{a}_E\times\boldsymbol{a}_y = -\boldsymbol{a}_x \longrightarrow \boldsymbol{a}_E = \boldsymbol{a}_z$$

$$\boldsymbol{E} = \underline{\underline{1.047\sin(2\times10^8 t + 6x)\boldsymbol{a}_z}} \text{ V/m}$$

Prob. 10.18 (a) $\dfrac{\sigma}{\omega\varepsilon} = \dfrac{10^{-6}}{2\pi\times10^7\times5\times\dfrac{10^{-9}}{36\pi}} = 3.6\times10^{-4} << 1$

Thus, the material is <u>lossless</u> at this frequency.

(b) $\beta = \omega\sqrt{\mu\varepsilon} = \dfrac{2\pi\times10^7}{3\times10^8}\sqrt{5\times750} = \underline{\underline{12.83}}$ rad/m

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{12.83} = \underline{\underline{0.49}} \text{ m}$$

(c) Phase difference = $\beta l = \underline{\underline{25.66 \text{ rad}}}$

(d) $\eta = \sqrt{\mu / \varepsilon} = 120\pi\sqrt{\frac{\mu_r}{\varepsilon_r}} = 120\pi\sqrt{\frac{750}{5}} = \underline{\underline{4.62 \text{ k}\Omega}}$

Prob. 10.19

(a) $\boldsymbol{E} = \text{Re}[\boldsymbol{E}_s e^{j\omega t}] = (5\boldsymbol{a}_x + 12\boldsymbol{a}_y)e^{-0.2z}\cos(\omega t - 3.4z)$

At $z = 4$m, $t = T/8$, $\omega t = \frac{2\pi}{T}\frac{T}{8} = \frac{\pi}{4}$

$$\boldsymbol{E} = (5\boldsymbol{a}_x + 12\boldsymbol{a}_y)e^{-0.8}\cos(\pi/4 - 13.6)$$

$$|E| = 13e^{-0.8}|\cos(\pi/4 - 13.6)| = \underline{\underline{5.662}} \text{ V/m}$$

(b) loss $= \alpha\Delta z = 0.2(3) = 0.6$ Np. Since 1 Np = 8.686 dB,

loss = 0.6 x 8.686 = $\underline{\underline{5.212 \text{ dB}}}$

(c) Let $x = \sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2}$

$$\frac{\alpha}{\beta} = \left(\frac{x-1}{x+1}\right)^{1/2} = 0.2/3.4 = \frac{1}{17}$$

$$\frac{x-1}{x+1} = 1/289 \longrightarrow x = 1.00694$$

$$\alpha = \omega\sqrt{\mu\varepsilon/2}\sqrt{x-1} = \frac{\omega}{c}\sqrt{\varepsilon_r/2}\sqrt{x-1}$$

$$\sqrt{\frac{\varepsilon_r}{2}} = \frac{\alpha c}{\omega\sqrt{x-1}} = \frac{0.2 \times 3 \times 10^8}{10^8\sqrt{0.00694}} = 7.2 \longrightarrow \varepsilon_r = 103.68$$

$$|\eta| = \frac{\sqrt{\frac{\mu_o}{\varepsilon_o}} \cdot \frac{1}{\sqrt{\varepsilon_r}}}{\sqrt{x}} = \frac{120\pi}{\sqrt{103.68 \times 1.00694}} = 36.896$$

$$\tan 2\theta_\eta = \frac{\sigma}{\omega\varepsilon} = \sqrt{x^2 - 1} = 0.118 \quad \longrightarrow \quad \theta_\eta = 3.365^o$$

$$\eta = \underline{\underline{36.896\angle 3.365^o \ \Omega}}$$

Prob. 10.20

This is a lossless material.

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = 377\sqrt{\frac{\mu_r}{\varepsilon_r}} = 105 \qquad (1)$$

$$u = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{\mu\varepsilon}} = \frac{c}{\sqrt{\mu_r\varepsilon_r}} = 7.6\times 10^7 \qquad (2)$$

From (1),

$$\sqrt{\frac{\mu_r}{\varepsilon_r}} = \frac{105}{377} = 0.2785 \qquad (1)a$$

From (2),

$$\frac{1}{\sqrt{\mu_r\varepsilon_r}} = \frac{7.6\times 10^7}{3\times 10^8} = 0.2533 \qquad (2)a$$

Multiplying (1)a by (2)a,

$$\frac{1}{\varepsilon_r} = 0.2785\times 0.2533 = 0.07054 \quad \longrightarrow \quad \varepsilon_r = \underline{\underline{14.175}}$$

Dividing (1)a by (2)a,

$$\mu_r = \frac{0.2785}{0.2533} = \underline{\underline{1.0995}}$$

Prob. 10.21

(a) $$\nabla\times\boldsymbol{E} = \begin{vmatrix} \boldsymbol{a}_x & \boldsymbol{a}_y & \boldsymbol{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x(z,t) & E_y(z,t) & 0 \end{vmatrix} = -\frac{\partial E_y}{\partial z}\boldsymbol{a}_x + \frac{\partial E_x}{\partial z}\boldsymbol{a}_y$$

$$= -6\beta\cos(\omega t - \beta z)\boldsymbol{a}_x + 8\beta\sin(\omega t - \beta z)\boldsymbol{a}_y$$

But $\nabla\times\boldsymbol{E} = -\mu\frac{\partial\boldsymbol{H}}{\partial t} \quad \longrightarrow \quad \boldsymbol{H} = -\frac{1}{\mu}\int\nabla\times\boldsymbol{E}dt$

$$\boldsymbol{H} = \underline{\underline{\frac{6\beta}{\mu\omega}\sin(\omega t - \beta z)\boldsymbol{a}_x + \frac{8\beta}{\mu\omega}\cos(\omega t - \beta z)\boldsymbol{a}_y}}$$

(b) $\beta = \omega\sqrt{\mu\varepsilon} = \dfrac{2\pi f}{c}\sqrt{4.5} = \dfrac{2\pi \times 40 \times 10^6}{3\times 10^8}\sqrt{4.5} = \underline{\underline{1.777 \text{ rad/m}}}$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{1.777} = \underline{\underline{3.536 \text{ m}}}$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} = \frac{120\pi}{\sqrt{4.5}} = \underline{\underline{177.72\ \Omega}}$$

$$u = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{\sqrt{4.5}} = \frac{3\times 10^8}{\sqrt{4.5}} = \underline{\underline{1.4142\times 10^8 \text{ m/s}}}$$

Prob. 10.22

$$0.4E_o = E_o e^{-\alpha z} \quad \longrightarrow \quad \frac{1}{0.4} = e^{2\alpha}$$

Or $\quad \alpha = \dfrac{1}{2}\ln\dfrac{1}{0.4} = 0.4581 \quad \longrightarrow \quad \delta = 1/\alpha = \underline{\underline{2.183}}$ m

$$\lambda = 2\pi/\beta = 2\pi/1.6$$

$$u = f\lambda = 10^7 \times \frac{2\pi}{1.6} = \underline{\underline{3.927\times 10^7}} \text{ m/s}$$

Prob. 10.23

(a)

$$\beta = \omega\sqrt{\mu\varepsilon} = \frac{\omega}{c} = \frac{10^8\pi}{3\times 10^8} = \frac{\pi}{3} = \underline{\underline{1.0472 \text{ rad/m}}}$$

(b)

$$E = 0 \quad \longrightarrow \quad \sin(10^8\pi t_o - \beta x_o) = 0 = \sin(n\pi), n = 1,2,3,...$$

$$10^8\pi t_o - \beta x_o = \pi$$

$$10^8\pi \times 5\times 10^{-3} - \frac{\pi}{3}x_o = \pi \quad \longrightarrow \quad x_o \simeq \underline{\underline{5\times 10^5 \text{ m}}}$$

(c)

$$\boldsymbol{H} = H_o \sin(10^8\pi t - \beta x)\boldsymbol{a}_H$$

$$H_o = \frac{E_o}{\eta} = \frac{50\times 10^{-3}}{120\pi} = 132.63\ \mu\text{A/m}$$

$$\boldsymbol{a}_H = \boldsymbol{a}_k \times \boldsymbol{a}_E = \boldsymbol{a}_x \times \boldsymbol{a}_z = -\boldsymbol{a}_y$$

$$\underline{\underline{\boldsymbol{H} = -132.63\sin(10^8\pi t - 1.0472x)\boldsymbol{a}_y\ \mu\text{A/m}}}$$

Prob. 10.24

$$\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} \longrightarrow \sigma = \frac{2\alpha^2}{\omega\mu} = \frac{2\times 12^2}{2\pi\times 10^6\times 4\pi\times 10^{-7}} = \underline{\underline{36.48}}$$

$$\eta = |\eta|\angle\theta_\eta = \sqrt{\frac{\omega\mu}{\sigma}}\angle 45^o$$

$$|\eta| = \sqrt{\frac{\omega\mu}{\sigma}} = \sqrt{\frac{2\pi\times 10^6\times 4\pi\times 10^{-7}}{36.48}} = 0.4652$$

$$E_o = |\eta| H_o = 0.4652\times 20\times 10^{-3} = 9.305\times 10^{-3}$$

$$\mathbf{a}_E = \mathbf{a}_H \times \mathbf{a}_k = \mathbf{a}_y \times (-\mathbf{a}_z) = -\mathbf{a}_x$$

$$\mathbf{E} = E_o e^{-\alpha z}\sin(\omega t + \beta z)\mathbf{a}_E$$

$$\underline{\underline{= -9.305 e^{-12z}\sin(2\pi\times 10^6 t + 12z + 45^o)\mathbf{a}_x \text{ mV/m}}}$$

Prob. 10.25 For a good conductor, $\frac{\sigma}{\omega\varepsilon} >> 1$, say $\frac{\sigma}{\omega\varepsilon} > 100$

(a) $$\frac{\sigma}{\omega\varepsilon} = \frac{10^{-2}}{2\pi\times 8\times 10^6\times 15\times\frac{10^{-9}}{36\pi}} = 1.5 \longrightarrow \text{lossy}$$

<u>No</u>, not conducting.

(b) $$\frac{\sigma}{\omega\varepsilon} = \frac{0.025}{2\pi\times 8\times 10^6\times 16\times\frac{10^{-9}}{36\pi}} = 3.515 \longrightarrow \text{lossy}$$

<u>No</u>, not conducting.

(c) $$\frac{\sigma}{\omega\varepsilon} = \frac{25}{2\pi\times 8\times 10^6\times 81\times\frac{10^{-9}}{36\pi}} = 694.4 \longrightarrow \text{conducting}$$

<u>Yes</u>, conducting.

Prob. 10.26

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} - 1\right]} = \frac{2\pi f}{c}\sqrt{\frac{\mu_r\varepsilon_r}{2}\left[\sqrt{1.0049} - 1\right]} = \frac{2\pi\times 6\times 10^6}{3\times 10^8}\sqrt{\frac{4}{2}\times 2.447\times 10^{-3}}$$

$$\alpha = 8.791\times 10^{-3}$$

$\delta = 1/\alpha = \underline{\underline{113.75 \text{ m}}}$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]} = \frac{4\pi}{100}\sqrt{\frac{4}{2}\left[\sqrt{1.0049}+1\right]} = 0.2515$$

$$u = \omega/\beta = \frac{2\pi\times 6\times 10^6}{0.2515} = \underline{\underline{1.5\times 10^8}} \text{ m/s}$$

Prob. 10.27 (a)

$$R_{dc} = \frac{l}{\sigma S} = \frac{l}{\sigma\pi a^2} = \frac{600}{5.8\times 10^7\times\pi\times(1.2)^2\times 10^{-6}} = \underline{\underline{2.287\ \Omega}}$$

(b) $R_{ac} = \dfrac{l}{\sigma 2\pi a\delta}$. At 100 MHz, $\delta = 6.6\times 10^{-3}$ mm $=6.6\times10^{-6}$ m mm for copper (see Table 10.2).

$$R_{ac} = \frac{600}{5.8\times 10^7\times 2\pi\times(1.2\times 10^{-3})\times 6.6\times 10^{-6}} = \underline{\underline{207.61\ \Omega}}$$

(c) $\dfrac{R_{ac}}{R_{dc}} = \dfrac{a}{2\delta} = 1 \quad\longrightarrow\quad \delta = a/2 = \dfrac{66.1\times 10^{-3}}{\sqrt{f}}$

$$\sqrt{f} = \frac{66.1\times 2\times 10^{-3}}{a} = \frac{66.1\times 2}{1.2} \quad\longrightarrow\quad f = \underline{\underline{12.137}} \text{ kHz}$$

Prob. 10.28

(a) $\tan\theta = \dfrac{\sigma}{\omega\varepsilon} = \dfrac{3.5\times 10^7}{2\pi\times 150\times 10^6\times\dfrac{10^{-9}}{36\pi}} = \dfrac{3.5\times 18\times 10^9}{15} >> 1$

$$\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f\mu\sigma} = \sqrt{150\pi\times 10^6\times 4\pi\times 10^{-7}\times 3.5\times 10^7} = 143{,}965.86$$

$\gamma = \alpha + j\beta = \underline{\underline{1.44(1+j)\times 10^5 \text{ /m}}}$

(b) $\delta = 1/\alpha = \underline{\underline{6.946\times 10^{-6} \text{ m}}}$

(c) $u = \dfrac{\omega}{\beta} = \dfrac{150\times 2\pi\times 10^6}{1.44\times 10^5} = \underline{\underline{6547 \text{ m/s}}}$

Prob. 10.29

$$\frac{\sigma}{\omega\varepsilon} = \frac{4}{2\pi\times 2\times 10^9 \times 24\times \dfrac{10^{-9}}{36\pi}} = 1.5$$

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right]} = 2\pi\times 2\times 10^9\sqrt{\frac{4\pi\times 10^{-7}\times 24\times \dfrac{10^{-9}}{36\pi}}{2}\left[\sqrt{1+(1.5)^2}-1\right]}$$

$$= 130.01 \text{ Np/m}$$

$$10^{-5}E_o = E_o e^{-\alpha d}$$

Taking the log of both sides gives

$$\text{-}5\ln 10 = -\alpha d \quad \longrightarrow \quad d = \frac{5\ln 10}{\alpha} = \frac{5\ln 10}{130.01} = \underline{\underline{0.0886 \text{ m}}}$$

Prob. 10.30

$$\alpha = \beta = 1/\delta$$

$$\lambda = 2\pi/\beta = 2\pi\delta = 6.283\delta \quad \longrightarrow \quad \delta = 0.1591\lambda$$

showing that δ is shorter than λ.

Prob. 10.31

$$t = 5\delta = \frac{5}{\sqrt{\pi f \mu\sigma}} = \frac{5}{\sqrt{\pi\times 12\times 10^9\times 4\pi\times 10^{-7}\times 6.1\times 10^7}} = \underline{\underline{2.94\times 10^{-6} \text{ m}}}$$

Prob. 10.32

$$\delta = \frac{1}{\sqrt{\pi f\mu\sigma}} \quad \longrightarrow \quad f = \frac{1}{\delta^2\pi\mu\sigma}$$

$$f = \frac{1}{4\times 10^{-6}\times\pi\times 4\pi\times 10^{-7}\times 6.1\times 10^7} = \underline{\underline{1.038 \text{ kHz}}}$$

Prob. 10.33

(a) Linearly polarized along $\mathbf{a}_z$

(b) $\omega = 2\pi f = 2\pi \times 10^7 \longrightarrow f = 10^7 = \underline{\underline{10 \text{ MHz}}}$

(c)
$$\beta = \omega\sqrt{\mu\varepsilon} = \omega\sqrt{\mu_o \varepsilon_o}\sqrt{\varepsilon_r} = \frac{\omega}{c}\sqrt{\varepsilon_r}$$

$$\sqrt{\varepsilon_r} = \frac{\beta c}{\omega} = \frac{3 \times 3 \times 10^8}{2\pi \times 10^7} = 14.32 \longrightarrow \varepsilon_r = \underline{\underline{205.18}}$$

Let $\boldsymbol{H} = H_o \sin(\omega t - 3y)\boldsymbol{a}_H$

$$H_o = \frac{E_o}{\eta}, \quad \eta = \sqrt{\frac{\mu}{\varepsilon}} = \frac{\eta_o}{\sqrt{\varepsilon_r}} = \frac{120\pi}{14.32} = 26.33$$

(d) $H_o = \dfrac{12}{26.33} = 0.456$

$$\boldsymbol{a}_H = \boldsymbol{a}_k \times \boldsymbol{a}_E = \boldsymbol{a}_y \times \boldsymbol{a}_z = \boldsymbol{a}_x$$

$$\boldsymbol{H} = \underline{\underline{0.456\sin(2\pi \times 10^7 t - 3y)\boldsymbol{a}_x \text{ A/m}}}$$

Prob. 10.34

$\boldsymbol{E} = (2\boldsymbol{a}_y - 5\boldsymbol{a}_z)\sin(\omega t - \beta x)$

The ratio E_y / E_z remains the same as t changes. Hence the wave is $\underline{\underline{\text{linearly}}}$ polarized

Prob. 10.35

(a)

$E_x = E_o \cos(\omega t + \beta y), \quad E_y = E_o \sin(\omega t + \beta y)$

$$E_x(0,t) = E_o \cos\omega t \longrightarrow \cos\omega t = \frac{E_x(0,t)}{E_o}$$

$$E_y(0,t) = E_o \sin\omega t \longrightarrow \sin\omega t = \frac{E_y(0,t)}{E_o}$$

$$\cos^2\omega t + \sin^2\omega t = 1 \longrightarrow \left(\frac{E_x}{E_o}\right)^2 + \left(\frac{E_y}{E_o}\right)^2 = 1$$

Hence, we have $\underline{\underline{\text{circular polarization.}}}$

(b)

$E_x = E_o \cos(\omega t - \beta y), \quad E_y = -3E_o \sin(\omega t - \beta y)$

In the y=0 plane,

$$E_x(0,t) = E_o \cos \omega t \quad \longrightarrow \quad \cos \omega t = \frac{E_x(0,t)}{E_o}$$

$$E_y(0,t) = E_o \sin \omega t \quad \longrightarrow \quad \sin \omega t = \frac{-E_y(0,t)}{3E_o}$$

$$\cos^2 \omega t + \sin^2 \omega t = 1 \quad \longrightarrow \quad \left(\frac{E_x}{E_o}\right)^2 + \frac{1}{9}\left(\frac{E_y}{E_o}\right)^2 = 1$$

Hence, we have elliptical polarization.

Prob. 10.36

(a) We can write

$$\boldsymbol{E} = \text{Re}(E_s e^{j\omega t}) = (40\boldsymbol{a}_x + 60\boldsymbol{a}_y)\cos(\omega t - 10z)$$

Since E_x / E_y does not change with time, the wave is linearly polarized.

(b) This is elliptically polarized.

Prob. 10.37

(a)

When $\phi = 0$,

$$\boldsymbol{E}(y,t) = (E_{o1}\boldsymbol{a}_x + E_{o2}\boldsymbol{a}_z)\cos(\omega t - \beta y)$$

The two components are in phase and the wave is linearly polarized.

(b)

When $\phi = \pi / 2$,

$$E_z = E_{o2}\cos(\omega t - \beta y + \pi / 2) = -E_{o2}\sin(\omega t - \beta y)$$

We can combine E_x and E_z to show that the wave is elliptically polarized.

(c)

When $\phi = \pi$,

$$\boldsymbol{E}(y,t) = E_{o1}\cos(\omega t - \beta y)\boldsymbol{a}_x + E_{o2}\cos(\omega t - \beta y + \pi)\boldsymbol{a}_z$$
$$= (E_{o1}\,\boldsymbol{a}_x - E_{o2}\boldsymbol{a}_y)\cos(\omega t - \beta y)$$

Thus, the wave is linearly polarized.

Prob. 10.38
We can write $\mathbf{E}_s$ as

$$\boldsymbol{E}_s = \boldsymbol{E}_1(z) + \boldsymbol{E}_2(z)$$

where

$$\boldsymbol{E}_1(z) = \frac{1}{2}E_o(\boldsymbol{a}_x - j\boldsymbol{a}_y)e^{-j\beta z}$$

$$\boldsymbol{E}_2(z) = \frac{1}{2}E_o(\boldsymbol{a}_x + j\boldsymbol{a}_y)e^{-j\beta z}$$

We recognize that $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ are circularly polarized waves. The problem is therefore proved.

Prob. 10.39

(a) The wave is <u>elliptically</u> polarized.
(b)

Let $\boldsymbol{E} = \boldsymbol{E}_1 + \boldsymbol{E}_2$,

where $\boldsymbol{E}_1 = 40\cos(\omega t - \beta z)\boldsymbol{a}_x$, $\quad \boldsymbol{E}_2 = 60\sin(\omega t - \beta z)\boldsymbol{a}_y$

$$\boldsymbol{H}_1 = H_{o1}\cos(\omega t - \beta z)\boldsymbol{a}_{H1}$$

$$H_{o1} = \frac{40}{\eta_o} = \frac{40}{120\pi} = 0.106$$

$$\boldsymbol{a}_{H1} = \boldsymbol{a}_k \times \boldsymbol{a}_E = \boldsymbol{a}_z \times \boldsymbol{a}_x = \boldsymbol{a}_y$$

$$\boldsymbol{H}_1 = 0.106\cos(\omega t - \beta z)\boldsymbol{a}_y$$

$$\boldsymbol{H}_2 = H_{o2}\sin(\omega t - \beta z)\boldsymbol{a}_{H2}$$

$$H_{o1} = \frac{60}{\eta_o} = \frac{60}{120\pi} = 0.1592$$

$$\boldsymbol{a}_{H2} = \boldsymbol{a}_k \times \boldsymbol{a}_E = \boldsymbol{a}_z \times \boldsymbol{a}_y = -\boldsymbol{a}_x$$

$$\boldsymbol{H}_2 = -0.1592\sin(\omega t - \beta z)\boldsymbol{a}_x$$

$$\boldsymbol{H} = \boldsymbol{H}_1 + \boldsymbol{H}_2 = \underline{\underline{-159.2\sin(\omega t - \beta z)\boldsymbol{a}_x + 106\cos(\omega t - \beta z)\boldsymbol{a}_y \text{ mA/m}}}$$

Prob. 10.40
Let $\boldsymbol{E}_s = \boldsymbol{E}_r + j\boldsymbol{E}_i$ and $\boldsymbol{H}_s = \boldsymbol{H}_r + j\boldsymbol{H}_i$

$$\boldsymbol{E} = \text{Re}(\boldsymbol{E}_s e^{j\omega t}) = \boldsymbol{E}_r\cos\omega t - \boldsymbol{E}_i\sin\omega t$$

Similarly,

$$\boldsymbol{H} = \boldsymbol{H}_r\cos\omega t - \boldsymbol{H}_i\sin\omega t$$

$$\mathscr{P} = \boldsymbol{E}\times\boldsymbol{H} = \boldsymbol{E}_r\times\boldsymbol{H}_r\cos^2\omega t + \boldsymbol{E}_i\times\boldsymbol{H}_i\sin^2\omega t - \frac{1}{2}(\boldsymbol{E}_r\times\boldsymbol{H}_i + \boldsymbol{E}_i\times\boldsymbol{H}_r)\sin\ 2\omega t$$

$$\mathscr{P}_{\text{ave}} = \frac{1}{T}\int_0^T \mathscr{P}dt = \frac{1}{T}\int_0^T \cos^2\omega dt(\boldsymbol{E}_r\times\boldsymbol{H}_r) + \frac{1}{T}\int_0^T \sin^2\omega dt(\boldsymbol{E}_i\times\boldsymbol{H}_i) - \frac{1}{2T}\int_0^T \sin 2\omega dt(\boldsymbol{E}_i\times\boldsymbol{H}_i + \boldsymbol{E}_i\times\boldsymbol{H}_r)$$

$$= \frac{1}{2}(\boldsymbol{E}_r\times\boldsymbol{H}_r + \boldsymbol{E}_i\times\boldsymbol{H}_i) = \frac{1}{2}\text{Re}[(\boldsymbol{E}_r + j\boldsymbol{E}_i)\times(\boldsymbol{H}_r - j\boldsymbol{H}_i)]$$

$$\mathscr{P}_{\text{ave}} = \frac{1}{2}\ \text{Re}(\boldsymbol{E}_s\times\boldsymbol{H}_s^*)$$

as required.

Prob. 10.41

(a)

$$\beta = \omega\sqrt{\mu\varepsilon} = \omega\sqrt{\mu_o\varepsilon_o}\sqrt{\varepsilon_r} = \frac{\omega}{c}\sqrt{\varepsilon_r}$$

$$\sqrt{\varepsilon_r} = \frac{\beta c}{\omega} = \frac{8\times 3\times 10^8}{10^9} = 2.4$$

$$\varepsilon_r = \underline{\underline{5.76}}$$

(b) $\eta = \sqrt{\dfrac{\mu}{\varepsilon}} = \sqrt{\dfrac{\mu_o}{\varepsilon_o}}\dfrac{1}{\sqrt{\varepsilon_r}} = \dfrac{377}{2.4} = \underline{\underline{157.1\ \Omega}}$

(c) $u = \dfrac{\omega}{\beta} = \dfrac{10^9}{8} = \underline{\underline{1.25\times 10^8\ \text{m/s}}}$

(d)

Let $\boldsymbol{H} = H_o\cos(10^9 t + 8x)\boldsymbol{a}_H$

$$H_o = \frac{E_o}{\eta} = \frac{150}{157.1} = 0.955$$

$$\boldsymbol{a}_H = \boldsymbol{a}_k\times\boldsymbol{a}_E = -\boldsymbol{a}_x\times\boldsymbol{a}_z = \boldsymbol{a}_y$$

$$\boldsymbol{H} = \underline{\underline{0.955\cos(10^9 t + 8x)\boldsymbol{a}_y\ \text{A/m}}}$$

(e)

$$\mathscr{P} = \boldsymbol{E}\times\boldsymbol{H} = -150(0.955)\cos^2(10^9 t + 8x)\boldsymbol{a}_x$$

$$= \underline{\underline{-143.25\cos^2(10^9 t + 8x)\boldsymbol{a}_x\ \text{W/m}^2}}$$

Prob. 10.42

$$\boldsymbol{P} = \boldsymbol{E} \times \boldsymbol{H} = \begin{vmatrix} E_x & 0 & E_z \\ 0 & H_y & 0 \end{vmatrix} = -E_z H_y \boldsymbol{a}_x + E_x H_y \boldsymbol{a}_z$$

$$= -H_o E_{o2} \sin\alpha x \cos\alpha x \sin(\omega t - \beta x)\cos(\omega t - \beta z)\boldsymbol{a}_x$$
$$+ H_o E_{o1} \cos^2 \alpha x \cos^2(\omega t - \beta z)\boldsymbol{a}_z$$

$$= \quad -\frac{1}{4} H_o E_{o2} \sin 2\alpha x \sin 2(\omega t - \beta x)\boldsymbol{a}_x$$
$$+ H_o E_{o1} \cos^2 \alpha x \cos^2(\omega t - \beta z)\boldsymbol{a}_z$$

$$\boldsymbol{P}_{ave} = \frac{1}{T}\int_0^T \boldsymbol{P}dt = 0 + \frac{1}{2} H_o E_{o1} \cos^2 \alpha x \boldsymbol{a}_z$$

$$= \underline{\underline{\frac{1}{2} E_{o1} H_o \cos^2 \alpha x \boldsymbol{a}_z}}$$

Prob. 10.43

(a)

Let $\quad \boldsymbol{H}_s = \dfrac{H_o}{r} \sin\theta e^{-j3r} \boldsymbol{a}_H$

$$H_o = \frac{E_o}{\eta_o} = \frac{10}{120\pi} = \frac{1}{12\pi}$$

$$\boldsymbol{a}_H = \boldsymbol{a}_k \times \boldsymbol{a}_E = \boldsymbol{a}_r \times \boldsymbol{a}_\theta = \boldsymbol{a}_\phi$$

$$\underline{\underline{\boldsymbol{H}_s = \frac{1}{12\pi r} \sin\theta e^{-j3r} \boldsymbol{a}_\phi \text{ A/m}}}$$

(b)

$$\mathscr{P}_{ave} = \frac{1}{2}\text{Re}(\boldsymbol{E}_s \times \boldsymbol{H}_s) = \frac{10}{2 \times 12\pi r^2} \sin^2\theta \boldsymbol{a}_r$$

$$P_{ave} = \int_S \mathscr{P}_{ave} \bullet d\boldsymbol{S}, \qquad d\boldsymbol{S} = r^2 \sin\theta d\theta d\phi \boldsymbol{a}_r$$

$$P_{ave} = \frac{10}{24\pi} \int_{\phi=0}^{\pi} \int_{\theta=0}^{\pi/6} r^2 \sin^3\theta d\theta \Big|_{r=2} = \frac{5}{8} - \frac{5\sqrt{3}}{32} = 0.007145$$

$$= \underline{\underline{7.145 \text{ mW}}}$$

Prob. 10.44

(a) $\quad P_{ave} = \dfrac{1}{2}\text{Re}(E_s H_s^*) = \dfrac{1}{2}\text{Re}(\dfrac{|E_s|}{|\eta|}) = \dfrac{8^2}{2|\eta|} e^{-0.2z}$

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}-1\right]}$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]}$$

Let $\quad x = \sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}$

$$\frac{\alpha}{\beta} = \frac{\sqrt{x-1}}{\sqrt{x+1}} = 0.1/0.3 = 1/3$$

$$\frac{x-1}{x+1} = \frac{1}{9} \quad \longrightarrow \quad x = 5/4$$

$$\frac{5}{4} = \sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2} \quad \longrightarrow \quad \frac{\sigma}{\omega\varepsilon} = 3/4$$

$$|\eta| = \frac{\sqrt{\frac{\mu}{\varepsilon}}}{\sqrt[4]{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}} = \frac{120\pi/\sqrt{81}}{\sqrt{\frac{5}{4}}} = 37.4657$$

$$P_{ave} = \frac{64}{2(37.4657)}e^{-0.2z} = \underline{\underline{0.8541e^{-0.2z}\ \text{W/m}^2}}$$

(b) $20dB = 10\log\frac{P_1}{P_2} \quad \longrightarrow \quad \frac{P_1}{P_2} = 100$

$$\frac{P_2}{P_1} = e^{-0.2z} = \frac{1}{100} \quad \longrightarrow \quad e^{0.2z} = 100$$

$$z = 5\log 100 = \underline{\underline{23\ \text{m}}}$$

Prob. 10.45

(a) $u = \omega/\beta \quad \longrightarrow \quad \omega = u\beta = \frac{\beta c}{\sqrt{4.5}} = \frac{2\times 3\times 10^8}{\sqrt{4.5}} = \underline{\underline{2.828\times 10^8}}$ rad/s

$$\eta = \frac{120\pi}{\sqrt{4.5}} = 177.7\Omega$$

$$\boldsymbol{H} = \boldsymbol{a}_k \times \frac{\boldsymbol{E}}{\eta} = \frac{\boldsymbol{a}_z}{\eta} \times \frac{40}{\rho}\sin(\omega t - 2z)\boldsymbol{a}_\rho = \underline{\underline{\frac{0.225}{\rho}\sin(\omega t - 2z)\boldsymbol{a}_\phi \text{ A/m}}}$$

(b) $$\boldsymbol{P} = \boldsymbol{E} \times \boldsymbol{H} = \underline{\underline{\frac{9}{\rho^2}\sin^2(\omega t - 2z)\boldsymbol{a}_z \text{ W/m}^2}}$$

(c) $$\mathscr{P}_{\text{ave}} = \frac{4.5}{\rho^2}\boldsymbol{a}_z, \ \mathrm{d}\mathbf{S} = \rho \mathrm{d}\phi \mathrm{d}\rho \boldsymbol{a}_z$$

$$\mathrm{P}_{\text{ave}} = \int \mathbf{P}_{\text{ave}} \bullet dS = 4.5 \int_{2\text{mm}}^{3\text{mm}} \frac{\mathrm{d}\rho}{\rho} \int_0^{2\pi} \mathrm{d}\phi = 4.5\ln(3/2)(2\pi) = \underline{\underline{11.46 \text{ W}}}$$

Prob. 10.46

$$\mathscr{P} = \boldsymbol{E} \times \boldsymbol{H} = \frac{E_o^2}{\eta r^2}\sin^2\theta \sin^2\omega(t - r/c)\boldsymbol{a}_r$$

$$\boldsymbol{P}_{ave} = \frac{1}{T}\int_0^T \mathscr{P}\, dt = \underline{\underline{\frac{E_o^2}{2\eta r^2}\sin^2\theta \boldsymbol{a}_r}}$$

Prob. 10.47

$$\beta = \frac{\omega}{c} \longrightarrow \omega = \beta c = 40(3 \times 10^8) = \underline{\underline{12 \times 10^9 \text{ rad/s}}}$$

$$\boldsymbol{E} = \frac{1}{\varepsilon}\int \nabla \times \boldsymbol{H} dt$$

$$\nabla \times \boldsymbol{H} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 10\sin(\omega t - 40x) & -20\sin(\omega t - 40x) \end{vmatrix}$$

$$= -800\cos(\omega t - 40x)\boldsymbol{a}_y - 400\cos(\omega t - 40x)\boldsymbol{a}_z$$

$$\boldsymbol{E} = \frac{1}{\varepsilon}\int \nabla \times \boldsymbol{H} dt = -\frac{800}{\omega\varepsilon}\sin(\omega t - 40x)\boldsymbol{a}_y - \frac{400}{\omega\varepsilon}\sin(\omega t - 40x)\boldsymbol{a}_z$$

$$= -\frac{800}{12 \times 10^9 \times \frac{10^{-9}}{36\pi}}\sin(\omega t - 40x)\boldsymbol{a}_y - \frac{400}{12 \times 10^9 \times \frac{10^{-9}}{36\pi}}\sin(\omega t - 40x)\boldsymbol{a}_z$$

$$\underline{\underline{= -7.539\sin(\omega t - 40x)\boldsymbol{a}_y - 3.77\sin(\omega t - 40x)\boldsymbol{a}_z \text{ kV/m}}}$$

$$\boldsymbol{P} = \boldsymbol{E}\times\boldsymbol{H} = \begin{vmatrix} 0 & E_y & E_z \\ 0 & H_y & H_z \end{vmatrix} = (E_yH_z - E_zH_y)\boldsymbol{a}_x$$

$$= \left[20(7.537)\sin^2(\omega t - 40x) + 37.7\sin^2(\omega t - 40x)\right]\boldsymbol{a}_x 10^3$$

$$\boldsymbol{P}_{\text{ave}} = \frac{1}{2}\left[20(7.537) + 37.7\right]\boldsymbol{a}_x 10^3 = \underline{\underline{94.23\,\boldsymbol{a}_x\ \text{kW/m}^2}}$$

Prob. 10.48

$$P = \frac{E_o^2}{2\eta_o} \quad\longrightarrow\quad E_o^2 = 2\eta_o P = 2(120\pi)10\times10^{-3} = 7.539$$

$$E_o = \underline{\underline{2.746\ \text{V/m}}}$$

Prob. 10.49

Let $T = \omega t - \beta z$.

$$-\frac{\partial \boldsymbol{B}}{\partial t} = \nabla\times\boldsymbol{E} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos T & \sin T & 0 \end{vmatrix}$$

$$-\mu\frac{\partial \boldsymbol{H}}{\partial t} = \beta\cos T\boldsymbol{a}_x + \beta\sin T\boldsymbol{a}_y$$

$$\boldsymbol{H} = -\frac{\beta}{\mu}\int\left[\cos T\boldsymbol{a}_x + \sin T\boldsymbol{a}_y\right]dt = -\frac{\beta}{\mu\omega}\sin T\boldsymbol{a}_x + \frac{\beta}{\mu\omega}\cos T\boldsymbol{a}_y$$

$$\mathscr{P} = \boldsymbol{E}\times\boldsymbol{H} = \begin{vmatrix} cosT & sinT & 0 \\ -\frac{\beta}{\mu\omega}\sin T & \frac{\beta}{\mu\omega}\cos T & 0 \end{vmatrix} = \frac{\beta}{\mu\omega}(\cos^2 T + \sin^2 T)\boldsymbol{a}_z$$

$$= \frac{\beta}{\mu\omega}\boldsymbol{a}_z = \sqrt{\frac{\varepsilon}{\mu}}\boldsymbol{a}_z$$

which is constant everywhere.

Prob. 10.50

$$\mathcal{P} = \frac{E_o^2}{2\eta_o}$$

$$P = \mathcal{P}S = \frac{E_o^2 S}{2\eta_o} = \frac{(2.4\times10^3)^2\times450\times10^{-4}}{2\times377} = \underline{\underline{343.8\ \text{W}}}$$

Prob. 10.51

$$\mathscr{P} = \boldsymbol{E} \times \boldsymbol{H} = \frac{V_o I_o}{2\pi\rho^2 \ln(b/a)} \sin^2(\omega t - \beta z)\boldsymbol{a}_z$$

(a) $$\mathscr{P}_{ave} = \frac{1}{T}\int_0^T \mathscr{P} dt = \frac{V_o I_o}{2\pi\rho^2 \ln(b/a)} \frac{1}{T}\int_0^T \sin^2(\omega t - \beta z) dt \boldsymbol{a}_z = \frac{V_o I_o}{2\pi\rho^2 \ln(b/a)} \frac{1}{2}\boldsymbol{a}_z$$

$$= \underline{\underline{\frac{V_o I_o}{4\pi\rho^2 \ln(b/a)}\boldsymbol{a}_z}}$$

(b)

$$P_{ave} = \int_S \mathscr{P}_{ave} \bullet d\boldsymbol{S}, \qquad d\boldsymbol{S} = \rho\, d\rho d\phi \boldsymbol{a}_z$$

$$= \frac{V_o I_o}{4\pi \ln(b/a)} \int_{\phi=0}^{2\pi} \int_{\rho=a}^{b} \frac{1}{\rho^2} \rho d\rho d\phi = \frac{V_o I_o}{4\pi \ln(b/a)} (2\pi) \ln(b/a)$$

$$= \underline{\underline{\frac{1}{2} V_o I_o}}$$

Prob. 10.52

(a) $$P_{i,ave} = \frac{E_{io}^2}{2\eta_1}, \qquad P_{r,ave} = \frac{E_{ro}^2}{2\eta_1}, \qquad P_{t,ave} = \frac{E_{to}^2}{2\eta_2}$$

$$R = \frac{P_{r,ave}}{P_{i,ave}} = \frac{E_{ro}^2}{E_{io}^2} = \Gamma^2 = \underline{\underline{\left(\frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}\right)^2}}$$

$$R = \left(\frac{\sqrt{\frac{\mu_o}{\varepsilon_2}} - \sqrt{\frac{\mu_o}{\varepsilon_1}}}{\sqrt{\frac{\mu_o}{\varepsilon_2}} + \sqrt{\frac{\mu_o}{\varepsilon_1}}} \right)^2 = \left(\frac{\sqrt{\mu_o \varepsilon_1} - \sqrt{\mu_o \varepsilon_2}}{\sqrt{\mu_o \varepsilon_1} + \sqrt{\mu_o \varepsilon_2}} \right)^2$$

Since $n_1 = c\sqrt{\mu_1 \varepsilon_1} = c\sqrt{\mu_o \varepsilon_1}, \qquad n_2 = c\sqrt{\mu_o \varepsilon_2},$

$$R = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2$$

$$T = \frac{P_{t,ave}}{P_{i,ave}} = \frac{\eta_1}{\eta_2}\frac{E_{to}^2}{E_{io}^2} = \frac{\eta_1}{\eta_2}\tau^2 = \frac{\eta_1}{\eta_2}(1+\Gamma)^2 = \frac{4n_1 n_2}{(n_1 + n_2)^2}$$

(b) If $P_{r,ave} = P_{t,ave} \longrightarrow RP_{i,ave} = TP_{i.ave} \longrightarrow R = T$

i.e. $(n_1 - n_2)^2 = 4n_1n_2 \quad \longrightarrow \quad n_1^{\,2} - 6n_1n_2 + n_2^{\,2} = 0$

or $\left(\dfrac{n_1}{n_2}\right)^2 - 6\left(\dfrac{n_1}{n_2}\right) + 1 = 0,$ so

$$\frac{n_1}{n_2} = 3 \pm \sqrt{8} = \underline{\underline{5.828 \quad \text{or} \quad 0.1716}}$$

(Note that these values are mutual reciprocals, reflecting the inherent symmetry of the problem.)

Prob. 10.53 (a) $\eta_1 = \eta_o, \quad \eta_2 = \sqrt{\dfrac{\mu}{\varepsilon}} = \eta_o / 2$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{\eta_o/2 - \eta_o}{3\eta_o/2} = \underline{\underline{-1/3}}, \qquad \tau = \frac{2\eta_2}{\eta_2 + \eta_1} = \frac{\eta_o}{3\eta_o/2} = \underline{\underline{2/3}}$$

$$s = \frac{1+|\Gamma|}{1-|\Gamma|} = \frac{1+1/3}{1-1/3} = \underline{\underline{2}}$$

(b) $E_{or} = \Gamma E_{oi} = -\dfrac{1}{3}\times(30) = -10$

$$\boldsymbol{E}_r = -10\cos(\omega t + z)\boldsymbol{a}_x \quad \text{V/m}$$

Let $\boldsymbol{H}_r = H_{or}\cos(\omega t + z)\boldsymbol{a}_H$

$\boldsymbol{a}_E \times \boldsymbol{a}_H = \boldsymbol{a}_k \longrightarrow -\boldsymbol{a}_x \times \boldsymbol{a}_H = -\boldsymbol{a}_z \longrightarrow \boldsymbol{a}_H = \boldsymbol{a}_y$

$$\boldsymbol{H}_r = \frac{10}{120\pi}\cos(\omega t + z)\boldsymbol{a}_y = \underline{\underline{26.53\cos(\omega t + z)\boldsymbol{a}_y}} \quad \text{mA/m}$$

Prob. 10.54

$$\beta_1 = \frac{\omega}{c} = \frac{2\pi\times10^8}{3\times10^8} = \frac{2\pi}{3}$$

$$\eta_1 = \eta_o, \qquad \eta_2 = 0$$

$$\Gamma = \frac{\eta_2-\eta_1}{\eta_2+\eta_1} = \frac{0-\eta_o}{0+\eta_o} = -1, \qquad \tau = 1+\Gamma = 0$$

$$\boldsymbol{E}_t = \boldsymbol{0}$$

$$\boldsymbol{E}_r = \underline{\underline{-50\sin(2\pi\times10^8 t+\beta_1 x)\boldsymbol{a}_z \text{ V/m}}}$$

Prob. 10.55 (a) Medium 1 is free space. Given that $\beta = 1$,

$$\beta = 1 = \frac{\omega}{c} \longrightarrow \omega = c = \underline{\underline{3\times10^8 \text{ rad/s}}}$$

(b) $\eta_1 = \eta_o, \qquad \eta_2 = \eta_o\sqrt{\frac{\mu_r}{\varepsilon_r}} = \eta_o\sqrt{\frac{3}{12}} = \eta_o/2$

$$\Gamma = \frac{\eta_2-\eta_1}{\eta_2+\eta_1} = -1/3, \qquad \tau = 1+\Gamma = 2/3$$

$$s = \frac{1+|\Gamma|}{1-|\Gamma|} = \frac{1+1/3}{1-1/3} = \underline{\underline{2}}$$

(c) Let $\boldsymbol{H}_r = H_{or}\cos(\omega t+z)\boldsymbol{a}_H$, where

$$\boldsymbol{E}_r = -\frac{1}{3}(30)\cos(\omega t+z)\boldsymbol{a}_y = -10\cos(\omega t+z)\boldsymbol{a}_y, \qquad H_{or} = \frac{10}{\eta_o} = \frac{10}{120\pi}$$

$$\boldsymbol{a}_E\times\boldsymbol{a}_H = \boldsymbol{a}_k \longrightarrow -\boldsymbol{a}_y\times\boldsymbol{a}_H = -\boldsymbol{a}_z \longrightarrow \boldsymbol{a}_H = -\boldsymbol{a}_x$$

$$\boldsymbol{H}_r = -\frac{10}{120\pi}\cos(3\times10^8 t+z)\boldsymbol{a}_x \text{ A/m} = \underline{\underline{-26.53\cos(3\times10^8 t+z)\boldsymbol{a}_x \text{ mA/m}}}$$

(d) $\boldsymbol{P}_t = \frac{E_{ot}^2}{2\eta_2}\boldsymbol{a}_z, \qquad E_{ot} = \tau E_{oi} = \frac{2}{3}(30) = 20, \qquad \eta_2 = 60\pi$

$$\boldsymbol{P}_t = \frac{20^2}{120\pi}(\boldsymbol{a}_z) = \underline{\underline{1.061\boldsymbol{a}_z \text{ W/m}^2}}$$

Prob. 10.56 $\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}} = \eta_o / 2, \qquad \eta_2 = \eta_o$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = 1/3, \qquad \tau = 1 + \Gamma = 4/3$$

$$E_{or} = \Gamma E_{io} = (1/3)(5) = 5/3, \qquad E_{ot} = \tau E_{io} = 20/3$$

$$\beta = \frac{\omega}{c}\sqrt{\mu_r \varepsilon_r} = \frac{10^8}{3\times 10^8}\sqrt{4} = 2/3$$

(a) $\boldsymbol{E}_r = \frac{5}{3}\cos(10^8 t - 2y/3)\boldsymbol{a}_z$

$$\boldsymbol{E}_1 = \boldsymbol{E}_i + \boldsymbol{E}_r = \underline{\underline{5\cos(10^8 t + \frac{2}{3}y)\boldsymbol{a}_z + \frac{5}{3}\cos(10^8 t - \frac{2}{3}y)\boldsymbol{a}_z}} \text{ V/m}$$

(b) $\mathcal{P}_{\text{ave1}} = \frac{E_{io}^{\ 2}}{2\eta_1}(-\boldsymbol{a}_y) + \frac{E_{ro}^{\ 2}}{2\eta_1}(+\boldsymbol{a}_y) = \frac{25}{2(60\pi)}(1 - \frac{1}{9})(-\boldsymbol{a}_y) = \underline{\underline{-0.0589\boldsymbol{a}_y \text{ W/m}^2}}$

(c) $\mathcal{P}_{\text{ave2}} = \frac{E_{to}^{\ 2}}{2\eta_2}(-\boldsymbol{a}_y) = \frac{400}{9(2)(120\pi)}(-\boldsymbol{a}_y) = \underline{\underline{-0.0589\boldsymbol{a}_y \text{ W/m}^2}}$

Prob. 10.57

(a) $\eta_1 = \eta_o$

$$\boldsymbol{E}_i = E_{io}\sin(\omega t - 5x)\boldsymbol{a}_E$$

$$E_{io} = H_{io}\eta_o = 120\pi \times 4 = 480\pi$$

$$\boldsymbol{a}_E \times \boldsymbol{a}_H = \boldsymbol{a}_k \longrightarrow \boldsymbol{a}_E \times \boldsymbol{a}_y = \boldsymbol{a}_x \longrightarrow \boldsymbol{a}_E = -\boldsymbol{a}_z$$

$$\boldsymbol{E}_i = -480\pi\sin(\omega t - 5x)\boldsymbol{a}_z$$

$$\eta_2 = \sqrt{\frac{\mu_o}{4\varepsilon_o}} = \frac{120\pi}{\sqrt{4}} = 60\pi$$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{60\pi - 120\pi}{60\pi + 120\pi} = -1/3, \qquad \tau = 1 + \Gamma = 2/3$$

$E_{ro} = \Gamma E_{io} = (-1/3)(-480\pi) = 160\pi$

$$\boldsymbol{E}_r = 160\pi \sin(\omega t + 5x)\boldsymbol{a}_z$$

$$\boldsymbol{E}_1 = \boldsymbol{E}_i + \boldsymbol{E}_r = \underline{\underline{-1.508\sin(\omega t - 5x)\boldsymbol{a}_z + 0.503\sin(\omega t + 5x)\boldsymbol{a}_z}} \quad \text{kV/m}$$

(b) $E_{to} = \tau E_{io} = (2/3)(480\pi) = 320\pi$

$$\mathcal{P} = \frac{E_{to}^{\ 2}}{2\eta_2}\boldsymbol{a}_x = \frac{(320\pi)^2}{2(60\pi)}\boldsymbol{a}_x = \underline{\underline{2.68\boldsymbol{a}_x \text{ kW/m}^2}}$$

(c) $s = \dfrac{1+|\Gamma|}{1-|\Gamma|} = \dfrac{1+1/3}{1-1/3} = \underline{\underline{2}}$

Prob. 10.58

(a) In air, $\beta_1 = 1, \lambda_1 = 2\pi/\beta_1 = 2\pi = \underline{\underline{6.283 \text{ m}}}$

$$\omega = \beta_1 c = \underline{\underline{3\times 10^8 \text{ rad/s}}}$$

In the dielectric medium, ω is the same .

$$\omega = \underline{\underline{3\times 10^8 \text{ rad/s}}}$$

$$\beta_2 = \frac{\omega}{c}\sqrt{\varepsilon_{r2}} = \beta_1\sqrt{\varepsilon_{r2}} = \sqrt{3}$$

$$\lambda_2 = \frac{2\pi}{\beta_2} = \frac{2\pi}{\sqrt{3}} = \underline{\underline{3.6276 \text{ m}}}$$

(b) $H_o = \dfrac{E_o}{\eta_o} = \dfrac{10}{120\pi} = 0.0265$

$$\boldsymbol{a}_H = \boldsymbol{a}_k \times \boldsymbol{a}_E = \boldsymbol{a}_z \times \boldsymbol{a}_y = -\boldsymbol{a}_x$$

$$\boldsymbol{H}_i = \underline{\underline{-26.5\cos(\omega t - z)\boldsymbol{a}_x \text{ mA/m}}}$$

(c) $\eta_1 = \eta_o, \quad \eta_2 = \eta_o/\sqrt{3}$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{(1/\sqrt{3}) - 1}{(1/\sqrt{3}) + 1} = \underline{\underline{-0.268}}, \qquad \tau = 1 + \Gamma = \underline{\underline{0.732}}$$

(d) $E_{to} = \tau E_{io} = 7.32, \qquad E_{ro} = \Gamma E_{io} = -2.68$

$$\mathbf{E}_1 = \mathbf{E}_i + \mathbf{E}_r = \underline{\underline{10\cos(\omega t - z)\mathbf{a}_y - 2.68\cos(\omega t + z)\mathbf{a}_y \text{ V/m}}}$$

$$\mathbf{E}_2 = \mathbf{E}_t = \underline{\underline{7.32\cos(\omega t - z)\mathbf{a}_y \text{ V/m}}}$$

$$\mathcal{P}_{\text{ave1}} = \frac{1}{2\eta_1}(\mathbf{a}_z)[E_{io}{}^2 - E_{ro}{}^2] = \frac{1}{2(120\pi)}(\mathbf{a}_z)(10^2 - 2.68^2) = \underline{\underline{0.1231\mathbf{a}_z \text{ W/m}^2}}$$

$$\mathcal{P}_{\text{ave2}} = \frac{E_{to}{}^2}{2\eta_2}(\mathbf{a}_z) = \frac{\sqrt{3}}{2\times 120\pi}(7.32)^2(\mathbf{a}_z) = \underline{\underline{0.1231\mathbf{a}_z \text{ W/m}^2}}$$

Prob. 10.59

$\eta_1 = \eta_o = 120\pi$

For seawater (lossy medium),

$$\eta_2 = \sqrt{\frac{j\omega\mu_o}{\sigma + j\omega\varepsilon}} = \sqrt{\frac{j2\pi\times 10^8\times 4}{4 + j2\pi\times 10^8\times 81\times\dfrac{10^{-9}}{36\pi}}} = 10.44 + j9.333$$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = 0.9461\angle 177.16$$

$|\Gamma|^2 = 0.8952, \qquad 1 - |\Gamma| = 0.1084$

$$\frac{P_r}{P_i} = \underline{\underline{89.51\%}}, \qquad \frac{P_t}{P_i} = \underline{\underline{10.84\%}},$$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{7.924\angle 43.975 - 377}{7.924\angle 43.975 + 377} = 0.9702\angle 178.2^o$$

The fraction of the incident power reflected is

$$\frac{P_r}{P_i} = |\Gamma|^2 = 0.9702^2 = \underline{\underline{0.9413}}$$

The transmitted fraction is

$$\frac{P_t}{P_i} = 1 - |\Gamma|^2 = 1 - 0.9702^2 = \underline{\underline{0.0587}}$$

Prob. 10.60
(a)

$$\eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}} = \frac{120\pi}{\sqrt{4}} = 188.5, \qquad \eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}} = \frac{120\pi}{\sqrt{3.2}} = 210.75$$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{210.75 - 188.5}{210.75 + 188.5} = 0.0557, \qquad \tau = \frac{2\eta_2}{\eta_2 + \eta_1} = \frac{2\times 210.75}{210.75 + 188.5} = 1.0557$$

$$E_{ro} = \Gamma E_{io} = (0.0557)(12) = 0.6684 \qquad E_{to} = \tau E_{io} = 1.0557(12) = 12.668$$

$$\beta_1 = \omega\sqrt{\mu_1\varepsilon_1} = \frac{\omega}{c}\sqrt{4} \quad \longrightarrow \quad \omega = \frac{\beta_1 c}{2} = \frac{40\pi(3\times 10^8)}{2} = \underline{\underline{6\pi\times 10^9 \text{ rad/s}}}$$

(b)

$$\beta_2 = \omega\sqrt{\mu_2\varepsilon_2} = \frac{\omega}{c}\sqrt{3.2} = \frac{6\pi\times 10^9\sqrt{3.2}}{3\times 10^8} = 112.4$$

$$\boldsymbol{E}_r = E_{ro}\cos(\omega t + 40\pi x)\boldsymbol{a}_z = \underline{\underline{0.6684\cos(6\pi\times 10^9 t + 40\pi x)\boldsymbol{a}_z \text{ V/m}}}$$

$$\boldsymbol{E}_t = E_{to}\cos(\omega t - \beta_2 x)\boldsymbol{a}_z = \underline{\underline{12.668\cos(6\pi\times 10^9 t - 112.4x)\boldsymbol{a}_z \text{ V/m}}}$$

Prob. 10.61 (a) $\omega = \beta c = 3\times 3\times 10^8 = \underline{\underline{9\times 10^8 \text{ rad/s}}}$

(b) $\lambda = 2\pi / \beta = 2\pi / 3 = \underline{\underline{2.094 \text{ m}}}$

(c) $$\frac{\sigma}{\omega\varepsilon} = \frac{4}{9\times 10^8 \times 80\times 10^{-9} / 36\pi} = 2\pi = \underline{\underline{6.288}}$$

$$\tan 2\theta_\eta = \frac{\sigma}{\omega\varepsilon} = 6.288 \quad \longrightarrow \quad \theta_\eta = 40.47^o$$

$$|\eta_2| = \frac{\sqrt{\mu_2 / \varepsilon_2}}{\sqrt[4]{1 + \left(\dfrac{\sigma_2}{\omega\varepsilon_2}\right)^2}} = \frac{377/\sqrt{80}}{\sqrt[4]{1+4\pi^2}} = 16.71$$

$$\eta_2 = \underline{\underline{16.71\angle 40.47^o \ \Omega}}$$

(d) $$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{16.71\angle 40.47^o - 377}{16.71\angle 40.47^o + 377} = 0.935\angle 179.7^o$$

$$E_{or} = \Gamma E_{oi} = 9.35\angle 179.7^o$$

$$\boldsymbol{E}_r = \underline{\underline{9.35\sin(\omega t - 3z + 179.7)\boldsymbol{a}_x \text{ V/m}}}$$

$$\alpha_2 = \frac{\omega}{c}\sqrt{\frac{\mu_{r2}\varepsilon_{r2}}{2}\left[\sqrt{1+\left(\frac{\sigma_2}{\omega\varepsilon_2}\right)^2}-1\right]} = \frac{9\times10^8}{3\times10^8}\sqrt{\frac{80}{2}\left[\sqrt{1+4\pi^2}-1\right]} = 43.94 \text{ Np/m}$$

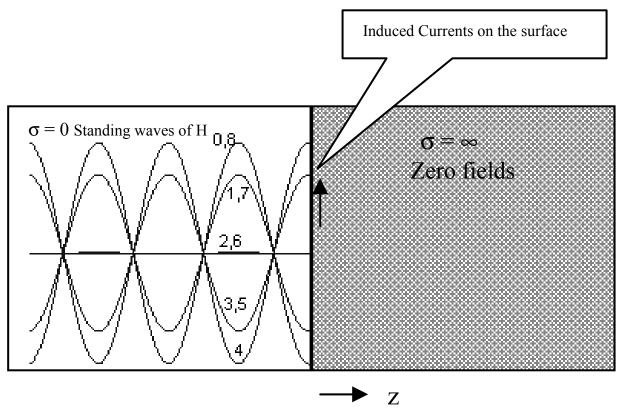
$$\beta_2 = \frac{9\times10^8}{3\times10^8}\sqrt{\frac{80}{2}\left[\sqrt{1+4\pi^2}+1\right]} = 51.48 \text{ rad/m}$$

$$\tau = \frac{2\eta_2}{\eta_2+\eta_1} = \frac{2\times16.71\angle40.47^o}{16.71\angle40.47^o+377} = 0.0857\angle38.89^o$$

$$E_{ot} = \tau E_o = 0.857\angle38.89^o$$

$$\boldsymbol{E}_t = 0.857e^{43.94z}\sin(9\times10^8 t + 51.48z + 38.89^o)\boldsymbol{a}_x \text{ V/m}$$

Prob. 10.62



Curve 0 is at t = 0; curve 1 is at t = T/8; curve 2 is at t = T/4; curve 3 is at t = 3T/8, etc.

Prob. 10.63 Since $\mu_o = \mu_1 = \mu_2$,

$$\sin\theta_{t1} = \sin\theta_i\sqrt{\frac{\varepsilon_o}{\varepsilon_1}} = \frac{\sin 45^o}{\sqrt{4.5}} = 0.3333 \longrightarrow \underline{\underline{\theta_{t1} = 19.47^o}}$$

$$\sin\theta_{t2} = \sin\theta_{t1}\sqrt{\frac{\varepsilon_1}{\varepsilon_2}} = \frac{1}{3}\sqrt{\frac{4.5}{2.25}} = 0.4714 \longrightarrow \underline{\underline{\theta_{t2} = 28.13^o}}$$

Prob. 10.64

$$\boldsymbol{E}_s = \frac{20(e^{jk_x x} - e^{-jk_x x})}{j2}\frac{(e^{jk_y y} + e^{-jk_y y})}{2}\boldsymbol{a}_z$$

$$= -j5\left[e^{j(k_x x + k_y y)} + e^{j(k_x x - k_y y)} - e^{-j(k_x x - k_y y)} - e^{-j(k_x x + k_y y)}\right]\boldsymbol{a}_z$$

which consists of four plane waves.

$$\nabla\times\boldsymbol{E}_s = -j\omega\mu_o\boldsymbol{H}_s \longrightarrow \boldsymbol{H}_s = \frac{j}{\omega\mu_o}\nabla\times\boldsymbol{E}_s = \frac{j}{\omega\mu_o}\left(\frac{\partial E_z}{\partial y}\boldsymbol{a}_x - \frac{\partial E_z}{\partial x}\boldsymbol{a}_y\right)$$

$$\boldsymbol{H}_s = -\frac{j20}{\omega\mu_o}\left[k_y\sin(k_x x)\sin(k_y y)\boldsymbol{a}_x + k_x\cos(k_x x)\cos(k_y y)\boldsymbol{a}_y\right]$$

Prob. 10.65

$\eta_1 = \eta_o = 377\Omega$

For η_2,

$$\frac{\sigma_2}{\omega\varepsilon_2} = \frac{4}{2\pi\times1.2\times10^9\times50\times\frac{10^{-9}}{36\pi}} = 1.2$$

$$\tan 2\theta_{\eta_2} = \frac{\sigma_2}{\omega\varepsilon_2} = 1.2 \longrightarrow \theta_{\eta_2} = 25.1^o$$

$$|\eta_2| = \frac{\sqrt{\mu/\varepsilon}}{\sqrt[4]{1+\left(\frac{\sigma_2}{\omega\varepsilon_2}\right)^2}} = \frac{120\pi\sqrt{1/50}}{\sqrt[4]{1+1.2^2}} = 42.658$$

$\eta_2 = 42.658\angle 25.1^o$

$$\Gamma = \frac{\eta_2-\eta_1}{\eta_2+\eta_1} = \frac{42.658\angle25.1^o - 377}{42.658\angle25.1^o + 377} = \underline{\underline{0.8146\angle174.4^o}}$$

Prob. 10.66

(a)

$P_t = (1-|\Gamma|^2)P_i$

$$s = \frac{1+|\Gamma|}{1-|\Gamma|} \longrightarrow |\Gamma| = \frac{s-1}{s+1}$$

$$\frac{P_t}{P_i} = 1-\left(\frac{s-1}{s+1}\right)^2 = \underline{\underline{\frac{4s}{(s+1)^2}}}$$

(b) $P_i = P_r + P_t \quad \longrightarrow \quad \frac{P_r}{P_i} = 1 - \frac{P_t}{P_i} = \underline{\underline{\left(\frac{s-1}{s+1}\right)^2}}$

Prob. 10.67
If **A** is a uniform vector and $\Phi(r)$ is a scalar,

$$\nabla \times (\Phi \boldsymbol{A}) = \nabla\Phi \times \boldsymbol{A} + \Phi(\nabla \times \boldsymbol{A}) = \nabla\Phi \times \boldsymbol{A}$$

since $\nabla \times \boldsymbol{A} = \boldsymbol{0}$.

$$\nabla \times \boldsymbol{E} = (\frac{\partial}{\partial x}\boldsymbol{a}_x + \frac{\partial}{\partial y}\boldsymbol{a}_y + \frac{\partial}{\partial z}\mathrm{a}_z) \times \boldsymbol{E}_o e^{j(k_x x + k_y y + k_z z - \omega t)} = j(k_x \boldsymbol{a}_x + k_y \boldsymbol{a}_y + k_z \boldsymbol{a}_z) e^{j(\boldsymbol{k}\bullet\boldsymbol{r} - \omega t)} \times \boldsymbol{E}_o$$

$$= j\boldsymbol{k} \times \boldsymbol{E}_o e^{j(\boldsymbol{k}\bullet\boldsymbol{r} - \omega t)} = j\boldsymbol{k} \times \boldsymbol{E}$$

Also, $-\frac{\partial \boldsymbol{B}}{\partial t} = j\omega\mu\boldsymbol{H}$. Hence $\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$ becomes $\boldsymbol{k} \times \boldsymbol{E} = \omega\mu\boldsymbol{H}$

From this, $\boldsymbol{a}_k \times \boldsymbol{a}_E = \boldsymbol{a}_H$

Prob. 10.68

$$k = |\mathbf{k}| = \sqrt{124^2 + 124^2 + 263^2} = 316.1$$

$$\lambda = \frac{2\pi}{k} = \underline{\underline{19.88 \text{ mm}}}$$

$$k = \omega\sqrt{\mu\varepsilon} = \frac{2\pi f}{c} \quad \longrightarrow \quad f = \frac{kc}{2\pi} = \frac{316.1 \times 3 \times 10^8}{2\pi} = \underline{\underline{15.093 \text{ GHz}}}$$

$$\boldsymbol{k} \bullet \boldsymbol{a}_x = k\cos\theta_x \quad \longrightarrow \quad \cos\theta_x = \frac{124}{316.1} \quad \longrightarrow \quad \theta_x = 66.9^o = \theta_y$$

$$\theta_z = \cos^{-1}\frac{263}{361.1} = 33.69^o$$

Thus,

$$\underline{\underline{\theta_x = \theta_y = 66.9^o, \ \theta_z = 33.69^o}}$$

Prob. 10.69

$\boldsymbol{k} = -3.4\boldsymbol{a}_x + 4.2\boldsymbol{a}_y$

$\boldsymbol{k} \bullet \boldsymbol{E} = 0 \quad \longrightarrow \quad 0 = -3.4E_o + 4.2$

$E_o = \dfrac{4.2}{3.4} = \underline{\underline{1.235}}$

$k = |\boldsymbol{k}| = \beta = \sqrt{(-3.4)^2 + (4.2)^2} = 5.403$

$\lambda = \dfrac{2\pi}{\beta} = \dfrac{2\pi}{5.403} = 1.162$

$f = \dfrac{c}{\lambda} = \dfrac{3\times 10^8}{1.162} = \underline{\underline{258 \text{ MHz}}}$

$$\boldsymbol{H}_s = \frac{1}{\mu\omega}\boldsymbol{k}\times\boldsymbol{E}_s = \frac{1}{\mu kc}\boldsymbol{k}\times\boldsymbol{E}_s$$

$$= \frac{1}{4\pi\times 10^{-7}\times 5.403\times 3\times 10^8}\begin{vmatrix} -3.4 & 4.2 & 0 \\ E_o & 1 & 3+j4 \end{vmatrix} A_o$$

where $A_o = e^{-j3.4x+4.2y}$

$$\boldsymbol{H}_s = 4.91A_o\times 10^{-4}\left[4.2(3+j4)\boldsymbol{a}_x + 3.4(3+j4)\boldsymbol{a}_y + (-3.4-4.2E_o)\boldsymbol{a}_z\right]$$

$$\underline{\underline{= 0.491\left[(12.6+j16.8)\boldsymbol{a}_x + (10.2+j13.6)\boldsymbol{a}_y - 8.59\boldsymbol{a}_z\right]e^{-j3.4x+4.2y} \text{ mA/m}}}$$

Prob.10.70

$$\nabla\bullet\boldsymbol{E} = \left(\frac{\partial}{\partial x}\boldsymbol{a}_x + \frac{\partial}{\partial y}\boldsymbol{a}_y + \frac{\partial}{\partial z}\boldsymbol{a}_z\right)\bullet\boldsymbol{E}_o e^{j(k_x x + k_y y + k_z z - \omega t)} = j(k_x\boldsymbol{a}_x + k_y\boldsymbol{a}_y + k_z\boldsymbol{a}_z)e^{j(\boldsymbol{k}\bullet\boldsymbol{r}-\omega t)}\bullet\boldsymbol{E}_o$$

$$= j\boldsymbol{k}\bullet\boldsymbol{E}_o e^{j(\boldsymbol{k}\bullet\boldsymbol{r}-\omega t)} = j\boldsymbol{k}\bullet\boldsymbol{E} = 0 \quad \longrightarrow \quad \boldsymbol{k}\bullet\boldsymbol{E} = 0$$

Similarly,

$$\nabla\bullet\boldsymbol{H} = j\boldsymbol{k}\bullet\boldsymbol{H} = 0 \quad \longrightarrow \quad \boldsymbol{k}\bullet\boldsymbol{H} = 0$$

It has been shown in the previous problem that

$$\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t} \quad \longrightarrow \quad \boldsymbol{k}\times\boldsymbol{E} = \omega\mu\boldsymbol{H}$$

Similarly,

$$\nabla\times\boldsymbol{H} = \frac{\partial\boldsymbol{D}}{\partial t} \quad \longrightarrow \quad kxH = -\varepsilon\omega E$$

From $\mathbf{k}\times\mathbf{E}=\omega\mu\mathbf{H}$, $\quad \mathbf{a}_k\times\mathbf{a}_E=\mathbf{a}_H$ and

From $\mathbf{k}\times\mathbf{H}=-\varepsilon\omega\mathbf{E}$, $\quad \mathbf{a}_k\times\mathbf{a}_H=-\mathbf{a}_E$

Prob. 10.71

If $\mu_o=\mu_1=\mu_2,\quad \eta_1=\dfrac{\eta_o}{\sqrt{\varepsilon_{r1}}},\eta_2=\dfrac{\eta_o}{\sqrt{\varepsilon_{r2}}}$

$$\Gamma_{\backslash\backslash}=\frac{\dfrac{1}{\sqrt{\varepsilon_{r2}}}\cos\theta_t-\dfrac{1}{\sqrt{\varepsilon_{r1}}}\cos\theta_i}{\dfrac{1}{\sqrt{\varepsilon_{r2}}}\cos\theta_t+\dfrac{1}{\sqrt{\varepsilon_{r1}}}\cos\theta_i}$$

$$\sqrt{\varepsilon_{r1}}\sin\theta_i=\sqrt{\varepsilon_{r2}}\sin\theta_t \quad\longrightarrow\quad \frac{\sqrt{\varepsilon_{r2}}}{\sqrt{\varepsilon_{r1}}}=\frac{\sin\theta_i}{\sin\theta_t}$$

$$\Gamma_{\backslash\backslash}=\frac{\cos\theta_t-\dfrac{\sin\theta_i}{\sin\theta_t}\cos\theta_i}{\cos\theta_t+\dfrac{\sin\theta_i}{\sin\theta_t}\cos\theta_i}=\frac{\sin\theta_t\cos\theta_t-\sin\theta_i\cos\theta_i}{\sin\theta_t\cos\theta_t+\sin\theta_i\cos\theta_i}$$

$$=\frac{\sin 2\theta_t-\sin 2\theta_i}{\sin 2\theta_t+\sin 2\theta_i}=\frac{\sin(\theta_t-\theta_i)\cos(\theta_t+\theta_i)}{\cos(\theta_t-\theta_i)\sin(\theta_t+\theta_i)}=\frac{\tan(\theta_t-\theta_i)}{\tan(\theta_t+\theta_i)}$$

Similarly,

$$\tau_{\backslash\backslash}=\frac{\dfrac{2}{\sqrt{\varepsilon_{r2}}}\cos\theta_i}{\dfrac{1}{\sqrt{\varepsilon_{r2}}}\cos\theta_t+\dfrac{1}{\sqrt{\varepsilon_{r1}}}\cos\theta_i}=\frac{2\cos\theta_i}{\cos\theta_t+\dfrac{\sin\theta_i}{\sin\theta_t}\cos\theta_i}$$

$$=\frac{2\cos\theta_i\sin\theta_t}{\sin\theta_t\cos\theta_t(\sin^2\theta_i+\cos^2\theta_i)+\sin\theta_i\cos\theta_i(\sin^2\theta_t+\cos^2\theta_t)}$$

$$=\frac{2\cos\theta_i\sin\theta_t}{(\sin\theta_i\cos\theta_t+\sin\theta_t\cos\theta_i)(\cos\theta_i\cos\theta_t+\sin\theta_i\sin\theta_t)}$$

$$=\frac{2\cos\theta_i\sin\theta_t}{\sin(\theta_i+\theta_t)\cos(\theta_i-\theta_t)}$$

$$\Gamma_\perp = \frac{\frac{1}{\sqrt{\varepsilon_{r2}}}\cos\theta_i - \frac{1}{\sqrt{\varepsilon_{r1}}}\cos\theta_t}{\frac{1}{\sqrt{\varepsilon_{r2}}}\cos\theta_i + \frac{1}{\sqrt{\varepsilon_{r1}}}\cos\theta_t} = \frac{\cos\theta_i - \frac{\sin\theta_i}{\sin\theta_t}\cos\theta_t}{\cos\theta_i + \frac{\sin\theta_i}{\sin\theta_t}\cos\theta_t} = \frac{\sin(\theta_t - \theta_i)}{\sin(\theta_t + \theta_i)}$$

$$\tau_\perp = \frac{\frac{2}{\sqrt{\varepsilon_{r2}}}\cos\theta_i}{\frac{1}{\sqrt{\varepsilon_{r2}}}\cos\theta_i + \frac{1}{\sqrt{\varepsilon_{r1}}}\cos\theta_t} = \frac{2\cos\theta_i}{\cos\theta_i + \frac{\sin\theta_i}{\sin\theta_t}\cos\theta_t} = \frac{2\cos\theta_i \sin\theta_i}{\sin(\theta_t + \theta_i)}$$

Prob. 10.72

(a) $n_1 = 1, \quad n_2 = c\sqrt{\mu_2\varepsilon_2} = c\sqrt{6.4\varepsilon_o \times \mu_o} = \sqrt{6.4} = 2.5298$

$$\sin\theta_t = \frac{n_1}{n_2}\sin\theta_i = \frac{1}{2.5298}\sin 12^o = 0.082185 \quad \longrightarrow \quad \theta_t = 4.714^o$$

$$\eta_1 = 120\pi, \qquad \eta_2 = 120\pi\sqrt{\frac{1}{6.4}} = 47.43\pi$$

$$\frac{E_{ro}}{E_{io}} = \Gamma = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i} = \frac{47.43\pi\cos 4.714^o - 120\pi\cos 12^o}{47.43\pi\cos 4.714^o + 120\pi\cos 12^o}$$

$$= \frac{47.27 - 117.38}{47.27 + 117.38} = \underline{\underline{-0.4258}}$$

$$\frac{E_{to}}{E_{io}} = \tau = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i} = \frac{2x47.43\cos 12^o}{47.27 + 117.33} = \frac{92.787}{164.65} = \underline{\underline{0.5635}}$$

Prob. 10.73

(a) $\boldsymbol{k}_i = 4\boldsymbol{a}_y + \mathbf{3}a_z$

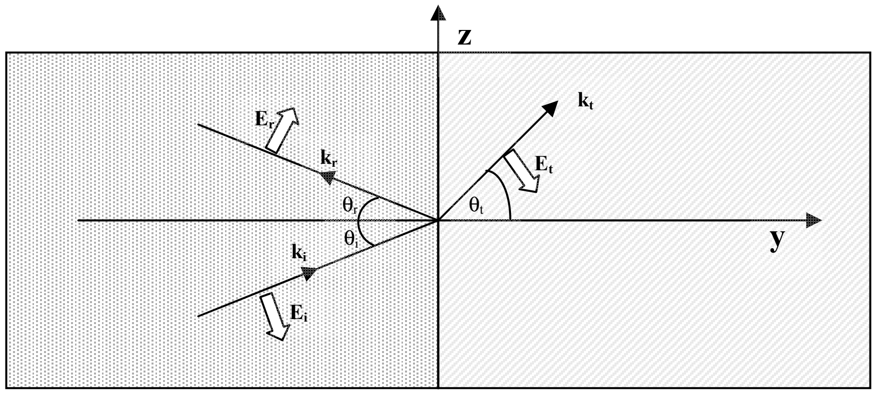
$$\boldsymbol{k}_i \bullet \boldsymbol{a}_n = k_i\cos\theta_i \quad \longrightarrow \quad \cos\theta_i = 4/5 \quad \longrightarrow \quad \underline{\underline{\theta_i = 36.87^o}}$$

(b)

$$\boldsymbol{P}_{\text{ave}} = \frac{1}{2}\,\text{Re}(\,\boldsymbol{E}_s \times \boldsymbol{H}_s^*\,) = \frac{E_o^{\,2}}{2\eta}\boldsymbol{a}_k = \frac{(\sqrt{8^2+6^2})^2}{2\times 120\pi}\frac{(4\boldsymbol{a}_y + 3\boldsymbol{a}_z)}{5} = \underline{\underline{106.1\boldsymbol{a}_y + 79.58\boldsymbol{a}_z \text{ mW/m}^2}}$$

(c) $\theta_r = \theta_i = 36.87^o$. Let

$$\boldsymbol{E}_r = (E_{ry}\boldsymbol{a}_x + E_{rz}\boldsymbol{a}_z)\sin(\omega t - \boldsymbol{k}_r \bullet \boldsymbol{r})$$



From the figure, $\boldsymbol{k}_r = k_{rz}\boldsymbol{a}_z - k_{ry}\boldsymbol{a}_y$. But $k_r = k_i = 5$

$$k_{rz} = k_r \sin\theta_r = 5(3/5) = 3, \qquad k_{ry} = k_r \cos\theta_r = 5(4/5) = 4,$$

Hence, $\boldsymbol{k}_r = -4\boldsymbol{a}_y + 3\boldsymbol{a}_z$

$$\sin\theta_t = \frac{n_1}{n_2}\sin\theta_i = \frac{c\sqrt{\mu_1\varepsilon_1}}{c\sqrt{\mu_2\varepsilon_2}}\sin\theta_i = \frac{3/5}{\sqrt{4}} = 0.3$$

$$\theta_t = 17.46, \quad \cos\theta_t = 0.9539, \qquad \eta_1 = \eta_o = 120\pi, \eta_2 = \eta_o / 2 = 60\pi$$

$$\Gamma_{//} = \frac{E_{ro}}{E_{io}} = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i} = \frac{\frac{\eta_o}{2}(0.9539) - \eta_o(0.8)}{\frac{\eta_o}{2}(0.9539) + \eta_o(0.8)} = -0.253$$

$$E_{ro} = \Gamma_{//}E_{io} = -0.253(10) = -2.53$$

But $(E_{ry}\boldsymbol{a}_y + E_{rz}\boldsymbol{a}_z) = E_{ro}(\sin\theta_r\boldsymbol{a}_y + \cos\theta_r\boldsymbol{a}_z) = -2.53(\frac{3}{5}\boldsymbol{a}_y + \frac{4}{5}\boldsymbol{a}_z)$

$$\underline{\underline{\boldsymbol{E}_r = -(1.518\boldsymbol{a}_y + 2.024\boldsymbol{a}_z)\sin(\omega t + 4y - 3z)}} \text{ V/m}$$

Similarly, let

$$\boldsymbol{E}_t = (E_{ty}\boldsymbol{a}_y + E_{tz}\boldsymbol{a}_z)\sin(\omega t - \boldsymbol{k}_t \bullet \boldsymbol{r})$$

$$k_t = \beta_2 = \omega\sqrt{\mu_2\varepsilon_2} = \omega\sqrt{4\mu_o\varepsilon_o}$$

But $k_i = \beta_1 = \omega\sqrt{\mu_o\varepsilon_o}$

$$\frac{k_t}{k_i} = 2 \longrightarrow k_t = 2k_i = 10$$

$$k_{ty} = k_t\cos\theta_t = 9.539, \qquad k_{tz} = k_t\sin\theta_t = 3,$$

$$k_t = 9.539a_y + 3a_z$$

Note that $k_{iz} = k_{rz} = k_{tz} = 3$

$$\tau_{\backslash\backslash} = \frac{E_{to}}{E_{io}} = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i} = \frac{\eta_o(0.8)}{\frac{\eta_o}{2}(0.9539) + \eta_o(0.8)} = 0.6265$$

$$E_{to} = \tau_{\backslash\backslash}E_{io} = 6.265$$

But

$$(E_{ty}\boldsymbol{a}_y + E_{tz}\boldsymbol{a}_z) = E_{to}(\sin\theta_t\boldsymbol{a}_y - \cos\theta_t\boldsymbol{a}_z) = 6.256(0.3\boldsymbol{a}_y - 0.9539\boldsymbol{a}_z)$$

Hence,

$$\underline{\underline{\boldsymbol{E}_t = (1.879\boldsymbol{a}_y - 5.968\boldsymbol{a}_z)\sin(\omega t - 9.539y - 3z)}} \text{ V/m}$$

Prob. 10.74

(a) $n = \frac{c}{u} = \sqrt{\mu_r\varepsilon_r} = \sqrt{2.1\times 1} = \underline{\underline{1.45}}$

(b) $n = \sqrt{\mu_r\varepsilon_r} = \sqrt{1\times 81} = \underline{\underline{9}}$

(c) $n = \sqrt{\varepsilon_r} = \sqrt{2.7} = \underline{\underline{1.643}}$

Prob. 10.75

(a) From air to seawater,

$\varepsilon_{r1} = 1, \quad \varepsilon_{r2} = 81$

$\tan\theta_B = \sqrt{\dfrac{\varepsilon_2}{\varepsilon_1}} = \sqrt{\dfrac{81}{1}} = 9 \quad \longrightarrow \quad \underline{\underline{\theta_B = 83.66^o}}$

(b) From seawater to air,

$\varepsilon_{r1} = 81, \quad \varepsilon_{r2} = 1$

$\tan\theta_B = \sqrt{\dfrac{\varepsilon_2}{\varepsilon_1}} = \sqrt{\dfrac{1}{81}} = \dfrac{1}{9} \quad \longrightarrow \quad \underline{\underline{\theta_B = 6.34^o}}$

Prob. 10.76

(a)

$\tan\theta_i = \dfrac{k_{ix}}{k_{iz}} = \dfrac{1}{\sqrt{8}} \quad \longrightarrow \quad \underline{\underline{\theta_i = \theta_r = 19.47^o}}$

$\sin\theta_t = \sin\theta_i\sqrt{\dfrac{\varepsilon_{r1}}{\varepsilon_{r2}}} = \dfrac{1}{3}(3) = 1 \quad \longrightarrow \quad \underline{\underline{\theta_t = 90^o}}$

(b) $\beta_1 = \dfrac{\omega}{c}\sqrt{\varepsilon_{r1}} = \dfrac{10^9}{3\times10^8}\times 3 = 10 = k\sqrt{1+8} = 3k \quad \longrightarrow \quad \underline{\underline{k = 3.333}}$

(c) $\lambda = 2\pi/\beta, \quad \lambda_1 = 2\pi/\beta_1 = 2\pi/10 = \underline{\underline{0.6283}}\,\text{m}$

$\beta_2 = \omega/c = 10/3, \quad \lambda_2 = 2\pi/\beta_2 = 2\pi\times 3/10 = \underline{\underline{1.885}}\,\text{m}$

(d)
$$\begin{aligned} \boldsymbol{E}_i &= \eta_1 \boldsymbol{H}_x \times \boldsymbol{a}_k = 40\pi(0.2)\cos(\omega t - \boldsymbol{k}\bullet\boldsymbol{r})\boldsymbol{a}_y \times \frac{(\boldsymbol{a}_x + \sqrt{8}\boldsymbol{a}_z)}{3} \\ &= \underline{\underline{(23.6954\boldsymbol{a}_x - 8.3776\boldsymbol{a}_z)\cos(10^9 t - kx - k\sqrt{8}z)\ \ \text{V/m}}} \end{aligned}$$

(e) $\tau_{//} = \dfrac{2\cos\theta_i\sin\theta_t}{\sin(\theta_i+\theta_t)\cos(\theta_t-\theta_i)} = \dfrac{2\cos 19.47^o\sin 90^o}{\sin 19.47^o\cos 19.47^o} = 6$

$\Gamma_{//} = -\dfrac{\cot 19.47^o}{\cot 19.47^o} = -1$

Let $\quad \boldsymbol{E}_t = -E_{io}(\cos\theta_t\boldsymbol{a}_x - \sin\theta_t\boldsymbol{a}_z)\cos(10^9 t - \beta_2 x\sin\theta_t - \beta_2 z\cos\theta_t)$

where

$$\boldsymbol{E}_t = -E_{io}(\cos\theta_i \boldsymbol{a}_x - \sin\theta_i \boldsymbol{a}_z)\cos(10^9 t - \beta_1 x \sin\theta_i - \beta_1 z\cos\theta_i)$$

$$\sin\theta_t = 1, \quad \cos\theta_t = 0, \quad \beta_2 \sin\theta_t = 10/3$$

$$E_{to}\sin\theta_t = \tau_{\backslash\backslash} E_{io} = 6(24\pi)(3)(1) = 1357.2$$

Hence,

$$\boldsymbol{E}_t = 1357\cos(10^9 t - 3.333x)\boldsymbol{a}_z \quad \text{V/m}$$

Since $\Gamma = -1, \quad \theta_r = \theta_i$

$$\underline{\underline{\boldsymbol{E}_r = (213.3\boldsymbol{a}_x + 75.4\boldsymbol{a}_z)\cos(10^9 t - kx + k\sqrt{8}z)}} \text{ V/m}$$

(f) $\tan\theta_{B//} = \sqrt{\dfrac{\varepsilon_2}{\varepsilon_1}} = \sqrt{\dfrac{\varepsilon_o}{9\varepsilon_o}} = 1/3 \longrightarrow \underline{\underline{\theta_{B//} = 18.43^o}}$

Prob.10.77
Microwave is used:

(1) For surveying land with a piece of equipment called the *tellurometer.* This radar system can precisely measure the distance between two points.
(2) For guidance. The guidance of missiles, the launching and homing guidance of space vehicles, and the control of ships are performed with the aid of microwaves.
(3) In semiconductor devices. A large number of new microwave semiconductor devices have been developed for the purpose of microwave oscillator, amplification, mixing/detection, frequency multiplication, and switching. Without such achievement, the majority of today's microwave systems could not exist.

Prob.10.78
(a) In terms of the S-parameters, the T-parameters are given by

$T_{11} = 1/S_{21}$, $T_{12} = -S_{22}/S_{21}$, $T_{21} = S_{11}/S_{21}$, $T_{22} = S_{12} - S_{11} S_{22}/S_{21}$

(b) $T_{11} = 1/0.4 = 2.5$, $T_{12} = -0.2/0.4$,

$T_{21} = 0.2/0.4$, $T_{22} = 0.4 - 0.2 \text{ x } 0.2/0.4 = 0.3$

Hence,

$$T = \begin{bmatrix} 2.5 & -0.5 \\ 0.5 & 0.3 \end{bmatrix}$$

Prob. 10.79
Since $Z_L = Z_o$, $\Gamma_L = 0$.

$$\Gamma_i = S_{11} = \underline{0.33 - j0.15}$$

$$\Gamma_g = (Z_g - Z_o)/(Z_g + Z_o) = (2 - 1)/(2 + 1) = 1/3$$

$$\Gamma o = S_{22} + S_{12}S_{21}\Gamma_g /(1 - S_{11} \Gamma_g)$$

$$= 0.44 - j0.62 + 0.56x0.56\ x(1/3)/[1 - (0.11 - j0.05)]$$

$$= \underline{0.5571 - j0.6266}$$

Prob. 10.80 The microwave wavelengths are of the same magnitude as the circuit components. The wavelength in air at a microwave frequency of 300 GHz, for example, is 1 mm. The physical dimension of the lumped element must be in this range to avoid interference. Also, the leads connecting the lumped element probably have much more inductance and capacitance than is needed.

CHAPTER 11

P.E. 11.1 Since Z_o is real and $\alpha \neq 0$, this is a distortionless line.

$$Z_o = \sqrt{\frac{R}{G}} \qquad (1)$$

or $$\frac{L}{R} = \frac{C}{G} \qquad (2)$$

$$\alpha = \sqrt{RG} \qquad (3)$$

$$\beta = \omega L \sqrt{\frac{G}{R}} = \frac{\omega L}{Z_o} \qquad (4)$$

$$(1) \times (3) \rightarrow R = \alpha Z_o = 0.04 \times 80 = \underline{\underline{3.2\Omega / \text{m}}},$$

$$(3) \div (1) \rightarrow G = \frac{\alpha}{Z_o} = \frac{0.04}{80} = \underline{\underline{5 \times 10^{-4} S / \text{m}}}$$

$$L = \frac{\beta Z_o}{\omega} = \frac{1.5 \times 80}{2\pi \times 5 \times 10^8} = \underline{\underline{38.2 \text{ nH/m}}}$$

$$C = \frac{LG}{R} = \frac{12}{\pi} \times 10^{-8} \times \frac{0.04}{80} \times \frac{1}{0.04 \times 80} = \underline{\underline{5.97 \text{ pF/m}}}$$

P.E. 11.2

(a) $$Z_o = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{0.03 + j2\pi \times 0.1 \times 10^{-3}}{0 + j2\pi \times 0.02 \times 10^{-6}}}$$

$$= 70.73 - j1.688 = \underline{\underline{70.75\angle -1.367^o \Omega}}$$

(b) $$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \sqrt{(0.03 + j0.2\pi)(j0.4 \times 10^{-4}\pi)}$$

$$\underline{\underline{= 2.121 \times 10^{-4} + j8.888 \times 10^{-3} \text{ / m}}}$$

(c) $$u = \frac{\omega}{\beta} = \frac{2\pi \times 10^3}{8.888 \times 10^{-3}} = \underline{\underline{7.069 \times 10^5 \text{ m/s}}}$$

P.E. 11.3

(a) $Z_o = Z_l \rightarrow Z_{in} = Z_o = \underline{\underline{30 + j60\Omega}}$

(b) $V_{in} = V_o = \dfrac{Z_{in}}{Z_{in} + Z_o} V_g = \dfrac{V_g}{2} = \underline{\underline{7.5\angle 0^o \text{ V}_{\text{rms}}}}$

$$I_{in} = I_o = \frac{V_g}{Z_g + Z_{in}} = \frac{V_g}{2Z_o} = \frac{15\angle 0^o}{2(30 + j60)}$$

$$= 0.2236\angle -63.43^o A$$

assuming that $Z_g = 0$.

(c) Since $Z_o = Z_r$, $\Gamma = 0 \rightarrow V_o^- = 0, V_o^+ = V_o$

The load voltage is $V_L = V_s(z = l) = V_o^+ e^{-\gamma l}$

$$e^{-\gamma l} = \frac{V_o^+}{V_L} = \frac{7.5\angle 0^o}{5\angle -48^o} 1.5\angle 48^o$$

$$e^{\alpha l} e^{j\beta l} = 1.5\angle 48^o$$

$$e^{\alpha l} = 1.5 \rightarrow \alpha = \frac{1}{l}\ln(1.5) = \frac{1}{40}\ln(1.5) = 0.0101$$

$$e^{j\beta l} = e^{j48^o} \rightarrow \beta = \frac{1}{l}\frac{48^o}{180^o}\pi rad = 0.02094$$

$$\underline{\underline{\gamma = 0.0101 + j0.02094 \text{ /m}}}$$

assuming that $Z_g = 0$.

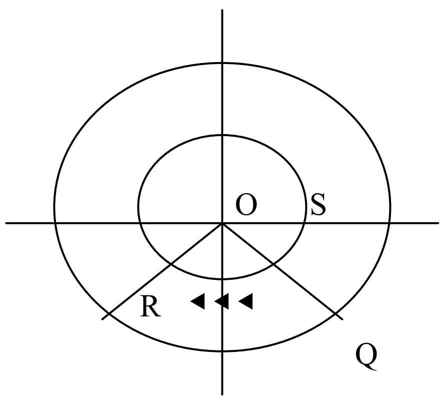
P.E. 11.4

(a) Using the Smith chart, locate S at s = 1.6. Draw a circle of radius OS. Locate P where $\theta_\Gamma = 300^o$. At P,

$$|\Gamma| = \frac{OP}{OQ} = \frac{2.1cm}{9.2cm} = 0.228$$

$$\underline{\underline{\Gamma = 0.228\angle 300^o}}$$

Also at P, z_L=1.15-j0.48,



$Z_L = Z_o z_L = 70(1.15 - j0.48) = \underline{\underline{80.5 - j33.6\Omega}}$

$\ell = 0.6\lambda \rightarrow 0.6 \times 720^o = 432^o = \underline{\underline{360^o + 73^o}}$

From P, move 432^o to R. At R, $z_{in} = 0.68 - j025$

$$Z_{in} = Z_o Z_{in} = 70(0.68 - j0.25) = \underline{\underline{47.6 - j17.5\Omega}}$$

(b) The minimum voltage (the only one) occurs at $\theta_\Gamma = 180^o$; its distance from the load is $\frac{180-60}{720}\lambda = \frac{\lambda}{6} = \underline{\underline{0.1667\lambda}}$

Values obtained using formulas are as follows:

$$\Gamma = \frac{s-1}{s+1}\angle 300^o = 0.2308\angle 300^o$$

$Z_L = 80.5755 - j34.018\ \Omega$

$Z_{in} = 48.655 - j17.63\ \Omega$

These are pretty close.

P.E. 11.5

(a) $\Gamma = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{60 + j60 - 60}{60 + j60 + 60} = \frac{j}{2+j} = \underline{\underline{0.4472\angle 63.43^o}}$

$$s = \frac{1+|\Gamma|}{1-|\Gamma|} = \frac{1+0.4472}{1-0.4472} = \underline{\underline{2.618}}$$

Let $x = \tan(\beta l) = \tan\frac{2\pi l}{\lambda}$

$$Z_{in} = Z_o\left[\frac{Z_L + jZ_o\tan(\beta l)}{Z_o + jZ_L\tan(\beta l)}\right]$$

$$120 - j60 = 60\left[\frac{60 + j60 + j60x}{60 + j(60 + j60)x}\right]$$

Or $2 - j = \frac{1 + j(1+x)}{1 - x + jx} \rightarrow 1 - x + j(2x - 2) = 0$

Or $x = 1 = \tan(\beta l)$

$$\frac{\pi}{4} + n\pi = \frac{2\pi l}{\lambda}$$

i.e $\underline{\underline{l = \frac{\lambda}{8}(1+4n), n = 0,1,2,3...}}$

(b) $z_L = \frac{Z_L}{Z_o} \frac{60 + j60}{60} = 1 + j$

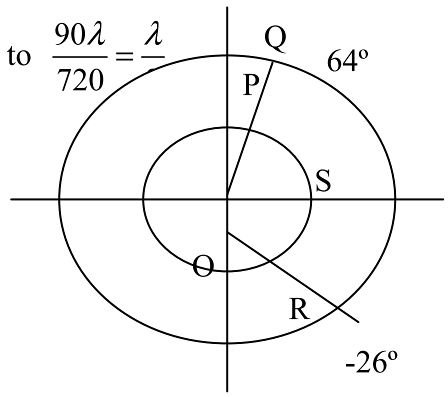
Locate the load point P on the Smith chart.

$$|\Gamma| = \frac{OP}{OQ} = \frac{4.1cm}{9.2cm} = 0.4457, \theta_\Gamma = 62^o$$

$$\Gamma = 0.4457\angle 62^o$$

Locate the point S on the Smith chart. At S, r = $\underline{\underline{s = 2.6}}$

$Z_{in} = \frac{Z_{in}}{Z_o} = \frac{120 + j60}{60} = 2 - j$, which is located at R on the chart. The angle between OP and OR is 64°-(-25°) = 90° which is equivalent to $\frac{90\lambda}{720} = \frac{\lambda}{8}$



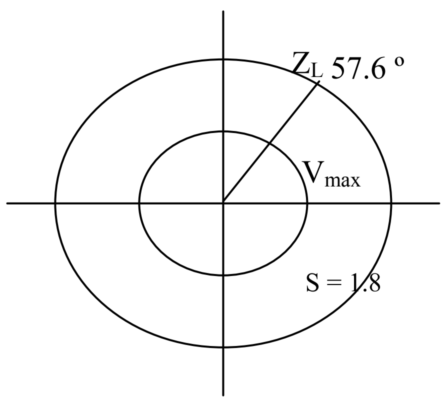
Hence $l = \frac{\lambda}{8} + n\frac{\lambda}{2} = \frac{\lambda}{8}(1+4n), n = 0,1,2........$

$$(Z_{in})_{\max} = sZ_o = 2.618(60) = 157.08\Omega$$

$$(Z_{in})_{\min} = Z_o / s = 60 / 2.618 = \underline{\underline{22.92\ \Omega}}$$

$$l = \frac{62^o}{720^o}\lambda = 0.0851\lambda$$

P.E. 11.6



$$\frac{\lambda}{2} = 37.5 - 25 = 12.5cm \text{ or } \lambda = 25cm$$

$$l = 37.5 - 35.5 = 2cm = \frac{2\lambda}{25}$$

$$l = 0.08\lambda \rightarrow 57.6^o$$

$$z_L = 1.184 + j0.622$$

$$Z_L = Z_o z_L = 50(1.184 - j0.622)$$

$$= 59.22 + j31.11\Omega$$

P.E. 11.7 See the Smith chart

$$z_L = \frac{100 - j80}{75} = 1.33 - j1.067$$

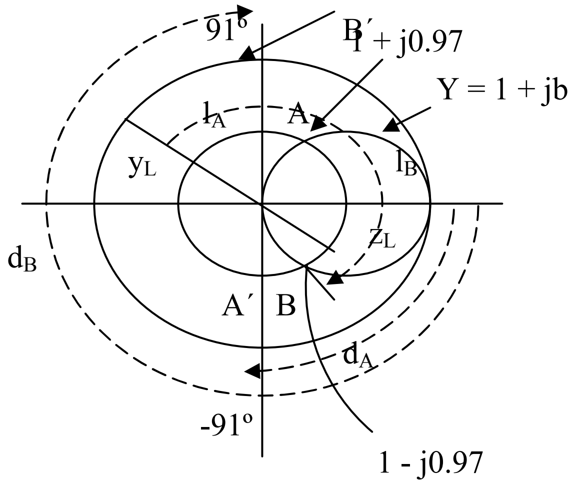
$$l_A = \frac{132^o - 65}{72}\lambda = \underline{\underline{0.093\lambda}}$$

$$l_B = \frac{132^o + 64^o}{720^o} = \underline{\underline{0.272\lambda}}$$

$$d_A = \frac{91}{720}\lambda = 0.126\lambda$$

$$d_B = 0.5\lambda - d_A \quad = 0.374\lambda$$

$$Y_s = \pm\frac{j0.95}{75} = \underline{\underline{\pm j12.67 \text{ mS}}}$$

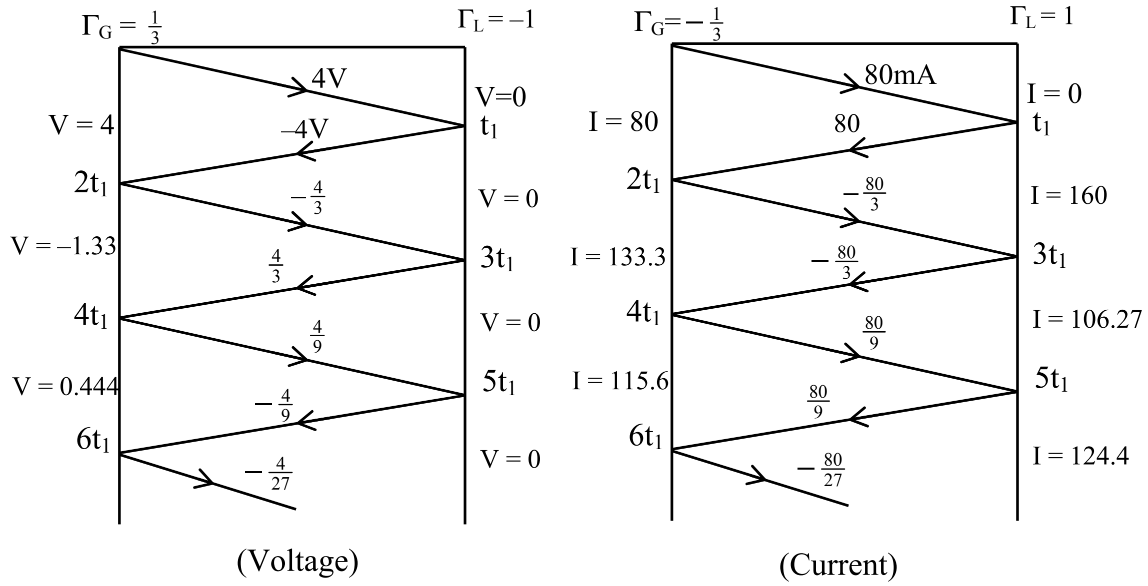


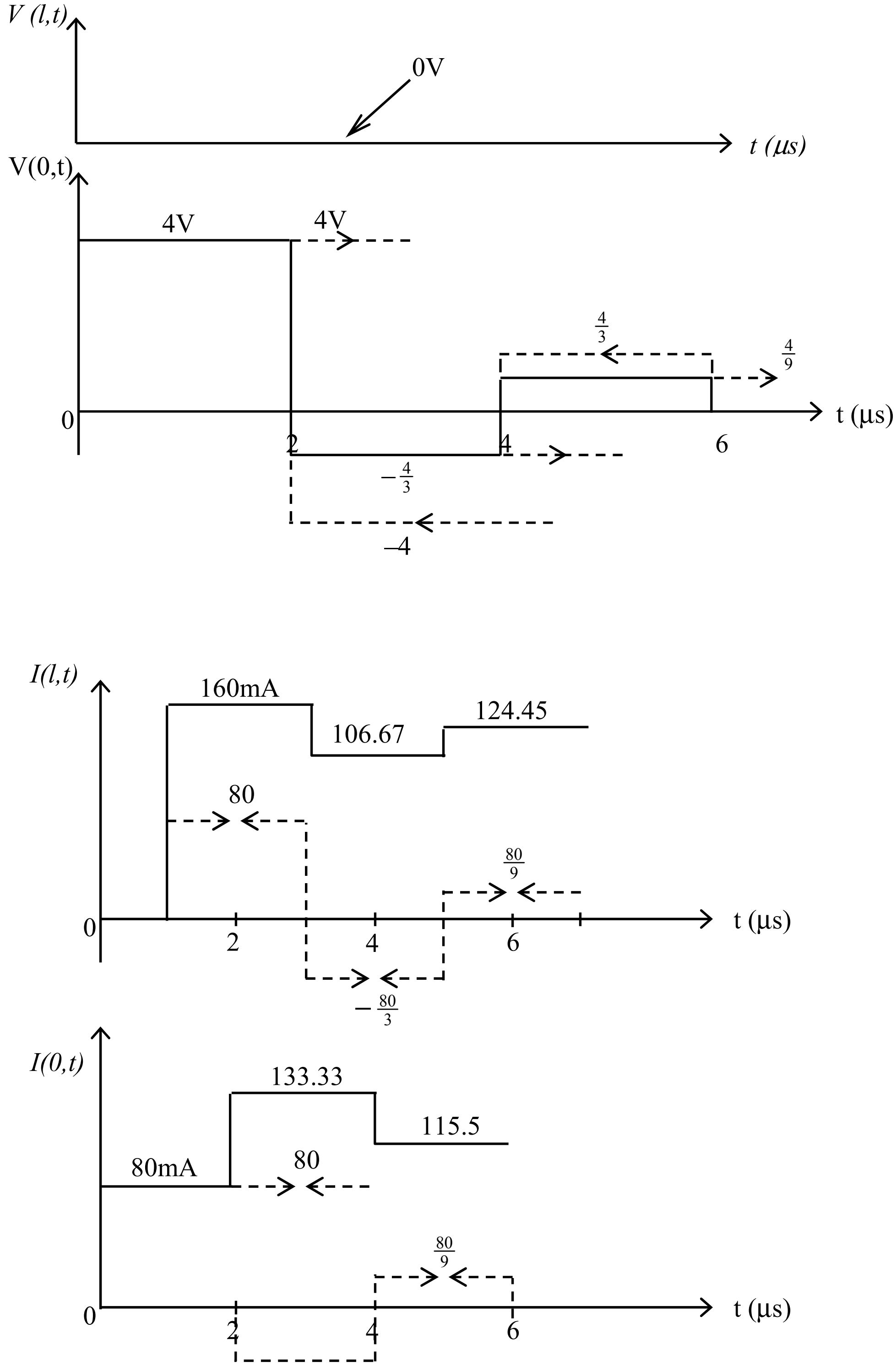
P.E. 11.8

(a) $\Gamma_G = \frac{1}{3}, \Gamma_L = \lim_{Z_L \to 0} \frac{Z_L - Z_o}{Z_L + Z_o} = -1$

$$V_\infty = \lim_{z_L \to 0} \frac{Z_L}{Z_L + Z_g} V_g = 0, \qquad I_\infty = \lim_{z_L \to 0} \frac{V_g}{Z_g + Z_L} = \frac{V_g}{Z_g} = \frac{12}{100} = 120mA$$

Thus the bounce diagrams for current and waves are as shown below.

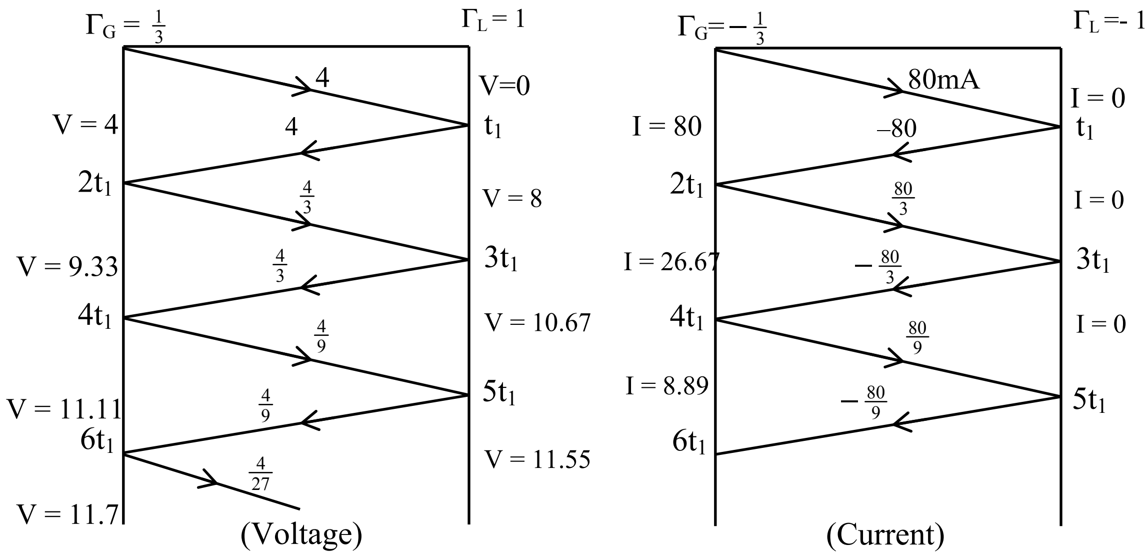


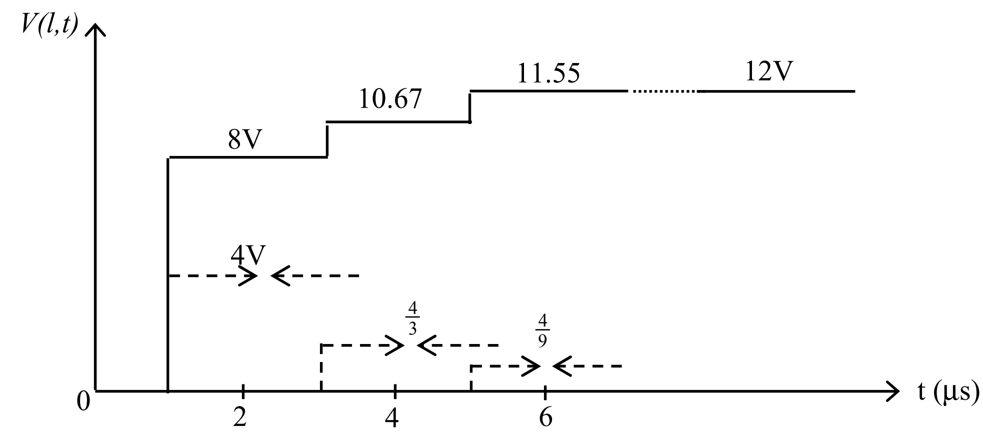

V (l,t)
0V
t (μs)
V(0,t)
4V
4V
4/3
4/9
0
t (μs)
2
4
6
−4/3
−4
I(l,t)
160mA
106.67
124.45
80
80/9
0
2
4
6
t (μs)
−80/3
I(0,t)
133.33
115.5
80mA
80
80/9
0
2
4
6
t (μs)

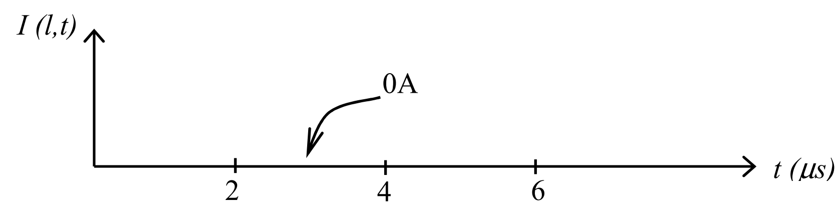
(b) $\Gamma_G = \frac{1}{3}, \Gamma_L = \lim_{Z_L \to \infty} \frac{Z_L - Z_o}{Z_L + Z_o} = 1$

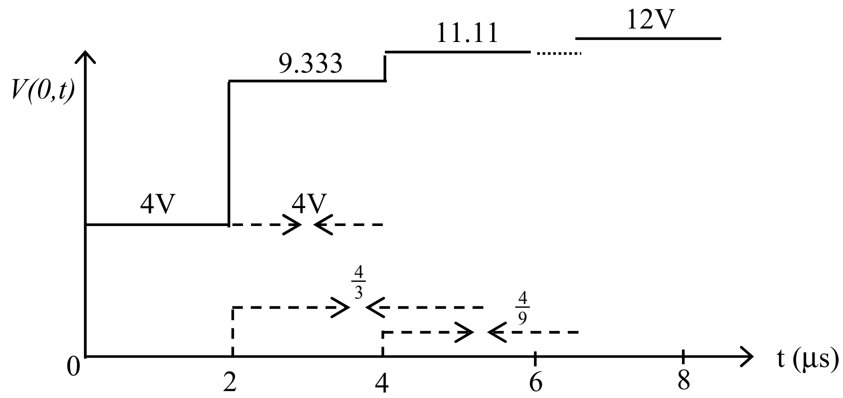
$$V_\infty = \lim_{z_L \to \infty} \frac{Z_L}{Z_L + Z_g} V_g = V_g = 12V, \qquad I_\infty = \lim_{z_L \to \infty} \frac{V_g}{Z_L + Z_g} = 0$$

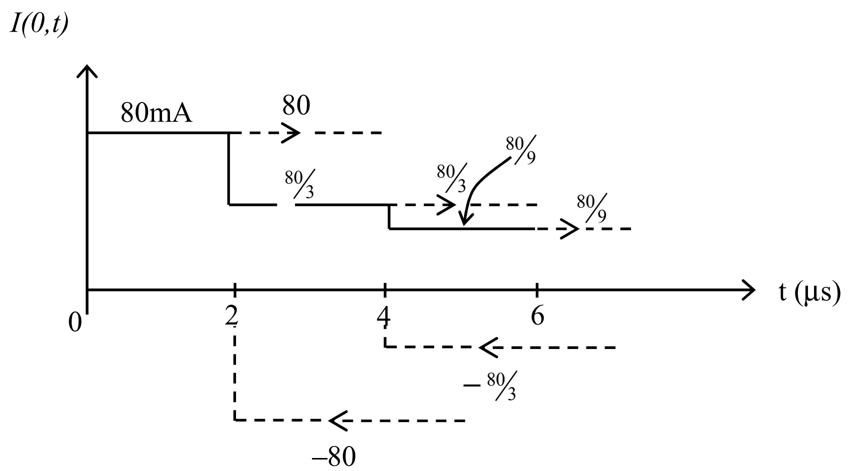
The bounce diagrams for current and voltage waves are as shown below.









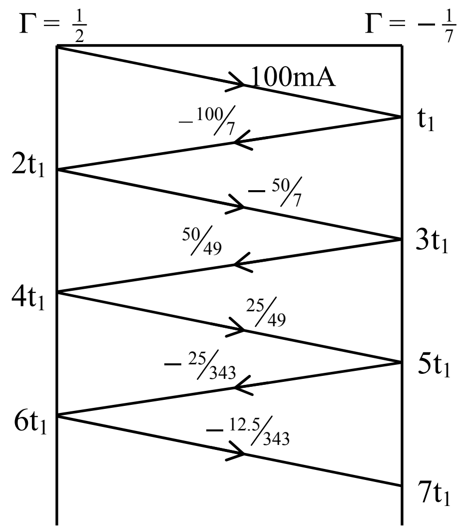


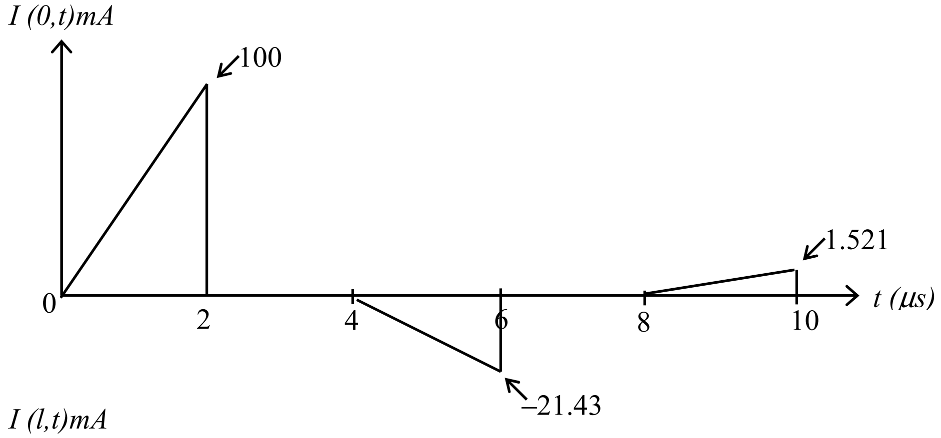
P.E. 11.9

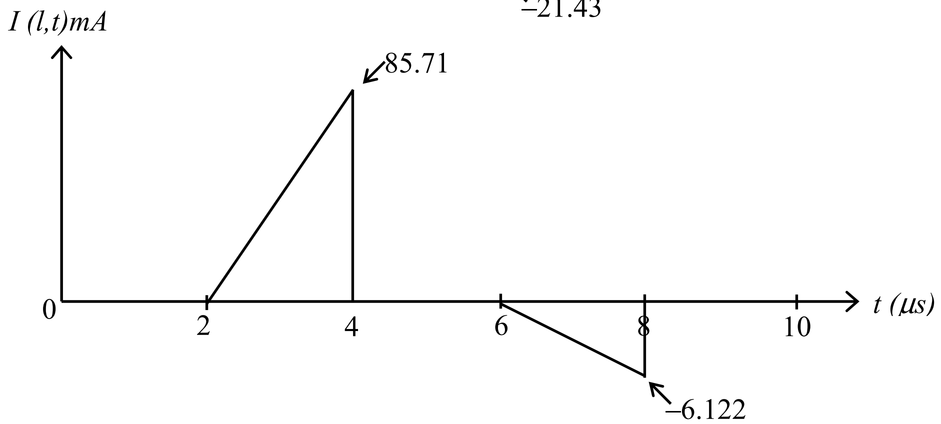
$$\Gamma_G = -\frac{1}{2}, \Gamma_L = \frac{1}{7}, t_1 = 2\mu s$$

$$\left(I_o\right)_{\max} = \frac{\left(V_g\right)_{\max}}{Z_g + Z_o} = \frac{10}{100} = 100mA$$

The bounce diagrams for maximum current are as shown below.







P.E. 11.10

(a) For $w/h = 0.8$, $\varepsilon_{eff} = \dfrac{4.8}{2} + \dfrac{2.8}{2}\left[1 + \dfrac{12}{0.8}\right]^{-1/2} = \underline{\underline{2.75}}$

(b) $Z_o = \dfrac{60}{\sqrt{2.75}} \ln\left(\dfrac{8}{0.8} + \dfrac{0.8}{4}\right) = 36.18 \ln 10.2 = \underline{\underline{84.03\Omega}}$

(c) $\lambda = \dfrac{3\times10^8}{10^{10}\sqrt{2.75}} = \underline{\underline{18.09 \text{ mm}}}$

P.E. 11.11

$$R_s = \sqrt{\frac{\pi f \mu_o}{\sigma_c}} = \sqrt{\frac{\pi\times20\times10^9\times4\pi\times10^{-7}}{5.8\times10^7}}$$

$$= 3.69\times10^{-2}$$

$$\alpha_c = 8.685\frac{R_s}{wZ_o} = \frac{8.686\times3.69\times10^{-2}}{2.5\times10^{-3}\times50}$$

$$= \underline{\underline{2.564 \text{ dB/m}}}$$

P.E. 11.12

$Z_L = (1+j2)Z_o \quad \longrightarrow \quad z_L = \dfrac{Z_L}{Z_o} = 1+j2$

We locate z_L on the Smith chart.

$\dfrac{\lambda}{4} \quad \longrightarrow \quad \dfrac{720^o}{4} = 180^o$

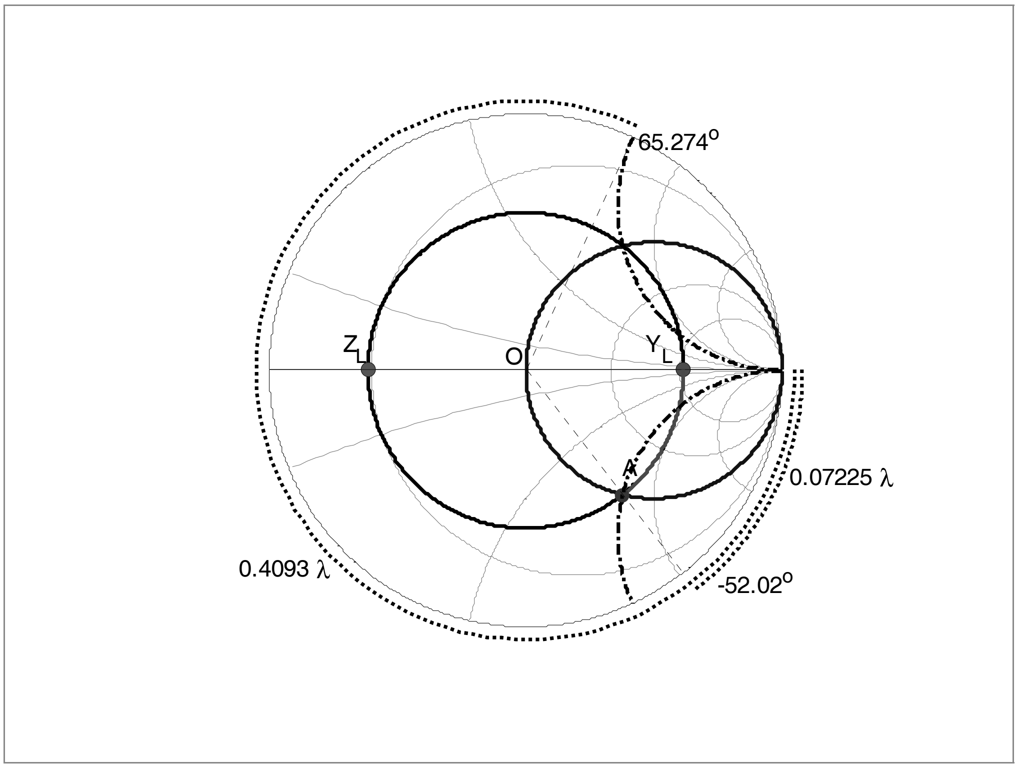
We move 180^o toward the generator and locate point Q at which

$z = 0.2 - j0.4$

$Z = zZ_o = \underline{\underline{(0.2 - j0.4)Z_o}}$

P.E. 11.13

$\dfrac{\lambda}{4} \quad \rightarrow \quad \dfrac{720^o}{4} = 180^o$



At A, $\quad y_{in} = 1 - j1.561$

$\qquad\qquad y_{stub} = j1.5614$

Position of the stub = $\underline{\underline{0.0723\lambda}}$

Length of the stub = $\underline{\underline{0.4093\lambda}}$

Prob. 11.1

$$C = \frac{\pi \varepsilon l}{\cosh^{-1}(d/2a)} \cong \frac{\pi \varepsilon l}{\ln(d/a)}$$

since $(d/2a)^2 = 11.11 >> 1$.

$$C = \frac{\pi \times \dfrac{10^{-9}}{36\pi} \times 16 \times 10^{-3}}{\ln(2/0.3)} = \underline{\underline{0.2342\text{ pF}}}$$

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma_c}} = \frac{1}{\sqrt{\pi \times 10^7 \times 4\pi \times 10^{-7} \times 5.8 \times 10^7}} = 2.09 \times 10^{-5}\text{ m} << a$$

$$R_{ac} = \frac{l}{\pi a \delta \sigma_c} = \frac{16 \times 10^{-3}}{\pi \times 0.3 \times 10^{-3} \times 2.09 \times 10^{-5} \times 5.8 \times 10^7} = \underline{\underline{1.4 \times 10^{-2}\ \Omega}}$$

Prob. 11.2

$$L = \frac{\mu}{2\pi}\ln(b/a), \qquad C = \frac{2\pi\varepsilon}{\ln(b/a)}$$

$$\frac{L}{C} = \frac{\mu}{2\pi}\frac{\left(\ln\dfrac{b}{a}\right)^2}{2\pi\varepsilon}$$

$$Z_o = \sqrt{\frac{L}{C}} = \frac{\ln(b/a)}{2\pi}\sqrt{\frac{\mu}{\varepsilon}} = 50$$

$$50 = \frac{\ln(b/a)}{2\pi}\sqrt{\frac{\mu_o}{\varepsilon_o \varepsilon_r}}$$

$$\varepsilon_r = \frac{\left(\ln\dfrac{b}{a}\right)^2}{4\pi^2}\frac{\eta_o^2}{50^2} = \frac{\left(\ln\dfrac{20}{6}\right)^2}{4\pi^2}\frac{377^2}{50^2} = \underline{\underline{2.0874}}$$

Prob. 11.3

Method 1:
Assume a charge per unit length Q on the surface of the inner conductor and –Q on the surface of the outer conductor. Using Gauss's law,

$$E_\rho = \frac{Q}{2\pi\varepsilon\rho}, \qquad a < \rho < b$$

$$V = -\int_a^b \mathbf{E} \bullet d\mathbf{l} = \frac{Q}{2\pi\varepsilon}\ln(b/a)$$

$$J = \sigma E = \frac{\sigma Q}{2\pi\varepsilon\rho}$$

$$I = \int_S \mathbf{J} \bullet d\mathbf{S} = \int_{\phi=0}^{2\pi} \frac{\sigma Q}{2\pi\varepsilon\rho}(1)\rho d\phi = \frac{\sigma Q}{\varepsilon}$$

$$G = \frac{I}{V} = \frac{\dfrac{\sigma Q}{\varepsilon}}{\dfrac{Q}{2\pi\varepsilon}\ln(b/a)} = \underline{\underline{\frac{2\pi\sigma}{\ln(b/a)}}}$$

Method 2:

Consider a section of unit length, Assume that a total current of I flows from inner conductor to outer conductor. At any radius ρ between a and b,

$$\mathbf{J} = \frac{I}{2\pi\rho}\mathbf{a}_\rho, \qquad a < \rho < b$$

$$\mathbf{E} = \frac{\mathbf{J}}{\sigma} = \frac{I}{2\pi\sigma\rho}\mathbf{a}_\rho$$

$$V = -\int_a^b \mathbf{E} \bullet d\mathbf{l} = \frac{I}{2\pi\sigma}\ln(b/a)$$

$$G = \frac{I}{V} = \frac{2\pi\sigma}{\ln(b/a)}$$

Prob. 11.4

$$\delta = \frac{1}{\sqrt{\pi f \mu_c \sigma_c}} = \frac{1}{\sqrt{\pi \times 80 \times 10^6 \times 4\pi \times 10^{-7} \times 5.28 \times 10^7}} = 7.744 \times 10^{-6}$$

$$R = \frac{1}{2\pi\delta\sigma_c}\left[\frac{1}{a} + \frac{1}{b}\right] = \frac{\left[\frac{1}{0.8\times10^{-3}} + \frac{1}{2.6\times10^{-3}}\right]}{2\pi \times 7.744\times10^{-6} \times 5.28\times10^7} = \frac{10^3(1.25 + 0.3836)}{2569.09} = \underline{\underline{0.6359\ \Omega/\text{m}}}$$

$$L = \frac{\mu}{2\pi}\ln\frac{b}{a} = \frac{4\pi\times10^{-7}}{2\pi}\ln\frac{2.6}{0.8} = \underline{\underline{2.357\times10^{-7}\ \text{H/m}}}$$

$$G = \frac{2\pi\sigma}{\ln\frac{b}{a}} = \frac{2\pi\times10^{-5}}{\ln\frac{2.6}{0.8}} = \underline{\underline{5.33\times10^{-5}\ \text{S/m}}}$$

$$C = \frac{2\pi\varepsilon}{\ln\frac{b}{a}} = \frac{2\pi\times3.5 * \frac{10^{-9}}{36\pi}}{\ln\frac{2.6}{0.8}} = \underline{\underline{1.65\times10^{-10}\ \text{F/m}}}$$

Prob. 11.5

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma_c}} = \frac{1}{\sqrt{\pi \times 500\times10^6 \times 4\pi\times10^{-7} \times 7\times10^7}}$$

$$\delta = 2.6902\times10^{-6}$$

$$R = \frac{2}{w\delta\sigma_c} = \frac{2}{0.3\times2.6902\times10^{-6}\times7\times10^7} = \underline{\underline{0.0354\Omega\,/\,\text{m}}}$$

$$L = \frac{\mu_o d}{w} = \frac{4\pi\times10^{-7}\times1.2\times10^{-2}}{0.3} = \underline{\underline{50.26\ \text{nH/m}}}$$

$$C = \frac{\varepsilon_o w}{d} = \frac{10^{-9}}{36\pi}\times\frac{0.3}{1.2\times10^{-2}} = \underline{\underline{221\ \text{pF/m}}}$$

Since $\sigma = 0$ for air,

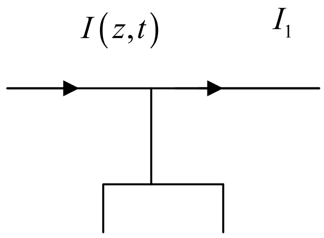
$$G = \frac{\sigma w}{d} = \underline{\underline{0}}$$

Prob. 11.6

(a) Applying Kirchhoff's voltage law to the loop yields

$$V(z+\Delta z,t)=V(z,t)-R\Delta z I_1 - L\Delta z\frac{\partial I_1}{\partial t}$$

But $I_1 = I(z,t)-\frac{C}{2}\Delta z\frac{\partial V(z,t)}{\partial t}-\frac{G}{2}\Delta z V(z,t)$



Hence,

$$V(z+\Delta z,t)=V(z,t)-R\Delta z\left[I(z,t)-\frac{C}{2}\Delta z\frac{\partial V}{\partial t}-\frac{G}{2}\Delta z V\right]-L\Delta z\left[\frac{\partial I}{\partial t}-\frac{C}{2}\Delta z\frac{\partial^2 V}{\partial t^2}-\frac{G}{2}\Delta z\frac{\partial V}{\partial t}\right]$$

Dividing by Δz and taking limits as $\Delta t \to 0$ give

$$\lim_{\Delta z\to 0}\frac{V(z+\Delta z,t)-V(z,t)}{\Delta z}=\lim_{\Delta z\to 0}\left[-RI-L\frac{\partial I}{\partial t}+\frac{RC}{2}\Delta z\frac{\partial V}{\partial t}+\frac{RG}{2}\Delta z V+\frac{LC}{2}\Delta z\frac{\partial^2 V}{\partial t^2}+\frac{LG}{2}\Delta z\frac{\partial V}{\partial t}\right]$$

or $-\frac{\partial V}{\partial z}=RI+L\frac{\partial I}{\partial t}$

Similarly, applying Kirchhoff's law to the node leads to

$$I(z+\Delta z,t)-I(z,t)=-G\Delta z\left(\frac{V(z,t)+V(z+\Delta z)}{2}\right)-C\Delta z\frac{\partial}{\partial t}\left(\frac{V(z,t)+V(z+\Delta z,t)}{2}\right)$$

Let $\Delta z \longrightarrow 0$, we get

$$-\frac{\partial I}{\partial z}=GV+C\frac{\partial V}{\partial t}$$

(b) Applying Kirchhoff's voltage law,

$$V(z,t)=R\frac{\Delta l}{2}I(z,t)+L\frac{\Delta l}{2}\frac{\partial I}{\partial t}(z,t)+V(z+\Delta l/2,t)$$

or

$$-\frac{V(z+\Delta l/2,t)-V(z,t)}{\Delta l/2}=RI+L\frac{\partial I}{\partial t}$$

As $\Delta l \to 0$, $\quad -\frac{\partial V}{\partial z}=RI+L\frac{\partial I}{\partial t}$

Here, we take $\Delta l = \Delta z$. Applying Kirchhoff's current law,

$$I(z,t) = I(z+\Delta l,t) + G\Delta l V(z+\Delta l/2,t) + C\Delta l \frac{\partial V\left(z+\Delta l/2,t\right)}{\partial t}$$

or

$$-\frac{I(z+\Delta l,t) - I(z,t)}{\Delta l} = GV(z+\Delta l/2,t) + C\frac{\partial V(z+\Delta l/2,t)}{\partial t}$$

As $\Delta l \to 0$, $\quad -\dfrac{\partial I(z,t)}{\partial z} = GV(z,t) + C\dfrac{\partial V(z,t)}{\partial t}$

Prob. 11.7

(a)

$$\gamma = \sqrt{(R+j\omega L)(G+j\omega C)} = j\omega\sqrt{LC}\sqrt{(1+\frac{R}{j\omega L})(1+\frac{G}{j\omega C})}$$

$$= j\omega\sqrt{LC}\sqrt{1-\frac{RG}{\omega^2 LC}+\frac{R}{j\omega L}+\frac{G}{j\omega C}}$$

As R<< ωL and G<<ωC, dropping the ω^2 term gives

$$\gamma \cong j\omega\sqrt{LC}\sqrt{1+\frac{R}{j\omega L}+\frac{G}{j\omega C}} \cong j\omega\sqrt{LC}\left[1+\frac{R}{2j\omega L}+\frac{G}{2j\omega C}\right]$$

$$= \underline{\underline{\left(\frac{R}{2}\sqrt{\frac{C}{L}}+\frac{G}{2}\sqrt{\frac{L}{C}}\right)+j\omega\sqrt{LC}}}$$

(b)

$$Z_o = \sqrt{\frac{R+j\omega L}{G+j\omega C}} = \sqrt{\frac{L}{C}}\sqrt{\frac{1+\dfrac{R}{j\omega L}}{1+\dfrac{G}{j\omega C}}} = \sqrt{\frac{L}{C}}\left(1+\frac{R}{j\omega L}\right)^{1/2}\left(1+\frac{G}{j\omega C}\right)^{-1/2}$$

$$\cong \sqrt{\frac{L}{C}}\left(1+\frac{R}{2j\omega L}+\ldots\right)\left(1-\frac{G}{j2\omega C}+\ldots\right) = \sqrt{\frac{L}{C}}\left(1-j\frac{R}{2\omega L}+j\frac{G}{2\omega C}+\ldots\right)$$

$$\cong \underline{\underline{\sqrt{\frac{L}{C}}\left[1+j\left(\frac{G}{2\omega C}-\frac{R}{2\omega L}\right)\right]}}$$

Prob. 11.8

For a lossless line,

$$u = \frac{1}{\sqrt{LC}}, \qquad Z_o = \sqrt{\frac{L}{C}}$$

$$uZ_o = \frac{1}{C}$$

$$u = \underline{\underline{\frac{1}{Z_o C}}}$$

Prob. 11.9

$$Z_o = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \qquad (1)$$

$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)} \qquad (2)$$

$$\alpha = 0.04 \text{ dB/m} = \frac{0.04}{8.686} \text{ Np/m} = 0.00461 \text{ Np/m}$$

Multiplying (1) and (2),

$$Z_o(\alpha + j\beta) = R + j\omega L \longrightarrow 50(0.00461 + j2.5) = R + j\omega L$$

$$R = 50 \times 0.00461 = \underline{\underline{0.2305 \ \Omega/\text{m}}}$$

$$L = \frac{50 \times 2.5}{2\pi \times 60 \times 10^6} = \underline{\underline{0.3316 \ \mu\text{H/m}}}$$

Dividing (2) by (1),

$$\frac{\alpha + j\beta}{Z_o} = G + j\omega C$$

$$G = \frac{\alpha}{Z_o} = \frac{0.00461}{50} = \underline{\underline{92.2 \ \mu\text{S/m}}}$$

$$C = \frac{\beta}{\omega Z_o} = \frac{2.5}{2\pi \times 60 \times 10^6 \times 50} = \underline{\underline{0.1326 \ \text{nF/m}}}$$

Prob. 11.10

$$Z_o = \sqrt{\frac{L}{c}} = \sqrt{\frac{\mu d}{w} . \frac{d}{\varepsilon w}} = \frac{d}{w}\sqrt{\frac{\mu}{\varepsilon}}$$

$$Z_o = \eta_o \frac{d}{w} = 78$$

$$Z_o' = \eta_o \frac{d}{w'} = 75$$

$$\frac{78}{75} = \frac{w'}{w} \rightarrow w' = 1.04w$$

i.e. the width must be increased by 4%.

Prob. 11.11

(a) $$Z_o = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{6.8 + j2\pi \times 10^3 \times 3.4 \times 10^{-3}}{0.42 \times 10^{-6} + j2\pi \times 10^3 \times 8.4 \times 10^{-9}}}$$

$$= 10^3 \sqrt{\frac{6.8 + j21.36}{0.42 + j52.78}} = \underline{\underline{644.3 - j97\ \Omega}}$$

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = 10^{-3}\sqrt{(6.8 + j21.36)(0.42 - j52.78)}$$

$$\underline{\underline{= (5.415 + j33.96) \times 10^{-3}\ \ /\text{mi}}}$$

(b) $$u = \frac{\omega}{\beta} = \frac{2\pi \times 10^3}{33.96 \times 10^{-3}} = \underline{\underline{1.85 \times 10^5\ \ \text{mi/s}}}$$

(c) $$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{33.96 \times 10^{-3}} = \underline{\underline{185.02\ \ \text{mi}}}$$

Prob. 11.12

Using eq. (11.42a),

$$Z_{in} = -jZ_o \cot \beta\ell$$

$$Z_o = 250, \qquad \lambda = \frac{c}{f} = \frac{3 \times 10^8}{400 \times 10^6} = 0.75 \text{ m}$$

$$\beta\ell = \frac{2\pi}{\lambda}\ell = \frac{2\pi \times 0.1}{0.75} = 48^o$$

$$Z_{in} = -j(250)\cot 48^o = \underline{\underline{-j225.1\ \Omega}}$$

Prob. 11.13
Assume that the line is lossless.

$$Z_o = \sqrt{\frac{L}{C}}$$

From Table 11.1, $L = \frac{\mu}{2\pi}\ln\frac{b}{a}, \quad C = \frac{2\pi\varepsilon}{\ln\frac{b}{a}}$

$$\frac{L}{C} = \frac{\mu}{\varepsilon}\left(\frac{1}{2\pi}\ln\frac{b}{a}\right)^2$$

$$Z_o = \sqrt{\frac{L}{C}} = \frac{1}{2\pi}\ln\frac{b}{a}\times\sqrt{\frac{\mu}{\varepsilon}} = \frac{\eta_o}{2\pi\sqrt{\varepsilon_r}}\ln\frac{b}{a}$$

$$\ln\frac{b}{a} = 2\pi\sqrt{\varepsilon_r}\frac{Z_o}{\eta_o} = 2\pi\sqrt{2.25}\frac{75}{120\pi} = 1.875$$

$$\frac{b}{a} = e^{1.875} \longrightarrow a = be^{-1.875} = 3e^{-1.875} \text{ mm} = \underline{\underline{0.46 \text{ mm}}}$$

Prob. 11.14

(a) For a lossless line, R = 0 = G.

$$\gamma = j\omega\sqrt{LC} \longrightarrow \beta = \omega\sqrt{LC} = \omega\sqrt{\mu_o c_o} = \frac{\omega}{c}$$

$$u = \frac{\omega}{\beta} = c = \frac{1}{\sqrt{LC}}$$

(b) For lossless line, R = 0 =G

$$L = \frac{\mu}{\pi}\cosh^{-1}\frac{d}{2a}, C = \frac{\pi\varepsilon}{\cosh^{-1}\frac{d}{2a}}$$

$$Z_o = \sqrt{\frac{L}{C}} = \sqrt{\frac{\mu}{\pi}.\frac{1}{\pi\varepsilon}}\cosh^{-1}\frac{d}{2a} = \frac{120\pi}{\pi\sqrt{\varepsilon_r}}\cosh^{-1}\frac{d}{2a}$$

$$\underline{\underline{= \frac{120}{\sqrt{\varepsilon_r}}\cosh^{-1}\frac{d}{2a}}}$$

Yes, true for other lossless lines.

Prob. 11.15

$$L = \frac{\mu}{\pi}\cosh^{-1}\frac{d}{2a} = 4\times10^{-7}\cosh^{-1}\frac{0.32}{0.12}$$

$$\underline{\underline{L = 0.655\ \mu\text{H/m}}}$$

$$C = \frac{\pi\varepsilon}{Cosh^{-1}\dfrac{d}{2a}} = \frac{\pi\times\dfrac{10^{-9}}{36\pi}\times3.5}{Cosh^{-1}2.667}$$

$$\underline{\underline{C = 59.4\ \text{pF/m}}}$$

$$Z_o = \sqrt{\frac{L}{C}} = \sqrt{\frac{0.655\times10^{-6}}{59.4\times10^{-12}}} = \underline{\underline{105\Omega}}$$

or

$$Z_o = \frac{120}{\sqrt{3.5}}\cosh^{-1}2.667 = \underline{\underline{105\Omega}}$$

Prob. 11.16

For a distortionless cable,

$$\frac{R}{L} = \frac{G}{C} \quad\longrightarrow\quad RC = LG \qquad (1)$$

$$Z_o = \sqrt{\frac{L}{C}} = 60 \qquad (2)$$

$$u = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} = \frac{\ell}{t_o} = \frac{4}{80\times10^{-6}} \qquad (3)$$

$$\alpha\ell = 0.24dB = \frac{0.24}{8.686}Np = 0.0276$$

$$\alpha = \sqrt{RG} = 0.00069 \qquad (4)$$

From (2) and (3),

$$\frac{1}{\text{C}} = \frac{60\times4}{80\times10^{-6}} \quad\longrightarrow\quad C = \frac{8\times10^{-5}}{240} = \underline{\underline{333.3\ \text{nF/m}}}$$

From (2),

$$L = (60)^2C = 3600\times333.3\times10^{-9} = \underline{\underline{1.20\ \text{mH/m}}}$$

From (1) and (4),

$$\frac{C}{G}=\frac{LG}{0.00069^2} \longrightarrow G^2=\frac{0.00069^2}{60^2}$$

$$G=\frac{0.00069}{60}=\underline{\underline{11.51\ \mu\text{S/m}}}$$

From (4),

$$R=\frac{0.00069^2}{G}=0.00069\times 60=\underline{\underline{0.0414\ \Omega/\text{m}}}$$

Prob. 11.17

(a) $\dfrac{R}{L}=\dfrac{G}{C}\rightarrow G=\dfrac{R}{L}C=\dfrac{20\times 63\times 10^{-12}}{0.3\times 10^{-6}}$

$$G=4.2\times 10^{-3}\ \text{S/m}$$

$$\alpha=\sqrt{RG}=\sqrt{20\times 4.2\times 10^{-3}}=0.2898$$

$$\beta=\omega\sqrt{LC}=2\pi\times 120\times 10^6\sqrt{0.3\times 10^{-6}\times 63\times 10^{-12}}=3.278$$

$$\underline{\underline{\gamma=0.2898+j3.278\ \ /\text{m}}}$$

$$u=\frac{\omega}{\beta}=\frac{2\pi\times 120\times 10^6}{3.278}=\underline{\underline{2.3\times 10^8\ \text{m/s}}}$$

$$Z_o=\sqrt{\frac{L}{C}}=\sqrt{\frac{0.3\times 10^{-6}}{63\times 10^{-12}}}=\underline{\underline{69\ \Omega}}$$

(b) Let V_o be its original magnitude

$$V_o e^{-\alpha z}=0.2V_o\rightarrow e^{\alpha z}=5$$

$$z=\frac{1}{\alpha}\ln 5=\underline{\underline{5.554\ \text{m}}}$$

(c) $\beta l=45^o={\pi}/{4}\rightarrow l=\dfrac{\pi}{4\beta}=\dfrac{4}{4\times 3.278}=\underline{\underline{0.3051\ \text{m}}}$

Prob. 11.18

(a) $\alpha = \underline{\underline{0.0025 \text{ Np/m}}}, \quad \beta = \underline{\underline{2 \text{ rad/m}}},$

$$u = \frac{\omega}{\beta} = \frac{10^8}{2} = \underline{\underline{5\times10^7 \text{ m/s}}}$$

(b) $\Gamma = \dfrac{V_o}{V_o^+} = \dfrac{60}{120} = \dfrac{1}{2}$

But $\Gamma = \dfrac{Z_L - Z_o}{Z_L + Z_o} \rightarrow \dfrac{1}{2} = \dfrac{300 - Z_o}{300 + Z_o} \rightarrow \underline{\underline{Z_o = 100\Omega}}$

$$I(l') = \frac{120}{Z_o} e^{0.0025l'} \cos(10^8 + 2l') - \frac{60}{Z_o} e^{-0.0025l'} \cos(10^8 t - 2l')$$

$$= 1.2e^{0.0025l'} \cos(10^8 + 2l') - 0.6e^{-0.0025l'} \cos(10^8 t - 2l') A$$

Prob. 11.19

$\alpha = 10^{-3}, \qquad \beta = 0.01$

$$\gamma = \alpha + j\beta = 0.001 + j0.01 = \underline{\underline{(1 + j10)\times10^{-3} \text{ /m}}}$$

$$u = \frac{\omega}{\beta} = \frac{2\pi\times10^4}{0.01} = \underline{\underline{6.283\times10^6 \text{ m/s}}}$$

Prob. 11.20

$$Z_o = \sqrt{\frac{L}{C}} = \sqrt{\frac{0.6\times10^{-6}}{82\times10^{-12}}} = \underline{\underline{85.54\ \Omega}}$$

$$RC = LG \quad \longrightarrow \quad G = \frac{RC}{L}$$

$$\alpha = \sqrt{RG} = \sqrt{R\frac{RC}{L}} = \frac{R}{Z_o} = \frac{10\times10^{-3}}{85.54} = 1.169\times10^{-4} \text{ Np/m}$$

$$\beta = \omega\sqrt{LC} = 2\pi\times80\times10^6\sqrt{0.6\times10^{-6}\times82\times10^{-12}}$$

$$= 3.5258 \text{ rad/m}$$

$$\gamma = \underline{\underline{1.169\times10^{-4} + j3.5258 \text{ /m}}}$$

Prob. 11.21

$$R + j\omega L = 6.5 + j2\pi \times 2\times 10^6 \times 3.4\times 10^{-6} = 6.5 + j42.73$$

$$G + j\omega C = 8.4\times 10^{-3} + j2\pi \times 2\times 10^6 \times 21.5\times 10^{-12} = (8.4 + j0.27)\times 10^{-3}$$

$$Z_o = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{6.5 + j42.73}{(8.4 + j0.27)\times 10^{-3}}}$$

$$Z_o = 71.71\angle 39.75^o = \underline{\underline{55.12 + j45.85\Omega}}$$

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} \qquad = \sqrt{\left(43.19\angle 81.34^o\right)\left(8.4\times 10^{-3}\angle 1.84^o\right)}$$

$$= \underset{\alpha}{0.45} + \underset{\beta}{j0.4} \ \ /\mathrm{m}$$

$$t = \frac{l}{u}, \text{ but } u = \frac{\omega}{\beta},$$

$$t = \frac{\beta l}{\omega} = \frac{0.39\times 5.6}{2\pi \times 2\times 10^6} = \underline{\underline{0.1783\mu s}}$$

Prob. 11.22

$$Z_{in} = Z_o\left[\frac{Z_L + Z_o \tanh \gamma\ell}{Z_0 + Z_L \tanh \gamma\ell}\right]$$

$$\gamma\ell = \alpha\ell + j\beta\ell = 1.4\times 0.5 + j2.6\times 0.5 = 0.7 + j1.3$$

$$\tanh \gamma\ell = 1.4716 + j0.3984$$

$$Z_{in} = (75 + j60)\left[\frac{200 + (75 + j60)(1.4716 + j0.3984)}{(75 + j60) + 200(1.4716 + j0.3984)}\right]$$

$$= \underline{\underline{57.44 + j48.82\ \Omega}}$$

Prob. 11.23

(a) For a lossy line,

$$Z_{in} = Z_o\left[\frac{Z_L + Z_o \tanh\gamma\ell}{Z_0 + Z_L \tanh\gamma\ell}\right]$$

For a short-circuit, $Z_L = 0.$

$$Z_{sc} = Z_{in}\Big|_{Z_L=0} = Z_o \tanh\gamma\ell$$

$$\tanh\gamma\ell = \frac{Z_{sc}}{Z_o} = \frac{30-j12}{80+j60} = 0.168 - j0.276$$

$$\gamma\ell = \alpha\ell + j\beta\ell = \tanh^{-1}(0.168 - j0.276) = 0.1571 - j0.2762$$

$$\alpha = \frac{0.1571}{2.1} = \underline{\underline{0.0748\ \text{Np/m}}}$$

$$\beta = \frac{0.2762}{2.1} = \underline{\underline{0.1316\ \text{rad/m}}}$$

(b)

$$Z_{in} = (80+j60)\left[\frac{(40+j30)+(80+j60)(0.168-j0.276)}{(80+j60)+(40+j30)(0.168-j0.276)}\right]$$

$$= \underline{\underline{61.46 + j24.43\ \Omega}}$$

Prob. 11.24

(a) $$T_L = \frac{V_L}{V_o^+} = \frac{Z_L I_L}{\tfrac{1}{2}\left(V_L + Z_o I_L\right)} = \frac{2Z_L I_L}{Z_L I_L + Z_o I_L}$$

$$= \frac{2Z_L}{Z_L + Z_o}$$

$$1 + \Gamma_L = 1 + \frac{Z_L - Z_o}{Z_L + Z_o} = \underline{\underline{\frac{2Z_L}{Z_L + Z_o}}}$$

(b) (i) $$\tau_L = \frac{2nZ_o}{nZ_o + Z_o} = \frac{2n}{n+1}$$

(ii) $$\tau_L = {}_{Y_L} \xrightarrow{\lim} {}_0 = \frac{2}{1 + Z_o/Z_L} = 2$$

(iii) $$\tau_L = {}_{Z_L} \xrightarrow{\lim} {}_0 = \frac{2Z_L}{Z_L + Z_o} = 0$$

(iv) $$\tau_L = \frac{2Z_o}{2Z_o} = 1$$

Prob. 11.25

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)} = \sqrt{(3.5 + j2\pi \times 400 \times 10^6 \times 2 \times 10^{-6})(0 + j2\pi \times 400 \times 10^6 \times 120 \times 10^{-12})}$$

$$= \sqrt{(3.5 + j5026.55)(j0.3016)} = 0.0136 + j38.94$$

$$\alpha = \underline{\underline{0.0136 \text{ Np/m}}}, \qquad \beta = \underline{\underline{38.94 \text{ rad/m}}}$$

$$u = \frac{\omega}{\beta} = \frac{2\pi \times 400 \times 10^6}{38.94} = \underline{\underline{6.452 \times 10^7 \text{ m/s}}}$$

$$Z_o = \sqrt{\frac{R + j\omega L}{G + j\omega C}} = \sqrt{\frac{3.5 + j5026.55}{j0.3016}} = \underline{\underline{129.1 - j0.045\ \Omega}}$$

Prob. 11.26

From eq. (11.33)

$$Z_{sc} = Z_{in}\Big|_{Z_L=0} = Z_o \tanh \gamma l$$

$$Z_{oc} = Z_{in}\Big|_{Z_L=\infty} = \frac{Z_o}{\tanh \gamma l} = Z_o \coth(\gamma l)$$

For lossless line, $\gamma = j\beta, \tan(\gamma l) = \tanh(j\beta l) = j\tan(\beta l)$

$$Z_{sc} = jZ_o \tan(\beta l), Z_{oc} = -jZ_o \cot(\beta l)$$

Prob. 11.27

$$(a) \quad \beta l = \frac{1}{4} \times 100 = 25 \text{ rad} = 1432.4^o = 352.4^o$$

$$Z_{in} = 60\left[\frac{j40 + j60 \tan 352.4^o}{60 - 40 \tan 352.4^o}\right] = \underline{\underline{j29.375\Omega}}$$

$$V(z=0) = V_o = \frac{Z_{in}}{Z_{in} + Z_g} V_g = \frac{j29.375(10\angle 0^o}{j29.375 + 50 - j40}$$

$$= \frac{293.75\angle 90^o}{51.116\angle -12^o} = \underline{\underline{5.75\angle 102^o}}$$

$$(b) \quad Z_{in} = Z_L = \underline{\underline{j40\Omega.}}$$

$$V_o^+ = \frac{V_g}{(e^{j\beta l} + \Gamma e^{-j\beta l})} \qquad \text{(l is from the load)}$$

$$V_L = \frac{V_g(1+\Gamma)}{(e^{j\beta l} + \Gamma e^{-j\beta l})} = \underline{\underline{12.62\angle 0^o \text{ V}}}$$

$(c) \quad \beta l' = \frac{1}{4} \times 4 = 1 rad = 57.3^o$

$$Z_{in} = 60\left[\frac{j40 + j60\tan 57.3^o}{60 - 40\tan 57.3^o}\right] = \underline{\underline{-j3471.88\Omega.}}$$

$$V = \frac{V_g(e^{j} + \Gamma e^{-j})}{(e^{j25} + \Gamma e^{-j25})} = \underline{\underline{22.74\angle 0^o \;\; \text{V}}}$$

(d) 3m from the source is the same as 97m from the load., i.e.

$$l' = 100 - 3 = 97m, \quad \beta l' = \frac{1}{4} \times 97 = 24.25 rad = 309.42^o$$

$$Z_{in} = 60\left[\frac{j40 + j60\tan 309.42^o}{60 - 40\tan 309.42^o}\right] = \underline{\underline{-j18.2\Omega}}$$

$$V = \frac{V_g(e^{j97/4} + \Gamma e^{-j97/4})}{(e^{j25} + \Gamma e^{-j25})} = \underline{\underline{6.607\angle 180^o \;\; \text{V}}}$$

Prob. 11.28

(a) $\Gamma = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{120 - 50}{170} = \underline{\underline{0.4118}}$

For resistive load, $s = \frac{Z_L}{Z_o} = \underline{\underline{2.4}}$

(b) $Z_{in} = Z_o \frac{Z_L + jZ_o\tan(\beta l)}{Z_o + jZ_L\tan(\beta l)}$

$$\beta l = \frac{2\pi}{\lambda} . \frac{\lambda}{6} = 60^o$$

$$Z_{in} = 50\left[\frac{120 + j50\tan(60^o)}{50 + j120\tan(60^o)}\right] = \underline{\underline{34.63\angle -40.65^o \,\Omega}}$$

Prob. 11.30

$$V_1 = V_s(z=0) = V_o^+ + V_o^- \qquad (1)$$

$$V_2 = V_s(z=l) = V_o^+ e^{-\gamma l} + V_o^- \mathrm{e}^{\gamma l} \qquad (2)$$

$$\mathrm{I}_1 = I_s(z=0) = \frac{V_o^+}{Z_o} - \frac{V_o^-}{Z_o} \qquad (3)$$

$$I_2 = -I_s(z=l) = -\frac{V_o^+}{Z_o} e^{-\gamma l} + \frac{V_o^-}{Z_o} \mathrm{e}^{\gamma l} \qquad (4)$$

$$(1) + (3) \rightarrow V_o^+ = \frac{1}{2}(V_1 + Z_o I_1)$$

$$(1) - (3) \rightarrow V_o^- = \frac{1}{2}(V_1 - Z_o I_1)$$

Substituting V_o^+ and V_o^- in (2) gives

$$V_2 = \frac{1}{2}(V_1 + Z_o I_1)e^{-\gamma l} + \frac{1}{2}(V_1 - Z_o I_1)e^{\gamma l}$$

$$= \frac{1}{2}(e^{\gamma l} + e^{-\gamma l})V_1 + \frac{1}{2}Z_o(e^{-\gamma l} - e^{\gamma l})I_1$$

$$V_2 = \cosh\gamma l V_1 - Z_o \sinh\gamma l I_1 \qquad (5)$$

Substituting V_o^+ and V_o^- in (4),

$$I_2 = -\frac{1}{2Z_o}(V_1 + Z_o I_1)e^{-\gamma l} + \frac{1}{2Z_o}(V_1 - Z_o I_1)e^{\gamma l}$$

$$= \frac{1}{2Z_o}(e^{\gamma l} - e^{-\gamma l})V_1 + \frac{1}{2}(e^{\gamma l} + e^{-\gamma l})I_1$$

$$I_2 = \frac{1}{Z_o}\sinh\gamma l V_1 - \cosh\gamma l I_1 \qquad (6)$$

From (5) and (6)

$$\begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} = \begin{bmatrix} \cosh\gamma l & -Z_o \sinh\gamma l \\ -\frac{1}{Z_o}\sinh\gamma l & \cosh\gamma l \end{bmatrix} \begin{bmatrix} V_1 \\ I_1 \end{bmatrix}$$

But

$$\begin{bmatrix} \cosh\gamma l & -Z_o \sinh\gamma l \\ -\frac{1}{Z_o}\sinh\gamma l & \cosh\gamma l \end{bmatrix}^{-1} = \begin{bmatrix} \cosh\gamma l & Z_o \sinh\gamma l \\ \frac{1}{Z_o}\sinh\gamma l & \cosh\gamma l \end{bmatrix}$$

Thus

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} \cosh\gamma l & Z_o \sinh\gamma l \\ \frac{1}{Z_o}\sinh\gamma l & \cosh\gamma l \end{bmatrix} \begin{bmatrix} V_2 \\ -I_2 \end{bmatrix}$$

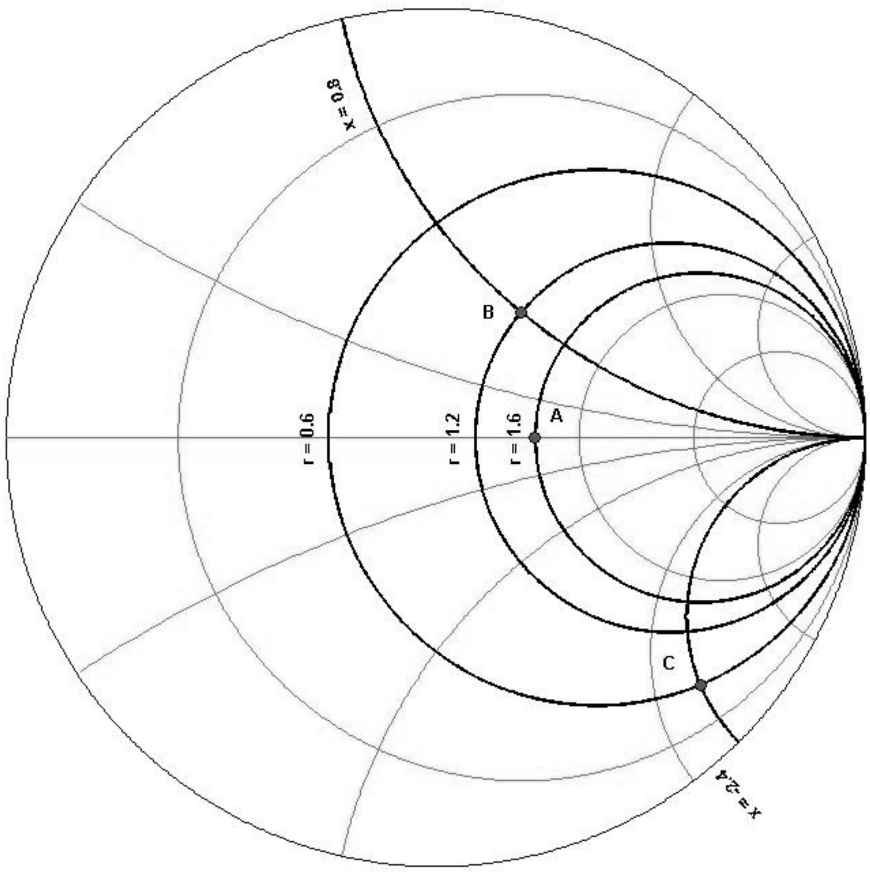
Prob. 11.31

(a) $z_a = \dfrac{Z_a}{Z_o} = \dfrac{80}{50} = 1.6$

(b) $z_b = \dfrac{Z_b}{Z_o} = \dfrac{60 + j40}{50} = 1.2 + j0.8$

(c) $z_c = \dfrac{Z_c}{Z_o} = \dfrac{30 - j120}{50} = 0.6 - j2.4$

The three loads are located on the Smith chart, as A, B, and C as shown next.



Prob. 11.32

$$z_L = \frac{Z_L}{Z_O} = \frac{210}{100} = 2.1 = s$$

Or $\Gamma = \dfrac{Z_L - Z_O}{Z_L + Z_O} = \dfrac{110}{310},$

$$s = \frac{1+|\Gamma|}{1-|\Gamma|} = 2.1$$

But $s = \dfrac{V_{\max}}{V_{\min}} \rightarrow V_{\max} = sV_{\min}$

Since the line is $\dfrac{\lambda}{4}$ long, $\dfrac{\lambda}{4} \rightarrow \dfrac{720^o}{4} = 120^o$

Hence the sending end will be $V_{\min}$, while the receiving end at $V_{\max}$

$V_{\min} = V_{\max} / s = 80 / 2.1 = 38.09$

$V_{sending} = \underline{\underline{38.09\angle 90^o}}$

Prob. 11.33

(a) Method 1: At Y,

$$Z_{in} = Z_o\left[\frac{Z_L + jZ_o \tan\beta\ell}{Z_o + jZ_L \tan\beta\ell}\right]$$

$$\beta\ell = \frac{2\pi}{\lambda}\frac{\lambda}{2} = \pi, \qquad \tan\pi = 0$$

$$Z_{in} = Z_o \frac{Z_L}{Z_o} = Z_L = 150\ \Omega$$

At X, $Z_L' = 150\Omega$

$$\beta\ell = \frac{2\pi}{\lambda}\frac{\lambda}{4} = \pi/2, \qquad \tan\pi/2 = \infty$$

$$Z_{in} = \lim_{\tan\beta\ell\rightarrow\infty} Z_o\left[\frac{jZ_o + \dfrac{Z_L'}{\tan\beta\ell}}{jZ_L' + \dfrac{Z_o}{\tan\beta\ell}}\right] = \frac{Z_o^2}{Z_L'} = \frac{(75)^2}{150} = 37.5\Omega$$

Method 2: Using the Smith chart,

$$z_L = \frac{Z_L'}{Z_o} = \frac{150}{50} = 3$$

Since $\ell=\lambda/2$, we must move 360^o toward the generator. We arrive at the same point. Hence,

$$Z_{in} = Z_L = 150$$

$$z_L' = \frac{150}{75} = 2$$

$$\ell = \frac{\lambda}{4} \rightarrow 180^o$$

We move 180^o toward the generator. $z_{in} = 0.5$

$$Z_{in} = 75(0.6) = \underline{\underline{37.5\Omega}}$$

(b) From the Smith chart,

s = 3 for section XY

s = 2 for section YZ

(c) $\Gamma = \frac{Z_L - Z_o}{Z_L - Z_o} = \frac{150-50}{150+50} = \underline{\underline{0.5}}$

Prob. 11.34

$$Z_{in} = Z_o\left[\frac{Z_L + jZ_o \tan\beta\ell}{Z_o + jZ_L \tan\beta\ell}\right]$$

$$\lambda = \frac{c}{f} = \frac{3\times10^8}{4\times10^8} = 0.75$$

$$\beta\ell = \frac{2\pi}{\lambda}\ell = \frac{2\pi\times3.2}{0.75} = 26.81$$

$$Z_{in} = 50\left[\frac{(30-j50)+j50\tan(26.81)}{50+j(30-j50)\tan(26.81)}\right] = \underline{\underline{26.13 + j44.23\ \Omega}}$$

Prob. 11.35

$z_L = \dfrac{Z_L}{Z_o} = \dfrac{40 - j25}{50} = 0.8 - j0.5$

We locate this at point P on the Smith chart shown below

$|\Gamma_L| = \dfrac{OP}{OQ} = \dfrac{2.4 \text{ cm}}{8 \text{ cm}} = 0.3, \qquad \theta_\Gamma = -96^o$

$\underline{\underline{\Gamma_L = 0.3\angle -96^o}}$

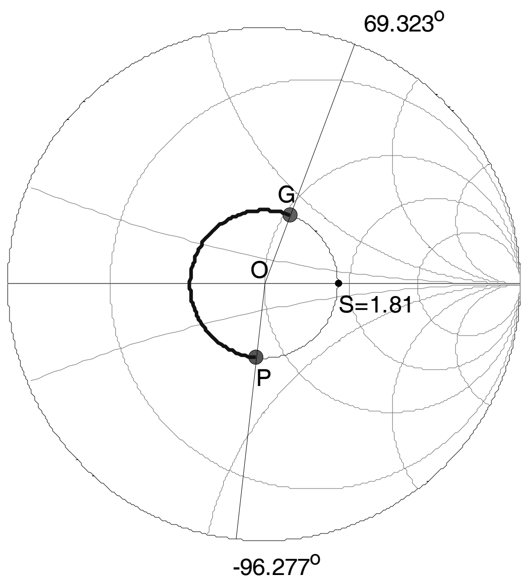
At S, $\quad$ s = r = $\underline{\underline{1.81}}$

$\ell = 0.27\lambda \quad \longrightarrow \quad 0.27 \times 720^o = 194.4^o$

From P, we move 194.4^o toward the generator to G. At G,

$z_{in} = 1.0425 + j0.6133$

$Z_{in} = Z_o z_{in} = 50(1.0425 + j0.6133) = \underline{\underline{52.13 - j30.66\ \Omega}}$

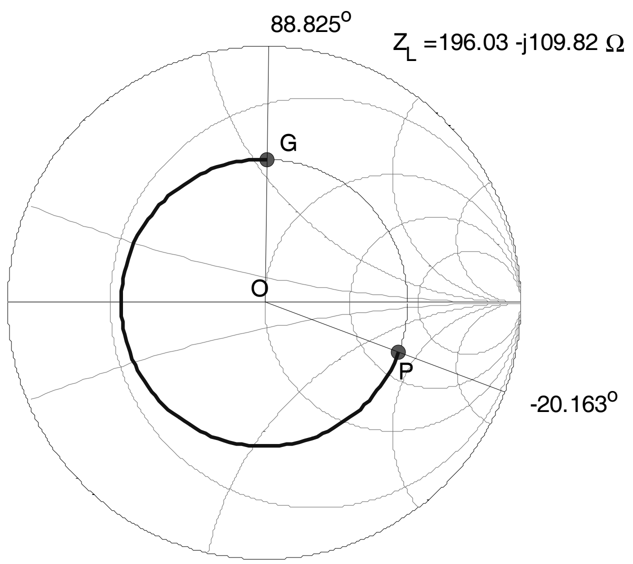


Prob. 11.36

$$u = \frac{c}{\sqrt{\varepsilon_r}} = \frac{3\times10^8}{\sqrt{3.62}} = 1.5767\times10^8$$

$$\lambda = \frac{u}{f} = \frac{1.5767\times10^8}{400\times10^6} = 39.42 \text{ cm}$$

$$\ell = 132 \text{ cm} \longrightarrow 3.3485\lambda \longrightarrow \theta = 720^o \times 0.3485 = 250.92^o$$



$$z_{in} = \frac{Z_{in}}{Z_o} = \frac{40 + j65}{75} = 0.5333 + j0.8666$$

We move toward the load from G (corresponding to Z_{in}) to P (corresponding to the load). At P,

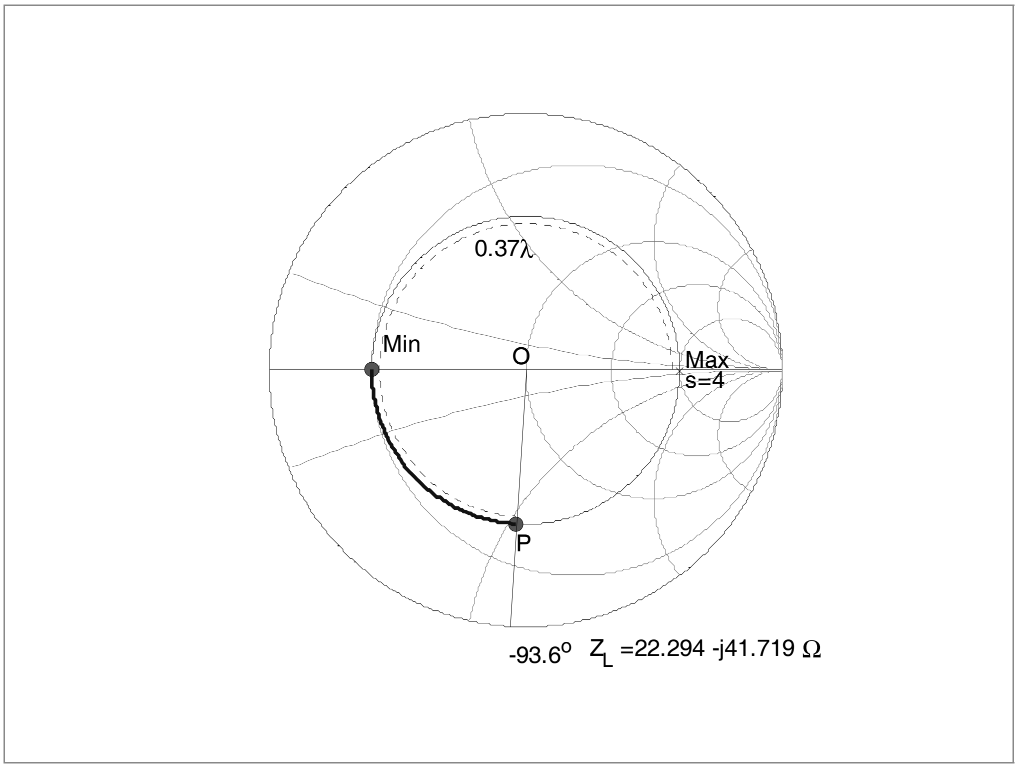
$z_L = 2.6137 - j1.4643$

$Z_L = Z_o z_L = 75(2.6137 - j1.4643) = \underline{\underline{196.03 - j109.82\ \Omega}}$

Prob. 11.37

(a) $0.12\lambda \quad \longrightarrow \quad 0.12 \times 720^{o} = 86.4^{o}$

We draw the s=4 circle and locate V_{min}. We move from that location 86.4^{o} toward the load.



At P, $z_L = 0.45 - j0.83$

$Z = 50(0.45 - j0.83) = \underline{\underline{22.3 - j41.72\ \Omega}}$

(b) the load is $\underline{\underline{\text{capacitive.}}}$

(c) V_{min} and V_{max} are $\lambda/4$ apart. Hence the first maximum occurs at

$0.12\lambda + 0.25\lambda = \underline{\underline{0.37\lambda}}$

Prob. 11.38

(a) $z_L = \dfrac{Z_L}{Z_o} = \dfrac{75 + j60}{50} = 1.5 + j1.2$

$$|\Gamma| = \frac{OP}{OQ} = \frac{3.8\text{cm}}{8\text{cm}} = 0.475, \qquad \theta_\Gamma = 42^o$$

$$\Gamma = \underline{\underline{0.475\angle 42^0}}$$

(Exact value = 0.4688 $\angle 41.76^o$)

(b) s=2.8
(Exact value = 2.765)

(c) $0.2\lambda \to 0.2x720^o = 144^o$

$$z_{in} = 0.55 - j0.65$$

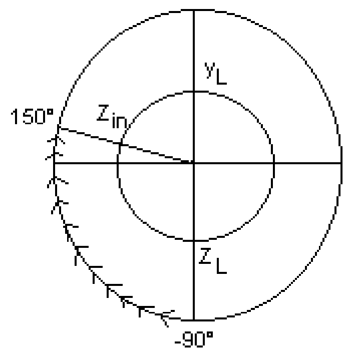
$$Z_{in} = Z_o z_{in} = 50(0.55 + j0.65) = \underline{\underline{27.5 + j32.5\ \Omega}}$$

(d) Since $\theta_\Gamma = 42^o$, V_{min} occurs at

$$\frac{42}{720}\lambda = \underline{\underline{0.05833\lambda}}$$

(e) same as in (d), i.e.. $\underline{\underline{0.05833\lambda}}$

Prob. 11.39



If $\lambda \to 720^o$, then $\dfrac{\lambda}{6} \to 120^o$

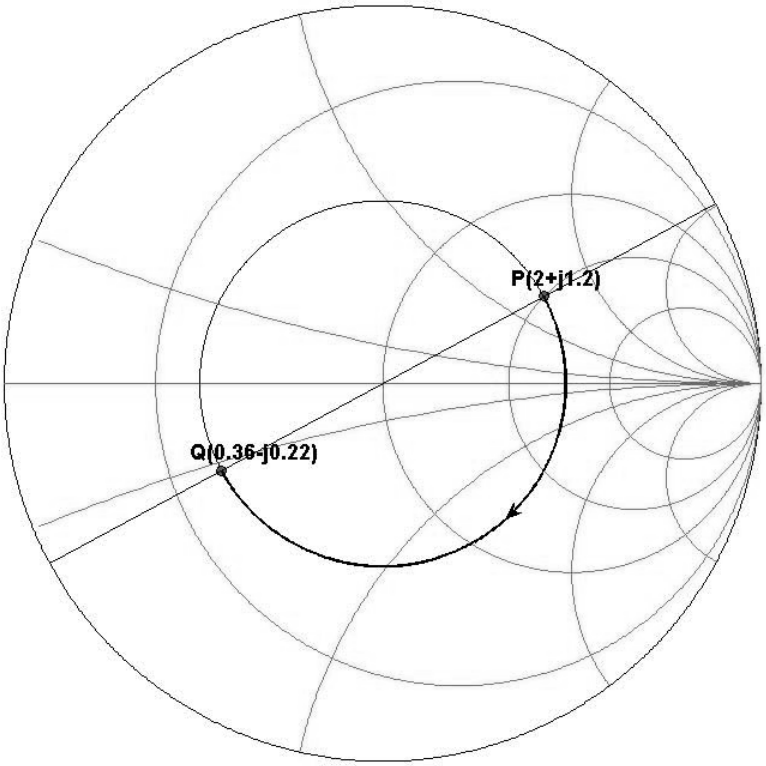
$z_{in} = \underline{\underline{0.35 + j0.24}}$

Prob. 11.40

$$z = \frac{Z}{Z_o} = \frac{100 + j60}{50} = 2 + j1.2$$

We locate z on the Smith chart. We move 180^o toward the generator to reach point Q. At Q, y = 0.36-j0.22

$$Y = yY_o = \frac{1}{50}(0.36 - j0.22) = \underline{\underline{7.4 - j4.4 \text{ mS}}}$$



Prob. 11.41

$$\lambda = \frac{u}{f} = \frac{0.5 \times 3 \times 10^8}{160 \times 10^6} = 0.9375 \text{ m}$$

$$z_L = \frac{Z_L}{Z_o} = \frac{50 + j30}{50} = 1 + j0.6$$

We locate z at P on the Smith chart. We draw a circle that passes through P. We locate point Q as the point where the circle crosses the Γ_r – axis. At Q,

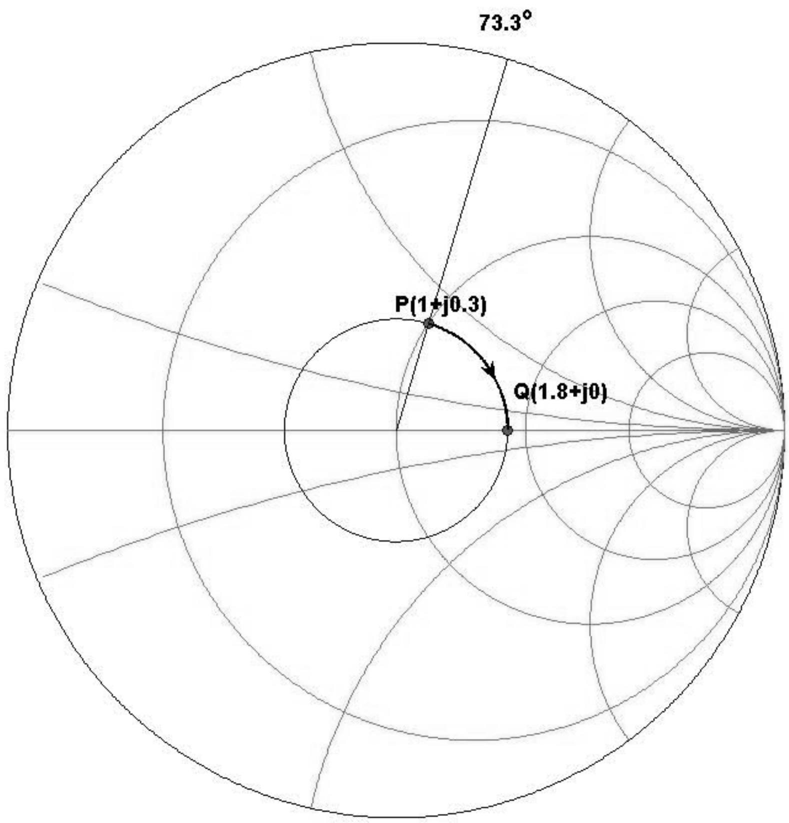
$$z_{in} = 1.8$$

$$Z_{in} = Z_o z_{in} = 50(1.8) = \underline{\underline{90\ \Omega}}$$

The angular distance between P and Q is $\theta_\Gamma = 73.3^o$.

If $\quad \lambda \to 720^o, \qquad 73.3^o \to \frac{\lambda}{720^o} 73.3^o$

$$\ell = \frac{73.3^o}{720^o} \times 0.9375 = \underline{\underline{0.0954 \text{ m}}}$$



Prob. 11.42

$$z_L = \frac{Z_L}{Z_o} = \frac{40 - j30}{50} = 0.8 - j0.6$$

Locate this load at point P on the Smith chart.

$$\frac{\lambda}{4} \to \frac{720^o}{4} = 180^o$$

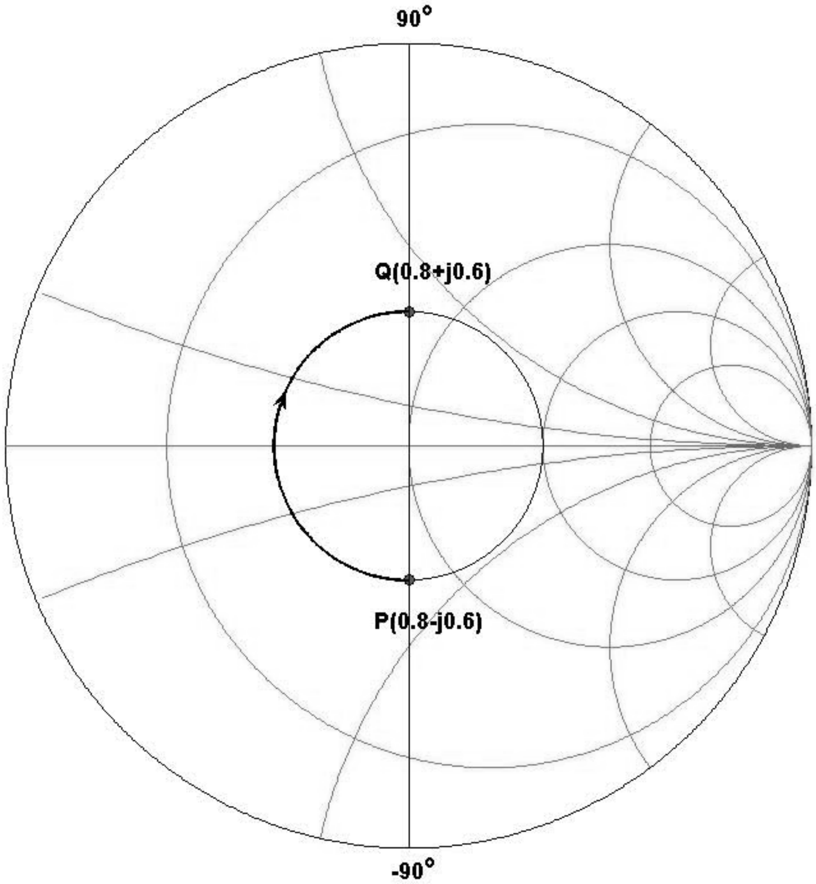
Draw a circle that passes through P and move 180^o toward the generator.

At point Q,

$$z_{in} = 0.8 + j0.6$$

$$Z_{in} = Z_o z_{in} = 50(0.8 + j0.6) = 40 + j30$$

$$Y_{in} = \frac{1}{Z_{in}} = 0.016 - j0.012 = \underline{\underline{16 - j12 \text{ mS}}}$$



Prob. 11.43

(a) $\Gamma = \dfrac{Z_L - Z_o}{Z_L + Z_o} = \dfrac{0.5 - j - 1}{0.5 - j + 1} = 0.0769 - j0.6154 = \underline{\underline{0.6202\angle -82.87^o}}$

(b)

$$\Gamma = \frac{z_L - 1}{z_L + 1} \quad \longrightarrow \quad z_L = \frac{1+\Gamma}{1-\Gamma} = \frac{1 + 0.4\angle 25^o}{1 - 0.4\angle 25^o}$$

$z_L = 1.931 + j0.7771$

$Z_L = \underline{\underline{(1.931 + j0.7771)Z_o}}$

Prob. 11.44

$$z_L = \frac{Z_L}{Z_o} = \frac{40 + j25}{50} = 0.8 + j0.5$$

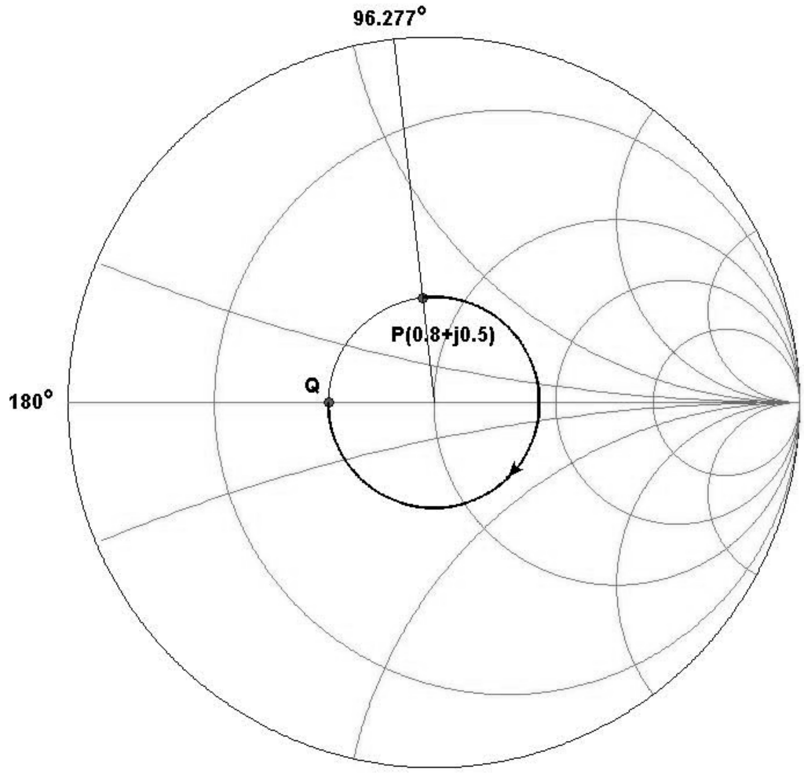
Locate this load at point P on the Smith chart. Draw a circle that passes through P. Locate point Q where the negative Γ_r – axis crosses the circle.

$\theta_P = 97^o$. The angular distance between P and Q is

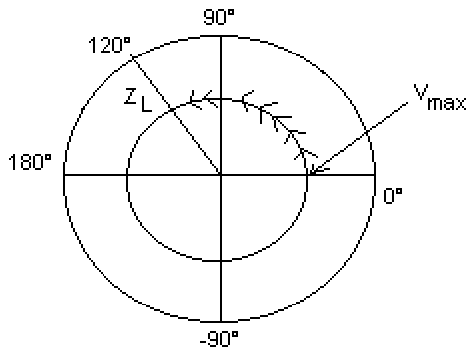
$\theta = 180 + \theta_P = 277^o$.

$720^o \rightarrow \lambda$

$277^o \rightarrow \ell = \dfrac{\lambda}{720^o} 277^o = \underline{\underline{0.3847\lambda}}$



Prob. 11.45



(a) $\frac{\lambda}{2} = 120\text{cm} \rightarrow \lambda = 2.4\text{m}$

$u = f\lambda \rightarrow f = \frac{u}{\lambda} = \frac{3\times10^8}{2.4} = \underline{\underline{125\text{MHz}}}$

(b) 40cm $= \frac{40\lambda}{240} = \frac{\lambda}{6} \rightarrow \frac{720°}{6} = 120°$

$Z_L = Z_o z_L = 150(0.48 + j0.48$

$= \underline{\underline{72 + j72}}\ \Omega$

(Exact value = 73.308+j70.324 Ω)

(c) $|\Gamma| = \frac{s-1}{s+1} = \frac{1.6}{3.9} = 0.444,$

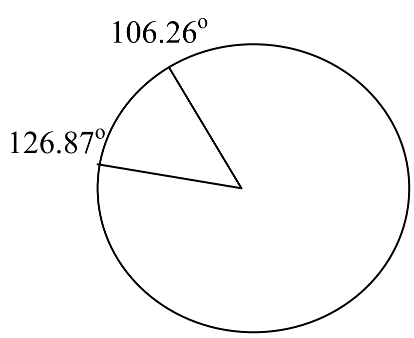
$\Gamma = \underline{\underline{0.444\angle 120°}}$

Prob. 11.46

(a)

$z_L = \frac{Z_L}{Z_o} = \frac{j60}{80} = j0.75, \qquad z_{in} = \frac{Z_{in}}{Z_o} = \frac{j40}{80} = j0.5$

The two loads fall on the r=0 circle, the outermost resistance circle. The shortest distance between them is



$$\frac{360^o - (126.87^o - 106.26^o)\lambda}{720^o} = \underline{\underline{0.4714\lambda}}$$

(b) $\underline{\underline{s = \infty}}$, $\Gamma_L = \underline{\underline{1\angle 106.26^o}}$

Prob. 11.47

(a) $Z_{in} = \dfrac{Z_{in}}{Z_o} = \dfrac{100 - j120}{80} = 1.25 - j1.5$

$\lambda = \dfrac{u}{f} = \dfrac{0.8\times 3\times 10^8}{12\times 10^6} = 20\text{m}$

$l_1 = 22\text{m} = \dfrac{22\lambda}{20} = 1.1\lambda \rightarrow 720° + 72°$

$l_2 = 28\text{m} = \dfrac{28\lambda}{20} = 1.4\lambda \rightarrow 720° + 72° + 216°$

To locate P(the load), we move 2 revolutions plus 72° toward the load. At P,

$|\Gamma_L| = \dfrac{OP}{OQ} = \dfrac{5.1\text{cm}}{9.2\text{cm}} = 0.5543$

$\theta_\Gamma = 72° - 47° = 25°$

$\Gamma_L = \underline{\underline{0.5543\angle 25°}}$

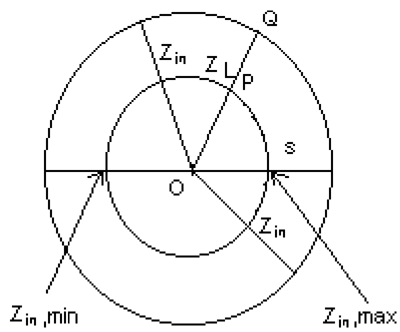
(Exact value = $0.5624\angle 25.15^o$)

$Z_{in}, \max = sZ_o = 3.7(80) = \underline{\underline{296\Omega}}$

(Exact value = 285.59 Ω)

$Z_{in}, \min = \dfrac{Z_o}{s} = \dfrac{80}{3.7} = \underline{\underline{21.622\Omega}}$

(Exact value =22.41 Ω)



(b) Also, at P, $Z_L = 2.3 + j1.55$

$$Z_L = 80(2.3 + j1.55) = \underline{\underline{184 + j124\Omega}}$$

(Exact value = 183.45+j128.25 Ω)

At S, $s = \underline{\underline{3.7}}$

To Locate $Z_{in}^{'}$, we move 216° from Z_{in} toward the geneator.

At $Z_{in}^{'}$,

$z_{in}^{'} = 0.48 + j0.76$

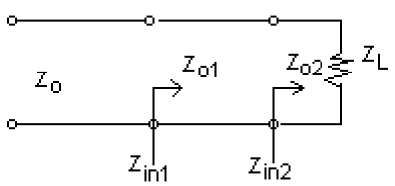
$Z_{in}^{'} = 80(0.48 + j0.76) = \underline{\underline{38.4 + j60.8\Omega}}$

(Exact=37.56+j61.304 Ω)

(c) Between Z_L and Z_{in}, we move 2 revolutions and 72°. During the movement, we pass through $Z_{in,\max}$ 3 times and $Z_{in,\min}$ twice. Thus there are:

$$\underline{\underline{3\ Z_{in,\max} \text{ and } 2\ Z_{in,\min}}}$$

Prob. 11.48



(a) From Eq. (11.43), $Z_{in2} = \dfrac{Z_{o2}^2}{Z_L}$

$Z_{in1} = \dfrac{Z_{o1}^2}{Z_{in2}} = Z_o$, i.e. $Z_{in2} = \dfrac{Z_{o1}^2}{Z_o} = \dfrac{Z_{o2}^2}{Z_L}$

$Z_{o1} = Z_{o2}\sqrt{\dfrac{Z_o}{Z_L}} = 30\sqrt{\dfrac{50}{75}} = \underline{\underline{24.5\Omega.}}$

(b) Also, $\dfrac{Z_o}{Z_{o1}} = \left(\dfrac{Z_{o2}}{Z_L}\right) \rightarrow Z_{o2} = \dfrac{Z_o Z_L}{Z_{o1}}$ (1)

Also, $\dfrac{Z_{o1}}{Z_{o2}} = \left(\dfrac{Z_{o2}}{Z_L}\right)^2 \rightarrow \left(Z_{o2}\right)^3 = Z_{o1} Z_L^2$ (2)

From (1) and (2), $(Z_{o2})^3 = Z_{o1}Z_L^2 = \dfrac{Z_o^3 Z_L^3}{Z_{o1}^3}$ (3)

or $Z_{o1} = \sqrt[4]{Z_o^3 Z_L} = \sqrt[4]{(50)^3(75)} = \underline{\underline{55.33\Omega}}$

From (3), $Z_{o2} = \sqrt[3]{Z_{o1}Z_L^2} = \sqrt[3]{(55.33)(75)^2} = \underline{\underline{67.74\Omega}}.$

Prob. 11.49

$$l_1 = \frac{\lambda}{4} \rightarrow Z_{in1} = \frac{Z_o^2}{Z_L} \quad or \quad y_{in1} = \frac{Z_L}{Z_o^2}$$

$$y_{in1} = \frac{200 + j150}{(100)^2} = \underline{\underline{20 + j15 \text{ mS}}}$$

$$l_2 = \frac{\lambda}{8} \rightarrow Z_{in2} =_{Z_L} \underline{\lim}_0 Z_o \left(\frac{Z_L + jZ_o \tan\dfrac{\pi}{4}}{Z_o + jZ_L \tan\dfrac{\pi}{4}} \right) = jZ_o$$

$$y_{in2} = \frac{1}{jZ_o} = \frac{1}{j100} = \underline{\underline{-j10 \text{ mS}}}$$

$$l_3 = \frac{7\lambda}{8} \rightarrow Z_{in3} = Z_o \frac{\left(Z_i + jZ_o \tan\dfrac{7\pi}{4} \right)}{\left(Z_o + jZ_i \tan\dfrac{7\pi}{4} \right)} = \frac{Z_o\left(Z_i - jZ_o\right)}{\left(Z_o - jZ_i\right)}$$

But

$$y_i = y_{in1} + y_{in2} = 20 + j5 \text{ mS}$$

$$z_i = \frac{1}{y_i} = \frac{1000}{20 + j5} = 47.06 - j11.76$$

$$y_{in3} = \frac{Z_o - jZ_{in}}{Z_o\left(Z_{in} - jZ_o\right)} = \frac{\text{100-j47.06-11.76}}{100\left(\text{47.06-j11.76-j100}\right)}$$

$$= \underline{\underline{6.408 + j5.189 \text{ mS}}}$$

If the shorted section were open,

$$y_{in1} = \underline{\underline{20 + j15 \text{ mS}}}$$

$$y_{in2} = \frac{1}{Z_{in2}} = \frac{j\tan \pi/4}{Z_o} = \frac{j}{100} = \underline{\underline{j10 \text{ mS}}}$$

$$l_3 = \frac{7\lambda}{8} \rightarrow Z_{in3} = Z_o \frac{\left(Z_i + jZ_o \tan\frac{7\pi}{4}\right)}{\left(Z_o + jZ_i \tan\frac{7\pi}{4}\right)} = \frac{Z_o\left(Z_i - jZ_o\right)}{\left(Z_o - jZ_i\right)}$$

$$y_i = y_{in1} + y_{in2} = 20 + j15 + j10 = 20 + j25 \text{ mS}$$

$$Z_i = \frac{1}{y_i} = \frac{1000}{20 + j25} = 19.51 - j24.39\Omega$$

$$y_{in3} = \frac{Z_o - jZ_i}{Z_o\left(Z_i - jZ_o\right)} = \frac{75.61\text{-j}19.51}{100\left(19.51 - j124.39\right)}$$

$$= \underline{\underline{2.461 + j5.691 \text{ mS}}}$$

Prob. 11.50

From the previous problem, $Z_{in} = 148\Omega$

$$I_{in} = \frac{V_g}{Z_g + Z_{in}} = \frac{120}{80 + 148} = 0.5263A$$

$$P_{ave} = \frac{1}{2}\left|I_{in}\right|^2 R_{in} = \frac{1}{2}(0.5263)^2(148) = 20.5W$$

Since the lines are lossless, the average power delivered to either antenna is $\underline{\underline{10.25W}}$

Prob. 11.51

(a) $\beta l = \frac{2\pi}{4}.\frac{\lambda}{4} = \frac{\pi}{2}, \qquad \tan\beta l = \infty$

$$Z_{in} = Z_o\left(\frac{Z_L + jZ_o \tan\beta l}{Z_o + jZ_L \tan\beta l}\right) = Z_o \frac{\left(\frac{Z_L}{\tan\beta l} + jZ_o\right)}{\left(\frac{Z_o}{\tan\beta l} + jZ_L\right)}$$

As $\tan\beta l \rightarrow \infty$,

$$Z_{in} = \frac{Z_o^2}{Z_L} = \frac{(50)^2}{200} = \underline{\underline{12.5\Omega}}$$

(b) If $Z_L = 0$,

$$Z_{in} = \frac{Z_o^2}{0} = \underline{\underline{\infty}} \qquad \text{(open)}$$

(c) $Z_L = 25 // \infty = \dfrac{25\times\infty}{25+\infty} = \dfrac{25}{1+\dfrac{25}{\infty}} = 25\Omega$

$$Z_{in} = \frac{(50)^2}{12.5} = \underline{\underline{200\Omega}}$$

Prob. 11.52

$$\frac{\lambda}{4} \to 180°, \quad z_L = \frac{74}{50} = 1.48, \quad \frac{1}{z_L} = 0.6756$$

This acts as the load to the left line. But there are two such loads in parallel due to the two lines on the right. Thus

$$Z_L^{'} = 50\frac{\left(\dfrac{1}{Z_L}\right)}{2} = 25(0.6756) = 16.892$$

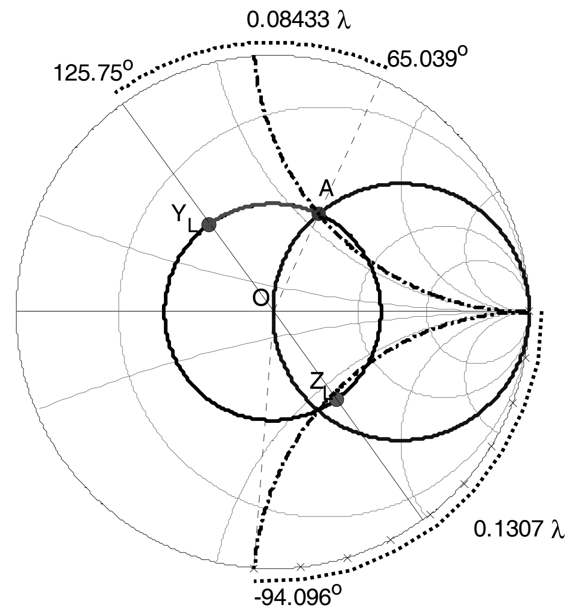
$$z_L^{'} = \frac{16.892}{50} = 0.3378, \quad z_{in} = \frac{1}{z_L^{'}} = 2.96$$

$$Z_{in} = 50(2.96) = \underline{\underline{148\Omega.}}$$

Prob. 11.53

$$z_L = \frac{Z_L}{Z_o} = \frac{60 - j50}{50} \quad = 1.2 - j1$$

$$y_L = \frac{1}{z_L}$$



At A, $y = 1 + j0.92$, $y_s = -j0.92$

$$Y_s = Y_o y_s = \frac{-j0.92}{50} = \underline{\underline{-j18.4}} \text{ mS}$$

Stub length = $\underline{\underline{0.1307\lambda}}$

Stub position = $\underline{\underline{0.0843\lambda}}$

Prob. 11.54

$d_A = 0.12\lambda \quad \rightarrow \quad 0.12 \times 720^o = 86.4^o$

$l_A = 0.3\lambda \quad \rightarrow \quad 0.3 \times 720^o = 216^o$

(a) From the Smith Chart below,

$z_L = 0.57 + j0.69$

$Z_L = 60(0.57 + j0.69)$

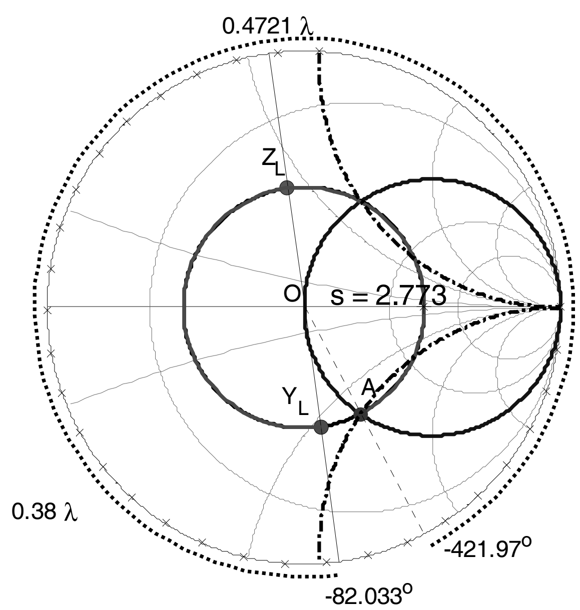
$= \underline{\underline{34.2 + j41.4\Omega}}$

(b) $d_B = \dfrac{360^o - 86.4^o}{720^o}\lambda \;=\; \underline{\underline{0.38\lambda}}$

$l_B = \dfrac{\lambda}{2} - \dfrac{(-62.4^o - -82^o)}{720^o}\lambda \;=\; \underline{\underline{0.473\lambda}}$

(c) $\underline{\underline{s \;= 2.65}}$

(Exact value = 2.7734)



Prob. 11.55

$$z_L = \frac{Z_L}{Z_o} = \frac{120 + j220}{50} = 2.4 + j4.4$$

We follow Example 11.7. At A, y_s=-j3 and at B, y_s=+j3. The required stub admittance is

$$Y_s = Y_o y_s = \frac{\pm j3}{50} = \pm j0.06 \text{ S}$$

The distance between the load and the stub is determined as follows. For A,value = 0.2308λ)

For B,

$$l_B = \frac{180 + 10 + 17}{720}\lambda = \underline{\underline{0.2875\lambda}}$$

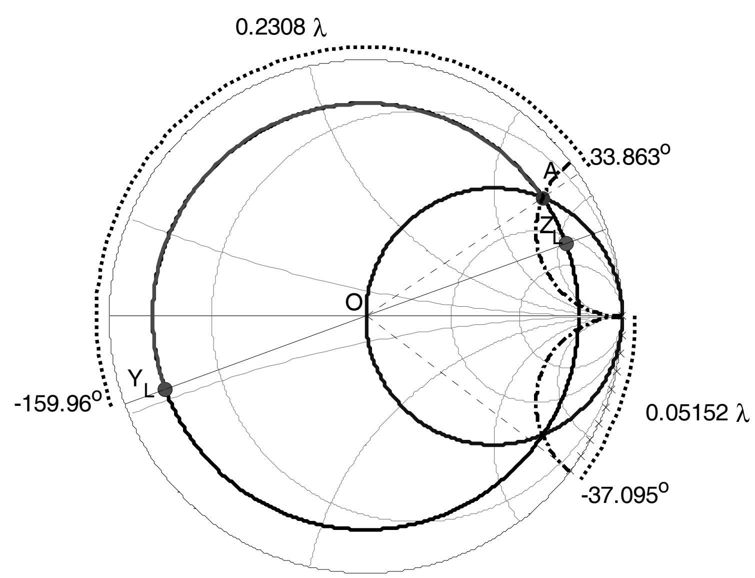
The length of the stub line is determined as follows.

$$d_A = \frac{19}{720}\lambda = \underline{\underline{0.0264\lambda}}$$

(Exact value = 0.0515 λ)

$$d_B = \frac{360 - 19}{720}\lambda = \underline{\underline{0.4736\lambda}}$$

(Exact value = 0.4485λ)



Prob. 11.56

$$s = \frac{V_{max}}{V_{min}} = \frac{4V}{1V} = \underline{\underline{4}}$$

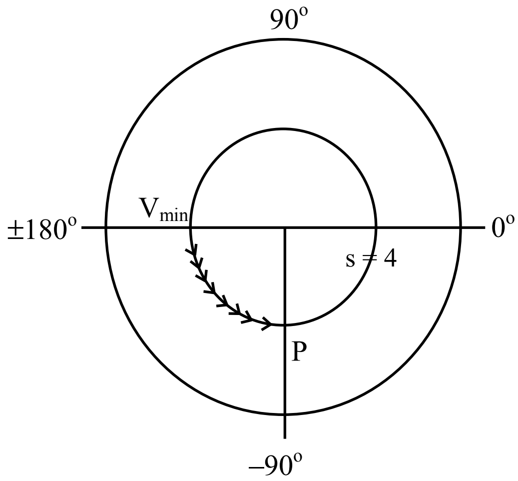
$$|\Gamma| = \frac{s-1}{s+1} = \frac{3}{5} = 0.6$$

$$\frac{\lambda}{2} = 25 \text{ cm} - 5 \text{ cm} = 20 \text{ cm}$$

$$\rightarrow \quad \lambda = 40 \text{ cm}$$

The load is l=5cm from V_{min}, i.e.

$$l = \frac{5\lambda}{40} = \frac{\lambda}{8} \quad \rightarrow \quad 90^o$$



On the s = 4 circle, move 90° from V_{min} towards the load and obtain $Z_L = 0.46 - j0.88$ at P.

$$Z_L = Z_o z_L = 60(0.46 - j0.88) = \underline{\underline{27.6 - j52.8\ \Omega}}$$

(Exact value = 28.2353 –j52.9412 Ω)

$$\theta_\Gamma = 270^o \text{ or } -90^o$$

$$\Gamma = \underline{\underline{0.6\angle -90^o}}$$

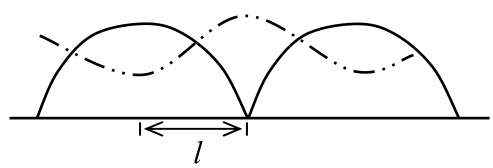
Prob. 11.57

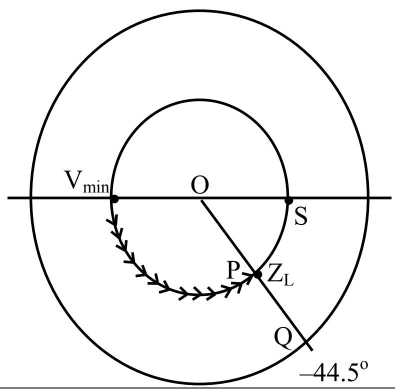
$$s = \frac{V_{max}}{V_{min}} = \frac{0.95}{0.45} = \underline{\underline{2.11}}$$

$$\frac{\lambda}{2} = 22.5 - 14 = 8.5 \quad \rightarrow \quad \lambda = 17 \text{ cm}$$

$$f = \frac{c}{\lambda} = \frac{3\times10^8}{0.17} = \underline{\underline{1.764\ \text{GHz}}}$$

$$l = 3.2 \text{ cm} = \frac{3.2}{17}\lambda \quad \rightarrow \quad 135.5^o$$





At P, $z_L = 1.4 - j0.8$

$$Z_L = 50(1.4 - j0.8) = \underline{\underline{70 - j40\Omega}}$$

(Exact value = 70.606-j40.496 Ω)

$$|\Gamma| = \frac{s-1}{s+1} = \frac{1.11}{3.11} = 0.357, \quad \theta_\Gamma = -44.5^o$$

$$\Gamma = \underline{\underline{0.357\angle -44.5^o}}$$

(Exact value = 0.3571∠-44.471°)

Prob. 11.58

$$\Gamma_s = \frac{R_g - R_o}{R_g + R_o} = \frac{0-50}{0+50} = \underline{\underline{-1}}$$

$$\Gamma_L = \frac{R_L - R_o}{R_L + R_o} = \frac{80-50}{80+50} = \underline{\underline{0.231}}$$

Prob. 11.59

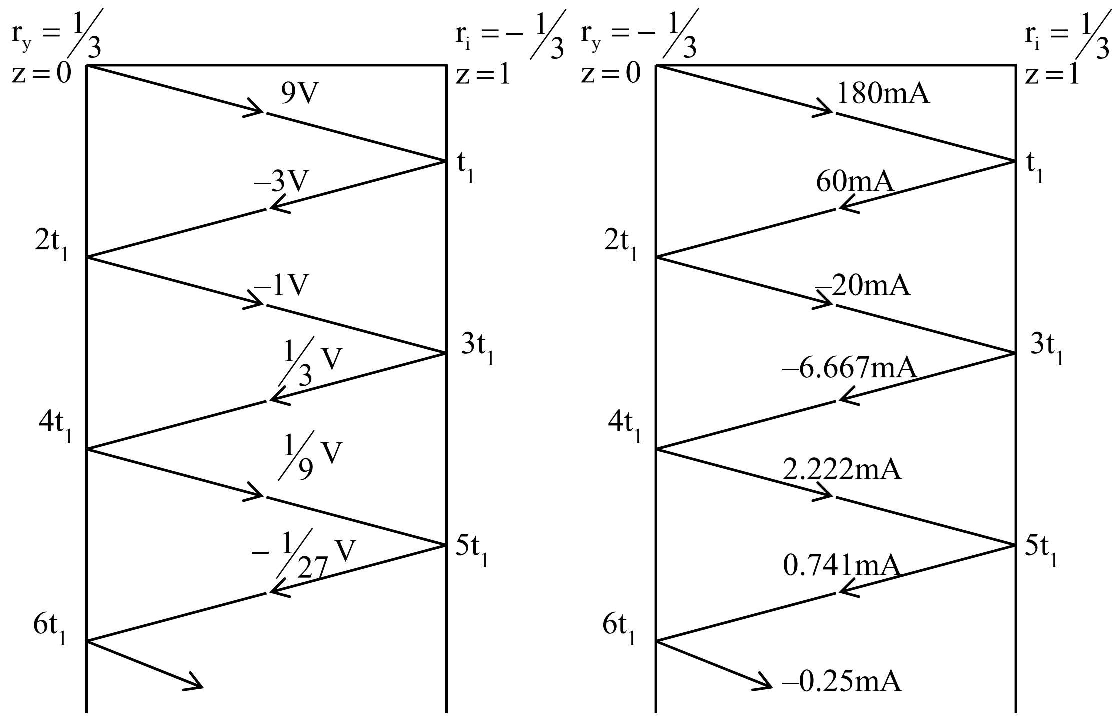
$$\Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{0.5Z_o - Z_o}{1.5Z_o} = -\frac{1}{3}$$

$$\Gamma_g = \frac{Z_g - Z_o}{Z_g + Z_o} = \frac{Z_o}{3Z_o} = \frac{1}{3}$$

$$t_1 = \frac{l}{u} = 2\mu s, \quad V_o = \frac{Z_o}{3Z_o}(27) = 9 \text{ V}, \quad I_o = \frac{V_o}{Z_o} = 180 \text{ mA}$$

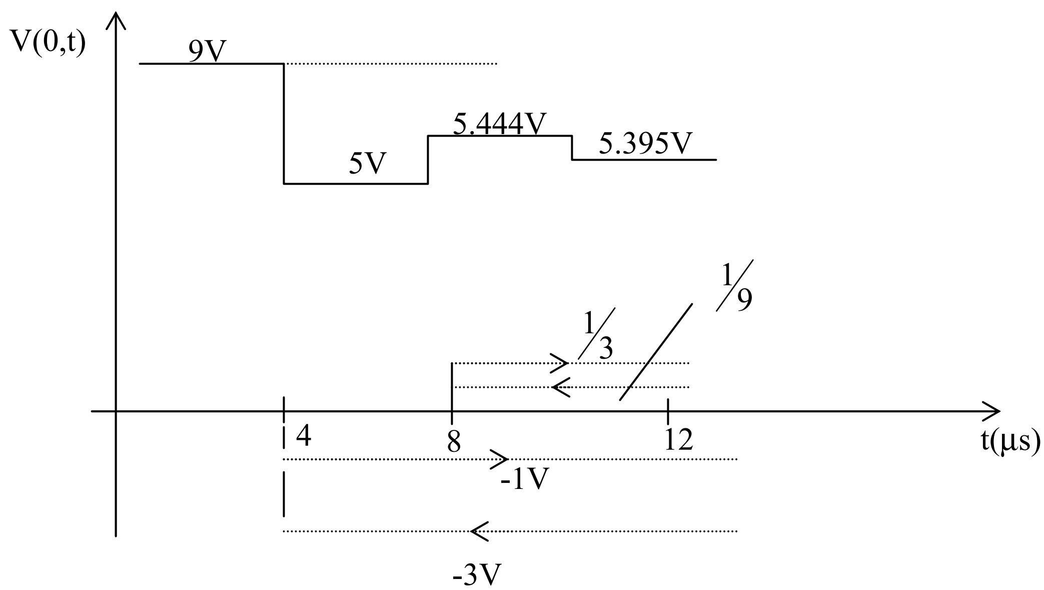
$$V_\infty = \frac{Z_L}{Z_g + Z_L}V_g = \frac{0.5}{2.5}(27) = 5.4 \text{ V}, \quad I_\infty = \frac{V_\infty}{Z_L} = 216 \text{ mA}$$

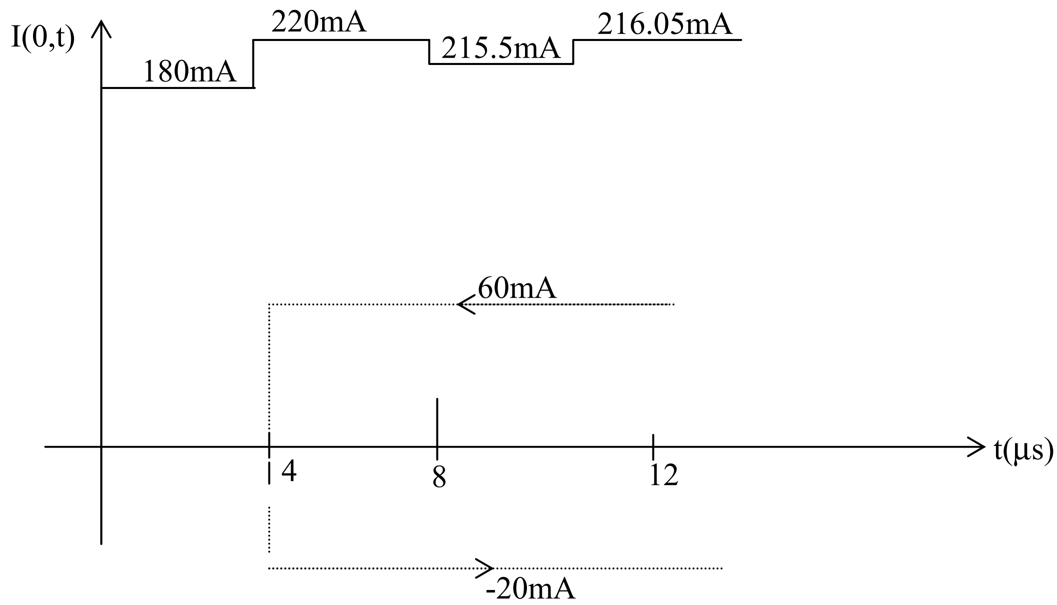
The voltage and current bounce diagrams are shown below



(Voltage bounce diagram) (Current bounce diagram)

From the bounce diagrams, we obtain V(0,t) and I(0,t) as shown below:





Prob. 11.60

Using Thevenin equivalent at $z = 0$ gives

$R_g = R_s = 4Z_o = 200\Omega$

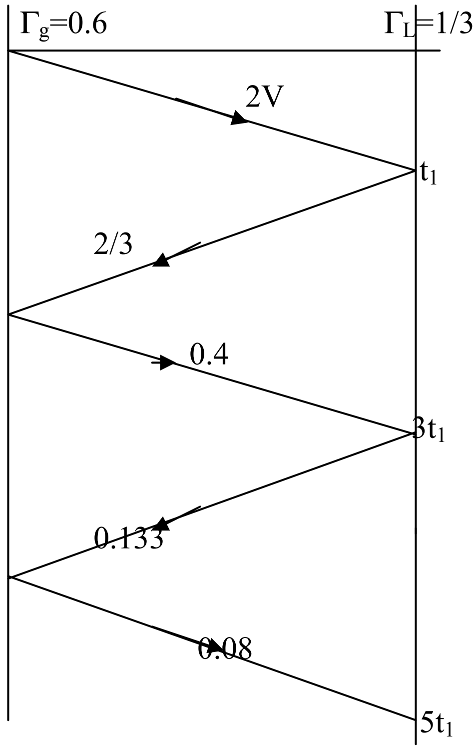
$V_g = I_s R_s = 10 \times 200 \times 10^{-3} = 2V$

$$\Gamma_g = \frac{Z_g - Z_o}{Z_g + Z_o} = \frac{4Z_o - Z_o}{4Z_o + Z_o} = \frac{3}{5}$$

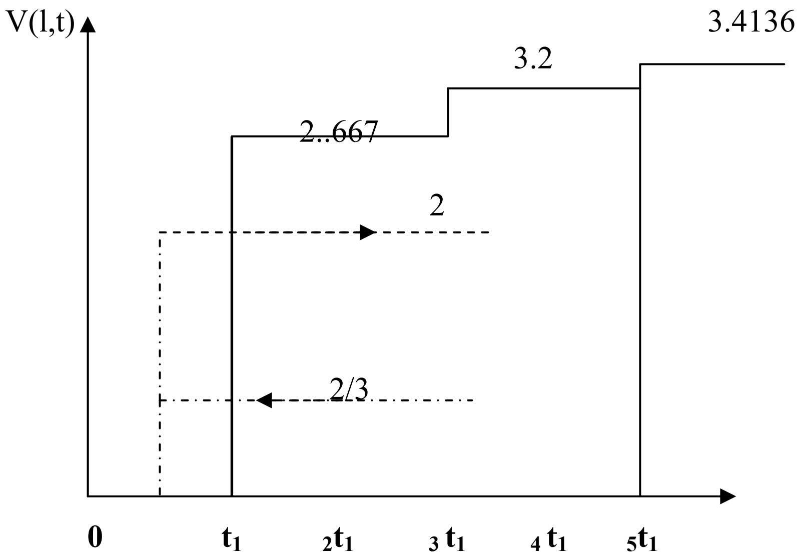
$$\Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{2Z_o - Z_o}{2Z_o + Z_o} = \frac{1}{3}$$

$$t_1 = \frac{\ell}{u} = \frac{10}{2 \times 10^8} = 50 \text{ ns}$$

The bounce diagram is shown below.



The load voltage is sketched below.



$$I(\ell,t)=\frac{V(\ell,t)}{Z_L}=\frac{V(\ell,t)}{100}$$

To get I(I,t), we just scale down V(l,t) by 100.

Prob. 11.61

$$\Gamma_g = \frac{Z_g - Z_o}{Z_g + Z_o} = \frac{32-75}{32+75} = 0.4019$$

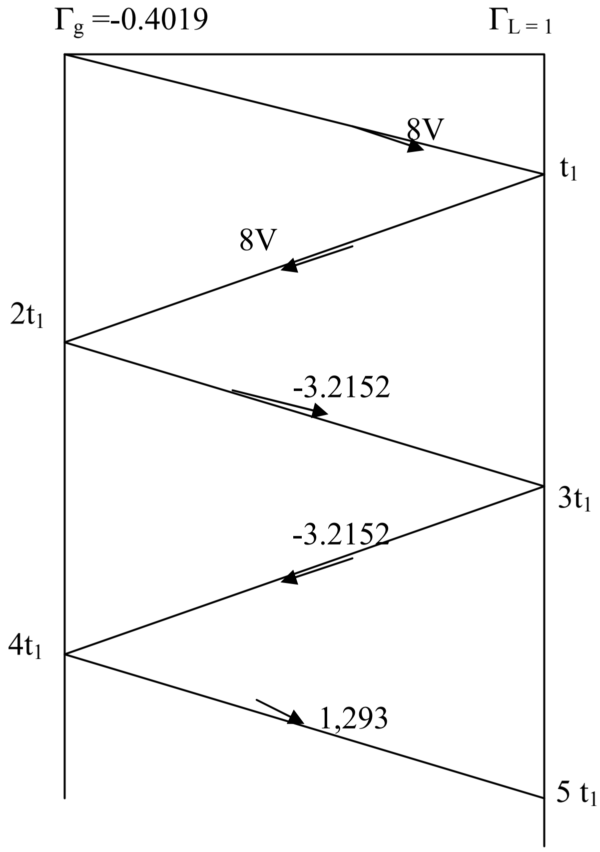
$$\Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{2\times10^6 - 75}{2\times10^6 + 75} \simeq 1$$

$$t_1 = \frac{\ell}{u} = \frac{50\times10^{-2}}{2\times10^8} = 2.5ns$$

The bounce diagram is shown below.

At t =20 ns = $4t_1$,

$V = 8+8-3.2152-3.2152 = \underline{\underline{9.57\text{ V}}}$

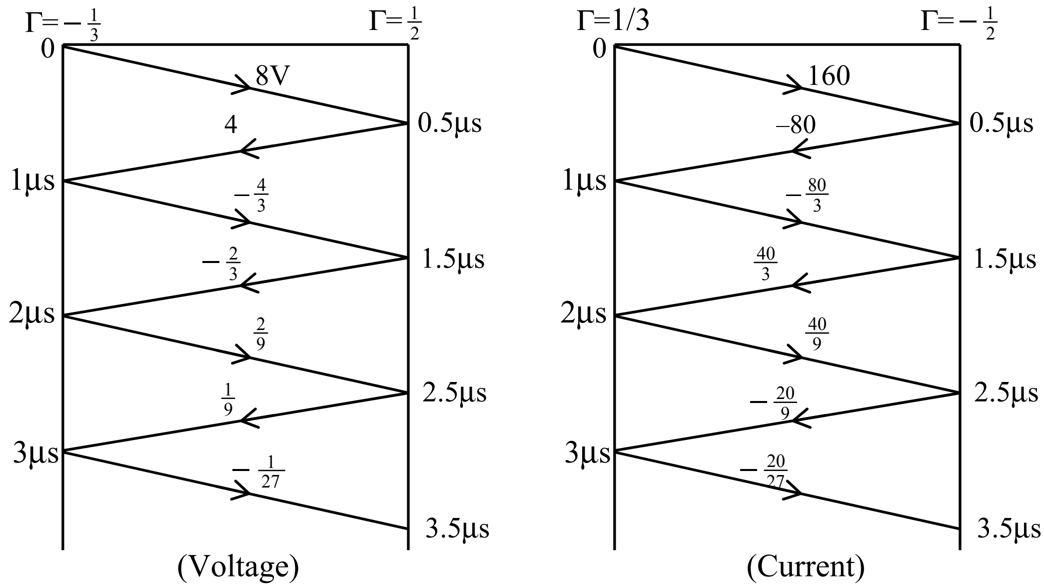


Prob. 11.62

$$(a) \qquad t_1 = \frac{l}{u} = \frac{150}{3\times10^8} = 0.5\mu s,$$

$$\Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{150-50}{150+150} = \frac{1}{2}, \qquad \Gamma_g = \frac{Z_g - Z_o}{Z_g + Z_o} = \frac{25-50}{75} = -\frac{1}{3},$$

$$V_o = \frac{Z_o V_g}{Z_o + Z_g} = \frac{50(12)}{75} = 8V, \qquad I_o = \frac{V_g}{Z_g + Z_o} = \frac{12}{75} = 160\text{ mA}$$

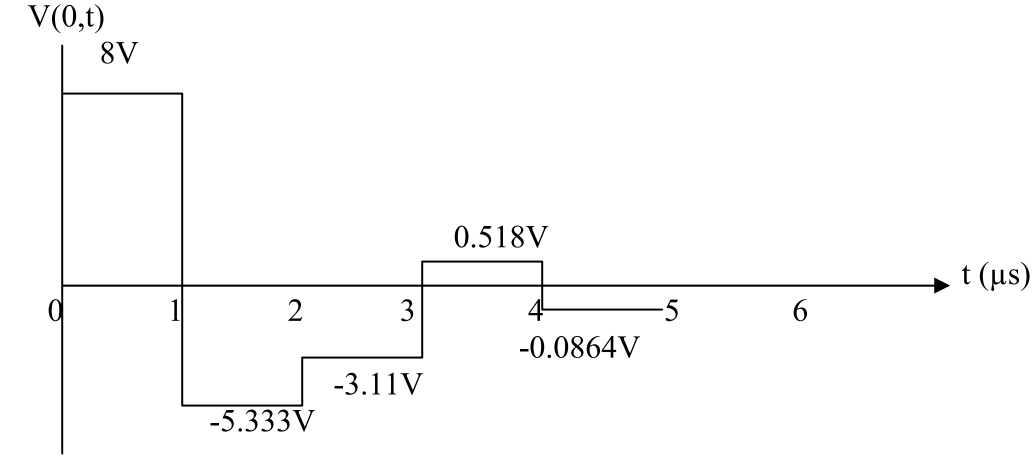


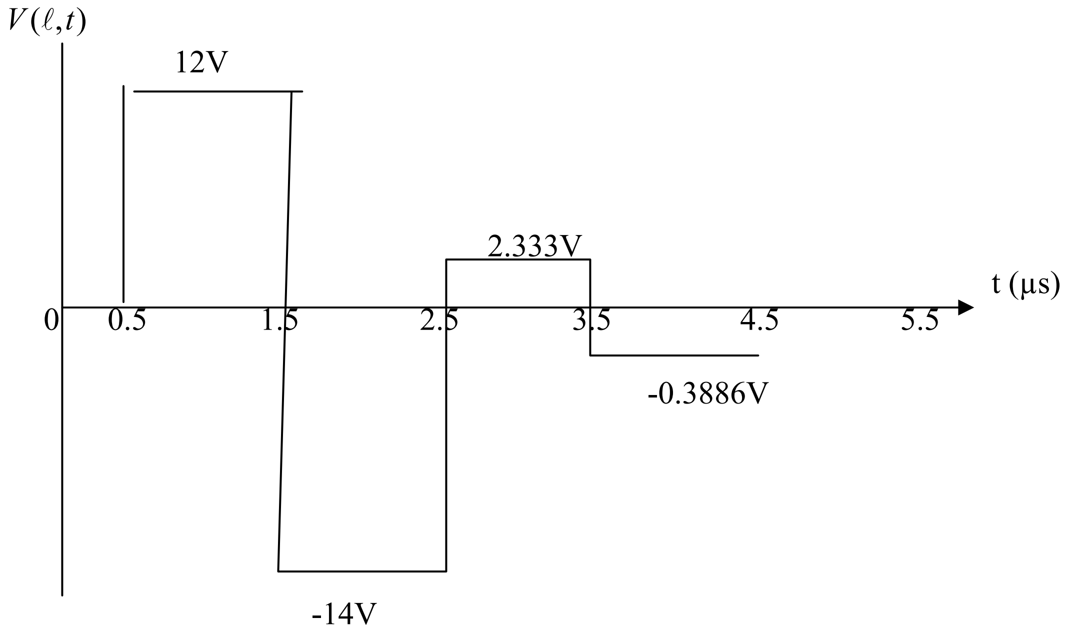
The bounce diagrams are for the leading pulse. The bounce diagrams for the second pulse is delayed by 1μs and negated because of -12V.

(b) For each time interval, we add the contributions of the two pulses together.
For $0 < t < 1\mu s$, V(0,t) = 8V
For $1 < t < 2\mu s$, V(0,t) = -8 + 4 – 4/3 = -5.331V
For $2 < t < 3\mu s$, V(0,t) = -(4-4/3)-2/3 + 2/9 = -2.667 -0.444 = -3.11V
For $3 < t < 4\mu s$, V(0,t) = 0.444 + 1/9 -1/27 = 0.444+0.0741 = 0.518V
For $4 < t < 5\mu s$, V(0,t)= -0.0741 – 0.0124 = -0.0864V

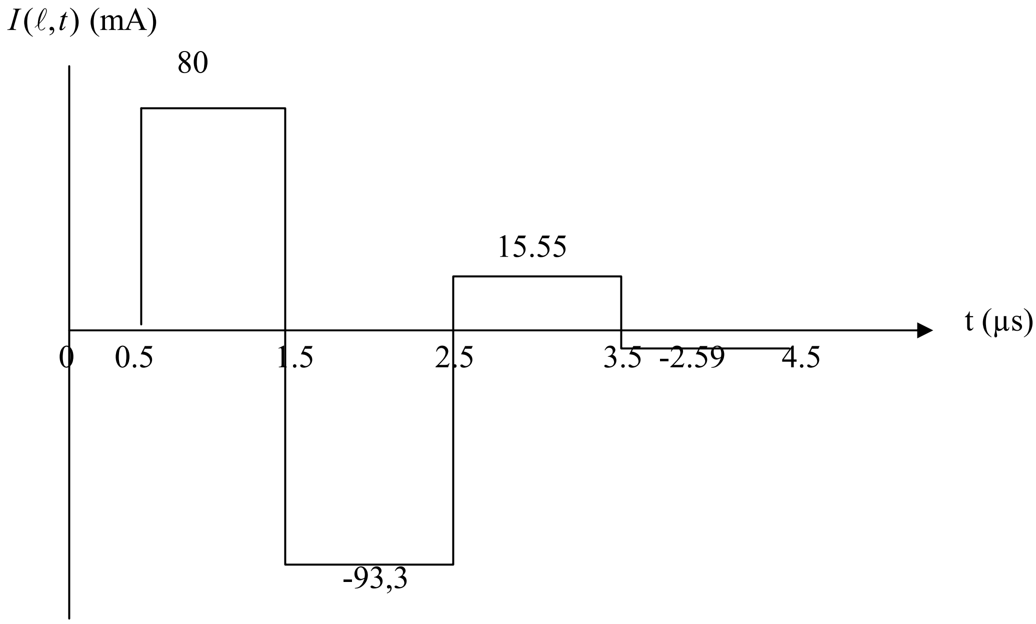
We do the same thing at the load end.
For $0 < t < 0.5\,\mu s$, $V(\ell,t)=0$
For $0.5 < t < 1.5\mu s$, V(ℓ,t)=8+4 = 12
For $1.5 < t < 2.5\mu s$, V(ℓ,t)=-12 + (-4/3 -2/3)=-12-2=-14
For $2.5 < t < 3.5\mu s$, V(ℓ,t)=2+2/9 +1/9=2.333
For $3.5 < t < 4.5\mu s$, V(ℓ,t)=-0.333-1/27-1/54=-0.3886V

The results are shown below.





Since $I(\ell,t) = \dfrac{V(\ell,t)}{Z_L} = \dfrac{V(\ell,t)}{150}$, we scale $V(\ell,t)$ by a factor of 1/150 as shown below.



Prob. 11.63

$$V_o = 8V = \frac{Z_o}{Z_o + Z_g}V_g = \frac{50}{50+60}V_g \longrightarrow V_g = \frac{8\times 110}{50} = \underline{\underline{17.6 \text{ V}}}$$

$$2t_1 = 4\mu s \longrightarrow t_1 = 2\mu s = \frac{\ell}{u}$$

$$\ell = ut_1 = 3\times 10^8 \times 2\times 10^{-6} = \underline{\underline{600 \text{ m}}}$$

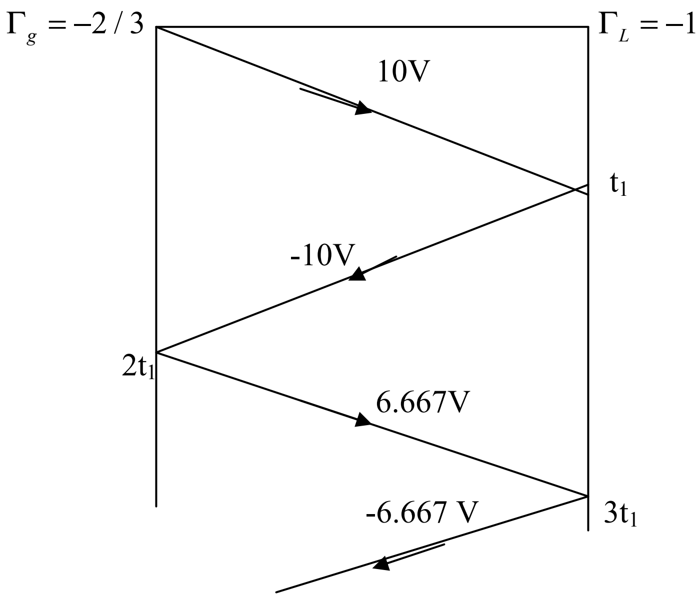
Prob. 11.64

$$t_1 = \frac{20}{2x10^8} = 10^{-7} = 0.1\mu s, \qquad V_o = \frac{Z_o}{Z_o + Z_g}V_g = \frac{50}{60}(12) = 10V$$

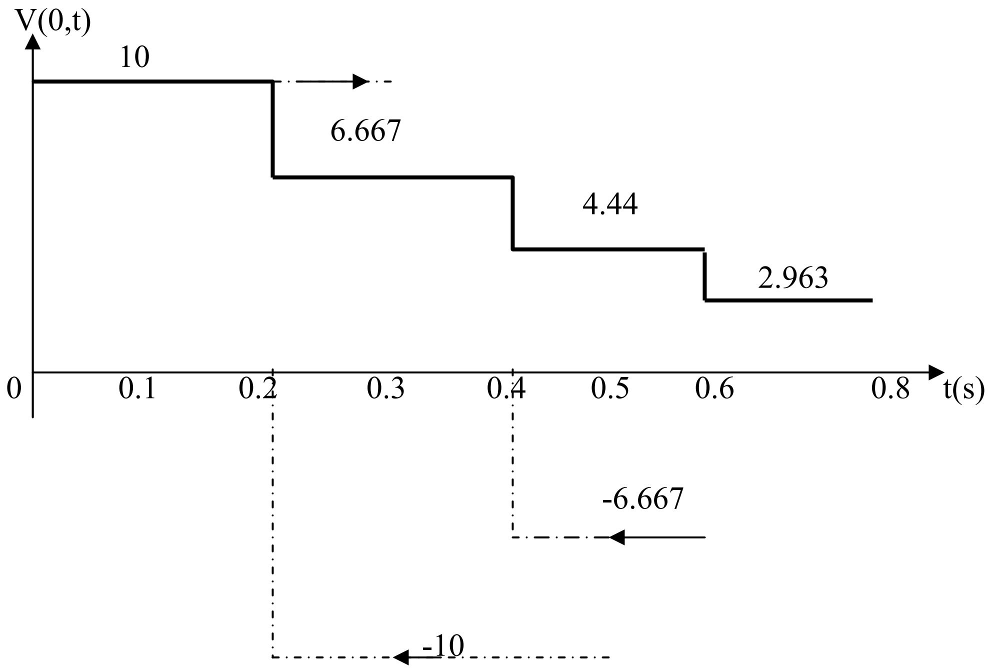
$$\Gamma_g = \frac{Z_g - Z_o}{Z_g + Z_o} = \frac{10-50}{10+50} = -2/3$$

$$\Gamma_L = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{0-50}{0+50} = -1$$

The voltage bounce diagram is shown below



From the bounce diagram, we obtain V(0,t) as shown. V(l,t) =0 due to the short circuit.



Prob. 11.65
The initial pulse on the line is

$$V_o = \frac{Z_o}{Z_o + Z_g} V_g \qquad (1)$$

The reflection coefficient is

$$\Gamma = \frac{Z_g - Z_o}{Z_g + Z_o} \qquad (2)$$

The first reflected wave has amplitude ΓV_o, while the second reflected wave has amplitude $\Gamma^2 V_o$, etc. For a very long time, the total voltage is

$$V_T = V_o + \Gamma V_o + \Gamma^2 V_o + \ldots$$

$$= V_o(1 + \Gamma + \Gamma^2 + \Gamma^3 + \ldots) = V_o\left(\frac{1}{1-\Gamma}\right) \qquad (3)$$

Substituting (1) and (2) into (3) gives

$$V_T = \frac{Z_o}{Z_o + Z_g} V_g \left(\frac{1}{1 - \dfrac{Z_g - Z_o}{Z_g + Z_o}} \right) = Z_o V_g \frac{1}{Z_g + Z_o - Z_g + Z_o} = \underline{\underline{\frac{V_g}{2}}}$$

Prob. 11.66
For w = 0.4 mm, $\frac{w}{h} = \frac{0.4\text{ mm}}{2\text{ m}} = 0.2 \rightarrow$ narrow strip

For $\frac{w}{h} = 0.2$, $\varepsilon_{\text{eff}} = 5.851$, $Z_o = 91.53\Omega$

For $\frac{w}{h} = 0.4$, $\varepsilon_{\text{eff}} = 6.072$, $Z_o = 73.24\Omega$

Hence,

$$\underline{\underline{73.24\Omega < Z_o < 91.53\Omega}}$$

Prob. 11.67

(a) Let $x = w/h$. If $x < 1$,

$$50 = \frac{60}{\sqrt{4.6}}\ln\left(\frac{8}{x} + x\right)$$

$$5\sqrt{4.6} - 6\ln\left(\frac{8}{x} + x\right) = 0$$

we solve for x (e.g using Maple) and get x = 2.027 or 3.945

which contradicts our assumptiom that $x < 1$. If $x > 1$,

$$50 = \frac{120\pi}{\sqrt{4.6}\left[x + 1.393 + 0.667\ln(x + 1.444)\right]}$$

We solve this iteratively and obtain:

$x = 1.8628, w = xh = 14.9024$ mm

For this w and h,

$$\varepsilon_{eff} = 3.4598$$

(b) $$\beta = \frac{\omega\sqrt{\varepsilon_{\text{eff}}}}{c}$$

$$\beta\ell = 45^0 = \frac{\pi}{4} = \frac{\omega\ell\sqrt{\varepsilon_{\text{eff}}}}{c}$$

$$\ell = \frac{\pi c}{4\sqrt{\varepsilon_{\text{eff}}}\,2\pi f} = \frac{3 \times 10^8}{8 \times \sqrt{3.4598} \times 8 \times 10^9}$$

$$\underline{\underline{\ell = 0.00252 \text{ m}}}$$

Prob. 11.68

$w = 1.5cm, \quad \text{h} = 1\text{cm}, \quad \frac{w}{h} = 1.5$

(a) $\varepsilon_{\text{eff}} = \left(\frac{6 \; +1}{2}\right) + \frac{\varepsilon_{\text{r}} - 1}{2\sqrt{1 \; +12\text{h/w}}} = 1.6 \; + \frac{0.6}{\sqrt{1 \; + \; 12/1.5}} = \underline{\underline{1.8}}$

$$\text{Z}_0 = \frac{377}{\sqrt{1.8}\;(1.5 \; +1.393 \; +0.667 \text{ In } (2.944))} = \frac{281}{3.613} = \underline{\underline{77.77\Omega}}$$

(b) $\alpha_c = 8.686 \frac{\text{R}_{\text{s}}}{\text{wZ}_{\text{o}}}$

$$\text{R}_{\text{s}} = \frac{1}{\sigma_c \; \sigma} = \sqrt{\frac{\mu\pi\text{f}}{\sigma_{\text{c}}}} = \sqrt{\frac{19 \times 2.5 \times 10^9 \times 4\pi \times 10^{-3}}{1.1 \times 10^7}}$$

$$= 2.995 \times 10^{-2}$$

$$\alpha_{\text{c}} = \frac{8.686 \times 2.995 \times 10^{-2}}{1.5 \times 10^{-2} \times 77.77} = \underline{\underline{0.223\text{dB/m}}}$$

$$\text{u} = \frac{\text{c}}{\sqrt{\varepsilon_{\text{eff}}}} \rightarrow \lambda = \frac{\text{u}}{\text{f}} = \frac{\text{c}}{\text{f}\sqrt{\varepsilon_{\text{eff}}}} = \frac{3 \times 10^8}{2.5 \times 10^9 \; \sqrt{1.8}} = 8.944 \times 10^{-2}$$

$$\alpha_{\text{d}} = 27.3 \times \frac{0.8}{1.2} \frac{(2.2)}{\sqrt{1.8}} \; \frac{2 \times 10^{-2}}{8.944 \times 10^{-2}} = \frac{96.096}{14.3996}$$

$$\alpha_{\text{d}} = \underline{\underline{6.6735 \text{ dB/m}}}$$

(c) $\alpha = \alpha_{\text{c}} + \alpha_{\text{d}} = 6.8965 \text{ dB/m}$

$$\alpha\ell = 20\text{dB} \rightarrow \ell = \frac{20}{\alpha} = \frac{20}{6.8965} = \underline{\underline{2.9 \text{ m}}}$$

Prob. 11.69

$$w' = 0.5 + \frac{0.1}{3.2}\ln\left(\frac{5\times1.2}{0.1}\right) = 0.5 + 0.1279 = 0.6279$$

$$Z_o = \frac{377}{2\pi\sqrt{2}}\ln\left\{1 + \frac{4\times1.2}{\pi\times0.6177}\left[\frac{8\times1.2}{4\times0.6279} + 3.354\right]\right\}$$

$$= 42.43\ln\{1 + 2.49(3.822 + 3.354\} = 42.43\ln(20.255)$$

$$= \underline{\underline{127.64\ \Omega}}$$

Prob. 11.70

Suppose we guess that w/h < 2

$$A = \frac{75}{60}\sqrt{\frac{3.3}{2}} + \frac{1.3}{3.3}\left(0.23 + \frac{0.11}{2.3}\right) = 1.117$$

$$\frac{w}{h} = \frac{8e^A}{e^{2A} - 2} = \frac{24.44}{7.337} = 3.331 \rightarrow w = 3.331h = \underline{\underline{4mm}}$$

If we guess that w/h > 2,

$$B = \frac{60\pi^2}{Z_o\sqrt{\varepsilon_r}} = \frac{60\pi^2}{75\sqrt{2.3}} = 5.206$$

$$\frac{w}{h} = \frac{2}{\pi}\left[4.266 - \ln 9.412 + \frac{1.3}{4.6}\left(\ln 4.206 + 0.39 - \frac{0.61}{2.3}\right)\right]$$

$$= 1.665 < 2$$

$$\text{Thus } \frac{w}{h} = 3.331 > 2$$

$$\varepsilon_{\text{eff}} = \frac{3.3}{2} + \frac{1.3}{2\sqrt{1 + \frac{12}{3.331}}} = 1.953$$

$$u = \frac{3 \times 10^8}{\sqrt{1.953}} = \underline{\underline{2.1467 \times 10^8 \text{ m/s}}}$$

Prob. 11.71

$$\Gamma = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{100 - 150}{250} = -0.2$$
$$RL = -20\log|\Gamma| = \underline{\underline{13.98 \text{ dB}}}$$

Prob. 11.72

$$\Gamma = \frac{Z_L - Z_o}{Z_L + Z_o} = \frac{120 - 50}{170} = 0.4118$$
$$RL = -20\log_{10}|\Gamma| = -20\log_{10}(0.4118) = \underline{\underline{7.706 \text{ dB}}}$$

CHAPTER 12

P. E. 12.1 (a) For TE_{10}, $f_c = 3$ GHz,

$$\sqrt{1-(f_c/f)^2} = \sqrt{1-(3/15)^2} = \sqrt{0.96}, \quad \beta_o = \omega / u_o = 4\pi f / c$$

$$\beta = \frac{4\pi f}{c}\sqrt{0.96} = \frac{4\pi \times 15 \times 10^9}{3 \times 10^8}\sqrt{0.96} = \underline{\underline{615.6}} \text{ rad/m}$$

$$u = \frac{\omega}{\beta} = \frac{2\pi \times 15 \times 10^9}{615.6} = \underline{\underline{1.531 \times 10^8}} \text{ m/s}$$

$$\eta' = \sqrt{\frac{\mu}{\varepsilon}} = 60\pi, \quad \eta_{TE} = \frac{60\pi}{\sqrt{0.96}} = \underline{\underline{192.4\Omega}}$$

(b) For TM_{11}, $f_c = 3\sqrt{7.25}$ GHz, $\sqrt{1-(f_c/f)^2} = 0.8426$

$$\beta = \frac{4\pi f}{c}(0.8426) = \frac{4\pi \times 15 \times 10^9 (0.8426)}{3 \times 10^8} = \underline{\underline{529.4}} \text{ rad/m}$$

$$u = \frac{\omega}{\beta} = \frac{2\pi \times 15 \times 10^9}{529.4} = \underline{\underline{1.78 \times 10^8}} \text{ m/s}$$

$$\eta_{TM} = 60\pi(0.8426) = \underline{\underline{158.8\Omega}}$$

P. E. 12.2 (a) Since $E_z \neq 0$, this is a TM mode

$$E_{zs} = E_o \sin(m\pi x / a)\sin(n\pi y / b)e^{-j\beta z}$$

$E_o = 20$, $\dfrac{m\pi}{a} = 40\pi \longrightarrow$ m=2, $\dfrac{n\pi}{b} = 50\pi \longrightarrow$ n=1

i.e. $\underline{TM_{21}\text{ mode}}$.

(b) $f_c = \dfrac{u'}{2}\sqrt{(m/a)^2 + (n/b)^2} = \dfrac{3 \times 10^8}{2}\sqrt{40^2 + 50^2} = 1.5\sqrt{41}$ GHz

$$\beta = \omega\sqrt{\mu\varepsilon}\sqrt{1-(f_c/f)^2} = \frac{2\pi f}{c}\sqrt{f^2 - f_c^2} = \frac{2\pi \times 10^9}{3 \times 10^8}\sqrt{225 - 92.25} = \underline{241.3 \text{ rad/m.}}$$

(c)

$$E_{xs} = \frac{-j\beta}{h^2}(40\pi)20\cos 40\pi x \sin 50\pi y e^{-j\beta z}$$

$$E_{ys} = \frac{-j\beta}{h^2}(50\pi)20\sin 40\pi x \cos 50\pi y e^{-j\beta z}$$

$$\frac{E_y}{E_x} = \underline{\underline{1.25\tan 40\pi x \cot 50\pi y}}$$

P. E. 12.3 If TE_{13} mode is assumed, f_c and β remain the same.

$\underline{\underline{f_c = 28.57 \text{ GHz}, \ \beta = 1718.81 \text{ rad/m}, \ \gamma = j\beta}}$

$$\eta_{TE13} = \frac{377/2}{\sqrt{1-(28.57/50)^2}} = \underline{\underline{229.69\ \Omega}}$$

For m=1, n=3, the field components are:

$E_z = 0$

$$H_z = H_o \cos(\pi x/a)\cos(3\pi y/b)\cos(\omega t - \beta z)$$

$$E_x = -\frac{\omega\mu}{h^2}\left(\frac{3\pi}{b}\right)H_o \cos(\pi x/a)\sin(3\pi y/b)\sin(\omega t - \beta z)$$

$$E_y = \frac{\omega\mu}{h^2}\left(\frac{\pi}{a}\right)H_o \sin(\pi x/a)\cos(3\pi y/b)\sin(\omega t - \beta z)$$

$$H_x = -\frac{\beta}{h^2}\left(\frac{\pi}{a}\right)H_o \sin(\pi x/a)\cos(3\pi y/b)\sin(\omega t - \beta z)$$

$$H_y = -\frac{\beta}{h^2}\left(\frac{3\pi}{a}\right)H_o \cos(\pi x/a)\sin(3\pi y/b)\sin(\omega t - \beta z)$$

Given that $H_{ox} = 2 = -\frac{\beta}{h^2}(\pi/a)H_o$,

$$H_{oy} = -\frac{\beta}{h^2}(3\pi/b)H_o = 6a/b = 6(1.5)/8 = 11.25$$

$$H_{oz} = H_o = -\frac{2h^2 a}{\beta\pi} = \frac{-2\times 14.51\pi^2 \times 10^4 \times 1.5\times 10^{-2}}{1718.81\pi} = -7.96$$

$$E_{oy} = \frac{\omega\mu}{h^2}\left(\frac{\pi}{a}\right)H_o = -\frac{2\omega\mu}{\beta} = 2\eta_{TE} = -459.4$$

$$E_{ox} = -E_{oy}\frac{3a}{b} = 459.4(4.5/0.8) = 2584.1$$

$E_x = 2584.1\cos(\pi x / a)\sin(3\pi y / b)\sin(\omega t - \beta z)$ V/m,

$E_y = -459.4\sin(\pi x / a)\cos(3\pi y / b)\sin(\omega t - \beta z)$ V/m,

$E_z = 0$,

$H_y = 11.25\cos(\pi x / a)\sin(3\pi y / b)\sin(\omega t - \beta z)$ A/m,

$H_z = -7.96\cos(\pi x / a)\cos(3\pi y / b)\cos(\omega t - \beta z)$ A/m

P. E. 12.4

$$f_{c11} = \frac{u'}{2}\sqrt{\frac{1}{a^2} + \frac{1}{b^2}} = \frac{3\times10^8\times10^2}{2}\sqrt{1/8.636^2 + 1/4.318^2} = 3.883 \text{ GHz}$$

$$u_p = \frac{3\times10^8}{\sqrt{1-(3.883/4)^2}} = \underline{\underline{12.5\times10^8}} \text{ m/s,}$$

$$u_g = \frac{9\times10^{16}}{12.5\times10^8} = \underline{\underline{7.2\times10^7}} \text{ m/s}$$

P. E. 12.5 The dominant mode becomes TE_{01} mode

$$f_{c01} = \frac{c}{2b} = 3.75 \text{ GHz}, \quad \eta_{TE} = 406.7\Omega$$

From Example 12.2,

$$E_x = -E_o\sin(3\pi y / b)\sin(\omega t - \beta z), \quad \text{where } E_o = \frac{\omega\mu b}{\pi}H_o.$$

$$\mathcal{P}_{ave} = \int_{x=0}^{a}\int_{y=0}^{b}\frac{|E_{xs}|^2}{2\eta}dxdy = \frac{E_o^{\ 2}\, ab}{4\eta}$$

Hence $E_o = 63.77$ V/m as in Example 12.5.

$$H_o = \frac{\pi E_o}{\omega\mu b} = \frac{\pi\times63.77}{2\pi\times10^{10}\times4\pi\times10^{-7}\times4\times10^{-2}} = \underline{\underline{63.34}} \text{ mA/m}$$

P. E. 12.6 (a) For m=1, n=0, f_c = u'/(2a)

$$\frac{\sigma}{\omega\varepsilon}=\frac{10^{-15}}{2\pi\times 9\times 10^{9}\times 2.6\times 10^{-9}/(36\pi)}=\frac{10^{-15}}{1.3}<<1$$

Hence,

$$u'\cong\frac{1}{\sqrt{\mu\varepsilon}}=c/\sqrt{2.6},\quad f_c=\frac{3\times 10^{8}}{2\times 2.4\times 10^{-2}\sqrt{2.6}}=2.2149\ \text{ GHz}$$

$$\alpha_d=\frac{\sigma\eta'}{2\sqrt{1-(f_c/f)^2}}=\frac{10^{-15}\times 377/\sqrt{2.6}}{2\sqrt{1-(2.2149/9)^2}}=1.205\times 10^{-13}\ \text{Np/m}$$

For n = 0, m=1,

$$\alpha_c=\frac{2R_s}{b\eta'\sqrt{1-(f_c/f)^2}}[\frac{1}{2}+\frac{b}{a}(f_c/f)^2]$$

=

$$\frac{2\sqrt{2.6}\sqrt{\pi\times 9\times 10^{9}\times 1.1\times 10^{7}\times 4\pi\times 10^{-7}}}{377\times 1.5\times 10^{-2}\times 1.1\times 10^{7}\sqrt{1-(2.2149/9)^2}}[0.5+(2.4/1.5)(2.2148/9)^2]=\underline{\underline{2\times 10^{-2}}}\ \text{Np/m}$$

(b) Since $\alpha_c >> \alpha_d, \alpha=\alpha_c+\alpha_d\cong\alpha_c=2x10^{-2}$

loss = $\alpha l=2\times 10^{-2}\times 0.4=0.8\times 10^{-2}$ Np = $\underline{\underline{0.06945\ \text{dB}}}$

P. E. 12.7 For TE_{11} , m = 1 = n,

$$H_{zs}=H_o\cos(\pi x/a)\cos(\pi y/b)e^{-\gamma z}$$

$$E_{xs}=\frac{j\omega}{h^2}(\pi/b)H_o\cos(\pi x/a)\sin(\pi y/b)e^{-\gamma z}$$

$$E_{ys}=-\frac{j\omega\mu}{h^2}(\pi/a)H_o\sin(\pi x/a)\cos(\pi y/b)e^{-\gamma z}$$

$$H_{xs}=\frac{j\beta}{h^2}(\pi/a)H_o\sin(\pi x/a)\cos(\pi y/b)e^{-\gamma z}$$

$$H_{ys}=\frac{j\beta}{h^2}(\pi/b)H_o\cos(\pi x/a)\sin(\pi y/b)e^{-\gamma z}$$

$$E_{zs}=0$$

For the electric field lines,

$$\frac{dy}{dx} = \frac{E_y}{E_x} = (a/b)\tan(\pi x/a)\cot(\pi y/b)$$

For the magnetic field lines

$$\frac{dy}{dx} = \frac{H_y}{H_x} = -(a/b)\cot(\pi x/a)\tan(\pi y/b)$$

Notice that $(\frac{E_y}{E_x})(\frac{H_y}{H_x}) = -1$

showing that the electric and magnetic field lines are mutually orthogonal. The field lines are as shown in Fig. 12.14.

P. E. 12.8

$$u' = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{\sqrt{\varepsilon_r}}$$

$$f_{TE101} = \frac{1.5\times10^{10}}{\sqrt{3}}\sqrt{1/25+0+1/100} = \underline{\underline{1.936}} \text{ GHz}$$

$$Q_{TE101} = \frac{1}{61\delta}, \text{ where}$$

$$\delta = \frac{1}{\sqrt{\pi f_{101}\mu\sigma_c}} = \frac{1}{\sqrt{\pi\times1.936\times10^9\times4\pi\times10^{-7}\times5.8\times10^7}} = 1.5\times10^{-6}$$

$$Q_{TE101} = \frac{10^6}{61\times1.5} = \underline{\underline{10{,}929}}$$

P. E. 12.9

(a) By Snell's law, $n_1 \sin\theta_1 = n_2 \sin\theta_2$. Thus
$\theta_2 = 90^\circ \longrightarrow \sin\theta_2 = 1$

$\sin\theta_1 = n_2/n_1$, $\theta_1 = \sin^{-1} n_2/n_1 = \sin^{-1} 1.465/1.48 = \underline{\underline{81.83^\circ}}$

(b) $\text{NA} = \sqrt{n_1^2 - n_2^2} = \sqrt{1.48^2 - 1.465^2} = \underline{\underline{0.21}}$

P. E. 12.10

αl = 10 log P(0)/P(l) = 0.2 X 10 = 2

P(0)/P(l) = $10^{0.2}$, i.e. P(l) = P(0) $10^{-0.2}$ = 0.631 P(0)

i.e. 63.1 %

Prob. 12.1

$$f_c = \frac{u'}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} = \frac{3\times10^8}{2\sqrt{2.25}\times10^{-2}}\left[\left(\frac{m}{2.28}\right)^2 + \left(\frac{n}{1.01}\right)^2\right]^{1/2}$$

$$= \frac{15}{\sqrt{2.25}}\left[\left(\frac{m}{2.28}\right)^2 + \left(\frac{n}{1.01}\right)^2\right]^{1/2} \text{ GHz}$$

Using this formula, we obtain the cutoff frequencies for the given modes as shown below.

Mode	f_c (GHz)
TE_{01}	9.901
TE_{10}	4.386
TE_{11}	10.829
TE_{02}	19.802
TE_{22}	21.658
TM_{11}	10.829
TM_{12}	20.282
TM_{21}	13.228

Prob. 12.2

$$f_{cmn}=\frac{u'}{2}\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}=\frac{3\times10^8}{2}\times10^2\sqrt{\left(\frac{m}{6}\right)^2+\left(\frac{n}{4}\right)^2}=15\sqrt{\left(\frac{m}{6}\right)^2+\left(\frac{n}{4}\right)^2}\text{ GHz}$$

$$f_{10}=\frac{15}{6}=2.5\text{ GHz}$$

$$f_{c01}=\frac{15}{4}=3.75\text{ GHz}$$

$$f_{c20}=15\times\frac{2}{6}=5\text{ GHz}$$

$$f_{c11}=15\sqrt{\frac{1}{6^2}+\frac{1}{4^2}}=15\times0.3005=4.51\text{ GHz}$$

Possible modes are $\text{TE}_{10},\text{TE}_{01},\text{TE}_{20},\text{TE}_{11}$ and TM_{11}.

Prob. 12.3

(a) For TE_{10}mode, $f_c=\frac{u'}{2a}=\frac{3\times10^3}{2\times6\times10^{-2}}=\underline{\underline{2.5\text{ GHz}}}$

$(b)\ f=3f_c=7.5$ GHz

$$f_{cmn}=\frac{u^1}{2}\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}\quad=\frac{3\times10^2\times10^2}{2}\sqrt{\left(\frac{m}{6}\right)^2+\left(\frac{n}{4}\right)^2}$$

$$=15\sqrt{\left(\frac{m}{6}\right)^2+\left(\frac{n}{4}\right)^2}\text{ GHz}$$

$$f_{c20}=15\times\frac{2}{6}=5\text{ GHz}$$

$f_{c01}=3.75$ GHz, $\quad f_{c02}=7.5$ GHz

$f_{c10}=2.5$ GHz, $\quad f_{c20}=5.0$ GHz

$f_{c21}=6.25$ GHz, $\quad f_{c30}=7.5$ GHz

$$f_{12}=15\sqrt{\left(\frac{1}{6}\right)^2+\left(\frac{2}{4}\right)^2}=7.91\text{ GHz}$$

$$f_{11}=15\sqrt{\left(\frac{1}{6}\right)^2+\left(\frac{1}{4}\right)^2}=4.507\text{ GHz}$$

The following modes are transmitted

$TE_{01},TE_{02},TE_{10},TE_{11},TE_{20},TE_{21},TE_{30}$

TM_{11},TM_{21}

i.e. $\underline{\underline{\text{7 TE modes and 2 TM modes}}}$

Prob. 12.4

(a)

For TE_{10} mode,

$$f_c = \frac{u'}{2}\sqrt{\left(\frac{1}{a}\right)^2} = \frac{3\times10^8}{2\times2.4\times10^{-2}} = \underline{\underline{6.25 \text{ GHz}}}$$

For TE_{01} mode,

$$f_c = \frac{u'}{2}\sqrt{\left(\frac{1}{b}\right)^2} = \frac{3\times10^8}{2\times1.2\times10^{-2}} = \underline{\underline{12.5 \text{ GHz}}}$$

For TE_{20} mode,

$$f_c = \frac{u'}{2}\sqrt{\left(\frac{2}{a}\right)^2} = 2\times6.25 = \underline{\underline{12.5 \text{ GHz}}}$$

For TE_{02} mode,

$$f_c = \frac{u'}{2}\sqrt{\left(\frac{2}{b}\right)^2} = 2\times12.5 = \underline{\underline{25 \text{ GHz}}}$$

(b) Since f = 12 GHz, $\underline{\underline{\text{only } TE_{10} \text{ mode}}}$ will propagate.

Prob. 12.5 (a) For TE_{10} mode, $f_c = \dfrac{u'}{2a}$

Or $a = \dfrac{u'}{2f_c} = \dfrac{3\times10^8}{2\times5\times10^9} = \underline{\underline{3 \text{ cm}}}$

For TE_{01} mode, $f_c = \dfrac{u'}{2b}$

Or $b = \dfrac{u'}{2f_c} = \dfrac{3\times10^8}{2\times12\times10^9} = \underline{\underline{1.25 \text{ cm}}}$

(b) Since a > b, 1/a < 1/b, the next higher modes are calculated as shown below.

Mode	f_c (GHz)
TE_{10}	5
*TE_{20}	10
TE_{30}	15
TE_{40}	20
*TE_{01}	12
TE_{02}	24
*TE_{11}	13
TE_{21}	15.62

The next three higher modes are starred ones, i.e. $\underline{\underline{TE_{20}, TE_{01}, TE_{11}}}$

(c) $u' = \dfrac{1}{\sqrt{\mu\varepsilon}} = \dfrac{c}{\sqrt{2.25}} = 2\times10^{8}$ m/s

For TE_{11} modes,

$$f_c = \frac{3\times10^{8}}{2\times10^{-2}\sqrt{2.25}}\sqrt{\frac{1}{3^2}+\frac{1}{1.25^2}} = \underline{\underline{8.67}} \text{ GHz}$$

Prob. 12.6 For the dominant mode,

$$f_c = \frac{c}{2a} = \frac{3\times10^{8}}{2\times8} = 18.75 \text{ MHz}$$

(a) It <u>will not pass</u> the AM signal, (b) it <u>will pass</u> the FM signal.

Prob. 12.7 a/b = 3 $\longrightarrow$ a = 3b

$$f_{c10} = \frac{u'}{2a} \longrightarrow a = \frac{u'}{2f_{c10}} = \frac{3\times10^{8}}{2\times18\times10^{9}} \text{ m} = 0.833\text{cm}$$

A design could be <u>a = 9mm, b = 3mm.</u>

Prob. 12.8

Let $F_{12} = \sqrt{1-\left(\dfrac{f_{c12}}{f}\right)^2} = \sqrt{1-\left(\dfrac{25}{40}\right)^2} = 0.7806$

$$\lambda' = \frac{c}{f} = \frac{3\times10^{8}}{40\times10^{9}} = 0.0075 \text{ m} = 7.5\times10^{-3} \text{ m}$$

$$\lambda_{12} = \frac{\lambda'}{F_{12}} = \frac{7.5\times10^{-3}\text{ m}}{0.7806} = \underline{\underline{9.608\times10^{-3}\text{ m}}}$$

$$u_{12} = \frac{u'}{F_{12}} = \frac{3\times10^{8}}{0.7806} = \underline{\underline{3.843\times10^{8}\text{ m/s}}}$$

$$\beta_{12} = \frac{2\pi}{\lambda_{12}} = \frac{2\pi}{9.608\times10^{-3}} = \underline{\underline{653.95 \text{ rad/m}}}$$

$$\eta_{TE12} = \frac{\eta'}{F_{12}} = \frac{120\pi}{0.7806} = \underline{\underline{482.95\Omega}}$$

Prob. 12.9

$$u = \frac{\omega}{\beta} = \frac{u'}{\sqrt{1-(f_c/f)^2}} = \frac{3\times 10^8}{\sqrt{1-(6.5/7.2)^2}} = 6.975\times 10^8 \text{ m/s}$$

$$u_g = \frac{9\times 10^{16}}{u} = 1.2903\times 10^8 \text{ m/s}$$

$$t = \frac{2l}{u_g} = \frac{300}{1.2903\times 10^8} = \underline{\underline{2.325}}\ \mu\text{s}$$

Prob. 12.10

$$f = 1.12 f_{c10} = 1.12\frac{c}{2a} \longrightarrow a = 1.12\frac{c}{2f} = \frac{1.12\times 3\times 10^8}{2\times 4\times 10^9} = \underline{\underline{4.2 \text{ cm}}}$$

The next higher-order mode is $f_{c01} = c/2b$.

$$f = 0.85 f_{c01} = 0.85\frac{c}{2b} \longrightarrow b = \frac{0.85c}{2f} = \frac{0.85\times 3\times 10^8}{2\times 4\times 10^9} = \underline{\underline{3.187 \text{ cm}}}$$

Prob. 12.11

$$(a) f_{c10} = \frac{u'}{2a} = \frac{c}{2a\sqrt{\varepsilon_r}}$$

$$= \frac{3\times 10^8}{2\times 1.067\times 10^{-2}\sqrt{6.8}}$$

$$= \frac{30}{2\times 1.067\sqrt{6.8}} \text{ GHz}$$

$$\underline{\underline{= 5.391 \text{ GHz}}}$$

(b)

$$F = \sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

$$= \sqrt{1-\left(\frac{5.391}{6}\right)^2}$$

$$= 0.439$$

$$u_\rho = \frac{u'}{F} = \frac{c}{F\sqrt{\varepsilon_r}} = \frac{3\times10^8}{0.439\times\sqrt{6.8}} = \underline{\underline{2.62\times10^8 \text{ m/s}}}$$

(c) $$\lambda = \frac{\lambda'}{F} = \frac{u'/f}{F} = \frac{c}{fF\sqrt{\varepsilon_r}} = \frac{3\times10^8}{0.439\times6\times10^9\times\sqrt{6.8}}$$

$$= \frac{10^{-1}}{2\times0.439\sqrt{6.8}} = 0.04368 \text{ m} = \underline{\underline{4.368 \text{ cm}}}$$

Prob. 12.12
In evanescent mode,

$$k^2 = \omega^2\mu\varepsilon < \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

$$\beta = 0, \qquad \gamma = \alpha = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - k^2} = \sqrt{4\pi^2\mu\varepsilon f_c^2 - \omega^2\mu\varepsilon}$$

$$\alpha = \sqrt{\mu\varepsilon}\sqrt{4\pi^2 f_c^2 - 4\pi^2 f^2} = \underline{\underline{2\pi\sqrt{\mu\varepsilon}\, f_c\sqrt{1-\left(\frac{f}{f_c}\right)^2}}}$$

Prob. 12.13
$E_z \neq 0$. This must be TM_{23} mode (m=2, n=3). Since a= 2b,

$$f_c = \frac{c}{4b}\sqrt{m^2+4n^2} = \frac{3\times10^8}{4\times3\times10^{-2}}\sqrt{4+36} = 15.81 \text{ GHz}, \quad f = \frac{\omega}{2\pi} = \frac{10^{12}}{2\pi} = 159.2 \text{ GHz}$$

$$\eta_{TM} = 377\sqrt{1-(15.81/159.2)^2} = \underline{\underline{375.1\ \Omega}}$$

$$\mathcal{P}_{ave} = \frac{|E_{xs}|^2 + |E_{ys}|^2}{2\eta_{TM}}\boldsymbol{a}_z$$

$$= \frac{\beta^2 E_o^{\,2}}{2h^4\eta_{TM}}\left[(2\pi/a)^2\cos^2(2\pi x/a)\sin^2(3\pi y/b) + (3\pi/b)^2\sin^2(2\pi x/a)\cos^2(3\pi y/b)\right]\boldsymbol{a}_z$$

$$P_{ave} = \int \mathcal{P}_{ave}.d\boldsymbol{S} = \int_{x=0}^{a}\int_{y=0}^{b} \mathcal{P}_{ave}\bullet dxdy\boldsymbol{a}_z$$

$$= \frac{\beta^2 E_o^{\,2}}{2h^4\eta_{TM}}\frac{ab}{4}\left[\frac{4\pi^2}{a^2} + \frac{9\pi^2}{b^2}\right] = \frac{\beta^2 E_o^{\,2} ab}{8h^2\eta_{TM}}$$

But

$$\beta = \frac{\omega}{c}\sqrt{1-(f_c/f)^2} = \frac{10^{12}}{3\times 10^8}\sqrt{1-(15.81/159.2)^2} = 3.317\times 10^3$$

$$h^2 = \frac{4\pi^2}{a^2} + \frac{9\pi^2}{b^2} = \frac{10\pi^2}{b^2} = 1.097\times 10^5$$

$$P_{ave} = \frac{(3.317)^2\times 10^6\times 5^2\times 18\times 10^{-4}}{8\times(1.098\times 10^5)\times 375.1} = \underline{\underline{1.5 \text{ mW}}}$$

Prob. 12.14 (a) Since m=2 and n=1, we have $\underline{\underline{TE_{21}}}$ mode

(b) $\beta = \beta'\sqrt{1-(f_c/f)^2} = \omega\sqrt{\mu_o\varepsilon_o}\sqrt{1-(\omega_c/\omega)^2}$

$$\beta c = \sqrt{\omega^2 - \omega^2_c} \longrightarrow \omega_c^2 = \sqrt{\omega^2 - \beta^2 c^2}$$

$$f_c = \frac{\omega_c}{2\pi} = \sqrt{f^2 - \frac{\beta^2 c^2}{4\pi^2}} = \sqrt{36\times 10^{18} - \frac{144\times 9\times 10^{16}}{4\pi^2}} = \underline{\underline{5.973}} \text{ GHz}$$

(c) $\eta_{TE} = \dfrac{\eta}{\sqrt{1-(f_c/f)^2}} = \dfrac{377}{\sqrt{1-(5.973/6)^2}} = \underline{\underline{3978\Omega}}$

(d) For TE mode,

$$E_y = \frac{\omega\mu}{h^2}(m\pi/a)H_o \sin(m\pi x/a)\cos(n\pi y/b)\sin(\omega t - \beta z)$$

$$H_x = \frac{-\beta}{h^2}(m\pi/a)H_o \sin(m\pi x/a)\cos(n\pi y/b)\sin(\omega t - \beta z)$$

$\beta = 12$, m = 2, n =1

$$E_{oy} = \frac{\omega\mu}{h^2}(m\pi/a)H_o, \quad H_{ox} = \frac{\beta}{h^2}(m\pi/a)H_o$$

$$\eta_{TE} = \frac{E_{oy}}{H_{ox}} = \frac{\omega\mu}{\beta} = \frac{2\pi\times 6\times 10^9\times 4\pi\times 10^{-7}}{12} = 4\pi^2\times 100$$

$$H_{ox} = \frac{E_{oy}}{\eta_{TE}} = \frac{5}{4\pi^2\times 100} = 1.267 \text{ mA/m}$$

$$\underline{\underline{H_x = -1.267\sin(m\pi x/a)\cos(n\pi y/b)\sin(\omega t - \beta z)}} \text{ mA/m}$$

Prob. 12.15 (a) Since m=2, n=3, the mode is $\underline{\underline{TE_{23}}}$.

(b) $\beta = \beta' \sqrt{1-(f_c/f)^2} = \dfrac{2\pi f}{c}\sqrt{1-(f_c/f)^2}$

But

$$f_c = \frac{u'}{2}\sqrt{(m/a)^2+(n/b)^2} = \frac{3\times10^8}{2\times10^{-2}}\sqrt{(2/2.86)^2+(3/1.016)^2} = 46.19\,\text{GHz},\ \text{f} = 50\ \text{GHz}$$

$$\beta = \frac{2\pi\times50\times10^9}{3\times10^8}\sqrt{1-(46.19/50)^2} = 400.68\ \text{rad/m}$$

$$\gamma = j\beta = \underline{\underline{j400.7}}\ /\text{m}$$

(c) $\eta = \dfrac{\eta'}{\sqrt{1-(f_c/f)^2}} = \dfrac{377}{\sqrt{1-(46.19/50)^2}} = \underline{\underline{985.3\Omega}}$

Prob. 12.16 In free space,

$$\eta_1 = \frac{\eta_o}{\sqrt{1-(f_c/f)^2}}, \qquad f_c = \frac{c}{2a} = \frac{3\times10^8}{2\times5\times10^{-2}} = 3\ \text{GHz}$$

$$\eta_1 = \frac{377}{\sqrt{1-(3/8)^2}} = 406.7\Omega$$

$$\eta_2 = \frac{\eta'_1}{\sqrt{1-(f_c/f)^2}}, \eta' = \frac{120\pi}{\sqrt{2.25}} = 80\pi, f_c = \frac{u'}{2a}, u' = \frac{c}{\sqrt{\varepsilon_r}}$$

$$f_c = \frac{3\times10^8}{2\times5\times10^{-2}\sqrt{2.25}} = 2\ \text{GHz}, \quad \eta_2 = \frac{80\pi}{\sqrt{1-(2/8)^2}} = 259.57\Omega$$

$$\Gamma = \frac{\eta_2-\eta_1}{\eta_2+\eta_1} = -0.2208$$

$$s = \frac{1+|\Gamma|}{1-|\Gamma|} = \underline{\underline{1.5667}}$$

Prob. 12.17 Substituting $E_z = R\Phi Z$ into the wave equation,

$$\frac{\Phi Z}{\rho}\frac{d}{d\rho}(\rho R') + \frac{RZ}{\rho^2}\Phi'' + R\Phi Z'' + k^2 R\Phi Z = 0$$

Dividing by $R\Phi Z$,

$$\frac{1}{R\rho}\frac{d}{d\rho}(\rho R') + \frac{\Phi''}{\Phi\rho^2} + k^2 = -\frac{Z''}{Z} = -k_z^2$$

i.e. $\underline{\underline{Z'' - k_z^2 Z = 0}}$

$$\frac{1}{R\rho}\frac{d}{d\rho}(\rho R') + \frac{\Phi''}{\Phi\rho^2} + (k^2 + k_z^2) = 0$$

$$\frac{\rho}{R}\frac{d}{d\rho}(\rho R') + (k^2 + k_z^2)\rho^2 = -\frac{\Phi''}{\Phi} = k_\phi^2$$

or

$$\underline{\underline{\Phi'' + k_\phi^2 \Phi = 0}}$$

$\rho\frac{d}{d\rho}(\rho R') + (k_\rho^2\rho^2 - k_\phi^2)R = 0$, where $k_\rho^2 = k^2 + k_z^2$. Hence

$$\underline{\underline{\rho^2 R'' + \rho R' + (k_\rho^2 \rho^2 - k_\phi^2)R = 0}}$$

Prob. 12.18

(a)

For TE_{10} mode,

$$f_c = \frac{u'}{2a} = \frac{3\times10^8}{2\times7.2\times10^{-2}} = 2.083 \text{ GHz}$$

$$\text{Let} \quad F = \sqrt{1-\left(\frac{f_c}{f}\right)^2} = \sqrt{1-\left(\frac{2.083}{6.2}\right)^2} = 0.942$$

$$\beta = \omega\sqrt{\mu\varepsilon}F = \frac{\omega F}{c} = \frac{2\pi\times6.2\times10^9\times0.942}{3\times10^8} = \underline{\underline{122.32 \text{ rad/m}}}$$

$$u_p = \frac{\omega}{\beta} = \frac{c}{F} = \frac{3\times10^8}{0.942} = \underline{\underline{3.185\times10^8 \text{ m/s}}}$$

$$u_g = u'F = 3\times10^8(0.942) = \underline{\underline{2.826\times10^8 \text{ m/s}}}$$

$$\eta_{TE} = \frac{\eta'}{F} = \frac{377}{0.942} = \underline{\underline{400.21\ \Omega}}$$

(b)

$$u' = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{\sqrt{\varepsilon_r}} = \frac{3\times10^8}{\sqrt{2.25}} = 2\times10^8$$

$$f_c = \frac{u'}{2a} = \frac{2\times10^8}{2\times7.2\times10^{-2}} = 1.389 \text{ GHz}$$

$$\text{Let} \quad F = \sqrt{1-\left(\frac{f_c}{f}\right)^2} = \sqrt{1-\left(\frac{1.389}{6.2}\right)^2} = 0.9746$$

$$\beta = \omega\sqrt{\mu\varepsilon}F = \frac{\omega F\sqrt{\varepsilon_r}}{c} = \frac{2\pi\times6.2\times10^9\times0.9746\times1.5}{3\times10^8} = \underline{\underline{189.83 \text{ rad/m}}}$$

$$u_p = \frac{\omega}{\beta} = \frac{2\pi\times6.2\times10^9}{189.83} = \underline{\underline{2.052\times10^8 \text{ m/s}}}$$

$$u_g = u'F = 2\times10^8(0.9746) = \underline{\underline{1.949\times10^8 \text{ m/s}}}$$

$$\eta_{TE} = \frac{\eta'}{F} = \frac{377}{1.5\times0.9746} = \underline{\underline{257.88\ \Omega}}$$

Prob. 12.19

$$f_{c10} = \frac{u'}{2a} = \frac{\frac{1}{\sqrt{\mu\varepsilon}}}{2a} = \frac{c}{2a\sqrt{\varepsilon_r}} = \frac{3\times10^8}{2\times7.214\times10^{-2}\sqrt{2.5}} = 1.315 \text{ GHz}$$

Let $F = \sqrt{1-\left(\frac{f_c}{f}\right)^2} = \sqrt{1-\left(\frac{1.315}{4}\right)^2} = 0.9444$

$$\beta = \omega\sqrt{\mu\varepsilon}F = \frac{\omega\sqrt{\varepsilon_r}F}{c}$$

$$u_p = \frac{\omega}{\beta} = \frac{c}{F\sqrt{\varepsilon_r}} = \frac{3\times10^8}{0.9444\times\sqrt{2.5}} = \underline{\underline{2.009\times10^8 \text{ m/s}}}$$

$$u_g = u'F = \frac{cF}{\sqrt{\varepsilon_r}} = \frac{3\times10^8\times0.9444}{\sqrt{2.5}} = \underline{\underline{1.792\times10^8 \text{ m/s}}}$$

Prob. 12.20

$$f_c = \frac{u'}{2a}$$

$$u_g = u'\sqrt{1-\left(\frac{f_c}{f}\right)^2} \longrightarrow \left(\frac{f_c}{f}\right)^2 = 1-\left(\frac{u_g}{u'}\right)^2 = 1-\left(\frac{1.8\times10^8}{\frac{3\times10^8}{\sqrt{2.2}}}\right)^2 = 0.208$$

$$f_c = \sqrt{0.208}f = 2.0523 \text{ GHz}$$

$$a = \frac{u'}{2f_c} = \frac{3\times10^8}{2\sqrt{2.2}\times2.053\times10^9} = \underline{\underline{4.927 \text{ cm}}}$$

Prob. 12.21

Let $F = \sqrt{1-(f_c/f)^2} = \sqrt{1-(16/24)^2} = 0.7453$

$u' = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{3\times10^8}{\sqrt{2.25}} = 2\times10^8$, $\quad u_p = \frac{u'}{F}$, $\quad u_g = u'F = 2\times10^8\times0.7453 = \underline{\underline{1.491\times10^8}}$ m/s

$$\eta_{TE} = \eta'/F = \frac{377}{1.5x0.7453} = \underline{\underline{337.2\Omega}}$$

Prob. 12.22

$$f_c = \frac{3\times10^8}{2}\sqrt{(m/0.025)^2 + (n/0.01)^2} = 15\sqrt{n^2 + (m/2.5)^2} \quad \text{GHz}$$

f_{c10} = 6 GHz, f_{c20} = 12 GHz, f_{c01} = 15 GHz.

Since f_{c20} , f_{c10} > 11 GHz, only the dominant TE_{10} mode is propagated.

(a) $$\frac{u_p}{u} = \frac{1}{\sqrt{1-(f_c/f)^2}} = \frac{1}{\sqrt{1-(6/11)^2}} = \underline{\underline{1.193}}$$

(b) $$\frac{u_g}{u} = \sqrt{1-(6/11)^2} = \underline{\underline{0.8381}}$$

Prob. 12.23

For the TE_{10} mode,

$$H_{zs} = H_o \cos\left(\frac{\pi x}{a}\right)e^{-j\beta z}$$

$$H_{xs} = \frac{j\beta a}{\pi}H_o \sin\left(\frac{\pi x}{a}\right)e^{-j\beta z}$$

$$E_{ys} = -\frac{j\omega\mu a}{\pi}H_o \sin\left(\frac{\pi x}{a}\right)e^{-j\beta z}$$

$$E_{xs} = 0 = E_{zs} = H_{ys}$$

$$\boldsymbol{E}_s \times \boldsymbol{H}_s^* = \begin{vmatrix} 0 & E_{ys} & 0 \\ H_{xs}^* & 0 & H_{zs}^* \end{vmatrix} = E_{ys}H_{zs}^*\boldsymbol{a}_x - E_{ys}H_{xs}^*\boldsymbol{a}_z$$

$$= -\frac{j\omega\mu a}{\pi}H_o^2 \cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi x}{a}\right)\boldsymbol{a}_x + \frac{\omega\mu\beta a^2}{\pi^2}H_o^2 \sin^2\left(\frac{\pi x}{a}\right)\boldsymbol{a}_z$$

$$\mathscr{P}_{ave} = \frac{1}{2}\mathrm{Re}\left[\boldsymbol{E}_s \times \boldsymbol{H}_s^*\right] = \underline{\underline{\frac{\omega\mu\beta a^2}{2\pi^2}H_o^2 \sin^2\left(\frac{\pi x}{a}\right)\boldsymbol{a}_z}}$$

Prob. 12.24

$$\boldsymbol{P}_{\text{ave}} = \frac{|E_{xs}|^2 + |E_{ys}|^2}{2\eta}\boldsymbol{a}_z = \underline{\underline{\frac{\omega^2\mu^2\pi^2}{2\eta b^2 h^4}H_o{}^2 \sin^2 \pi y/b\boldsymbol{a}_z}}$$

where $\eta = \eta_{TE10}$.

$$\mathrm{P}_{\text{ave}} = \int \boldsymbol{P}_{\text{ave}}.d\boldsymbol{S} = \frac{\omega^2\mu^2\pi^2}{2\eta b^2 h^4}H_o{}^2 \int_{x=0}^{a}\int_{y=0}^{b} \sin^2 \pi y/b\,dxdy$$

$$\mathrm{P}_{\text{ave}} = \frac{\omega^2\mu^2\pi^2}{2\eta b^2 h^4}H_o{}^2 ab/2$$

But $h^2 = (m\pi / a)^2 + (n\pi / b)^2 = \dfrac{\pi^2}{b^2}$,

$$\underline{\underline{P_{ave} = \frac{\omega^2 \mu^2 a b^3 H_o{}^2}{4\pi^2 \eta}}}$$

Prob. 12.25

$$R_s = \sqrt{\frac{\pi \mu f}{\sigma_c}} = \sqrt{\frac{\pi \times 12 \times 10^9 \times 4\pi \times 10^{-7}}{5.8 \times 10^7}} = 2.858 \times 10^{-2}$$

$$f_{c10} = \frac{u'}{2a} = \frac{3 \times 10^8}{2\sqrt{2.6} \times 2 \times 10^{-2}} = 4.651 \text{ GHz}$$

$$f_{c11} = \frac{u'}{2}\left[\frac{1}{a^2} + \frac{1}{b^2}\right]^{1/2} = 10.4 \text{ GHz}$$

$$\eta' = \sqrt{\frac{\mu}{\varepsilon}} = \frac{377}{\sqrt{2.6}} = 233.81\Omega$$

(a) For TE_{10} mode, eq.(12.57) gives

$$\alpha_d + j\beta_d = \sqrt{-\omega^2 \mu\varepsilon + k_x{}^2 + k_y{}^2 + j\omega\mu\sigma_d}$$

$$= \sqrt{-\omega^2 / u^2 + \frac{\pi^2}{a^2} + j\omega\mu\sigma_d}$$

$$= \sqrt{-\left(\frac{2\pi \times 12 \times 10^9}{3 \times 10^8}\right)^2 (2.6) + \frac{\pi^2}{(2 \times 10^{-2})^2} + j2\pi \times 12 \times 10^9 \times 4\pi \times 10^{-7} \times 10^{-4}}$$

$$= 0.012682 + j373.57$$

$$\underline{\underline{\alpha_d = 0.012682}} \text{ Np/m}$$

$$\alpha_c = \frac{2R_s}{b\eta' \sqrt{1 - (f_c / f)^2}}\left[\frac{1}{2} + \frac{b}{a}\left(\frac{f_c}{f}\right)^2\right]$$

$$= \frac{2x2.858x10^{-2}}{10^{-2}(233.81)\sqrt{1 - (4.651 \;/ 12)^2}}\left[\frac{1}{2} + \frac{1}{2}\left(\frac{4.651}{12}\right)^2\right] = \underline{\underline{0.0153}} \text{ Np/m}$$

(b) For TE_{11} mode,

$$\alpha_d + j\beta_d = \sqrt{-\omega^2 / u^2 + 1/a^2 + 1/b^2 + j\omega\mu\sigma_d}$$

$$= \sqrt{-139556.21 + \frac{\pi^2}{(10^{-2})^2} + j9.4748} = 0.02344 + j202.14$$

$$\underline{\underline{\alpha_d = 0.02344}} \text{ Np/m}$$

$$\alpha_c = \frac{2R_s}{b\eta'\sqrt{1-(f_c/f)^2}}\left[\frac{(b/a)^3+1}{(b/a)^2+1}\right] = \frac{2\times 2.858\times 10^{-2}}{10^{-2}(233.81)\sqrt{1-(10.4/12)^2}}\left[\frac{(1/8)+1}{(1/4)\ \ +1}\right]$$

$$\underline{\underline{\alpha_c = 0.0441}} \text{ Np/m}$$

Prob. 12.26 $\varepsilon_c = \varepsilon' - j\varepsilon'' = \varepsilon - j\dfrac{\sigma}{\omega}$

Comparing this with

$$\varepsilon_c = 16\varepsilon_o(1 - j10^{-4}) = 16\varepsilon_o - j16\varepsilon_o \times 10^{-4}$$

$$\varepsilon = 16\varepsilon_o, \qquad \frac{\sigma}{\omega} = 16\varepsilon_o x10^{-4}$$

For TM_{21} mode,

$$f_c = \frac{u'}{2}\left[\frac{m^2}{a^2} + \frac{n^2}{b^2}\right]^{1/2} = 2.0963 \text{ GHz}, \quad f = 1.1 f_c = 2.3059 \text{ GHz}$$

$$\sigma = 16\varepsilon_o\omega\times 10^{-4} = 16\times 2\pi\times 2.3059\times 10^9 \times \frac{10^{-9}}{36\pi}\times 10^{-4} = 2.0525\times 10^{-4}$$

$$\eta' = \sqrt{\frac{\mu}{\varepsilon}} = 30\pi$$

$$\alpha_d = \frac{\sigma\eta'}{2\sqrt{1-(f_c/f)^2}} = \frac{4.1\times 10^{-4}\times 30\pi}{2\sqrt{1-1/1.12}} = \underline{\underline{0.0231}} \text{ Np/m}$$

$$E_o e^{-\alpha_d z} = 0.8E_o \longrightarrow z = \frac{1}{\alpha_d}\ln(1/0.8) = \underline{\underline{9.66 \text{ m}}}$$

Prob. 12.27
For TM_{21} mode,

$$\alpha_c = \frac{2R_s}{b\eta'\sqrt{1-(f_c/f)^2}}$$

$$R_s = \frac{1}{\sigma_c \delta} = \sqrt{\frac{\pi f \mu}{\sigma_c}} = \sqrt{\frac{\pi \times 2.3059\times10^9 \times 4\pi\times10^{-7}}{1.5\times10^7}} = 0.0246$$

$$\alpha_c = \frac{2\times0.0246}{4\pi\times10^{-2}\times30\pi\times0.4166} = 0.0314 \text{ Np/m}$$

$$E_o e^{-(\alpha_c+\alpha_d)z} = 0.7E_o \longrightarrow z = \frac{1}{\alpha_c+\alpha_d}\ln(1/0.7) = \underline{\underline{6.5445 \text{ m}}}$$

Prob. 12.28

$$f_{c10} = \frac{u'}{2a} = \frac{3\times10^8}{2\times6\times10^{-2}} = 2.5 \text{ GHz}$$

$$\eta_{\mathrm{TE}} = \frac{\eta'}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}} = \frac{377}{\sqrt{1-\left(\frac{2.5}{4}\right)^2}} = 483$$

From Example 12.5,

$$P_{ave} = \frac{E_o^2 ab}{2\eta} = \frac{(2.2)^2\times10^6\times6\times3\times10^{-4}}{2\times483} = \underline{\underline{9.0196 \text{ mW}}}$$

Prob. 12.29 For TE_{10} mode,

$$f_c = \frac{u'}{2a} = \frac{3\times10^8}{2\sqrt{2.11}\times4.8\times10^{-2}} = 2.151 \text{ GHz}$$

(a) loss tangent $= \dfrac{\sigma}{\omega\varepsilon} = d$

$$\sigma = d\omega\varepsilon = 3\times10^{-4}\times2\pi\times4\times10^9\times2.11\times\frac{10^{-9}}{36\pi} = 1.4086\times10^{-4}$$

$$\eta' = \frac{120\pi}{\sqrt{2.11}} = 259.53$$

$$\alpha_d = \frac{\sigma\eta'}{2\sqrt{1-(f_c/f)^2}} = \frac{1.4067\times10^{-4}\times259.53}{2\sqrt{1-(2.151/4)^2}} = \underline{\underline{2.165\times10^{-2}\text{ Np/m}}}$$

(b) $R_s = \sqrt{\dfrac{\mu f\pi}{\sigma_c}} = \sqrt{\dfrac{\pi\times4\times10^9\times4\pi\times10^{-7}}{4.1\times10^7}} = 1.9625\times10^{-2}$

$$\alpha_c = \frac{2R_s}{b\eta'\sqrt{1-(f_c/f)^2}}\left[\frac{1}{2}+\frac{b}{a}(f_c/f)^2\right] = \frac{3.925\times10^{-2}(0.5+0.5\times0.2892)}{2.4\times10^{-2}\times259.53\times0.8431}$$

$$= \underline{\underline{4.818\times10^{-3}\text{ Np/m}}}$$

Prob. 12.30

$$\alpha_c = \frac{2R_s}{b\eta'\sqrt{1-\left(\frac{f_c}{f}\right)^2}}\left[\frac{1}{2}+\frac{b}{a}\left(\frac{f_c}{f}\right)^2\right] = \frac{2\sqrt{\frac{\pi f\mu}{\sigma_c}}}{b\eta'\sqrt{1-\left(\frac{f_c}{f}\right)^2}}\left[\frac{1}{2}+\frac{1}{2}\left(\frac{f_c}{f}\right)^2\right]$$

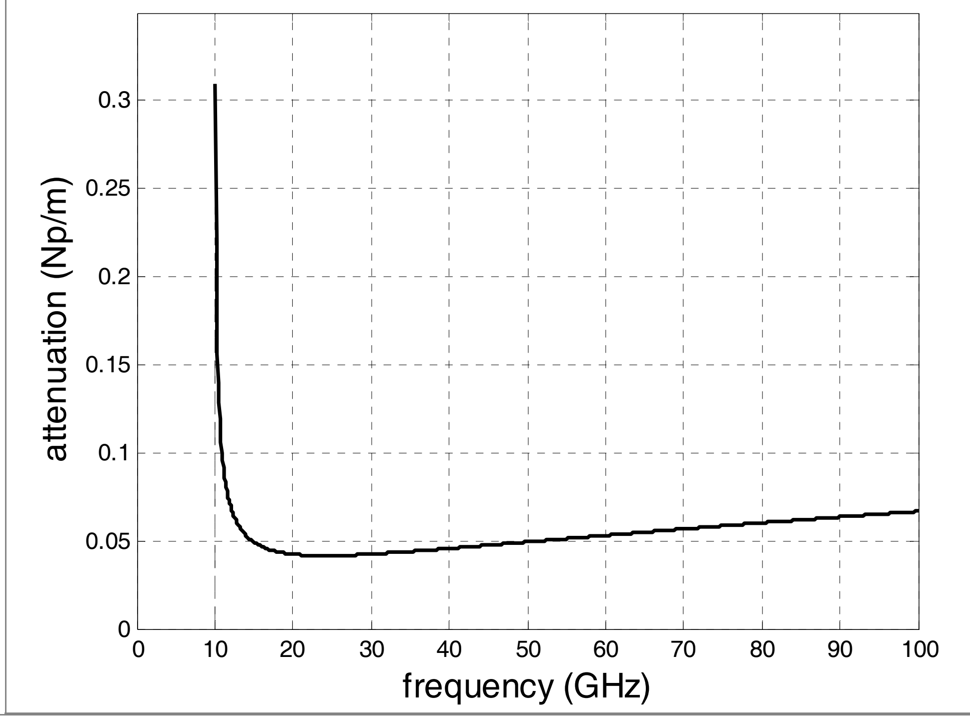
$$= \frac{2\sqrt{4\pi\times10^{-7}\times\pi}\sqrt{f}\times\frac{1}{2}}{0.5\times10^{-2}\times(120\pi/\sqrt{2.25})\sqrt{5.8\times10^7}\sqrt{1-\left(\frac{f_c}{f}\right)^2}}\left[1+\left(\frac{f_c}{f}\right)^2\right]$$

$$= \frac{10^{-5}\sqrt{f}}{30\sqrt{(5.8/2.25)}\sqrt{1-\left(\frac{f_c}{f}\right)^2}}\left[1+\left(\frac{f_c}{f}\right)^2\right]$$

The MATLAB code is shown below

```
k=10^(-5)/(30*sqrt(5.8/2.25));
fc=10^10;
for  n=1:1000
   f(n)=fc*(n/100+1);
   fn=f(n);
   num=sqrt(fn)*(1 +(fc/fn)^2);
   den=sqrt(1- (fc/fn)^2);
   alpha(n) =k*num/den;
end
plot(f/10^9,alpha)
xlabel('frequency (GHz)')
ylabel('attenuation')
grid
```

The plot of attenuation versus frequency is shown below.



Prob. 12.31 (a) For TE_{10} mode,

$$f_c = \frac{u'}{2a}, \quad u' = \frac{c}{\sqrt{2.11}}$$

$$f_c = \frac{3\times10^8}{\sqrt{2.11}(2\times2.25\times10^{-2})} = \underline{\underline{4.589 \text{ GHz}}}$$

(b) $$\alpha_{cTE10} = \frac{2R_s}{b\eta'\sqrt{1-(f_c/f)^2}}\left[\frac{1}{2}+\frac{b}{a}(f_c/f)^2\right]$$

$$R_s = \sqrt{\frac{\pi f \mu}{\sigma_c}} = \sqrt{\frac{\pi\times5\times10^9\times4\pi\times10^{-7}}{1.37\times10^7}} = 3.796\times10^{-2}$$

$$\eta' = \frac{377}{\sqrt{2.11}} = 259.54$$

$$\alpha_c = \frac{2\times3.796\times10^{-2}[0.5+\dfrac{1.5}{2.25}(4.589/5)^2]}{1.5\times10^{-4}(259.54)\sqrt{1-(4.589/5)^2}} = \underline{\underline{0.05217}} \text{ Np/m}$$

Prob. 12.32

(a) $f_{c10} = \dfrac{u'}{2a} = \dfrac{3\times10^8}{2\times3.8\times10^{-2}} = 3.947 \text{ GHz}$

$$u_g = u'\sqrt{1-\left(\frac{f_c}{f}\right)^2} = 3\times10^8\sqrt{1-(0.3947)^2} = \underline{\underline{2.756\times10^8 \text{ m/s}}}$$

(b) $\alpha = \alpha_d + \alpha_c$

$\alpha_d = 0$ since the guide is air-filled.

$$R_s = \sqrt{\frac{\pi f\mu}{\sigma_c}} = \sqrt{\frac{\pi\times10^{10}\times4\pi\times10^{-7}}{5.8\times10^7}} = 2.609\times10^{-2}\,\Omega$$

$$\alpha_c = \frac{2R_s}{b\eta'\sqrt{1-\left(\dfrac{f_c}{f}\right)^2}}\left[0.5+\frac{b}{a}\left(\frac{f_c}{f}\right)^2\right]$$

$$= \frac{2\times2.609\times10^{-2}}{1.6\times10^{-2}(377)\sqrt{1-(0.3947)^2}}\left[0.5+\frac{1.6}{3.8}(0.3947)^2\right] = \frac{5.218\times0.5656}{554.23}$$

$$\underline{\underline{= 5.325\times10^{-3} \text{ Np/m}}}$$

$$\alpha_c(dB) = 8.686\times5.325\times10^{-3} = 0.04626 \text{ dB/m}$$

Prob. 12.33

$$f_{c10} = \frac{u'}{2a} = \frac{c}{2a\sqrt{\varepsilon_r\mu_r}} = \frac{3\times10^8}{2\times2.5\times10^{-2}\sqrt{2.26}} = 3.991 \text{ GHz}$$

$$\beta' = \frac{\omega}{u'} = \frac{2\pi f\sqrt{\varepsilon_r}}{c}$$

$$F = \sqrt{1-\left(\frac{f_c}{f}\right)^2} = \sqrt{1-\left(\frac{3.991}{7.5}\right)^2} = 0.8467$$

$$\beta = \beta' F = \frac{2\pi\times7.5\times10^9\sqrt{2.26}}{3\times10^8}0.8467 = \underline{\underline{199.94 \text{ rad/m}}}$$

$$\alpha_d = \frac{\sigma\eta'}{2F} = \frac{\sigma\eta_o}{2F\sqrt{\varepsilon_r}} = \frac{10^{-4}(377)}{2\times 0.8467\sqrt{2.26}} = \underline{\underline{1.481\times 10^{-2}\ \text{Np/m}}}$$

$$\alpha_c = \frac{2R_s}{b\eta'\sqrt{1-\left(\frac{f_c}{f}\right)^2}}\left[0.5+\frac{b}{a}\left(\frac{f_c}{f}\right)^2\right]$$

$$R_s = \sqrt{\frac{\pi f\mu}{\sigma_c}} = \sqrt{\frac{\pi\times 7.5\times 10^9\times 4\pi\times 10^{-7}}{1.1\times 10^7}} = 0.0519$$

$$\alpha_c = \frac{2\times 0.0519\left[0.5+\frac{1.5}{2.5}\left(\frac{3.991}{7.5}\right)^2\right]}{1.5\times 10^{-2}\times\frac{377}{\sqrt{2.66}}\times 0.8467} = \frac{0.1038\times 0.6698}{3.1848}$$

$$\underline{\underline{= 0.02183\ \text{Np/m}}}$$

$$u_p = \frac{u'}{F} = \frac{c}{F\sqrt{\varepsilon_r}} = \frac{3\times 10^8}{0.8467\sqrt{2.26}} = \underline{\underline{2.357\times 10^8\ \text{m/s}}}$$

$$u_g = u'F = \frac{3\times 10^8\times 0.8467}{\sqrt{2.26}} = \underline{\underline{1.689\times 10^8\ \text{m/s}}}$$

$$\lambda_c = \frac{u'}{f_c} = \frac{c}{f_c\sqrt{\varepsilon_r}} = \frac{3\times 10^8}{3.991\times 10^9\sqrt{2.26}} = 0.05\ \text{m} = \underline{\underline{5\ \text{cm}}}(= 2a,\ \text{as expected})$$

Prob. 12.34

The cutoff frequency of the dominant mode is

$$f_{c10} = \frac{u}{2a} = \frac{3\times 10^8}{4.576\times 10^{-2}} = 6.56\ \text{GHz}$$

The surface resistance is

$$R_s = \sqrt{\frac{\pi f\mu}{\sigma_c}} = \sqrt{\frac{\pi\times 8.4\times 10^9\times 4\pi\times 10^{-7}}{5.8\times 10^7}} = 23.91\times 10^{-3}$$

For TE_{10} mode,

$$\alpha_c = \frac{2R_s}{b\eta'\sqrt{1-\left(\frac{f_c}{f}\right)^2}}\left[0.5+\frac{b}{a}\left(\frac{f_c}{f}\right)^2\right]$$

$$\frac{f_c}{f} = \frac{6.56}{8.4} = 0.781, \qquad \eta' = \eta_o = 377$$

$$\alpha_c = \frac{2\times 23.91\times 10^{-3}}{1.016\times 10^{-2}\times 377\sqrt{1-0.781^2}}\left[0.5+\frac{1.016}{2.286}(0.781)^2\right]$$

$$= \frac{47.82\times 10^{-3}(0.5+0.2711)}{3.83\times 0.6245} = 15.42\times 10^{-3} \text{ Np/m}$$

$$=15.42\times 10^{-3}\times 8.686 \text{ dB/m} = \underline{\underline{0.1339 \text{ dB/m}}}$$

Prob. 12.35 For TE_{10} mode,

$$\alpha_c = \frac{2R_s}{b\eta'\sqrt{1-(f_c/f)^2}}\left[\frac{1}{2}+\frac{b}{a}(\frac{f_c}{f})^2\right]$$

But a = b, $R_s = \dfrac{1}{\sigma_c\delta} = \sqrt{\dfrac{\pi f\mu}{\sigma_c}}$

$$\alpha_c = \frac{2\sqrt{\dfrac{\pi f\mu}{\sigma_c}}}{a\eta'\sqrt{1-(f_c/f)^2}}\left[\frac{1}{2}+(\frac{f_c}{f})^2\right] = \frac{k\sqrt{f}\left[\dfrac{1}{2}+(\dfrac{f_c}{f})^2\right]}{\sqrt{1-(f_c/f)^2}}$$

where k is a constant.

$$\frac{d\alpha_c}{df} = \frac{k[1-(\frac{f_c}{f})^2]^{1/2}[\frac{1}{4}f^{-1/2}-\frac{3}{2}f_c^{\,2}f^{-5/2}]-\frac{k}{2}[\frac{1}{2}f^{1/2}+f_c^{\,2}f^{-3/2}](2f_c^{\,2}f^{-3})[1-(\frac{f_c}{f})^2]^{-1/2}}{1-(f_c/f)^2}$$

For minimum value, $\dfrac{d\alpha_c}{df} = 0$. This leads to $\underline{\underline{f = 2.962\, f_c}}$.

Prob. 12.36 For the TE mode to z,

$$E_{zs} = 0, H_{zs} = H_o\cos(m\pi x/a)\cos(n\pi y/b)\sin(p\pi z/c)$$

$$E_{ys} = -\frac{\gamma}{h^2}\frac{\partial E_{zs}}{\partial y} + \frac{j\omega\mu}{h^2}\frac{\partial H_{zs}}{\partial x} = -\frac{j\omega\mu}{h^2}(m\pi/a)H_o\sin(m\pi x/a)\cos(n\pi y/b)\sin(p\pi z/c)$$

as required.

$$E_{xs} = -\frac{\gamma}{h^2}\frac{\partial E_{zs}}{\partial x} - \frac{j\omega\mu}{h^2}\frac{\partial H_{zs}}{\partial y} = \frac{j\omega\mu}{h^2}(n\pi/b)H_o\cos(m\pi x/a)\sin(n\pi y/b)\sin(p\pi z/c)$$

From Maxwell's equation,

$$-j\omega\mu \boldsymbol{H}_s = \nabla \times \boldsymbol{E}_s = \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_{xs} & E_{ys} & 0 \end{vmatrix}$$

$$H_{xs} = \frac{1}{j\omega\mu}\frac{\partial E_{ys}}{\partial z} \quad = -\frac{1}{h^2}(m\pi / a)(p\pi / c)H_o \sin(m\pi x / a)\cos(n\pi y / b)\cos(p\pi z / c)$$

Prob. 12.37 Maxwell's equation can be written as

$$H_{xs} = \frac{j\omega\varepsilon}{h^2}\frac{\partial E_{zs}}{\partial y} - \frac{\gamma}{h^2}\frac{\partial H_{zs}}{\partial x}$$

For a rectangular cavity,

$$h^2 = k_x{}^2 + k_y{}^2 = (m\pi / a)^2 + (n\pi / b)^2$$

For TM mode, $H_{zs} = 0$ and

$$E_{zs} = E_o \sin(m\pi x / a)\sin(n\pi y / b)\cos(p\pi z / c)$$

Thus

$$H_{xs} = \frac{j\omega\varepsilon}{h^2}\frac{\partial E_{zs}}{\partial y} \quad = \frac{j\omega\varepsilon}{h^2}(n\pi / b)E_o \sin(m\pi x / a)\cos(n\pi y / b)\cos(p\pi z / c)$$

as required.

$$H_{xs} = -\frac{j\omega\varepsilon}{h^2}\frac{\partial E_{zs}}{\partial x} - \frac{\gamma}{h^2}\frac{\partial H_{zs}}{\partial y}$$

$$= -\frac{j\omega\varepsilon}{h^2}(m\pi / a)E_o \cos(m\pi x / a)\sin(n\pi y / b)\cos(p\pi z / c)$$

From Maxwell's equation,

$$j\omega\varepsilon \boldsymbol{E}_s = \nabla \times \boldsymbol{H}_s = \begin{vmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ H_{xs} & H_{ys} & 0 \end{vmatrix}$$

$$E_{ys} = \frac{1}{j\omega\varepsilon}\frac{\partial H_{xs}}{\partial z} \quad = \frac{1}{h^2}(n\pi / b)(p\pi / c)E_o \sin(m\pi x / a)\cos(n\pi y / b)\cos(p\pi z / c)$$

Prob. 12.38

$$f_r = \frac{u'}{2}\sqrt{(m/a)^2 + (n/b)^2 + (p/c)^2}$$

where for TM mode to z, m = 1, 2, 3,…, n=1, 2, 3, …., p = 0, 1, 2, ….

and for TE mode to z, m = 0,1, 2, 3,…, n=0,1, 2, 3, …., p = 1, 2, 3, … , $(m+n) \neq 0$.

(a) If a < b < c, 1/a > 1/b > 1/c,

The lowest TM mode is TM_{110} with $f_r = \frac{u'}{2}\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$

The lowest TE mode is TE_{011} with $f_r = \frac{u'}{2}\sqrt{\frac{1}{b^2} + \frac{1}{c^2}} < \frac{u'}{2}\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$

Hence the dominant mode is TE_{011}.

(b) If a > b > c, 1/a < 1/b < 1/c,

The lowest TM mode is TM_{110} with $f_r = \frac{u'}{2}\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$

The lowest TE mode is TE_{101} with $f_r = \frac{u'}{2}\sqrt{\frac{1}{a^2} + \frac{1}{c^2}} > \frac{u'}{2}\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$

Hence the dominant mode is TM_{110}.

(c) If a = c > b, 1/a = 1/c < 1/b,

The lowest TM mode is TM_{110} with $f_r = \frac{u'}{2}\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$

The lowest TE mode is TE_{101} with $f_r = \frac{u'}{2}\sqrt{\frac{1}{a^2} + \frac{1}{c^2}} < \frac{u'}{2}\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$

Hence the dominant mode is TE_{101}.

Prob. 12.39

(a) Since a > b < c, the dominant mode is TE_{101}.

$$f_r = \frac{c}{2}\sqrt{\frac{1}{a^2}+\frac{1}{c^2}} = \frac{3\times10^8}{2}\sqrt{\frac{1}{(2\times10^{-2})^2}+\frac{1}{(3\times10^{-2})^2}} = \underline{\underline{9.014\text{ GHz}}}$$

(b) $$\delta = \frac{1}{\sqrt{\pi f_{r101}\mu_o\sigma}} = \frac{1}{\sqrt{\pi\times9.014\times10^9\times4\pi\times10^{-7}\times5.8\times10^7}}$$

$$= \frac{1}{\sqrt{4\pi^2\times5.8\times9.014\times10^9}}$$

$$Q = \frac{(a^2+c^2)abc}{\delta\left[2b(a^3+c^3)+ac(a^2+c^2)\right]} = \frac{(4+9)6\times10^{-2}}{\delta\left[2(8+27)+6(4+9)\right]} = \frac{0.78}{148\delta}$$

$$= \frac{0.78}{148}\sqrt{4\pi^2\times5.8\times9.014\times10^9} = \underline{\underline{7571.5}}$$

Prob. 12.40

$$u' = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{\sqrt{2.5}} = \frac{3\times10^8}{\sqrt{2.5}} = 1.897\times10^8$$

$$f_c = \frac{u'}{2}\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2+\left(\frac{p}{c}\right)^2} = \frac{1.897\times10^8\times10^2}{2}\sqrt{\left(\frac{m}{1}\right)^2+\left(\frac{n}{2}\right)^2+\left(\frac{p}{3}\right)^2}$$

$$= 9.485\sqrt{m^2+0.25n^2+0.111p^2}\quad\text{GHz}$$

$$f_{r101} = 9.485\sqrt{1+0+0.111} = 10\text{ GHz}$$

$$f_{r011} = 9.485\sqrt{0+0.25+0.111} = 5.701\text{ GHz}$$

$$f_{r012} = 9.485\sqrt{0+0.25+0.444} = 7.906\text{ GHz}$$

$$f_{r013} = 9.485\sqrt{0+0.25+0.999} = 10.61\text{ GHz}$$

$$f_{r021} = 9.485\sqrt{0+1+0.111} = 10\text{ GHz}$$

Thus, the first five resonant frequencies are:

5.701 GHz(TE_{011})

7.906 GHz (TE_{012})

10 GHz (TE_{101} and TE_{021})

10.61 GHz (TE_{013} or TM_{110})

11.07 GHz (TE_{111} or TM_{111})

Prob. 12.41

$$Q = \frac{(a^2 + c^2)abc}{\delta\left[2b(a^3 + c^3) + ac(a^2 + c^2)\right]}$$

When a = b = c,

$$Q = \frac{2a^2a^3}{\delta\left[2a \times 2a^3 + a^2 \times 2a^2\right]} = \frac{2a^5}{6\delta a^4} = \underline{\underline{\frac{a}{3\delta}}}$$

Prob. 12.42

(a) Since a > b < c, the dominant mode is TE_{101}

$$f_{r101} = \frac{u'}{2}\sqrt{\frac{1}{a^2} + 0 + \frac{1}{c^2}} = \frac{3\times10^8 \times 10^2}{2}\sqrt{\frac{1}{2^2} + \frac{1}{1^2}} = \underline{\underline{16.77 \text{ GHz}}}$$

(b) $$Q_{TE101} = \frac{(a^2 + c^2)abc}{\delta\left[2b(a^3 + c^3) + ac(a^2 + c^2)\right]}$$

$$= \frac{(400+100)20\times8\times10\times10^{-3}}{\delta[16(8000+1000)+200(400+100)]} = \frac{3.279\times10^{-3}}{\delta}$$

But $$\delta = \frac{1}{\sqrt{\pi f_{r101}\mu_o\sigma}} = \frac{1}{\sqrt{\pi 16.77\times10^9 \times 4\pi\times10^{-7}\times6.1\times10^7}} = \frac{10^{-4}}{200.961}$$

$$Q_{TE101} = 3.279\times10^{-3}\frac{200.961}{10^{-4}} = \underline{\underline{6589.51}}$$

Prob. 12.43

$$f_r = \frac{c}{2a}\sqrt{m^2 + n^2 + p^2}$$

The lowest possible modes are TE_{101}, TE_{011}, and TM_{110}. Hence

$$f_r = \frac{c}{2a}\sqrt{2} \longrightarrow a = \frac{c}{f_r\sqrt{2}} = \frac{3\times10^8}{\sqrt{2}\times3\times10^9} = 7.071 \text{ cm}$$

$\underline{\underline{a = b = c = 7.071 \text{ cm}}}$

Prob. 12.44

(a) a = b = c

$$f_r = \frac{u'}{2a}\sqrt{m^2+n^2+p^2}$$

For the dominant mode TE_{101},

$$f_r = \frac{u'}{2a}\sqrt{1+1} = \frac{c}{2a}\sqrt{2}$$

$$a = \frac{c\sqrt{2}}{2f_r} = \frac{3\times10^8\sqrt{2}}{2\times5.6\times10^9} = 0.03788 \text{ m}$$

$$\underline{\underline{a = b = c = 3.788 \text{ cm}}}$$

(b)

For $\varepsilon_r = 2.05$, $\quad u' = \dfrac{c}{\sqrt{\varepsilon_r}}$

$$a = \frac{c\sqrt{2}}{2f_r\sqrt{\varepsilon_r}} = \frac{0.03788}{\sqrt{2.05}} = 0.02646$$

$$\underline{\underline{a = b = c = 2.646 \text{ cm}}}$$

Prob. 12.45

(a)

This is a TM mode to z. From Maxwell's equations,

$$\nabla\times\boldsymbol{E}_s = -j\omega\mu\boldsymbol{H}_s$$

$$\boldsymbol{H}_s = -\frac{1}{j\omega\mu}\nabla\times\boldsymbol{E}_s = \frac{j}{\omega\mu}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & E_{zs}(x,y) \end{vmatrix} \quad = \frac{j}{\omega\mu}\left(\frac{\partial E_{zs}}{\partial y}\boldsymbol{a}_x - \frac{\partial E_{zs}}{\partial x}\boldsymbol{a}_y\right)$$

But

$$E_{zs} = 200\sin30\pi x\sin30\pi y,\ \ \frac{1}{\omega\mu} = \frac{1}{6\times10^9\times4\pi\times10^{-7}} = \frac{10^{-2}}{24\pi}$$

$$\boldsymbol{H}_s = \frac{j10^{-2}}{24\pi}\times200\times30\pi\left\{\sin30\pi x\cos30\pi y\boldsymbol{a}_x - \cos30\pi x\sin30\pi y\boldsymbol{a}_y\right\}$$

$$\mathbf{H} = \text{Re}\,(\mathbf{H}_s\, e^{j\omega t})$$

$$\underline{\underline{\boldsymbol{H} = 2.5\left\{-\sin30\pi x\cos30\pi y\boldsymbol{a}_x + \cos30\pi x\sin30\pi y\boldsymbol{a}_y\right\}\sin6\times10^9\pi t}} \ \text{ A/m}$$

(b) $\boldsymbol{E} = E_z\boldsymbol{a}_z, \qquad \boldsymbol{H} = H_x\boldsymbol{a}_x + H_y\boldsymbol{a}_y$

$\boldsymbol{E} \bullet \boldsymbol{H} = 0$

Prob. 12.46

(a) $a = b = c \quad \longrightarrow \quad f_{r101} = \dfrac{3\times10^8}{a\sqrt{2}} = 12\times10^9$

$$a = \frac{3\times10^8}{\sqrt{2}\times12\times10^9} = \underline{\underline{1.77 \text{ cm}}}$$

(b) $Q_{TE101} = \dfrac{a}{3\delta} = \dfrac{a\sqrt{\pi f_{r101}\mu\sigma}}{3}$

$$= \frac{1.77\times10^{-2}\sqrt{\pi\times12\times10^9\times4\pi\times10^{-7}\times5.8\times10^7}}{3} = \underline{\underline{9767.61}}$$

Prob. 12.47

$$f_r = \frac{u'}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{c}\right)^2}$$

$$f_{r101} = \frac{3\times10^8}{2}\sqrt{\frac{1}{(10.2)^2} + \frac{1}{(3.6)^2}} = 44.186 \text{ MHz}$$

$$f_{r011} = 150\sqrt{\frac{1}{(8.7)^2} + \frac{1}{(3.6)^2}} \text{ MHz} = 45.093 \text{ MHz}$$

$$f_{r111} = 150\sqrt{\frac{1}{(10.2)^2} + \frac{1}{(8.7)^2} + \frac{1}{(3.6)^2}} \text{ MHz} = 47.43 \text{ MHz}$$

$$f_{r110} = 150\sqrt{\frac{1}{(10.2)^2} + \frac{1}{(8.7)^2}} \text{ MHz} = 22.66 \text{ MHz}$$

$$f_{r102} = 150\sqrt{\frac{1}{(10.2)^2} + \frac{4}{(3.6)^2}} \text{ MHz} = 84.62 \text{ MHz}$$

$$f_{r201} = 150\sqrt{\frac{4}{(10.2)^2} + \frac{1}{(3.6)^2}} \text{ MHz} = 51 \text{ MHz}$$

Thus, the resonant frequences below 50 MHz are

$\underline{\underline{f_{r110}, f_{r101,} f_{r011}, \text{ and } f_{r111}}}$

Prob. 12.48

$$\text{n} = \text{c/u}_\text{m} = \frac{3\times10^8}{2.1\times10^8} = \underline{1.4286}$$

Prob. 12.49 When an optical fiber is used as the transmission medium, cable radiation is eliminated. Thus, optical fibers offer total EMI isolation because they neither emit nor pick up EM waves.

Prob. 12.50

(a) NA = $\sqrt{n_1^2 - n_2^2} = \sqrt{1.62^2 - 1.604^2} = \underline{\underline{0.2271}}$

(b) NA = sin θ_a = 0.2271 or $\theta_a = \sin^{-1} 0.2271 = \underline{\underline{13.13^o}}$

(c) $V = \dfrac{\pi d}{\lambda}\text{ NA } = \dfrac{\pi \times 50\times 10^{-6} \times 0.2271}{1300\times 10^{-9}} = 27.441$

N = $V^2/2$ $\underline{\underline{6 \text{ modes}}}$

Prob. 12.51

$$V = \frac{\pi d}{\lambda}\sqrt{n_1^2 - n_2^3} = \frac{\pi \times 2\times 5\times 10^{-6}}{1300\times 10^{-9}}\sqrt{1.48^2 - 1.46^2} = 5.86$$

$$N = \frac{V^2}{2} = 17.17 \text{ or } \underline{\underline{17 \text{ modes}}}$$

Prob. 12.52

(a) NA = sin $\theta_a = \sqrt{n_1^2 - n_2^2} = \sqrt{1.53^2 - 1.45^2} = 0.4883$

$\theta_a = \sin^{-1} 0.4883 = \underline{\underline{29.23^o}}$

(b) P(l)/P(0) = $10^{-\alpha l/10} = 10^{-0.4X5/10} = 0.631$

i.e. $\underline{\underline{63.1\ \%}}$

Prob. 12.53

$$P(\ell) = P(0)10^{-\alpha\ell/10} = 10\times 10^{-0.5\times 0.85/10} = \underline{\underline{9.0678 \text{ mW}}}$$

Prob. 12.54 As shown in Eq. (10.35), $\log_{10} P_1/P_2 = 0.434 \ln P_1/P_2$,

1 Np = 20 $\log_{10}$ e = 8.686 dB or 1 Np/km = 8.686 dB/km,

or 1Np/m = 8686 dB/km. Thus,

$$\alpha_{12} = \underline{\underline{8686\alpha_{10}}}$$

Prob. 12.55

$$\alpha\ell = 10\log_{10}\frac{P_{in}}{P_{out}} = 10\log_{10}\frac{1.2\times10^{-3}}{1\times10^{-6}} = 30.792$$

$$\alpha = 0.4 \text{ dB/km} = \frac{0.4}{8.686} \quad \text{Np/km}$$

$$\ell = \frac{30.792}{\alpha} = \frac{30.392 \text{ dB}}{0.4 \text{ dB/km}} = \underline{\underline{76.98 \text{ km}}}$$

CHAPTER 13

P.E. 13.1

(a) For this case, r is at near field.

$$H_{\phi s} = \frac{I_o dl \sin\theta}{4\pi}\left(\frac{j\beta}{r} + \frac{1}{r^2}\right)e^{-j\beta r}, \quad \beta r = \frac{2\pi}{\lambda}.\frac{\lambda}{5} = 72^o$$

$$\lambda = \frac{2\pi c}{\omega} = \frac{2\pi \times 3\times 10^8}{10^8} = 6\pi, \qquad \beta = \frac{2\pi}{\lambda} = \frac{1}{3}$$

$$H_{\phi s} = \frac{(0.25)\dfrac{6\pi}{100}\sin 30^o}{4\pi}\left(\frac{j1/3}{6\pi/5} + \frac{1}{(6\pi/5)^2}\right)e^{-j72^o} = 0.2119\angle -20.511^o \text{ mA/m}$$

$\boldsymbol{H} = \text{Im}\left(H_{\phi s}e^{j\omega t}\,\boldsymbol{a}_\phi\right)$ Im is used since I = $I_o \sin \omega t$

$$= \underline{\underline{0.2119\sin(10^8 - 20.5^o)\boldsymbol{a}_\phi \text{ mA/m}}}$$

(b) For this case, r is at far field. $\beta = \frac{2\pi}{\lambda}\times 200\lambda = 0^o$

$$H_{\phi s} = \frac{j(0.25)(\dfrac{2\pi}{\lambda})\dfrac{\lambda}{100}Sin60^o e^{-j0^o}}{4\pi(6\pi\times 200)} = 0.2871e^{j90^o} \quad \mu Am$$

$$\boldsymbol{H} = \text{Im}\,(H_{\phi s}\boldsymbol{a}_\phi e^{j\omega t}) \quad = \underline{\underline{0.2871\sin(10^8 + 90^o)\boldsymbol{a}_\phi}} \quad \mu Am\,.$$

P. E. 13.2

(a) $l = \frac{\lambda}{4} = \underline{\underline{1.5m}}$,

(b) I_o = $\underline{\underline{83.3\text{mA}}}$

(c) R_{rad} = 36.56 Ω, $P_{rad} = \frac{1}{2}(0.0833)^2 36.56$

$= \underline{\underline{126.8 \text{ mW}}}$.

(d) Z_L = 36.5 + j21.25,

$$\Gamma = \frac{36.5 + j21.25 - 75}{36.5 + j21.25 + 75} = 0.3874\angle 140.3^o$$

$$s = \frac{1+0.3874}{1-0.3874} = \underline{\underline{2.265}}$$

P.E. 13.3

$$D = \frac{4\pi U_{max}}{P_{rad}}$$

(a) For the Hertzian monopole

$$U(\theta,\phi) = \sin^2\theta, \; 0<\theta<\pi/2, \quad 0<\phi<2\pi, \; U_{max} = 1$$

$$P_{rad} = \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} \sin^2 \sin\theta d\theta d\phi = \frac{4\pi}{3}$$

$$D = \frac{4\pi(1)}{4\pi/3} = \underline{\underline{3}}$$

(b) For the $\frac{\lambda}{4}$ monopole,

$$U(\theta,\phi) = \frac{\cos^2(\frac{\pi}{2}\cos\theta)}{\sin^2\theta}, \; U_{max} = 1$$

$$P_{rad} = \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} \frac{\cos^2(\frac{\pi}{2}\cos\theta)}{\sin^2\theta} \sin\theta d\theta d\phi = 2\pi(0.609)$$

$$D = \frac{4\pi(1)}{2\pi(0.609)} = \underline{\underline{3.28}}$$

P. E. 13.4

(a) $P_{rad} = \eta_r P_{in} = 0.95(0.4)$

$$D = \frac{4\pi U_{max}}{P_{rad}} = \frac{4\pi(0.5)}{0.4\times 0.95} = \underline{\underline{16.53}}$$

(b) $D = \frac{4\pi(0.5)}{0.3} = \underline{\underline{20.94}}$

P. E. 13.5

$$P_{rad} = \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} \sin\theta \sin\theta d\theta d\phi = \frac{\pi^2}{2}, \quad U_{max} = 1$$

$$D = \frac{4\pi(1)}{\pi^2/2} = \underline{\underline{2.546}}$$

P. E. 13.6

(a) $f(\theta) = |\cos\theta| \cos\left[\frac{1}{2}(\beta d\cos\theta + \alpha)\right]$

where $\alpha = \pi$, $\beta d = \frac{2\pi}{\lambda}.\frac{\lambda}{2} = \pi$

$$\underbrace{|\cos\theta|}_{\text{unit pattern}}\ \underbrace{\cos\left[\frac{1}{2}(\pi\cos\theta + \pi)\right]}_{\text{group pattern}} = f(\theta)$$

For the group pattern, we have nulls at

$$\pi/2(\cos\theta + 1) = \pm\pi/2 \longrightarrow \theta = \pm\pi/2$$

and maxima at

$$\pi/2(\cos\theta + 1) = 0, \pi \longrightarrow \cos\theta = -1, 1 \longrightarrow \theta = 0, \pi$$

Thus the group pattern and the resultant patterns are as shown in Fig.13.15(a)

(b) $f(\theta) = |\cos\theta| \cos\left[\frac{1}{2}(\beta d\cos\theta + \alpha)\right]$

where $\alpha = -\pi/2$, $\beta d = \pi/2$

$$f(\theta) = \underbrace{|\cos\theta|}_{\text{unit pattern}}\ \underbrace{\cos\left[\frac{1}{2}\left(\frac{\pi}{2}\cos\theta - \pi/2\right)\right]}_{\text{group pattern}}$$

For the group pattern, the nulls are at

$$\frac{\pi}{4}(\cos\theta - 1) = -\frac{\pi}{2} \quad \longrightarrow \quad \theta = 180^o$$

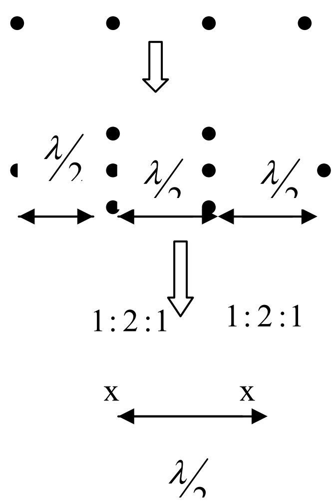
and maxima at

$$\cos\theta - 1 = 0 \quad \longrightarrow \quad \theta = 0$$

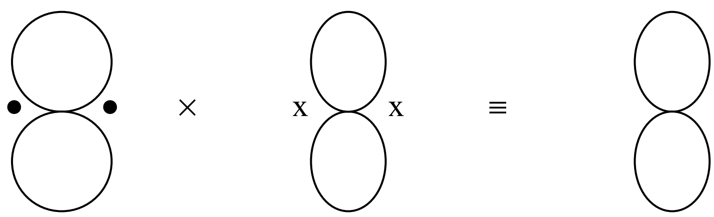
Thus the group pattern and the resultant patterns are as shown in Fig.13.15(b)

P. E. 13.7

(a)



Thus, we take a pair at a time and multiply the patterns as shown below.



(b) The group pattern is the normalized array factor, i.e.

$$(AF)_n = \frac{1}{\sum}\left|1 + Ne^{i\psi} + \frac{N(N-1)}{2!}e^{i2\psi} + \frac{N(N-1)(N-2)}{3!}e^{i3\psi} + \ldots\ldots\ldots\ldots + e^{i(N-1)\psi}\right|$$

where $\sum = \sum_{i-1}^{N-1}\binom{N}{i} = 1 + N + \frac{N-1}{2!} + \frac{N(N-1)(N-2)}{3!} + \ldots\ldots\ldots\ldots$

$$= (1+1)^{N-1} = 2^{N-1}$$

$$(AF)_n = \frac{1}{2^{N-1}}\left|1 + e^{j\psi}\right|^{N-1} = \frac{1}{2^{N-1}}\left|e^{j\psi/2}\right|\left|e^{-j\psi/2} + e^{j\psi/2}\right|^{N-1}$$

$$= \frac{1}{2^{N-1}} \left| 2\cos\frac{\psi}{2} \right|^{N-1} = \underline{\underline{\left| \cos\frac{\psi}{2} \right|^{N-1}}}$$

P. E. 13.8

$$A_e = \frac{\lambda^2}{4\pi} G_d, \quad \lambda = \frac{c}{f} = \frac{3\times 10^8}{10^8} = 3 \text{ m}$$

For the Hertzian dipole,

$$G_d = 1.5 \sin^2\theta$$

$$A_e = \frac{\lambda^2}{4\pi}(1.5\sin^2\theta)$$

$$A_{e,\max} = \frac{1.5\lambda^2}{4\pi} = \frac{1.5\times 9}{4\pi} = \underline{\underline{1.074 \text{ m}^2}}$$

By definition,

$$P_r = A_e P_{ave} \longrightarrow P_{ave} = \frac{P_r}{A_e} = \frac{3\times 10^{-6}}{1.074}$$

$$= \underline{\underline{2.793\ \mu\text{W / m}^2}}$$

P. E. 13.9

(a) $$G_d = \frac{4\pi r^2 P_{ave}}{P_{rad}} = \frac{4\pi r^2 \frac{1}{2}\frac{E^2}{\eta}}{P_{rad}} = \frac{2\pi r^2 E^2}{\eta P_{rad}}$$

$$= \frac{2\pi \times 400\times 10^6 \times 144\times 10^{-6}}{120\pi \times 100\times 10^3} = 0.0096$$

$$G = 10\log_{10} G_d = \underline{\underline{\text{-20.18 dB}}}$$

(b) $G = \eta_r G_d = 0.98\times 0.0096 = \underline{\underline{9.408\times 10^{-3}}}$

P. E. 13.10

$$r = \left[\frac{\lambda^2 G_d{}^2 \sigma}{(4\pi)^3} \frac{P_{rad}}{P_r} \right]^{1/4}$$

where $\lambda = \dfrac{c}{f} = \dfrac{3\times 10^8}{6\times 10^9} = 0.05m$

$$A_e = 0.7\pi a^2 = 0.7\pi(1.8)^2 = 7.125m^2$$

$$G_d = \frac{4\pi A_e}{\lambda^2} = \frac{4\pi(7.125)}{25\times10^{-4}} = 3.581\times10^4$$

$$r = \left[\frac{25\times10^{-4}\times(3.581)^2\times10^8\times5\times60\times10^3}{(4\pi)^3\times0.26\times10^{-3}}\right]^{1/4}$$

$$= 1168.4\text{m} = \underline{\underline{0.631\text{ nm}}}$$

At $r = {r_{\max}}/{2} = 584.2m,$

$$P = \frac{G_d P_{rad}}{4\pi r^2} = \frac{3.581\times10^4\times60\times10^3}{4\pi(584.2)^2} = \underline{\underline{501\text{ W/m}^2}}$$

Prob. 13.1

Using vector transformation,

$A_{rs} = A_{xs}\sin\theta\cos\phi$, $A_{\theta s} = A_{xs}\cos\theta\cos\phi$, $A_{\phi s} = -A_{xs}\sin\phi$

$$\boldsymbol{A}_s = \frac{50e^{-j\beta r}}{r}(\sin\theta\cos\phi\boldsymbol{a}_r + \cos\theta\cos\phi\boldsymbol{a}_\theta - \sin\phi\boldsymbol{a}_\phi)$$

$$\frac{\nabla\times\boldsymbol{A}_s}{\mu} = \boldsymbol{H}_s = \frac{100\cos\theta\sin\phi}{\mu r^2\sin\theta}e^{-j\beta r}\boldsymbol{a}_r - \frac{50}{\mu r^2}(1-j\beta r)\sin\phi e^{-j\beta r}\boldsymbol{a}_\theta$$
$$-\frac{50}{\mu r^2}\cos\theta\cos\phi(1+j\beta r)e^{-j\beta r}\boldsymbol{a}_\phi$$

At far field, only $\frac{1}{r}$ term remains. Hence

$$\boldsymbol{H}_s = \frac{j50}{\mu r}\beta e^{-j\beta r}(\sin\phi\boldsymbol{a}_\theta - \cos\theta\cos\phi\boldsymbol{a}_\phi)$$

$$\boldsymbol{E}_s = -\eta\boldsymbol{a}_r\times\boldsymbol{H}_s = \frac{-j50\beta\eta e^{-j\beta r}}{\mu r}(\sin\phi\boldsymbol{a}_\phi + \cos\theta\cos\phi\boldsymbol{a}_\theta)$$

$$\boldsymbol{E} = \mathrm{Re}\left[\boldsymbol{E}_s e^{j\omega t}\right] = \underline{\underline{\frac{50\eta\beta}{\mu r}\sin(\omega t - \beta r)(\sin\phi \boldsymbol{a}_\phi + \cos\theta\cos\phi \boldsymbol{a}_\theta)}} \text{ V/m}$$

$$\boldsymbol{H} = \mathrm{Re}\left[\boldsymbol{H}_s e^{j\omega t}\right] = \underline{\underline{\frac{-50}{\mu r}\beta\sin(\omega t - \beta r)(\sin\phi \boldsymbol{a}_\theta - \cos\theta\cos\phi \boldsymbol{a}_\phi)}} \text{ A/m}$$

Prob. 13.2

(a) $\lambda = \dfrac{c}{f} = \dfrac{3\times 10^8}{400\times 10^6} = 0.75 \text{ m}$

$$\mathrm{r}_{\min} = \frac{2d^2}{\lambda} = \frac{2(0.02\lambda)^2}{\lambda} \ll r$$

i.e. r is in the far field.

$$H_{\phi s} = \frac{jI_o\beta dl}{4\pi r}\sin\theta e^{-j\beta r}$$

$$|H_{\phi s}| = \frac{I_o\beta dl}{4\pi r}\sin\theta = \frac{3\times\dfrac{2\pi}{\lambda}\times 0.02\lambda\times\sin 90^o}{4\pi(60)} = 5\times 10^{-4} = 0.5 \text{ mA/m}$$

$$|E_{\theta s}| = \eta_o\,|H_{\phi s}| = \underline{\underline{0.1885 \text{ V/m}}}$$

(b) $|H_{\phi s}| = 0.5 \text{ mA/m}|$

(c) $\mathrm{R}_{\mathrm{rad}} = 80\pi^2\left(\dfrac{dl}{\lambda}\right)^2 = 80\pi^2(0.02)^2 = \underline{\underline{0.3158\ \Omega}}$

(d) $P_{rad} = \dfrac{1}{2}|I_o|^2\,\mathrm{R}_{\mathrm{rad}} = \dfrac{1}{2}(9)(0.3158) = \underline{\underline{1.421 \text{ W}}}$

Prob. 13.4

$$\lambda = \frac{c}{f} = \frac{3\times 10^8}{50\times 10^6} = 6 \text{ m}$$

$$\mathrm{r}_{\min} = \frac{2d^2}{\lambda} = \frac{2(5\times 10^{-3})^2}{6} = 8.333\times 10^{-6} \text{ m} \ll r = 15 \text{ cm}$$

i.e. r is in the far field.

$$H_{\phi s} = \frac{jI_o \beta dl}{4\pi r}\sin\theta e^{-j\beta r}$$

$$|H_{\phi s}| = \frac{I_o \beta dl}{4\pi r}\sin\theta = \frac{2\times\frac{2\pi}{6}\times 5\times 10^{-3}\times \sin 30^o}{4\pi\times 15\times 10^{-2}} = \frac{5}{18}\times 10^{-2} = \underline{\underline{2.778 \text{ mA/m}}}$$

$$|E_{\theta s}| = \eta_o |H_{\phi s}| = 377\times 2.778\times 10^{-3} = \underline{\underline{1.047 \text{ V/m}}}$$

Prob. 13.5

(a) $$A_{zs} = \frac{e^{-j\beta r}}{4\pi r}\int_{-l/2}^{l/2} I_o(1-\frac{2|z|}{l})e^{j\beta z\cos\theta}\partial z$$

$$= \frac{e^{-j\beta r}}{4\pi r}I_o\left[\int_{-l/2}^{l/2}(1-\frac{2|z|}{l})\cos(\beta z\cos\theta)dz + j\int_{-l/2}^{l/2}(1-\frac{2|z|}{l})\sin(\beta z\cos\theta)dz\right]$$

$$= \frac{e^{-j\beta r}}{4\pi r}2I_o\int_0^{l/2}(1-\frac{2z}{l})\cos(\beta z\cos\theta)dz$$

$$= \frac{I_o e^{-j\beta r}}{2\pi r\beta^2\cos^2\theta}.\frac{2}{l}\left[1-\cos(\beta l/2\cos\theta)\right]$$

$$\boldsymbol{E}_s = -j\omega\mu \boldsymbol{A}_s \longrightarrow \qquad E_{\theta s} = j\omega\mu\sin\theta A_{zs} = j\beta\eta\sin\theta A_{zs}$$

$$E_{\theta s} = \frac{j\eta I_o e^{-j\beta r}}{\pi r l}\frac{\sin\theta\left[1-\cos(\beta l/2\cos\theta)\right]}{\beta\cos^2\theta}$$

If $\beta l/2 \langle\langle 1$, $\cos(\beta l/2\cos\theta) = 1 - \dfrac{(\frac{\beta l}{2}\cos\theta)^2}{2!}$.

Hence

$$E_{\theta s} = \frac{j\eta I_o}{8\pi r}\beta l e^{-j\beta r}\sin\theta \ , \quad H_{\phi s} = E_{\theta s}/\eta$$

$$P_{ave} = \frac{|E_{\theta s}|^2}{2\eta}, \qquad P_{rad} = \int P_{ave}dS$$

$$P_{rad} = \int_0^{2\pi}\int_0^{\pi}\frac{n}{2}\left(\frac{I_o\beta l}{8\pi}\right)^2\frac{1}{r^2}\sin^2\theta r^2\sin\theta d\theta d\phi$$

$$= 10\pi^2 I_o^2\left(\frac{l}{\lambda}\right)^2 = 1/2\, I_o^2 R_{rad}$$

$$\text{or} \quad R_{rad} = 20\pi^2\left(\frac{l}{\lambda}\right)^2$$

(b) $0.5 = 20\pi^2\left(\frac{l}{\lambda}\right)^2 \longrightarrow \quad \underline{\underline{l = 0.05\lambda}}$

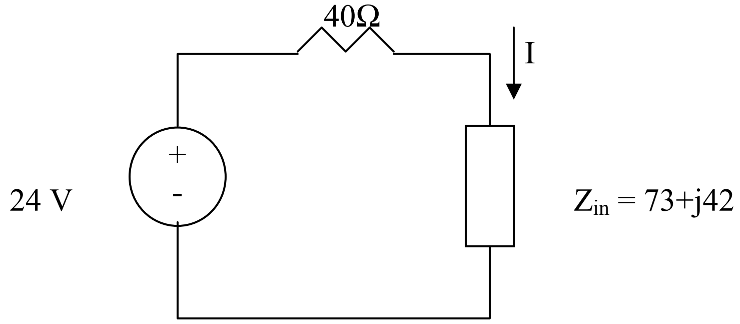
Prob. 13.6

$$\lambda = \frac{c}{f} = \frac{3\times10^8}{3\times10^6} = 100 \text{ m}$$

$\ell = 4$ m $\ll \lambda$. Hence the antenna is electrically short.

$$P_{rad} = \frac{1}{2}I_o^2 R_{rad} = 40\left(\frac{\pi I_o dl}{\lambda}\right)^2 = 40\left(\frac{\pi\times3\times4}{100}\right)^2 = \underline{\underline{5.685 \text{ W}}}$$

Prob. 13.8



$$I = \frac{V}{R_s + Z_{in}} = \frac{24}{40+73+j42} = 0.1866 - j0.0694$$

$$P_{rad} = \frac{1}{2}|I|^2 R_{rad}, \qquad R_{rad} = 73$$

$$P_{rad} = \frac{1}{2}(0.1991)^2\times73 = \underline{\underline{1.447 \text{ W}}}$$

Prob. 13.9

Change the limits in Eq. (13.16) to $\pm l/2$ i.e.

$$A_s = \frac{\mu I_o e^{j\beta z Cos\theta}}{4\pi r}\frac{\left(j\beta\cos\theta\cos\beta t+\beta\sin\beta t\right)}{-\beta^2\cos^2\theta+\beta^2}\Bigg|_{-l/2}^{l/2}$$

$$=\frac{\mu I_o e^{j\beta r}}{2\pi r}\frac{1}{\beta\sin^2\theta}\left[\sin\frac{\beta l}{2}\cos\left(\frac{\beta l}{2}\cos\theta\right)-\cos\theta\cos\frac{\beta l}{2}\sin\left(\frac{\beta l}{2}\cos\theta\right)\right]$$

But $\boldsymbol{B}=\mu\boldsymbol{H}=\nabla\times\boldsymbol{A}$

$$H_{\phi s}=\frac{1}{\mu r}\left[\frac{\partial}{\partial r}(rA_\theta)-\frac{\partial A_r}{\partial\theta}\right],$$

where $A_o = -A_z\sin\theta, A_r = A_z\cos\theta$

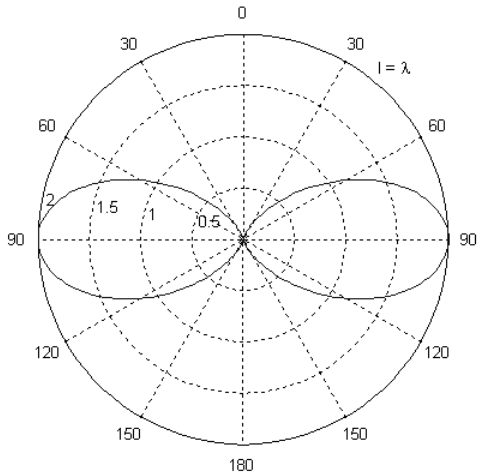
$$H_{\phi s}=\frac{I_o}{2\pi r}\frac{e^{-j\beta r}}{\beta}\left(\frac{j\beta}{\sin\theta}\right)\left[\sin\frac{\beta l}{2}\cos\left(\frac{\beta l}{2}\cos\theta\right)-\cos\theta\cos\frac{\beta l}{2}\sin\left(\frac{\beta l}{2}\cos\theta\right)\right]+\frac{I_o}{2\pi r^2}e^{-j\beta r}(......)$$

For far field, only the $\frac{1}{r}$-term remains. Hence

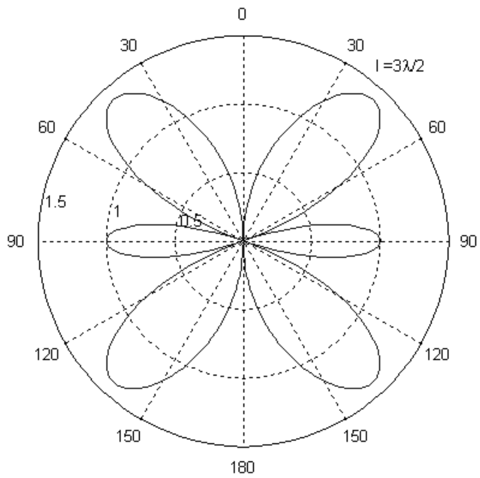
$$H_{\phi s}=\frac{jI_o}{2\pi r}e^{-j\beta r}\frac{\left[\sin\frac{\beta l}{2}\cos\left(\frac{\beta l}{2}\cos\theta\right)-\cos\theta\cos\frac{\beta l}{2}\sin\left(\frac{\beta l}{2}\cos\theta\right)\right]}{\sin\theta}$$

(b) $f(\theta)=\dfrac{\cos\left(\frac{\beta l}{2}\cos\theta\right)-\cos\frac{\beta l}{2}}{\sin\theta}$

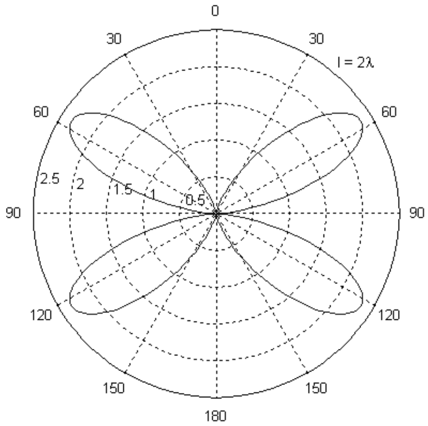
For $l=\lambda, f(\theta)=\dfrac{\cos(\pi\cos\theta)+1}{\sin\theta}$



For $l = \dfrac{3\lambda}{2}, f(\theta) = \dfrac{\cos\left(\dfrac{3\pi}{2}\cos\theta\right)}{\sin\theta}$



For $l = 2\lambda,\ f(\theta) = \dfrac{\cos\theta\sin\left(2\pi\cos\theta\right)}{\sin\theta}$



Prob. 13.10

(a)

$$\lambda = \frac{c}{f} = \frac{3\times10^8}{450\times10^6} = 0.6667 \text{ m}$$

$$\ell = \frac{\lambda}{2} = \underline{\underline{0.333 \text{ m}}}$$

(b)

$$\frac{\sigma}{\omega\varepsilon} = \frac{4}{2\pi\times450\times10^6\times81\times\dfrac{10^{-9}}{36\pi}} = 1.975$$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}+1\right]} = \frac{2\pi\times460\times10^6}{c}\sqrt{\frac{81}{2}\left[\sqrt{1+(1.975)^2}+1\right]}$$

$$= \frac{2\pi\times460\times10^6}{3\times10^8}\times11.4086 = 109.91$$

$$\lambda = \frac{2\pi}{\beta} = 0.0572$$

$$\ell = \frac{\lambda}{2} = \underline{\underline{28.58 \text{ mm}}}$$

Prob. 13.11

(a)

$$\lambda = \frac{c}{f} = \frac{3\times10^8}{1.150\times10^6} = 260.8 \text{ m}$$

$$\ell = \frac{\lambda}{4} = \underline{\underline{65.22 \text{ m}}}$$

(b)

$$\lambda = \frac{c}{f} = \frac{3\times10^8}{90\times10^6} = 3.333 \text{ m}$$

$$\ell = \frac{\lambda}{4} = \underline{\underline{0.8333 \text{ m}}}$$

(c)

$$\lambda = \frac{c}{f} = \frac{3\times10^8}{80\times10^6} = 3.75 \text{ m}$$

$$\ell = \frac{\lambda}{4} = \underline{\underline{0.9375 \text{ m}}}$$

(d)

$$\lambda = \frac{c}{f} = \frac{3\times10^8}{600\times10^6} = 0.5 \text{ m}$$

$$\ell = \frac{\lambda}{4} = \underline{\underline{0.125 \text{ m}}}$$

Prob. 13.13

(a) $\lambda = \frac{c}{f} = \frac{3\times10^8}{10\times10^6} = 30 \text{ m}$

$$E_{\max} = \frac{\eta\pi I_o S}{r\lambda^2} \longrightarrow I_o = \frac{E_{\max} r\lambda^2}{\eta\pi S}$$

$$I_o = \frac{50\times10^{-3}\times3\times30^2}{120\pi^2\pi(0.2)^2 100} = \underline{\underline{9.071 \text{ mA}}} \quad (S = N\pi r^2)$$

(b) $R_{rad} = \frac{320\pi^4 S^2}{\lambda^4} = \frac{320\pi^4\pi^2(0.2)^4\times10^4}{30^4} = 6.077\Omega$

$$P_{rad} = \frac{1}{2}I_o{}^2 R_{rad} = \frac{1}{2}(9.071)^2\times10^{-6}\times6.077$$

$$= \underline{\underline{0.25 \text{ mW}}}$$

Prob. 13.14

$$\lambda = \frac{c}{f} = \frac{3\times10^8}{80\times10^6} = 3.75 \text{ m}$$

$$S = N\pi\rho_o^2$$

$$R_{rad} = \frac{320\pi^4 S^2}{\lambda^4} = \frac{320\pi^4 N^2\pi^2\rho_o^4}{\lambda^4} \longrightarrow N^2 = \frac{\lambda^4 R_{rad}}{320\pi^6(1.2\times10^{-2})^4}$$

$$N^2 = \frac{(3.75)^4\times 8}{320\pi^6(1.2\times10^{-2})^4} = 248006 \longrightarrow N \simeq \underline{\underline{498}}$$

Prob. 13.15

(a) $$R_{rad} = \frac{320\pi^4 S^2}{\lambda^4}$$

$$S = \pi\rho_o^2 = \pi(0.4)^2 = 0.5027 \text{ m}^2$$

$$\lambda = \frac{c}{f} = \frac{3\times10^8}{6\times10^6} = 50 \text{ m}$$

$$R_{rad} = \frac{320\pi^4(0.5027)^2}{(50)^4} = \underline{\underline{1.26 \text{ m}\Omega}}$$

(b) $$P_{rad} = \frac{1}{2}I_o{}^2 R_{rad} = \frac{1}{2}(50)^2\times1.26\times10^{-3} = \underline{\underline{1.575 \text{ W}}}$$

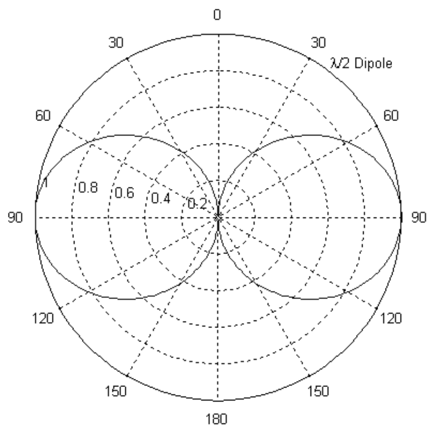
(c) $$\mathrm{R}_{\ell} = \frac{a}{2\delta}\,\mathrm{R}_{dc} = \frac{a}{2\delta}\frac{\ell}{\sigma S} = \frac{a\ell}{2\sigma\pi a^2}\sqrt{\pi f\mu\sigma} = \frac{2\pi R}{2\pi a}\sqrt{\frac{\mu f\pi}{\sigma}}$$

$$\mathrm{R}_{\ell} = \frac{R}{a}\sqrt{\frac{\mu f\pi}{\sigma}} = \frac{0.4}{4\times10^{-3}}\sqrt{\frac{4\pi\times10^{-7}\times6\times10^6\times\pi}{5.8\times10^7}} = 63.91 \text{ m}\Omega$$

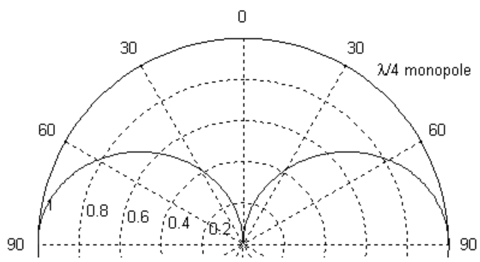
$$\eta = \frac{\mathrm{R}_{rad}}{\mathrm{R}_{rad}+R_\ell} = \frac{1.26}{1.26+63.91}\times100\% = \underline{\underline{1.933\%}}$$

Prob. 13.16

(a) $f(\theta) = \left| \dfrac{\cos\left(\dfrac{\pi}{2}\cos\theta\right)}{\sin\theta} \right|$



(b) The same as for $\dfrac{\lambda}{2}$ dipole except that the fields are zero for $\theta \rangle \dfrac{\pi}{2}$ as shown.



Prob. 13.17

Let P_{rad1} and P_{rad2} be the old and new radiated powers respectively.

Let P_{ohm1} and P_{ohm2} be the old and new ohmic powers respectively.

$$\eta_{r1} = 20\% = \frac{P_{rad1}}{P_{rad1} + P_{ohm1}} = \frac{1}{5} \longrightarrow 4P_{rad1} = P_{ohm1} \qquad (1)$$

But $$P_{ohm1} = \frac{1}{2} I^2 R_s \Delta z$$

$$P_{ohm2} = \frac{1}{2} I^2 R_s 2\Delta z = 2P_{ohm1} \qquad (2)$$

$$P_{rad1} = \frac{1}{2} I_o^2 R_{rad} = \frac{1}{2} I_o^2 \times 80\pi^2 \left(\frac{\Delta z}{\lambda}\right)^2$$

$$P_{rad2} = \frac{1}{2} I_o^2 R_{rad} = \frac{1}{2} I_o^2 \times 80\pi^2 \left(\frac{2\Delta z}{\lambda}\right)^2 = 4P_{rad1} \qquad (3)$$

From (1) to (3),

$$\eta_{r2} = \frac{P_{rad2}}{P_{rad2} + P_{ohm2}} = \frac{4P_{rad1}}{4P_{rad1} + 2P_{ohm1}} = \frac{P_{ohm1}}{3P_{ohm1}} = \underline{\underline{33.3\%}}$$

Prob. 13.18

(a) Let $\boldsymbol{H}_s = \dfrac{\cos 2\theta}{\eta_o r} e^{-j\beta r} \boldsymbol{a}_H$

$$\boldsymbol{a}_E \times \boldsymbol{a}_H = \boldsymbol{a}_k \longrightarrow \boldsymbol{a}_\theta \times \boldsymbol{a}_H = \boldsymbol{a}_r \longrightarrow \boldsymbol{a}_H = \boldsymbol{a}_\phi$$

$$\underline{\underline{\boldsymbol{H}_s = \frac{\cos 2\theta}{120\pi r} e^{-j\beta r} \boldsymbol{a}_\phi}}$$

(b) $\boldsymbol{P}_{ave} = \dfrac{|E_s|^2}{2\eta} \boldsymbol{a}_r = \dfrac{\cos^2(2\theta)}{2\eta r^2} \boldsymbol{a}_r$

$$P_{rad} = \frac{1}{2\eta} \iint \frac{\cos^2 2\theta}{r^2} r^2 \sin\theta d\theta d\phi = \frac{1}{240\pi}(2\pi)\int_0^\pi \cos^2 2\theta \sin\theta d\theta$$

But $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$

$$P_{rad} = -\frac{1}{120}\int_0^\pi (2\cos^2\theta - 1)^2 d(\cos\theta)$$

$$= -\frac{1}{120}\int_0^\pi (4\cos^4\theta - 4\cos^2\theta + 1) d(\cos\theta)$$

$$= -\frac{1}{120}\left(\frac{4\cos^5\theta}{5} - \frac{4\cos^3\theta}{3} + \cos\theta\right)\Bigg|_0^\pi$$

$$= -\frac{1}{120}[-\frac{4}{5} + \frac{4}{3} - 1 - \frac{4}{5} + \frac{4}{3} - 1] = \frac{1}{120}(\frac{14}{15})$$

$$= \underline{\underline{7.778 \text{ mW}}}$$

(c)

$$P_{rad} = -\frac{1}{120}\int_{60^o}^{120^o} (2\cos^2\theta - 1)^2 d(\cos\theta)$$

$$= -\frac{1}{120}\left(\frac{4\cos^5\theta}{5} - \frac{4\cos^3\theta}{3} + \cos\theta\right)\Bigg|_{60^o}^{120^o}$$

$$= -\frac{1}{120}[\frac{4}{5}(-\frac{1}{32}) - \frac{4}{3}(-\frac{1}{8}) - \frac{1}{2} - \frac{4}{5}(\frac{1}{32}) + \frac{4}{3}(\frac{1}{8}) - \frac{1}{2}] = \frac{1}{60}[\frac{1}{40} + \frac{1}{2} - \frac{1}{6}]$$

$$= 5.972 \text{ mW}$$

which is $\dfrac{5.972}{7.778} = 0.7678$ or $\underline{\underline{76.78\%}}$

Prob. 13.20

(a) $P_{rad} = \int \boldsymbol{P}_{rad} \cdot d\boldsymbol{S} = P_{ave}.2\pi r^2$ (hemisphere)

$$P_{ave} = \frac{P_{rad}}{2\pi r^2} = \frac{200\times10^3}{2\pi(2500\times10^6)} = 12.73\mu W / m^2$$

$$\boldsymbol{P}_{ave} = \underline{\underline{12.73\boldsymbol{a}_r \ \mu\text{W/m}^2}}.$$

(b) $$P_{ave} = \frac{(E_{\max})^2}{2\eta}$$

$$E_{\max} = \sqrt{2\eta P_{ave}} = \sqrt{240\pi\times12.73\times10^{-6}}$$

$$= \underline{\underline{0.098 \text{ V/m}}}$$

Prob. 13.21

$$G_d = \frac{U}{U_{ave}} = \frac{4\pi r^2 P_{ave}}{\int P_{ave}.\partial s} = \frac{8\pi\sin\theta\cos\phi}{\int P_{ave}.dS}$$

But $\int P_{ave}.dS = \int_{\theta=0}^{\pi}\int_{\phi=0}^{\pi/2} 2\sin\theta\cos\phi\sin\theta d\theta d\phi$

$$= 2\int_0^{\pi/2}\cos\phi d\phi\int_0^{\pi}\sin^2\theta d\theta = 2\sin\phi\Big|_0^{\pi/2}\left(\frac{\pi}{2}\right) = \pi$$

$$G_d = \underline{\underline{8\sin\theta\cos\phi}}$$

$$D = G_{d,\max} = \underline{\underline{8}}$$

Prob. 13.22

From Prob. 13.11, set $\ell = \dfrac{3\lambda}{2}, \qquad \beta\ell = \dfrac{2\pi}{\lambda}\dfrac{3\lambda}{2} = 3\pi$

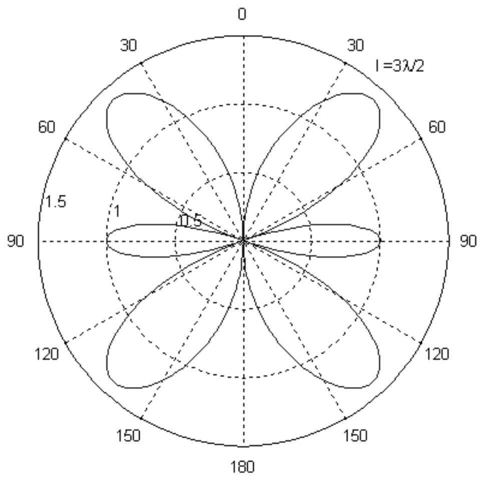
$$H_{\phi s} = \frac{jI_o e^{-\beta r}}{2\pi r}\frac{\left[\cos\left(\frac{2\pi}{\lambda}\frac{1}{2}\frac{3\lambda}{2}\cos\theta\right) - \cos\left(\frac{2\pi}{\lambda}\frac{1}{2}\frac{3\lambda}{2}\right)\right]}{\sin\theta}$$

$$= \frac{jI_o e^{-\beta r}}{2\pi r}\frac{\left[\cos\left(\frac{3\pi}{2}\cos\theta\right) - \cos\left(\frac{3\pi}{2}\right)\right]}{\sin\theta} = \frac{jI_o e^{-\beta r}}{2\pi r}\frac{\cos(1.5\pi\cos\theta)}{\sin\theta}$$

Hence, the normalized radiated field pattern is

$$f(\theta) = \frac{\cos(1.5\pi\cos\theta)}{\sin\theta}$$

which is plotted below.



Prob. 13.23
The MATLAB code is shown below

```
N=20;
del= 2*pi/N;
sum=0;
for k=1:N
  theta = del*k;
  term = (1 - cos(theta))/theta;
  sum = sum + term;
end
int = del*sum
```

When the program is run, it gives the value of 2.4335. The accuracy may be increased by increasing N.

Prob. 13.24

(a) $E_{\theta s} = \dfrac{j\eta I_o \beta dl}{4\pi r}\sin\theta e^{-j\beta r}$

$$R_{rad} = 80\pi^2\left(\frac{dl}{\lambda}\right)^2$$

$$G_d = \frac{4\pi r^2 P_{ave}}{P_{rad}} = \frac{4\pi r^2 \cdot \frac{1}{2\eta}\left|E_{\theta s}\right|^2}{\frac{1}{2}I_o^{\ 2}R_{rad}}$$

$$= \frac{4\pi r^2}{I_o^{\ 2}}\cdot\frac{1}{80\pi^2}\left(\frac{\lambda}{dl}\right)^2\cdot\frac{1}{\eta}\frac{\eta^2 I_o^{\ 2}\beta^2 (dl)^2\sin^2\theta}{16\pi^2 r^2}$$

$$G_d = \underline{\underline{1.5\sin^2\theta}}$$

(b) $D = G_{d,\max} = \underline{\underline{1.5}}$

(c) $A_e = \dfrac{\lambda^2}{4\pi}G_d = \underline{\underline{\dfrac{1.5\lambda^2\sin^2\theta}{4\pi}}}$

(d) $R_{rad} = 80\pi^2\left(\dfrac{1}{16}\right)^2 = \underline{\underline{3.084\ \Omega}}$

Prob. 13.25

(a) $$E_{\phi s} = \frac{120\pi^2 I_o}{r}\frac{S}{\lambda^2}\sin\theta e^{-j\beta r}$$

$$R_{rad} = \frac{320\pi^4 S^2}{\lambda^4}$$

$$G_d = \frac{4\pi U(\theta,\phi)}{P_{rad}} = \frac{4\pi r^2 P_{ave}}{\frac{1}{2}I_o^2 R_{rad}} = \frac{8\pi r^2}{I_o^{\ 2}}.\frac{1}{2\eta}\frac{\left|E_{\phi s}\right|^2}{R_{rad}}$$

$$= \frac{8\pi r^2}{I_o^{\ 2}}.\frac{1}{2\eta}.14400\pi^4\frac{I_o^{\ 2}}{r^2}\frac{S^2}{\lambda^4}\sin^2\theta\frac{\lambda^2}{320\pi^4 S^2}$$

$$G_d = \underline{\underline{1.5\sin^2\theta}}$$

(b) $\underline{D = 1.5}$

(c) $$A_e = \frac{\lambda^2 G_d}{4\pi} = \underline{\underline{\frac{\lambda^2}{4\pi}1.5\sin^2\theta}}$$

(d) $$S = \pi a^2 = \frac{\pi d^2}{4} = \frac{320\pi^6}{(576)^2}$$

$$R_{rad} = \underline{\underline{0.927\Omega}}$$

Prob. 13.26

(a)

$$\lambda = \frac{c}{f} = \frac{3\times10^8}{1.2\times10^6} = 250$$

$$\ell = \frac{\lambda}{4} = \underline{\underline{62.5\text{ m}}}$$

(b) From eq. (13.30), $R_{rad} = \underline{\underline{36.5\ \Omega}}$

(c)

For $\lambda/4$-monopole,

$$f(\theta)=\frac{\cos(\frac{\pi}{2}\cos\theta)}{\sin\theta}, \qquad 0<\theta<\pi/2$$

$$G_d(\theta,\phi)=\frac{4\pi f^2(\theta)}{\int f^2(\theta)d\Omega}=\frac{\dfrac{4\pi\cos^2(\frac{\pi}{2}\cos\theta)}{\sin^2\theta}}{\int\limits_0^{2\pi}\int\limits_0^{\pi/2}\dfrac{\cos^2(\frac{\pi}{2}\cos\theta)}{\sin\theta}d\theta d\phi}$$

$$=\frac{4\pi\cos^2(\frac{\pi}{2}\cos\theta)}{\sin^2\theta}\frac{1}{2\pi(0.6094)}=\frac{3.282\cos^2(\frac{\pi}{2}\cos\theta)}{\sin^2\theta}$$

$$D=G_{d,\max}=\underline{\underline{3.282}}$$

Prob. 13.27

(a) $U_{max}=1$

$$U_{ave}=\frac{P_{rad}}{4\pi}=\frac{\int Ud\Omega}{4\pi}$$

$$=\frac{1}{4\pi}\int\int\sin^2 2\theta\sin\theta d\theta d\phi$$

$$=\frac{1}{4\pi}(2\pi)\int_0^{\pi}\left(2\sin\theta\cos\theta\right)^2 d\left(-\cos\theta\right)$$

$$=2\int_0^{\pi}\left(\cos^4\theta-\cos^2\theta\right)d\left(\cos\theta\right)$$

$$=2\left[\frac{\cos^5\theta}{5}-\frac{\cos^3\theta}{3}\right]\Bigg|_0^{\pi}$$

$$=2\left[-\frac{2}{5}+\frac{2}{3}\right]=\frac{8}{15}$$

$$U_{ave}=\underline{\underline{0.5333}}$$

$$D = \frac{U_{\max}}{U_{ave}} = \underline{\underline{1.875}}$$

(b) $U_{\max} = 4$

$$U_{ave} = \frac{1}{4\pi}\int U d\Omega = \frac{4}{4\pi}\iint \frac{\sin\theta}{\sin^2\theta} d\theta d\phi$$

$$= \frac{1}{\pi}\int_0^{\pi} d\phi \int_{\pi/3}^{\pi/2} \operatorname{cosec}\theta d\theta = \frac{\pi}{\pi}\ln\sqrt{3}$$

$U_{ave} = \underline{\underline{0.5493}}$

$$D = \frac{U_{\max}}{U_{ave}} = \frac{16}{3\ln\sqrt{3}} = \underline{\underline{9.7092}}$$

(c) $U_{\max} = 2$

$$U_{ave} = \frac{1}{4\pi}\int U d\Omega = \frac{1}{4\pi}\iint 2\sin^2\theta \sin^2\phi \sin\theta d\theta d\phi$$

$$= \frac{1}{2\pi}\int_0^{\pi} \sin^2\phi d\phi \int_0^{\pi} \left(1-\cos^2\theta\right) d\left(-\cos\theta\right)$$

$$= \frac{1}{2\pi}.\frac{\pi}{2}\left(\frac{\cos^3\theta}{3} - \cos\theta\right)\Bigg|_0^{\pi} = \frac{1}{4}\left[-\frac{2}{3}+2\right] = \frac{1}{3}$$

$U_{ave} = \underline{\underline{0.333}}$

$$D = \frac{U_{\max}}{U_{ave}} = \underline{\underline{6}}$$

Prob. 13.28

(a) $$G_d(\theta,\phi)=\frac{U(\theta,\phi)}{U_{ave}}$$

$$U_{ave}=\frac{1}{4\pi}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}10\sin\theta\sin^2\phi\times\sin\theta d\theta d\phi$$

$$=\frac{10}{4\pi}\int_0^{2\pi}\sin^2\phi d\phi\int_0^{\pi}\sin^2\theta d\theta$$

$$=\frac{10}{4\pi}\frac{1}{2}\left(\theta-\frac{\sin 2\theta}{2}\right)\Bigg|_0^{2\pi}\frac{1}{2}\left(\phi-\frac{\sin 2\phi}{2}\right)\Bigg|_0^{\pi}$$

$$=\frac{10}{16\pi}(2\pi-0)(\pi-0)=\frac{5\pi}{4}$$

$$G_d(\theta,\phi)=\frac{40\sin\theta\sin^2\phi}{5\pi}=\underline{\underline{2.546\sin\theta\sin^2\phi}}$$

$$D=G_{d.\max}=\underline{\underline{2.546}}$$

(b) $$U_{ave}=\frac{1}{4\pi}\int_{\phi=0}^{\pi}\int_{\theta=0}^{\pi}2\sin^2\theta\sin^3\phi\times\sin\theta d\theta d\phi$$

$$=\frac{2}{4\pi}\int_0^{\pi}\sin^3\phi d\phi\int_0^{\pi}\sin^3\theta d\theta=\frac{2}{4\pi}\left[\int_0^{\pi}(1-\cos^2\phi)d(-\cos\phi)\right]^2$$

$$=\frac{1}{2\pi}\left[\left(\frac{\cos^3\phi}{3}-\cos\phi\right)\Bigg|_0^{\pi}\right]=\frac{1}{2\pi}\left(\frac{4}{3}\right)^2=\frac{16}{18\pi}$$

$$G_d(\theta,\phi)=\frac{18\pi}{16}2\sin^2\theta\sin^3\phi=\underline{\underline{2.25\pi\sin^2\theta\sin^3\phi}}$$

$$D=G_{d.\max}=\underline{\underline{7.069}}$$

(c) $$U_{ave} = \frac{1}{4\pi}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} 5(1+\sin^2\theta\sin^2\phi)\times\sin\theta d\theta d\phi$$

$$= \frac{5}{4\pi}\left[\int_0^{\pi}\sin\theta d\theta\int_0^{2\pi} d\phi + \int_0^{\pi}\sin^3\theta d\theta\int_0^{2\pi}\sin^2 d\phi\right]$$

$$= \frac{5}{4\pi}\left[2\pi(-\cos\theta)\Big|_0^{\pi} + \frac{4}{3}\left(\frac{\phi}{2} - \frac{\sin 2\phi}{4}\right)\Big|_0^{2\pi}\right]$$

$$= \frac{5}{4\pi}\left[4\pi + \frac{4}{3}\pi\right] = \frac{20}{3}$$

$$G_d(\theta,\phi) = \frac{3}{20}5(1+\sin^2\theta\sin^2\phi) = \underline{\underline{0.75(1+\sin^2\theta\sin^2\phi)}}$$

$$D = G_{d.\max} = \underline{\underline{1.5}}$$

Prob. 13.29

$$U_{\max} = 4$$

$$U_{ave} = \frac{1}{4\pi}\int U d\Omega = \frac{1}{4\pi}\iint 4\sin^2\theta\sin\frac{\phi}{2}\sin\theta d\theta d\phi$$

$$= \frac{1}{\pi}\int_0^{\pi}\sin^3\theta d\theta\int_0^{\pi}\sin\frac{\phi}{2}d\phi = \frac{1}{\pi}\int_0^{\pi}(1-\cos^2\theta)d(-\cos\theta)(-2\cos\frac{\phi}{2})\Big|_0^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{4}{3}\right)(2) = \frac{8}{3\pi}$$

$$D = \frac{U_{\max}}{U_{ave}} = 4\times\frac{3\pi}{8} = \underline{\underline{4.712}}$$

Prob. 13.30

$$\boldsymbol{P}_{ave} = \frac{|E_r|^2}{2\eta}\boldsymbol{a}_r = \frac{I_o^2\sin^2\theta}{2\eta r^2}\boldsymbol{a}_r$$

$$P_{rad} = \frac{I_o^2}{2\eta}\iint\frac{\sin^2\theta}{r^2}r^2\sin\theta d\theta d\phi = \frac{I_o^2}{240\pi}(2\pi)\int_0^{2\pi}(1-\cos^2\theta)d(-\cos\theta)$$

$$= \frac{I_o^2}{120}\left(\frac{\cos^3\theta}{3} - \cos\theta\right)\Big|_0^{\pi} = \frac{I_o^2}{120}(-1/3+1-1/3+1) = \frac{I_o^2}{90}$$

$$I_o^2 = 90P_{ave} = 90\times 50\times 10^{-3} \longrightarrow I_o = \underline{\underline{2.121\text{ A}}}$$

Prob. 13.31

$$P_{rad}=\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\frac{|\boldsymbol{E}|^2}{2\eta_o}r^2\sin\theta d\theta d\phi=\frac{100}{r^2}\frac{r^2}{2\eta_o}(2\pi)\int_0^{\pi}\sin^5\theta d\theta$$

$$=\frac{2\pi}{2\times 377}100\left(-\frac{5}{8}\cos\theta+\frac{5}{48}\cos 3\theta-\frac{1}{80}\cos 5\theta\right)\Bigg|_0^{\pi}$$

$$=\frac{100\pi}{377}\left(\frac{5}{8}+\frac{5}{8}-\frac{5}{48}-\frac{5}{48}+\frac{1}{80}+\frac{1}{80}\right)=\underline{\underline{0.889\text{ W}}}$$

Prob. 13.32

This is similar to Fig. 13.10 except that the elements are z-directed.

$$\boldsymbol{E}_s=\boldsymbol{E}_{s1}+\boldsymbol{E}_{s2}=\frac{j\eta\beta I_o dl}{4\pi}\left[\sin\theta_1\frac{e^{-j\beta r_1}}{r_1}\boldsymbol{a}_{\theta 1}+\sin\theta_2\frac{e^{-j\beta r_2}}{r_2}\boldsymbol{a}_{\theta 2}\right]$$

where $r_1\cong r-\frac{d}{2}\cos\theta,\qquad r_2\cong r+\frac{d}{2}\cos\theta,\qquad \theta_1\cong\theta_2\cong\theta,\qquad \boldsymbol{a}_{\theta 1}\cong\boldsymbol{a}_{\theta 2}=\boldsymbol{a}_\theta$

$$\boldsymbol{E}_s=\frac{j\eta\beta I_o dl}{4\pi}\sin\theta\,\boldsymbol{a}_\theta\left[e^{j\beta d\cos\theta/2}+e^{-j\beta d\cos\theta/2}\right]$$

$$\underline{\underline{\boldsymbol{E}_s=\frac{j\eta\beta I_o dl}{2\pi}\sin\theta\ \cos(\frac{1}{2}\beta d\cos\theta)\boldsymbol{a}_\theta}}$$

Prob. 13.33

Equation (13.33) applies except that cosθ must be replaced by sinθ. Hence, the radiation pattern is

$$f(\theta)=\sin\theta\ 2\cos\left[\frac{1}{2}(\beta d\cos\theta+\alpha)\right]$$

$$\alpha=0,\ \ d=2(\lambda/4)=\lambda/2,\qquad \beta\text{d}=\frac{2\pi}{\lambda}\frac{\lambda}{2}=\pi$$

$$f(\theta)=\underline{\underline{2\sin\theta\cos(\frac{1}{2}\pi\cos\theta)}}$$

Prob. 13.34

(a) $\text{AF}=2\cos\left[\frac{1}{2}\left(\beta d\cos\theta+\alpha\right)\right],\qquad \alpha=0,\qquad \beta d=\frac{2\pi}{\lambda}\lambda=2\pi$

AF = $\underline{\underline{2\cos(\pi\cos\theta)}}$

(b) Nulls occur when

$$\cos(\pi\cos\theta) = 0 \quad \longrightarrow \quad \pi\cos\theta = \pm\pi/2, \pm 3\pi/2, ...$$

or

$$\theta = \underline{\underline{60^o, 120^o}}$$

(c) Maxima and minima occur when

$$\frac{df}{d\theta} = 0 \quad \longrightarrow \quad \sin(\pi\cos\theta)\pi\sin\theta = 0$$

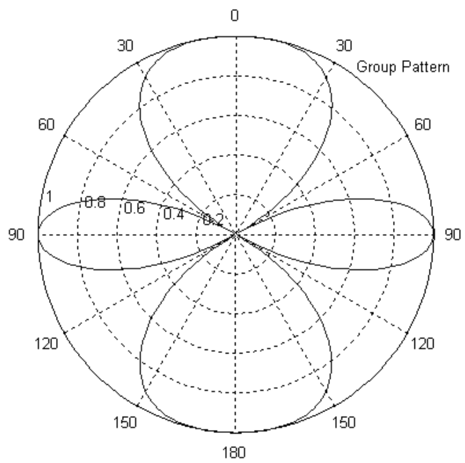
i.e. $\sin\theta = 0 \quad \longrightarrow \quad \theta = 0^o, 180^o$

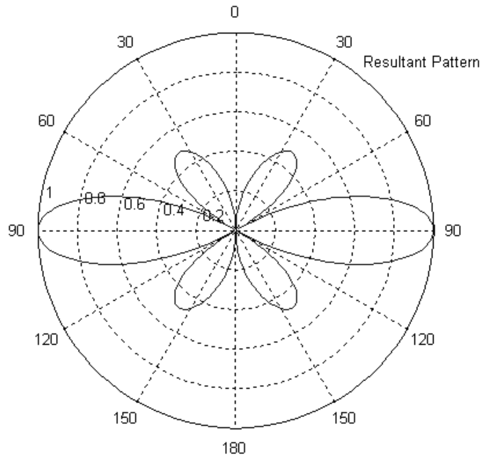
$\cos\theta = 0 \quad \longrightarrow \quad \theta = 90^o$

or

$$\theta = \underline{\underline{0^o, 90^o, 180^o}}$$

(d) The group pattern is sketched below.





Prob. 13.35

$$f(\theta) = \cos\left[\frac{1}{2}(\beta d \cos\theta + \alpha)\right]$$

(a) $\alpha = \pi/2, \beta d = \frac{2\pi}{\lambda}.\lambda = 2\pi$

$$f(\theta) = \cos\left(\pi\cos\theta + \pi/4\right)$$

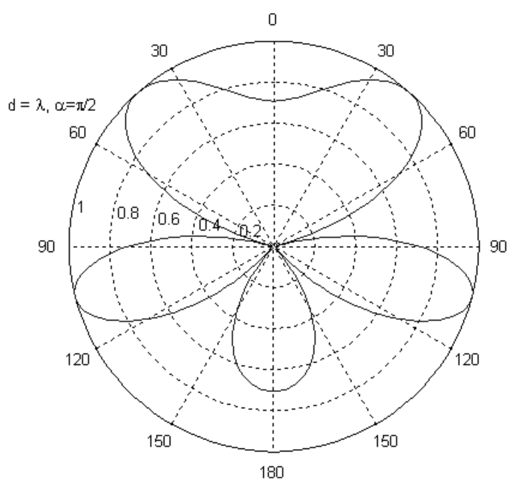
Nulls occur at $\pi\cos\theta + \pi/4 = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$ or $\theta = 75.5^o, 138.6^o$

Maxima occur at $\frac{\partial f}{\partial\theta} = 0 \longrightarrow \quad \sin\theta = 0 \longrightarrow \quad \theta = 0^o, 180^o$

Or $\sin\left(\pi\cos\theta + \frac{\pi}{4}\right) = 0 \longrightarrow \quad \theta = 41.4^o, 104.5^o$

With $f_{\max} = 0.71, 1$.

Hence the group pattern is sketched below.



θ = 41.0138°, abs(f) = 1,
θ = 104.6083°, abs(f) = 1,
θ = 180°, abs(f) = 1/√2=0.707

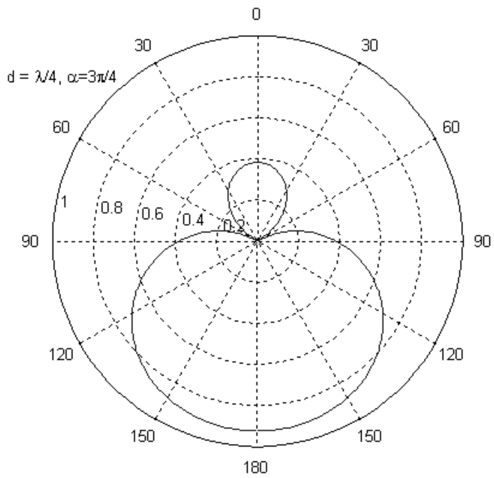
(b) $\alpha = \frac{3\pi}{4}, \beta d = \frac{2\pi}{\lambda}.\frac{\lambda}{4} = \frac{\pi}{2}$

$$f(\theta) = \left|\cos\left(\frac{\pi}{4}\cos\theta + \frac{3\pi}{8}\right)\right|$$

Nulls occur at $\frac{\pi}{4}\cos\theta + \frac{3\pi}{8} = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots \longrightarrow \theta = 60^o$

Minima and maxima occur at $\sin\theta\cos\left(\frac{\pi}{4}\cos\theta + \frac{3\pi}{8}\right) = 0$

i.e. $\theta = 0^o, 180^o \rightarrow f(\theta) = 0.383, 0.924$



θ = 0, abs(f) = 0.3827
θ = 180, abs(f) = 0.9239

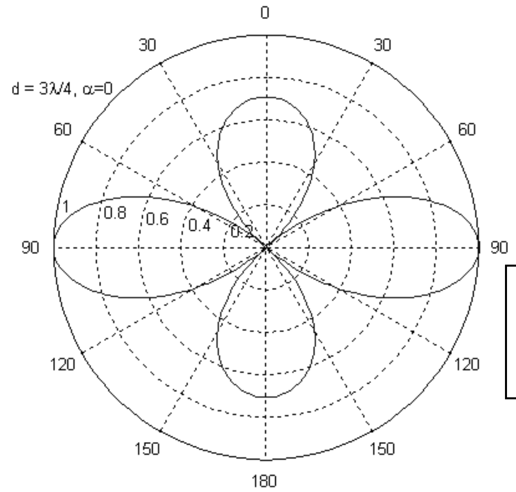
(c) $\alpha = 0, \beta d = \frac{2\pi}{\lambda}.\frac{3\lambda}{4} = \frac{3\pi}{2}$

$$f(\theta) = \left|\cos\left(\frac{3\pi}{4}\cos\theta\right)\right|$$

It has nulls at $\frac{3\pi}{4}\cos\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots \rightarrow \theta = 48.2^o, 131.8^o$

It has maxima and minima at $\frac{df}{d\theta} = 0 \rightarrow \sin\theta\sin\left(\frac{3\pi}{4}\cos\theta\right) = 0$

i.e. $\theta = 0^o, 180^o \rightarrow f(\theta) = 0.71, 1, \quad \theta = \pm 90^o, \rightarrow f(\theta) = 1$



θ = 0 °,	abs(f) = 1/√2=0.707,
θ = 90°,	abs(f) = 1,
θ = 180°,	abs(f) = 1/√2=0.707

Prob. 13.36

(a) For N = 2, $f(\theta)=\cos\left[\frac{1}{2}(\beta d\cos\theta+\alpha)\right]$

$\alpha=0, d=\frac{\lambda}{4}$

$$f(\theta)=\cos\left[\frac{1}{2}\left(\frac{2\pi}{\lambda}.\frac{\lambda}{4}\cos\theta+0\right)\right]=\cos\left(\frac{\pi}{4}\cos\theta\right)$$

Maxima and minima occur at

$$\frac{d}{d\theta}\left[\cos\left(\frac{\pi}{4}\cos\theta\right)\right]=0$$

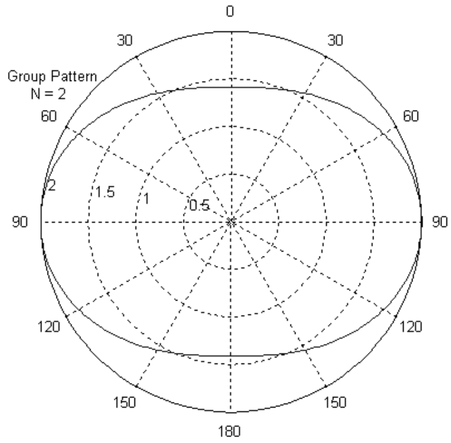
$$\sin\theta\sin\left(\frac{\pi}{4}\cos\theta\right)=0$$

$$\sin\theta=0\rightarrow\theta=\pi,0 \text{ and } f(\theta)=0.707$$

$$\sin\left(\frac{\pi}{4}\cos\theta\right)\rightarrow\cos=0\rightarrow\theta=90^o, f(\theta)=1$$

Nulls occur as $\frac{\pi}{4}cos\theta=\pm\frac{\pi}{2},\pm\frac{3\pi}{2},...$(No Solution)

The group pattern is sketched below.



(b) For N = 4,

$$AF = \frac{\sin 2\left(\beta d \cos\theta + 0\right)}{\sin\frac{1}{2}\left(\beta d \cos\theta + 0\right)}$$

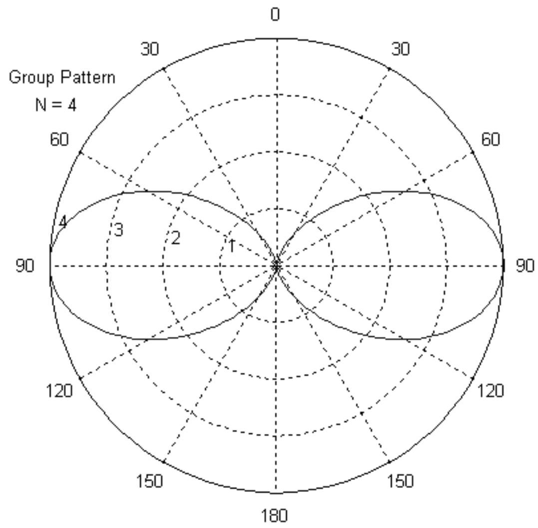
Now, $\dfrac{\sin 4\theta}{\sin\theta} = \dfrac{2\sin 2\theta \cos 2\theta}{\sin\theta} = 4\cos 2\theta \cos\theta$

$$AF = 4\cos\left(\beta d \cos\theta\right)\cos\left(\frac{1}{2}\beta d \cos\theta\right)$$

$$f\left(\theta\right) = \cos\left(\frac{2\pi}{\lambda}.\frac{\lambda}{4}\cos\theta\right)\cos\left(\frac{1}{2}\frac{2\pi}{\lambda}\frac{\lambda}{4}\right)\cos\theta$$

$$= \cos\left(\frac{\pi}{2}\cos\theta\right)\cos\left(\frac{\pi}{4}\cos\theta\right)$$

The plot is shown below.

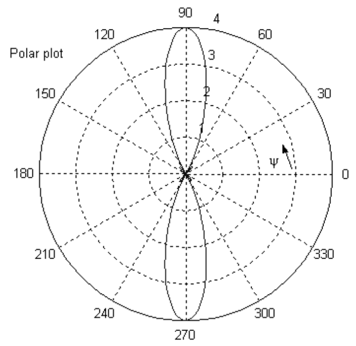


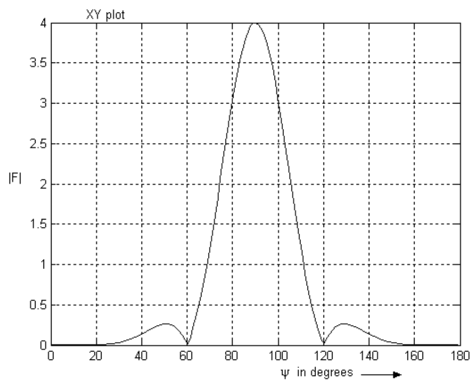
Prob. 13.37

The MATLAB code is shown below.

```
for n=1:180
    phi=n*pi/180;
    p(n)=n;
    sn=sin(2*pi*cos(phi));
    cn=cos(0.5*pi*cos(phi));
    sd=sin(0.5*pi*cos(phi));
    fun=sn*cn*cn/sd;
    f(n)= abs(fun);
end
polar(p,f)
```

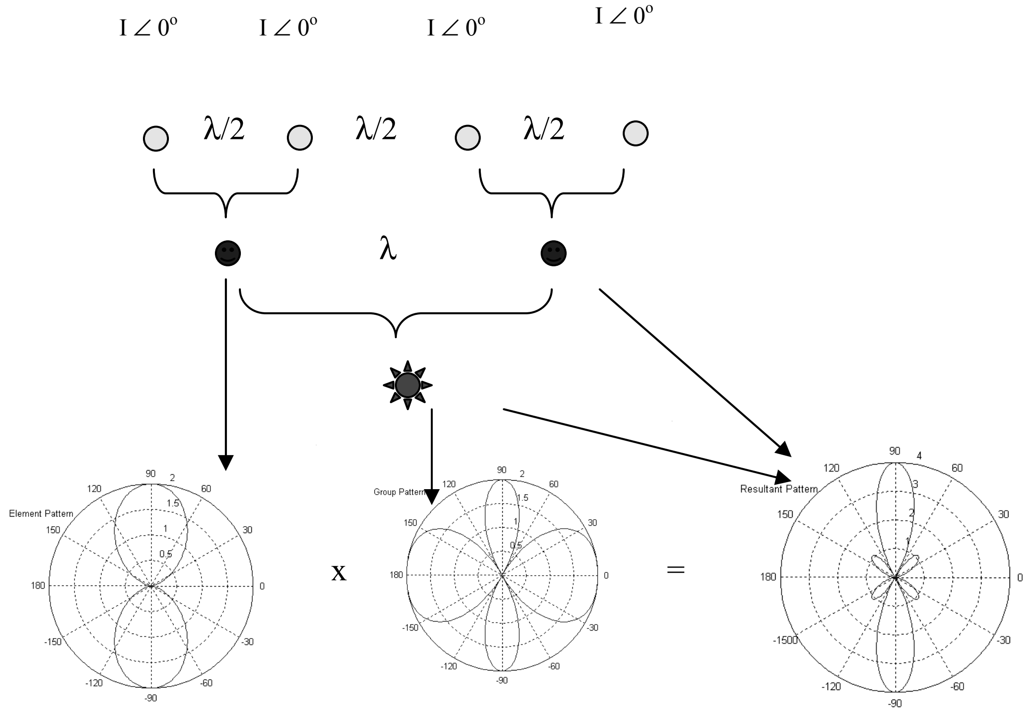
The polar plot and the xy plot are shown below.





Prob. 13.38

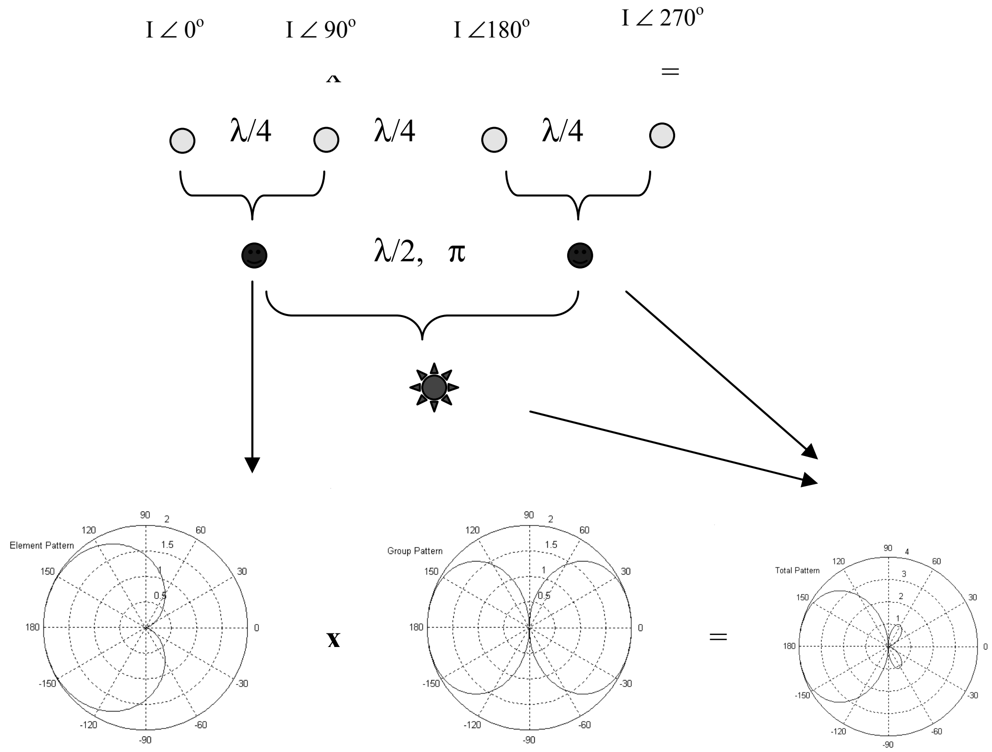
(a) The resultant pattern is obtained as follows.



(b) The array is replaced by $+ \quad \angle 0^o \quad \frac{\lambda}{4} \quad + \quad \angle \pi/2$

where + stands for $\angle 0^o \longleftrightarrow \angle \pi$

Thus the resultant pattern is obtained as shown.



Prob. 13.39

$$G_d(dB) = 20dB = 10\log_{10} G_d \quad \longrightarrow \quad G_d = 10^2 = 100$$

$$\lambda = \frac{c}{f} = \frac{3\times10^8}{10\times10^9} = 3\times10^{-2}$$

$$A_e = \frac{\lambda^2}{4\pi} G_d = \frac{9\times10^{-4}}{4\pi} 100 = \underline{\underline{7.162\times10^{-2}\ \text{m}^2}}$$

Prob. 13.40

$$A_e = \frac{P_r}{P_{ave}} = \frac{P_r}{|E_r|^2 / 2\eta} = \frac{2\eta P_r}{|E_r|^2}$$

$$= \frac{2\times 120\pi \times 2\times 10^{-6}}{25\times 10^2 \times 10^{-6}} = \frac{48\pi}{250} = \underline{\underline{0.6031}}$$

Prob. 13.41

Friis equation states that

$$\frac{P_r}{P_t} = G_r G_t \left(\frac{\lambda}{4\pi r}\right)^2$$

$$\lambda = \frac{c}{f} = \frac{3\times 10^8}{200\times 10^6} = 1.5 \text{ m}, \qquad r = 238{,}857 \times 1.609\times 10^3 = 3.843\times 10^8$$

$$G_t(dB) = 15dB = 10\log_{10} G_t \quad \longrightarrow \quad G_t = 10^{15/10} = 31.623$$

$$G_r = \left(\frac{4\pi r}{\lambda}\right)^2 \frac{P_r}{P_t} = \left(\frac{4\pi \times 3.843\times 10^8}{1.5}\right)^2 \frac{4\times 10^{-9}}{120\times 10^{-3}} = 34.55\times 10^{10}$$

$$G_r(dB) = 10\log_{10} G_r = 10\log_{10} 34.55\times 10^{10} = \underline{\underline{115.384 \text{ dB}}}$$

Prob. 13.42

Using Frii's equation,

$$P_r = G_r G_t \left[\frac{\lambda}{4\pi r}\right]^2 P_t$$

$$P_t = \left[\frac{4\pi r}{\lambda}\right]^2 \frac{P_r}{G_r G_t}$$

$$\lambda = \frac{c}{f} = \frac{3\times 10^8}{3\times 10^9} = 0.1, \qquad r = 42 \text{ km}$$

$$G_t(dB) = 10\log_{10} G_t = 25 \quad \longrightarrow \quad G_t = 10^{2.5} = 316.23$$

$$G_r(dB) = 10\log_{10} G_r = 20 \quad \longrightarrow \quad G_t = 10^2 = 100$$

$$P_t = \left(\frac{4\pi \times 42\times 10^3}{0.1}\right)^2 \frac{3\times 10^{-6}}{31623} = \underline{\underline{2.642 \text{ kW}}}$$

Prob. 13.43

$$G_{dt} = 10^4, G_{dr} = 10^{3.2} = 1585$$

$$\lambda = \frac{c}{f} = \frac{3\times10^8}{15\times10^9} = 0.02m = \frac{1}{50}$$

$$P_r = G_{dr}G_{dt}\left(\frac{\lambda}{4\pi r}\right)^2 P_t = 10^4(1585)\left(\frac{0.02}{4\pi\times2.456741\times10^7}\right)^2 320$$

$$= 2.129\times10^{-11}\text{ W} = \underline{\underline{21.29\text{ pW}}}$$

Prob. 13.44

Using Frii's equation,

$$P_r = G^2\left[\frac{\lambda}{4\pi r}\right]^2 P_t$$

$$P_t = \left[\frac{4\pi r}{\lambda}\right]^2 \frac{P_r}{G^2}$$

$$G(dB) = 25 \quad \longrightarrow \quad G = 10^{2.5} = 316.23$$

$$\lambda = \frac{c}{f} = \frac{3\times10^8}{600\times10^6} = 0.5$$

$$P_t = \left[\frac{4\pi\times450}{0.5}\right]^2 \frac{4\times10^{-3}}{(316.23)^2} = \underline{\underline{5.1163\text{ W}}}$$

Prob. 13.45

$$30dB = \log\frac{P_t}{P_r} \to \frac{P_t}{P_r} = 10^3 = 1000$$

But $P_r = (G_d)^2\left(\frac{3}{50\times4\pi\times12}\right)^2 P_t = P_t\left(\frac{G_d}{800\pi}\right)^2$

$$\left(\frac{G_d}{800\pi}\right)^2 = \frac{P_r}{P_t} = \frac{1}{1000} = \left(\frac{1}{10\sqrt{10}}\right)^2$$

or $G_d = \frac{800\pi}{10\sqrt{10}} = 79.476$

$$G_d = 10\log 79.476 = \underline{\underline{19\text{ dB}}}$$

Prob. 13.46

(a) $P_i = \dfrac{|E|^2}{2\eta_o} = \dfrac{P_{rad}G_d}{4\pi r^2} \rightarrow |E_i| = \sqrt{\dfrac{240\pi P_{rad}G_d}{4\pi r^2}}$

$$|E_i| = \frac{1}{r}\sqrt{60P_{rad}G_d} = \frac{1}{120\times10^3}\sqrt{60\times200\times10^3\times3500}$$

$$= \underline{\underline{1.708 \text{ V/m}}}$$

(b) $|E_s| = \sqrt{\dfrac{|E_i|^2\sigma}{4\pi r^2}} = \sqrt{\dfrac{1.708^2\times8}{4\pi\times14400\times10^6}} = \underline{\underline{11.36\ \mu\text{V/m}}}$

(c) $P_c = P_i\sigma = \dfrac{1.708^2}{240\pi}(8) = \underline{\underline{30.95 \text{ mW}}}$

(d) $P_i = \dfrac{|E|^2}{2\eta_o} = \dfrac{(11.36)^2\times10^{-12}}{240\pi} = 1.712\times10^{-13}\ \ \text{W/m}^2$

$$\lambda = \frac{3\times10^8}{15\times10^8} = 0.2m, A_{2r} = \frac{\lambda^2 G}{4\pi} = \frac{0.04\times3500}{4\pi}$$

$$P_r = P_a A_{er} = 1.712\times10^{-13}\times11.14 = 1.907\times10^{-12}$$

or $P_r = \dfrac{(\lambda G_d)^2\sigma P_{rad}}{(4\pi)^3 r^4} = \dfrac{(0.2\times3500)^2\times8\times2\times10^5}{(4\pi)^3\times12^4\times10^{16}}$

$$= \underline{\underline{1.91\times10^{-12}\ \text{W}}}$$

Prob. 13.47

$$P_r = \frac{(\lambda G_d)^2\sigma P_{rad}}{(4\pi)^3 r^4}$$

$$G_d(dB) = 30dB = 10\log_{10} G_d \longrightarrow G_d = 10^3$$

$$\lambda = \frac{c}{f} = \frac{3\times10^8}{4\times10^9} = 0.075 \text{ m}$$

$$P_r = \frac{(0.075\times10^3)^2\times12\times80\times10^3}{(4\pi)^3(10\times10^3)^4} = \underline{\underline{272.1 \text{ pW}}}$$

Prob. 13.48

$$P_{rad} = \frac{4\pi}{G_{dt}G_{dr}}\left(\frac{4\pi r_1 r_2}{\lambda}\right)^2 \frac{P_r}{\sigma}$$

But $G_{dt} = 36dB = 10^{3.6} = 3981.1$

$$G_{dr} = 20dB = 10^2 = 100$$

$$\lambda = \frac{c}{f} = \frac{3\times10^8}{5\times10^9} = 0.06$$

$$r_1 = 3km,\ r_2 = 5km$$

$$P_{rad} = \frac{4\pi}{3981.1\times100}\left(\frac{4\pi\times15\times10^6}{6\times10^{-2}}\right)^2 \frac{8\times10^{-12}}{2.4}$$

$$= \underline{\underline{1.038 \text{ kW}}}$$

Prob. 13.49

$$P_r = \frac{(\lambda G_d)^2 \sigma P_{rad}}{(4\pi)^3 r^4} \rightarrow P_{rad} = \frac{(4\pi)^3 r^4 P_r}{(\lambda G_d)^2 \sigma}$$

$$\lambda = \frac{c}{f} = \frac{3\times10^8}{6\times10^9} = \frac{1}{20} << r = 250m$$

$$40 = \log_{10} G_d \rightarrow G_d = 10^4$$

$$P_{rad} = \frac{(4\pi)^3 (0.25\times10^3)^4 \times 2\times10^{-6}}{\left(\frac{1}{20}\times10^4\right)^2 \times 0.8} \quad = \underline{\underline{77.52 \text{ W}}}$$

Prob. 13.50

(a)

$$F = \frac{\pi f L}{R} = \frac{\pi\times300\times10^6\times50\times10^{-9}}{20} = 2.356$$

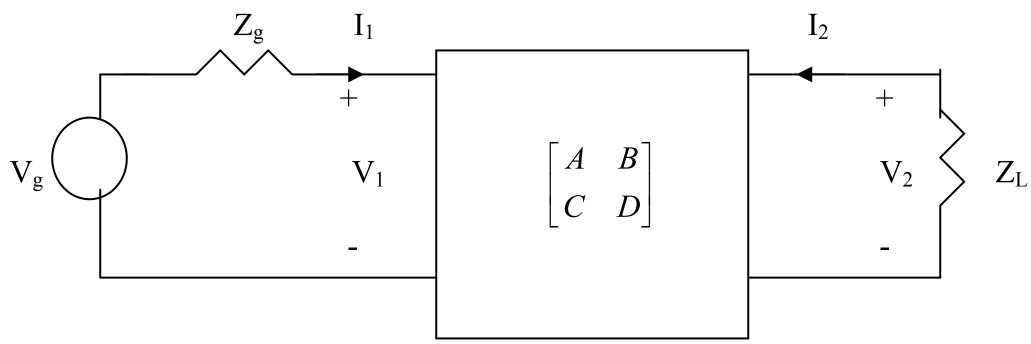
$$IL = 10\log_{10}(1+F^2) = 10\log_{10}(1+2.356^2) = \underline{\underline{8.164 \text{ dB}}}$$

(b)

$F = \pi fRC = \pi \times 300 \times 10^6 \times 10 \times 10^3 \times 60 \times 10^{-12} = 180\pi = 565.5$

$IL = 10\log_{10}(1+F^2) = 10\log_{10}(1+565.5^2) = \underline{\underline{55.05 \text{ dB}}}$

Prob. 13.51



By definition,

$$V_1 = AV_2 - BI_2 \qquad (1)$$
$$I_1 = CV_2 - DI_2 \qquad (2)$$

Let V_2 and $\bar{V}_2$ be respectively the load voltages when the filter circuit is present and when it is absent.

$$V_2 = -I_2 Z_L = \frac{I_1 Z_L}{CZ_L + D}$$

$$= \frac{V_g Z_L}{\left(Z_g + \frac{V_1}{I_1}\right)(CZ_L + D)} = \frac{V_g Z_L}{\left(Z_g + \frac{AV_2 - BI_2}{CV_2 - DI_2}\right)(CZ_L + D)}$$

$$= \frac{V_g Z_L}{\left(Z_g + \frac{AZ_L + B}{CZ_L + D}\right)(CZ_L + D)}$$

$$= \frac{V_g Z_L}{\left(Z_g (CZ_L + D) + AZ_L + B\right)}$$

$$\bar{V}_2 = \frac{V_g Z_L}{\left(Z_g + Z_L\right)}$$

Ratio and modulus give

$$\left|\frac{\bar{V}_2}{V_2}\right| = \frac{\left(Z_g\left(CZ_L + D\right) + AZ_L + B\right)}{Z_g + Z_L}$$

Insertion loss =

$$\text{IL} = 20\log_{10}\left|\frac{\bar{V}_2}{V_2}\right| = 20\log_{10}\left|\frac{\left(Z_g\left(CZ_L + D\right) + AZ_L + B\right)}{Z_g + Z_L}\right|$$

which is the required result

Prob. 13.52

$$SE = 20\log_{10}\frac{E_i}{E_o} = 20\log_{10}\frac{6}{20\times10^{-6}} = 20\log_{10}(3\times10^5)$$

$$= \underline{\underline{109.54 \text{ dB}}}$$

CHAPTER 14

P. E. 14.1 The program in Fig. 14.3 was used to obtain the plot in Fig. 14.5.

P. E. 14.2 For the exact solution,

$(D^2 + 1)\, y = 0 \longrightarrow \quad y = A\cos x + B\sin x$

$y(0) = 0 \longrightarrow \quad A = 0$

$y(1) = 1 \longrightarrow \quad 1 = B\sin 1$ or $B = 1/\sin 1$

Thus, $y = \sin x/\sin 1$

For the finite difference solution,

$$y'' + y = 0 \quad \longrightarrow \quad \frac{y(x+\Delta) - 2y(x) + y(x-\Delta)}{\Delta^2} + y = 0$$

or

$$y(x) = \frac{y(x+\Delta) + y(x-\Delta)}{2-\Delta^2}, y(0) = 0, y(1) = 1, \Delta = 1/4$$

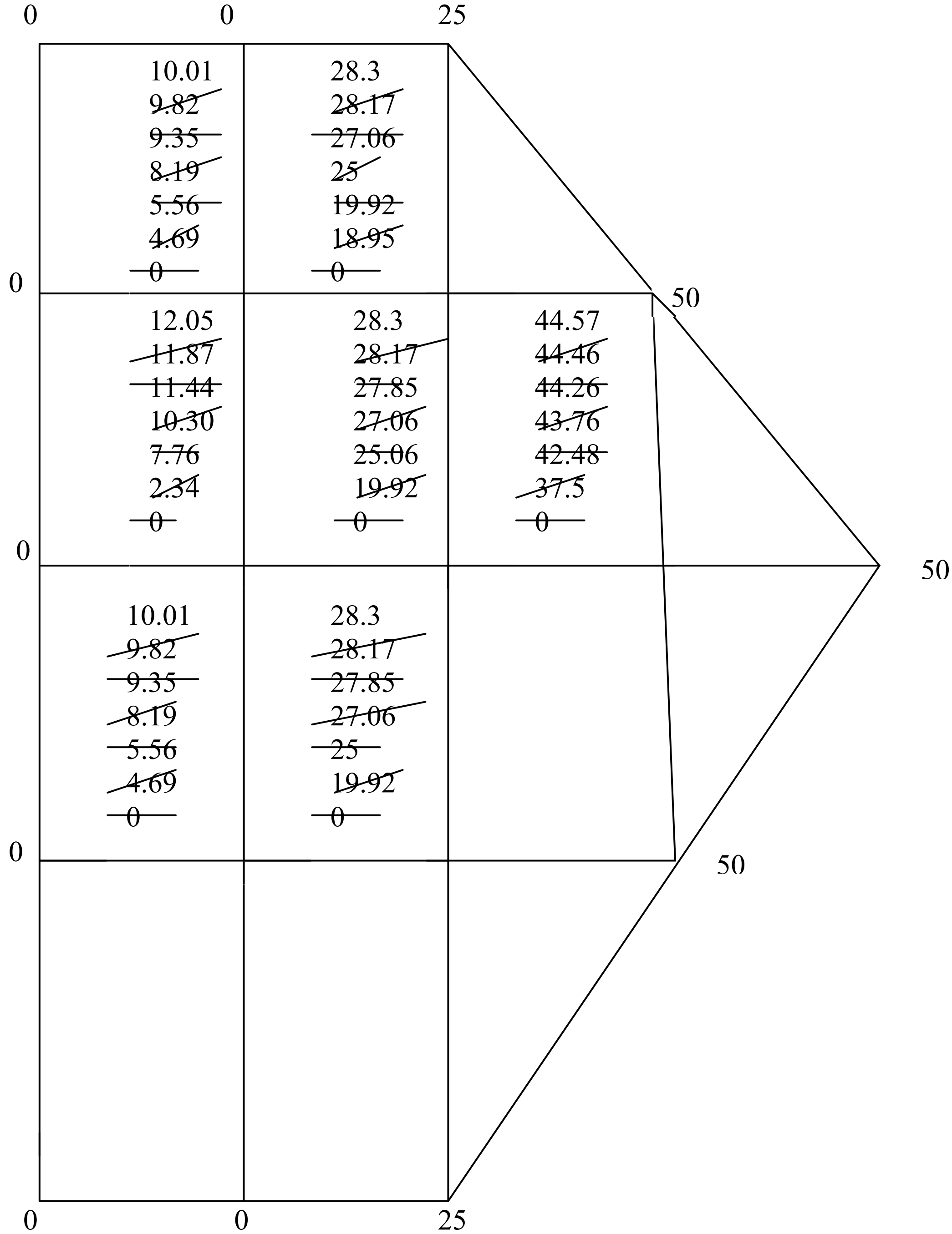
With the MATLAB program shown below, we obtain the exact result y_e and FD result y.

```
y(1)=0.0;
y(5)=1.0;
del=0.25;
for  n=1:20
   for k=2:4
      y(k)=(y(k+1) +y(k-1))/(2-del*del)
      x=(k-1)*del;
      ye=sin(x)/sin(1.0)
   end
end
```

The results are listed below.

y(x)	N=5	N=10	N=15	N=20	Exact $y_e(x)$
y(0.25)	0.2498	0.2924	0.2942	0.2943	0.2940
y(0.5)	0.5242	0.5682	0.5701	0.5702	0.5697
y(0.75)	0.7867	0.8094	0.8104	0.8104	0.8101

P. E. 14.3 By applying eq. (14.16) to each node as shown below, we obtain the following results after 5 iterations.



P. E. 14.4 (a) Using the program in Fig. 14.16 with nx = 4+1=5 and ny = 8+1=9, we obtain the potential at center as

$$V(3,5) = \underline{\underline{23.796\ \text{V}}}$$

(b) Using the same program with nx = 12+1=13 and ny = 24+1=25, the potential at the center is

V(7,13) = 23.883 V

P. E. 14.5 $V = 0.79$ (the solution improves further and gets closer to the expected value due to a finer grid).

Code: It is the same as in Example 14.5 except the following lines:

```
grid_size=0.05;
nt=90; %% Top plate at 90th grid row from the upper boundary
nB=110; %% Bottom plate at 110th grid row from the upper boundary
ns1=90; %% Both plates start from 90th grid column
ns2=110; %% Both plates ends at 110th grid column
```

$v = V((\text{nt}\times\text{lx})+100)$

P. E. 14.6 The code remains essentially the same as in Example 14.6 except for boundary conditions which need to be explicitly defined as explained at the end of Section 14.4 above Practice Exercise 14.6.

P. E. 14.7 By combining the ideas in Figs. 14.23 and 14.27, and dividing each wire into N segments, the results listed in Table 14.2 is obtained.

P.E. 14.8 To determine V and **E** at (-1,4,5), we use the program in Fig. 14.23.

$$V = \int_0^L \frac{\rho_L dl}{4\pi\varepsilon_o R}, \quad \text{where} \quad R = \sqrt{26 + (4 - y')^2}$$

$$V = \frac{\Delta}{4\pi\varepsilon} \sum_{k=1}^{N} \frac{\rho_k}{\sqrt{26 + (y - y_k)^2}}$$

$$E = \int_0^L \frac{\rho_L dl R}{4\pi\varepsilon_o R^3}$$

where **R** = **r** – **r'** = (-1, 4-y', 5), R = |**R**|

$$E_x \cong \frac{\Delta}{4\pi\varepsilon} \sum_{k=1}^{N} \frac{(-1)\rho_k}{[26 + (4 - y_k)^2\ ]^{3/2}}$$

$$E_y \cong \frac{\Delta}{4\pi\varepsilon} \sum_{k=1}^{N} \frac{(4 - y_k)\rho_k}{[26 + (4 - y_k)^2\ ]^{3/2}}$$

$E_z = -5E_x$

For N = 20, V_o = 1V, L = 1m, a = 1mm, the program in Fig. 14.23 is modified. The result is:

V = 12.47 mV, **E** = -0.3266 $\mathbf{a}_x$ + 1.1353$\mathbf{a}_y$ + 1.6331$\mathbf{a}_z$ mV/m

P. E. 14.9 Consider Poisson's equation as the partial differential equation and Dirac delta function as the forcing function.

$$\nabla^2 G = -g$$

Green's function is the impulse response of the given differential equation. So forcing function f in the above expression is Dirac-delta function.

$$\nabla^2 G = -g$$

By considering axial symmetry the variation in the direction of $\emptyset$ and Z is zero.

$$\nabla^2 G = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial G}{\partial\rho}\right)$$

Substituting and integrating over the surface of ρ and $\emptyset$

$$\iint \nabla^2 G\, \rho d\rho d\emptyset = \iint -\delta\, \rho d\rho d\emptyset$$

$$\iint \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial G}{\partial\rho}\rho d\rho d\emptyset = \iint \partial\left(\rho\frac{\partial G}{\partial\rho}\right) d\emptyset = -1$$

$$\int \partial\left(\rho\frac{\partial G}{\partial\rho}\right)\int_0^{2\pi} d\emptyset = \rho\frac{\partial G}{\partial\rho}2\pi = -1$$

$$\int \partial G = \int -\frac{1}{2\pi}\frac{\partial\rho}{\rho}$$

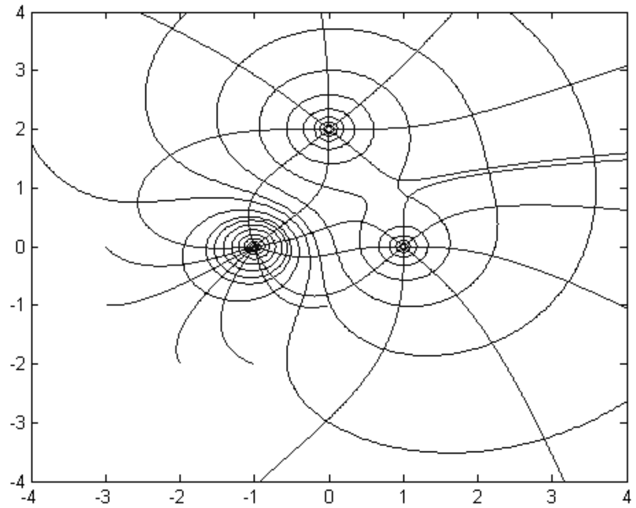
$$G = -\frac{1}{2\pi}ln(\rho)$$

ρ is distance between the source and observation points.

Prob. 14.1 (a) Using the Matlab code in Fig. 14.3, we input the data as:

>> plotit([-1 2 1], [-1 0; 0 2; 1 0], 1, 1, 0.01, 0.01, 8, 2, 5)

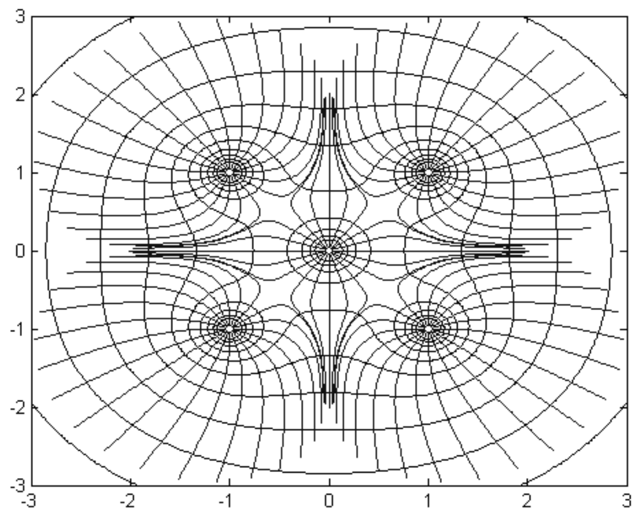
and the plot is shown below.



(b) Using the MATLAB code in Fig. 14.3, we input the required data as:

>> plotit([1 1 1 1 1], [-1 -1; -1 1; 1 –1; 1 1; 0 0], 1, 1, 0.02, 0.01, 6, 2, 5)

and obtain the plot shown below.



Prob. 14.2

$$\nabla^2 V = \frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial V}{\partial \rho} + \frac{\partial^2 V}{\partial z^2} = 0$$

The equivalent finite difference expression is

$$\frac{V(\rho_o + \Delta\rho, z_o) - 2V(\rho_o, z_o) + V(\rho_o - \Delta\rho, z_o)}{(\Delta\rho)^2} + \frac{1}{\rho_o}\frac{V(\rho_o + \Delta\rho, z_o) - V(\rho_o - \Delta\rho, z_o)}{2\Delta\rho}$$

$$+\frac{V(\rho_o, z_o + \Delta z) - 2V(\rho_o, z_o) + V(\rho_o, z_o - \Delta z)}{(\Delta z)^2} = 0$$

If $\Delta z = \Delta\rho = h$, rearranging terms gives

$$V(\rho_o, z_o) = \frac{1}{4}V(\rho_o, z_o + h) + \frac{1}{4}V(\rho_o, z_o - h) + (1 + \frac{h}{2\rho_o})V(\rho + h, z_o)$$

$$+(1 - \frac{h}{2\rho_o})V(\rho - h, z_o)$$

as expected.

Prob. 14.3 (a)

$$\frac{dV}{dx} = \frac{V(x + \Delta x) - V(x - \Delta x)}{2\Delta x}$$

For $\Delta x = 0.05$ and at x = 0.15,

$$\frac{dV}{dx} = \frac{2.0134 - 1.00}{0.05X2} = \underline{\underline{10.117}}$$

$$\frac{d^2V}{dx^2} = \frac{V(x + \Delta x) - 2V(x) + V(x - \Delta x)}{(\Delta x)^2} = \frac{2.0134 + 1.0017 - 2x1.5056}{(0.05)^2} = \underline{\underline{1.56}}$$

(b) V = 10 sinh x, dV/dx = 10 cosh x. At x = 0.15, dV/dx = $\underline{\underline{10.113}}$

which is close to the numerical estimate.

d^2V/dx^2 = 10 sinh x. At x = 0.15, d^2V/dx^2 = $\underline{\underline{1.5056}}$

which is slightly lower than the numerical value.

Prob.14.4

Exact solution:

$$(D^2+4)y=0 \longrightarrow y(x)=A\cos 2x + B\sin 2x$$

$$y(0)=0 \longrightarrow 0=A$$

$$y(1)=10 \longrightarrow 10=B\sin 2 \longrightarrow B=\frac{10}{\sin 2}$$

$$y(x)=10\frac{\sin 2x}{\sin 2}$$

$$y(0.25)=10\frac{\sin 0.5}{\sin 2}=\underline{\underline{5.272}}$$

Finite difference solution:

$$\frac{y(x+\Delta)-2y(x)+y(x-\Delta)}{\Delta^2}+4y(x)=0$$

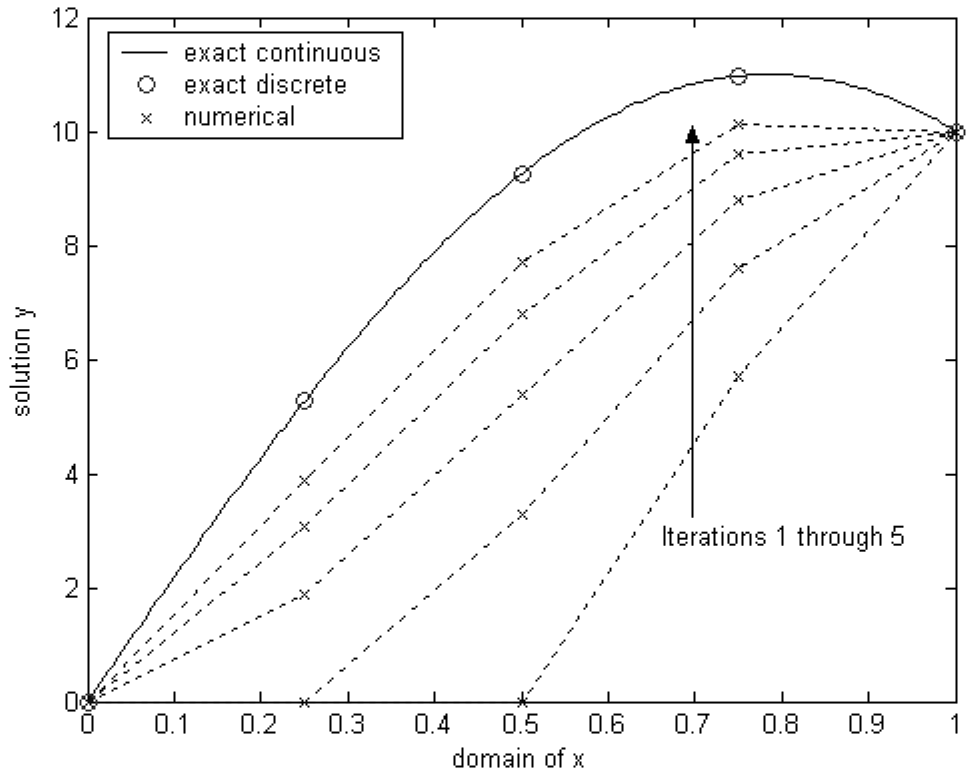
$$y(x+\Delta)+y(x-\Delta)=2y(x)-4\Delta^2 y(x)=(2-4\Delta^2)y(x)$$

or

$$y(x)=\frac{y(x+\Delta)+y(x-\Delta)}{(2-4\Delta^2)}, \qquad \Delta=0.25$$

Using this scheme, we obtain the result shown below. The number of iterations is not enough to get accurate result. The numerical results are compared with the exact solution as shown in the figure below.

Iteration	**0**	**0.25**	**0.5**	**0.75**	**1.0**
0	0	0	0	0	10
1	0	0	0	5.7143	10
2	0	0	3.2653	7.5802	10
3	0	1.8659	5.398	8.7987	10
4	0	3.0844	6.7904	9.5945	10
5	0	3.8802	7.7904	10.1142	10



Prob. 14.5

$$\nabla^2 V = \frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial V}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} = 0\,, \qquad (1)$$

$$\frac{\partial^2 V}{\partial \rho^2} = \frac{V_{m+1}{}^{n} - 2V_m{}^{n} + V_{m+1}{}^{n}}{(\Delta\rho)^2}\,, \qquad (2)$$

$$\frac{\partial^2 V}{\partial \phi^2} = \frac{V_m{}^{n+1} - 2V_m{}^{n} + V_m{}^{n-1}}{(\Delta\phi)^2}\,, \qquad (3)$$

$$\frac{\partial V}{\partial \rho}\Big|_{m,n} = \frac{V^n{}_{m+1} - V^n{}_{m-1}}{2\Delta\rho}\,. \qquad (4)$$

Substituting (2) to (4) into (1) gives

$$\nabla^2 V = \frac{V^n{}_{m+1} - V^n{}_{m-1}}{m\Delta\rho(2\Delta\rho)} + \frac{V_{m+1}{}^{n} - 2V_m{}^{n} + V_{m+1}{}^{n}}{(\Delta\rho)^2} + \frac{V_m{}^{n+1} - 2V_m{}^{n} + V_m{}^{n-1}}{(m\Delta\rho\Delta\phi)^2}$$

$$=\frac{1}{(\Delta\rho)^2}\left[(1-\frac{1}{2m})V_{m-1}{}^{n}-2V_m{}^{n}+(1+\frac{1}{2m})V_{m-1}{}^{n}+\frac{1}{(m\Delta\phi)^2}(V_m{}^{n+1}-2V_m{}^{n}+V_m{}^{n-1})\right]$$

as required.

Prob. 14.6

Iteration →	0	1	2	3	4	5
V_1	0.0000	25.0000	35.6250	38.9063	39.7266	39.9316
V_2	0.0000	26.2500	32.8125	34.4531	34.8633	34.9658
V_3	0.0000	16.2500	22.8125	24.4531	24.8633	24.9658
V_4	0.0000	15.6250	18.9063	19.7266	19.9316	19.9829

Prob. 14.7

$$V_o=\frac{V_1+V_2+V_3+V_4}{4}=\frac{10-40+50+80}{4}=\underline{\underline{25\text{V}}}$$

Prob. 14.8

$$V_1=\frac{1}{4}[20+20+40+V_4]=20+\frac{V_4}{4}$$

$$V_2=\frac{1}{4}[-10+20+0+V_3]=2.5+\frac{V_3}{4}$$

$$V_3=\frac{1}{4}[0+20+V_2+V_4]=5+\frac{1}{4}(V_2+V_4)$$

$$V_4=\frac{1}{4}[0+40+V_1+V_3]=10+\frac{1}{4}(V_1+V_3)$$

Using these relationships, we obtain the data in the table below.

Iteration	1st	2nd	3rd	4th	5th
V_1	20	24.1	24.64	24.72	24.73
V_2	2.5	3.91	5.02	5.22	5.26
V_3	5.625	10.08	10.89	11.03	11.05
V_4	16.406	18.54	18.88	18.94	18.95

Prob. 14.9

(a) We follow Example 6.5 with a=b.

$$V = V_1 + V_2 = \underline{\underline{\frac{4V_o}{\pi}\sum_{n=odd}^{\infty}\frac{\sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi y}{a}\right)}{n\sinh(n\pi)} + \frac{4V_o}{\pi}\sum_{n=odd}^{\infty}\frac{\sin\left(\frac{n\pi y}{a}\right)\sinh\left(\frac{n\pi x}{a}\right)}{n\sinh(n\pi)}}}$$

(b) At the center of the region, finite difference gives

$$V(a/2, a/2) = \frac{1}{4}(0+0+V_o+V_o) = \frac{V_o}{2} = \underline{\underline{25\text{ V}}}$$

Prob. 14.10

$$k = \frac{h^2\rho_s}{\varepsilon} = 10^{-4}\times\frac{50\times10^{-9}}{\frac{10^{-9}}{36\pi}} = 0.18\pi = 0.5655$$

At node 1,

$$V_1 = \frac{1}{4}[0+V_2+V_3+k] \quad \longrightarrow \quad 4V_1 - V_2 - V_3 = k \qquad (1)$$

At node 2,

$$V_2 = \frac{1}{4}[0+V_1+V_4+k] \quad \longrightarrow \quad 4V_2 - V_1 - V_4 = k \qquad (2)$$

At node 3,

$$V_3 = \frac{1}{4}[0+2V_1+V_4+k] \quad \longrightarrow \quad 4V_3 - 2V_1 - V_4 = k \qquad (3)$$

At nde 4,

$$V_4 = \frac{1}{4}[0+2V_2+V_3+k] \quad \longrightarrow \quad 4V_4 - 2V_2 - V_3 = k \qquad (4)$$

Putting (1) to (4) in matrix form,

$$\begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -2 & 0 & 4 & -1 \\ 0 & -2 & -1 & 4 \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} 0.5655 \\ 0.5655 \\ 0.5655 \\ 0.5655 \end{bmatrix}$$

Using a calculator or MATLAB, we obtain

$$\underline{\underline{V_1 = V_2 = 0.3231\text{ V}, \quad V_3 = V_4 = 0.4039\text{ V}}}$$

Prob. 14.11
(a)

$$\begin{bmatrix} -4 & 1 & 0 & 1 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 \\ 1 & 0 & 0 & -4 & 1 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 \\ 0 & 0 & 1 & 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} V_a \\ V_b \\ V_c \\ V_d \\ V_e \\ V_f \end{bmatrix} = \begin{bmatrix} -200 \\ -100 \\ -100 \\ -100 \\ 0 \\ 0 \end{bmatrix}$$

[A] [B]

(b)

$$\begin{bmatrix} -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -4 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \\ V_8 \end{bmatrix} = \begin{bmatrix} -30 \\ -15 \\ -30 \\ -7.5 \\ 0 \\ -7.5 \\ 0 \\ 0 \end{bmatrix}$$

[A] [B]

Prob. 14.12 (a) Matrix [A] remains the same. To each term of matrix [B], we add

$-h^2 \rho_v / \varepsilon$.

(b) Let $\Delta x = \Delta y = h = 0.25$ so that nx = 5= ny.

$$\frac{\rho_v}{\varepsilon} = \frac{x(y-1)10^{-9}}{10^{-9}/36\pi} = 36\pi x(y-1)$$

Modify the program in Fig. 14.16 as follows.

```
   H=0.25;
 for   I=1:nx -1
   for   J=1: ny-1
   X = H*I;
   Y=H*J;
   RO = 36.0*pi*X*(Y-1);
   V(I,J) = 0.25*(V(I+1,J) + V(I-1,J) + V(I,J+1) + V(I,J-1) + H*H*RO);
     end
  end
```

This is the major change. However, in addition to this, we must set

```
v1 = 0.0;
v2 = 10.0;
v3 = 20.0;
v4 = -10.0;
nx = 5;
ny = 5;
```

The results are:

V_a = 4.6095 V_b= 9.9440 V_c= 11.6577
V_d = -1.5061 V_e =3.5090 V_f= 6.6867
V_g= -3.2592 V_h = 0.2366 V_i = 3.3472

Prob. 14.13

$$V_1 = \frac{1}{4}(0+0+V_2+V_4) = \frac{1}{4}(V_2+V_4)$$

$$V_2 = \frac{1}{4}(0+50+V_1+V_3) = \frac{1}{4}(50+V_1+V_3)$$

$$V_3 = \frac{1}{4}(0+100+50+V_2) = \frac{1}{4}(150+V_2)$$

$$V_4 = \frac{1}{4}(0+50+V_1+V_5) = \frac{1}{4}(50+V_1+V_5)$$

$$V_5 = \frac{1}{4}(0+0+V_4+V_6) = \frac{1}{4}(V_4+V_6)$$

$$V_6 = \frac{1}{4}(0+50+V_5+V_7) = \frac{1}{4}(50+V_5+V_7)$$

$$V_7 = \frac{1}{4}(0+100+V_6+50) = \frac{1}{4}(150+V_6)$$

Initially set all free potentials equal to zero. Apply the seven formulas above iteratively and obtain the results shown below.

n	1	2	3	4	5
V_1	0	6.25	9.77	10.63	10.97
V_2	12.5	24.22	25.83	26.15	26.25
V_3	40.625	43.55	43.96	44.04	44.06
V_4	12.5	14.84	16.70	17.73	17.97
V_5	3.12	7.03	10.29	10.93	11.05
V_6	13.281	24.46	25.98	26.23	26.28
V_7	40.82	43.62	43.99	44.06	44.07

Prob. 14.14

$$\frac{1}{c^2}\frac{\Phi^{j+1}{}_{m,n} + \Phi^{j-1}{}_{m,n} - 2\Phi^{j}{}_{m,n}}{(\Delta t)^2} = \frac{\Phi^{j}{}_{m+1,n} + \Phi^{j}{}_{m-1,n} - 2\Phi^{j}{}_{m,n}}{(\Delta x)^2}$$
$$+\frac{\Phi^{j}{}_{m,n+1} + \Phi^{j}{}_{m,n-1} - 2\Phi^{j}{}_{m,n}}{(\Delta z)^2}$$

If $h = \Delta x = \Delta z$, then after rearranging we obtain

$$\Phi^{j+1}{}_{m,n} = 2\Phi^{j}{}_{m,n} - \Phi^{j-1}{}_{m,n} + \alpha(\Phi^{j}{}_{m+1,n} + \Phi^{j}{}_{m-1,n} - 2\Phi^{j}{}_{m,n)}$$
$$+\alpha(\Phi^{j}{}_{m,n+1} + \Phi^{j}{}_{m,n-1} - 2\Phi^{j}{}_{m,n})$$

where $\alpha = (c\Delta t / h)^2$.

Prob. 14.15

$$\frac{\partial^2 V}{\partial x^2} = \frac{\partial^2 V}{\partial t^2} \quad\longrightarrow\quad \frac{V(x+\Delta x,t) - 2V(x,t) + V(x-\Delta x,t)}{(\Delta x)^2} =$$
$$\frac{V(x,t+\Delta t) - 2V(x,t) + V(x,t-\Delta t)}{(\Delta t)^2}$$

$$V(x,t+\Delta t) = \left(\frac{\Delta t}{\Delta x}\right)^2 \left[V(x+\Delta x,t) - 2V(x,t) + V(x-\Delta x,t\right] + 2V(x,t) - V(x,t-\Delta t)$$

or

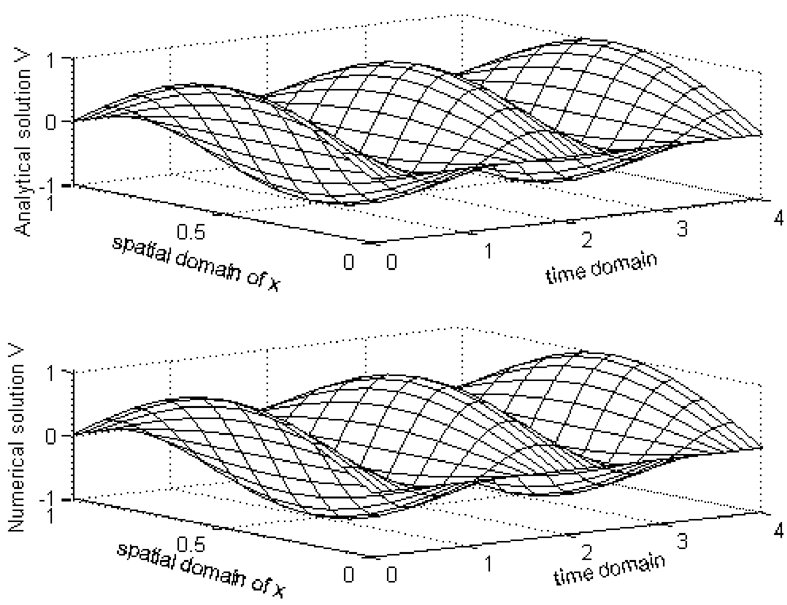
$$V(i,j+1) = \alpha\left[V(i+1,j) + v(i-1,j)\right] + 2(1-\alpha)V(i,j) - V(i,j-1)$$

where $\alpha = \left(\frac{\Delta t}{\Delta x}\right)^2$. Applying the finite difference formula derived above, the following programs was developed.

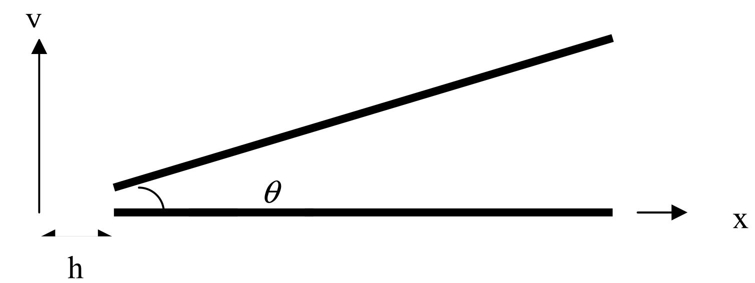
```
xd=0:.1:1;td=0:.1:4;
[t,x]=meshgrid(td,xd);
Va=sin(pi*x).*cos(pi*t);%Analytical result
subplot(211) ;mesh(td,xd,Va);colormap([0 0 0])
% Numerical result
N=length(xd);M=length(td);
v(:,1)=sin(pi*xd');
v(2:N-1,2)=(v(1:N-2,1)+v(3:N,1))/2;
for k=2:M-1
   v(2:N-1,k+1)=-v(2:N-1,k-1)+v(1:N-2,k)+v(3:N,k);
```

end
subplot(212);mesh(td,xd,v);colormap([0 0 0])

The results of the finite difference algorithm agree perfectly with the exact solution as shown below.



Prob. 14.16



To find C, take the following steps:

(1) Divide each line into N equal segments. Number the segments in the lower conductor as 1, 2, …, N and segments in the upper conductor as N+1, N+2, …, 2N,

(2) Determine the coordinate (x_k, y_k) for the center of each segment.

For the lower conductor, $y_k = 0$, k=1, …, N, $x_k = h + \Delta(k-1/2)$, k = 1,2,… N

For the upper conductor, $y_k = [h + \Delta(k-1/2)] \sin\theta$, k=N+1, N+2, …,2 N,

$x_k = [h + \Delta(k-1/2)] \cos\theta$, k = N+1,N+2,… 2N

where h is determined from the gap g as

$$h = \frac{g}{2\sin\theta/2}$$

(3) Calculate the matrices [V] and [A] with the following elements

$$V_k = \begin{cases} V_o, k = 1,..., N \\ -V_o, k = N+1,...2N \end{cases}$$

$$A_{ij} = \begin{cases} \dfrac{\Delta}{4\pi\varepsilon R_{ij}}, i \neq j \\ 2\ln\Delta / a, i = j \end{cases}$$

where $R_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$

(4) Invert matrix [A] and find $[\rho] = [A]^{-1} [V]$.

(5) Find the charge Q on one conductor

$$Q = \sum \rho_k \Delta = \Delta \sum_{k=1}^{N} \rho_k$$

(6) Find $C = |Q|/2V_o$

Taking N= 10, V_o = 1.0, a program was developed to obtain the following result.

θ	C (in pF)
10	8.5483
20	9.0677
30	8.893
40	8.606
50	13.004
60	8.5505
70	9.3711
80	8.7762
90	8.665
100	8.665
110	10.179
120	8.544
130	9.892
140	8.7449
150	9.5106
160	8.5488
170	11.32
180	8.6278

Prob. 14.17 Combining the ideas in the programs in Figs. 14.23 and 14.27, we develop a MATLAB code which gives

N = 20 $\longrightarrow$ C = 19.4 pF/m

N = 40 $\longrightarrow$ C = 13.55 pF/m

N = 100 $\longrightarrow$ $\underline{\underline{\text{C = 12.77 pF/m}}}$

For the exact value, d/2a = 50/10 = 5

$$C = \frac{\pi\varepsilon}{\cosh^{-1}\dfrac{d}{2a}} = \frac{\pi\times10^{-9}/36\pi}{\cosh^{-1}5} = \underline{\underline{12.12}}\ \text{pF/m}$$

Prob. 14.18 We may modify the program in Fig. 14.27 and obtain the result in the table below. $Z_o \cong \underline{\underline{100\ \Omega}}$.

N	Z_o, in Ω
10	97.2351
20	97.8277
30	98.0515
40	98.1739
50	98.2524

Prob. 14.19

We make use of the formulas in Problem 14.18.

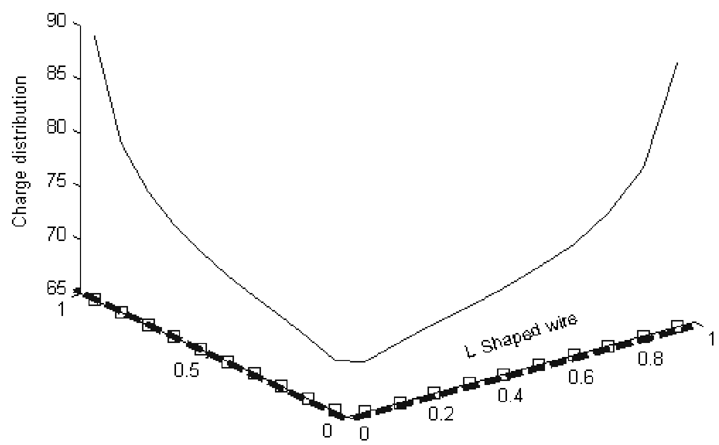
$$V_i = \sum_{j=1}^{2N} A_{ij}\rho_i$$

where N is the number of divisions on each arm of the conductor.

The MATLAB codes is as follows:

```
aa=0.001;
L=2.0;
N=10;  %no.of divisions on each arm
NT=N*2;
delta=L/(NT);
x=zeros(NT,1);
y=zeros(NT,1);
%Second calculate the elements of the coefficient matrix
for i=1:N-1
   y(i)=0;
   x(i)=delta*(i-0.5)
end
for i=N+1:NT
   x(i)=0;
   y(i)=delta*(i-N-0.5);
end
for  i=1:NT
   for j=1:NT
      if (i ~=j)
         R=sqrt((x(i)-x(j))^2 + (y(i)-y(j))^2)
      A(i,j)=-delta*R;
     else
      A(i,j)=-delta*(log(delta)-1.5);
      end
   end
end
%Determine the matrix of constant vector B and find rho
B=2*pi*eo*vo*ones(NT,1);
rho=inv(A)*B;
```

The result is presented below.



Segment	x	y	ρ in pC/m
1	0.9500		89.6711
2	0.8500	0	80.7171
3	0.7500	0	77.3794
4	0.6500	0	75.4209
5	0.5500	0	74.0605
6	0.4500	0	73.0192
7	0.3500	0	72.1641
8	0.2500	0	71.4150
9	0.1500	0	70.6816
10	0.0500	0	69.6949
11	0	0	69.6949
12	0	0.0500	70.6816
13	0	0.1500	71.4150
14	0	0.2500	72.1641
15	0	0.3500	73.0192
16	0	0.4500	74.0605
17	0	0.5500	75.4209
18	0	0.6500	77.3794
19	0	0.7500	80.7171
20	0	0.8500	89.6711

Prob. 14.20(a) Exact solution yields

$C = 2\pi\varepsilon / \ln(\Delta / a) = 8.02607 \times 10^{-11}$ F/m and $Z_o = 41.559\Omega$

where a = 1cm and Δ= 2cm. The numerical solution is shown below.

N	C (pF/m)	$Z_o(\Omega)$
10	82.386	40.486
20	80.966	41.197
40	80.438	41.467
100	80.025	41.562

(b)For this case, the numerical solution is shown below.

N	C (pF/m)	$Z_o(\Omega)$
10	109.51	30.458
20	108.71	30.681
40	108.27	30.807
100	107.93	30.905

Prob. 14.21 We modify the MATLAB code in Fig. 14.27 (for Example 14.7) by changing the input data and matrices [A] and [B]. We let

$x_i = h + \Delta(i-1/2)$, i = 1,2,… N, Δ = L/N

$y_i = h/2$, j = 1,2,… N, $z_k = t/2$, k = 1,2,… N

and calculate

$$R_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2}$$

We obtain matrices [A] and [B]. Inverting [A] gives

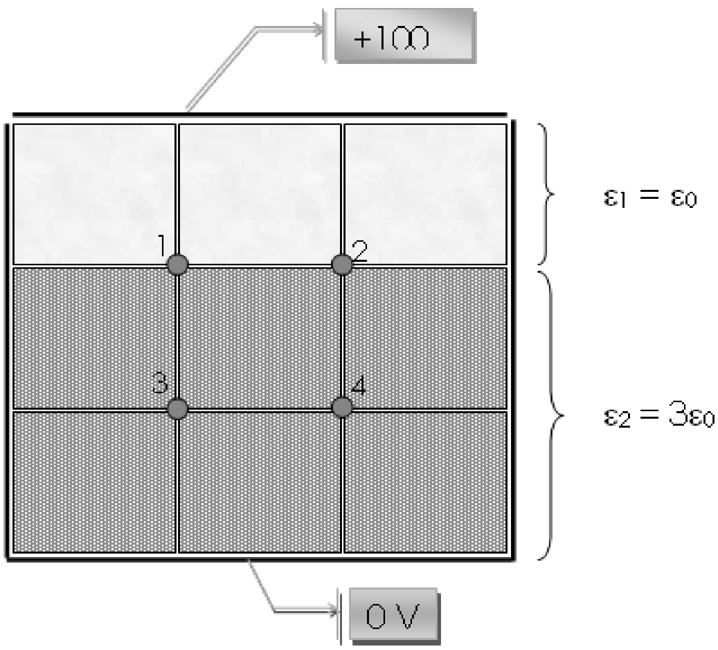
$[q] = [A]^{-1} [B]$, $[\rho_v] = [q]/(ht\Delta)$, $C = \dfrac{\sum_{i=1}^{N} q_i}{10}$

The computed values of $[\rho_v]$ and C are shown below.

Prob. 14.22

The MATLAB code is similar to the one in Fig.14.34. When the program is run, it gives $Z_o = \underline{\underline{40.587\ \Omega}}$.

Prob. 14.23



On the interface,

$$\frac{\varepsilon_1}{2(\varepsilon_1+\varepsilon_2)} = \frac{1}{8}, \qquad \frac{\varepsilon_2}{2(\varepsilon_1+\varepsilon_2)} = \frac{3}{8}$$

$$V_1 = \frac{V_2}{4} + \frac{3V_3}{8} + 12.5$$

$$V_2 = 12.5 + \frac{3V_4}{8} + \frac{V_1}{4}$$

$$V_3 = \frac{1}{4}(V_1 + V_4)$$

$$V_4 = \frac{1}{4}(V_2 + V_3)$$

Applying this iteratively, we obtain the results shown in the table below.

No. of iterations	0	1	2	3	4	5…	100
V_1	0	12.5	17.57	19.25	19.77	19.93	20
V_2	0	15.62	18.65	19.58	19.87	19.96	20
V_3	0	3.125	5.566	6.33	6.56	6.6634	6.667
V_4	0	4.688	6.055	6.477	6.608	6.649	6.667

Prob. 14.24

$$V_1 = \frac{1}{4}(0+0+100+V_2) = 25 + \frac{V_2}{4}$$

$$V_2 = \frac{1}{4}(0+100+V_1+V_3) = 25 + \frac{V_1+V_3}{4}$$

$$V_3 = \frac{1}{4}(0+0+100+V_2) = 25 + \frac{V_2}{4}$$

$$V_4 = \frac{1}{4}(0+0+100+V_5) = 25 + \frac{V_5}{4}$$

$$V_5 = \frac{1}{4}(0+0+100+V_4+V_6) = 25 + \frac{(V_4+V_6)}{4}$$

$$V_6 = \frac{1}{4}(0+0+100+V_5) = 25 + \frac{V_5}{4}$$

We initially set $V_1 = V_2 = V_3 = V_4 = V_5 = V_6 = 0$ and then apply above formulas iteratively. The solutions are presented in the table below.

iteration	1st	2nd	3rd	4th	5th
V_1	25	32.81	35.35	35.67	35.71
V_2	31.25	41.41	42.68	42.83	42.85
V_3	32.81	35.35	35.67	35.71	35.71
V_4	25	23.81	35.35	35.67	35.71
V_5	31.25	41.41	42.68	42.83	42.85
V_6	32.81	35.35	35.67	35.71	35.71

$$\underline{\underline{V_1 = V_4 = 35.71 \text{ V}, \quad V_2 = V_5 = 42.85\text{V}, \quad V_3 = V_6 = 35.71 \text{ V}}}$$

Alternatively, if we take advantage of the symmetry, $V_1 = V_3 = V_4 = V_6$ and $V_2 = V_5$. We need to find solve two equations, namely,

$V_1 = 25 + V_2 \ / 4$

$V_2 = 25 + V_1 / 2$

Solving these gives

$V_1 = 35.714$

$V_2 = 42.857$

Other node voltages follow.

Prob. 14.25

The finite difference solution is obtained by following the same steps as in Example 14.10. We obtain $\underline{\underline{Z_o = 43\ \Omega}}$

Prob.14.26

$$V_1 = \frac{1}{4}(V_2 + 100 + 100 + 100) = \frac{1}{4}V_2 + 75$$

$$V_2 = \frac{1}{4}(V_1 + V_4 + 2V_3)$$

$$V_3 = \frac{1}{4}(V_2 + V_5 + 200) = \frac{1}{4}(V_2 + V_5) + 50$$

$$V_4 = \frac{1}{4}(V_2 + V_7 + 2V_5)$$

$$V_5 = \frac{1}{4}(V_3 + V_4 + V_6 + V_8)$$

$$V_6 = \frac{1}{4}(V_5 + V_9 + 200) = \frac{1}{4}(V_5 + V_9) + 50$$

$$V_7 = \frac{1}{4}(V_4 + 2V_8 + 0) = \frac{1}{4}(V_4 + 2V_8)$$

$$V_8 = \frac{1}{4}(V_5 + V_7 + V_9)$$

$$V_9 = \frac{1}{4}(V_6 + V_8 + 100 + 0) = \frac{1}{4}(V_6 + V_8) + 25$$

Using these equations, we apply iterative method and obtain the results shown below.

	1st	2nd	3rd	4th	5th
V_1	75	79.687	87.11	89.91	92.01
V_2	18.75	48.437	59.64	68.06	74.31
V_3	54.69	65.82	73.87	79.38	82.89
V_4	4.687	19.824	34.57	46.47	53.72
V_5	14.687	35.14	49.45	57.24	61.78
V_6	53.71	68.82	74.2	77.01	78.6
V_7	1.172	6.958	18.92	26.08	30.194
V_8	4.003	20.557	28.93	33.53	36.153
V_9	39.43	47.34	50.78	52.63	53.69

Prob. 14.27

Applying the difference method,

$$V_1 = \frac{V_3}{4} + \frac{V_2}{2} + 25$$

$$V_2 = \frac{1}{4}(V_1 + V_4) + 50$$

$$V_3 = \frac{1}{4}(V_1 + 2V_4)$$

$$V_4 = \frac{1}{4}(V_2 + V_3 + V_5)$$

$$V_5 = \frac{V_4}{4} + 50$$

Applying these equations iteratively, we obtain the results below.

Iterations	0	1	2	3	4	5…	100
V_1	0	25.0	54.68	64.16	70.97	73.79	74.68
V_2	0	56.25	67.58	74.96	78.23	79.54	80.41
V_3	0	6.25	17.58	33.91	38.72	40.63	41.89
V_4	0	15.63	35.74	41.96	44.86	45.31	45.95
V_5	0	53.91	58.74	60.49	61.09	61.37	51.49

CHAPTER 15

P. E. 15.1 From Table 15.1, the functional for the two-dimensional diffusion equation is

$$\mathrm{F} = \frac{1}{2}\int_v \left(\left(\left(\frac{\partial A}{\partial x}\right)^2 + \left(\frac{\partial A}{\partial y}\right)^2 \right) + j\omega\mu\sigma\, A^2 \right) dxdy$$

The function of the functional is $f = \frac{1}{2}\left(\left(\left(\frac{\partial A}{\partial x}\right)^2 + \left(\frac{\partial A}{\partial y}\right)^2 \right) + j\omega\mu\sigma\, A^2 \right) = \frac{1}{2}({A'_x}^2 + {A'_x}^2 + j\omega\mu\sigma\, A^2)$

Putting this in the Euler-Lagrange equation,

$$\frac{\partial f}{\partial A} - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial A'_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial A'_y}\right) = 0$$

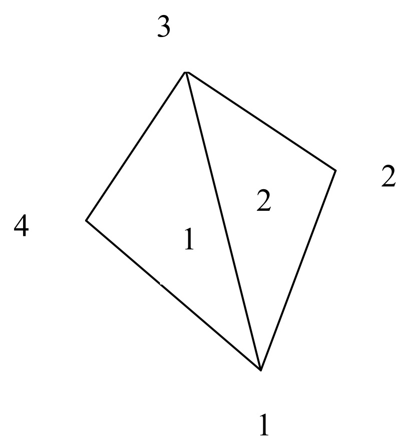
$$j\omega\mu\sigma\, A - (A''_x) - (A''_y) = 0$$

$$(A''_x) + (A''_y) - j\omega\mu\sigma\, A = 0$$

Hence, minimizing the above functional leads to the solution of the diffusion equation.

P. E. 15.2

(a)



For element 1, local 1-2-3 corresponds with global 1-3-4 so that $A_1 = 0.35$,

$P_1 = 0.8$, $P_2 = 0.6$, $P_3 = -1.4$, $Q_1 = -0.5$, $Q_2 = 0.5$, $Q_3 = 0$

$$C^{(1)} = \begin{bmatrix} 0.6357 & 0.1643 & -0.8 \\ 0.1643 & 0.4357 & -0.6 \\ -0.8 & -0.6 & 1.4 \end{bmatrix}$$

For element 2, local 1-2-3 corresponds with global 1-2-3 so that $A_2 = 0.7$,

$P_1 = 0.1$, $P_2 = 1.4$, $P_3 = -1.5$, $Q_1 = -1$, $Q_2 = 0$, $Q_3 = 1$

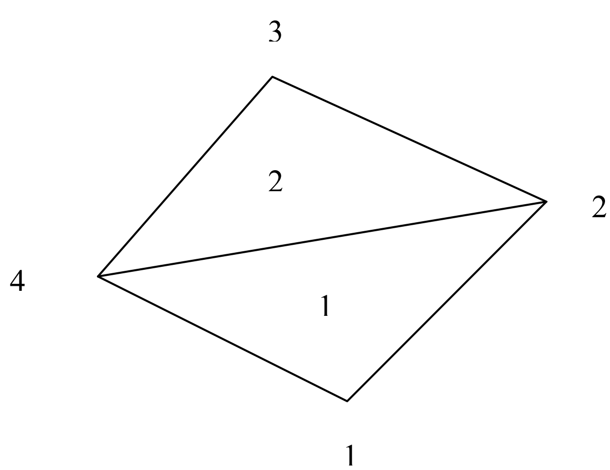
$$C^{(2)} = \begin{bmatrix} 0.3607 & 0.05 & -0.4107 \\ 0.05 & 0.7 & -0.75 \\ -0.4107 & -0.75 & 1.1607 \end{bmatrix}$$

The global coefficient matrix is given by

$$C = \begin{bmatrix} C^{(1)}_{11} + C_{11}^{(2)} & C_{12}^{(2)} & C_{12}^{(1)} + C_{13}^{(2)} & C_{13}^{(1)} \\ C_{21}^{(2)} & C_{22}^{(2)} & C_{23}^{(2)} & 0 \\ C_{21}^{(1)} + C_{31}^{(2)} & C_{32}^{(2)} & C_{22}^{(1)} + C_{33}^{(2)} & C_{23}^{(1)} \\ C_{31}^{(1)} & 0 & C_{32}^{(2)} & C_{33}^{(1)} \end{bmatrix}$$

$$= \begin{bmatrix} 0.9964 & 0.05 & -0.2464 & -0.8 \\ 0.05 & 0.7 & -0.75 & 0 \\ -0.2464 & -0.75 & 1.596 & -0.6 \\ -0.8 & 0 & -0.75 & 1.4 \end{bmatrix}$$

(b)



For element 1, local 1-2-3 corresponds with global 1-2-4 .
$P_1 = 0.9000$; $P_2 = 0.6000$; $P_3 = -1.5000$
$Q_1 = -1.5000$; $Q_2 = 0.5000$; $Q_3 = 1$;

$A_1 = 0.6750$;

$$C^{(1)} = \begin{bmatrix} 1.1333 & -0.0778 & -1.0556 \\ -0.0778 & 0.2259 & -0.1481 \\ -1.0556 & -0.1481 & 1.2037 \end{bmatrix}$$

For element 2, local numbering 1-2-3 corresponds with global numbering 2-3-4.
$P_1 =$ 0.8000; $P_2 =$ -0.9000 ; $P_3 =$ 0.1000 ;
$Q_1 =$ -0.5000 ; $Q_2 =$ 1.5000 ; $Q_3 =$ -1 ;

$A_2 =$ 0.3750 ;

$$C^{(2)} = \begin{bmatrix} 0.5933 & -0.9800 & 0.3867 \\ -0.9800 & 2.0400 & -1.0600 \\ 0.3867 & -1.0600 & 0.6733 \end{bmatrix}$$

The global coefficient matrix is

$$C = \begin{bmatrix} C^{(1)}_{11} & C_{12}^{(1)} & 0 & C_{13}^{(1)} \\ C_{21}^{(1)} & C_{22}^{(1)} + C_{11}^{(2)} & C_{12}^{(2)} & C_{23}^{(1)} + C_{13}^{(2)} \\ 0 & C_{12}^{(2)} & C_{22}^{(2)} & C_{23}^{(2)} \\ C_{31}^{(1)} & C_{32}^{(1)} + C_{31}^{(2)} & C_{32}^{(2)} & C_{33}^{(1)} + C_{33}^{(2)} \end{bmatrix}$$

$$C = \begin{bmatrix} 1.1333 & -0.0778 & 0 & -1.0556 \\ -0.0778 & 0.8193 & -0.9800 & 0.2385 \\ 0 & -0.9800 & 2.0400 & -1.0600 \\ -1.0556 & 0.2385 & -1.0600 & 1.8770 \end{bmatrix}$$

P.E. 15.3 We use the MATLAB program in Fig. 15.8. The input data for the region in Fig. 15.9 is as follows:

```
NE = 32;  ND =  26;  NP = 18;
NL = [ 1  2  4
     2  5  4
     2  3  5
     3  6  5
     4  5  9
     5  10  9
     5  6  10
```

```
6  11  10
7  8  12
 8  13  12
8  9  13
9  14  13
 9  10  14
10  15  14
10  11  15
11  16  15
12  13  17
 13  18  17
13  14  18
 14  19  18
 14  15  19
 15  20  19
 15  16  20
 16  21  20
 17  18  22
 18  23  22
 18  19  23
 19   24  23
  19  20  24
  20  25  24
 20  21  25
 21  26  25];
X = [ 1.0 1.5 2.0 1.0 1.5 2.0 0.0  0.5 1.0 1.5  2.0 0.0 0.5 1.0 1.5 2.0 0.0 0.5  1.0 1.5
     2    0.0  0.5 1.0 1.5 2.0];
 Y = [ 0.0  0.0  0.0 0.5 0.5  0.5 1.0 1.0  1.0 1.0  1.0  1.5  1.5  1.5  1.5  1.5  2.0  2.0  2.0
   2.0  2.0  2.5  2.5  2.5  2.5  2.5  ];
 NDP = [ 1   2  3  6  11  16  21  26  25  24  23  22  17  12  7  8  9  4];
 VAL = [0.0  0.0  15.0  30.0  30.0  30.0  30.0  25.0  20.0  20.0  20.0 10.0  0.0  0.0 0.0
        0.0  0.0  0.0];
```

With this data, the finite element (FEM) solution is compared with the finite difference (FD) solution as shown in the table below.

Node #	X	Y	FEM	FD
5	1.5	0.5	11.265	11.25
10	1.5	1.0	15.06	15.02
13	0.5	1.5	4.958	4.705
14	1.0	1.5	9.788	9.545
15	1.0	1.5	18.97	18.84
18	0.5	2.0	10.04	9.659
19	1.0	2.0	15.32	15.85
20	1.5	2.0	21.05	20.87

P. E. 15.4

Using Eq. (15.22):

Energy stored =

$$\frac{1}{2}\times 10^6 \times [4\ \ 5\ \ 6]\begin{bmatrix} 0.56 & -0.46 & -0.1 \\ -0.46 & 0.82 & -0.37 \\ -0.1 & -0.37 & 0.47 \end{bmatrix}\times 10^6 \times \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}\times 8.85\times 10^{-12} = 4.3\ \text{J/m}$$

P. E. 15.5

$$\begin{bmatrix} V_{e1} \\ V_{e2} \\ V_{e3} \end{bmatrix} = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$V_e = [1 \quad x \quad y]\frac{1}{(x_2-x_1)(x_3-x_1)(x_3-x_2)}$$
$$\begin{bmatrix} (x_3-x_2)x_3x_2 & (x_1-x_3)x_3x_1 & (x_2-x_1)x_2x_1 \\ -(x_3-x_2)(x_3+x_2) & -(x_1-x_3)(x_1+x_3) & -(x_2-x_1)(x_2+x_1) \\ (x_3-x_2) & (x_1-x_3) & (x_2-x_1) \end{bmatrix}\begin{bmatrix} V_{e1} \\ V_{e2} \\ V_{e3} \end{bmatrix}$$

$$V_e = \sum_{i=1}^{3}\alpha_i(x,y)\,V_{ei}$$

$$\alpha_1 = \frac{1}{(x_2-x_1)(x_3-x_1)(x_3-x_2)}[(x_3-x_2)x_3x_2 - (x_3-x_2)(x_3+x_2)x + (x_3-x_2)x^2] = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}$$

$$\alpha_2 = \frac{1}{(x_2-x_1)(x_3-x_1)(x_3-x_2)}[(x_1-x_3)x_3x_1 - (x_1-x_3)(x_1+x_3)x + (x_1-x_3)x^2] = \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)}$$

$$\alpha_3 = \frac{1}{(x_2-x_1)(x_3-x_1)(x_3-x_2)}[(x_2-x_1)x_2x_1 - (x_2-x_1)(x_2+x_1)x + (x_2-x_1)x^2] = \frac{(x-x_2)(x-x_1)}{(x_3-x_2)(x_3-x_1)}$$

Therefore, for an i^{th} node of an element having i, j and k nodes, the shape function is

$$\alpha_i = \frac{(x-x_j)(x-x_k)}{(x_i-x_j)(x_i-x_k)}$$

P. E. 15.6

$$\mathbf{E} = -\nabla V$$

Using eq. (15.38),

$$V_e = \alpha_1 V_{e1} + \alpha_2 V_{e2} + \alpha_3 V_{e3}$$

Using eq. (15.11), we obtain:

$$V_e = \frac{(x_2y_3 - x_3y_2) + P_1x + Q_1y}{2\Delta} V_{e1} + \frac{(x_3y_1 - x_1y_3) + P_2x + Q_2y}{2\Delta} V_{e2} + \frac{(x_1y_2 - x_2y_1) + P_3x + Q_3y}{2\Delta} V_{e3}$$

$$\mathbf{E}^e = -\left(\frac{P_1V_{e1} + P_2V_{e2} + P_3V_{e3}}{2\Delta}\right)\mathbf{a}_x - \left(\frac{Q_1V_{e1} + Q_2V_{e2} + Q_3V_{e3}}{2\Delta}\right)\mathbf{a}_y$$

$$\mathbf{E}^e = -\left(\frac{((0.5\times-1)+(0.25\times1)+0)\times10^{-6}}{2\times0.5\times10^{-6}}\right)\mathbf{a}_x - \left(\frac{(0+(0.25\times-1)+(0.16\times1))\times10^{-6}}{2\times0.5\times10^{-6}}\right)\mathbf{a}_y$$

$$\mathbf{E}^e = 0.25\mathbf{a}_x + 0.9\mathbf{a}_y$$

P. E. 15.7

Area of element 1 = 0.5×10^{-6} m^2
Area of element 2 = 0.5×10^{-6} m^2
Using equation (15.53),

$$T^{(1)} = T^{(2)} = \frac{\Delta}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} = \frac{0.5\times10^{-6}}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Assembling of elemental matrices $T^{(e)}$ into global matrix $[T]$ is similar to assembling the element coefficient matrix given in Section 15.3c.

Therefore global matrix $[T]$ is

$$[T] = \begin{bmatrix} \mathrm{T}_{11}^{(1)} & \mathrm{T}_{12}^{(1)} & \mathrm{T}_{13}^{(1)} & 0 \\ \mathrm{T}_{12}^{(1)} & \mathrm{T}_{22}^{(1)} + \mathrm{T}_{11}^{(2)} & \mathrm{T}_{23}^{(1)} + \mathrm{T}_{13}^{(2)} & \mathrm{T}_{12}^{(2)} \\ \mathrm{T}_{13}^{(1)} & \mathrm{T}_{23}^{(1)} + \mathrm{T}_{13}^{(2)} & \mathrm{T}_{33}^{(1)} + \mathrm{T}_{33}^{(2)} & \mathrm{T}_{23}^{(2)} \\ 0 & \mathrm{T}_{12}^{(2)} & \mathrm{T}_{23}^{(2)} & \mathrm{T}_{22}^{(2)} \end{bmatrix}$$

$$[\mathrm{T}] = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 4 & 2 & 1 \\ 1 & 2 & 4 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix} \frac{0.5\times10^{-6}}{12}$$

P. E. 15.8

For element 1 the local node numbers are assigned as 1-2-3 for the global node numbers 1-2-4.

From eq. (15.11b) shape function of node 2 in element 1 is

$$\alpha_2 = \frac{1}{2\Delta_1}[(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y]$$

From eq. (15.12), the area of element 1:

$$\Delta_1 = \frac{1}{2}[(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)] = \frac{1}{2}[|(0.6)(0.9) - (0.4)(-0.4)|]$$

$$\Delta_1 = 0.35 \text{ unit}^2$$

$$\alpha_2 = \frac{1}{2 \times 0.35}\left[\big((1.2)(1.8) - (0.8)(2.7)\big) + 0.9x - 0.4y\right]$$

$$\alpha_2 = \frac{1}{0.7}[0.9x - 0.4y] \quad (A)$$

Similarly for element 2 the local nodes are assigned as 1-2-3 for the global node numbers 2-3-4

From eq. (15.12), the area of element 2 is 0.525 unit^2

$$\alpha_2 = \frac{1}{2 \times 0.525}\left[\big((2.7)(2.1) - (1.2)(2.1)\big) - 0.6x - 0.9y\right]$$

$$\alpha_2 = \frac{1}{1.05}[3.15 - 0.6x - 0.9y] \quad (B)$$

Line equation of segment 4-2:

$$y - 1.4 = \frac{1.3}{-0.2}(x - 1.4)$$

$$y = -0.65x + 10.5 \quad (C)$$

The expression of the two shape functions of node 2 can be rewritten by eliminating y form eq. (*A*) and eq. (*B*) using eq. (*C*).

The expression of the shape function in element 1:

$$\alpha_2 = \frac{1}{0.7}[0.9x - 0.4(-0.65x + 10.5)] = \frac{1}{0.7}[3.5x - 4.2] = 5x - 6$$

$$\alpha_2 = 5x - 6$$

The expressionof the shape function in element 2:

$$\alpha_2 = \frac{1}{0.7}[3.15 - 0.6x - 0.9(-0.65x + 10.5)] = \frac{1}{1.05}[5.25x - 6.3] = 5x - 6$$

$$\alpha_2 = 5x - 6$$

Thus, we can conclude that the expressionof the shape function is same on the segment 4-2 for both elements.

P. E. 15.9

```
% The geometry is the same as that in Example 14.4
%  The geometry is drawn in the pde toolbox and p e t matrices are imported into
% the MATLAB code using procedure described in Section 15.7
clc;
clear all;
load meshdata;%% loading p, e, t matrices
[A3 n_elements]=size(t);% A3=4
[A2 n_edges]=size(e);% A2=7
[A1 n_nodes]=size(p);% A1=2
C=zeros(n_nodes,n_nodes);% Initialization of the global coefficient matrix
K=zeros(n_nodes,n_nodes);
V1=40;% Potential of the top  plate
V2=100;% Potential of the plate on the right hand side
V3=0;% Potential of the plate on the left hand side
V4=10;% Potential of the bottom plate
V=zeros(n_nodes,1);% Initialization of matrix of unknown potentials
B=zeros(n_nodes,1);% Initialization of the right hand side matrix
%contributed by boundary conditions
%Formation of coefficient matrices for all elements
for element=1:n_elements
    nodes=t(1:3,element);% 'nodes' is a 3 by 1 matrix containing global
    % node numbers of element under consideration
    Xc=p(1,nodes');  %   X coordinates of nodes
    Yc=p(2,nodes');   %  Y coordinates of nodes
    P=zeros(3,1);
    Q=zeros(3,1);
    P(1)=Yc(2)-Yc(3);
    P(2)=Yc(3)-Yc(1);
    P(3)=Yc(1)-Yc(2);
    Q(1)=Xc(3)-Xc(2);
    Q(2)=Xc(1)-Xc(3);
    Q(3)=Xc(2)-Xc(1);
    % eq. (15.21c)
    delta= 0.5*abs((P(2)*Q(3))-(P(3)*Q(2)));%Absolute value is taken since
    % the three nodes may not have been numbered in the anticlockwise direction

    for i=1:3
        for j=1:3
            % eq. (15.21b)
            c(element,i,j)=((P(i)*P(j))+(Q(i)*Q(j)))/(4*delta);
        end
     end
end
```

```
for element=1:n_elements
   nodes=t(1:3,element);
   % Formation of the global coefficient matrix
   for i=1:3
      for j=1:3
         C(nodes(i),nodes(j))=c(element,i,j)+C(nodes(i),nodes(j));
      end
   end
end
% Imposing boundary conditions
K=C;
for edge=1:n_edges
   if e(5,edge)==1
      node1=e(1,edge);
      node2=e(2,edge);
      B(node1)=V1;
      B(node2)=V1;
      K(node1,:)=zeros(1,n_nodes);
      K(node1,node1)=1;
      K(node2,:)=zeros(1,n_nodes);
      K(node2,node2)=1;
   end
   if e(5,edge)==2
      node1=e(1,edge);
      node2=e(2,edge);
      B(node1)=V2;
      B(node2)=V2;
      K(node1,:)=zeros(1,n_nodes);
      K(node1,node1)=1;
      K(node2,:)=zeros(1,n_nodes);
      K(node2,node2)=1;
   end
   if e(5,edge)==3
      node1=e(1,edge);
      node2=e(2,edge);
      B(node1)=V4;
      B(node2)=V4;
      K(node1,:)=zeros(1,n_nodes);
      K(node1,node1)=1;
      K(node2,:)=zeros(1,n_nodes);
      K(node2,node2)=1;
   end
```

```
   if e(5,edge)==4
      node1=e(1,edge);
      node2=e(2,edge);
      B(node1)=V3;
      B(node2)=V3;
      K(node1,:)=zeros(1,n_nodes);
      K(node1,node1)=1;
      K(node2,:)=zeros(1,n_nodes);
      K(node2,node2)=1;
   end

end
% When the goemetry of Fig. 14.14 is drawn using the "rectangle"
% option in the pdetoolbox, the four corner nodes get numbered as 1,2,3 and 4
% These nodes need to be made floating to represent gaps between the
% different plates.
for i=1:n_nodes
   if (i==1||i==2||i==3||i==4)
   K(i,:)=C(i,:);
   B(i)=0;
   end
end
%solving for unknown potentials
V=K\B;
figure(1);
pdeplot(p,e,t,'xydata',V,'mesh','off','colormap','jet','colorbar','on',
'contour','on','levels',20);
[Ex,Ey]=pdegrad(p,t,V);% Calculation of E field if required
E=sqrt(Ex.^2+Ey.^2);
figure(2)
pdeplot(p,e,t,'xydata',E,'colorbar','off')
disp('potential at the point (0.5, 0.5)')
 x=0.5;
 y=0.5;
 for i=1:n_nodes
    xc=p(1,i);
    yc=p(2,i);
    dist(i)=((x-xc)^2+(y-yc)^2)^0.5;
 end
 [d,node]=min(dist);
 V(node,1)
```

P. E. 15.10
The code remains essentially the same as in Example 15.10 except for boundary conditions which need to be explicitly defined as explained in the code for TM modes and at the end of Section 14.4 above Practice Exercise 14.6.

Prob. 15.1 From the given figure, we obtain

$$\alpha_1 = \frac{A_1}{A} = \frac{1}{2A}\begin{vmatrix} 1 & x & y \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \frac{1}{2A}[(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y]$$

as expected. The same applies for α_2 and α_3.

Prob. 15.2

(a) $P_1 = 1.5, P_2 = 0.5, P_3 = -2, Q_1 = -1, Q_2 = 1.5, Q_3 = -0.5$

$$A = \frac{1}{2}(P_2Q_3 - P_3Q_2) = 1.375$$

$$C_{ij} = \frac{1}{4A}[P_iP_j + Q_iQ_j]$$

$$C = \begin{bmatrix} 0.5909 & -0.1364 & -0.4545 \\ -0.1364 & 0.4545 & -0.3182 \\ -0.4545 & -0.3182 & 0.7727 \end{bmatrix}$$

(b)

$P_1 = -4, P_2 = 4, P_3 = 0, Q_1 = 0, Q_2 = -3, Q_3 = 3$

$$A = \frac{1}{2}(P_2Q_3 - P_3Q_2) = 6$$

$$C = \begin{bmatrix} 0.6667 & -0.6667 & 0 \\ -0.6667 & 1.042 & -0.375 \\ 0 & -0.375 & 0.375 \end{bmatrix}$$

Prob. 15.3 (a)

$$2A = \begin{vmatrix} 1 & 1/2 & 1/2 \\ 1 & 3 & 1/2 \\ 1 & 2 & 2 \end{vmatrix} = 15/4$$

$$\alpha_1 = \frac{4}{15}[(6-1)+(-1\frac{1}{2})x+(-1)y] = \frac{4}{15}(5-1.5x-y)$$

$$\alpha_2 = \frac{4}{15}[(1-1)+\frac{3}{2}x-\frac{3}{2}y] = \frac{4}{15}(1.5x-1.5y)$$

$$\alpha_3 = \frac{4}{15}[(1/4-3/2)+0x+\frac{5}{2}y] = \frac{4}{15}(-1.25+2.5y)$$

$$V = \alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3$$

Substituting V=80, V_1 = 100, V_2 = 50, V_3 = 30, α_1, α_2, and α_3 leads to

20 = 7.5x + 10y + 3.75

Along side 12, y=1/2 so that

20 = 15x/2 + 5 + 15/4 $\longrightarrow$ x=3/2, i.e (1.5, 0.5)

Along side 13, x =y

20 = 15x/2 + 10x + 15/4 $\longrightarrow$ x=13/4, i.e. (13/14, 13/14)

Along side 23, y = -3x/2 + 5

20 = 15x/2 – 15 + 50 + 15/4 $\longrightarrow$ x=-5/2 (not possible)

Hence intersection occurs at

<u>(1.5, 0.5) along 12 and (0.9286, 0.9286) along 13</u>

(b) At (2,1),

$$\alpha_1 = \frac{4}{15}, \quad \alpha_2 = \frac{6}{15}, \quad \alpha_3 = \frac{5}{15}$$

$V(2,1) = \alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 =$ (400 + 300 + 150)/15 = <u>56.67 V</u>

Prob. 15.4

$$2A = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 1 & 4 \end{vmatrix} = 9$$

$$\alpha_1 = \frac{1}{9}[(0-0)+(4-0)x+(0-1)y] = \frac{1}{9}(4x-y)$$

$$\alpha_2 = \frac{1}{9}[(0-0)+(0+1)x+(2-0)y] = \frac{1}{9}(x+2y)$$

$$\alpha_3 = \frac{1}{9}[(8+1)+(-1-4)x+(1-2)y] = \frac{1}{9}(9-5x-y)$$

$$V_e = \alpha_1 V_{e1} + \alpha_2 V_{e2} + \alpha_{31} V_{e3}$$

V(1,2) = 8(4-2)/9 + 12(1+4)/9 + 10(9-5-1)/9 = 96/9 = $\underline{\underline{10.667\text{ V}}}$

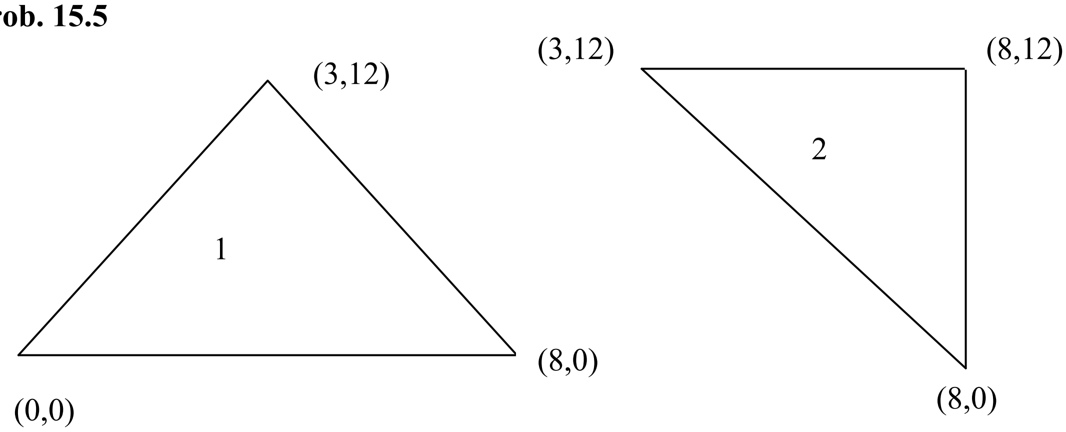
At the center $\alpha_1 = \alpha_2 = \alpha_3 = 1/3$ so that

V(center) = (8 + 12 + 10)/3 = 10

Or at the center, (x, y) = (0 + 1 + 2, 0 + 4 –1)/3 = (1,1)

V(1,1) = 8(3)/9 + 12(3)/9 + 10(3)/9 = $\underline{\underline{10\text{ V}}}$

Prob. 15.5



For element 1, local numbering 1-2-3 corresponds to global numbering 4-2-1.

$P_1 = 12$, $P_2 = 0$, $P_3 = -12$, $Q_1 = -3$, $Q_2 = 8$, $Q_3 = -5$,

A = (0 + 12 x 8)/2 = 48

$$C_{ij} = \frac{1}{4x48}[P_j P_i + Q_j Q_j]$$

$$C^{(1)} = \begin{bmatrix} 0.7969 & -0.125 & -0.6719 \\ -0.125 & 0.3333 & -0.2083 \\ -0.6719 & -0.2083 & 0.8802 \end{bmatrix}$$

For element 2, local numbering 1-2-3 corresponds to global numbering 2-4-3.

P_1 = -12, P_2 = 0, P_3 = 12, Q_1 = 0, Q_2 = -5, Q_3 = 5,

A = (0 + 60)/2 = 30

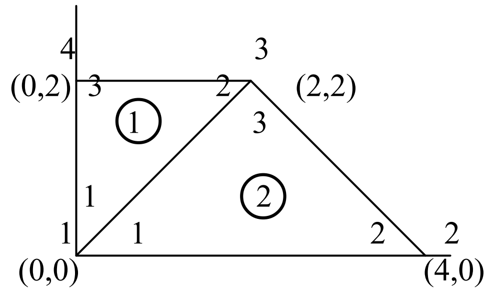
$$C_{ij} = \frac{1}{4x48}[P_j P_i + Q_j Q_j]$$

$$C^{(2)} = \begin{bmatrix} 1.2 & 0 & -1.2 \\ 0 & 0.208 & -0.208 \\ -1.2 & -0.208 & 1.408 \end{bmatrix}$$

$$C = \begin{bmatrix} C^{(1)}_{33} & C_{23}^{(1)} & 0 & C_{31}^{(1)} \\ C_{23}^{(1)} & C_{22}^{(1)} + C_{11}^{(2)} & C_{13}^{(2)} & C_{21}^{(1)} + C_{12}^{(2)} \\ 0 & C_{31}^{(2)} & C_{33}^{(2)} & C_{32}^{(2)} \\ C_{13}^{(1)} & C_{21}^{(1)} + C_{21}^{(2)} & C_{23}^{(2)} & C_{22}^{(2)} + C_{11}^{(1)} \end{bmatrix}$$

$$= \begin{bmatrix} 0.8802 & -0.2083 & 0 & -0.6719 \\ -0.2083 & 1.533 & -1.2 & -0.125 \\ 0 & -1.2 & 1.4083 & -0.2083 \\ -0.6719 & -0.125 & -0.2083 & 1.0052 \end{bmatrix}$$

Prob. 15.6



For element 1,

$$P_1 = 0, P_2 = 2, P_3 = -2, Q_1 = -2, Q_2 = 0, Q_3 = 2$$

$$A = \frac{1}{2}(4-0) = 2, \quad 4A = 8$$

$$C^{(1)} = \frac{1}{8}\begin{bmatrix} 4 & 0 & -4 \\ 0 & 4 & -4 \\ -4 & -4 & 8 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 & -0.5 \\ 0 & 0.5 & -0.5 \\ -0.5 & -0.5 & 1 \end{bmatrix}$$

For element 2,

$$P_1 = -2, P_2 = 2, P_3 = 0, Q_1 = -2, Q_2 = -2, Q_3 = 4$$

$$A = \frac{1}{2}(8-0) = 4, \quad 4A = 16$$

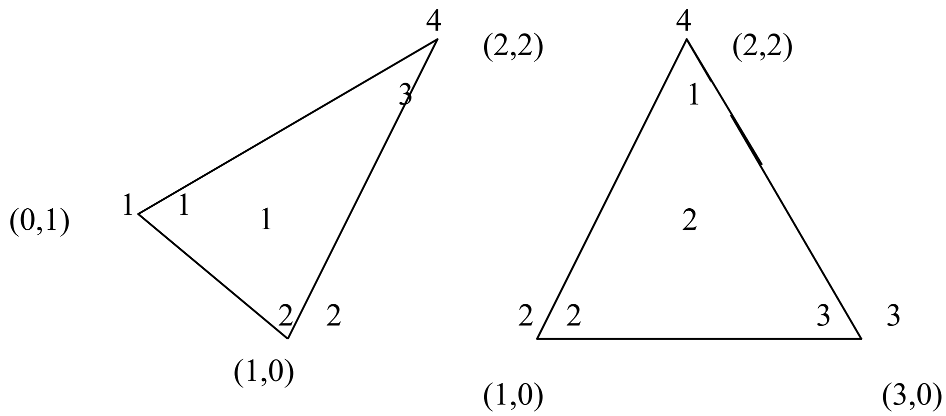
$$C^{(2)} = \frac{1}{16}\begin{bmatrix} 8 & 0 & -8 \\ 0 & 8 & -8 \\ -8 & -8 & 16 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 & -0.5 \\ 0 & 0.5 & -0.5 \\ -0.5 & -0.5 & 1 \end{bmatrix}$$

The global coefficient matrix is

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & C_{34} \\ C_{41} & C_{42} & C_{43} & C_{44} \end{bmatrix} = \begin{bmatrix} C_{11}^{(1)} + C_{11}^{(2)} & C_{12}^{(2)} & C_{12}^{(1)} + C_{13}^{(2)} & C_{13}^{(1)} \\ C_{21}^{(2)} & C_{22}^{(2)} & C_{23}^{(2)} & 0 \\ C_{21}^{(1)} + C_{31}^{(2)} & C_{32}^{(2)} & C_{22}^{(1)} + C_{33}^{(2)} & C_{23}^{(1)} \\ C_{31}^{(1)} & 0 & C_{32}^{(1)} & C_{33}^{(1)} \end{bmatrix}$$

$$= \underline{\underline{\begin{bmatrix} 1 & 0 & -0.5 & -0.5 \\ 0 & 0.5 & -0.5 & 0 \\ -0.5 & -0.5 & 1.5 & -0.5 \\ -0.5 & 0 & -0.5 & 1 \end{bmatrix}}}$$

Prob. 15.7



For element 1, local numbering 1-2-3 corresponds to global numbering 1-2-4.

$P_1 = -2$, $P_2 = 1$, $P_3 = 1$, $Q_1 = 1$, $Q_2 = -2$, $Q_3 = 1$,

$A = (P_2 Q_3 - P_3 Q_2)/2 = 3/2$, i.e. $4A = 6$

$$C_{ij} = \frac{1}{4A}[P_j P_i + Q_j Q_j]$$

$$C^{(1)} = \frac{1}{6}\begin{bmatrix} 5 & -4 & -1 \\ -4 & 5 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

For element 2, local numbering 1-2-3 corresponds to global numbering 4-2-3.

$P_1 = 0$, $P_2 = -2$, $P_3 = 2$, $Q_1 = 2$, $Q_2 = -1$, $Q_3 = -1$,

$A = 2$, $4A = 8$

$$C^{(2)} = \frac{1}{8}\begin{bmatrix} 4 & -2 & -2 \\ -2 & 5 & -3 \\ -2 & -3 & 5 \end{bmatrix}$$

The global coefficient matrix is

$$C = \begin{bmatrix} C^{(1)}_{11} & C_{12}^{(1)} & 0 & C_{13}^{(1)} \\ C_{12}^{(1)} & C_{22}^{(1)} + C_{22}^{(2)} & C_{23}^{(2)} & C_{23}^{(1)} + C_{21}^{(2)} \\ 0 & C_{23}^{(2)} & C_{33}^{(2)} & C_{31}^{(2)} \\ C_{13}^{(1)} & C_{23}^{(1)} + C_{21}^{(2)} & C_{31}^{(2)} & C_{33}^{(1)} + C_{11}^{(2)} \end{bmatrix}$$

$$= \begin{bmatrix} 0.8333 & -0.667 & 0 & -0.1667 \\ -0.6667 & 1.4583 & -0.375 & -0.4167 \\ 0 & -0.375 & 0.625 & -0.25 \\ -0.1667 & -0.4167 & -0.25 & 0.833 \end{bmatrix}$$

Prob. 15.8 We can do it by hand as in Example 15.2. However, it is easier to prepare an input file and use the program in Fig. 15.8. The MATLAB input data is

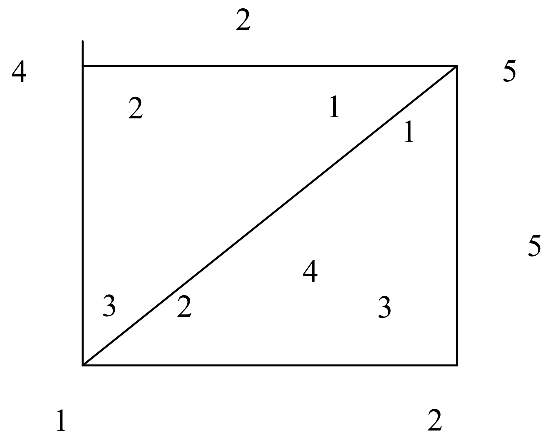
```
NE = 2;
ND = 4;
NP = 2;
NL = [1  2  4
      4    2   3];
X = [ 0.0    1.0   3.0   2.0];
Y = [ 1.0  0.0  0.0  2.0];
NDP= [ 1  3 ];
VAL = [ 10.0   30.0]
```

The result is $V = \begin{bmatrix} 10 \\ 18 \\ 30 \\ 20 \\ \end{bmatrix}$

From this,

$\underline{V_2 = 18 \text{ V}, \quad V_4 = 20 \text{ V}}$

Prob. 15.9



The local numbering 1-2-3 in element 3 corresponds with the global numbering 5-4-1, while the local number 1-2-3 in element 4 corresponds with the global numbering 5-1-2.

$C_{5,5} = C_{11}^{(2)} + C_{11}^{(3)} + C_{11}^{(4)} + C_{11}^{(5)}$, $A = 2$,

$C_{11}^{(2)} = (2 \times 2 + 2 \times 2)/8 = 1 = C_{11}^{(5)}$

$C_{11}^{(3)} = (2 \times 2 + 0)/8 = ½ = C_{11}^{(4)}$

$C_{5,5} = 1 + 1 + ½ + ½ = \underline{3}$

$C_{5,1} = C_{31}^{(3)} + C_{21}^{(4)}$

But $C_{31}^{(3)} = \dfrac{1}{8}(P_3P_1 + Q_3 Q_1) = 0$ since $P_3 = 0 = Q_3$

$C_{21}^{(4)} = \dfrac{1}{8}(P_2P_1 + Q_2 Q_1) = 0$ since $P_3 = 0 = Q_3$

$\underline{\underline{C_{5,1} = 0}}$

Prob. 15.10 As in P. E. 15.3, we use the program in Fig. 15.8. The input data based on Fig. 15.19 is as follows.

```
NE =50;  ND= 36;  NP= 20;
 NL = [1      8      7
       1      2      8
       2      9      8
       2      3      9
       3      10     9
       3      4      10
```

```
        4       11      10
        4       5       11
        5       12      11
        5       6       12
        7       14      13
        7       8       14
        8       15      14
        8       9       15
        9       16      15
        9       10      16
        10      17      16
        10      11      17
        11      18      17
        11      12      18
        13      20      19
        13      14      20
        14      21      20
        14      15      21
        15      22      21
        15      16      22
        16      23      22
        16      17      23
        17      24      23
        17      18      24
        19      26      25
        19      20      26
        20      27      26
        20      21      27
        21      28      27
        21      22      28
        22      29      28
        22      23      29
        23      30      29
        23      24      30
        25      32      31
        25      26      32
        26      33      32
        26      27      33
        27      34      33
        27      28      34
        28      35      34
        28      29      35
        29      36      35
        29      30      36];
X = [0.0  0.2  0.4  0.6  0.8  1.0  0.0  0.2  0.4  0.6  0.8  1.0  0.0  0.2  0.4  0.6 0.8  1.0
     0.0  0.2  0.4  0.6  0.8  1.0  0.0  0.2  0.4  0.6  0.8  1.0  0.2  0.4  0.6  0.8  1.0];
```

Y = [0.0 0.0 0.0 0.0 0.0 0.0 0.2 0.2 0.2 0.2 0.2 0.2 0.4 0.4 0.4 0.4 0.4 0.4 0.6 0.6 0.6 0.6 0.6 0.6 0.8 0.8 0.8 0.8 0.8 0.8 1.0 1.0 1.0 1.0 1.0 1.0];
NDP = [1 2 3 4 5 6 12 18 24 30 36 35 34 33 32 31 25 19 13 7];
VAL = [0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 50.0 100.0 100.0 100.0
100.0 50.0 0.0 0.0 0.0 0.0];

With this data, the potentials at the free nodes are compared with the exact values as shown below.

Node no.	FEM Solution	Exact Solution
8	4.546	4.366
9	7.197	7.017
10	7.197	7.017
11	4.546	4.366
14	10.98	10.60
15	17.05	16.84
16	17.05	16.84
17	10.98	10.60
20	22.35	21.78
21	32.95	33.16
22	32.95	33.16
23	22.35	21.78
26	45.45	45.63
27	59.49	60.60
28	59.49	60.60
29	45.45	45.63

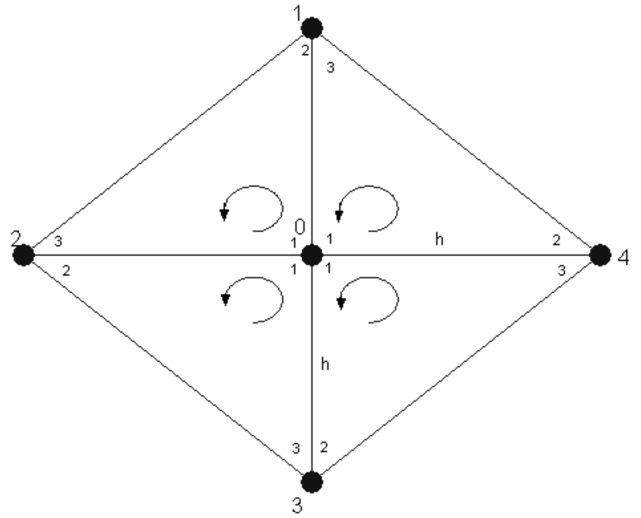
Prob. 15.11 We use exactly the same input data as in the previous problem except that the last few lines are replaced by the following lines.

VAL = [0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 0.0 29.4 58.8 95.1 95.1
58.8 29.4 0.0 0.0 0.0 0.0];

The potential at the free nodes obtained with the input data are compared with the exact solution as shown below.

Node no.	FEM Solution	Exact Solution
8	3.635	3.412
9	5.882	5.521
10	5.882	5.521
11	3.635	3.412
14	8.659	8.217
15	14.01	13.30
16	14.01	13.30
17	8.659	8.217
20	16.99	16.37
21	27.49	26.49
22	27.49	26.49
23	16.99	16.37
26	31.81	31.21
27	51.47	50.5
28	51.47	50.5
29	31.81	31.21

Prob. 15.12



For element 1, the local numbering 1-2-3 corresponds with nodes with V_1 , V_2 , and V_3.

$$V_o = -\frac{1}{C_{oo}}\sum_{i=1}^{4} V_i C_{io}$$

$$C_{oo} = \sum_{j=1}^{4} C_{oj}^{(e)} = \frac{1}{4h^2/2}(hh+hh)\times 2 + \frac{1}{4h^2/2}(hh+0)\times 4 = 4$$

$$C_{o1} = \frac{2\times 1}{2h^2}[P_3P_1 + Q_3Q_1] = \frac{2}{2h^2}[-hh-0] = -1$$

$$C_{o2} = \frac{2\times 1}{2h^2}[P_1P_2 + Q_1Q_2] = \frac{2}{2h^2}[-h\times 0 + h\times(-h)] = -1$$

Similarly, $C_{03} = -1 = C_{04}$. Thus

$V_o = (V_1 + V_2 + V_3 + V_4)/4$

which is the same result obtained using FDM.