MGF. $M_X(t) = E\left(e^{tX}\right)$

$$= \sum_{k=0}^{n} e^{tk} \binom{n}{k} p^k q^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \left(pe^t\right)^k q^{n-k}$$

$$= \left(q+pe^t\right)^n$$

<u>Ex</u> An airline knows that 5% of the people making reservations do not turn up for the flight. So it sells 52 tickets for a 50 seat flight. What is the prob that every passenger who turns up will get a seat?

<u>Sol^n</u>: $X \rightarrow$ no of passengers who turn up for the flight.

Then $X \sim \text{Bin}(52, 0.95)$

$$P(X \leq 50) = 1 - P(X=51) - P(X=52)$$

$$= 1 - \binom{52}{51}(.95)^{51}(0.05) - (0.95)^{52}$$

$$\approx 0.74$$

Geometric Distribution :

Suppose Bernoulli trials are conducted independently under identical conditions till the first success is achieved. Let X

denote the no of trials needed for the first success

$$\underbrace{f\,f\,\cdots\,f}_{(k-1)}\ s \rightarrow k^{th}$$

$$X \rightarrow 1, 2, 3, \cdots\cdots$$

$$P(X=k) = q^{k-1}p, \quad k=1,2,\cdots$$

$$\sum_{k=1}^{\infty} p_X(k) = \sum_{k=1}^{\infty} q^{k-1}p = p + pq + pq^2 + \cdots$$

$$= p(1+q+q^2+\cdots) = \frac{p}{1-q} = 1$$

$$\mu_1' = E(X) = \sum_{k=1}^{\infty} k q^{k-1} p$$

$$= p + 2pq + 3pq^2 + \cdots$$

$$= p\left(1 + 2q + 3q^2 + \cdots\right) = \frac{p}{(1-q)^2} = \frac{1}{p}$$

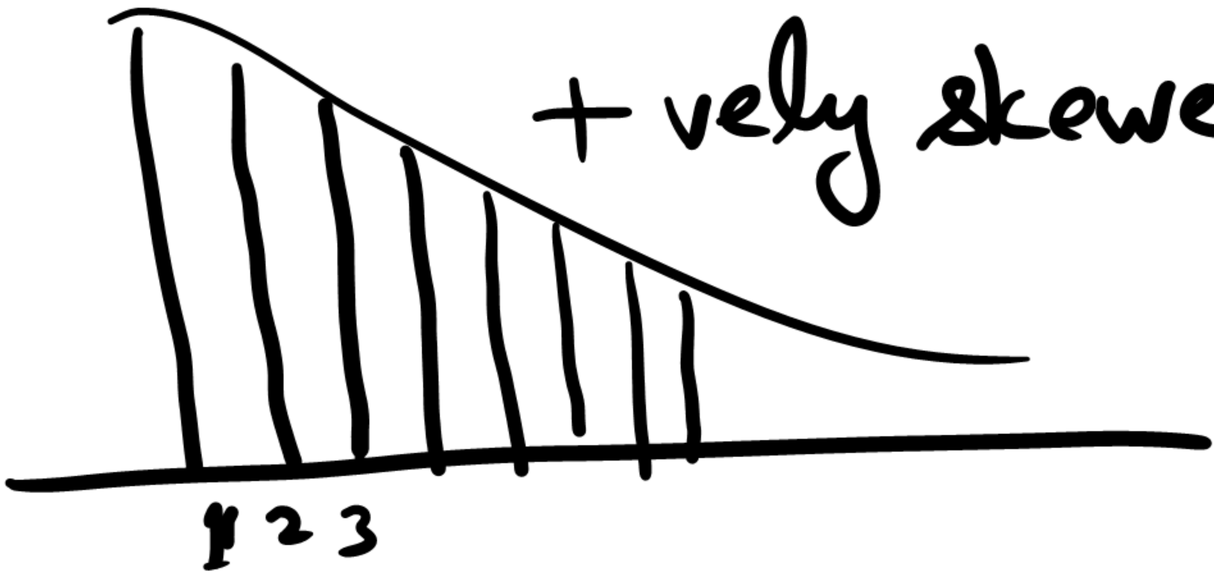
For calculating $EX(X-1)$, $EX(X-1)(X-2)$ etc. we can use formula

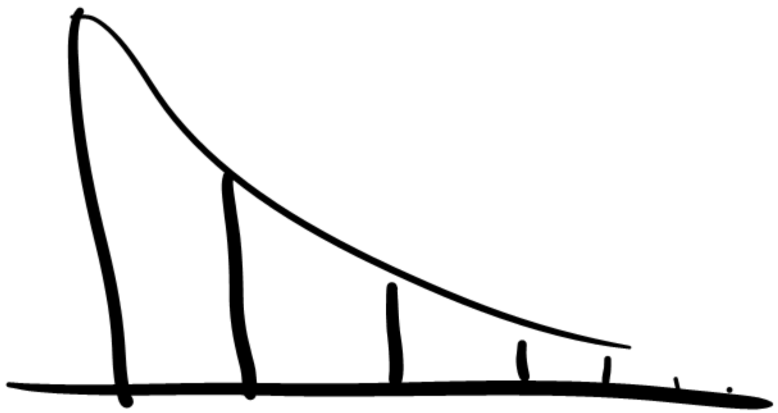
$$\sum_{j=k}^{\infty} \binom{j}{k} r^{j-k} = \sum_{i=0}^{\infty} \binom{k+i}{k} r^{i} = \frac{1}{(1-r)^{k+1}} \qquad (*)$$

$$0 < r < 1$$

$$\mu_2' = \frac{1+q}{p^2} . \quad Var(X) = \mu_2' - \mu_1'^2 = \frac{q}{p^2}$$

+vely skewed





$$M_X(t) = E(e^{tX}) = \sum_{k=1}^{\infty} e^{tk} q^{k-1} p$$

$$= pe^t \sum_{k=1}^{\infty} (qe^t)^{k-1} = \frac{pe^t}{1-qe^t},$$

if $qe^t < 1 \Rightarrow$ ie $t < -\log q$

<u>Ex</u> Suppose independent tests are conducted on mice while developing a vaccine. If the prob of success is 0.2 in each trial what is the prob that at

least 5 trials are needed to get the first success?

Sol^n $X \rightarrow$ no of trials needed to get the first success

$X \sim Geo(1/5)$

$$P(X \geq 5) = \sum_{k=5}^{\infty} \left(\frac{4}{5}\right)^{k-1} \left(\frac{1}{5}\right)$$

$$= \left(\frac{4}{5}\right)^4 \cdot \frac{1}{5} \left[1 + \frac{4}{5} + \left(\frac{4}{5}\right)^2 + \cdots\right]$$

$$= \left(\frac{4}{5}\right)^4 \cdot \frac{1}{5} \cdot \frac{1}{1-\frac{4}{5}} = \left(\frac{4}{5}\right)^4$$

$$= 0.4096$$

$X \sim Geo(p)$

$$P(X > m) = \sum_{k=m+1}^{\infty} q^{k-1} p$$

$$= q^m p + q^{m+1} p + \cdots$$

$$= q^m p(1+q+q^2+\cdots) = \frac{q^m p}{1-q} = q^m$$

$$P(\underbrace{X > m+n}_{A} \mid \underbrace{X > n}_{B}) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{P(X > m+n)}{P(X > n)} = \frac{q^{m+n}}{q^n} = q^m = P(X > m)$$

Memoryless property of the geometric dist^{n}

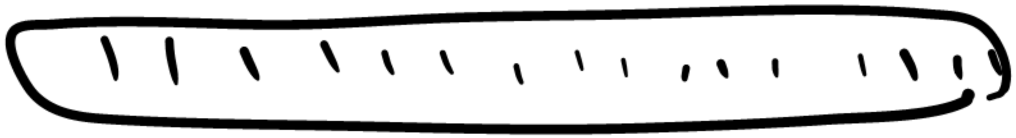
Negative Binomial dist^n

Suppose Bernoulli trials are conducted independently under identical conditions until the r^{th} success is observed.

$X \to$ no of trials needed for r^{th} success

$\to r, r+1, r+2, \ldots$

k

$(r-1)$  $\hookrightarrow s$

$$p_X(k) = \binom{k-1}{r-1} q^{k-r} p^r, \quad k = r, r+1, \ldots$$

$$\mu_1' = E(X) = \frac{r}{p}, \quad \mu_2 = Var(X) = \frac{rq}{p^2}$$

$$M_X(t) = E(e^{tX})$$

$$= \sum_{k=r}^{\infty} e^{tk} \binom{k-1}{r-1} q^{k-r} p^r$$

$$= \left(\frac{pe^t}{1-qe^t}\right)^r, \quad 0 < t < -\log q$$

<u>Ex</u> Suppose an airplane fails if 2 of its engines fail where the prob of failure of each engine is p

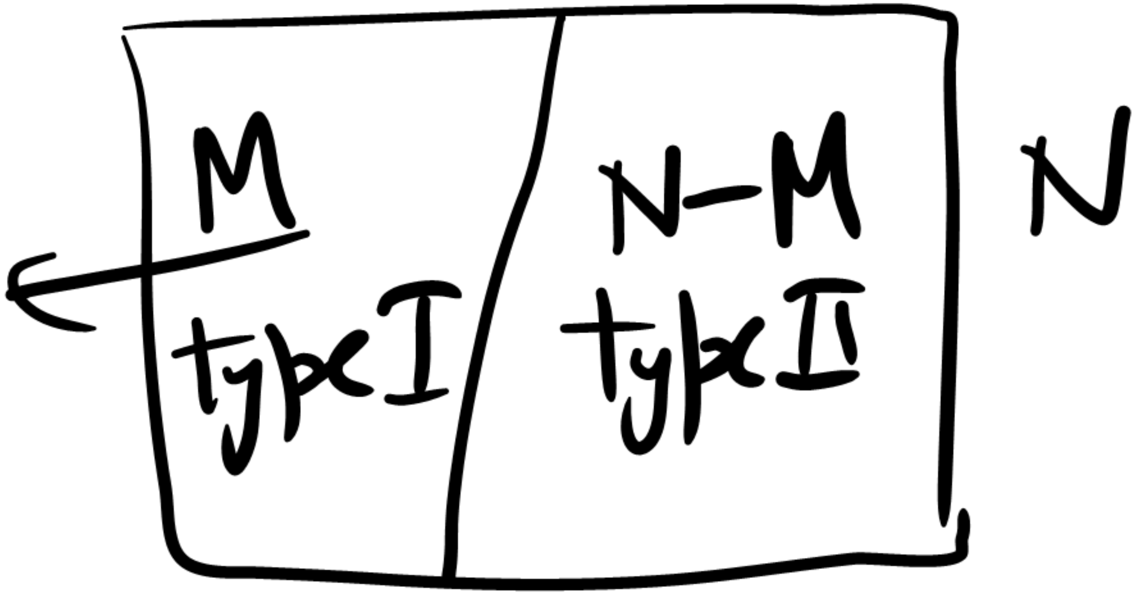
$X \to$ no of engines failing

$$p_X(k) = \binom{k-1}{2-1} q^{k-2} p^2.$$

$$= (k-1)\, q^{k-2} p^2, \quad k = 2, 3, \ldots$$

Hypergeometric Distribution

Suppose a pop^n has N elements



→ n items are selected at random without replacement (WOR)

X → no of items of type I in the selected sample.

$$p_X(k) = P(X=k) = \frac{\binom{M}{k}\binom{N-M}{n-k}}{\binom{N}{n}},$$

$k = 0, 1, \ldots, n$

$k \le M, \quad n-k \le N-M$

$$(1+x)^N = (1+x)^M (1+x)^{N-M}$$

The coefficient of x^n on both sides

we get

$$\binom{N}{n} = \sum_{k=0}^{n} \binom{M}{k}\binom{N-M}{n-k}$$

$$\mu_1' = E(X) = \sum_{k=1}^{n} \frac{k\binom{M}{k}\binom{N-M}{n-k}}{\binom{N}{n}}$$

$$= \sum_{k=1}^{n} \frac{\dfrac{M!}{(k-1)!\,(M-k)!}\;\dfrac{(N-M)!}{(n-k)!\,(N-M-\overline{n-k})!}}{\binom{N}{n}}$$

$$k-1=j$$

$$= \frac{nM}{N} \sum_{j=0}^{n-1} \left\{ \binom{M-1}{j}\binom{N-1-\overline{M-1}}{n-1-j} \Big/ \binom{N-1}{n-1} \right\}$$

$$= \frac{nM}{N}.$$

$$EX(X-1) = \frac{n(n-1)M(M-1)}{N(N-1)}$$

$$E(X^2) = EX(X-1) + E(X)$$

$$= \frac{nM(nM-M-n+N)}{N(N-1)}$$

$$\mu_2 = Var(X) = \left(\frac{N-n}{N-1}\right) \frac{nM}{N} \left(1 - \frac{M}{N}\right)$$

<u>Theorem</u> : Let $X \sim$ hypergeo(M, N, n).

If $M \to \infty$, $N \to \infty$ $\ni$ $\frac{M}{N} \to p$, then

$$p_X(k) \longrightarrow \binom{n}{k} p^k q^{n-k}, \quad k = 0, 1, \ldots n$$

<u>Pf</u>. $\circledast$ Do it yourself.

Suppose M (type I) is unknown

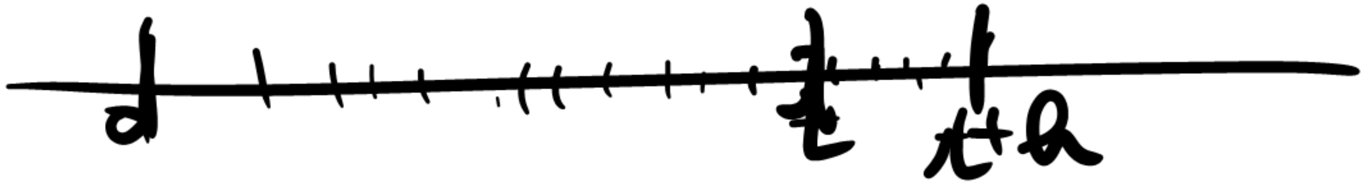
$$\frac{X}{n} \rightarrow \frac{M}{N}$$

$$M \approx \frac{NX}{n}$$

$$\left.N \approx \frac{Mn}{X}\right\} \checkmark$$

Poisson Process : We are observing events/ happenings over time/ area/ space etc.

Suppose we take time



We say that the occurrences/ happenings observed in a time scale follow a Poisson process provided they satisfy the following assumptions :

1. The number of occurrences in disjoint time intervals are independent.

2. The prob of a single occurrence during a small time interval is proportional to the length of the

interval.

$X(h) \rightarrow$ no of occurrence in interval of length h

$$P(X(h)=1) = \lambda h + o(h) = P_1(h)$$

3. Prob of more than one occurrence in a small time interval is negligible.

$$P_2(h) + P_3(h) + \cdots = 1 - P_0(h) - P_1(h) = o(h)$$

$$\frac{o(h)}{h} \rightarrow 0 \text{ as } h \rightarrow 0 \qquad \Rightarrow P_0(h) = 1 - \lambda h - o(h)$$

$X(t) \to$ no of occurrences in an interval of length t

$$P(X(t)=n) = P_n(t)$$

Under assumptions (1) – (3),

$$P_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \quad n=0,1,2,\cdots \quad \cdots (1)$$