Transformations of Random Vectors

Let $\underline{f} : \mathbb{R}^n \to \mathbb{R}^m$ be a measurable fn.

$\underline{X} = (X_1, \cdots X_n)$

$Y_1 = f_1(X_1, \cdots, X_n)$

$\vdots$

$Y_m = f_m(X_1, \cdots X_n)$

$\underline{Y} = (Y_1, \cdots, Y_m)$

$\underline{Y}$ is m-dimensional random vector

For specific types of functions sometimes mgf is useful. For example, in finding distributions of sums of independent r.v.'s.

<u>Example</u>: Let $X_1 \ldots X_n \overset{i.i.d.}{\sim} N(\mu, \sigma^2)$

$$Y = \sum_{i=1}^{n} X_i$$

$$M_Y(t) = \prod_{i=1}^{n} M_{X_i}(t) = \prod_{i=1}^{n} \left[e^{\mu t + \frac{1}{2}\sigma^2 t^2} \right]$$

$$= e^{n\mu t + \frac{1}{2} n\sigma^2 t^2}$$

ie $Y \sim N(n\mu, n\sigma^2)$

$$\bar{X} = \frac{Y}{n} = \frac{\sum X_i}{n} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

If r.v's are discrete, we may have to use pmf directly to derive pmf of the transformed variables.

Ex. Let $X, Y \overset{i.i.d}{\sim} B(n, p)$

$U = X + Y \sim Bin(2n, p)$.

$V = X - Y$, $\quad v \to -n, -(n-1), \ldots, -1, 0, 1, \ldots$

$\ldots, n$

$$P(V=v) = P(X-Y=v) = P(X=v+Y)$$

$$= \sum_{y=0}^{n} P(X=v+y,\ Y=y)$$

$$= \sum_{y=0}^{n} P(X=v+y)\, P(Y=y)$$

$0 \le v+y \le n$

$$= \sum \binom{n}{u+y} p^{u+y} (1-p)^{n-u-y} \binom{n}{y} p^{y} (1-p)^{n-y}$$

$$= \sum_{y=0}^{n} \binom{n}{u+y} \binom{n}{y} p^{u+2y} (1-p)^{2n-u-2y}$$

$$u+y = 0, 1, \ldots, n$$

2. $U = \dfrac{X}{Y+1}$, $V = Y+1$

(U, V) ?? (dist^n)

$$V \to 1, 2, \ldots, (n+1)$$

$$U \to \left.\begin{array}{l} 0, 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n+1} \\ 2, \frac{2}{2}=1, \frac{2}{3}, \ldots, \frac{2}{n+1} \\ \vdots \\ n, \frac{n}{2}, \ldots, \frac{n}{n+1} \end{array}\right\}$$

The joint pmf of (U, V)

$$P(U=u, V=v) = P(X=uv, Y=v-1)$$

$$= P(X=uv)\,P(Y=v-1)$$

$$= \binom{n}{uv} p^{uv} (1-p)^{n-uv} \cdot \binom{n}{v-1} p^{v-1} (1-p)^{n-v+1}$$

3. Let (X,Y) have joint pmf

$Y \backslash X$	-1	0	1
-2	$1/6$	$1/12$	$1/6$
1	$1/6$	$1/12$	$1/6$
2	$1/12$	0	$1/12$

$U = |X|$, $V = Y^2$

$(U,V) \to$?? joint pmf

$U \to 0, 1,$ $\quad V \to 1, 4$

$V \backslash U$	0	1
1	1/12	1/3
4	1/12	1/2

CDF Approach

Ex Let (X, Y) have joint pdf

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{1+xy}{4}, & |x|<1, \ |y|<1 \\ 0, & \text{ew} \end{cases}$$

$U = X^2, \quad V = Y^2$

The joint cdf of (U, V)

$$F_{U,V}(u, v) = P(U \le u, V \le v), \qquad \begin{matrix} 0 \le u < 1 \\ 0 \le v < 1 \end{matrix}$$

$$= P(-\sqrt{u} \le X \le \sqrt{u}, \ -v < Y \le \sqrt{v})$$

$$= \int_{-\sqrt{v}}^{\sqrt{v}} \int_{-\sqrt{u}}^{\sqrt{u}} \left(\frac{1 + xy}{4} \right) dx\, dy = \boxed{\sqrt{u}\sqrt{v}}$$

Theorem: Let $\underline{X} = (X_1, \ldots, X_n)$ be a continuous random vector with joint pdf

$$f_{\underline{X}}(\underline{x}), \quad \underline{x} = (x_1, \ldots, x_n).$$

(a) Let $u_i = g_i(\underline{x})$, $i = 1, \ldots, n$

$\underline{u} = (u_1 \cdots u_n) : \mathbb{R}^n \to \mathbb{R}^n$ be one-to-one

Let $x_i = h_i(\underline{u})$, $i = 1, \ldots, n$ be inverses.

(b) Let the function & inverse both be continuous

(c) Assume partial derivatives $\frac{\partial x_i}{\partial u_j}$, $i, j = 1 \ldots n$ exist and are continuous

(d) Assume that the Jacobian of transformation

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \cdots & \frac{\partial x_1}{\partial u_n} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} & \cdots & \frac{\partial x_2}{\partial u_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial x_n}{\partial u_1} & \frac{\partial x_n}{\partial u_2} & \cdots & \frac{\partial x_n}{\partial u_n} \end{vmatrix} \neq 0$$

in the range of the transformation.

Then the r. vector $\underline{U} = (U_1, \ldots, U_n)$ is continuous and has joint pdf given by

$$f_{\underline{U}}(\underline{u}) = f_{\underline{X}}(h_1(\underline{u}), \ldots, h_n(\underline{u}))\,|J|.$$

<u>Example</u>. 1. Let $X_1, X_2, X_3 \overset{i.i.d.}{\sim} \mathrm{Exp}(1)$

So the joint pdf of $\underline{X} = (X_1, X_2, X_3)$

$$f_{\underline{X}}(\underline{x}) = \prod_{i=1}^{3} f_{X_i}(x_i) = e^{-(x_1+x_2+x_3)}, \quad x_i > 0, \ i=1,2,3$$

Let $\underline{Y} = (Y_1, Y_2, Y_3)$, where

$Y_1 = X_1 + X_2 + X_3$, $Y_2 = \dfrac{X_1 + X_2}{X_1 + X_2 + X_3}$, $Y_3 = \dfrac{X_1}{X_1 + X_2}$

We want joint pdf of $\underline{Y}$ and marginal pdf's of Y_1, Y_2, Y_3.

$x_1 = y_1 y_2 y_3$

$x_2 = y_1 y_2 (1 - y_3)$

$x_3 = y_1 (1 - y_2)$

$y_1 > 0$

$0 < y_2 < 1$

$0 < y_3 < 1$

$$|J| = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \frac{\partial x_1}{\partial y_3} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \frac{\partial x_2}{\partial y_3} \\ \frac{\partial x_3}{\partial y_1} & \frac{\partial x_3}{\partial y_2} & \frac{\partial x_3}{\partial y_3} \end{vmatrix} = \begin{vmatrix} y_2 y_3 & y_1 y_3 & y_1 y_2 \\ y_2(1-y_3) & y_1(1-y_2) & -y_1 y_2 \\ 1-y_2 & -y_1 & 0 \end{vmatrix}$$

$$= -y_1^2 y_2$$

So the joint pdf of $\underline{Y} = (Y_1, Y_2, Y_3)$ is

$$f_{\underline{Y}}(\underline{y}) = \begin{cases} y_1^2 y_2 e^{-y_1}, & y_1 > 0,\ 0 < y_2 < 1,\ 0 < y_3 < 1 \\ 0, & \text{ew} \end{cases}$$

The marginal (joint) pdf of Y_1, Y_2 is

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} y_1^2 y_2 e^{-y_1}, & y_1 > 0,\ 0 < y_2 < 1 \\ 0, & \text{ew} \end{cases}$$

The marginal pdf of Y_1 is

$$f_{Y_1}(y_1) = \begin{cases} \frac{1}{2} y_1^2 e^{-y_1}, & y_1 > 0 \\ 0, & \text{ew} \end{cases}$$

$\boxed{\text{Gamma}(3, 1)}$

The marginal pdf of Y_2 is

$$f_{Y_2}(y_2) = \begin{cases} 2y_2, & 0 < y_2 < 1 \\ 0, & \text{ew} \end{cases}$$

$\boxed{\text{Beta}(2, 1)}$

The marginal pdf of Y_3 is

$$f_{Y_3}(y_3) = \begin{cases} 1, & 0 < y_3 < 1 \\ 0, & \text{ow} \end{cases} \qquad \boxed{U(0,1)}$$

Since $f_{\underline{Y}}(\underline{y}) = \prod_{i=1}^{3} f_{Y_i}(y_i) \quad \forall\ \underline{y} \in \mathbb{R}^3$,

we conclude that Y_1, Y_2, Y_3 are also independent.

2. Let $X, Y \overset{i.i.d.}{\sim} U(0,1)$

$U = X+Y, \quad V = X-Y.$

Find joint & marginal pdf's of U & V.

The joint pdf of (X,Y) is

$$f_{X,Y}(x,y) = \begin{cases} 1, & 0<x<1,\ 0<y<1 \\ 0, & \text{ew} \end{cases}$$

$$x = \frac{u+v}{2}, \quad y = \frac{u-v}{2}, \qquad J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{1}{2}$$

The joint pdf of (U, V) is

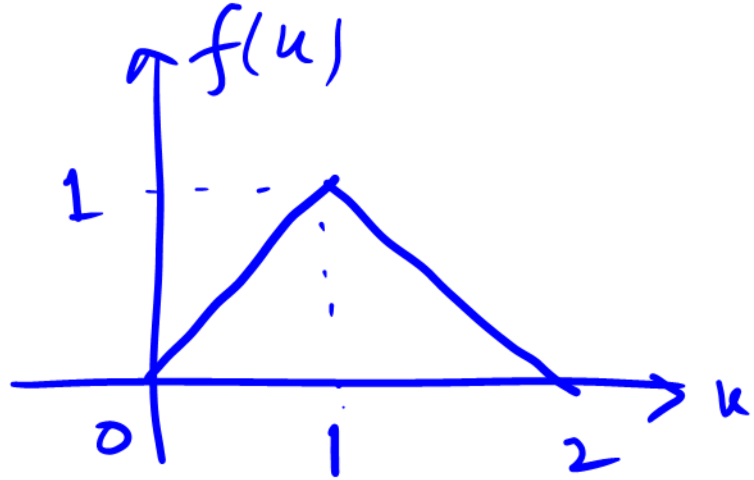
$$f_{U,V}(u,v) = \begin{cases} \frac{1}{2}, & 0<u+v<2,\ 0<u-v<2 \\ & \text{where } 0<u<2 \\ & \text{\& } -1<v<1 \\ 0, & \text{ew} \end{cases}$$

The marginal pdf of U is

$$f_U(u) = \begin{cases} \frac{1}{2}\int_{-u}^{u} dv, & 0<u<1 \end{cases}$$

$$\left(\frac{1}{2}\int_{u-2}^{2-u} dv \, , \qquad 1 \le u < 2 \right.$$

$$= \begin{cases} u, & 0 < u < 1 \\ 2-u, & 1 < u < 2 \\ 0, & \text{ow} \end{cases}$$

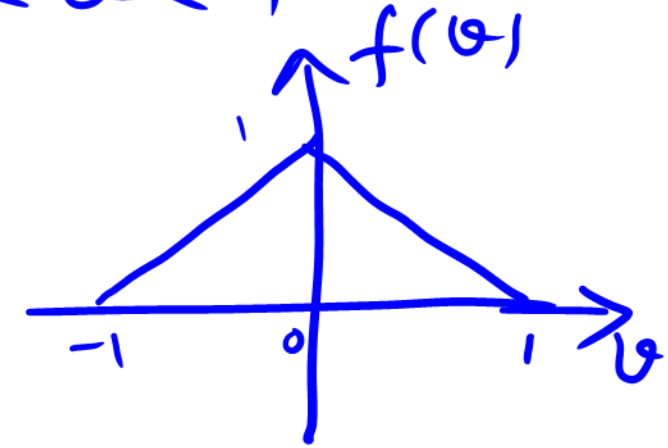


The marginal pdf of V is

$$f_V(v) = \begin{cases} \frac{1}{2}\int_{-v}^{v+2} du & -1 < v \le 0 \end{cases}$$

$$\frac{1}{2}\int_{v}^{2-v} du, \qquad 0<v<1$$

$$= \begin{cases} 1+v, & -1<v\le 0 \\ 1-v & 0<v<1 \\ 0, & \text{ew} \end{cases}$$



The sum and difference of two independent $U(0,1)$ r.v's have triangular distributions.