Magnetic Force

- **Force on a moving charged particle**, with a velocity $\vec{u}$ in a magnetic field $\vec{B}$.

$$\vec{F}_m = Q\,\vec{u} \times \vec{B}$$

$\vec{F}_m$ cannot perform work because $(\vec{F}_m \cdot d\vec{l} = 0)$ it is $\perp$ to motion.

- **Force on a current element:**

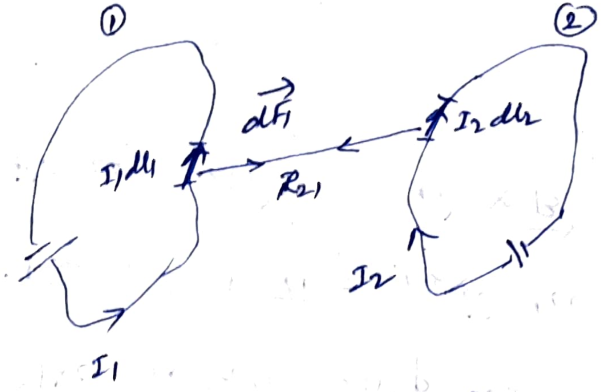
For convection current $\vec{J} = \rho_v \vec{u}$.

The relationship between current elements $I\,d\vec{l} = \vec{K}\,dS = \vec{J}\,dv$

Thus, $I\,d\vec{l} = \rho_v \vec{u}\,dv = dQ\,\vec{u}$.

Hence, $d\vec{F} = I\,d\vec{l} \times \vec{B} = \vec{K}\,dS \times \vec{B} = \vec{J}\,dv \times \vec{B}$.

• Force between two ~~charge~~ current elements.



Force $\vec{dF_1}$ on element $I_1 \vec{dl_1}$ due to field $\vec{dB_2}$.

$\vec{dB_2}$ is produced by element $I_2 \vec{dl_2}$.

$$\vec{dB_2} = \frac{\mu_0 I_2 \vec{dl_2} \times \hat{a}_{R_{21}}}{4\pi R_{21}^2}$$

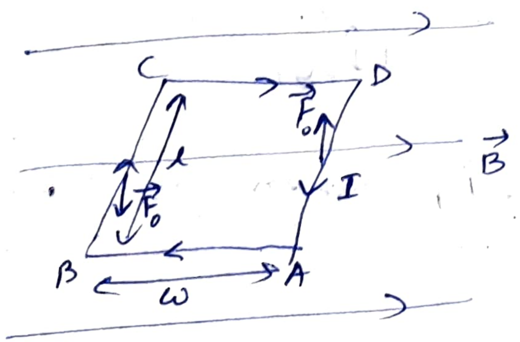
$$\vec{dF_1} = I_1 \vec{dl_1} \times \vec{dB_2}$$

Thus $$\vec{F_1} = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{L_1} \oint_{L_2} \frac{\vec{dl_1} \times (\vec{dl_2} \times \hat{a}_{R_{21}})}{R_{21}^2}$$ (Oersted's Law).

• $\vec{F_2} = -\vec{F_1}$ (i.e. Force on loop 2 due to magnetic field B_1 from loop 1).

Magnetic Torque & Magnetic Moment

• The torque $\vec{T}$ on (current) loop is $\vec{T} = \vec{r} \times \vec{F}$ (Newton-m)

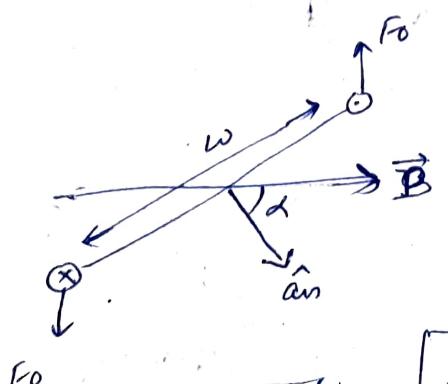


$\vec{r}$ = Moment arm.

$|\vec{F_0}| = BIl$

$|\vec{T}| = BIlw \sin\alpha$

$= BIS \sin\alpha$, S = Area of the loop

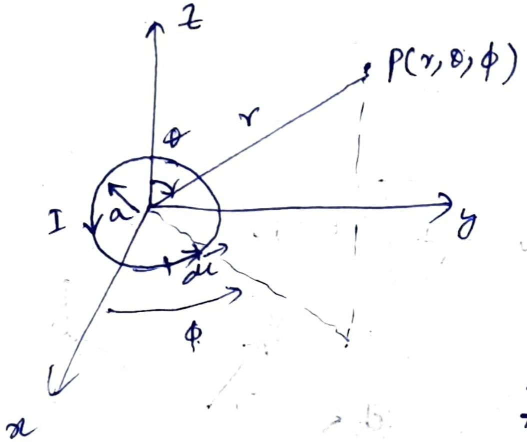


Define:- $IS\hat{a_n} = \vec{m}$ (Magnetic Dipole moment) in $A \cdot m^2$.

$\hat{a_n}$: Direction determined by right-hand rule.

Thus, $\boxed{\vec{T} = \vec{m} \times \vec{B}}$

Magnetic Dipole.

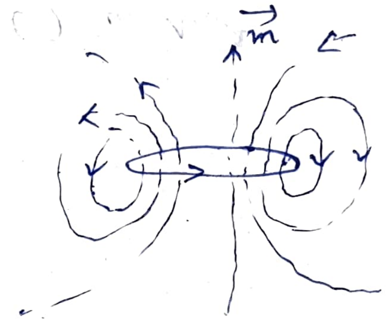


$$\vec{A}(P) = \frac{\mu_0 I}{4\pi} \oint \frac{d\vec{l}}{r}$$

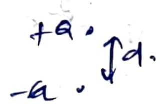
For a small loop, $r >> a$, thus, $\vec{A} = \frac{\mu_0 I a^2 \sin\theta}{4\pi r^2} \hat{a}_\phi$.

$$\vec{A} = \frac{\mu_0 \vec{m} \times \hat{a}_r}{4\pi r^2}, \quad \vec{m} = I\pi a^2 \hat{a}_z$$ (magnetic moment of the loop)

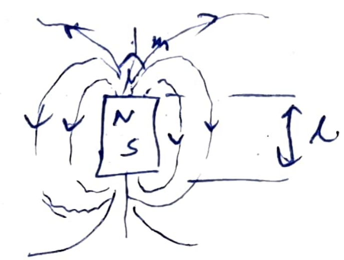
$$\vec{B} = \nabla \times \vec{A} = \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta \, \hat{a}_r + \sin\theta \, \hat{a}_\theta\right).$$



For an electric dipole, $\vec{E} = \frac{Qd}{4\pi\varepsilon_0 r^3}\left(2\cos\theta \, \hat{a}_r + \sin\theta \, \hat{a}_\theta\right)$.



Thus a small current loop is equivalent to a magnetic dipole.



A Bar magnet: Dipole moment = $Q_m l$, Q_m (is pole strength).

Magnetization

- When an external $\vec{B}$ is applied, the magnetic moments of electrons allign themselves with $\vec{B}$.

Magnetization ($\vec{M}$) in A/m is the net magnetic dipole moment per unit volume.

$$\vec{M} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{N} \vec{m}_k}{\Delta v}.$$

For a differential volume dv', the magnetic moment $\vec{dm} = \vec{M} dv'$

$$\vec{dA} = \frac{\mu_0 \vec{M} \times \hat{a}_R}{4\pi R^2} dv' \qquad \text{using, } \nabla' \frac{1}{R} = \frac{\hat{a}_R}{R^2}$$

$$\vec{A} = \frac{\mu_0}{4\pi} \iiint_{v'} \vec{M} \times \left(\nabla' \frac{1}{R}\right) dv' \qquad \text{using, } \nabla \times (f\vec{F}) = f \nabla \times \vec{F} + (\nabla f) \times \vec{F}$$

$$= \frac{\mu_0}{4\pi} \iiint_{v'} \frac{\nabla' \times \vec{M}}{R} dv' - \frac{\mu_0}{4\pi} \iiint_{v'} \nabla' \times \frac{\vec{M}}{R} dv'$$

Using the vector identity, $\iiint_{v'} \nabla' \times \vec{F} \, dv' = - \oiint_{S'} \vec{F} \times \vec{ds}$.

$$\vec{A} = \frac{\mu_0}{4\pi} \iiint_{v'} \frac{\nabla' \times \vec{M}}{R} \, dv' + \frac{\mu_0}{4\pi} \oint\!\!\oint_{S'} \frac{\vec{M} \times \hat{a_n}}{R} \, ds'$$

$$= \frac{\mu_0}{4\pi} \iiint_{v'} \frac{\vec{J_b} \, dv'}{R} + \frac{\mu_0}{4\pi} \oint\!\!\oint_{S'} \frac{\vec{K_b} \, ds'}{R}.$$

where, $\vec{J_b} = \nabla \times \vec{M}$ (bound volume current density) in A/m^2

$\vec{K_b} = \vec{M} \times \hat{a_n}$ (bound surface current density) in A/m.

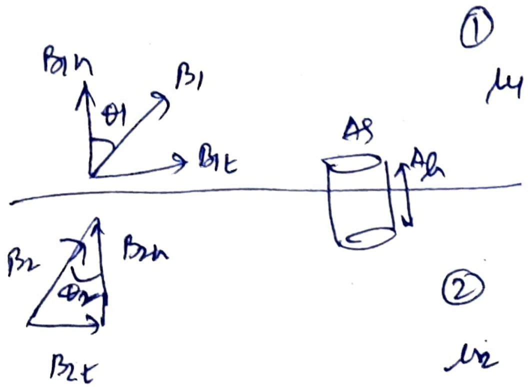
In free-space :- $\vec{M} = 0$, $\nabla \times \vec{H} = \vec{J_f}$ or, $\nabla \times \left(\frac{1}{\mu_0} \vec{B}\right) = \vec{J_f}$. (free volume current density)

In material medium:- $\vec{M} \neq 0$, $\nabla \times \left(\frac{1}{\mu_0} \vec{B}\right) = \vec{J_f} + \vec{J_b} = \nabla \times \vec{H} + \nabla \times \vec{M}$

Thus, $\vec{B} = \mu_0 (\vec{H} + \vec{M})$

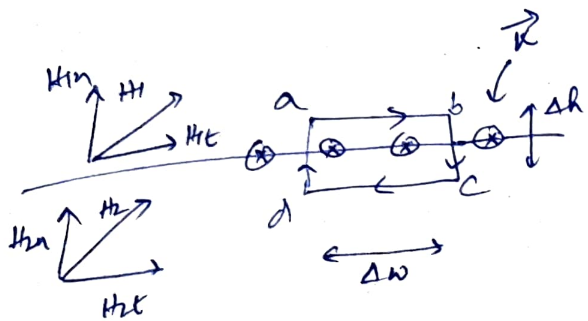
For <u>linear</u> materials, $\vec{M} = \chi_m \vec{H}$; (χ_m: Magnetic Susceptibility)
(& isotropic) Hence, $\vec{B} = \mu_0 \mu_r \vec{H}$, $\underline{\mu_r = 1 + \chi_m}$,

Magnetic Boundary Conditions



$$\oint \vec{B}\cdot d\vec{s} = 0 \Rightarrow B_{1n}\,\Delta S - B_{2n}\,\Delta S = 0$$

$$\Rightarrow \boxed{B_{1n} = B_{2n}} \quad \text{or} \quad \boxed{\mu_1 H_{1n} = \mu_2 H_{2n}}$$



$$\oint \vec{H}\cdot d\vec{l} = I \Rightarrow K\cdot\Delta w = H_{1t}\cdot\Delta w + H_{1n}\cdot\frac{\Delta h}{2} + H_{2n}\cdot\frac{\Delta h}{2}$$

$$- H_{2t}\cdot\Delta w - H_{2n}\cdot\frac{\Delta h}{2} - H_{1n}\cdot\frac{\Delta h}{2}$$

$$\Rightarrow H_{1t} - H_{2t} = K$$

i.e. $(\vec{H}_1 - \vec{H}_2)\times \hat{a}_{n12} = \vec{K}$, $\hat{a}_{n12}$: Unit vector normal to the interface, directed from medium-1 to medium-2.

Surface current $\vec{K}$ on the boundary interface.

For no surface currents at interface,

$$\left.\begin{aligned} B_1\cos\theta_1 &= B_2\cos\theta_2 \\ \& \quad \frac{B_1}{\mu_1}\sin\theta_1 &= \frac{B_2}{\mu_2}\sin\theta_2 \end{aligned}\right\} \Rightarrow \boxed{\frac{\tan\theta_1}{\tan\theta_2} = \frac{\mu_1}{\mu_2}}$$