Vector Algebra

Arijit De

Coordinate System (orthogonal)

- Cartesian : $\vec{A} = A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z$

$-\infty < x, y, z < \infty$

- Cylindrical : $\vec{A} = A_\rho \hat{a}_\rho + A_\phi \hat{a}_\phi + A_z \hat{a}_z$

$0 \leq \rho < \infty$, $0 \leq \phi < 2\pi$, $-\infty < z < \infty$

$\hat{a}_\rho \cdot \hat{a}_\phi = \hat{a}_\phi \cdot \hat{a}_z = \hat{a}_z \cdot \hat{a}_\rho = 0$

$\hat{a}_\rho \times \hat{a}_\phi = \hat{a}_z$

$\hat{a}_\phi \times \hat{a}_z = \hat{a}_\rho$

$\hat{a}_z \times \hat{a}_\rho = \hat{a}_\phi$

- Spherical : $\vec{A} = A_r \hat{a}_r + A_\theta \hat{a}_\theta + A_\phi \hat{a}_\phi$

$0 \leq r < \infty$, $0 \leq \theta \leq \pi$, $0 \leq \phi < 2\pi$

$\hat{a}_r \cdot \hat{a}_\theta = \hat{a}_\theta \cdot \hat{a}_\phi = \hat{a}_\phi \cdot \hat{a}_r = 0$

$\hat{a}_r \times \hat{a}_\theta = \hat{a}_\phi$

$\hat{a}_\theta \times \hat{a}_\phi = \hat{a}_r$

$\hat{a}_\phi \times \hat{a}_r = \hat{a}_\theta$

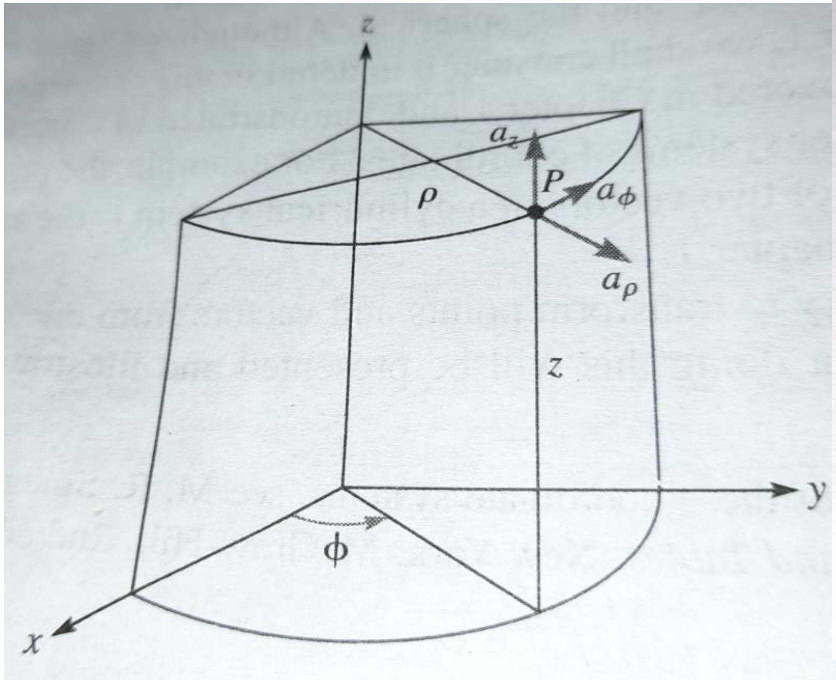
$$|\vec{A}| = \sqrt{A_\rho^2 + A_\phi^2 + A_z^2} = \sqrt{A_r^2 + A_\theta^2 + A_\phi^2}$$

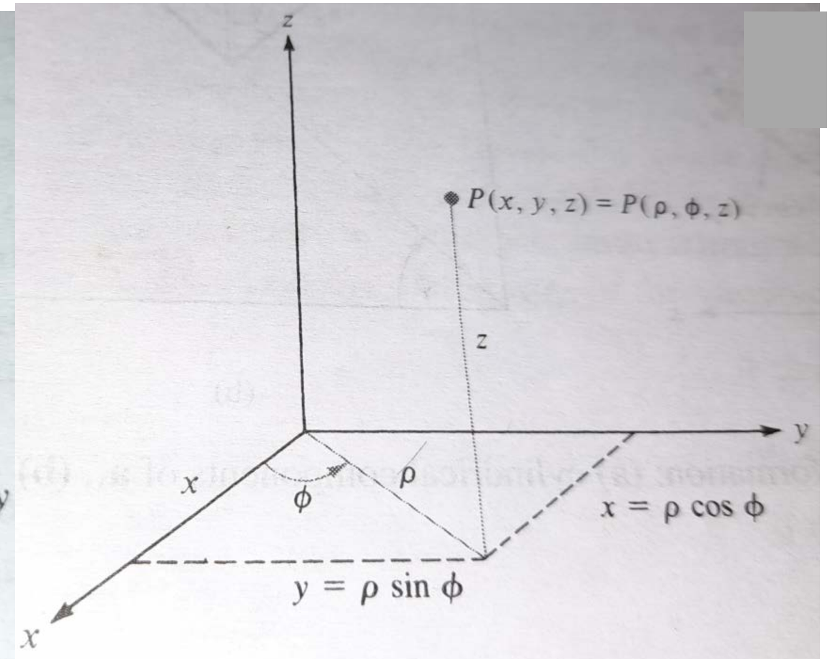
$\rho = \sqrt{x^2+y^2}$, $\phi = \tan^{-1} y/x$

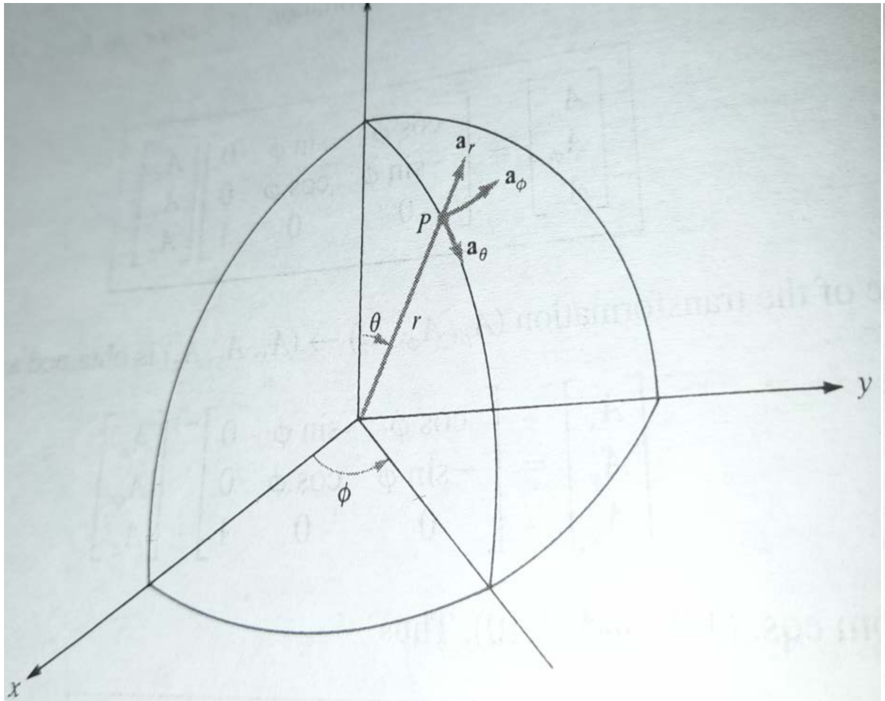
$x = \rho\cos\phi$, $y = \rho\sin\phi$

$r = \sqrt{x^2+y^2+z^2}$, $\theta = \tan^{-1} \frac{\sqrt{x^2+y^2}}{z}$, $\phi = \tan^{-1} y/x$

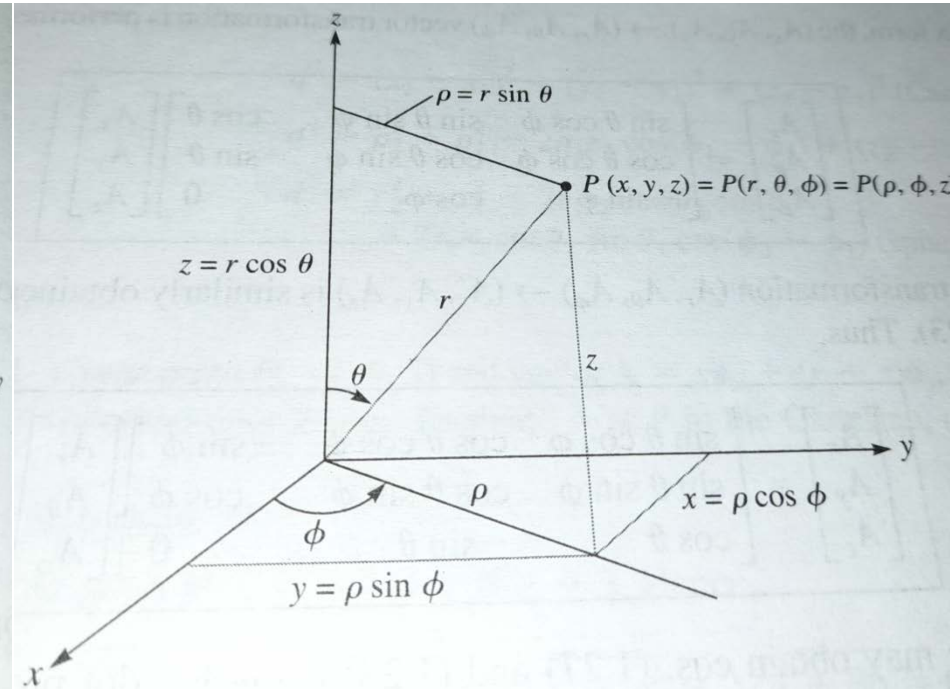
$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$.


z
a_z
P
a_ϕ
ρ
a_ρ
z
y
ϕ
x


z
$P(x, y, z) = P(\rho, \phi, z)$
z
y
ρ
x
ϕ
$x = \rho \cos \phi$
$y = \rho \sin \phi$
x



$\mathbf{a}_r$
$\mathbf{a}_\phi$
P
$\mathbf{a}_\theta$
θ
r
y
ϕ
x



z
$\rho = r \sin \theta$
$P\ (x, y, z) = P(r, \theta, \phi) = P(\rho, \phi, z)$
$z = r \cos \theta$
r
z
θ
y
ρ
$x = \rho \cos \phi$
ϕ
$y = \rho \sin \phi$
x

Unit Vector Transformation (Cylindrical)

$$\hat{a}_x = \cos\phi\,\hat{a}_\rho - \sin\phi\,\hat{a}_\phi$$

$$\hat{a}_y = \sin\phi\,\hat{a}_\rho + \cos\phi\,\hat{a}_\phi$$

$$\hat{a}_z = \hat{a}_z$$

$$\hat{a}_\rho = \cos\phi\,\hat{a}_x + \sin\phi\,\hat{a}_y$$

$$\hat{a}_\phi = -\sin\phi\,\hat{a}_x + \cos\phi\,\hat{a}_y$$

$$\hat{a}_z = \hat{a}_z$$

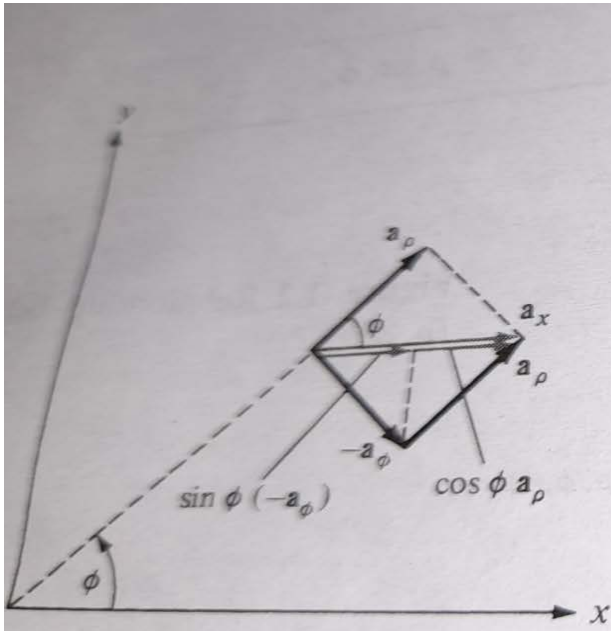
$$A_\rho = \vec{A}\cdot\hat{a}_\rho$$

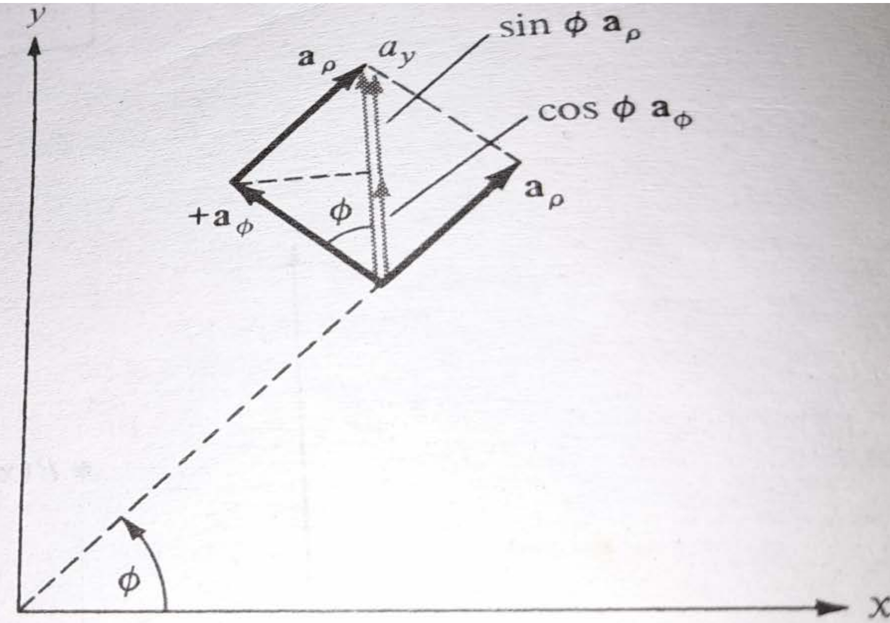
$$= A_x(\hat{a}_x\cdot\hat{a}_\rho) + A_y(\hat{a}_y\cdot\hat{a}_\rho) + A_z\cancelto{0}{(\hat{a}_z\cdot\hat{a}_\rho)}$$

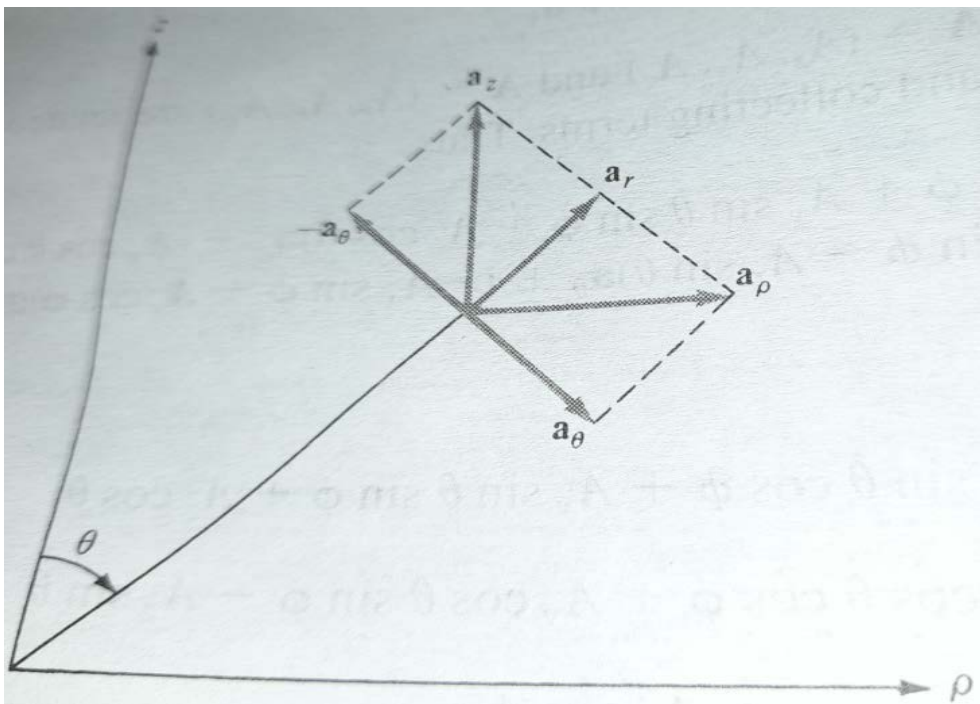
$$= A_x\cos\phi + A_y\sin\phi$$

$$A_\phi = \vec{A}\cdot\hat{a}_\phi$$

$$= -A_x\sin\phi + A_y\cos\phi.$$


y
$\mathbf{a}_\rho$
ϕ
$\mathbf{a}_x$
$\mathbf{a}_\rho$
$-\mathbf{a}_\phi$
$\sin\phi\,(-\mathbf{a}_\phi)$
$\cos\phi\,\mathbf{a}_\rho$
ϕ
x


y
$\sin\phi\,\mathbf{a}_\rho$
$\mathbf{a}_\rho$
a_y
$\cos\phi\,\mathbf{a}_\phi$
$\mathbf{a}_\rho$
$+\mathbf{a}_\phi$
ϕ
ϕ
x


z
$\mathbf{a}_z$
$\mathbf{a}_r$
$-\mathbf{a}_\theta$
$\mathbf{a}_\rho$
$\mathbf{a}_\theta$
θ
ρ

Unit Vector Transformation (Spherical)

$$\hat{a}_x = \sin\theta\cos\phi\,\hat{a}_r + \cos\theta\cos\phi\,\hat{a}_\theta - \sin\phi\,\hat{a}_\phi$$

$$\hat{a}_y = \sin\theta\sin\phi\,\hat{a}_r + \cos\theta\sin\phi\,\hat{a}_\theta + \cos\phi\,\hat{a}_\phi$$

$$\hat{a}_z = \cos\theta\,\hat{a}_r - \sin\theta\,\hat{a}_\theta$$

$$\hat{a}_r = \sin\theta\cos\phi\,\hat{a}_x + \sin\theta\sin\phi\,\hat{a}_y + \cos\theta\,\hat{a}_z$$

$$\hat{a}_\theta = \cos\theta\cos\phi\,\hat{a}_x + \cos\theta\sin\phi\,\hat{a}_y - \sin\theta\,\hat{a}_z$$

$$\hat{a}_\phi = -\sin\phi\,\hat{a}_x + \cos\phi\,\hat{a}_y$$

$$A_r = \vec{A}\cdot\hat{a}_r = A_x(\hat{a}_x\cdot\hat{a}_r) + A_y(\hat{a}_y\cdot\hat{a}_r) + A_z(\hat{a}_z\cdot\hat{a}_r)$$

$$= A_x\sin\theta\cos\phi + A_y\sin\theta\sin\phi + A_z\cos\theta$$

$$A_\theta = \vec{A}\cdot\hat{a}_\theta = A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta.$$

$$A_\phi = \vec{A}\cdot\hat{a}_\phi = -A_x\sin\phi + A_y\cos\phi.$$

Distance between 2 points.

$d = |\vec{r}_2 - \vec{r}_1|$.

Cartesian:- $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$

Cylindrical:- Convert $\vec{r}_2$ & $\vec{r}_1$ $\rightarrow$ Cartesian. and then compute

$$d^2 = \rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \cos(\phi_2 - \phi_1) + (z_2 - z_1)^2$$

Spherical:- Convert $\vec{r}_2$ & $\vec{r}_1$ $\rightarrow$ Cartesian and then compute,

$$d^2 = r_1^2 + r_2^2 - 2r_1r_2 \cos\theta_2 \cos\theta_1 - 2r_1r_2 \sin\theta_2 \sin\theta_1 \cos(\phi_2 - \phi_1).$$

Vector Calculus

- Differential displacement : $\vec{dl} = dx\,\hat{a}_x + dy\,\hat{a}_y + dz\,\hat{a}_z$ (Cartesian)

$\vec{dl} = d\rho\,\hat{a}_\rho + \rho d\phi\,\hat{a}_\phi + dz\,\hat{a}_z$ (cylindrical)

$\vec{dl} = dr\,\hat{a}_r + r d\theta\,\hat{a}_\theta + r\sin\theta\, d\phi\,\hat{a}_\phi$ (spherical)

- Differential surface : $\vec{ds} = dy\,dz\,\hat{a}_x\ ,\ dx\,dz\,\hat{a}_y\ ,\ dx\,dy\,\hat{a}_z$

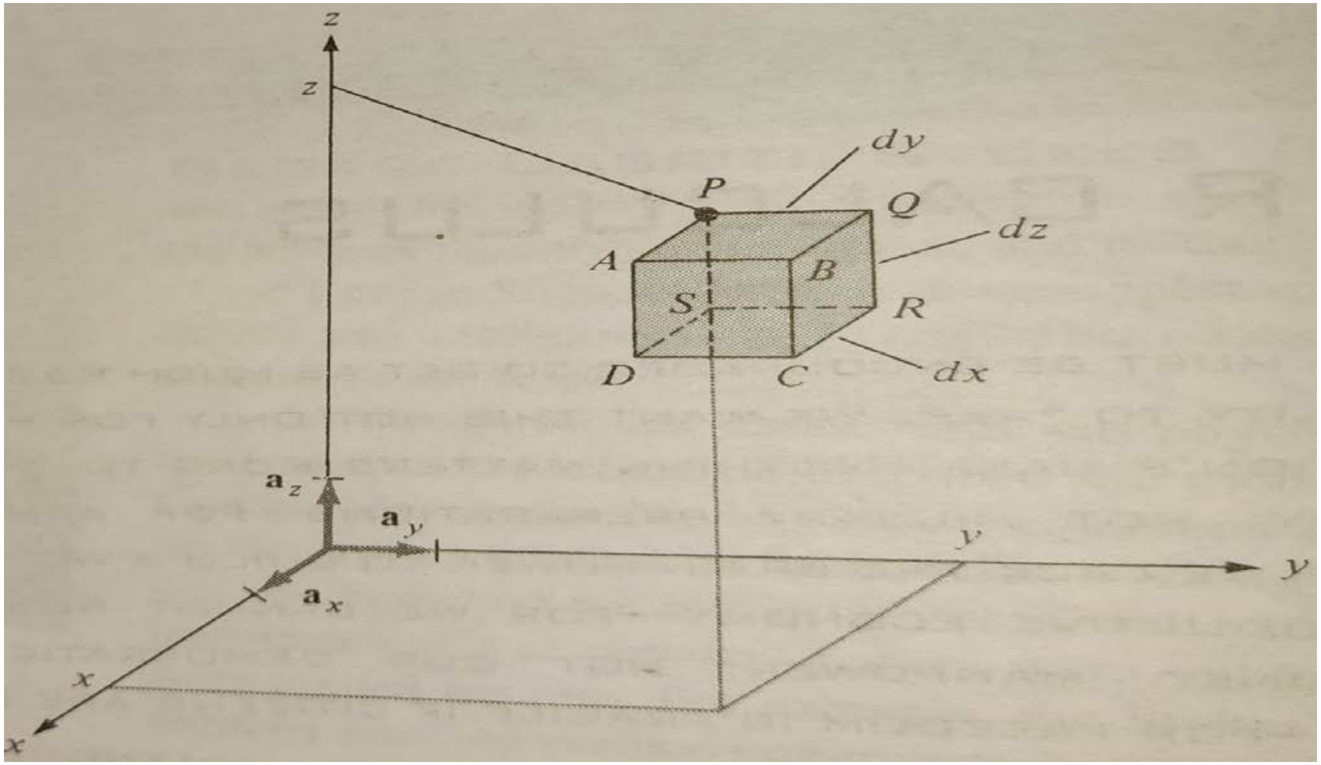
$= (\rho\,d\phi\,dz)\,\hat{a}_\rho\ ,\ d\rho\,dz\,\hat{a}_\phi\ ,\ \rho\,d\rho\,d\phi\,\hat{a}_z$

$= (r^2\sin\theta\,d\theta\,d\phi)\,\hat{a}_r\ ,\ (r\sin\theta\,dr\,d\phi)\,\hat{a}_\theta\ ,\ (r\,dr\,d\theta)\,\hat{a}_\phi$

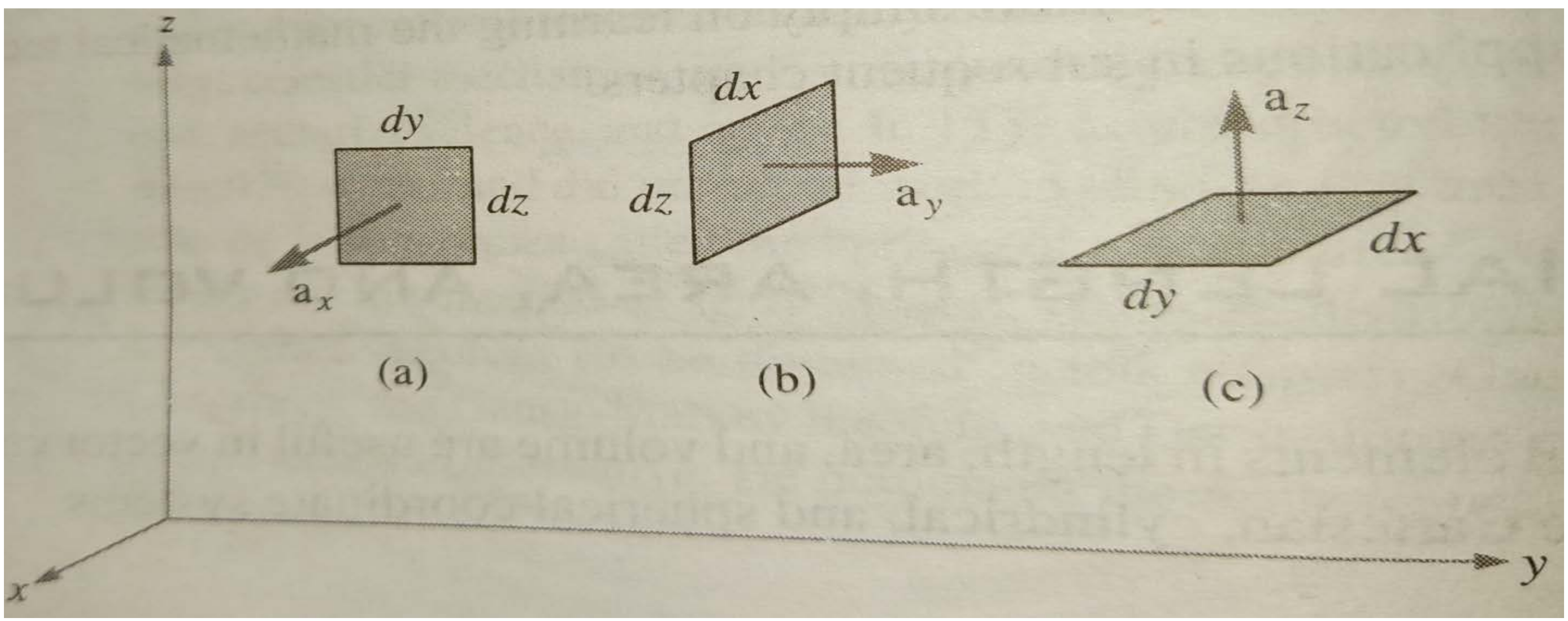
- Differential volume : $dv = dx\,dy\,dz$

$= \rho\,d\rho\,d\phi\,dz$

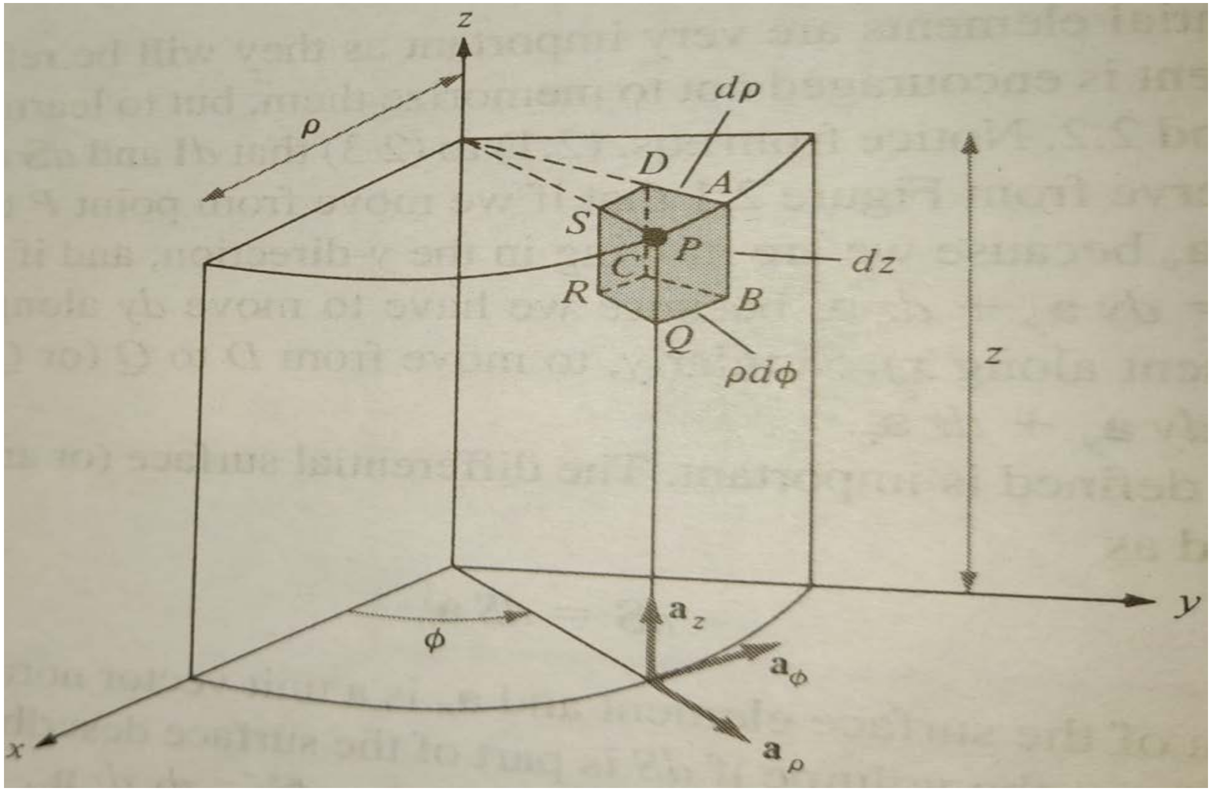
$= r^2\sin\theta\,dr\,d\theta\,d\phi$

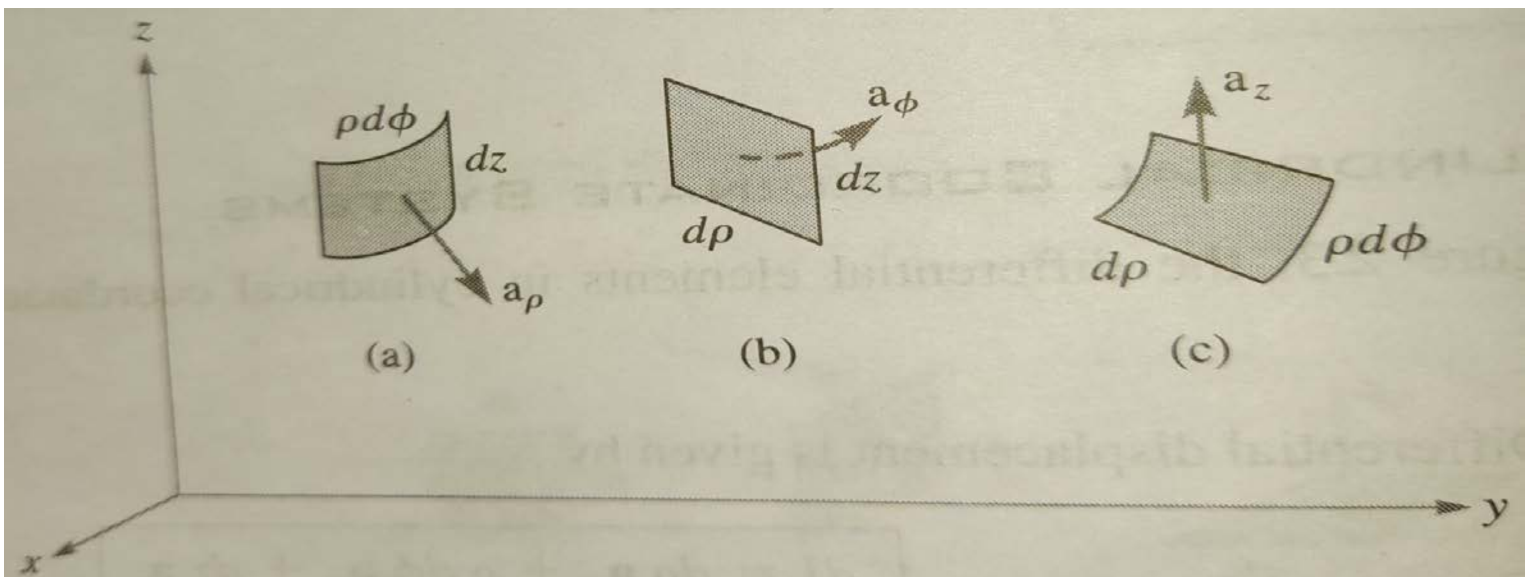


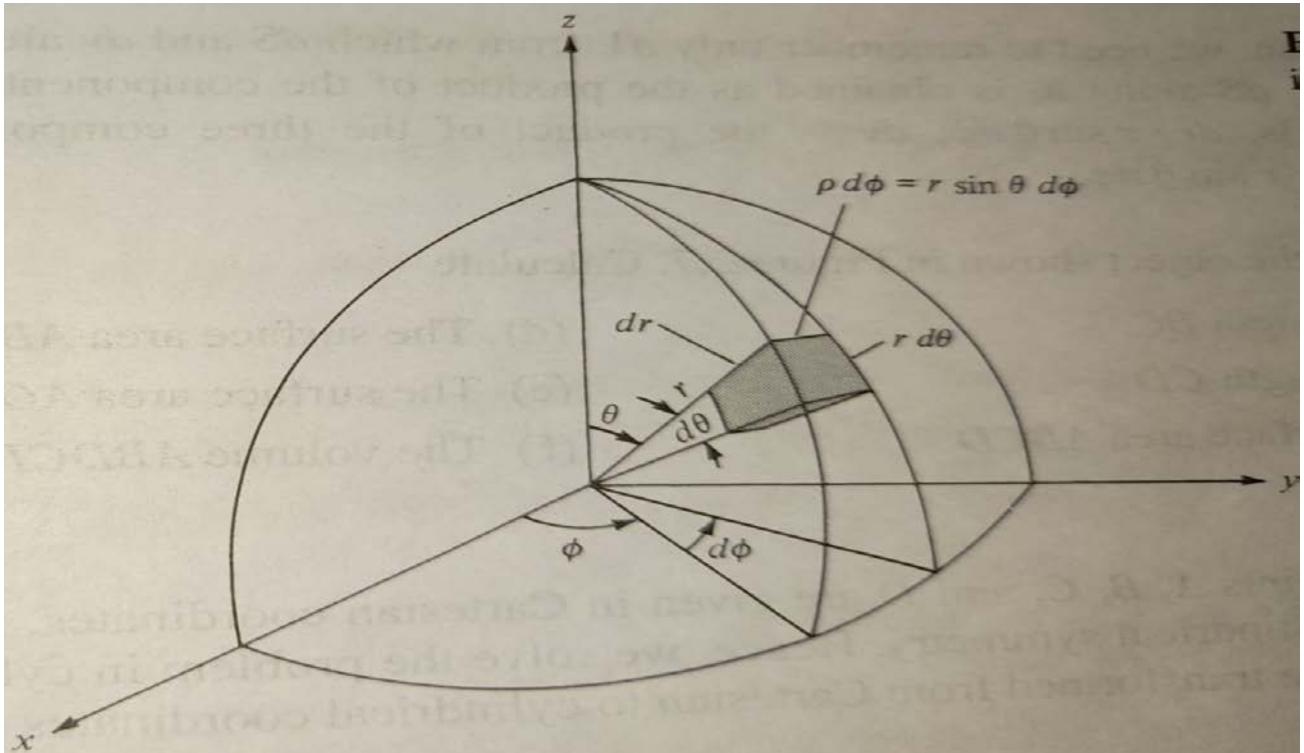
z
z
dy
P
Q
dz
A
B
S
R
D
C
dx
a_z
a_y
y
y
a_x
x
x

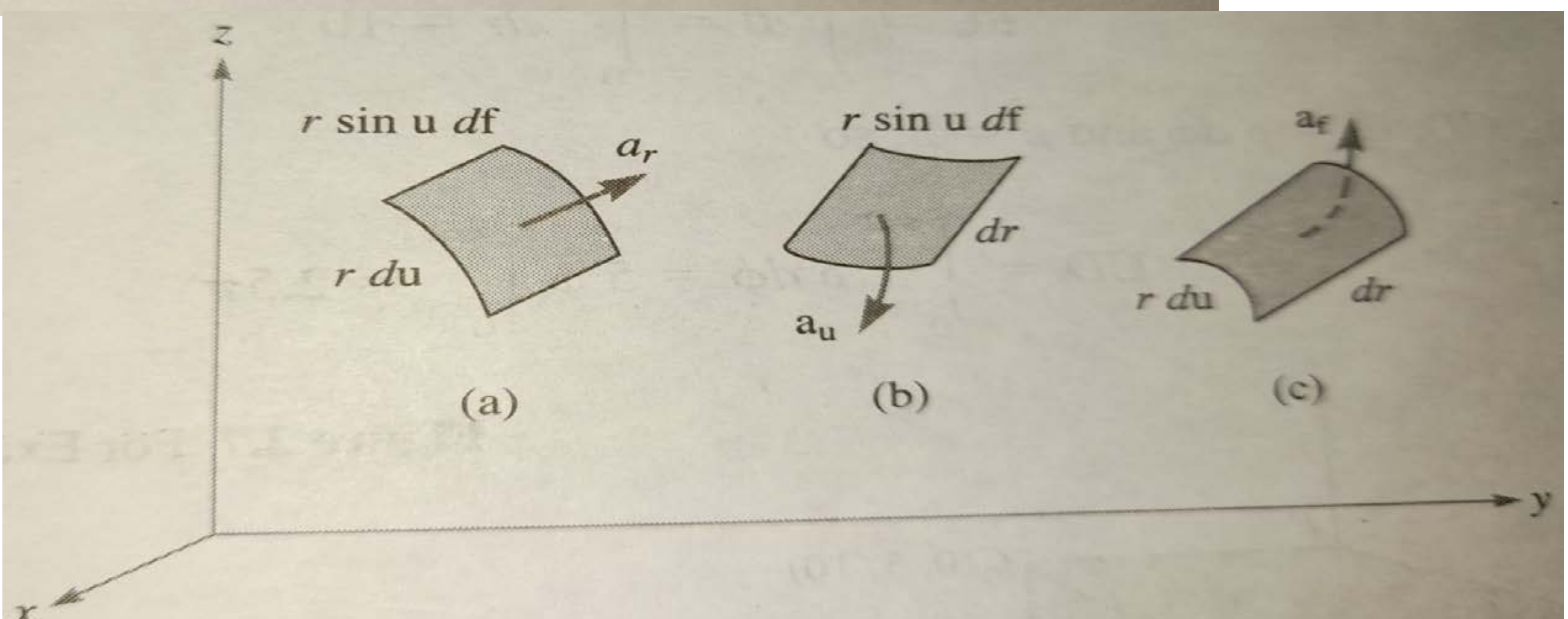


z
dy
dz
a_x
dx
dz
a_y
a_z
dx
dy
(a)
(b)
(c)
y
x


z
ρ
dρ
D
A
S
P
C
dz
R
B
Q
ρdφ
z
y
a_z
φ
a_φ
x
a_ρ


z
ρdφ
dz
a_ρ
a_φ
dz
dρ
a_z
ρdφ
dρ
y
x
(a)
(b)
(c)


z
ρ dφ = r sin θ dφ
dr
r dθ
r
θ
dθ
y
φ
dφ
x


z
r sin u df
a_r
r du
r sin u df
dr
a_u
a_f
r du
dr
(a)
(b)
(c)
y
x

Integrals.

- Line Integral :- $\int_L \vec{A} \cdot \vec{dl} = \int_a^b |A| \cos\theta \, dl$

closed contour integral - $\oint_L \vec{A} \cdot \vec{dl}$ (also called circulation of $\vec{A}$ around L)

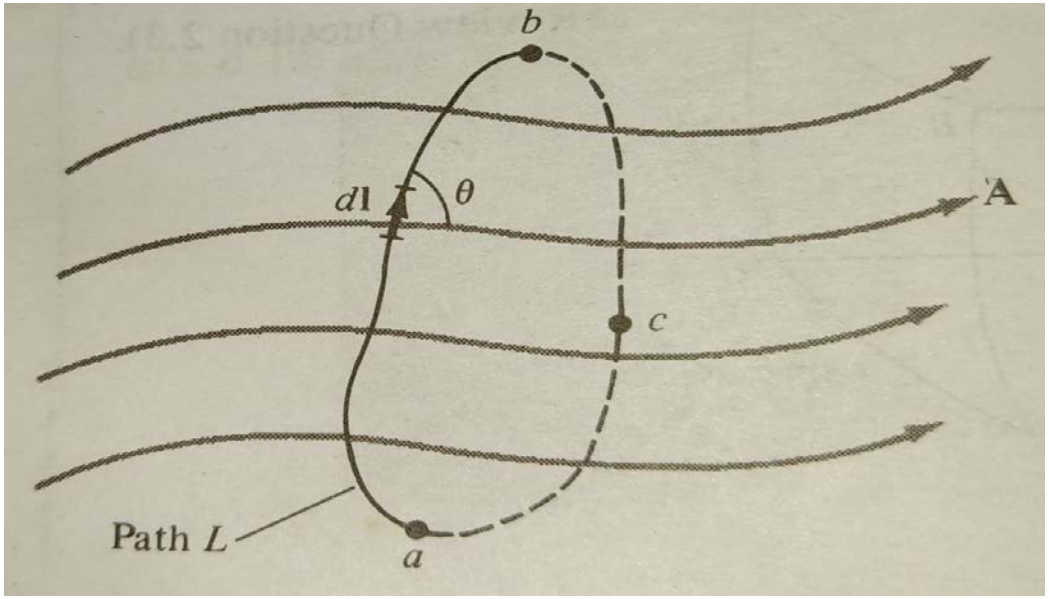
closed loop $\Longrightarrow$ open surface.

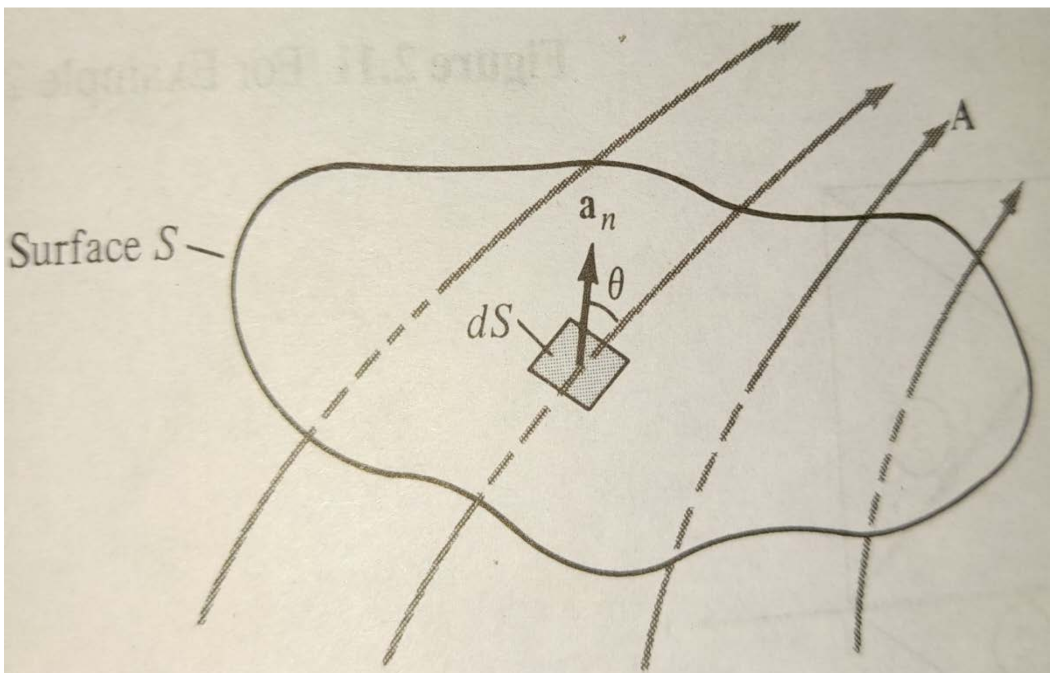
- Surface Integral :- $\psi = \int_S |A| \cos\theta \, ds = \iint_S \vec{A} \cdot \hat{a}_n \, ds$ (flux of $\vec{A}$ through S)

closed surface integral - $\oiint_S \vec{A} \cdot \vec{ds}$ — net outward flux of $\vec{A}$ from S.

closed surface $\Rightarrow$ volume.

- Volume Integral :- $= \iiint_V \rho_v \, dv$.


b
dl
θ
A
c
Path L
a


A
$\mathbf{a}_n$
Surface S
θ
dS

Calculus - Differential operator.

$\vec{\nabla}$ (operator) is a vector differential operator

In cartesian, $\vec{\nabla} = \boxed{\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z}$.

In Cylindrical, $\rho = \sqrt{x^2+y^2}$, $\tan\phi = y/x$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial}{\partial \phi}\frac{\partial \phi}{\partial x} = \cos\phi\frac{\partial}{\partial \rho} - \frac{\sin\phi}{\rho}\frac{\partial}{\partial \phi}.$$

$$\frac{\partial}{\partial y} = \sin\phi\frac{\partial}{\partial \rho} + \frac{\cos\phi}{\rho}\frac{\partial}{\partial \phi}.$$

Hence, $\vec{\nabla} = (\cos\phi\,\hat{a}_x + \sin\phi\,\hat{a}_y)\frac{\partial}{\partial \rho} + \left(-\frac{\sin\phi}{\rho}\hat{a}_x + \frac{\cos\phi}{\rho}\hat{a}_y\right)\frac{\partial}{\partial \phi} + \frac{\partial}{\partial z}\hat{a}_z$

$$= \boxed{\hat{a}_\rho\frac{\partial}{\partial \rho} + \hat{a}_\phi\frac{1}{\rho}\frac{\partial}{\partial \phi} + \hat{a}_z\frac{\partial}{\partial z}}.$$

In Spherical, $r = \sqrt{x^2+y^2+z^2}$, $\tan\theta = \frac{\sqrt{x^2+y^2}}{z}$, $\tan\phi = y/x$

$$\frac{\partial}{\partial x} = \sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial \theta} - \frac{\sin\phi}{\rho}\frac{\partial}{\partial \phi}.$$

$$\frac{\partial}{\partial y} = \sin\theta \sin\phi \frac{\partial}{\partial r} + \frac{\cos\theta \sin\phi}{r} \frac{\partial}{\partial \theta} + \frac{\cos\phi}{\rho} \frac{\partial}{\partial \phi}.$$

$$\frac{\partial}{\partial z} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r} \frac{\partial}{\partial \theta}.$$

$$\boxed{\vec{\nabla} = \hat{a}_r \frac{\partial}{\partial r} + \hat{a}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{a}_\phi \frac{1}{r\sin\theta} \frac{\partial}{\partial \phi}.}$$

Gradient:

- Gradient of a scalar field V, is a vector that represents both magnitude & direction of the maximum space rate of increase of V.

In cartesian:-

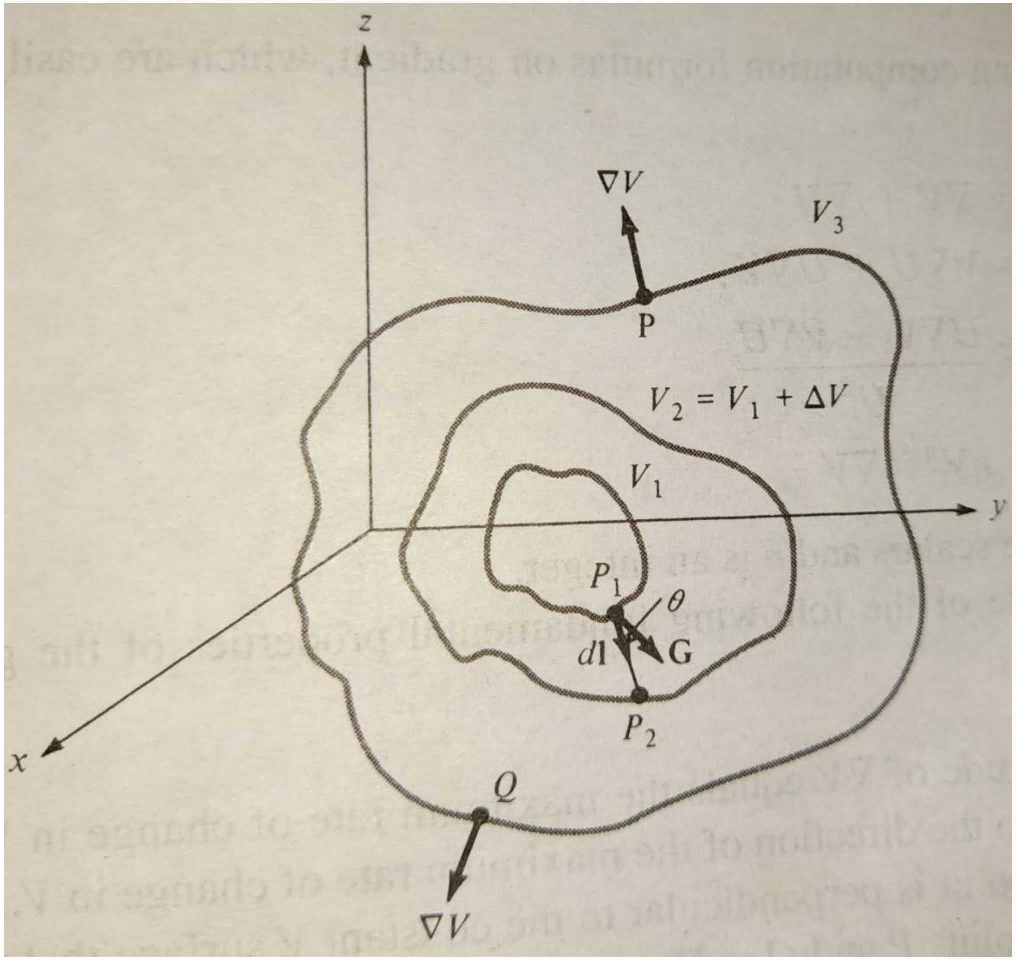
$$dV = \frac{\partial V}{\partial x}\,dx + \frac{\partial V}{\partial y}\,dy + \frac{\partial V}{\partial z}\,dz.$$

$$= \underbrace{\left(\frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z\right)}_{G} \cdot \underbrace{\left(dx\,\hat{a}_x + dy\,\hat{a}_y + dz\,\hat{a}_z\right)}_{\vec{dl}}$$

$$\frac{dV}{dl} = G\cos\theta \qquad \& \qquad \left.\frac{dV}{dl}\right|_{max} = \frac{dV}{dn} = G \quad \text{when } \theta = 0 \text{ i.e. } \vec{dl} \text{ is in the direction of } \vec{G}.$$

$\frac{dV}{dn}$ is the normal derivative., Thus G has the magnitude & direction as those of the maximum rate of change of V.

Hence, grad V = $\boxed{\nabla V = \frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z}$.


z
∇V
V₃
P
V₂ = V₁ + ΔV
V₁
y
P₁
θ
dl
G
P₂
x
Q
∇V

Cylindrical: $\nabla V = \boxed{\frac{\partial V}{\partial \rho}\hat{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\hat{a}_\phi + \frac{\partial V}{\partial z}\hat{a}_z}$

Spherical:- $\nabla V = \boxed{\frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{a}_\phi}$

Identities:-

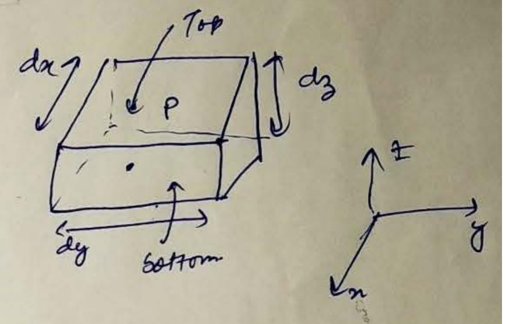
- $\nabla(V+U) = \nabla V + \nabla U$
- $\nabla(VU) = V\nabla U + U\nabla V$
- $\nabla\left(\frac{V}{U}\right) = \frac{U\nabla V - V\nabla U}{U^2}$
- $\nabla V^n = n V^{n-1}\nabla V$

- $(\nabla V \cdot \hat{a})$ = directional derivative of V along $\hat{a}$.
- ∇V is $\perp$ to the constant V surfaces.
- If $\vec{A} = \nabla V$, V is called scalar potential of $\vec{A}$.

DIVERGENCE

- Divergence of $\vec{A}$ at a given point P is the outward flux/vol. as the volume shrinks about P.

$$\text{div}\,\vec{A} = \nabla\cdot\vec{A} = \lim_{\Delta v \to 0} \frac{\oint \vec{A}\cdot\vec{ds}}{\Delta v}.$$



$$\oint_S \vec{A}\cdot\vec{ds} = \left(\iint_{\text{front}} + \iint_{\text{back}} + \iint_{\text{left}} + \iint_{\text{right}} + \iint_{\text{top}} + \iint_{\text{bottom}}\right)\vec{A}\cdot\vec{ds}$$

Consider front & back side :- $\vec{ds} = \pm\hat{a}_x\, dy\, dz$.

so, $A_x(x,y,z) \approx A_x(x_0, y_0, z_0) + (x-x_0)\frac{\partial A_x}{\partial x}\Big|_P + (y-y_0)\frac{\partial A_x}{\partial y}\Big|_P + (z-z_0)\frac{\partial A_x}{\partial z}\Big|_P$.

Front side :- $x = x_0 + dx/2$,
$\vec{ds} = dy\,dz\,\hat{a}_x$

Back side :- $x = x_0 - \frac{dx}{2}$.
$\vec{ds} = (-\hat{a}_x)\,dy\,dz$.

Then, $\iint_{\text{front}} \vec{A}\cdot\vec{ds} \simeq dy\,dz\left[A_x(x_0,y_0,z_0) + \frac{dx}{2}\frac{\partial A_x}{\partial x}\Big|_P\right]$

$\iint_{\text{back}} \vec{A}\cdot\vec{ds} \simeq -dy\,dz\left[A_x(x_0,y_0,z_0) - \frac{dx}{2}\frac{\partial A_x}{\partial x}\Big|_P\right]$

$$\iint_{front} \vec{A}\cdot\vec{ds} + \iint_{back} \vec{A}\cdot\vec{ds} \approx dx\,dy\,dz \left.\frac{\partial A_x}{\partial x}\right|_P$$

Similarly,

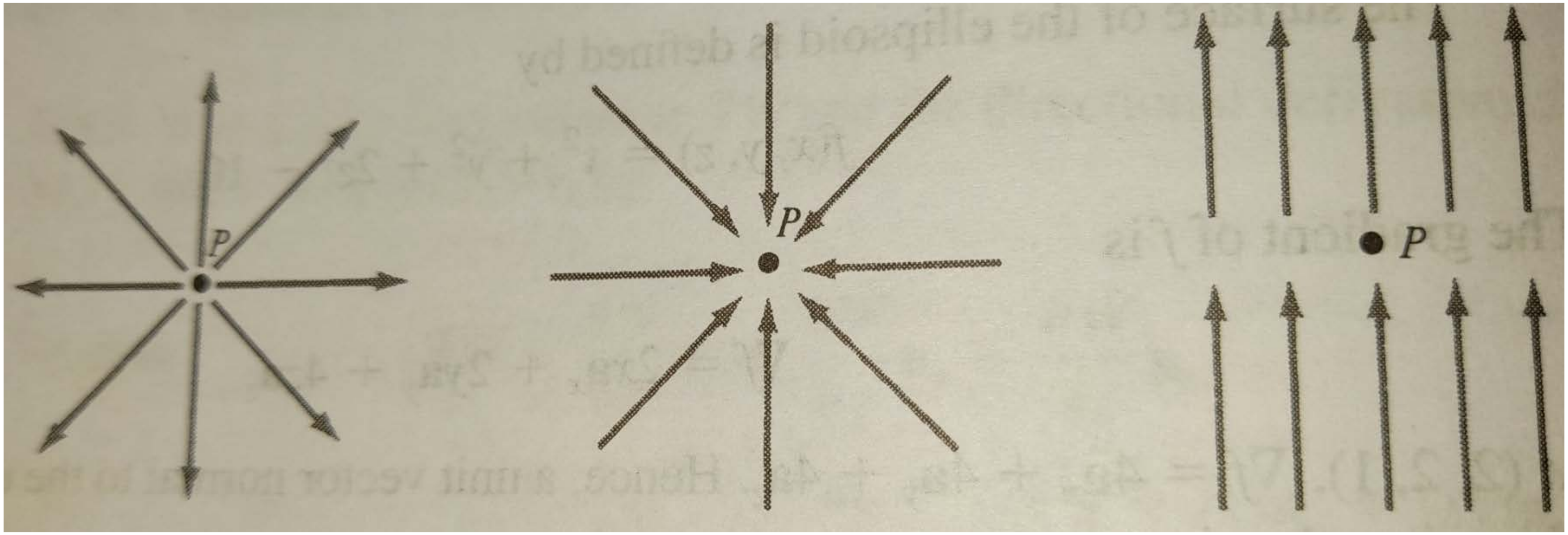
$$\left(\iint_{left} + \iint_{right}\right)\vec{A}\cdot\vec{ds} \approx dx\,dy\,dz \left.\frac{\partial A_y}{\partial y}\right|_P$$

$$\left(\iint_{top} + \iint_{bottom}\right)\vec{A}\cdot\vec{ds} \approx dx\,dy\,dz \left.\frac{\partial A_z}{\partial z}\right|_P$$

Thus, $\nabla\cdot\vec{A} = \lim_{\Delta v\to 0} \dfrac{\oint_S \vec{A}\cdot\vec{ds}}{\Delta v} = \boxed{\dfrac{\partial A_x}{\partial x} + \dfrac{\partial A_y}{\partial y} + \dfrac{\partial A_z}{\partial z}}\Bigg|_{\text{at } P}$

Cylindrical :- $\nabla\cdot\vec{A} = \boxed{\dfrac{1}{\rho}\dfrac{\partial}{\partial \rho}(\rho A_\rho) + \dfrac{1}{\rho}\dfrac{\partial A_\phi}{\partial \phi} + \dfrac{\partial A_z}{\partial z}}$

Spherical :- $\nabla\cdot\vec{A} = \boxed{\dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r^2 A_r) + \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial\theta}(A_\theta \sin\theta) + \dfrac{1}{r\sin\theta}\dfrac{\partial A_\phi}{\partial \phi}}$


P
P
P

Divergence Identities

- $\nabla\cdot\left(\frac{\vec{A}}{g}\right) = \frac{g(\nabla\cdot\vec{A}) - \vec{A}\cdot(\nabla g)}{g^2}$

- $\nabla\cdot(\vec{A}+\vec{B}) = \nabla\cdot\vec{A} + \nabla\cdot\vec{B}$

- $\nabla\cdot(V\vec{A}) = V\,\nabla\cdot\vec{A} + \vec{A}\cdot\nabla V$

- $\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B})$

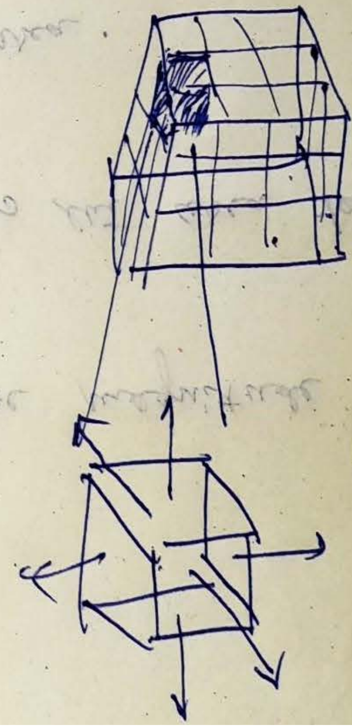
- Gauss-Ostrogradsky Theorem:- $\oiint_S \vec{A}\cdot d\vec{s} = \iiint (\nabla\cdot\vec{A})\,dv$

  (or Divergence theorem)

$$\oint_S \vec{A}\cdot d\vec{s} = \sum_k \oint_{S_k} \vec{A}\cdot d\vec{s} = \sum_k \frac{\oint_{S_k}\vec{A}\cdot d\vec{s}}{\Delta v_k}\,\Delta v_k$$

$$\underbrace{\frac{\oint_{S_k}\vec{A}\cdot d\vec{s}}{\Delta v_k}}_{\nabla\cdot\vec{A}}$$

$$= \iiint_V (\nabla\cdot\vec{A})\,dv$$



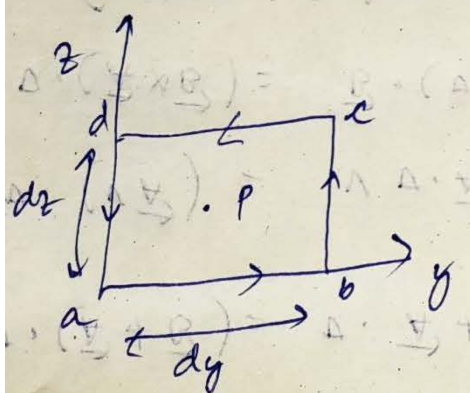
(Cancellation of flux at the interior surface)

Curl.

$$\text{curl } \vec{A} = \vec{\nabla} \times \vec{A} = \lim_{\Delta S \to 0} \frac{\oint_L \vec{A} \cdot \vec{dl}}{\Delta S} \, \hat{a}_n$$

Curl of $\vec{A}$ is an axial (or rotational) vector whose magnitude is the maximum circulation of $\vec{A}$ per unit area, as the area tends to zero.

The direction is the normal direction of the area.



$$\oint_L \vec{A} \cdot \vec{dl} = \left(\int_{ab} + \int_{bc} + \int_{cd} + \int_{da} \right) \vec{A} \cdot \vec{dl}.$$

$$\int_{ab} \vec{A} \cdot \vec{dl} = dy \left[A_y(x_0, y_0 z_0) - \frac{dz}{2} \frac{\partial A_y}{\partial z} \Big|_p \right]$$

(Here $\vec{dl} = dy \, \hat{a}_y$, $z = z_0 - dz/2$)

$$\int_{bc} \vec{A}\cdot\vec{dl} = dz\left[A_z(x_0, y_0, z_0) + \frac{dy}{2}\frac{\partial A_z}{\partial y}\Big|_P\right]$$

(Here $\vec{dl} = dz\,\hat{a}_z$, $y = y_0 + \frac{dy}{2}$)

$$\int_{cd} \vec{A}\cdot\vec{dl} = -dy\left[A_y(x_0, y_0, z_0) + \frac{dz}{2}\frac{\partial A_y}{\partial z}\Big|_P\right]$$

(Here $\vec{dl} = dy(-\hat{a}_y)$, $z = z_0 + dz/2$)

$$\int_{da} \vec{A}\cdot\vec{dl} = -dz\left[A_z(x_0, y_0, z_0) - \frac{dy}{2}\frac{\partial A_z}{\partial y}\Big|_P\right]$$

(Here $\vec{dl} = dz\,(-\hat{a}_z)$, $y = y_0 - dy/2$)

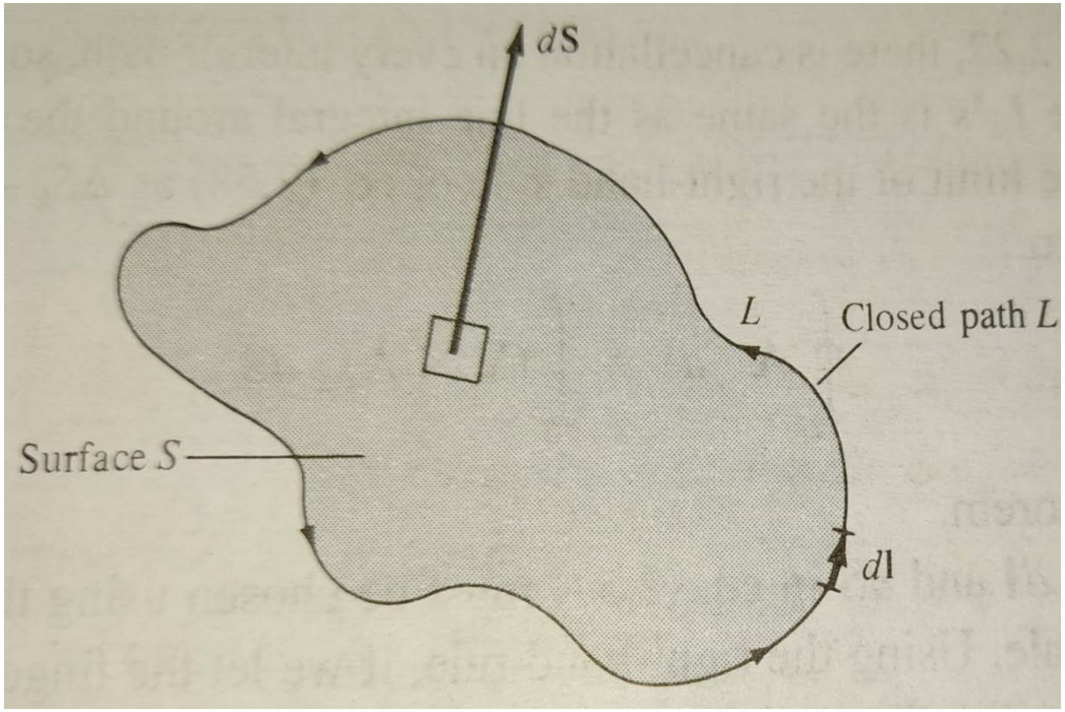
$$\lim_{\Delta S\to 0} \frac{\oint_L \vec{A}\cdot\vec{dl}}{\underset{dy\,dz}{\underset{\parallel}{\Delta S}}} = \boxed{\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}} \equiv (\text{curl } A)_x$$

Similarly, $(\text{curl } A)_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}$; $(\text{curl } A)_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$

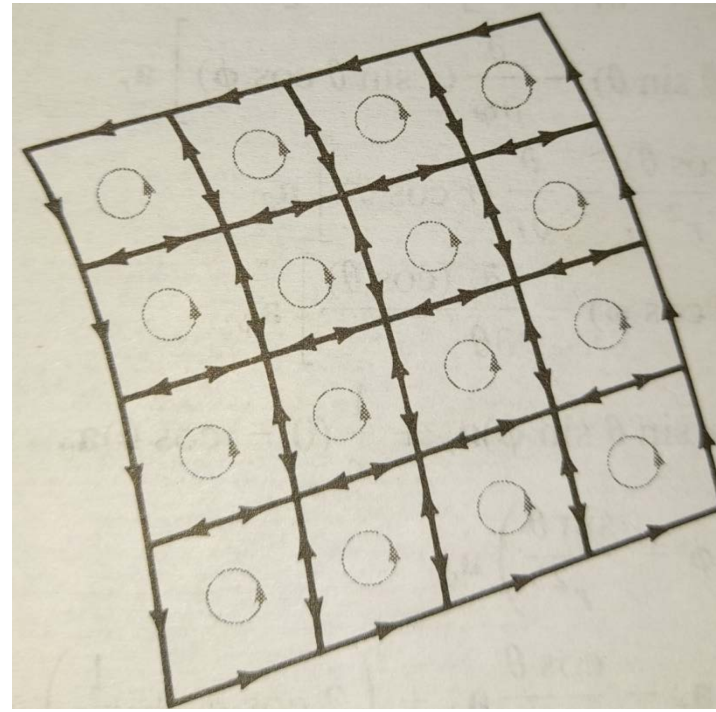
Combining, $\nabla \times \vec{A} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$. in Cartesian,

Cylindrical :- $\frac{1}{\rho} \begin{vmatrix} \hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$

Spherical :- $\frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{a}_r & r\hat{a}_\theta & r\sin\theta\,\hat{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & rA_\theta & r\sin\theta\, A_\phi \end{vmatrix}$.



dS
L
Closed path L
Surface S
dl



Identities of Curl.

- $\vec{\nabla} \times (\vec{A} + \vec{B}) = \nabla \times \vec{A} + \nabla \times \vec{B}$
- $\vec{\nabla} \times (\vec{A} \times \vec{B}) = \vec{A}(\nabla \cdot \vec{B}) - \vec{B}(\nabla \cdot \vec{A}) + (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B}$
- $\nabla \times \left(\frac{\vec{A}}{g}\right) = \frac{g(\nabla \times \vec{A}) + \vec{A} \times (\nabla g)}{g^2}$
- $\nabla \times (V\vec{A}) = V \nabla \times \vec{A} + \nabla V \times \vec{A}$

- $\nabla \cdot (\nabla \times \vec{A}) = 0$
- $\nabla \times \nabla V = 0$

$\left.\begin{matrix} \\ \\ \end{matrix}\right\}$ IMP.

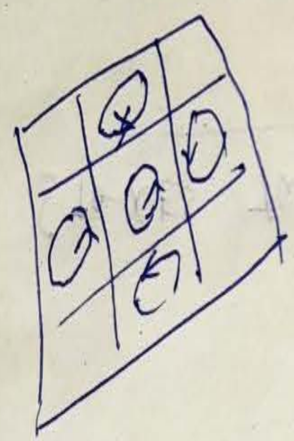
- STOKES Theorem:- $\oint_L \vec{A} \cdot \vec{dl} = \iint_S (\nabla \times \vec{A}) \cdot \vec{ds}$

The circulation of a vector field $\vec{A}$ around a (closed) path 'L', is equal to the surface integral of the curl of $\vec{A}$ over the open surface 'S' bounded by 'L', provided $\vec{A}$ and $\nabla \times \vec{A}$ are continuous on 'S'.

The surface 'S' is sub-divided into large number of cells, k^{th} cell bounded by path L_k with surface area ΔS_k

$$\oint_L \vec{A}\cdot\vec{dl} = \sum_k \oint_{L_k} \vec{A}\cdot\vec{dl} = \sum_k \frac{\oint_{L_k} \vec{A}\cdot\vec{dl}}{\Delta S_k}\,\Delta S_k.$$

(Note cancellation of the interior path)



Thus, $\oint_L \vec{A}\cdot\vec{dl} = \iint_S (\nabla\times\vec{A})\cdot\vec{ds}$.

. The direction of $\vec{dl}$ & $\vec{ds}$ must be chosen using right hand rule.

LAPLACIAN:

- Divergence of the gradient of a scalar V:

$$\nabla \cdot (\nabla V) = \nabla^2 V$$

Cartesian:- $\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$

Cylindrical:- $\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$

Spherical:- $\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$

- Vector Laplacian:- $\nabla^2 \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla \times \nabla \times \vec{A}$

ONLY in Cartesian System, $\nabla^2 \vec{A} = (\nabla^2 A_x)\,\hat{a}_x + (\nabla^2 A_y)\,\hat{a}_y + (\nabla^2 A_z)\,\hat{a}_z$

Tricks - Transfering Derivatives & Delta Fn.

- $\nabla\cdot(f\vec{A}) = f(\nabla\cdot\vec{A}) + \vec{A}\cdot(\nabla f)$

$$\iiint_V \nabla\cdot(f\vec{A})\,dv = \iiint_V f(\nabla\cdot\vec{A})\,dv + \iiint_V \vec{A}\cdot(\nabla f)\,dv = \oiint_S f\vec{A}\cdot d\vec{s}$$

Choose surface where, $\vec{A}$ or f vanishes,
(large enough)

$$\Rightarrow \iiint_V f(\nabla\cdot\vec{A})\,dv = -\iiint_V \vec{A}\cdot(\nabla f)\,dv.$$

- $\vec{v} = \frac{1}{r^2}\hat{a}_r$, $\nabla\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(1) = 0.$ (?)

Suppose we integrate over a sphere of radius R,

$$\oiint \vec{v}\cdot d\vec{s} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \frac{1}{r^2}\, r^2 \sin\theta\, d\theta\, d\phi = 4\pi = \iiint(\nabla\cdot\vec{v})\,dv.$$

Carefully consider $\nabla\cdot\vec{v}$ at $r=0$, $\vec{v}$ blows up, so entire volume integral contributio
comes from $r=0$. $\Rightarrow$ concept of delta fn. (in spatial volume).

$\delta^3(\vec{r}) = \delta(x)\,\delta(y)\,\delta(z)$. $\iiint_V \delta^3(\vec{r})\,dv = 1$. $\iiint_V f(\vec{r})\,\delta^3(\vec{r}-\vec{a})\,dv = f(\vec{a})$.

$\nabla\cdot\left(\frac{1}{r^2}\hat{a}_r\right) = 4\pi\,\delta^3(\vec{r})$. $\nabla\left(\frac{1}{r}\right) = -\hat{a}_r\cdot\frac{1}{r^2}$.

Helmholtz Theorem

- Consider $\vec{F} = yz\,\hat{a}_x + zx\,\hat{a}_y + xy\,\hat{a}_z$

  $\nabla\cdot\vec{F} = 0, \quad \nabla\times\vec{F} = 0$

  Hence, $\vec{F}$ is unique w.r.t. $\nabla\cdot\vec{F}$ & $\nabla\times\vec{F}$ provided it satisfies appropriate boundary conditions. (B.C.)

- In electrodynamics, fields typically go to zero "at infinity". With this B.C., Helmholtz Th^m. guarantees a field being uniquely determined by its div. & curl.

- "Irrotational fields" : $\nabla\times\vec{F} = 0$; $\int_a^b \vec{F}\cdot\vec{dl}$ is independent of path, for any given end pts.;

  $\oint \vec{F}\cdot\vec{dl} = 0$; $\vec{F} = -\nabla V$, V is not unique , $V = \tilde{V} + C$
  (scalar potential)

- "Solenoidal fields" : $\nabla\cdot\vec{F} = 0$; $\iint_S \vec{F}\cdot\vec{da}$ is independent of surface, for any given boundary line ;

  $\oiint \vec{F}\cdot da = 0$; $\vec{F} = \nabla\times\vec{A}$; $\vec{A}$ is not unique, $\vec{A} = \tilde{\vec{A}} + \nabla f$
  (vector potential).

- Any vector field $\vec{F} = -\nabla V + \nabla\times\vec{A}$

Summary of vector

- $\vec{dl} = f\,du\,\hat{u} + g\,dv\,\hat{v} + h\,dw\,\hat{w}$

- $\nabla t = \frac{1}{f}\frac{\partial t}{\partial u}\hat{u} + \frac{1}{g}\frac{\partial t}{\partial v}\hat{v} + \frac{1}{h}\frac{\partial t}{\partial w}\hat{w}$

- $\nabla \cdot \vec{A} = \frac{1}{fgh}\left[\frac{\partial}{\partial u}(gh A_u) + \frac{\partial}{\partial v}(fh A_v) + \frac{\partial}{\partial w}(fg A_w)\right]$, where, $\vec{A} = A_u\hat{u} + A_v\hat{v} + A_w\hat{w}$

- $\nabla \times \vec{A} = \frac{1}{gh}\left[\frac{\partial}{\partial v}(h A_w) - \frac{\partial}{\partial w}(g A_v)\right]\hat{u} + \frac{1}{fh}\left[\frac{\partial}{\partial w}(f A_u) - \frac{\partial}{\partial u}(h A_w)\right]\hat{v}$

$$+ \frac{1}{fg}\left[\frac{\partial}{\partial u}(g A_w) - \frac{\partial}{\partial v}(f A_u)\right]\hat{w}$$

- $\nabla^2 t = \frac{1}{fgh}\left[\frac{\partial}{\partial u}\left(\frac{gh}{f}\frac{\partial t}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{fh}{g}\frac{\partial t}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{fg}{h}\frac{\partial t}{\partial w}\right)\right]$.

System	u v w	f	g	h	
Cartesian	x y z	1	1	1	
Cylindrical	ρ ϕ z	1	ρ	1	
Spherical	r θ ϕ	1	r	$r\sin\theta$	

True for any orthogonal systems !!