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Module 5

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Limitations of Aristotelian Logic

1. The theory is partial. It understands predication only in terms of class inclusion or class membership. And that understanding is partial.

E.g. ‘**This leaf is green’**

It is not necessary to rely upon class-relations to understand the predication**.**

Also, the class-membership analysis does not capture all kinds of predication. Thereby, this logic leaves out other kinds of predication.

►E.g. It cannot explain why the argument below has to hold valid:

Premise: All horses are animals.

Conclusion: Therefore, All heads of a horse are heads of an animal.

Or, If anything is a head of a horse, then it is a head of an animal.

Also consider: Leibniz's example of a syllogism in which the predication cannot be understood within traditional Aristotelian logic:

Premise: Christ is God.

Conclusion: Hence, the mother of Christ is the mother of God.

Class-inclusion or class-membership would not correctly and adequately capture the predication.

So, Categorical Logic, because of its exclusive focus on class-inclusion leaves out many other kinds of predication.

2. Also, with its exclusive focus on certain kind of **class terms**, it leaves out the discussion on other kinds of terms:

E.g. **Singular Terms**: The Taj Mahal : Need not be understood as a class term

Negative Terms: Unkind, unavailable.

Compound terms: Damp and cold

Categorical Logic gives no clear principles which determine their use within the system.

These omissions were picked up as cues for further work in logic by Frege-Russell, and also by Lukasiewicz.

<u>3.One of the insights of modern logic</u>:

The verb *to be* (is, are) may have more than one logical meaning

That is, all 'is', 'are' may not stand for class membership.

E.g. Christ **is** God. (Leibniz)
Or, Morning star **is** evening star. (Frege)
Or, 117+136 **is** 253.

Peano said: **A is B** may mean any of the following:

(a) **Identity**, i.e. A = B
(b) **Membership**, $A \in B$
(c) **Inclusion**, $A \subset B$

De Morgan said: '**A is B'** may also be understood as A**R**B, where **R** may be an asymmetric, transitive relation. **Theory of Relational or n-place Predicates**.

Categorical logic understood ‘is/are’ only in terms of class-membership.

4. Moreover, there is the problem of **Existential Import** in Aristotelian logic

►What is the problem?

Existential import: A categorical statement is presumed to have **existential import.** That is, The class represented by its subject term is always **assumed** to be non-empty, or populated

E.g. (a) Some leaves are green.
(b) All triangles are geometrical figures.

Key question: Whenever we use a categorical statement, should we presume that it has existential import?

Aristotle, and traditional Logicians: YES.

This is evident in **traditional square of opposition**, and the ‘oppositions’ defined in it.

E.g. Consider subalternation relation: A-I, E-O. If superaltern is true, how would we know then the aubaltern also must be true, unless we assume that the subject class of these statements are presumed to be populated?

<u>Modern logicians</u>: NO. It is not logically justified to presume a class as non-empty, *unless there is scientific evidence* that it is a non-empty class..

<u>"Existence is not a predicate"</u>

Existence of members in a class should not be a matter of assumption or a choice of predication, but <u>must be borne by evidence</u>.

Existential fallacy: To unwarrantedly presume existence

So, a global assumption of Existential Import for all categorical statements, as Aristotle and old logicians did, is (a) not warranted, (b) not a safe assumption for Aristotelian Logic. Serious problems may arise.

What kind of problems?

Suppose: We **allow** a **blanket or global assumption** that for every categorical statement, its subject term will be presumed to refer to some non-empty class or other.

Then, consider these categorical statements:

(a) All **winged horses** are very fast creatures.

(b) Some **winged horses** are not very fast creatures.

► In reality, there are NO winged horses. That is, winged horses do NOT exist.

Note that (a) and (b) are contradictories. If you claim (a) is False, (b) must be true. But is it? You know (a) is false, does that make (b) true? : **No**

Are they then both False? Yes! But, if so, then you are violating the contradictory relation of the Traditional Square of Opposition.

What went wrong?

Consider the sub-contraries:

(c) Some **square-circles** are large objects.

(d) Some **square-circles** are not large objects.

Similar problem. Sub-contraries both may be true, but both cannot be false. But (c) and (d) both seem to be false, and neither seems to be true.

So, what to do?

May be Solution 1: Can we keep out all these problematic categorical statements the subject terms of which refer to empty classes? That is, **we allow the blanket presumption of existential import, but restrict the kind of categorical statements.**

That is, can we make it a rule that the subject term of a categorical statement must NEVER refer to an empty class?

Solution 1: A default existential import + categorical statements with empty classes as subject terms will be out of purview.

Objections from modern logicians: NO. Such a rule would serious affect our power of expression using the categorical statements.

There are powerful reasons why the following kinds of statements must be allowed without any assumption of their subject terms being non-empty:

E.G. 1. The class 'square-circle' is empty: Where discussion about empty classes is necessary.

2. All those who cheat in exams will be expelled: Prohibitive statements where the intention is to keep the subject class empty.

3. Scientific Statements about ideal entities: The Geometrical point, The Ideal Gas.

So, Solution 1 is NOT regarded as a satisfactory solution.

Solution 2 from modern logic

Boole's solution: Allow all kinds of categorical statements, but

(a) Restrict existential import only to I and O: As default

(b) A and E will NOT have existential import

(c) It is an existential fallacy to assume existence where no evidence is available or offered.

Russell: A and E statements will not have existential import. The **correct** way to read A and E is as a conditional:

All S are P: For any x, **if** x is S then x is P.

No S is P: For any x, **if** x is S then x is not P.

Note, this does NOT commit us to: There exists such S. A and E remain **conditional statements** about S and P classes.

If one wants to further add existence claim, then a conjunction is required:

(For any x, **if** x is S then x is P) • (There exists such x).

I and O will have default existential import. The way to read I and O:

Some S is P: <u>There exists at least one x</u> such that it is S and also P.

Some S is not P: <u>There exists at least one x</u> such that it is S but not P

Implication of Boolean solution for Aristotelian Logic:

Traditional Square of Opposition is seriously affected.

1. **Subalternation cannot be valid any more**:

For, the truth-conditions for A and E, and I and O, will not match any more. A and E can be <u>vacuously true</u> when their subject terms refer to empty classes, but their corresponding I and O will be false!

Consider: **All winged horses are fast creatures**

⇒ <u>**If**</u> <u>x is a winged horse</u> then x is a fast creature

Since there are no winged horses, antecedent is false. Hence, 'if-then' statement is true.

But ‘**some winged horses are fast creatures’** is:

<u>**There exists at least one x**</u> such that it is winged and a horse: FALSE

Similarly, for E and corresponding O.

So, truth of the universal does not entail the truth of the corresponding particular. Falsity of particular will not imply falsity of the universal: <u>Subalternation not valid.</u>

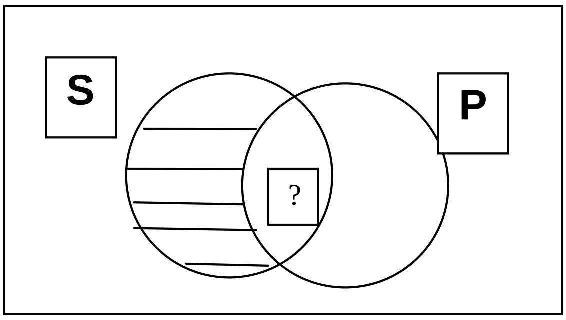
2. **Contrary relation also does not hold** between A and E. For, both will be (vacuously) true together when subject term refers to empty classes.
   So, in general, for A and E we have to give up contrary relation.

Consider: **All winged horses are fast creatures**

No winged horses are fast creatures

3. **Subcontrary also does not hold** between I and O. For, there are situations when both I and O may be false simultaneously.

E.g.: '**some winged horses are fast creatures'** and '**some winged horses are not fast creatures'** both FALSE.



Let us revisit the Venn Diagram with the existential import problem.

<u>All S are P</u>: We know that S $\bar{P}$ area is empty. But does that mean that the other non-shaded areas, such as SP area, $\bar{S}$P area, are non-empty?

Traditional logicians: Yes. It follows, as there is a default existential import.

Modern logicians: No. No such conclusion follows. All we know is S $\bar{P}$ area is empty. **Nothing else follows about the other areas.** In particular, it does not follow that the other areas are non-empty.

Boolean revision: It also does not follow that there is an S

But **Some S is P** and **Some S is not P** carry a default Existential commitment. By default, their plotting in the Venn Diagram asserts a non-empty region. The ‘x’ in the Venn Diagram is the mark of that existential commitment.

Predicate Logic

This logic is concerned with **predication** of properties to things / objects

E.g. This leaf is green

Greenness or **being green** is the property which is predicated to **this leaf**, which is **a thing**.

A different understanding of predication than Categorical Logic.

It is not necessary to rely exclusively upon class-relations to understand the predication.

But, Predicate Logic: Still a system of deductive logic

Its Syntax looks different, with additional symbols for:

- Predicate: Predicate symbols
- Quantity: Quantifiers.

Why is this Predicate logic called **First Order**?

Here, First refers to the most elementary level.

This logic is about the <u>most elementary</u> level of predication, where:

> Predication is understood as properties
>
> predicated to things / objects.

Example: This leaf (object) is **green** (property)

<u>Two major kinds:</u> (1) **Properties** which are predicated**,**, and (2) **things** to which the properties are predicated.

There can be more advanced level of predication: **Higher level** of predication:

E.g. The level of predication of properties to properties,

Quantification of the properties

So, there are and can be Higher Order of Predicate Logics

But we shall discuss only the **First Order Predicate Logic**

Syntax of First Order Predicate Logic

This syntax allows for individual constants

1. Individual Constants: These refer to the individual **things,** and they have fixed, constant reference

To the **particular things** / objects within the domain

They are **unique identifiers**, similar to proper names: E.g.: George Boole, The River Ganga.

Symbolic representation by individual constants: **Lowercase letters with or without numerical subscripts**:

Example:

Aristotle: **a**

Mount Everest: **e**

This leaf: $\mathbf{l_2}$

2.**Predicate symbols**

- Refer to predicates. E.g. ‘being green’’

Examples:

1. <u>Ordinary proposition</u>: **Marco Polo is an explorer**

<u>But, Predicate Logic reading of this proposition is:</u>

<u>Being explorer is a property predicated</u> to **<u>Marco Polo</u>**

► Predicates hold primary position before things / objects. <u>Prefix notation</u>.

Marco Polo belongs to the Things: Symbolized by individual constant. But ‘’being explorer’? .

- In symbolization key, we need **predicate symbols**
- **Predicate symbols**: Use suitable **<u>capital letters</u>**, e.g. ‘A’, ‘D’.

So, translation key:

Being explorer: E, Marco Polo: m

,

Marco Polo is an explorer is symbolized: **Em**

Ordinary proposition: **This leaf is green**

Being green: G

This leaf: t_2

Symbolized: $\mathbf{Gt_2}$

Note:

- Each of the propositions above is a structurally simple proposition
- We are using the lens of predication to find logically significant information in them.

3. Individual Variables:

They refer to unspecified individuals in the domain. E.g. 'some people', 'any executive'

Individual Variable Symbols: Use lowercase alphabets from the end of the alphabet series: **x, y, z**

Some people : **Px**

Note:

Some Predicates may require more than one constant or variable, and in an ordered sequence

E.g. 'is greater than' is a predicate that requires 2 variables or constants.

Gxy (2-place Predicate)

Example: Hundred (h) is greater than ninety (n): **Ghn**

'Is in between' is 3 place Predicate: Bxyz

Example: Kharagpur (g) is in between Kolkata (k) and Bhubaneswar (b): **Bgkb**

4.Predicate logic still includes the **truth-functional compound statements**:

E.g. **Aristotle is a philosopher and he taught Alexander the Great.**

Key:

Aristotle: a Alexander the Great: g

Px: x is a philosopher **Txy: x taught y**

Symbolized: Pa • Tag

5. Quantifiers: Terms / expressions in Predicate Logic which state how many or the quantity of individuals or things.

- Note: They do NOT state which individuals

Quantifier symbols:

- **Universal quantifier:** Stands for 'all', 'everyone'**.**

  Symbol: Inverted A **:** ∀
- **Existential quantifier:** Stands for 'some', 'there exists at least one'.

  Symbol: Inverted E**:** ∃

But, Quantifier symbols also require <u>individual variables</u> and <u>parentheses</u>:

A quantifier ranges over a variable. **Note:** Constants do not require quantifiers. Because, they are like names. There is no need to ask **<u>how many</u>** of them.

Example: $(\forall \boldsymbol{y})$, $(\exists \boldsymbol{x})$

$(\forall \boldsymbol{y})$ reads as: For all y

$(\exists \boldsymbol{x})$ reads as: There exists at least one x

The total quantifier symbol includes the parentheses and the variables.

Predicate Logic reading of basic 4 types of Categorical Propositions:

We need to remember Boole's reading of a Universal proposition: Universals are to be read as **conditionals**

1. **A**: All logicians are smart

Predicate logic reading: Two properties, 'being a logician' (Lx), and 'being smart' (Sx). All things that have Lx, also have Sx.

Translation:

For all x, <u>if</u> x is a logician, <u>then</u> x is smart.

($\forall x$), if x is a logician, then x is smart

($\forall x$) (Lx $\supset$ Sx)

► **Note**: No commitment about X's existence.

2. **E:** No logicians are smart.

Translation:

For all x, if x is a logician, then x is <u>not</u> smart.

($\forall x$), if x is a logician, then x is <u>not</u> smart

($\forall x$) (Lx $\supset$ ~Sx)

Boole: Existentials are to be read as with strong existential commitment

3. **I**: Some logicians are smart

Predicate logic reading:

There **exists** at least one x, such that x is logician **and** it is smart

($\exists x$) such that x is logician and x is smart

($\exists x$) (Lx • Sx)

4. **O**: Some logicians are not smart

Predicate logic reading:

There **exists** at least one x, such that x is logician **and** x is not smart

($\exists x$) such that x is logician and x is not smart

($\exists x$) (Lx • ~Sx)

Schematic form of A, E, I,O:

Where ϕ **and** ψ are any two properties,

A: $(\forall x)(\phi x \supset \psi x)$

E: $(\forall x)(\phi x \supset \sim\psi x)$

I: $(\exists x)(\phi x \bullet \psi x)$

O: $(\exists x)(\phi x \bullet \sim\psi x)$

Also: From the Square of opposition, the equivalences:

$(\forall x)(\phi x \supset \psi x) \equiv \sim(\exists x)(\phi x \bullet \sim\psi x)$

$(\exists x)(\phi x \bullet \sim\psi x) \equiv \sim(\forall x)(\phi x \supset \psi x)$

$(\forall x)(\phi x \supset \sim\psi x) \equiv \sim(\exists x)(\phi x \bullet \psi x)$

$(\exists x)(\phi x \bullet \psi x) \equiv \sim(\forall x)(\phi x \supset \sim\psi x)$

Translation using Quantifiers, and Predicate symbols:

Bx: x is a book	Ix: x is interesting
Ex: x is expensive	Gx: x is good
Rx: x is boring	k: The Keepers

1. Some books are expensive, but not interesting.

First, the reading: There exists at least one y, such that y is a book, and such that y is expensive but is not interesting.

$(\exists y)$ **(By • (Ex • ~Iy))**

2. No expensive things are good.

Reading: For all x, if x is expensive, x is not good.

$(\forall x)$ **(Ex ⊃ ~Gx)**

3. No good books are boring, but *The Keepers* is not good.

$(\forall z)$ **((Bz • Gz) ⊃ ~Rz) • ~Gk**

The scope of a quantifier?

- By convention, the **individual variable** used as part of the quantifier symbol falls within the scope. This is default scope.

  E.g. $(\forall x) \Longleftarrow$ 'x' is within the scope

- Also, the variable immediately adjacent to the quantifier falls within the scope, if it is the same variable as used in the quantifier symbol.

E.g. $(\forall \underline{x})\ F\underline{x}$ both the 'x's are within scope of the quantifier

- If a quantifier is followed by a parenthesis / bracket, the scope will continue till the matching parenthesis / bracket.

E.g. $(\forall \underline{x})\ (F\underline{x} \supset M\underline{x})$

Or, $(\exists \underline{x})\ (F\underline{x} \bullet {\sim}M\underline{x})$

Compare:

$(\forall \underline{x})\ (F\underline{x} \supset M\underline{x})$ and $(\forall \underline{x})\ F\underline{x} \supset M\underline{x}$

Bound variables: Occurrence of a variable is bound *iff* it is within the scope of a quantifier. : **DESIRABLE**

Free variables: Occurrence of a variable is free *iff* it is NOT within the scope of a quantifier:**NOT DESIRABLE**

$(\forall \underline{x})\ (F\underline{x} \supset M\underline{x})$: 3 occurrences of 'x'', each one is bound.

$(\forall \underline{x})\ F\underline{x} \supset M\underline{x}$: 3 occurrences of 'x', first two are bound, but the 3rd occurrence of 'x' is free.

►**Note**: Propositions with free variables are not syntactically acceptable in First Order Predicate Logic.

Difference between Quantified propositions and truth-functional compound propositions is also to be understood in terms of the scope of a quantifier.

Quantified propositions: Where the Quantifier is the main logical operator, and the whole proposition lies within the scope of a quantifier.

E.g. $(\forall x)\ (Fx \supset Mx)$

Truth-functional Compound: Where a truth-functional connective is the main logical operator , and the whole proposition lies within the scope of that connective.

E.g.: $(\forall x)\ (Fx \supset Mx)\ \mathbf{v}\ (\forall y)\ (Ay \supset By)$

So, three kinds of statements / propositions in Predicate Logic:

1. The atomic statements: Ga, Bmnd

2. The truth-functional statements:

$Bk \supset (\forall z)\ Tz$

3. The Quantified statements / propositions

$(\exists y)\ [\ Cy \bullet (\forall x)Jxy]$

Using quantifiers with n-place predicates in translation

Px: x is a person Kxy: x knows y, n: Nelson Mandela, p: Pratul

1. Everyone knows Nelson Mandela

Paraphrase: For all x, if x is a person, x knows Nelson Mandela

$$(\forall x)\ (Px \supset Kxn)$$

2. Pratul knows everyone

Paraphrase: For all x, if is a person, Pratul knows x.

$$(\forall x)\ (Px \supset Kpx)$$

Bx: x is a book Axy: x is authored by y a: Amartya Sen, p: Pratul, Rxy: x reads y

3. Pratul reads some books by Amartya Sen.

Paraphrase: There exists at least one y, such that it is a book, and it is read by Pratul and is authored by Amartya Sen

$(\exists y)\ (By \bullet (Rpy \bullet Aya))$

When using quantifiers, the **Universe of Discourse (U.D.)** helps to understand and interpret the individuals represented by the quantifier

De Morgan explained the effect of U.D on interpretation:

Unrestricted U.D.: Includes everything. The common context.

Where Ix: x is important

(∀x) Ix in an unrestricted U.D. means : Everything is important

Here, if you wish to mean in this U.D., every book is important, you need to specify with property: Bx: x is a book. The property has to be available in your U.D.

Restricted U.D.: Further specified U.D.

E.g. U.D.: The papers on this table

In that restricted U.D., (∀x) Ix means : Every paper on this table is important

►**Note:** If nothing is mentioned in translation, U.D. is to be understood as unrestricted.

Handling a non-standard quantity term: 'Any'

1. If anything is a story, then the Tale of Tarzan too is. (Sx, t)

Paraphrase: If at least one thing is a story, then..

$(\exists y)\ Sy \supset St$

2. If everything is a story, then the Tale of Tarzan is a story. (Sx, t).

$(\forall y)\ Sy \supset St$

Now consider:

3. If anything is a dog, then it is a mammal. (Dx, Mx)

4. Anything that is a dog is a mammal. (Dx, Mx)

Both to be read as: For all x, if x is a dog, x is a mammal

$(\forall z)\ (Dz \supset Mz)$

Now:

5. If <u>something</u> is a dog, then <u>it</u> is a mammal (Dx, Mx)

$(\exists x)\ Dx \supset Mx$ ✗ WRONG!

$(\exists x)\ (Dx \supset Mx)$ ✗ WRONG!

Correct: $(\forall x)\ (Dx \supset Mx)$

<u>Pronominal Cross-reference</u>: Reference and scope of a quantity term carried till the end of the statement by a <u>pronoun</u>.

5. If <u>something</u> is a dog, then <u>it</u> is a mammal (Dx, Mx)

If <u>anything</u> is a dog, then <u>it</u> is mammal. (Dx, Mx)

$\mathbf{(\forall x)\ (Dx \supset Mx)}$

Universal quantifier for ‘any’ with the pronominal reference

Overlapping Quantifiers, sharing predicate and scope

Consider: $(\forall y)\ (\exists z)$ Gzy

One quantifier with the scope of another. The predicate is shared by both variables.

Main quantifier: Quantifier with maximum scope, Placed first.

►In overlapping quantifier cases, sometimes the sequence of quantifiers is not important. And sometimes it becomes very important.

E.g. Some natural number is greater than some other natural number

U.D.: Natural numbers Gxy: x is greater than y

In this case: $(\exists y)\ (\exists z)$ Gyz OR $(\exists z)\ (\exists y)$Gzy: The sequence does not matter

But, $(\forall y)\ (\exists z)$ Gyz $\neq$ $(\exists z)\ (\forall y)$ Gzy. Sequence matters. The meaning, the emphasis change with the sequence.

Example: U.D.: Natural numbers Gxy: x is greater than y

(∀y) (∃z) Gyz: All natural numbers are such that it is greater than some number (or other). T

(∃z) (∀y) Gzy: There exists some natural number which is greater than all other natural numbers. F

Multiple quantifier reading:

(i) (∃y) (∃z): There is some y and some z such that…

(ii) (∀y) (∃z): For every y, there is some y or other such that…

(iii) (∃z) (∀y): There exists a z such that every y is…

(iv) (∀x) (∀y): For every x and every y such that…

Multiple quantifiers and Relational Predicates in a statement

More than one quantifier is required: When there are more than one subject terms, i.e. when more than one things or group of things are being referred to

Remember to avoid scope conflict among the quantifiers.

Preferable:

- Use a different variable for each quantifier
- Use the parentheses, brackets to indicate the scope clearly

Example:

UD: Unrestricted

Rx: x is a rule Bx: x is broken

Hx: x is human Px: x will be punished

If <u>any</u> <u>rule</u> is broken, <u>someone</u> will be punished.

Note: <u>Two</u> kinds of subjects: **Rules, Humans**

<u>Two</u> quantity terms: **Any, someone**

Paraphrase: If there exists at least one rule and it is broken, then there exists at least one human who will be punished.

Trans:

$(\exists x)\ (Rx \bullet Bx) \supset (\exists y)\ (Hy \bullet Py)$

Or, since there is no chance of scope conflict:

$(\exists x)\ (Rx \bullet Bx) \supset (\exists x)\ (Hx \bullet Px)$

Note:

- It is crucial to keep the scope separate when the scope of one quantifier may be within the scope of another quantifier

Example:

2. **If <u>anything</u> is a <u>rule</u>, then if <u>there are</u> <u>humans</u>, <u>it</u> will be <u>broken</u>**.

Paraphrase: For any x, if x is a rule, then if there exists at least one y such that y is a human, then x will be broken.

Trans: $(\forall x)\ (Rx \supset (\ (\exists y)\ Hy \supset Bx))$

Scope conflict must be avoided by choice of a different variable.

Changing scope of a quantifier

General rule: Where 'C' is any statement, note:

$(\exists x)\ Lx \supset C \equiv (\forall x)\ (Lx \supset C)$

$(\forall x)\ Lx \supset C \equiv (\exists x)\ (Lx \supset C)$

►Change of scope in these cases will mean a change of quantifier.

But not in these cases:

E.g. 1. $C \supset (\exists x)\ Lx \equiv (\exists x)\ (C \supset Lx)$

2. $C \supset (\forall x)\ Lx \equiv (\forall x)\ (C \supset Lx)$

3. $(\exists x)\ Lx \vee C \equiv (\exists x)\ (Lx \vee C)$

4. $(\forall x)\ Lx \vee C \equiv (\forall x)\ (Lx \vee C)$

5. $(\exists x)\ Lx \bullet C \equiv (\exists x)\ (Lx \bullet C)$

6. $(\forall x)\ Lx \bullet C \equiv (\forall x)\ (Lx \bullet C)$ etc.

But for '$\equiv$': $(\forall x)\ Lx \equiv C$ NOT equivalent to $(\forall x)\ (Lx \equiv C)$, similarly for $(\exists x)\ Lx \equiv C$

Translate: Everyone loves a lover

UD: Persons Lxy: x loves y

⟶ Every x (person) is such that x loves a lover.

a lover: any or all lovers

⟶ Every x (person) is such that if y is a lover, x loves y: All people love all lovers

a lover: A person who loves someone or other

y is a lover: There is at least one z such that y loves z

$(\exists z)\ Lyz$

Trans: $(\forall x)\ (\forall y)\ [\ (\exists z)\ Lyz \supset Lxy]$

Translate: The sages never tell a lie.

Lx: x is a lie Sx: x is a sage

Tx: x is a time Txyz: x tells y at z

The sages never tell a lie

3 kinds of things: Sages, Time, Lies

3 quantity terms: The, never, a

If starting with 'the sages',

Possible paraphrase: For any x, if x is a sage, then x never tells a lie.

Further: For any x, if x is a sage, then for every y, if y is a time, then x does not tell a lie at y.

Further: For any x, if x is a sage, then for every y, if y is a time, then for all z, if z is a time, x does not tell z at y;

OR, it is not the case that there is at least one z, such that is a lie and x tells z at y.

$$(\forall x) (Sx \supset (\forall y) (Ty \supset (\forall z) (Lz \supset {\sim}Txzy)))$$

The sages never tell a lie.

<u>If starting with the time:</u>

Possible paraphrase: There is no time when / such that <u>the</u> sages tell <u>a</u> lie at that time.

Further: It is not the case that there is a time x such that for any y, if y is a sage, then there is at least one z such that z is a lie and y tells z at x.

Trans.: $\sim [(\exists x)\ (Tx \bullet [(\forall y)\ (Sy \supset (\exists z)\ (Lz \bullet Tyzx))])]$

Or its equivalent: E statement

Predicate Logic Derivation

Total set of rules:

1. The 19 Rules of Inference and Replacement from Propositional Logic

2. **The quantifier negation rules (QN)** rules or Equivalence Rules:

(i) $(\forall x)\ \phi x \equiv \sim (\exists x) \sim\phi x$
(ii) $(\forall x) \sim \phi x \equiv \sim (\exists x)\ \phi x$
(iii) $(\exists x)\ \phi x \equiv \sim (\forall x)\sim\phi x$
(iv) $(\exists x) \sim\phi x \equiv \sim (\forall x)\phi x$

3. **Four NEW Quantifier Rules**:

- UI
- UG
- EI
- EG

Of these new four, **two rules** are for operating on the **Universal Quantifier**:

UI: Universal Instantiation

UG: Universal Generalization

And **two rules** for operating on the **Existential Quantifier**:

EI: Existential Instantiation

EG: Existential Generalization

Instantiation:

- The rule for removing the quantifier
- It allows the variable, freed by the removal of the quantifier, to be substituted by a chosen individual symbol

For example: UI allows removal of Universal Quantifier, EI allows removal of Existential Quantifier.

$(\forall x)\ Lx \longrightarrow Lz$

Generalization:

- The rule for inserting a quantifier
- By inserting a quantifier, it allows formerly free variables to be bound by the quantifier

For example: UG allows insertion of Universal Quantifier, EG allows insertion of Existential Quantifier.

Among these, UI, UG, EI, apply only to whole quantified statement.

But, EG may be applied to part as well as whole of quantified statement.

Universal Instantiation (UI):

$$\frac{(\forall\mu)\ \phi\mu}{\therefore\ \phi\nu} \quad \textbf{OR} \quad \frac{(\forall\mu)\ (\phi\mu \supset \psi\mu)}{\therefore \phi\nu \supset \psi\nu}$$

ϕ and ψ stand for: Any two Predicate letters

μ, and ν stand for: Any individual symbol (variable or constant)

UI application involves:

► Remove the universal quantifier

►Replace the freed variable by a suitable individual symbol: Variable or constant

Example: Given: $(\forall x)$ $(Mx \supset Fx)$

By application of UI, we may infer:

(i) $Ma \supset Fa$ (instantiation by a constant)

Or,

(ii) $My \supset Fy$ (instantiation by a variable)

Or

(iii) $Mx \supset Fx$ (instantiation by the freed variable)

All are legitimate applications of UI

Example of formal derivation using UI:

1. $(\forall x)\ (Mx \supset Fx)$
2. Mc / $\therefore Fc$

3. $Mc \supset Fc$ 1, UI
4. Fc 3,2, M.P.

UI should <u>not</u> be used on part of a statement. For example:

$(\forall x)\ Mx \supset (\forall y)\ Fy$

$Ma \supset (\forall y)\ Fy$ ✖ WRONG!

Or, $(\forall x)\ Mx \supset Fb$ ✖ WRONG!

Universal Generalization (UG)

$$\frac{\phi\nu}{\therefore (\forall\mu)\phi\mu} \quad \textbf{or} \quad \frac{\phi\nu \supset \psi\nu}{\therefore (\forall\mu)\ (\phi\mu \supset \psi\mu)}$$

UG: makes the move from a substitution instance to a universally quantified statement.

►Application of UG requires:

- Insertion of universal quantifier
- Ensuring every occurrence of the variable generalized on is bound by the quantifier. Once applied, there should not be any free occurrence of the generalized variable.

Restrictions on UG application:

1. The individual symbol generalized upon by UG, i.e. the ν, must be **a variable**, and NOT an individual constant.

So, Given **Db ⊃ Cb**,

UG should **not** be applied to obtain:

∴ $(\forall y)\ (Dy \supset Cy)$ ✖ Wrong! **Illicit generalization!**

Illicit generalization fallacy: Also called a hasty generalization, or over-generalization fallacy. It is a faulty reasoning, in which a sweeping general conclusion is derived from one, or very few (which may be non-representative samples).

So, **1st restriction**: To avoid illicit generalization, application of UG on individual constants is prohibited.

It has to be **variable**, on which UG can be applied.

2. 2nd Restriction on UG: Application of UG is prohibited on a variable that is free on a line that (a) either is an assumption, (b) or, falls within the scope of an assumption.

Example: 1. (∃x) Lx

2. Lx

3.(∀y) Ly 2, UG ✕ Wrong!

4.(∀y) Ly 1, 2-3, EI

Line 3 is obtained by UG applied on variable 'x' on line 2. On line 2, Lx is an assumption. The bent arrow marks the beginning of an assumption. And, every assumption is within its own scope. The unwarranted '($\forall$y) Ly' is a result of wrong application of UG.

The variable generalized on must not be an assumed one, or part of an assumption.

Moreover, it must be a **randomly selected variable**.

Example: Let ABC be any triangle.

Similarly, let 'y' be any individual: 'y' is a randomly selected individual.

So, given **By $\supset$ Cy**, we may infer by UG:

(i) $(\forall z)(Bz \supset Cz)$ (Choice of another variable)

(ii) $(\forall y)(By \supset Cy)$ (Choice of same variable)

1.1. UG cannot be applied to part of a statement:

Example: Given, 1. $Fx \supset (\exists y)\, My$

2. $\underline{(\forall x)\, Fx} \supset (\exists y)\, My$ 1, UG ✗ Wrong!

Note ► given 1. $Fx \supset (\exists y)\, My$

By UG application, you will get:

$$\mathbf{(\forall x)\,(Fx \supset (\exists y)\, My)}$$

3. **<u>3rd restriction on UG</u>**: The variable on which UG is applied, must not be allowed to remain free in the resulting universally quantified statement.

Obeying all the restrictions, UG can be applied:

Example: 1. $(\forall x)\,(Mx \supset Nx)$

2. $(\forall x)\, Mx$ / ∴ $(\forall x)\, Nx$

3. $My \supset Ny$ 1, UI ←

4. My 2, UI

5. Ny 3,4, MP

6. $\mathbf{(\forall x)\, Nx}$ **5, UG**

Existential Generalization (EG)

$\phi\nu$

$\therefore (\exists\mu)\ \phi\ \mu$

Application requires:

- Insertion of existential quantifier
- Ensuring that at least one occurrence of the variable generalized is bound by this quantifier.

Given, $Gw \bullet {\sim}Kw$, we may infer by EG

(i) $(\exists z)\ (Gz \bullet {\sim}Kz)$

OR

(ii) $(\exists z)\ (Gz \bullet {\sim}Kw)$

OR

(iii) $(\exists z)\ (Gw \bullet {\sim}Kz)$

Example in a proof using EG:

1. $(\forall x)\ {\sim}Dx$ $/ \therefore (\exists x) \sim Dx$
2. ${\sim}Db$ 1, UI
3. $(\exists x) \sim Dx$ 2, EG

Existential Instantiation

Basic move is: Given an Existentially Quantified statement, to Instantiate with an individual symbol.

$$\frac{(\exists\mu)\ \phi\mu}{\therefore \phi\nu}$$

But actually, the matter is not at all that simple.

All you know: Something in your world has ϕ.

Your conclusion is: 'So-and-so' has ϕ.

The conclusion does not follow, because the premise does not warrant the conclusion.

So, for every application of EI, we are enjoined to follow certain caveats, restrictions.

Actual format of EI is:

1. $(\exists\mu)\ \phi\mu$

   2. $\phi\nu$

   .

   .

   .

   n. p (your objective in the proof)

n+1. p **1, 2-n, EI**

EI format is similar to a CP or a proof with limited scope assumption, with <u>bent arrow format</u>. But it is <u>not</u> a CP.

But the ν is part of a limited scope assumption: **Let us assume that ν has ϕ**

No commitment that ν actually has ϕ

A <u>limited scope assumpt</u>ion: If we suppose for the time being that since something has ϕ, it is ν which has ϕ, then p.

<u>Restrictions on application of EI:</u>

1. EI by an individual constant is prohibited. A constant picks a specific individual. No such information is provided or warranted by $(\exists\mu)\ \phi\mu$. Non-specific information cannot validly lead to a conclusion about a very specific individual.

So, the instantiating individual symbol must be a **variable.**

2. Every fresh application of EI must introduce a fresh free variable: A new free variable each time.

This is to avoid undesirable bias with the instantiating variable.

(a) Why use a variable? and (b) Why use a <u>fresh</u> variable each time we apply EI?

Explaining with an example:

UD: Integers Ex: x is even Ox: x is odd, b: 3

Version 1:

1. $(\exists x)\ Ox$
2. $(\exists x)\ Ex$ / ∴ $(\exists y)\ (Ey \bullet Oy)$
3. Ob ← ×
4. Eb ← ×
5. $Eb \bullet Ob$ 3,4, Conj
6. $(\exists y)\ (Ey \bullet Oy)$ 5, EG
7. $(\exists y)\ (Ey \bullet Oy)$ 2, 4-6, EI ×
8. $(\exists y)\ (Ey \bullet Oy)$ 1, 3-7, EI ×

Version 2:

1.	(∃x) Ox	
2.	(∃x) Ex	/ ∴ (∃y) (Ey • Oy)
3.	Oy	
4.	Ey	← ×
5.	Ey • Oy	4,3, Conj
6.	(∃y) (Ey • Oy)	5, EG
7.	(∃y) (Ey • Oy)	2, 4-6, EI ×
8.	(∃y) (Ey • Oy)	1, 3-7, EI ×

► NOTE: Had a new free variable been used on line 4, the conclusion would NOT have followed.

1.	(∃x) Ox	
2.	(∃x) Ex	/ ∴ (∃y) (Ey • Oy)
3.	Oy	
4.	Ew	← ×
5.	(∃y) Ey	4, EG
6.	(∃x) Ox	3, EG
7.	(∃y) Ey • (∃x) Ox	5,6, Conj
8.	(∃y) Ey • (∃x) Ox	2, 4-7, EI
9.	(∃y) Ey • (∃x) Ox	1, 3-7, EI

This conclusion is unobjectionable, but it is NOT the same as $(\exists y)\ (Ey \bullet Oy)$

3. **The third restriction** is that the free variable introduced by EI must not occur free on line p, or on any line preceding $\phi\nu$.

Why?

Ans: To avoid the risk of illicit generalization.

Explaining with an example:

1. $(\exists x)\ Lx$
2. Ly
3. $Ly \vee Ly$ 2, Add
4. Ly 4, Taut ← ✗
5. Ly 1,2-4, EI ✗
6. **$(\forall x)\ Lx$** 5, UG

From $(\exists x)\ Lx$, $(\forall x)\ Lx$ as a conclusion should NOT follow. The mistake came from flouting the 3rd Restriction. 'y', the instantiating variable was free on line p, Line 4.

Then how to use EI?

Example:

1. $(\exists y)\ (Ky \bullet Jy)$ / $\therefore$ $(\exists x)\ Kx$

2. $Kx \bullet Jx$

3. Kx 2, Simp

4. $(\exists x)\ Kx$ 3, EG

5. $(\exists x)\ Kx$ 1, 2-4, EI

Ex. 1. $(\forall z)\ (Tz \supset Sz)$

2. $(\exists y)\ Ty$

3. $(\exists x)\ Sx \supset (\exists y)\ Ry$ / $\therefore$ $(\exists y)\ Ry$

4. Tw

5. $Tw \supset Sw$ 1, UI

6. Sw 5,4, MP

7. $(\exists x)\ Sx$ 6, EG

8. $(\exists x)\ Sx$ **2, 4-7, EI**

9. $(\exists y)\ Ry$ 3,8, MP

Let us try to do this proof:

1. $(\exists y)\ By \equiv (\forall x)\ Cx \quad / \therefore (\forall x)\ [Bx \supset (\forall y)\ Cy]$

2. Bx	
3. $(\exists y)\ By$	2, EG
4. $[(\exists y)\ By \supset (\forall x)\ Cx] \bullet [(\forall x)\ Cx \supset [(\exists y) By]$	1, Equiv
5. $(\exists y)\ By \supset (\forall x)\ Cx$	4, Simp
6. $(\forall x)\ Cx$	5,3, MP
7. Cy	6, UI
8. $(\forall y)\ Cy$	7, UG
9. $Bx \supset (\forall y)\ Cy$	2-8, C.P.
10. $(\forall x)\ [Bx \supset (\forall y)\ Cy]$	9, UG

Proof with multiple quantifiers and relational predicates:

1. $(\forall y)(\forall z)\ Dyz$
2. $(\exists y)\ (\forall z)\ (Dyz \supset Hzy) \quad / \therefore (\exists x)\ (\forall z)\ Hzx$

1. $(\forall y)(\forall z)\ Dyz$
2. $(\exists y)\ (\forall z)\ (Dyz \supset Hzy)$ $/ \therefore (\exists x)\ (\forall z)\ Hzx$

→ 3. $(\forall z)\ (Dyz \supset Hzy)$

4. $(\forall z)\ Dyz$	1, UI
5. Dyw	4, UI
6. $Dyw \supset Hwy$	3, UI
7. Hwy	6,5, MP
8. $(\forall z)\ Hzy$	7, UG
9. $(\exists x)\ (\forall z)\ Hzx$	8, EG

10. $(\exists x)\ (\forall z)\ Hzx$ 2, 3-9, EI

<u>Proof of tautology:</u>

Formal proof of validity or formal derivation is the only only available methof for proving tautologies or logical truths of Predicate Logic.

Once more: Zero-premise proof.

For example:

Show that the given is a tautology:

$(\forall y)\ (By \supset (\exists x)\ Bx)$

1. Bw

2. $(\exists x)\ Bx$ 1, EG

3. $Bw \supset (\exists x)\ Bx$ 1-2, CP

4. $(\forall y)\ (By \supset (\exists x)\ Bx)$ 3, UG