POISSON & LAPLACE EQUATION

- Coulomb's Law — $\vec{E}$ from known charge distribution
- Gauss's Law — $\vec{E}$ " and symmetry applies
- $\vec{E} = -\nabla V$ — when potential V is known throughout

- $\nabla \cdot \vec{D} = \nabla \cdot \varepsilon \vec{E} = \rho_v$, $\vec{E} = -\nabla V$

$\Rightarrow \nabla \cdot (-\varepsilon \nabla V) = \rho_v$

For homogeneous medium, $\boxed{\nabla^2 V = -\rho_v/\varepsilon}$ ← POISSON'S Eqn.

For charge-free region, $\rho_v = 0$, $\boxed{\nabla^2 V = 0}$ ← LAPLACE Eqn

$$\nabla^2 :- \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad , V(x,y,z)$$

$$\frac{1}{\rho}\left(\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right)\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad , V(\rho,\phi,z)$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0 \quad , V(r,\theta,\phi)$$

UNIQUENESS Theorem

Any solution of Laplace's Eqn. that satisfies the same boundary conditions must be the only solution regardless of the method used.

Let V_1, V_2 be two solutions, both satisfies the prescribed boundary conditions.

Thus, $\nabla^2 V_1 = 0$, $\nabla^2 V_2 = 0$, $V_1 = V_2$ on the boundary.

Let, $V_d \triangleq V_2 - V_1$, $\nabla^2 V_d = 0$, $V_d = 0$ on the boundary.

Let, $\vec{A} = V_d \nabla V_d$, Using $\nabla \cdot \vec{A} = \nabla \cdot (V_d \nabla V_d) = \cancel{V_d \nabla^2 V_d}^{\,0} + \nabla V_d \cdot \nabla V_d$.

Using Divergence Th$^{\text{m}}$, $\iiint_V \nabla \cdot \vec{A}\, dv = \oiint_S \vec{A} \cdot \vec{ds} \Rightarrow \iiint_V (\nabla V_d) \cdot (\nabla V_d)\, dv = \cancel{\oiint_S V_d \nabla V_d \cdot \vec{ds}}^{\,0}$ (S being the boundary)

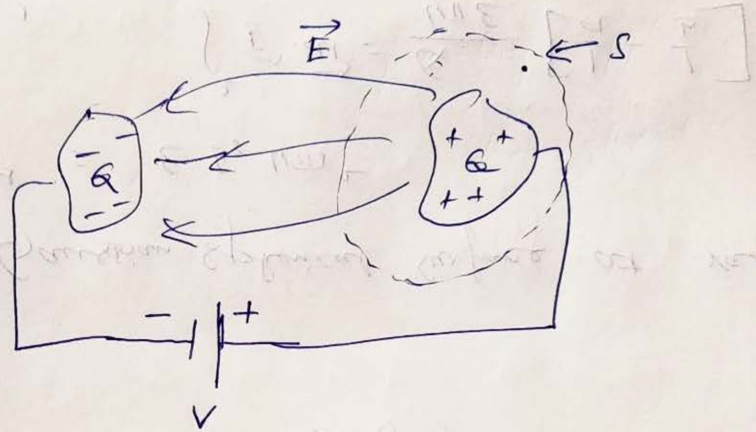
$\Rightarrow \iiint_V \underbrace{|\nabla V_d|^2}_{\text{positive}} dv = 0 \quad \Rightarrow \nabla V_d = 0$ everywhere in v

$\Rightarrow V_2 - V_1 = \text{constant}$ "

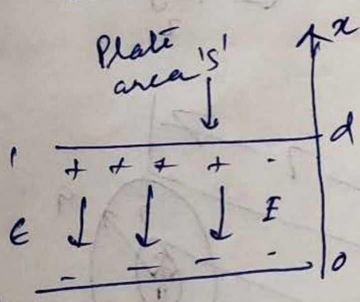
Since, $V_d = 0$ on the boundary $\Rightarrow V_1 = V_2$ everywhere.

$$R = \frac{V}{I} = \frac{\int \vec{E}\cdot d\vec{l}}{\oint_S \sigma \vec{E}\cdot d\vec{S}}$$, Solve Laplace Eqn. $\nabla^2 V$ to obtain V, Determine $\vec{E} = -\nabla V$, then Calculate R.

$$C = \frac{Q}{V} = \frac{\varepsilon \oint_S \vec{E}\cdot d\vec{s}}{\int \vec{E}\cdot d\vec{l}}$$



· Parallel-plate capacitor



$\rho_s = \frac{Q}{S}$

Ignoring fringing fields at the edges.

$$\vec{D} = -\rho_s \hat{a}_x$$

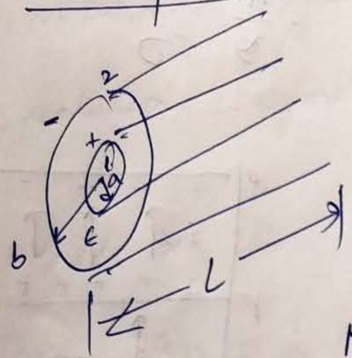
$$\vec{E} = -\frac{Q}{\varepsilon S}\hat{a}_x \quad , \quad V = \int_1^2 \vec{E}\cdot d\vec{l} = \frac{Qd}{\varepsilon S}.$$

$$C = \frac{Q}{V} = \frac{\varepsilon S}{d}$$

Energy stored in capacitor $(W_E) = \frac{1}{2}CV^2 = \frac{Q^2}{2C}$.

$$\left[W_E = \frac{1}{2}\iiint_V \varepsilon E^2 dv = \frac{\varepsilon Q^2 S d}{2\varepsilon^2 S^2} = \frac{Q^2}{2}\left(\frac{d}{\varepsilon S}\right) \right]$$

• Coaxial capacitor



Choose a Gaussian cylindrical surface at radius ρ

$$Q = \varepsilon \oiint \vec{E}\cdot\vec{ds} = \varepsilon E_\rho 2\pi\rho L$$

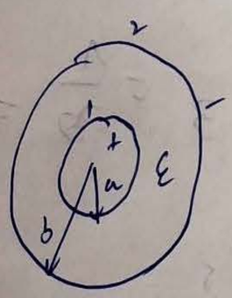
$$\vec{E} = \frac{Q}{2\pi\varepsilon\rho L}\,\hat{a}_\rho$$

Neglecting fringing flux at cylinder ends,

$$\int_1^2 \vec{E}\cdot\vec{dl} = \frac{Q}{2\pi\varepsilon L}\ln(b/a) \quad \Rightarrow \quad C = \frac{2\pi\varepsilon L}{\ln(b/a)}$$

Spherical capacitor

Choose a Gaussian spherical surface at radius 'r'.



$$Q = \varepsilon \oiint_S \vec{E}\cdot ds = \varepsilon E_r 4\pi r^2$$

$$\vec{E} = \frac{Q}{4\pi r^2}\hat{a}_r, \qquad \int_1^2 \vec{E}\cdot\vec{dl} = \frac{Q}{4\pi\varepsilon}\left[\frac{1}{a} - \frac{1}{b}\right]$$

$$\Rightarrow C = \frac{4\pi\varepsilon}{\left[\frac{1}{a} - \frac{1}{b}\right]}$$

Lt $C = 4\pi\varepsilon a$ (Capacitance of isolated sphere)
$b\to\infty$

Lossy Capacitor :- $R = \dfrac{\int_l \vec{E} \cdot \vec{dl}}{\oiint_S \sigma \vec{E} \cdot \vec{ds}}$, $C = \dfrac{\varepsilon \oiint_S \vec{E} \cdot \vec{ds}}{\int_l \vec{E} \cdot \vec{dl}}$, $\boxed{RC = \dfrac{\varepsilon}{\sigma}}$

(Homogeneous medium)

Relaxation Time (T_r) of the medium separating conductors.

- Parallel-plate : $C = \dfrac{\varepsilon S}{d}$, $R = \dfrac{d}{\sigma S}$
- Cylindrical : $C = \dfrac{2\pi \varepsilon L}{\ln(b/a)}$, $R = \dfrac{\ln(b/a)}{2\pi \sigma L}$.
- Spherical : $C = \dfrac{4\pi \varepsilon}{\frac{1}{a} - \frac{1}{b}}$, $R = \dfrac{\frac{1}{a} - \frac{1}{b}}{4\pi \sigma}$.

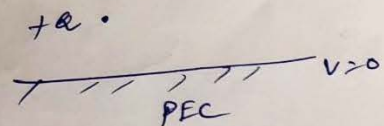
σ : conductivity of dielectric medium.

R: leakage resistance between plates

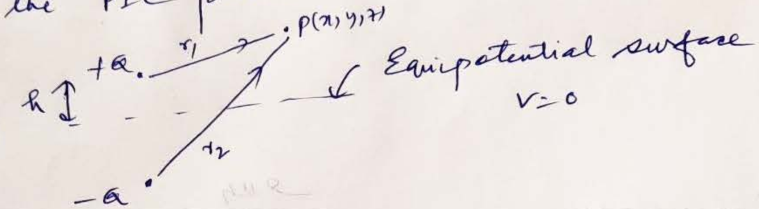
Additional 'R' due to plates themselves.

Method of Images

Image Theory: A given charge configuration above an infinite grounded PEC plane may be replaced by the charge configuration itself and its image, in place of the PEC plane.



$\Rightarrow$



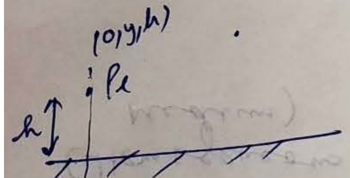
$\vec{r}_1 = (x, y, z-h)$, $V = V_+ + V_- = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_1} - \frac{1}{r_2}\right]$ for $z \geq 0$

$\vec{r}_2 = (x, y, z+h)$

Surface charge density $\rho_s = D_n = \varepsilon_0 E_n \Big|_{z=0} = \frac{-Qh}{2\pi\left[x^2+y^2+h^2\right]^{3/2}}$

Total induced charge on the conducting plane,

$$Q_i = \iint \rho_s \, ds = -Q.$$



$V = V^+ + V^- = -\frac{\rho_L}{2\pi\varepsilon_0}\ln\rho_1 - \frac{-\rho_L}{2\pi\varepsilon_0}\ln\rho_2 = -\frac{\rho_L}{2\pi\varepsilon_0}\ln\frac{\rho_1}{\rho_2}$.

Line charge

$\rho_s = D_n = \varepsilon_0 E_n \Big|_{z=0} = \frac{-\rho_L h}{\pi(x^2+h^2)}$

$\vec{\rho}_1 = (x, y, z) - (0, y, h) = (x, 0, z-h)$

$\vec{\rho}_2 = (x, 0, z+h)$

Induced charge/length $\rho_i = \int \rho_s dx = -\rho_L$.

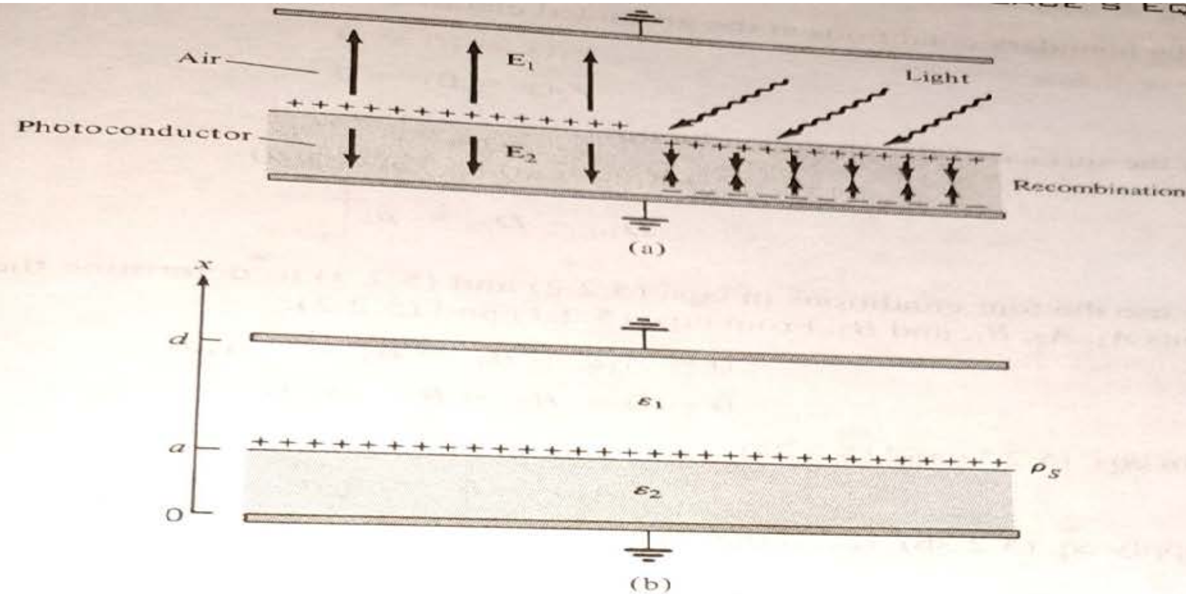


Figure 5.2 For Example 5.2.

surface combine with those on the upper surface to neutralize each other. The image is developed by pouring a charged black powder over the surface of the photoconductor. The electric field attracts the charged powder, which is later transferred to paper and melted to form a permanent image. We want to determine the electric field below and above the surface of the photoconductor.

Solution:
Consider the modeled version of Figure 5.2(a) shown in Figure 5.2(b). Since $\rho_v = 0$ in this case, we apply Laplace's equation. Also the potential depends only on x. Thus

$$\nabla^2 V = \frac{d^2 V}{dx^2} = 0$$

Integrating twice gives

$$V = Ax + B$$

Let the potentials above and below $x = a$ be V_1 and V_2, respectively:

$$V_1 = A_1 x + B_1, \qquad x > a \tag{5.2.1a}$$

$$V_2 = A_2 x + B_2, \qquad x < a \tag{5.2.1b}$$

The boundary conditions at the grounded electrodes are

$$V_1(x = d) = 0 \tag{5.2.2a}$$

$$V_2(x = 0) = 0 \tag{5.2.2b}$$

At the surface of the photoconductor,

$$V_1(x = a) = V_2(x = a) \tag{5.2.3a}$$

$$D_{1n} - D_{2n} = \rho_S \Big|_{x=a} \tag{5.2.3b}$$

We use the four conditions in eqs. (5.2.2) and (5.2.3) to determine the four unknown constants A_1, A_2, B_1, and B_2. From eqs. (5.2.1) and (5.2.2),

$$0 = A_1 d + B_1 \rightarrow B_1 = -A_1 d \tag{5.2.4a}$$

$$0 = 0 + B_2 \rightarrow B_2 = 0 \tag{5.2.4b}$$

From eqs. (5.2.1) and (5.2.3a),

$$A_1 a + B_1 = A_2 a \tag{5.2.5}$$

To apply eq. (5.2.3b), recall that $\mathbf{D} = \varepsilon \mathbf{E} = -\varepsilon \nabla V$ so that

$$\rho_S = D_{1n} - D_{2n} = \varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = -\varepsilon_1 \frac{dV_1}{dx} + \varepsilon_2 \frac{dV_2}{dx}$$

or

$$\rho_S = -\varepsilon_1 A_1 + \varepsilon_2 A_2 \tag{5.2.6}$$

Solving for A_1 and A_2 in eqs. (5.2.4) to (5.2.6), we obtain

$$\mathbf{E}_1 = -A_1 \mathbf{a}_x = \frac{\rho_S \mathbf{a}_x}{\varepsilon_1 \left[1 + \dfrac{\varepsilon_2}{\varepsilon_1}\dfrac{d}{a} - \dfrac{\varepsilon_2}{\varepsilon_1}\right]}, \qquad a \leq x \leq d$$

$$\mathbf{E}_2 = -A_2 \mathbf{a}_x = \frac{-\rho_S \left(\dfrac{d}{a} - 1\right) \mathbf{a}_x}{\varepsilon_1 \left[1 + \dfrac{\varepsilon_2}{\varepsilon_1}\dfrac{d}{a} - \dfrac{\varepsilon_2}{\varepsilon_1}\right]}, \qquad 0 \leq x \leq a$$

PRACTICE EXERCISE 5.2

For the model of Figure 5.2(b), if $\rho_S = 0$ and the upper electrode is maintained at V_o while the lower electrode is grounded, show that

$$\mathbf{E}_1 = \frac{-V_o \mathbf{a}_x}{d - a + \dfrac{\varepsilon_1}{\varepsilon_2} a}, \qquad \mathbf{E}_2 = \frac{-V_o \mathbf{a}_x}{a + \dfrac{\varepsilon_2}{\varepsilon_1} d - \dfrac{\varepsilon_2}{\varepsilon_1} a}$$

LE 5.3

Semi-infinite conducting planes at $\phi = 0$ and $\phi = \pi/6$ are separated by an infinitesimal insulating gap as shown in Figure 5.3. If $V(\phi = 0) = 0$ and $V(\phi = \pi/6) = 100$ V, calculate V and $\mathbf{E}$ in the region between the planes.

Solution:

Since V depends only on ϕ, Laplace's equation in cylindrical coordinates becomes

$$\nabla^2 V = \frac{1}{\rho^2}\frac{d^2 V}{d\phi^2} = 0$$

Since $\rho = 0$ is excluded owing to the insulating gap, we can multiply by ρ^2 to obtain

$$\frac{d^2 V}{d\phi^2} = 0$$

which is integrated twice to give

$$V = A\phi + B$$

We apply the boundary conditions to determine constants A and B. When $\phi = 0$, $V = 0$,

$$0 = 0 + B \rightarrow B = 0$$

When $\phi = \phi_o$, $V = V_o$,

$$V_o = A\phi_o \rightarrow A = \frac{V_o}{\phi_o}$$

Hence:

$$V = \frac{V_o}{\phi_o}\phi$$

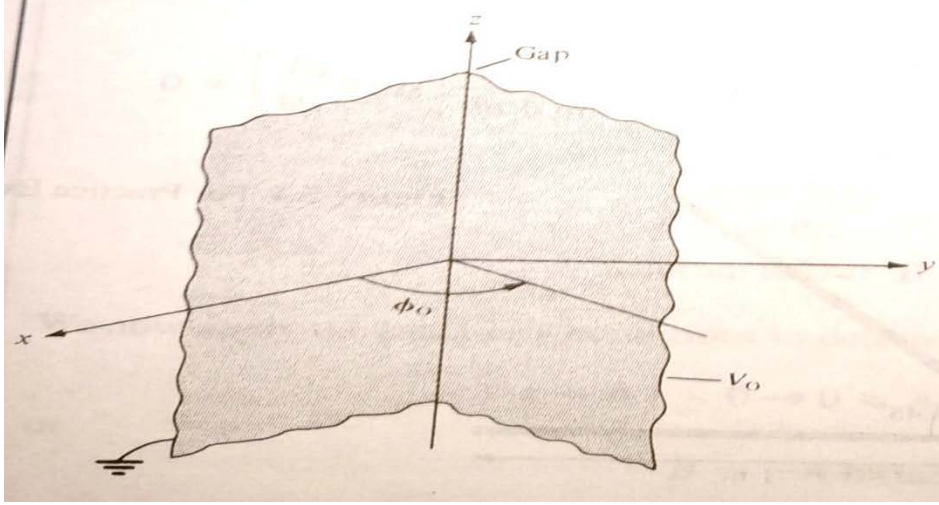


Figure 5.3 Potential $V(\phi)$ due to semi-infinite conducting planes.

and

$$\mathbf{E} = -\nabla V = -\frac{1}{\rho}\frac{dV}{d\phi}\mathbf{a}_\phi = -\frac{V_o}{\rho\phi_o}\mathbf{a}_\phi$$

Substituting $V_o = 100$ and $\phi_o = \pi/6$ gives

$$V = \frac{600}{\pi}\phi \qquad \text{and} \qquad \mathbf{E} = \frac{-600}{\pi\rho}\mathbf{a}_\phi$$

Check: $\nabla^2 V = 0$, $V(\phi = 0) = 0$, $V(\phi = \pi/6) = 100$.

PRACTICE EXERCISE 5.3

Two conducting plates of size 1×5 m are inclined at 45° to each other with a gap of width 4 mm separating them as shown in Figure 5.4. Determine an approximate value of the charge per plate if the plates are maintained at a potential difference of 50 V. Assume that the medium between them has $\varepsilon_r = 1.5$.

Answer: 22.2 nC.

5.4 Two conducting cones ($\theta = \pi/10$ and $\theta = \pi/6$) of infinite extent are separated by an infinitesimal gap at $r = 0$. If $V(\theta = \pi/10) = 0$ and $V(\theta = \pi/6) = 50$ V, find V and $\mathbf{E}$ between the cones.

Solution:

Consider the coaxial cone of Figure 5.5, where the gap serves as an insulator between the two conducting cones. Here V depends only on θ, so Laplace's equation in spherical coordinates becomes

$$\nabla^2 V = \frac{1}{r^2 \sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{dV}{d\theta}\right] = 0$$

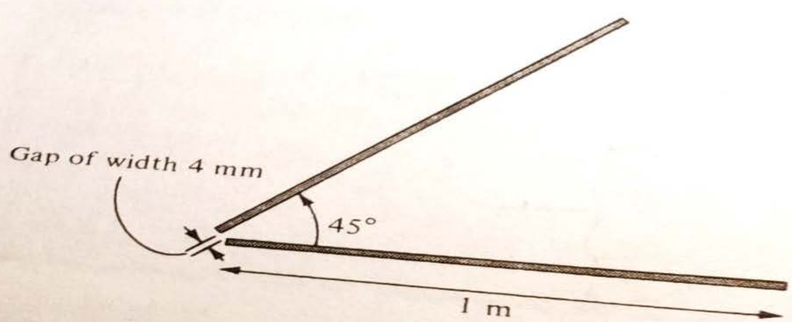


Figure 5.4 For Practice Exercise 5.3.

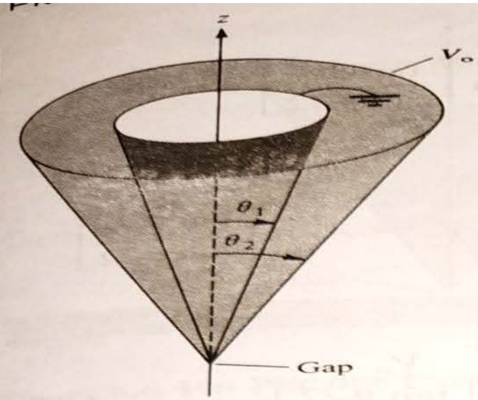


Figure 5.5 Potential $V(\theta)$ due to conducting cones.

Since $r = 0$ and $\theta = 0, \pi$ are excluded, we can multiply by $r^2 \sin\theta$ to get

$$\frac{d}{d\theta}\left[\sin\theta \frac{dV}{d\theta}\right] = 0$$

Integrating once gives

$$\sin\theta \frac{dV}{d\theta} = A$$

or

$$\frac{dV}{d\theta} = \frac{A}{\sin\theta}$$

Integrating this results in

$$\begin{aligned} V &= A \int \frac{d\theta}{\sin\theta} = A \int \frac{d\theta}{2\cos\theta/2 \sin\theta/2} \\ &= A \int \frac{1/2 \sec^2\theta/2 \, d\theta}{\tan\theta/2} \\ &= A \int \frac{d(\tan\theta/2)}{\tan\theta/2} \\ &= A \ln(\tan\theta/2) + B \end{aligned}$$

We now apply the boundary conditions to determine the integration constants A and B.

$$V(\theta = \theta_1) = 0 \rightarrow 0 = A \ln(\tan\theta_1/2) + B$$

or

$$B = -A \ln(\tan\theta_1/2)$$

Hence

$$V = A \ln \left[\frac{\tan \theta/2}{\tan \theta_1/2} \right]$$

Also

$$V(\theta = \theta_2) = V_o \rightarrow V_o = A \ln \left[\frac{\tan \theta_2/2}{\tan \theta_1/2} \right]$$

or

$$A = \frac{V_o}{\ln \left[\dfrac{\tan \theta_2/2}{\tan \theta_1/2} \right]}$$

Thus

$$V = \frac{V_o \ln \left[\dfrac{\tan \theta/2}{\tan \theta_1/2} \right]}{\ln \left[\dfrac{\tan \theta_2/2}{\tan \theta_1/2} \right]}$$

$$\mathbf{E} = -\nabla V = -\frac{1}{r}\frac{dV}{d\theta}\,\mathbf{a}_\theta = -\frac{A}{r \sin \theta}\,\mathbf{a}_\theta$$

$$= -\frac{V_o}{r \sin \theta \ln \left[\dfrac{\tan \theta_2/2}{\tan \theta_1/2} \right]}\,\mathbf{a}_\theta$$

Taking $\theta_1 = \pi/10$, $\theta_2 = \pi/6$, and $V_o = 50$ gives

$$V = \frac{50 \ln \left[\dfrac{\tan \theta/2}{\tan \pi/20} \right]}{\ln \left[\dfrac{\tan \pi/12}{\tan \pi/20} \right]} = 95.1 \ln \left[\frac{\tan \theta/2}{0.1584} \right] \text{V}$$

and

$$\mathbf{E} = -\frac{95.1}{r \sin \theta}\,\mathbf{a}_\theta \text{ V/m}$$

Check: $\nabla^2 V = 0$, $V(\theta = \pi/10) = 0$, $V(\theta = \pi/6) = V_o$.

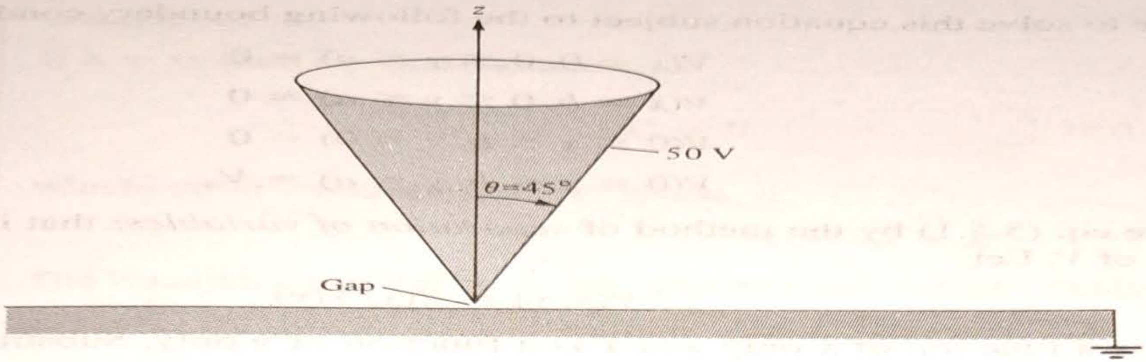


Figure 5.6 For Practice Exercise 5.4.

PRACTICE EXERCISE 5.4

A large conducting cone ($\theta = 45°$) is placed on a conducting plane with a tiny gap separating it from the plane as shown in Figure 5.5. If the cone is connected to a 50 V source, find V and $\mathbf{E}$ at $(-3, 4, 2)$.

Answer: 27.87 V, $-11.35\mathbf{a}_\theta$ V/m.

(a) Determine the potential function for the region inside the rectangular trough of infinite length whose cross section is shown in Figure 5.7.
(b) For $V_o = 100$ V and $b = 2a$, find the potential at $x = a/2$, $y = 3a/4$.

Solution:
(a) The potential V in this case depends on x and y. Laplace's equation becomes

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \tag{5.5.1}$$

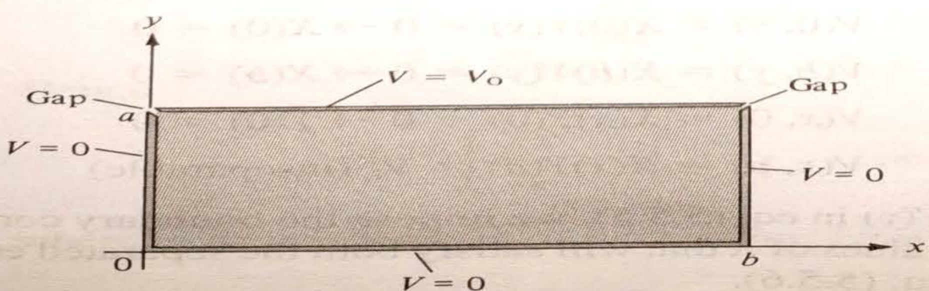


Figure 5.7 Potential $V(x, y)$ due to a conducting rectangular trough; for Example 5.5.

We have to solve this equation subject to the following boundary conditions:

$$V(x = 0, 0 \leq y \leq a) = 0 \tag{5.5.2a}$$
$$V(x = b, 0 \leq y \leq a) = 0 \tag{5.5.2b}$$
$$V(0 \leq x \leq b, y = 0) = 0 \tag{5.5.2c}$$
$$V(0 \leq x \leq b, y = a) = V_o \tag{5.5.2d}$$

We solve eq. (5.5.1) by the method of *separation of variables;* that is, we seek a product solution of V. Let

$$V(x, y) = X(x)\,Y(y) \tag{5.5.3}$$

when X is a function of x only and Y is a function of y only. Substituting eq. (5.5.3) into eq. (5.5.1) yields

$$X''Y + Y''X = 0$$

Dividing through by XY and separating X from Y gives

$$-\frac{X''}{X} = \frac{Y''}{Y} \tag{5.5.4a}$$

Since the left-hand side of this equation is a function of x only and the right-hand side is a function of y only, for the equality to hold, both sides must be equal to a constant λ; that is,

$$-\frac{X''}{X} = \frac{Y''}{Y} = \lambda \tag{5.5.4b}$$

The constant λ is known as the *separation constant*. From eq. (5.5.4b), we obtain

$$X'' + \lambda X = 0 \tag{5.5.5a}$$

and

$$Y'' - \lambda Y = 0 \tag{5.5.5b}$$

Thus the variables have been separated at this point and we refer to eq. (5.5.5) as *separated equations*. We can solve for $X(x)$ and $Y(y)$ separately and then substitute our solutions into eq. (5.5.3). To do this requires that the boundary conditions in eq. (5.5.2) be separated, if possible. We separate them as follows:

$$V(0, y) = X(0)Y(y) = 0 \rightarrow X(0) = 0 \tag{5.5.6a}$$
$$V(b, y) = X(b)Y(y) = 0 \rightarrow X(b) = 0 \tag{5.5.6b}$$
$$V(x, 0) = X(x)Y(0) = 0 \rightarrow Y(0) = 0 \tag{5.5.6c}$$
$$V(x, a) = X(x)Y(a) = V_o \text{ (inseparable)} \tag{5.5.6d}$$

To solve for $X(x)$ and $Y(y)$ in eq. (5.5.5), we impose the boundary conditions in eq. (5.5.6). We consider possible values of λ that will satisfy both the separated equations in eq. (5.5.5) and the conditions in eq. (5.5.6).

CASE 1.

If $\lambda = 0$, then eq. (5.5.5a) becomes

$$X'' = 0 \qquad \text{or} \qquad \frac{d^2X}{dx^2} = 0$$

which, upon integrating twice, yields

$$X = Ax + B \tag{5.5.7}$$

The boundary conditions in eqs. (5.5.6a) and (5.5.6b) imply that

$$X(x = 0) = 0 \rightarrow 0 = 0 + B \qquad \text{or} \qquad B = 0$$

and

$$X(x = b) = 0 \rightarrow 0 = A \cdot b + 0 \qquad \text{or} \qquad A = 0$$

because $b \neq 0$. Hence our solution for X in eq. (5.5.7) becomes

$$X(x) = 0$$

which makes $V = 0$ in eq. (5.5.3). Thus we regard $X(x) = 0$ as a trivial solution and we conclude that $\lambda \neq 0$.

CASE 2.

If $\lambda < 0$, say $\lambda = -\alpha^2$, then eq. (5.5.5a) becomes

$$X'' - \alpha^2 X = 0 \qquad \text{or} \qquad (D^2 - \alpha^2)X = 0$$

where $D = \dfrac{d}{dx}$, that is,

$$DX = \pm\,\alpha X \tag{5.5.8}$$

showing that we have two possible solutions corresponding to the plus and minus signs. For the plus sign, eq. (5.5.8) becomes

$$\frac{dX}{dx} = \alpha X \qquad \text{or} \qquad \frac{dX}{X} = \alpha\, dx$$

Hence

$$\int \frac{dX}{X} = \int \alpha\, dx \qquad \text{or} \qquad \ln X = \alpha x + \ln A_1$$

where $\ln A_1$ is a constant of integration. Thus

$$X = A_1 e^{\alpha x} \tag{5.5.9}$$

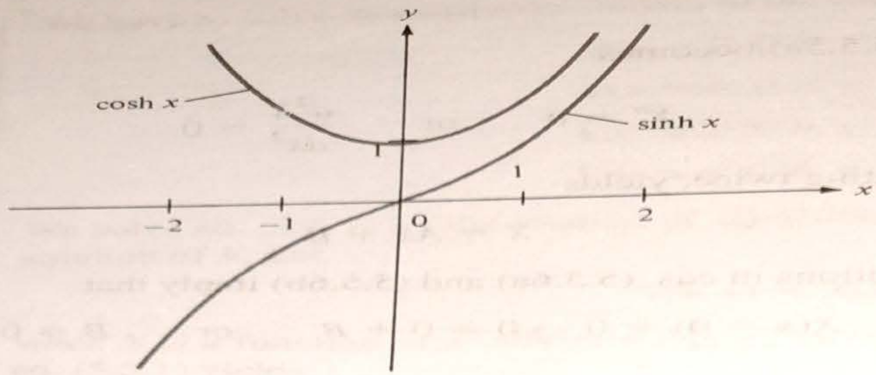


Figure 5.8 Sketch of cosh x and sinh x showing that sinh $x = 0$ if and only if $x = 0$; for Case 2 of Example 5.5.

Similarly, for the minus sign, solving eq. (5.5.8) gives

$$X = A_2 e^{-\alpha x} \tag{5.5.9b}$$

The total solution consists of what we have in eqs. (5.5.9a) and (5.5.9b); that is,

$$X(x) = A_1 e^{\alpha x} + A_2 e^{-\alpha x} \tag{5.5.10}$$

Since $\cosh \alpha x = (e^{\alpha x} + e^{-\alpha x})/2$ and $\sinh \alpha x = (e^{\alpha x} - e^{-\alpha x})/2$ or $e^{\alpha x} = \cosh \alpha x + \sinh \alpha x$ and $e^{-\alpha x} = \cosh \alpha x - \sinh \alpha x$, eq. (5.5.10) can be written as

$$X(x) = B_1 \cosh \alpha x + B_2 \sinh \alpha x \tag{5.5.11}$$

where $B_1 = A_1 + A_2$ and $B_2 = A_1 - A_2$. In view of the given boundary conditions, we prefer eq. (5.5.11) to eq. (5.5.10) as the solution. Again, eqs. (5.5.6a) and (5.5.6b) require that

$$X(x = 0) = 0 \rightarrow 0 = B_1 \cdot (1) + B_2 \cdot (0) \qquad \text{or} \qquad B_1 = 0$$

and

$$X(x = b) = 0 \rightarrow 0 = 0 + B_2 \sinh \alpha b$$

Since $\alpha \neq 0$ and $b \neq 0$, $\sinh \alpha b$ cannot be zero. This is due to the fact that $\sinh x = 0$ if and only if $x = 0$ as shown in Figure 5.8. Hence $B_2 = 0$ and

$$X(x) = 0$$

This is also a trivial solution and we conclude that λ cannot be less than zero.

CASE 3.

If $\lambda > 0$, say $\lambda = \beta^2$, then eq. (5.5.5a) becomes

$$X'' + \beta^2 X = 0$$

that is,

$$(D^2 + \beta^2)X = 0 \qquad \text{or} \qquad DX = \pm j\beta X \tag{5.5.12}$$

where $j = \sqrt{-1}$. From eqs. (5.5.8) and (5.5.12), we notice that the difference between Cases 2 and 3 is the replacement of α by $j\beta$. By taking the same procedure as in Case 2, we obtain the solution as

$$X(x) = C_o e^{j\beta x} + C_1 e^{-j\beta x} \tag{5.5.13a}$$

Since $e^{j\beta x} = \cos \beta x + j \sin \beta x$ and $e^{-j\beta x} = \cos \beta x - j \sin \beta x$, eq. (5.5.13a) can be written as

$$X(x) = g_o \cos \beta x + g_1 \sin \beta x \tag{5.5.13b}$$

where $g_o = C_o + C_1$ and $g_1 = C_o - jC_1$.

In view of the given boundary conditions, we prefer to use eq. (5.5.13b). Imposing the conditions in eqs. (5.5.6a) and (5.5.6b) yields

$$X(x = 0) = 0 \rightarrow 0 = g_o \cdot (1) + 0 \qquad \text{or} \qquad g_o = 0$$

and

$$X(x = b) = 0 \rightarrow 0 = 0 + g_1 \sin \beta b$$

Suppose $g_1 \neq 0$ (otherwise we get a trivial solution), then

$$\sin \beta b = 0 = \sin n\pi \rightarrow \beta b = n\pi$$

$$\beta = \frac{n\pi}{b}, \qquad n = 1, 2, 3, 4, \ldots \tag{5.5.14}$$

Note that, unlike $\sinh x$, which is zero only when $x = 0$, $\sin x$ is zero at an infinite number of points as shown in Figure 5.9. It should also be noted that $n \neq 0$ because $\beta \neq 0$; we have already considered the possibility $\beta = 0$ in Case 1, where we ended up with a trivial solution. Also we do not need to consider $n = -1, -2, -3, -4, \ldots$ because $\lambda = \beta^2$

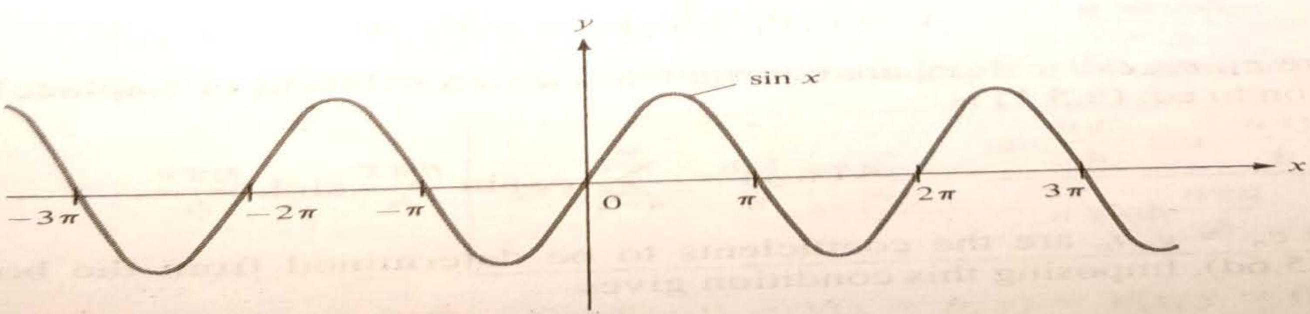


Figure 5.9 Sketch of $\sin x$ showing that $\sin x = 0$ at infinite number of points; for Case 3 of Example 5.5.

would remain the same for positive and negative integer values of n. Thus for a given n, eq. (5.5.13b) becomes

$$X_n(x) = g_n \sin \frac{n\pi x}{b} \tag{5.5.15}$$

Having found $X(x)$ and

$$\lambda = \beta^2 = \frac{n^2\pi^2}{b^2} \tag{5.5.16}$$

we solve eq. (5.5.5b) which is now

$$Y'' - \beta^2 Y = 0$$

The solution to this is similar to eq. (5.5.11) obtained in Case 2, that is,

$$Y(y) = h_o \cosh \beta y + h_1 \sinh \beta y$$

The boundary condition in eq. (5.5.6c) implies that

$$Y(y = 0) = 0 \rightarrow 0 = h_o \cdot (1) + 0 \qquad \text{or} \qquad h_o = 0$$

Hence our solution for $Y(y)$ becomes

$$Y_n(y) = h_n \sinh \frac{n\pi y}{b} \tag{5.5.17}$$

Substituting eqs. (5.5.15) and (5.5.17), which are the solutions to the separated equations in eq. (5.5.5), into the product solution in eq. (5.5.3) gives

$$V_n(x, y) = g_n h_n \sin \frac{n\pi x}{b} \sinh \frac{n\pi y}{b}$$

This shows that there are many possible solutions V_1, V_2, V_3, V_4, and so on, for $n =$ 1, 2, 3, 4, and so on.

By the *superposition theorem*, if $V_1, V_2, V_3, \ldots, V_n$ are solutions of Laplace's equation, the linear combination

$$V = c_1 V_1 + c_2 V_2 + c_3 V_3 + \cdots + c_n V_n$$

(where $c_1, c_2, c_3 \ldots, c_n$ are constants) is also a solution of Laplace's equation. Thus the solution to eq. (5.5.1) is

$$V(x, y) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{b} \sinh \frac{n\pi y}{b} \tag{5.5.18}$$

where $c_n = g_n h_n$ are the coefficients to be determined from the boundary condition in eq. (5.5.6d). Imposing this condition gives

$$V(x, y = a) = V_o = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{b} \sinh \frac{n\pi a}{b} \tag{5.5.19}$$

which is a Fourier series expansion of V_o. Multiplying both sides of eq. (5.5.19) by $\sin m\pi x/b$ and integrating over $0 < x < b$ gives

$$\int_0^b V_o \sin \frac{m\pi x}{b}\, dx = \sum_{n=1}^{\infty} c_n \sinh \frac{n\pi a}{b} \int_0^b \sin \frac{m\pi x}{b} \sin \frac{n\pi x}{b}\, dx \tag{5.5.20}$$

By the orthogonality property of the sine or cosine function (see Appendix A.9).

$$\int_0^{\pi} \sin mx \sin nx\, dx = \begin{cases} 0, & m \neq n \\ \pi/2, & m = n \end{cases}$$

Incorporating this property in eq. (5.5.20) means that all terms on the right-hand side of eq. (5.5.20) will vanish except one term in which $m = n$. Thus eq. (5.5.20) reduces to

$$\int_0^b V_o \sin \frac{n\pi x}{b}\, dx = c_n \sinh \frac{n\pi a}{b} \int_0^b \sin^2 \frac{n\pi x}{b}\, dx$$

$$-V_o \frac{b}{n\pi} \cos \frac{n\pi x}{b}\Big|_0^b = c_n \sinh \frac{n\pi a}{b} \frac{1}{2} \int_0^b \left(1 - \cos \frac{2n\pi x}{b}\right) dx$$

$$\frac{V_o b}{n\pi}(1 - \cos n\pi) = c_n \sinh \frac{n\pi a}{b} \cdot \frac{b}{2}$$

or

$$c_n \sinh \frac{n\pi a}{b} = \frac{2V_o}{n\pi}(1 - \cos n\pi)$$
$$= \begin{cases} \dfrac{4V_o}{n\pi}, & n = 1, 3, 5, \ldots \\ 0, & n = 2, 4, 6, \ldots \end{cases}$$

that is,

$$c_n = \begin{cases} \dfrac{4V_o}{n\pi \sinh \dfrac{n\pi a}{b}}, & n = \text{odd} \\ 0, & n = \text{even} \end{cases} \tag{5.5.21}$$

Substituting this into eq. (5.5.18) gives the complete solution as

$$\boxed{V(x, y) = \frac{4V_o}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{\sin \dfrac{n\pi x}{b} \sinh \dfrac{n\pi y}{b}}{n \sinh \dfrac{n\pi a}{b}}} \tag{5.5.22}$$

Check: $\nabla^2 V = 0$, $V(x = 0, y) = 0 = V(x = b, y) = V(x, y = 0)$, $V(x, y = a) = V_o$. The solution in eq. (5.5.22) should not be a surprise; it can be guessed by mere observation of the potential system in Figure 5.7. From this figure, we notice that along x, V varies from

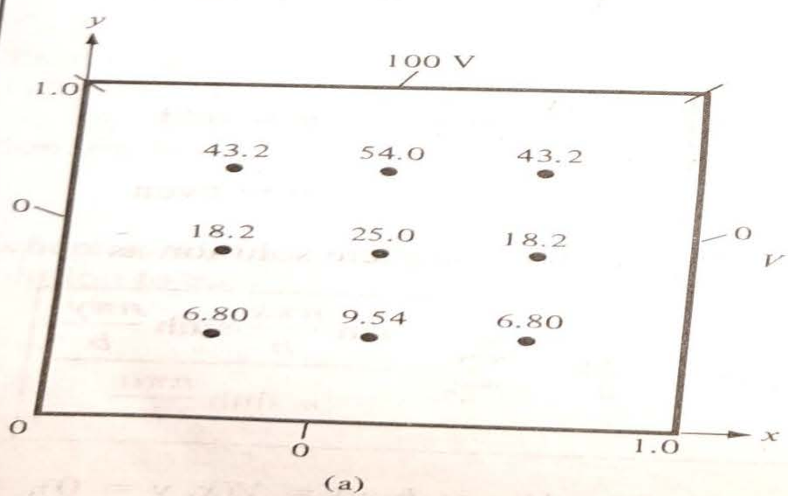
0 (at $x = 0$) to 0 (at $x = b$) and only a sine function can satisfy this requirement. Similarly, along y, V varies from 0 (at $y = 0$) to V_o (at $y = a$) and only a hyperbolic sine function can satisfy this. Thus we should expect the solution as in eq. (5.5.22).

To determine the potential for each point (x, y) in the trough, we take the first few terms of the convergent infinite series in eq. (5.5.22). Taking four or five terms may be sufficient.

(b) For $x = a/2$ and $y = 3a/4$, where $b = 2a$, we have

$$
\begin{aligned}
V\left(\frac{a}{2}, \frac{3a}{4}\right) &= \frac{4V_o}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{\sin n\pi/4 \sinh 3n\pi/8}{n \sinh n\pi/2} \\
&= \frac{4V_o}{\pi}\left[\frac{\sin \pi/4 \sinh 3\pi/8}{\sinh \pi/2} + \frac{\sin 3\pi/4 \sinh 9\pi/8}{3 \sinh 3\pi/2}\right. \\
&\qquad \left. + \frac{\sin 5\pi/4 \sinh 15\pi/8}{5 \sinh 5\pi/2} + \cdots\right] \\
&= \frac{4V_o}{\pi}(0.4517 + 0.0725 - 0.01985 - 0.00645 + 0.00229 + \cdots) \\
&= 0.6374V_o
\end{aligned}
$$

It is instructive to consider a special case of $a = b = 1$ m and $V_o = 100$ V. The potentials at some specific points are calculated by using eq. (5.5.22), and the result is displayed in Figure 5.10(a). The corresponding flux lines and equipotential lines are shown in Figure 5.10(b). A simple Matlab program based on eq. (5.5.22) is displayed in Figure 5.11. This self-explanatory program can be used to calculate $V(x, y)$ at any point within the trough. In Figure 5.11, $V(x = b/4, y = 3a/4)$ is typically calculated and found to be 43.2 V.



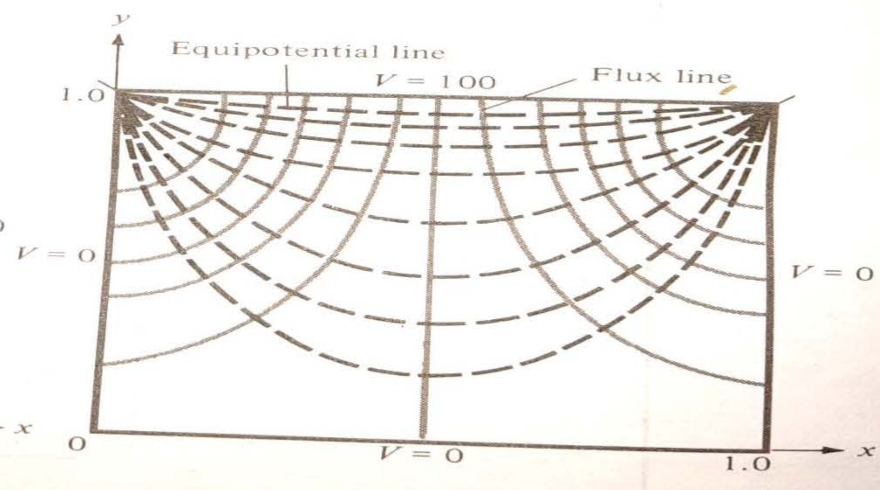


Figure 5.10 For Example 5.5: **(a)** $V(x, y)$ calculated at some points, **(b)** sketch of flux lines and equipotential lines.

Find the potential distribution in Example 5.5 if V_o is not constant but

(a) $V_o = 10 \sin 3\pi x/b,\ y = a,\ 0 \le x \le b$

(b) $V_o = 2 \sin \dfrac{\pi x}{b} + \dfrac{1}{10} \sin \dfrac{5\pi x}{b},\ y = a,\ 0 \le x \le b$

Solution:

(a) In Example 5.5, every step before eq. (5.5.19) remains the same; that is, the solution is of the form

$$V(x, y) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{b} \sinh \frac{n\pi y}{b} \tag{5.6.1}$$

in accordance with eq. (5.5.18). But instead of eq. (5.5.19), we now have

$$V(y = a) = V_o = 10 \sin \frac{3\pi x}{b} = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{b} \sinh \frac{n\pi a}{b}$$

By equating the coefficients of the sine terms on both sides, we obtain

$$c_n = 0, \qquad n \neq 3$$

For $n = 3$,

$$10 = c_3 \sinh \frac{3\pi a}{b}$$

or

$$c_3 = \frac{10}{\sinh \dfrac{3\pi a}{b}}$$

Thus the solution in eq. (5.6.1) becomes

$$V(x, y) = 10 \sin \frac{3\pi x}{b} \frac{\sinh \dfrac{3\pi y}{b}}{\sinh \dfrac{3\pi a}{b}}$$

(b) Similarly, instead of eq. (5.5.19), we have

$$V_o = V(y = a)$$

or

$$2 \sin \frac{\pi x}{b} + \frac{1}{10} \sin \frac{5\pi x}{b} = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{b} \sinh \frac{n\pi a}{b}$$

Equating the coefficient of the sine terms:

$$c_n = 0, \qquad n \neq 1, 5$$

For $n = 1$,

$$2 = c_1 \sinh \frac{\pi a}{b} \quad \text{or} \quad c_1 = \frac{2}{\sinh \frac{\pi a}{b}}$$

For $n = 5$,

$$\frac{1}{10} = c_5 \sinh \frac{5\pi a}{b} \quad \text{or} \quad c_5 = \frac{1}{10 \sinh \frac{5\pi a}{b}}$$

Hence,

$$V(x, y) = \frac{2 \sin \frac{\pi x}{b} \sinh \frac{\pi y}{b}}{\sinh \frac{\pi a}{b}} + \frac{\sin \frac{5\pi x}{b} \sinh \frac{5\pi y}{b}}{10 \sinh \frac{5\pi a}{b}}$$

PRACTICE EXERCISE 5.6

In Example 5.5, suppose everything remains the same except that V_o is replaced by $V_o \sin \frac{7\pi x}{b}$, $0 \leq x \leq b$, $y = a$. Find $V(x, y)$.

Answer: $\dfrac{V_o \sin \frac{7\pi x}{b} \sinh \frac{7\pi y}{b}}{\sinh \frac{7\pi a}{b}}$.

Obtain the separated differential equations for potential distribution $V(\rho, \phi, z)$ in a charge-free region.

Solution:

This example, like Example 5.5, further illustrates the method of separation of variables. Since the region is free of charge, we need to solve Laplace's equation in cylindrical coordinates; that is,

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0 \tag{5.7.1}$$

We let

$$V(\rho, \phi, z) = R(\rho)\, \Phi(\phi)\, Z(z) \tag{5.7.2}$$

where R, Φ, and Z are, respectively, functions of ρ, ϕ, and z. Substituting eq. (5.7.2) into eq. (5.7.1) gives

$$\frac{\Phi Z}{\rho}\frac{d}{d\rho}\left(\frac{\rho\, dR}{d\rho}\right) + \frac{RZ}{\rho^2}\frac{d^2\Phi}{d\phi^2} + R\Phi\frac{d^2Z}{dz^2} = 0 \tag{5.7.3}$$

We divide through by $R\Phi Z$ to obtain

$$\frac{1}{\rho R}\frac{d}{d\rho}\left(\frac{\rho\, dR}{d\rho}\right) + \frac{1}{\rho^2\Phi}\frac{d^2\Phi}{d\phi^2} = -\frac{1}{Z}\frac{d^2Z}{dz^2} \tag{5.7.4}$$

The right-hand side of this equation is solely a function of z, whereas the left-hand side does not depend on z. For the two sides to be equal, they must be constant; that is,

$$\frac{1}{\rho R}\frac{d}{d\rho}\left(\frac{\rho\, dR}{d\rho}\right) + \frac{1}{\rho^2\Phi}\frac{d^2\Phi}{d\phi^2} = -\frac{1}{Z}\frac{d^2Z}{dz^2} = -\lambda^2 \tag{5.7.5}$$

where $-\lambda^2$ is a separation constant. Equation (5.7.5) can be separated into two parts:

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = \lambda^2 \tag{5.7.6}$$

or

$$Z'' - \lambda^2 Z = 0 \tag{5.7.7}$$

and

$$\frac{\rho}{R}\frac{d}{d\rho}\left(\frac{\rho\, dR}{d\rho}\right) + \lambda^2\rho^2 + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = 0 \tag{5.7.8}$$

Equation (5.7.8) can be written as

$$\frac{\rho^2}{R}\frac{d^2R}{d\rho^2} + \frac{\rho}{R}\frac{dR}{d\rho} + \lambda^2\rho^2 = -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = \mu^2 \tag{5.7.9}$$

where μ^2 is another separation constant. Equation (5.7.9) is separated as

$$\Phi'' + \mu^2\Phi = 0 \tag{5.7.10}$$

and

$$\rho^2R'' + \rho R' + (\rho^2\lambda^2 - \mu^2)R = 0 \tag{5.7.11}$$

Equations (5.7.7), (5.7.10), and (5.7.11) are the required separated differential equations. Equation (5.7.7) has a solution similar to the solution obtained in Case 2 of Example 5.5; that is,

$$Z(z) = c_1 \cosh \lambda z + c_2 \sinh \lambda z \tag{5.7.12}$$

The solution to eq. (5.7.10) is similar to the solution obtained in Case 3 of Example 5.5; that is,

$$\Phi(\phi) = c_3 \cos \mu\phi + c_4 \sin \mu\phi \tag{5.7.13}$$

Equation (5.7.11) is known as the *Bessel differential equation* and its solution is beyond the scope of this text.[1]

PRACTICE EXERCISE 5.7

Repeat Example 5.7 for $V(r, \theta, \phi)$.

Answer: If $V(r, \theta, \phi) = R(r)\, F(\theta)\, \Phi(\phi)$, $\Phi'' + \lambda^2\Phi = 0$, $R'' + \dfrac{2}{r}R' - \dfrac{\mu^2}{r^2}R = 0$, $F'' + \cos\theta\, F' + (\mu^2 \sin\theta - \lambda^2 \operatorname{cosec}\theta)\, F = 0$.

A metal bar of conductivity σ is bent to form a flat 90° sector of inner radius a, outer radius b, and thickness t as shown in Figure 5.17. Show that (a) the resistance of the bar between the vertical curved surfaces at $\rho = a$ and $\rho = b$ is

$$R = \frac{2 \ln \dfrac{b}{a}}{\sigma \pi t}$$

and (b) the resistance between the two horizontal surfaces at $z = 0$ and $z = t$ is

$$R' = \frac{4t}{\sigma\pi(b^2 - a^2)}$$

Solution:

(a) Between the vertical curved ends located at $\rho = a$ and $\rho = b$, the bar has a nonuniform cross section and hence eq. (5.16) does not apply. We have to use eq. (5.16). Let a potential difference V_o be maintained between the curved surfaces at $\rho = a$ and $\rho = b$ so that

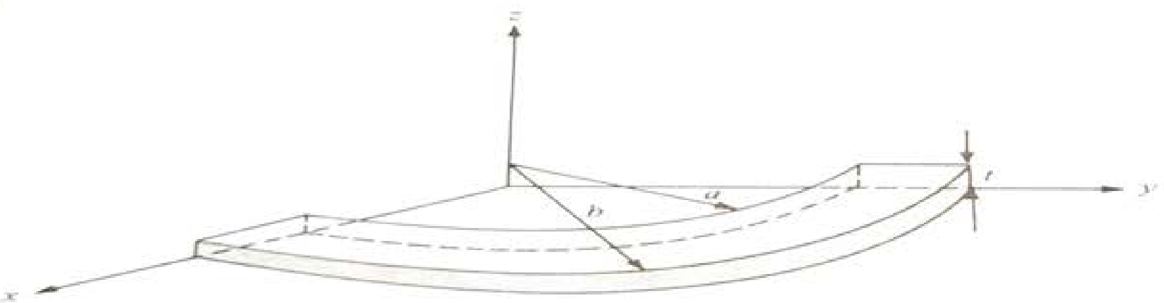


Figure 5.17 Bent metal bar for Example 5.8.

$V(\rho = a) = 0$ and $V(\rho = b) = V_o$. We solve for V in Laplace's equation $\nabla^2 V = 0$ in cylindrical coordinates. Since $V = V(\rho)$,

$$\nabla^2 V = \frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{dV}{d\rho}\right) = 0$$

As $\rho = 0$ is excluded, upon multiplying by ρ and integrating once, this becomes

$$\rho \frac{dV}{d\rho} = A$$

or

$$\frac{dV}{d\rho} = \frac{A}{\rho}$$

Integrating once again yields

$$V = A \ln \rho + B$$

where A and B are constants of integration to be determined from the boundary conditions.

$$V(\rho = a) = 0 \rightarrow 0 = A \ln a + B \qquad \text{or} \qquad B = -A \ln a$$

$$V(\rho = b) = V_o \rightarrow V_o = A \ln b + B = A \ln b - A \ln a = A \ln \frac{b}{a} \qquad \text{or} \qquad A = \frac{V_o}{\ln \frac{b}{a}}$$

Hence,

$$V = A \ln \rho - A \ln a = A \ln \frac{\rho}{a} = \frac{V_o}{\ln \frac{b}{a}} \ln \frac{\rho}{a}$$

$$\mathbf{E} = -\nabla V = -\frac{dV}{d\rho}\mathbf{a}_\rho = -\frac{A}{\rho}\mathbf{a}_\rho = -\frac{V_o}{\rho \ln \frac{b}{a}}\mathbf{a}_\rho$$

$$\mathbf{J} = \sigma \mathbf{E}, \qquad d\mathbf{S} = -\rho \, d\phi \, dz \, \mathbf{a}_\rho$$

$$I = \int \mathbf{J} \cdot d\mathbf{S} = \int_{\phi=0}^{\pi/2} \int_{z=0}^{t} \frac{V_o \sigma}{\rho \ln \frac{b}{a}} dz \, \rho \, d\phi = \frac{\pi}{2} \frac{t V_o \sigma}{\ln \frac{b}{a}}$$

Thus

$$R = \frac{V_o}{I} = \frac{2 \ln \frac{b}{a}}{\sigma \pi t}$$

as required.

(b) Let V_o be the potential difference between the two horizontal surfaces so that $V(z = 0) = 0$ and $V(z = t) = V_o$. $V = V(z)$, so Laplace's equation $\nabla^2 V = 0$ becomes

$$\frac{d^2V}{dz^2} = 0$$

Integrating twice gives

$$V = Az + B$$

We apply the boundary conditions to determine A and B:

$$V(z = 0) = 0 \rightarrow 0 = 0 + B \qquad \text{or} \qquad B = 0$$

$$V(z = t) = V_o \rightarrow V_o = At \qquad \text{or} \qquad A = \frac{V_o}{t}$$

Hence,

$$V = \frac{V_o}{t} z$$

$$\mathbf{E} = -\nabla V = -\frac{dV}{dz}\mathbf{a}_z = -\frac{V_o}{t}\mathbf{a}_z$$

$$\mathbf{J} = \sigma\mathbf{E} = -\frac{\sigma V_o}{t}\mathbf{a}_z, \qquad d\mathbf{S} = -\rho\, d\phi\, d\rho\, \mathbf{a}_z$$

$$I = \int \mathbf{J} \cdot d\mathbf{S} = \int_{\rho=a}^{b} \int_{\phi=0}^{\pi/2} \frac{V_o\sigma}{t} \rho\, d\phi\, d\rho$$

$$= \frac{V_o\sigma}{t} \cdot \frac{\pi}{2} \frac{\rho^2}{2}\bigg|_a^b = \frac{V_o\, \sigma\, \pi\, (b^2 - a^2)}{4t}$$

Thus

$$R' = \frac{V_o}{I} = \frac{4t}{\sigma\pi(b^2 - a^2)}$$

Alternatively, for this case, the cross section of the bar is uniform between the horizontal surfaces at $z = 0$ and $z = t$ and eq. (4.16) holds. Hence,

$$R' = \frac{\ell}{\sigma S} = \frac{t}{\sigma \dfrac{\pi}{4}(b^2 - a^2)}$$

$$= \frac{4t}{\sigma\pi(b^2 - a^2)}$$

as required.

A coaxial cable contains an insulating material of conductivity σ. If the radius of the central wire is a and that of the sheath is b, show that the conductance of the cable per unit length is (see eq. (5.37))

$$G = \frac{2\pi\sigma}{\ln \dfrac{b}{a}}$$

Solution:

Consider length L of the coaxial cable as shown in Figure 5.14. Let V_o be the potential difference between the inner and outer conductors so that $V(\rho = a) = 0$ and $V(\rho = b) = V_o$, which allows V and $\mathbf{E}$ to be found just as in part (a) of Example 5.8. Hence:

$$\mathbf{J} = \sigma\mathbf{E} = \frac{-\sigma V_o}{\rho \ln \dfrac{b}{a}}\,\mathbf{a}_\rho, \qquad d\mathbf{S} = -\rho\, d\phi\, dz\, \mathbf{a}_\rho$$

$$I = \int \mathbf{J}\cdot d\mathbf{S} = \int_{\phi=0}^{2\pi}\int_{z=0}^{L} \frac{V_o\sigma}{\rho \ln \dfrac{b}{a}}\,\rho\, dz\, d\phi$$

$$= \frac{2\pi L\sigma V_o}{\ln \dfrac{b}{a}}$$

The resistance per unit length is

$$R = \frac{V_o}{I}\cdot\frac{1}{L} = \frac{\ln \dfrac{b}{a}}{2\pi\sigma}$$

and the conductance per unit length is

$$G = \frac{1}{R} = \frac{2\pi\sigma}{\ln \dfrac{b}{a}}$$

as required.

Conducting spherical shells with radii $a = 10$ cm and $b = 30$ cm are maintained at a potential difference of 100 V such that $V(r = b) = 0$ and $V(r = a) = 100$ V. Determine V and $\mathbf{E}$ in the region between the shells. If $\varepsilon_r = 2.5$ in the region, determine the total charge induced on the shells and the capacitance of the capacitor.

Solution:

Consider the spherical shells shown in Figure 5.18 and assume that V depends only on r. Hence Laplace's equation becomes

$$\nabla^2 V = \frac{1}{r^2}\frac{d}{dr}\left[r^2 \frac{dV}{dr}\right] = 0$$

Since $r \neq 0$ in the region of interest, we multiply through by r^2 to obtain

$$\frac{d}{dr}\left[r^2 \frac{dV}{dr}\right] = 0$$

Integrating once gives

$$r^2 \frac{dV}{dr} = A$$

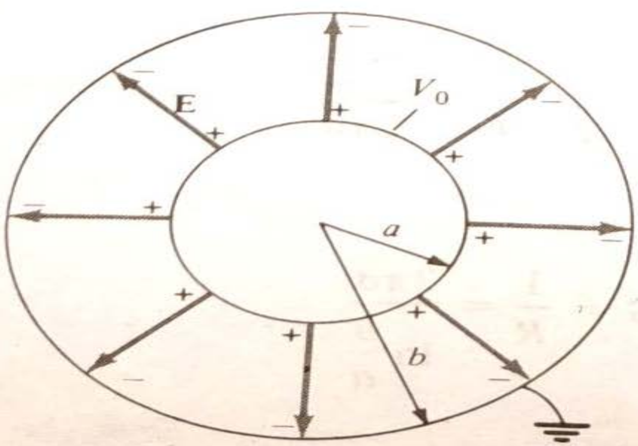


Figure 5.18 Potential $V(r)$ due to conducting spherical shells.

$$\frac{dV}{dr} = \frac{A}{r^2}$$

Integrating again gives

$$V = -\frac{A}{r} + B$$

As usual, constants A and B are determined from the boundary conditions.

When $r = b$, $V = 0 \rightarrow 0 = -\frac{A}{b} + B$ or $B = \frac{A}{b}$

Hence

$$V = A\left[\frac{1}{b} - \frac{1}{r}\right]$$

Also when $r = a$, $V = V_o \rightarrow V_o = A\left[\frac{1}{b} - \frac{1}{a}\right]$

or

$$A = \frac{V_o}{\frac{1}{b} - \frac{1}{a}}$$

Thus

$$V = V_o \frac{\left[\frac{1}{r} - \frac{1}{b}\right]}{\frac{1}{a} - \frac{1}{b}}$$

$$\mathbf{E} = -\nabla V = -\frac{dV}{dr}\mathbf{a}_r = -\frac{A}{r^2}\mathbf{a}_r$$

$$= \frac{V_o}{r^2\left[\frac{1}{a} - \frac{1}{b}\right]}\mathbf{a}_r$$

$$Q = \int \varepsilon \mathbf{E} \cdot d\mathbf{S} = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{\varepsilon_o \varepsilon_r V_o}{r^2\left[\frac{1}{a} - \frac{1}{b}\right]} r^2 \sin\theta \, d\phi \, d\theta$$

$$= \frac{4\pi\varepsilon_o\varepsilon_r V_o}{\frac{1}{a} - \frac{1}{b}}$$

The capacitance is easily determined as

$$C = \frac{Q}{V_o} = \frac{4\pi\varepsilon}{\dfrac{1}{a} - \dfrac{1}{b}}$$

which is the same as we obtained in eq. (5.32); there in Section 5.5, we assumed Q and found the corresponding V_o, but here we assumed V_o and found the corresponding Q to determine C. Substituting $a = 0.1$ m, $b = 0.3$ m, $V_o = 100$ V yields

$$V = 100 \frac{\left[\dfrac{1}{r} - \dfrac{10}{3}\right]}{10 - 10/3} = 15\left[\frac{1}{r} - \frac{10}{3}\right] \text{ V}$$

Check: $\nabla^2 V = 0$, $V(r = 0.3 \text{ m}) = 0$, $V(r = 0.1 \text{ m}) = 100$.

$$\mathbf{E} = \frac{100}{r^2\,[10 - 10/3]}\,\mathbf{a}_r = \frac{15}{r^2}\,\mathbf{a}_r \text{ V/m}$$

$$\begin{aligned} Q &= \pm 4\pi \cdot \frac{10^{-9}}{36\pi} \cdot \frac{(2.5)\cdot(100)}{10 - 10/3} \\ &= \pm 4.167 \text{ nC} \end{aligned}$$

The positive charge is induced on the inner shell; the negative charge is induced on the outer shell. Also

$$C = \frac{|Q|}{V_o} = \frac{4.167 \times 10^{-9}}{100} = 41.67 \text{ pF}$$

PRACTICE EXERCISE 5.10

If Figure 5.19 represents the cross sections of two spherical capacitors, determine their capacitances. Let $a = 1$ mm, $b = 3$ mm, $c = 2$ mm, $\varepsilon_{r1} = 2.5$, and $\varepsilon_{r2} = 3.5$.

Answer: (a) 0.53 pF, (b) 0.5 pF.

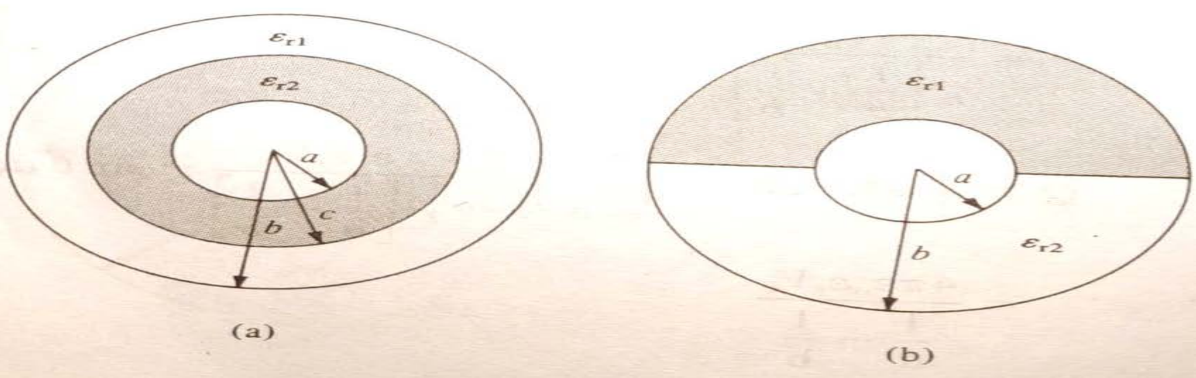


Figure 5.19 For Practice Exercises 5.9, 5.10, and 5.12.