Probability: Basic Concepts & Definitions:

Experiment:
Observing something happen or conducting something under certain conditions

Deterministic Expts

Non-deterministic / Random / Stochastic Expts

Sample Space : The set of all possible outcomes of a random expt is called a sample space. Ω, S

Examples: 1. Tossing of a Coin

$$\Omega = \{H, T\}$$

2. Tossing of a Dice

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

3. Drawing of a card
We want to see the colour

$$\Omega_1 = \{B, R\}$$

$$\Omega_2 = \{\text{Heart}, \text{Diamond}, \text{Club}, \text{Spade}\}$$

$$\Omega_3 = \{1, 2, 3, \ldots 10, J, Q, K\}$$

$$\Omega_4 = \{H_1, H_2, \ldots, H_{13}, D_1, \ldots, D_{13}, C_1, \ldots, C_{13}, S_1, \ldots, S_{13}\}$$

4. Birth of a child

gender $\Omega_1 = \{B, G\}$

Weight at birth $\Omega_2 = (0.5, 3.5)$ in kg.

height at birth $\Omega_3 = \ldots$

5. Longevity of an adult human

$\Omega = (18, 120)$ (years)

6. Amount of rainfall during monsoon in Mumbai (in cm)

$(30, 200)$

<u>Event</u> : Any subset of a sample <u>space</u> is an event.

$\phi \longrightarrow$ Impossible event

$S, \Omega \longrightarrow$ Sure event

Union $\rightarrow E \cup F$

$\rightarrow$ happening of either E or F or both

$$A_1 \cup A_2 \cup \dots \cup A_n = \bigcup_{i=1}^{n} A_i$$

$\rightarrow$ Occurrence of at least one A_i

$$\bigcup_{i=1}^{\infty} A_i$$

Intersection $E \cap F$

→ Simultaneous Occurrence of both E and F

$$A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i$$

→ Simultaneous occurrence of all A_i's.

$\bigcap_{i=1}^{\infty} A_i \rightarrow$ simultaneous occurrence

If $\bigcup_{i=1}^{n} A_i = \Omega$

then $A_1, \ldots, A_n$ are called

exhaustive events

$A \cap B = \phi$, then A and B

are called disjoint or mutually exclusive events.

If $A_1, \ldots, A_n$ are any events and $A_i \cap A_j = \phi$, $i \neq j$ then these are called pairwise disjoint events.

$A^c \rightarrow$ not happening of A.

Laplace (1812)

Classical Definition of Probability

Suppose a random expt has N possible outcomes which are mutually exclusive, exhaustive and equally likely. Let M of these be favourable to happenning of an event E. Then the prob of event E is defined by

$$P(E) = \frac{M}{N}.$$

<u>Drawbacks of the def^n</u>

1. N need not be infinite

2. The definition is circular in nature because it uses the term 'equally likely' which means outcomes are with equal prob.

Relative frequency / Empirical (Von. Mises) Defn of Prob.

If a random expt is repeated

n times and an event E occurs a_n times in these n trials. then

$$P(E) = \lim_{n\to\infty} \frac{a_n}{n} .$$

Example: HHHTHHHTHHHT...

$A = \{H\}$. Want $P(A)$!!

a_n/n $\frac{1}{1}, \frac{2}{2}, \frac{3}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{6}{8}, 7/9, 8/10, 9/11, 9/12, \dots$

$\longrightarrow$

$$\frac{a_n}{n} = \begin{cases} \dfrac{3k}{4k}, & n = 4k \\[2ex] \dfrac{3k}{4k-1}, & n = 4k-1 \\[2ex] \dfrac{3k-1}{4k-2}, & n = 4k-2 \\[2ex] \dfrac{3k-2}{4k-3}, & n = 4k-3 \end{cases}$$

$k = 1, 2, \ldots$

$$\lim_{n \to \infty} \frac{a_n}{n} = \frac{3}{4} = P(\text{Head})$$

Drawbacks: 1. Actual observations on the random expt may not be available.

2. The prob of an event may be zero but the event may be occurring. eg. $a_n = \log n$

Then $\frac{a_n}{n} \longrightarrow 0$

Similarly an event may not always happen but the prob may be one, eg $a_n = n \sin\left(\frac{1}{n}\right)$

Axiomatic Def^n (Kolmogorov 1933)

Let Ω be a sample space

Let $\mathcal{B}$ be a set (class) of events

ie $\mathcal{B}$ is a class of subsets of Ω satisfying the two conditions

(i) $E \in \mathcal{B} \Rightarrow E^c \in \mathcal{B}$

(ii) If $E_1, E_2, \ldots \in \mathcal{B}$

then $\bigcup_{i=1}^{\infty} E_i \in \mathcal{B}$

Then $\mathcal{B}$ is called a σ-field or σ-algebra of subsets of Ω.

A consequence of this def^n is that $\mathcal{B}$ is closed under the operation of countable intersections, differences etc.

$\Omega \rightarrow$ sample

$\mathcal{B} \rightarrow$ σ-algebra / σ-field of subsets of Ω

$\rightarrow$ event space.