Assignment

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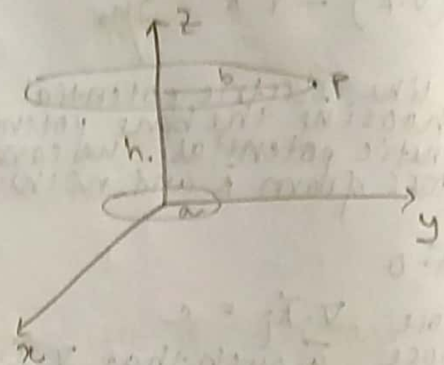
19EC10088

1. $a = 88$
$b = 2.5a$
$h = \frac{a+b}{2} = 1.75a$

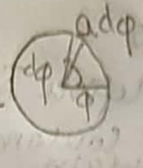
i) $\vec{A} = \oint \frac{\mu_0 I_1}{4\pi} \frac{\vec{dl}}{R}$



By symmetry, we expect the vector potential to be same everywhere on the loop. so we take point $P(0, b, h)$

$\vec{dl} = a\,d\varphi(-\sin\varphi\,\hat{a}_x + \cos\varphi\,\hat{a}_y)$
$\vec{R} = (-a\cos\theta,\ b - a\sin\theta,\ h)$

$$\vec{A} = \int_0^{2\pi} \frac{\mu_0 I_1}{4\pi} \frac{a\,d\varphi\,(-\sin\varphi\,\hat{a}_x + \cos\varphi\,\hat{a}_y)}{\sqrt{a^2\cos^2\varphi + (b - a\sin\varphi)^2 + h^2}}$$



$$\text{or } \vec{A} = \frac{\mu_0 I_1}{4\pi} a \int_0^{2\pi} \frac{(-\sin\varphi\,\hat{a}_x + \cos\varphi\,\hat{a}_y)\,d\varphi}{\sqrt{a^2\cos^2\varphi + h^2 + b^2 + a^2\sin^2\varphi - 2ab\sin\varphi}}$$

$$\text{or } \vec{A} = \frac{\mu_0 I_1}{4\pi} a \int_0^{2\pi} \frac{(-\sin\varphi\,\hat{a}_x + \cos\varphi\,\hat{a}_y)\,d\varphi}{\sqrt{a^2 + b^2 + h^2 - 2ab\sin\varphi}}$$

$$\text{or } \vec{A} = \frac{\mu_0 I_1}{4\pi} \int_0^{2\pi} \frac{(-\sin\varphi\,\hat{a}_x + \cos\varphi\,\hat{a}_y)\,d\varphi}{\sqrt{1 + \left(\frac{b}{a}\right)^2 + \left(\frac{h}{a}\right)^2 - 2\frac{b}{a}\sin\varphi}}$$

$$\text{or } \vec{A} = \frac{\mu_0 I_1}{4\pi} \int_0^{2\pi} \frac{(-\sin\varphi\,\hat{a}_x + \cos\varphi\,\hat{a}_y)\,d\varphi}{\sqrt{1 + 2.5^2 + 1.75^2 - 5\sin\varphi}}$$

$$\text{or } \vec{A} = \frac{\mu_0 I_1}{4\pi} \int_0^{2\pi} \frac{(-\sin\varphi\,\hat{a}_x + \cos\varphi\,\hat{a}_y)}{\sqrt{10.3125 - 5\sin\varphi}}\,d\varphi$$

The exact result was obtained using matlab and is presented below

$$\vec{A}(\vec{P}) = \frac{\mu_0 I_1}{4\pi}(0.2681969)(-\hat{a}_x) \quad [\text{At } \vec{P}\ \hat{a}_\varphi = -\hat{a}_x]$$

Hence the general vector potential for any point on loop is

$$\vec{A} = \frac{\mu_0 I_1}{4\pi}(0.2681969)\,\hat{a}_\varphi$$

This holds good only when we need to find $\vec{A}$ on the wire. Although the magnitude of $\vec{A}$ remains constant on the surface of wire, it does change when we decide to find it at some other point. In other words magnitude of $\vec{A}$ depends on ρ and z. For calculating field anywhere with slight errors, we make the dipole assumption.

<u>Assumption</u>
The loop behaves as a dipole everywhere at far enough distances.

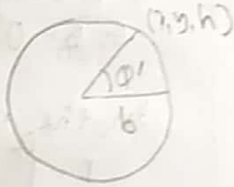
ii) using this assumption

$$\vec{A} = \frac{\mu_0 I a^2}{4r^2}\sin\theta\,\hat{a}_\phi$$

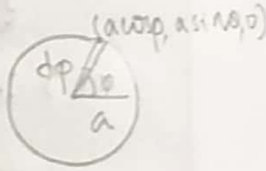
$$\vec{B} = \nabla\times\vec{A}$$
$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\phi)\,\hat{a}_z - \frac{\partial A_\phi}{\partial z}\,\hat{a}_\rho$$
$$= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(r\sin\theta\,A_\phi) + \frac{1}{r}\cancel{\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi}}\left(-\frac{\partial}{\partial r}(rA_\phi)\right)$$
$$= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\mu_0 I a^2}{4r^2}\sin^2\theta\right) - \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\mu_0 I a^2}{4r}\sin\theta\right)$$
$$= \frac{\mu_0 I a^2}{4r^3}\left(2\cos\theta\,\hat{a}_r + \sin\theta\,\hat{a}_\theta\right)$$

without these assumptions

$$\vec{A} = \frac{\mu_0 I_1}{4\pi}\int_0^{2\pi}\frac{a\,d\phi\,(-\sin\phi\,\hat{a}_x + \cos\phi\,\hat{a}_y)}{\sqrt{(x-a\cos\phi)^2+(y-a\sin\phi)^2+z^2}}$$



$$\text{or } \vec{A} = \frac{\mu_0 I_1}{4\pi}a\int_0^{2\pi}\frac{(-\sin\phi\,\hat{a}_x + \cos\phi\,\hat{a}_y)\,d\phi}{\sqrt{x^2+y^2+z^2-2ax\cos\phi-2ay\sin\phi}}$$



$$\text{or } \vec{A} = \frac{\mu_0 I_1}{4\pi}a\int_0^{2\pi}\frac{(-\sin\phi\,\hat{a}_x + \cos\phi\,\hat{a}_y)\,d\phi}{\sqrt{x^2+y^2+z^2-2a(x\cos\phi+y\sin\phi)}}$$

$$\text{or } \vec{A} = \frac{\mu_0 I_1}{4\pi}a\int_0^{2\pi}\frac{(-\sin\phi\,\hat{a}_x + \cos\phi\,\hat{a}_y)}{\sqrt{r^2-2ar\sin\theta\cos(\phi-\phi')}}\,d\phi$$

Clearly this integral is very tedious to compute. Hence the approximation was used. However numeric integration is possible, though it would not help in calculating the $\vec{B}$ field.

iii) $\Psi_{12} = \int_S \vec{B} \cdot d\vec{S}$

$= \int_S \nabla \times \vec{A} \cdot d\vec{S} = \oint \vec{A} \cdot d\vec{l_2}$

$= \int_0^{2\pi} A_\varphi \hat{a_\varphi} \cdot b\,d\varphi\, \hat{a_\varphi}$ [A_φ has already been computed $\vec{A} = A_\varphi \hat{a_\varphi}$]

$= bA_\varphi \int_0^{2\pi} d\varphi$

$= 2\pi b A_\varphi$

$A_\varphi = \frac{\mu_0 I_1}{4\pi} (0.2681969)$

$\Psi_{12} = \frac{\mu_0 I_1}{4\pi} (0.2681969) \times 2\pi b$

$= \frac{\mu_0}{2} I_1 (0.2682 b) = (\mu_0\, 0.1341 b)\, I_1$

$M = \frac{\Psi_{12}}{I_1} = \frac{\mu_0 b}{2} (0.2681969)$

$= 29.502 \mu_0 = 2.95\ \mu H$

iv) The assumption taken was that the field due to loop acts as dipole everywhere. This is true only at very large distances from loop. So we can check the error in the approximation

$\vec{A} = \frac{\mu_0 I_1}{4\pi} \left(\frac{\pi a^2}{r^2} \sin\theta \right) \hat{a_\varphi}$

or $\vec{A} = \frac{\mu_0 I_1}{4\pi} (0.276369)$

error $= 8.1721 \times 10^{-3} \frac{\mu_0 I_1}{4\pi}$

error % $= \frac{8.1721 \times 10^{-3}}{0.2681969} \times 100\%$

$= 3.047\%$

Thus even though $h \not\gg a$, the error with this approximation is just around 3%. The approximation thus holds good.

Scanned with CamScanner

17 Using directly the formula for force between two loops

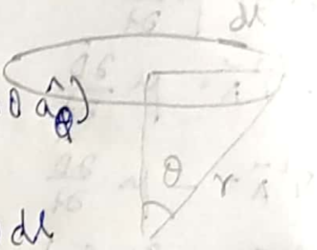
$$\vec{F} = \frac{\mu_0 I^2}{4\pi} \oint_{L_1} \oint_{L_2} \frac{\vec{dl_1} \times (\vec{dl_2} \times \hat{a}_{R_{21}})}{R_{21}^2}$$

$P_1 (a\cos\phi_1, a\sin\phi_1, 0)$
$P_2 (b\cos\phi_2, b\sin\phi_2, h)$

$R_{21}^2 = h^2 + a^2 + b^2 - 2ab\cos(\phi_2 - \phi_1)$

$\vec{dl_1} \times (\vec{dl_2} \times \vec{a}_{R_2})$
$= -ab(a+b)(\cos\phi_1 \hat{a_x} + \sin\phi_1 \hat{a_y})$

This integral would be tedious to solve. So instead we use the $\vec{B}$ approximate on loop 2



$d\vec{F} = I\vec{dl} \times \vec{B}$

or $d\vec{F} = \frac{\mu_0 I^2 a^2}{4r^3} (dl\, \hat{a_\phi}) \times (2\cos\theta\, \hat{a_r} + \sin\theta\, \hat{a_\theta})$

or $d\vec{F} = \frac{\mu_0 I^2 a^2}{4r^3} (2\cos\theta\, \hat{a_\theta} - \sin\theta\, \hat{a_r})\, dl$

Now for different dl, the $\hat{a_r}$ and $\hat{a_\theta}$ directions are different. So we need to convert to $\hat{a_x}$, $\hat{a_y}$ and $\hat{a_z}$

$\hat{a_\theta} = \cos\theta(\cos\phi\, \hat{a_x} + \sin\phi\, \hat{a_y}) - \sin\theta\, \hat{a_z}$
$\hat{a_r} = \sin\theta(\cos\phi\, \hat{a_x} + \sin\phi\, \hat{a_y}) + \cos\theta\, \hat{a_z}$

$2\cos\theta\, \hat{a_\theta} - \sin\theta\, \hat{a_r}$
$= (2\cos^2\theta - \sin^2\theta)(\cos\phi\, \hat{a_x} + \sin\phi\, \hat{a_y}) - 3\sin\theta\cos\theta\, \hat{a_z}$

$\therefore \int d\vec{F} = \frac{\mu_0 I^2 a^2}{4r^3} \int_0^{2\pi} (3\cos^2\theta - 1)\cos\phi\, \hat{a_x} + \sin\phi\, \hat{a_y} - 3\sin\theta\cos\theta\, \hat{a_z}(b$

Clearly the x and y components vanish

$$\vec{F} = \frac{\mu_0 I^2 a^2}{4r^3} \int_0^{2\pi} (-3\sin\theta\cos\theta)\, b\, d\phi\, \hat{a_z}$$

or $\vec{F} = \frac{\mu_0 I^2 a^2}{4r^3} (-1.5\sin 2\theta) \times 2\pi b\, \hat{a_z}$

or $\vec{F} = -\frac{3\mu_0 I^2 \pi}{4} \left(\frac{a^2 b}{r^3}\right) \sin 2\theta\, \hat{a_z}$

or $\vec{F} = -0.2447\, \hat{a_z}$ MN

The following MATLAB code was used in order to determine the integral in computer.

```
a = 88;
b = 2.5 * a;
h = (a + b) / 2;

syms theta
deno = sqrt(h*h + a*a + b*b -2*a*b*cos(theta));
num = [-sin(theta); cos(theta)];
expr = num ./ deno;
result = vpaintegral(expr, theta, 0, 2*pi);
disp(a*result);

syms phi1 phi2;
deno = h*h + a*a + b*b -2*a*b*cos(phi1 - phi2);
num = [cos(phi1); sin(phi1)];
expr = num ./ deno;
result = vpaintegral(expr, phi1, [0 2*pi], phi2, [0 2*pi]);
disp(result);
```