3.i) The Maxwell's eq^n in differential form is

$$\nabla \cdot \vec{D} = \rho_v$$
$$\nabla \cdot \vec{B} = 0$$
$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

For deriving the phasor form, we make the following substitutions

$\vec{D} = \vec{D_s} e^{j\omega t}$ $\quad \vec{E} = \vec{E_s} e^{j\omega t}$ $\quad \rho_v = \rho_{vs} e^{j\omega t}$

$\vec{B} = \vec{B_s} e^{j\omega t}$ $\quad \vec{H} = \vec{H_s} e^{j\omega t}$ $\quad \vec{J} = \vec{J_s} e^{j\omega t}$

Hence the Maxwell's eq^n in differential form in phasor notation would be

$$\nabla \cdot \vec{D_s} = \rho_{vs}$$
$$\nabla \cdot \vec{B_s} = 0$$
$$\nabla \times \vec{E_s} = -j\omega \vec{B_s}$$
$$\nabla \times \vec{H_s} = \vec{J_s} + j\omega \vec{D_s}$$

The phasor notation is useful because:-

i. It converts time dependencies to simple algebraic product.

ii. Sinusoidal or ac currents can be easily handled.

iii. In case of varying electric and magnetic field, it provides a link how they vary with time in addition to phase.

(ii) $\nabla \times \vec{E}_s = -j\omega\mu\vec{H}$ $\quad \vec{E} = \vec{E}_s e^{j\omega t}$

$\nabla \times \vec{H}_s = (\sigma + j\omega\epsilon)\vec{E}_s$ $\quad \vec{H} = \vec{H}_s e^{j\omega t}$

The Poynting vector in phasor form is defined as $\vec{E} \times \vec{H}^*$

so first we can use the vector identity

$$\nabla \cdot (\vec{H}^* \times \vec{E}) = \vec{E} \cdot (\nabla \times \vec{H}_s^*) - \vec{H}^* \cdot (\nabla \times \vec{E})$$

or $\nabla \cdot (\vec{H}^* \times \vec{E}) = \vec{E} \cdot (\sigma - j\omega\epsilon)\vec{E}^* - \vec{H}^* \cdot (-j\omega\mu)\vec{H}$

or $\nabla \cdot (\vec{H}^* \times \vec{E}) = (\sigma - j\omega\epsilon)E^2 + j\omega\mu H^2$

or $\nabla \cdot (\vec{H}^* \times \vec{E}) = \sigma E^2 + j\omega(\mu H^2 - \epsilon E^2)$

or $\nabla \cdot (\vec{E} \times \vec{H}^*) = -\sigma E^2 - j\omega(\mu H^2 - \epsilon E^2)$

or $$\iiint_V \nabla \cdot (\vec{E} \times \vec{H}^*)\, dv = -\iiint_V \sigma E^2\, dv - j\omega \iint_V \mu H^2 - \epsilon E^2\, dv$$

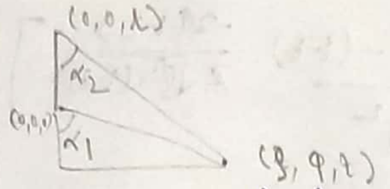
or $$\oint_S (\vec{E} \times \vec{H}^*) \cdot d\vec{s} = -\iiint_V \sigma E^2\, dv - j\omega \iiint_V \mu H^2 - \epsilon E^2\, dv$$

Here as usual the left side is the total power leaving the volume as given by Poynting vector. The first term in rhs is the ohmic loss whereas the second term shows how much energy decreases in electric and magnetic fields. This is just a statement of conservation of power or energy.

iii) $l = 1\,mm$ $\quad I = 3\cos(\omega t + \varphi)$

$f = 1\,GHz$

a.



$$\vec{H} = \int \frac{I_\rho\, d\vec{l} \times \hat{R}}{4\pi R^2} = \int_0^l \frac{I\rho\, dz}{4\pi(\rho^2+z^2)^{3/2}}\, \hat{a}_\varphi$$

$$= \frac{I\rho}{4\pi}\int_{\alpha_1}^{\alpha_2} \frac{-\rho\,\mathrm{cosec}^2\alpha\, d\alpha}{\rho^3\,\mathrm{cosec}^3\alpha}\, \hat{a}_\varphi \qquad z = \rho\cot\alpha$$

$$= \frac{I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1)\, \hat{a}_\varphi$$

$$= \frac{I}{4\pi\rho}\left[\frac{z}{\sqrt{\rho^2+z^2}} + \frac{l-z}{\sqrt{\rho^2+(l-z)^2}}\right] \hat{a}_\varphi = H_\varphi\, \hat{a}_\varphi$$

b.

$$\nabla \times \vec{H} = \frac{1}{\rho}\begin{vmatrix} \hat{a}_\rho & \rho\hat{a}_\varphi & \hat{a}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ 0 & \rho H_\varphi & 0 \end{vmatrix} = \frac{1}{\rho}\left[\hat{a}_\rho\left(-\frac{\partial}{\partial z}(\rho H_\varphi)\right) + \hat{a}_z\left(\frac{\partial}{\partial\rho}\rho H_\varphi\right)\right]$$

$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho H_\varphi)\, \hat{a}_z - \frac{\partial}{\partial z}H_\varphi\, \hat{a}_\rho$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho H_\varphi)$$

$$= \frac{1}{\rho}\frac{I}{4\pi}\frac{\partial}{\partial\rho}\left[\frac{z}{\sqrt{\rho^2+z^2}} + \frac{l-z}{\sqrt{\rho^2+(l-z)^2}}\right]$$

$$= \frac{I}{4\pi\rho}\left[-\frac{1}{2}\frac{z \times 2\rho}{(\rho^2+z^2)^{3/2}} + \frac{(l-z)\left(-\frac{1}{2}\right) \times 2\rho}{[\rho^2+(l-z)^2]^{3/2}}\right]$$

$$= \frac{-I}{4\pi\rho}\left[\frac{\rho z}{(\rho^2+z^2)^{3/2}} + \frac{\rho(l-z)}{[\rho^2+(l-z)^2]^{3/2}}\right]$$

$$= -\frac{I}{4\pi}\left[\frac{z}{(\rho^2+z^2)^{3/2}} + \frac{l-z}{[\rho^2+(l-z)^2]^{3/2}}\right]$$

$$\frac{\partial}{\partial z}H_\varphi$$

$$= \frac{\partial}{\partial z}\frac{I}{4\pi\rho}\left[\frac{z}{\sqrt{\rho^2+z^2}} + \frac{l-z}{\sqrt{\rho^2+(l-z)^2}}\right]$$

$$= \frac{I}{4\pi\rho}\frac{\partial}{\partial z}\left[\frac{z}{(\rho^2+z^2)^{1/2}} + \frac{l-z}{[\rho^2+(l-z)^2]^{1/2}}\right]$$

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$$= \frac{I}{4\pi\rho}\frac{\partial}{\partial z}\left(\frac{\rho z}{\sqrt{\rho^2+z^2}} + \frac{l-z}{\sqrt{\rho^2+(l-z)^2}}\right)$$

$$= \frac{I}{4\pi\rho}\left[\frac{\sqrt{\rho^2+z^2} - z\frac{2z}{2\sqrt{\rho^2+z^2}}}{\rho^2+z^2} + \frac{-\sqrt{\rho^2+(l-z)^2} - (l-z)\frac{-2(l-z)}{2\sqrt{\rho^2+(l-z)^2}}}{\rho^2+(l-z)^2}\right]$$

$$= \frac{I}{4\pi\rho}\left[\frac{\rho^2+z^2-z^2}{(\rho^2+z^2)^{3/2}} + \frac{(l-z)^2-\rho^2-(l-z)^2}{(\rho^2+(l-z)^2)^{3/2}}\right]$$

$$= \frac{I}{4\pi\rho}\left[\frac{\rho^2}{(\rho^2+z^2)^{3/2}} - \frac{\rho^2}{[\rho^2+(l-z)^2]^{3/2}}\right]$$

$$= \frac{I\rho}{4\pi}\left[\frac{1}{(\rho^2+z^2)^{3/2}} - \frac{1}{[\rho^2+(l-z)^2]^{3/2}}\right]$$

c. $\frac{\partial \vec{D}}{\partial t} = \nabla\times\vec{H}$

or $$\frac{\partial \vec{D}}{\partial t} = \frac{I\rho}{4\pi}\left[\frac{1}{(\rho^2+(l-z)^2)^{3/2}} - \frac{1}{(\rho^2+z^2)^{3/2}}\right]\hat{a}_\rho - \frac{I}{4\pi}\left[\frac{z}{(\rho^2+z^2)^{3/2}} + \frac{l-z}{[\rho^2+(l-z)^2]^{3/2}}\right]\hat{a}_z$$

or $$\frac{\partial \vec{D}}{\partial t} = \frac{I}{4\pi}\left[\left(\frac{\rho}{(\rho^2+(l-z)^2)^{3/2}} - \frac{\rho}{(\rho^2+z^2)^{3/2}}\right)\hat{a}_\rho - \left(\frac{z}{(\rho^2+z^2)^{3/2}} + \frac{l-z}{(\rho^2+(l-z)^2)^{3/2}}\right)\hat{a}_z\right]$$

or $$\vec{E} = \frac{\int I\,dt}{4\pi\epsilon}\left[\frac{\rho\hat{a}_\rho - (l-z)\hat{a}_z}{[\rho^2+(l-z)^2]^{3/2}} - \frac{\rho\hat{a}_\rho + z\hat{a}_z}{(\rho^2+z^2)^{3/2}}\right]$$

or $$\vec{E} = -\frac{3\sin(\omega_0 t+\varphi)}{4\pi\epsilon\omega_0}\left[\frac{\rho\hat{a}_\rho - (l-z)\hat{a}_z}{[\rho^2+(l-z)^2]^{3/2}} - \frac{\rho\hat{a}_\rho + z\hat{a}_z}{(\rho^2+z^2)^{3/2}}\right]$$

d. $\vec{P} = \vec{E}\times\vec{H}$

or $\vec{P} = (E_\rho\hat{a}_\rho + E_z\hat{a}_z)\times H_\varphi\hat{a}_\varphi$ $\vec{E} = E_\rho\hat{a}_\rho + E_z\hat{a}_z$, $\vec{H} = H_\varphi\hat{a}_\varphi$

or $\vec{P} = H_\varphi E_\rho\hat{a}_z - E_z H_\varphi\hat{a}_\rho$

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$$H_\varphi = \frac{3\cos(\omega_0 t+\varphi)}{4\pi\rho}\left[\frac{1}{\sqrt{\rho^2+z^2}} + \frac{l-z}{\sqrt{\rho^2+(l-z)^2}}\right]$$

$$E_\rho = -\frac{3\sin(\omega_0 t+\varphi)\rho}{4\pi\epsilon\omega_0}\left[\frac{1}{(\rho^2+(l-z)^2)^{3/2}} - \frac{1}{(\rho^2+z^2)^{3/2}}\right]$$

$$E_z = \frac{3\sin(\omega_0 t+\varphi)}{4\pi\epsilon\omega_0}\left[\frac{z}{(\rho^2+z^2)^{3/2}} + \frac{l-z}{(\rho^2+(l-z)^2)^{3/2}}\right]$$

$$\vec{P} = H_\varphi\,(E_\rho\hat{a}_z - E_z\hat{a}_\rho)$$

e. $\nabla\times\vec{H} = \frac{\partial\vec{D}}{\partial t}$

or $\nabla\times\vec{B} = \mu\frac{\partial\vec{D}}{\partial t}$

or $-\nabla^2\vec{A} = \mu\frac{\partial\vec{D}}{\partial t}$

or $\nabla^2\vec{A} = -\mu\frac{\partial\vec{D}}{\partial t}$

or $\nabla^2\vec{A} = \frac{3\mu\rho}{4\pi}\left[\frac{1}{(\rho^2+z^2)^{3/2}} - \frac{1}{(\rho^2+(l-z)^2)^{3/2}}\right]\hat{a}_\rho$

$\frac{I\mu}{4\pi}\left[\frac{z}{(\rho^2+z^2)^{3/2}} + \frac{l-z}{(\rho^2+(l-z)^2)^{3/2}}\right]\hat{a}_z$

$\therefore \nabla^2 A_x = \frac{\mu I x}{4\pi}\left[\frac{1}{(\rho^2+z^2)^{3/2}} - \frac{1}{(\rho^2+(l-z)^2)^{3/2}}\right]$

$\nabla^2 A_y = \frac{\mu I y}{4\pi}\left[\frac{1}{(\rho^2+z^2)^{3/2}} - \frac{1}{(\rho^2+(l-z)^2)^{3/2}}\right]$

$\nabla^2 A_z = \frac{\mu I}{4\pi}\left[\frac{z}{(\rho^2+z^2)^{3/2}} + \frac{l-z}{(\rho^2+(l-z)^2)^{3/2}}\right]$

The solutions to these non-homogenous Laplace eqn would give the three components of $\vec{A}$ field in cartesian coordinates (A_x, A_y, A_z)