CHAPTER 1

P.E. 1.1

(a) $\vec{A}+\vec{B} = (1,0,3) + (5,2,-6) = (6,2,-3)$

$|\vec{A}+\vec{B}| = \sqrt{36+4+9} = \underline{\underline{7}}$

(b) $5\vec{A} - \vec{B} = (5,0,15) - (5,2,-6) = \underline{\underline{(0,-2,21)}}$

(c) The component of $\vec{A}$ along $\vec{a}_y$ is

$A_y = \underline{\underline{0}}$.

(d) $3\vec{A} + \vec{B} = (3,0,9) + (5,2,-6) = (8,2,3)$

A unit vector parallel to this vector is

$$\vec{a}_{\parallel} = \frac{(8,2,3)}{\sqrt{64+4+9}}$$

$$= \underline{\underline{\pm(0.9117\,\vec{a}_x + 0.2279\,\vec{a}_y + 0.3419\,\vec{a}_z)}}$$

P.E. 1.2

(a) The distance vector

$\vec{r}_{QR} = \vec{r}_R - \vec{r}_Q = (0,3,8) - (2,4,6)$

$= \underline{\underline{-2\vec{a}_x - \vec{a}_y + 2\vec{a}_z}}$

(b) The distance between Q and R is

$|\vec{r}_{QR}| = \sqrt{4+1+4} = \underline{\underline{3}}$

(c) Vector $\vec{r}_{QP} = \vec{r}_P - \vec{r}_Q = (1,-3,5) - (2,4,6) = (-1,-7,-1)$

$$\cos\theta_{PQR} = \frac{\vec{r}_{QR}\cdot\vec{r}_{QP}}{|\vec{r}_{QR}||\vec{r}_{QP}|} = \frac{7}{3\sqrt{51}}$$

$$\theta_{PQR} = \underline{\underline{70.93^\circ}}.$$

(d) $\text{Area} = \frac{1}{2}|\vec{r}_{QR}\times\vec{r}_{QP}| = \frac{1}{2}|(15,-4,13)|$

$$= \underline{\underline{10.12}}.$$

$\boxed{\text{P.E. 1.3}}$

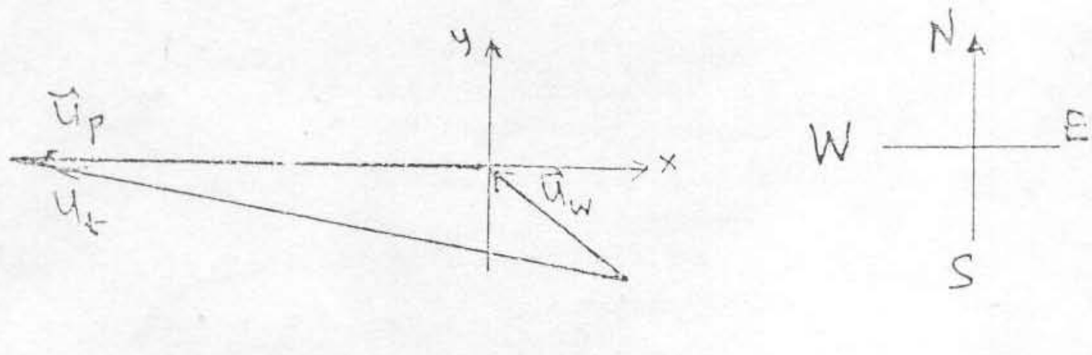
Consider the figure shown below.

$$\vec{u}_t = \vec{u}_p + \vec{u}_w = -350\vec{a}_x + \frac{40}{\sqrt{2}}(-\vec{a}_x + \vec{a}_y)$$

$$= -378\vec{a}_x + 28.28\vec{a}_y$$

or $u = 379.3 \angle -4.275^\circ$

i.e. 379.3 km/hr at 4.275° north of west.



$\boxed{\text{P.E. 1.4}}$

At point (1,0), $\vec{G} = \vec{a}_y$;
at point (0,1), $\vec{G} = -\vec{a}_x$;
at point (2,0), $\vec{G} = \vec{a}_y$;
at point (1,1), $\vec{G} = \dfrac{-\vec{a}_x + \vec{a}_y}{\sqrt{2}}$; and so on

It is evident that $\vec{G}$ is a unit vector at each point. Thus the vector field $\vec{G}$ is as sketched in fig. 1.8.

$\boxed{\text{P.E. 1.5}}$

Using dot product,

$$\cos\theta_{AB} = \frac{\vec{A}\cdot\vec{B}}{AB} = \frac{-13}{\sqrt{10}\sqrt{65}} = -\sqrt{\frac{13}{50}}$$

or using cross product,

$$\sin\theta_{AB} = \frac{|\vec{A}\times\vec{B}|}{AB} = \sqrt{\frac{481}{650}}$$

Either way, $\underline{\underline{\theta_{AB} = 120.56^\circ}}$

$\boxed{\text{P.E. 1.6}}$

(a) $\vec{E}_F = (\vec{E}\cdot\vec{a}_F)\vec{a}_F = \dfrac{(\vec{E}\cdot\vec{F})\vec{F}}{|\vec{F}|^2} = \dfrac{-10(4,-10,5)}{141}$

$$= \underline{\underline{-0.2837\vec{a}_x + 0.7092\vec{a}_y - 0.3546\vec{a}_z}}$$

(b) $\vec{E} \times \vec{F} = \begin{vmatrix} 0 & 3 & 4 \\ 4 & -10 & 5 \end{vmatrix} = (55, 16, -12)$

$$\vec{a}_{E \times F} = \pm \frac{(55, 16, -12)}{\sqrt{55^2 + 16^2 + 12^2}}$$

$$= \pm \underline{\underline{(0.9398, 0.2734, -0.205)}}.$$

$\boxed{\text{P.E. 1.7}}$

$$\vec{a} + \vec{b} + \vec{c} = 0$$

showing that $\vec{a}$, $\vec{b}$, and $\vec{c}$ form the sides of a triangle.

$$\vec{a} \cdot \vec{b} = 0,$$

hence it is a right angle triangle.

$$\text{Area} = \tfrac{1}{2} |\vec{a} \times \vec{b}| = \tfrac{1}{2} |\vec{b} \times \vec{c}| = \tfrac{1}{2} |\vec{c} \times \vec{a}|$$

$$\tfrac{1}{2} |\vec{a} \times \vec{b}| = \tfrac{1}{2} \begin{vmatrix} 4 & 0 & -1 \\ 1 & 3 & 4 \end{vmatrix} = \tfrac{1}{2} |(3, -17, 12)|$$

$$\text{Area} = \tfrac{1}{2} \sqrt{9 + 289 + 144} = \underline{\underline{10.51}}$$

$\boxed{\text{P.E. 1.8}}$

(a) $P_1 P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

$$= \sqrt{25 + 4 + 64} = \underline{\underline{9.644}}$$

(b) $\vec{r}_P = \vec{r}_{P_1} + \lambda (\vec{r}_{P_2} - \vec{r}_{P_1})$

$$= (1, 2, -3) + \lambda (-5, -2, 8)$$

$$= \underline{\underline{(1-5\lambda,\ 2-2\lambda,\ -3+8\lambda)}}.$$

(c) The shortest distance is

$$d = P_1P_3 \sin\theta = \left| \overrightarrow{P_1P_3} \times \vec{a}_{P_1P_2} \right|$$

$$= \frac{1}{\sqrt{93}} \begin{vmatrix} 6 & -3 & 5 \\ -5 & -2 & 8 \end{vmatrix}$$

$$= \frac{1}{\sqrt{93}} \left| (-14, -73, -27) \right| = \underline{\underline{8.2}}$$

$\boxed{\text{Prob. 1.1}}$

(a) $\vec{A} + 2\vec{B} = (2,5,-3) + (6,-8,0) = \underline{\underline{8\vec{a}_x - 3\vec{a}_y - 3\vec{a}_z}}$

(b) $\vec{A} - 5\vec{C} = (2,5,-3) - (5,5,5) = (-3, 0, -8)$

$|\vec{A} - 5\vec{C}| = \sqrt{9+0+64} = \underline{\underline{8.544}}$

(c) $k\vec{B} = 3k\vec{a}_x - 4k\vec{a}_y$

$|k\vec{B}| = \sqrt{9k^2 + 16k^2} = \pm 5k = 2$

i.e. $\underline{\underline{k = \pm 0.4}}$

(d) $\vec{A} \cdot \vec{B} = (2,5,-3) \cdot (3,-4,0) = 6-20+0 = 14$

$$\vec{A} \times \vec{B} = \begin{vmatrix} 2 & 5 & -3 \\ 3 & -4 & 0 \end{vmatrix} = (-12, -9, -23)$$

$$\frac{\vec{A} \times \vec{B}}{\vec{A} \cdot \vec{B}} = \left(\frac{12}{14}, \frac{9}{14}, \frac{23}{14} \right) = \underline{\underline{0.8571\vec{a}_x + 0.6428\vec{a}_y + 1.642\vec{a}_z}}$$

Prob. 1.2

(a) $\vec{A}\cdot\vec{B} = 6+5+20 = \underline{\underline{31}}$

(b) $\vec{A}\times\vec{B} = \begin{vmatrix} 2 & 5 & 4 \\ 3 & 1 & 5 \end{vmatrix} = \underline{\underline{21\vec{a}_x + 2\vec{a}_y - 13\vec{a}_z}}$

(c) $\vec{A}\cdot(\vec{B}-\vec{C}) = (2,5,4)\cdot(2,1,11) = \underline{\underline{53}}$

(d) $\vec{A}\times(\vec{B}+\vec{C}) = \begin{vmatrix} 2 & 5 & 4 \\ 4 & 1 & -1 \end{vmatrix} = \underline{\underline{-9\vec{a}_x + 18\vec{a}_y - 18\vec{a}_z}}$

Prob. 1.3

(a) $\vec{P}\cdot\vec{Q} = 3-10+12 = \underline{\underline{5}}$

(b) $\cos\theta_{PQ} = \dfrac{\vec{P}\cdot\vec{Q}}{PQ} = \dfrac{5}{\sqrt{35}\sqrt{29}} = \underline{\underline{0.1569}}$

(c) $\vec{Q}\times\vec{R} = \begin{vmatrix} 3 & 2 & 4 \\ 1 & -1 & 0 \end{vmatrix} = \underline{\underline{4\vec{a}_x + 4\vec{a}_y - 5\vec{a}_z}}$

(d) $\sin\theta_{PQ} = \dfrac{|\vec{Q}\times\vec{R}|}{QR} = \dfrac{\sqrt{57}}{\sqrt{2}\sqrt{29}} = \underline{\underline{0.9913}}$

(e) $\vec{P}\cdot(\vec{Q}\times\vec{R}) = (1,-5,3)\cdot(4,4,-5) = 4-20-15 = \underline{\underline{-31}}$

(f) $\vec{Q}\cdot(\vec{R}\times\vec{P}) = \vec{P}\cdot(\vec{Q}\times\vec{R}) = \underline{\underline{-31}}$

(g) $\vec{P}\times(\vec{Q}\times\vec{R}) = \begin{vmatrix} 1 & -5 & 3 \\ 4 & 4 & -5 \end{vmatrix} = \underline{\underline{13\vec{a}_x + 17\vec{a}_y + 24\vec{a}_z}}$

(h) $\vec{P}\times\vec{Q} = \begin{vmatrix} 1 & -5 & 3 \\ 3 & 2 & 4 \end{vmatrix} = (-26, 5, 17)$

$(\vec{P}\times\vec{Q})\times\vec{R} = \begin{vmatrix} -26 & 5 & 17 \\ 1 & -1 & 0 \end{vmatrix} = \underline{\underline{17\vec{a}_x + 17\vec{a}_y + 21\vec{a}_z}}$

Prob. 1.4

$\vec{A}\cdot\vec{B} = 0 \longrightarrow 4\alpha^2 + 16 - 20\alpha = 0$

$$\alpha^2 - 5\alpha + 4 = 0$$

$$\alpha = \frac{5 \pm \sqrt{25-16}}{2} = \underline{\underline{1, 4}}.$$

Prob. 1.5

If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are mutually orthogonal,

$$\vec{u}\cdot\vec{v} = 0 \longrightarrow 2u_x - 5v_y - 3 = 0 \quad — (1)$$

$$\vec{u}\cdot\vec{w} = 0 \longrightarrow 6u_x + 5 - w_z = 0 \quad — (2)$$

$$\vec{v}\cdot\vec{w} = 0 \longrightarrow 12 - v_y + 3w_z = 0 \quad — (3)$$

Solving (1) to (3) gives

$$\underline{\underline{u_x = -\frac{69}{44}, \quad v_y = -\frac{27}{22}, \quad w_z = -\frac{97}{22}}}$$

Prob. 1.6

(a) $T_s = \vec{T}\cdot\vec{a}_s = \dfrac{\vec{T}\cdot\vec{S}}{S} = \dfrac{(2,-6,-3)\cdot(1,2,1)}{\sqrt{6}}$

$$= \frac{-7}{\sqrt{6}} = \underline{\underline{-2.8577}}$$

(b) $\vec{S}_T = (\vec{S}\cdot\vec{a}_T)\vec{a}_T = \dfrac{(\vec{S}\cdot\vec{T})\vec{T}}{T^2} = \dfrac{-7(2,-6,3)}{(7)^2}$

$$= \underline{\underline{-0.2857\vec{a}_x + 0.8571\vec{a}_y - 0.4286\vec{a}_z}}$$

(c) $\sin\theta_{TS} = \dfrac{|\vec{T}\times\vec{S}|}{TS} = \begin{vmatrix} 2 & -6 & 3 \\ 1 & 2 & 1 \end{vmatrix} = \dfrac{|(-12, 1, 10)|}{7\sqrt{6}}$

$$= \frac{\sqrt{245}}{7\sqrt{6}} = 0.9129$$

$$\theta_{TS} = \underline{\underline{65.91^\circ}}$$

$\boxed{\text{Prob. 1.7}}$

Let $\vec{P} = \vec{A}\times\vec{B} = \begin{vmatrix} 2 & 3 & -4 \\ -6 & -4 & 1 \end{vmatrix} = (-13, 22, 10)$

Scalar component $= \vec{P}\cdot\vec{a}_c = \dfrac{\vec{P}\cdot\vec{C}}{C} = \dfrac{-25}{\sqrt{3}}$

$$= \underline{\underline{-14.43}}$$

Vector component $= (\vec{P}\cdot\vec{a}_c)\,\vec{a}_c = -\dfrac{25}{\sqrt{3}}\,\dfrac{(1,-1,1)}{\sqrt{3}}$

$$= \underline{\underline{-8.33\,\bar{a}_x + 8.33\,\bar{a}_y - 8.33\,\bar{a}_z}}$$

$\boxed{\text{Prob. 1.8}}$

$\cos\theta_x = \dfrac{\bar{H}\cdot\bar{a}_x}{H} = \dfrac{3}{\sqrt{9+25+64}} = \dfrac{3}{\sqrt{98}}$

$$\underline{\underline{\theta_x = 72.36^\circ}}$$

$\cos\theta_y = \dfrac{\bar{H}\cdot\bar{a}_y}{H} = \dfrac{5}{\sqrt{98}}$

$$\underline{\underline{\theta_y = 59.66^\circ}}$$

$\cos\theta_z = \dfrac{\bar{H}\cdot\bar{a}_z}{H} = -\dfrac{8}{\sqrt{98}} \longrightarrow \underline{\underline{\theta_z = 143.91^\circ}}$

Prob. 1.9

$$\vec{Q} \times \vec{R} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 3 \end{vmatrix} = (3, -1, -2)$$

$$\vec{P} \cdot (\vec{Q} \times \vec{R}) = (2, -1, 1) \cdot (3, -1, 2) = 6 + 1 - 2 = \underline{\underline{5}}$$

Prob. 1.10

(a) method 1: $\vec{A} \cdot (\vec{A} \times \vec{B}) = \begin{vmatrix} A_x & A_y & A_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = 0$

Since two rows are repeated. For the same reason, $\vec{B} \cdot (\vec{A} \times \vec{B}) = 0$.

method 2:

$$\vec{A} \cdot (\vec{A} \times \vec{B}) = A_x (A_y B_z - A_z B_y) + A_y (A_z B_x - A_x B_z) + A_z (A_x B_y - A_y B_x) = 0$$

The same idea applies for $\vec{B} \cdot (\vec{A} \times \vec{B}) = 0$.

method 3:

Since $\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{C} \cdot (\vec{A} \times \vec{B})$,

$$\vec{A} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{A} \times \vec{A}) = 0$$

$$\vec{B} \cdot (\vec{A} \times \vec{B}) = \vec{A} \times (\vec{B} \times \vec{B}) = 0$$

(b) $\vec{A} \cdot \vec{B} = AB \cos \theta_{AB}$, $|\vec{A} \times \vec{B}| = AB \sin \theta_{AB}$

$$|\vec{A}\cdot\vec{B}|^2 + |\vec{A}\times\vec{B}|^2 = (AB)^2(\cos^2\theta_{AB} + \sin^2\theta_{AB})$$
$$= (AB)^2$$

(c) Since $A_x = \vec{A}\cdot\vec{a}_x$, $A_y = \vec{A}\cdot\vec{a}_y$ and $A_z = \vec{A}\cdot\vec{a}_z$,

$$\vec{A} = A_x\vec{a}_x + A_y\vec{a}_y + A_z\vec{a}_z$$
$$= (A_x\cdot\vec{a}_x)\vec{a}_x + (\vec{A}\cdot\vec{a}_y)\vec{a}_y + (\vec{A}\cdot\vec{a}_z)\vec{a}_z.$$

Prob. 1.11

$$\overrightarrow{P_1P_2} = \vec{r}_{P_2} - \vec{r}_{P_1} = (-6, 0, -3)$$

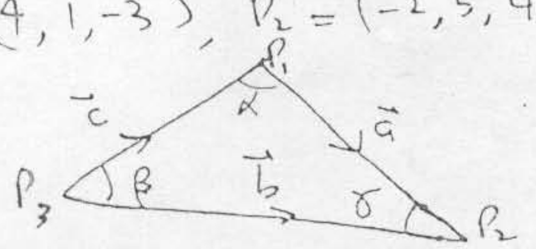
$$\overrightarrow{P_1P_3} = \vec{r}_{P_3} - \vec{r}_{P_1} = (1, 5, -6)$$

$$\overrightarrow{P_1P_2}\times\overrightarrow{P_1P_3} = \begin{vmatrix} -6 & 0 & -3 \\ 1 & 5 & -6 \end{vmatrix} = (15, 39, -30)$$

$$\text{Area of the triangle} = \frac{1}{2}\left|\overrightarrow{P_1P_2}\times\overrightarrow{P_1P_3}\right|$$
$$= \frac{1}{2}\sqrt{15^2 + 39^2 + 30^2}$$
$$= \underline{\underline{25.72}}$$

Prob. 1.12

Let $P_1 = (4, 1, -3)$, $P_2 = (-2, 5, 4)$, and $P_3 = (0, 1, 6)$



$$\vec{a} = \vec{r}_{p_2} - \vec{r}_{p_1} = (-2,5,4) - (4,1,-3) = (-6,4,7)$$

$$\vec{b} = \vec{r}_{p_3} - \vec{r}_{p_2} = (0,1,6) - (-2,5,4) = (2,-4,2)$$

$$\vec{c} = \vec{r}_{p_1} - \vec{r}_{p_3} = (4,1,-3) - (0,1,6) = (4,0,-9)$$

Note that $\vec{a} + \vec{b} + \vec{c} = 0$.

$$\vec{a}\cdot\vec{b} = ab\cos(180-\gamma) \rightarrow -\cos\gamma = \frac{\vec{a}\cdot\vec{b}}{ab}$$

$$= \frac{-12-16+14}{\sqrt{101}\,\sqrt{24}}$$

$$\gamma = \cos^{-1}\frac{14}{\sqrt{101}\,\sqrt{24}} = \underline{\underline{73.47^\circ}}$$

$$\vec{b}\cdot\vec{c} = bc\cos(180-\beta) \rightarrow -\cos\beta = \frac{\vec{b}\cdot\vec{c}}{bc}$$

$$= \frac{8+0-18}{\sqrt{24}\,\sqrt{97}}$$

$$\beta = \cos^{-1}\frac{10}{\sqrt{24}\,\sqrt{97}} = \underline{\underline{78.04^\circ}}$$

$$\vec{a}\cdot\vec{c} = ac\cos(180-\alpha) \rightarrow -\cos\alpha = \frac{\vec{a}\cdot\vec{c}}{ac}$$

$$= \frac{-24+0-63}{\sqrt{101}\,\sqrt{97}}$$

$$\alpha = \cos^{-1}\frac{87}{\sqrt{101}\,\sqrt{97}} = \underline{\underline{28.48^\circ}}$$

Prob. 1.13

(a) Let $\vec{A} = (A, B, C)$ and $\vec{r} = (x, y, z)$

$$(\vec{r}-\vec{A})\cdot\vec{A} = (x-A)A + (y-B)B + (z-C)C$$
$$= Ax + By + Cz + D$$

where $D = -A^2 - B^2 - C^2$. Hence

$$(\vec{r}-\vec{A})\cdot\vec{A} = 0 \longrightarrow Ax + By + Cz + D = 0$$

which is the equation of a plane.

(b) $$(\vec{r}-\vec{A})\cdot\vec{r} = (x-A)x + (y-B)y + (z-C)z$$
$$= x^2 + y^2 + z^2 - Ax - By - Cz$$

If $(\vec{r}-\vec{A})\cdot\vec{r} = 0$, then

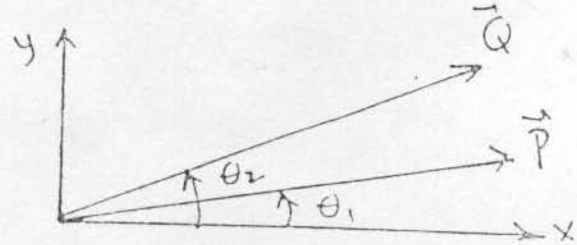
$$x^2 + y^2 + z^2 - Ax - By - Cz = 0$$

which is the equation of a sphere whose surface touches the origin.

(c) See parts (a) and (b).

Prob. 1.14

(a) Let $\vec{P}$ and $\vec{Q}$ be as shown below.



$|\vec{P}| = \cos^2\theta_1 + \sin^2\theta = 1$, $|\vec{Q}| = \cos^2\theta_2 + \sin^2\theta_2 = 1$

Hence $\vec{P}$ and $\vec{Q}$ are unit vectors.

(b) $\vec{P}\cdot\vec{Q} = (1)(1)\cos(\theta_2 - \theta_1)$

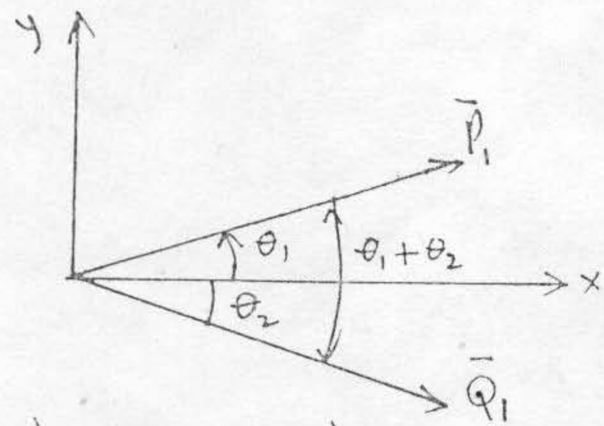
But $\vec{P}\cdot\vec{Q} = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2$. Thus

$$\underline{\underline{\cos(\theta_2 - \theta_1) = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2}}$$

Let $\vec{P}_1 = \vec{P} = \cos\theta_1\, a_x + \sin\theta_1\, \vec{a}_y$ and

$\vec{Q}_1 = \cos\theta_2\, \vec{a}_x - \sin\theta_2\, \vec{a}_y$,

$\vec{P}_1$ and $\vec{Q}_1$ are unit vectors shown below



$\vec{P}_1\cdot\vec{Q}_1 = (1)(1)\cos(\theta_1 + \theta_2)$

But $\vec{P}_1\cdot\vec{Q}_1 = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$, thus

$$\underline{\underline{\cos(\theta_2 + \theta_1) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2}}$$

Alternatively, we can obtain this formula from the previous one by replacing θ_2 by $-\theta_2$ in $\vec{Q}$.

(c) $\frac{1}{2}|\vec{P}-\vec{Q}| = \frac{1}{2}\Big| (\cos\theta_1 - \cos\theta_2)\,\vec{a}_x + (\sin\theta_1 - \sin\theta_2)\,\vec{a}_y \Big|$

$$= \frac{1}{2}\sqrt{\cos^2\theta_1 + \sin^2\theta_1 + \cos^2\theta_2 + \sin^2\theta_2 - 2\cos\theta_1\cos\theta_2 - 2\sin\theta_1\sin\theta_2}$$

$$= \frac{1}{2}\sqrt{2 - 2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)}$$

$$= \frac{1}{2}\sqrt{2 - 2\cos(\theta_2 - \theta_1)}$$

Let $\theta_2 - \theta_1 = \theta$, the angle between $\hat{P}$ and $\vec{Q}$.

$$\frac{1}{2}|\vec{P}-\vec{Q}| = \frac{1}{2}\sqrt{2 - 2\cos\theta}$$

But $\cos 2A = 1 - 2\sin^2 A$

$$\frac{1}{2}|\hat{P}-\vec{Q}| = \frac{1}{2}\sqrt{2 - 2 + 4\sin^2\theta/2} = \sin\theta/2.$$

$$\therefore \quad \frac{1}{2}|\hat{P}-\vec{Q}| = \left|\sin\frac{\theta_2-\theta_1}{2}\right|.$$

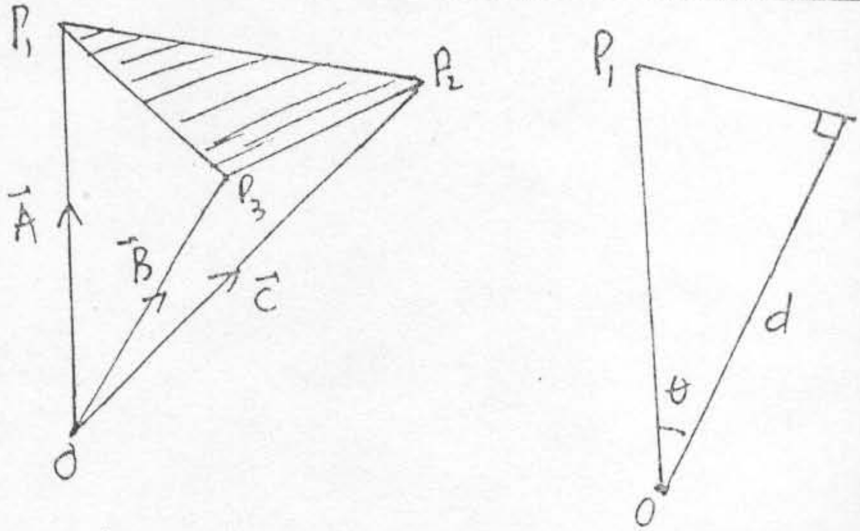
Prob. 1.15

(a) Consider the figure on the next page.

$$\overrightarrow{P_2P_1} = \vec{A} - \vec{B}, \quad \overrightarrow{P_3P_1} = \vec{A} - \vec{C}$$

A unit normal to the plane is

$$\vec{a}_n = \frac{\overrightarrow{P_2P_1} \times \overrightarrow{P_3P_1}}{|\overrightarrow{P_2P_1} \times \overrightarrow{P_3P_1}|} = \frac{(\vec{A}-\vec{B}) \times (\vec{A}-\vec{C})}{|(\vec{A}-\vec{B}) \times (\vec{A}-\vec{C})|}$$



The shortest distance from plane $P_1P_2P_3$ is the perpendicular distance from the plane to the origin, i.e.

$$d = OP_1 \cos\theta = \vec{A}\cdot\vec{a}_n = \vec{B}\cdot\vec{a}_n = \vec{C}\cdot\vec{a}_n.$$

Hence

$$d = \frac{\vec{A}\cdot\left[(\vec{A}-\vec{B})\times(\vec{A}-\vec{C})\right]}{\left|(\vec{A}-\vec{B})\times(\vec{A}-\vec{C})\right|}$$

as required.

(b) In (a), it is shown that a vector normal to the plane is

$$\vec{N} = (\vec{A}-\vec{B})\times(\vec{A}-\vec{C})$$
$$= \underbrace{\vec{A}\times\vec{A}}_{0} - \vec{A}\times\vec{C} - \vec{B}\times\vec{A} + \vec{B}\times\vec{C}$$
$$= \vec{A}\times\vec{B} + \vec{B}\times\vec{C} + \vec{C}\times\vec{A}$$

(c) For this case,

$$\vec{A}-\vec{B} = (3,-3,-3), \quad \vec{A}-\vec{C} = (0,-2,5)$$

$$(\vec{A}-\vec{B}) \times (\vec{A}-\vec{C}) = \begin{vmatrix} 3 & -3 & -3 \\ 0 & -2 & 5 \end{vmatrix} = -3(7,5,2)$$

$$d = \frac{(2,1,0)\cdot(-21,-15,-6)}{\left|21^2+15^2+6^2\right|^{1/2}} = \frac{-57}{\sqrt{702}}$$

$$= \underline{\underline{-2.151}}$$

$\boxed{\text{Prob. 1.16}}$

$$\vec{w} = \frac{w(1,-2,2)}{3} = (1,-2,2)$$

$$\vec{r} = \vec{r}_p - \vec{r}_0 = (1,3,4)-(2,-3,1) = (-1,6,3)$$

$$\vec{u} = \vec{w}\times\vec{r} = \begin{vmatrix} 1 & -2 & 2 \\ -1 & 6 & 3 \end{vmatrix} = (-18,-5,4)$$

$$\underline{\underline{\vec{u} = -18\vec{a}_x - 5\vec{a}_y + 4\vec{a}_z}}$$

$\boxed{\text{Prob. 1.17}}$

Let $\vec{A} = (10,-5,4)$, $\vec{B} = (3,1,0)$

$$\vec{A}_B = A\cos\theta_{AB}\,\vec{a}_B = \vec{A}\cdot\frac{\vec{B}}{B}\,\frac{\vec{B}}{B} = \frac{(\vec{A}\cdot\vec{B})\vec{B}}{B^2}$$

$$= \frac{(25)(3,1,0)}{10} = \underline{\underline{7.5\vec{a}_x + 2.5\vec{a}_y}}$$

$$A_\perp = \vec{A} - \vec{A}_B = \underline{\underline{2.5\vec{a}_x - 7.5\vec{a}_y + 4\vec{a}_z}}.$$

Prob. 1.18

(a) At T, $\vec{A} = (-4, 3, -9)$

$$|\vec{A}| = \sqrt{16+9+81} = \sqrt{106} = \underline{\underline{10.3}}$$

(b) Let $\vec{r}_{TS} = \vec{B} = B\,\vec{a}_B$

$$B = 5.6, \quad \vec{a}_B = \vec{a}_A = \frac{(-4,3,-9)}{10.3}$$

$$\vec{r}_{TS} = \vec{B} = \frac{5.6(-4,3,9)}{10.3}$$

$$= \underline{\underline{-2.175\,\vec{a}_x + 1.631\,\vec{a}_y - 4.893\,\vec{a}_z}}$$

(c) $\vec{r}_{TS} = \vec{r}_S - \vec{r}_T \longrightarrow \vec{r}_S = \vec{r}_T + \vec{r}_{TS}$

$$\therefore \vec{r}_S = \underline{\underline{-0.175\,\vec{a}_x + 0.631\,\vec{a}_y - 1.893\,\vec{a}_z}}$$

Prob. 1.19

(a) At $P(1,2,1)$, $\vec{Q} = (0,9,2)$ so that

$$\vec{a}_Q = \frac{\vec{Q}}{Q} = \frac{(0,9,2)}{\sqrt{85}} = \underline{\underline{(0, 0.9762, 0.2169)}}$$

(b) The component of $\vec{Q}$ along PT is

$$\vec{Q}_{PT} = |\vec{Q}|\cos\theta_{PT}\;\vec{a}_{PT} = (\vec{Q}\cdot\vec{a}_{PT})\,\vec{a}_{PT}$$

But $\vec{PT} = \vec{r}_T - \vec{r}_P = (5,3,-4) - (1,2,1)$

$$= (4,1,5)$$

$$\vec{a}_{PT} = \frac{(4,1,-5)}{\sqrt{42}}$$

Hence $$\bar{Q}_{PT} = \frac{[(0,9,2)\cdot(4,1,-5)]}{\sqrt{42}} \frac{(4,1,-5)}{\sqrt{42}}$$

$$= \frac{(-1)(4,1,-5)}{42}$$

$$= \underline{\underline{-0.09523\,\vec{a}_x - 0.02381\,\vec{a}_y + 0.119\,\vec{a}_z}}$$

(c) When $\vec{Q} = (1, 11, 10)$.

$$2x - y = 1 \quad \text{— (1)}$$
$$4y + z = 11 \quad \text{— (2)}$$
$$4x - 2z = 10 \quad \text{— (3)}$$

Solving for x, y, and z in (1) to (3) gives

$$(x, y, z) = \underline{\underline{(2, 3, -1)}}.$$

$\boxed{\text{Prob. 1.20}}$

(a) At $(1,2,3)$, $\bar{E} = (2, 1, 6)$

$$|\bar{E}| = \sqrt{4+1+36} = \sqrt{41} = \underline{\underline{6.403}}$$

(b) At $(1,2,3)$, $\vec{F} = (2, -4, 6)$

$$\hat{E}_F = (\vec{E}\cdot\vec{a}_F)\,\vec{a}_F = \frac{(\vec{E}\cdot\vec{F})\,\vec{F}}{F^2} = \frac{36}{56}(2,-4,6)$$

$$= \underline{\underline{1.286\,\vec{a}_x - 2.571\,\vec{a}_y + 3.857\,\vec{a}_z}}$$

(c) At $(0,1,-3)$, $\vec{E} = (0,1,-3)$, $\vec{F} = (0,-1,0)$

$$\vec{E}\times\vec{F} = \begin{vmatrix} 0 & 1 & -3 \\ 0 & -1 & 0 \end{vmatrix} = (-3,0,0)$$

$$\vec{a}_{E\times F} = \pm \frac{\vec{E}\times\vec{F}}{|E\times F|} = \underline{\underline{\pm\, \vec{a}_x}}$$

Prob. 1.21

In the bac-cab rule,

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B}),$$

let $\vec{A} = \vec{C}$ be replaced by $\vec{a}_B$ and $\vec{B}$ by $\vec{A}$., We obtain

$$\vec{a}_B \times (\vec{A}\times\vec{a}_B) = \vec{A} - \vec{a}_B(\vec{a}_B\cdot\vec{A})$$

or

$$\vec{A} = (\vec{A}\cdot\vec{a}_B)\,\vec{a}_B + \vec{a}_B\times(\vec{A}\times\vec{a}_B).$$

CHAPTER 2

P.E. 2.1

(a) At $P(1,3,5)$, $x=1$, $y=3$, $z=5$,

$$\rho = \sqrt{x^2+y^2} = \sqrt{10},\quad z=5,\quad \phi = \tan^{-1}\frac{y}{x} = 3$$

$$P(\rho,\phi,z) = P(\sqrt{10},\tan^{-1}3,5) = \underline{\underline{P(3.162, 71.6^\circ, 5)}}.$$

$r = \sqrt{x^2+y^2+z^2} = \sqrt{35} = 5.916,$

$\theta = \tan^{-1}\dfrac{\sqrt{x^2+y^2}}{z} = \tan^{-1}\dfrac{\sqrt{10}}{5} = \tan^{-1} 0.6325 = 32.31$

$P(r,\theta,\phi) = \underline{\underline{P(5.916, 32.31^\circ, 71.56^\circ)}}$

At $T(0,-4,3)$, $x=0$, $y=-4$, $z=3$,

$\rho = \sqrt{x^2+y^2} = 4$, $z = 3$, $\phi = \tan^{-1}\dfrac{y}{x} = \tan^{-1}\dfrac{-4}{0} = 270^\circ$

$T(\rho,\phi,z) = \underline{\underline{T(4, 270^\circ, 3)}}.$

$r = \sqrt{x^2+y^2+z^2} = 5$, $\theta = \tan^{-1}\dfrac{\rho}{z} = \tan^{-1}\dfrac{4}{3} = 53.13^\circ$

$T(r,\theta,\phi) = \underline{\underline{T(5, 53.13^\circ, 270^\circ)}}.$

At $S(-3,-4,-10)$, $x=-3$, $y=-4$, $z=-10$,

$\rho = \sqrt{x^2+y^2} = 5$, $\phi = \tan^{-1}\dfrac{-4}{-3} = 233.1^\circ$

$S(\rho,\phi,z) = \underline{\underline{S(5, 233.1^\circ, -10)}}.$

$r = \sqrt{x^2+y^2+z^2} = 5\sqrt{5} = 11.18,$

$\theta = \tan^{-1}\dfrac{\rho}{z} = \tan^{-1}\dfrac{5}{-10} = 153.43^\circ$

$S(r,\theta,\phi) = \underline{\underline{S(11.18, 153.43^\circ, 233.1^\circ)}}.$

(b) In cylindrical system,

$\sqrt{x^2+y^2} = \rho$, $yz = z\rho\sin\phi,$

$$Q_x = \frac{\rho}{\sqrt{\rho^2+z^2}}, \quad Q_y = 0, \quad Q_z = -\frac{z\rho\sin\phi}{\sqrt{\rho^2+z^2}}$$

$$\begin{bmatrix} Q_\rho \\ Q_\phi \\ Q_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Q_x \\ 0 \\ Q_z \end{bmatrix}$$

$$Q_\rho = Q_x\cos\phi = \frac{\rho\cos\phi}{\sqrt{\rho^2+z^2}}, \quad Q_\phi = -Q_x\sin\phi = \frac{-\rho\sin\phi}{\sqrt{\rho^2+z^2}}$$

Hence

$$\vec{Q} = \frac{\rho}{\sqrt{\rho^2+z^2}}\left(\cos\phi\,\vec{a}_\rho - \sin\phi\,\vec{a}_\phi - z\sin\phi\,\vec{a}_z\right).$$

In spherical coordinates,

$$Q_x = \frac{r\sin\phi}{r} = \sin\phi, \quad Q_z = \frac{-r\sin\phi\sin\theta\, r\cos\theta}{r}$$
$$= -r\sin\theta\cos\theta\sin\phi$$

$$\begin{bmatrix} Q_r \\ Q_\theta \\ Q_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} Q_x \\ 0 \\ Q_z \end{bmatrix}$$

$$Q_r = Q_x\sin\theta\cos\phi + Q_z\cos\theta$$
$$= \sin^2\theta\cos\phi - r\sin\theta\cos^2\theta\sin\phi$$

$$Q_\theta = Q_x\cos\theta\cos\phi - Q_z\sin\theta$$
$$= \sin\theta\cos\theta\cos\phi + r\sin^2\theta\cos\theta\sin\phi$$

$$Q_\phi = -Q_x \sin\phi = -\sin\theta \sin\phi$$

$$\vec{Q} = \sin\theta\,(\sin\theta\cos\phi - r\cos^2\theta\sin\phi)\,\vec{a}_r$$
$$+ \sin\theta\cos\phi\,(\cos\phi + r\sin\theta\sin\phi)\,\vec{a}_\theta$$
$$- \sin\theta\sin\phi\,\vec{a}_\phi$$

At T,

$$\vec{Q}(x,y,z) = \frac{4}{5}\vec{a}_x + \frac{12}{5}\vec{a}_z = \underline{\underline{0.8\vec{a}_x + 2.4\,a_z}},$$

$$\vec{Q}(\rho,\phi,z) = \frac{4}{5}\left(\cos 270^\circ\,\vec{a}_\rho - \sin 270^\circ\,\vec{a}_\phi - 3\sin 270\,\vec{a}_z\right)$$
$$= \underline{\underline{0.8\vec{a}_\phi + 2.4\vec{a}_z}},$$

$$\vec{Q}(r,\theta,\phi) = \frac{4}{5}\left(0 - \frac{45}{25}(-1)\right)\vec{a}_r + \frac{4}{5}\left(\frac{3}{5}\right)\left(0 - \frac{20}{5}(-1)\right)\vec{a}_\theta$$
$$- \frac{4}{5}(-1)\,\vec{a}_\phi = \frac{36}{25}\vec{a}_r + \frac{48}{25}\vec{a}_\theta + \frac{4}{5}\vec{a}_\phi$$
$$= \underline{\underline{1.44\,\vec{a}_r + 1.92\,\vec{a}_\theta + 0.8\,\vec{a}_\phi}}.$$

Note that $|\vec{Q}| = 2.53$ in the three cases.

$\boxed{\text{P.E. 2.2}}$

(a)
$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \rho z\sin\phi \\ 3\rho\cos\phi \\ \rho\cos\phi\sin\phi \end{bmatrix}$$

$$\vec{A} = (\rho z\cos\phi\sin\phi - 3\rho\cos\phi\sin\phi)\,\vec{a}_x$$
$$+ (\rho z\sin^2\phi + 3\rho\cos^2\phi)\,\vec{a}_y + \rho\cos\phi\sin\phi\, a_z.$$

But $\rho = \sqrt{x^2+y^2}$, $\tan\phi = \frac{y}{x} \rightarrow \cos\phi = \frac{x}{\sqrt{x^2+y^2}}$, $\sin\phi = \frac{y}{\sqrt{x^2+y^2}}$. Substituting all this yields

$$\vec{A} = \frac{1}{\sqrt{x^2+y^2}}\left[(xyz - 3xy)\,\bar{a}_x + (zy^2+3x^2)\,\bar{a}_y + xy\,\bar{a}_z\right].$$

(b) $$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\theta \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\theta \\ \cos\theta & -\sin\theta & 0 \end{bmatrix}\begin{bmatrix} r^2 \\ 0 \\ \sin\theta \end{bmatrix}$$

Since $r = \sqrt{x^2+y^2+z^2}$, $\tan\theta = \frac{\sqrt{x^2+y^2}}{z}$, $\tan\phi = \frac{y}{x}$,

$$\sin\theta = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}, \quad \cos\theta = \frac{z}{\sqrt{x^2+y^2+z^2}},$$

$$\sin\phi = \frac{y}{\sqrt{x^2+y^2}}, \quad \cos\phi = \frac{x}{\sqrt{x^2+y^2}}$$

$$B_x = r^2\sin\theta\cos\phi - \sin\theta\sin\phi = rx - \frac{y}{r} = \frac{1}{r}(r^2x - y)$$

$$B_y = r^2\sin\theta\sin\phi + \cos\phi\sin\theta = ry + \frac{x}{r} = \frac{1}{r}(r^2y + x)$$

$$B_z = r^2\cos\theta = rz = \frac{1}{r}(zr^2).$$ Hence

$$\vec{B} = \frac{1}{\sqrt{x^2+y^2+z^2}}\left[\{x(x^2+y^2+z^2) - y\}\,\bar{a}_x + \{y(x^2+y^2+z^2)+x\}\,a_y + z(x^2+y^2+z^2)\,\bar{a}_z\right].$$

$\boxed{P.E.2.3}$

(a) At $(1, \pi/3, 0)$, $\vec{H} = (0, 0.5, 1)$

$$\vec{a}_x = \cos\phi\,\vec{a}_\rho - \sin\phi\,\vec{a}_\phi = \frac{1}{2}(\vec{a}_\rho - \sqrt{3}\,\vec{a}_\phi)$$

$$\vec{H}\cdot\vec{a}_x = -\frac{\sqrt{3}}{4} = \underline{\underline{-0.433}}.$$

(b) At $(1, \pi/3, 0)$, $\vec{a}_\theta = \cos\theta\,\vec{a}_\rho - \sin\theta\,a_z = -a_z$

$$\vec{H}\times\vec{a}_\theta = \begin{vmatrix} 0 & \frac{1}{2} & 1 \\ 0 & 0 & -1 \end{vmatrix} = \underline{\underline{-0.5\,\vec{a}_\rho}}.$$

(c) $(\vec{H}\cdot\vec{a}_\rho)\,\vec{a}_\rho = \underline{\underline{0\,\vec{a}_\rho}}.$

(d) $$\vec{H}\times\vec{a}_z = \begin{vmatrix} 0 & \frac{1}{2} & 1 \\ 0 & 0 & 1 \end{vmatrix} = 0.5\,\vec{a}_\rho$$

$$|\vec{H}\times\vec{a}_z| = \underline{\underline{0.5}}.$$

$\boxed{P.E.2.4}$

(a) $\vec{A}\cdot\vec{B} = (3, 2, -6)\cdot(4, 0, 3) = \underline{\underline{-6}}.$

(b) $$|\vec{A}\times\vec{B}| = \begin{vmatrix} 3 & 2 & -6 \\ 4 & 0 & 3 \end{vmatrix} = |6\,\vec{a}_r - 33\,\vec{a}_\theta - 8\,\vec{a}_\phi|$$
$$= \underline{34.48}.$$

(c) At $(1, \pi/3, 5\pi/4)$, $\theta = \pi/3$,

$$\vec{a}_z = \cos\theta\,\vec{a}_r - \sin\theta\,\vec{a}_\theta = \frac{1}{2}\vec{a}_r - \frac{\sqrt{3}}{2}\vec{a}_\theta$$

$$(\vec{A}\cdot\vec{a}_z)\vec{a}_z = \left(\frac{3}{2} - \sqrt{3}\right)\left(\frac{1}{2}\vec{a}_r - \frac{\sqrt{3}}{2}\vec{a}_\theta\right)$$
$$= \underline{\underline{-0.116\,\vec{a}_r + 0.201\,\vec{a}_\theta}}$$

Prob. 2.1

(a) $x = \rho\cos\phi = 5\cos 120° = -2.5$

$y = \rho\sin\phi = 5\sin 120° = 4.33, \quad z = 0$

$P_1(x, y, z) = \underline{\underline{P_1(-2.5, 4.33, 0)}}.$

(b) $x = 1\cos 30° = 0.866, y = 1\sin 30° = 0.5, \quad z = 10$

$P_2(x, y, z) = \underline{\underline{P_2(0.866, 0.5, -10)}}.$

(c) $x = r\sin\theta\cos\phi = 10\sin 135°\cos 90° = 0$

$y = r\sin\theta\sin\phi = 10\sin 135°\sin 90° = 7.071$

$z = r\cos\theta = 10\cos 135° = -7.071$

$P_3(x, y, z) = \underline{\underline{P_3(0, 7.071, -7.071)}}$

(d) $x = 3\sin 30°\cos 240° = -0.75$

$y = 3\sin 30°\sin 240° = -1.299$

$z = r\cos\theta = 3\cos 30° = 2.598$

$P_4(x, y, z) = \underline{\underline{P_4(-0.75, -1.299, 2.598)}}.$

Prob. 2.2

(a) $\rho = \sqrt{x^2 + y^2} = \sqrt{17} = 4.123, \ \phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{-4}{1}$
$= 284.04°$

$P(\rho, \phi, z) = \underline{\underline{P(4.123, 284.04°, -3)}}.$

$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{1 + 16 + 9} = 5.099, \ \theta = \tan^{-1}\frac{\rho}{z}$
$= \tan^{-1}\frac{\sqrt{17}}{-3} = 126.04°$

$P(r,\theta,\phi) = \underline{\underline{P(5.099,\ 126.04°,\ 284.04°)}}$.

(b) $\rho = 3$, $\phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{0}{3} = 0°$

$Q(\rho,\phi,z) = \underline{\underline{Q(3,\ 0°,\ 5)}}$.

$r = \sqrt{9+0+25} = 5.831$, $\theta = \tan^{-1}\frac{\rho}{z} = \tan^{-1}\frac{3}{5} = 30.96°$

$Q(r,\theta,\phi) = \underline{\underline{Q(5.831,\ 30.96°,\ 0°)}}$.

(c) $\rho = \sqrt{4+36} = 6.325$, $\phi = \tan^{-1}\frac{6}{-2} = 108.4°$

$R(\rho,\phi,z) = \underline{\underline{R(6.325,\ 108.4°,\ 0)}}$.

$r = \rho = 6.325$, $\theta = \tan^{-1}\frac{\rho}{z} = \tan^{-1}\frac{6.325}{0} = 90°$

$R(r,\theta,\phi) = \underline{\underline{R(6.325,\ 90°,\ 108.4°)}}$.

Prob. 2.3

(a)
$$\begin{bmatrix} P_\rho \\ P_\phi \\ P_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y+z \\ 0 \\ 0 \end{bmatrix}.$$

$$\vec{P} = (y+z)\left(\cos\phi\,\vec{a}_\rho - \sin\phi\,\vec{a}_\phi\right)$$

But $y = \rho\sin\phi$, $z = z$

$$\vec{P} = \underline{\underline{(\rho\sin\phi + z)(\cos\phi\,\vec{a}_\rho - \sin\phi\,\vec{a}_\phi)}}.$$

$$\begin{bmatrix} P_r \\ P_\theta \\ P_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} y+z \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{P} = (y+z)\left(\sin\theta\cos\phi\,\vec{a}_r + \cos\theta\cos\phi\,\vec{a}_\theta - \sin\phi\,\vec{a}_\phi\right)$$

But $y = r\sin\theta\sin\phi$, $z = r\cos\theta$,

$$\vec{P} = r\left(\sin\theta\sin\phi + \cos\theta\right)\left(\sin\theta\cos\phi\,\vec{a}_r + \cos\theta\cos\phi\,\vec{a}_\theta - \sin\phi\,\vec{a}_\phi\right).$$

(b) $\vec{Q}(\rho,\phi,z) = (y\cos\phi + xz\sin\phi)\vec{a}_\rho + (-y\sin\phi + xz\cos\phi)\vec{a}_\phi + (x+y)\vec{a}_z.$

But $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$,

$$\vec{Q} = \frac{1}{2}\rho\sin 2\phi\,(1+z)\vec{a}_\rho + \rho\left(z\cos^2\phi - \sin^2\phi\right)\vec{a}_\phi + \rho\left(\cos\phi + \sin\phi\right)\vec{a}_z.$$

$$\vec{Q}(r,\theta,\phi) = \left[y\sin\theta\cos\phi + xz\sin\theta\sin\phi + (x+y)\cos\theta\right]\vec{a}_r + \left[y\cos\theta\cos\phi + xz\cos\theta\sin\phi - (x+y)\sin\theta\right]\vec{a}_\theta + \left[-y\sin\phi + xz\cos\phi\right]\vec{a}_\phi$$

But $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$.

$$\vec{Q} = \left[\frac{1}{2}r\sin 2\phi\sin^2\theta + \frac{1}{2}r^2\sin^2\theta\cos\theta\sin 2\phi + \frac{1}{2}r(\cos\phi + \sin\phi)\cos 2\theta\right]\vec{a}_r$$

$$+\left[\frac{1}{4} r \sin 2\theta \sin 2\phi + \frac{1}{2} r^2 \cos^2\theta \sin\theta \sin 2\phi\right.$$

$$\left. - \frac{r}{2}(\cos\phi + \sin\phi)\sin^2\theta\right]\vec{a}_\theta$$

$$+ \underline{\underline{\left(-r\sin\theta\sin^2\phi + \frac{1}{2} r^2 \sin 2\theta \cos^2\phi\right)\vec{a}_\phi}}.$$

(c) $$\vec{T}(\rho,\phi,z) = \left[\left(\frac{x^2}{x^2+y^2} - y^2\right)\cos\phi + \left(\frac{xyz}{x^2+y^2} + xy\right)\sin\phi\right]\vec{a}_\rho$$

$$+\left[\left(\frac{-x^2}{x^2+y^2} + y^2\right)\sin\phi + \left(\frac{xyz}{x^2+y^2} + xy\right)\cos\phi\right]\vec{a}_\phi + \vec{a}_z$$

But $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$,

$$\vec{T} = \left(\cos^3\phi - \cancel{\rho^2\sin^2\phi\cos\phi} + \sin^2\phi\cos\phi + \cancel{\rho^2\sin^2\phi\cos\phi}\right)\vec{a}_\rho$$

$$+ \left(-\sin\phi\cancel{\cos^2\phi} + \rho^2\sin^3\phi + \sin\phi\cancel{\cos^2\phi}\right.$$

$$\left. + \rho^2\sin\phi\cos^2\phi\right)\vec{a}_\phi + \vec{a}_z$$

$$\vec{T}(\rho,\phi,z) = \underline{\underline{\cos\phi\,\vec{a}_\rho + \rho^2\sin\phi\,\vec{a}_\phi + \vec{a}_z}}.$$

$$\begin{bmatrix} T_r \\ T_\theta \\ T_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\cos\phi & \cos\theta \\ \cos\theta\sin\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} \frac{x^2}{x^2+y^2} - y^2 \\ \frac{xy}{x^2+y^2} + xy \\ 1 \end{bmatrix}$$

Since $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$;

$$T_r = \left(\frac{r^2 \sin^2\theta \cos^2\phi}{r^2 \sin^2\theta} - r^2 \sin^2\theta \sin^2\phi \right) \sin\theta \cos\phi$$

$$+ \left(\cos\phi \sin\phi + r^2 \sin^2\theta \sin\phi \cos\phi \right) \sin\theta \sin\phi + \cos\theta$$

$$= \sin\theta \cos\phi + \cos\theta ,$$

$$T_\theta = \left(\cos^2\phi - r^2 \sin^2\theta \sin^2\phi \right) \cos\theta \sin\phi$$

$$+ \left(\cos\phi \sin\phi + r^2 \sin^2\theta \sin\phi \cos\phi \right) \cos\theta \sin\phi - \sin\theta$$

$$= \cos\theta \cos\phi - \sin\theta$$

$$T_\phi = - \left(\cos^2\phi - r^2 \sin^2\theta \sin^2\phi \right) \sin\phi$$

$$+ \left(\cos\phi \sin\phi + r^2 \sin^2\theta \sin\phi \cos\phi \right) \cos\phi$$

$$= - \cos^2\phi \sin\phi + r^2 \sin^2\theta \sin^3\phi + \sin\phi \cos\phi$$

$$+ r^2 \sin^2\theta \sin\phi \cos^2\phi = r^2 \sin^2\theta \sin\phi$$

$$\vec{T}(r,\theta,\phi) = (\sin\theta \cos\phi + \cos\theta)\vec{a}_r$$

$$+ (\cos\theta \cos\phi - \sin\theta)\vec{a}_\theta$$

$$\underline{\underline{+ r^2 \sin^2\theta \sin\phi \, \vec{a}_\phi .}}$$

(d) $$\begin{bmatrix} S_\rho \\ S_\phi \\ S_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{y}{x^2+y^2} \\ \frac{-x}{x^2+y^2} \\ 10 \end{bmatrix}$$

$$= \left(\cancel{\frac{\rho \sin\phi}{\rho^2}} \cos\phi - \cancel{\frac{\rho \cos\phi}{\rho^2}} \sin\phi \right) \vec{a}_\rho$$

$$+\left(-\frac{\rho\sin\phi}{\rho^2}\sin\phi-\frac{\rho\cos\phi}{\rho^2}\cos\phi\right)\vec{a}_\phi+10\vec{a}_z$$

$$\underline{\underline{\vec{S}(\rho,\phi,z)=-\frac{1}{\rho}\vec{a}_\phi+10\vec{a}_z.}}$$

$$\begin{bmatrix}S_r\\S_\theta\\S_\phi\end{bmatrix}=\begin{bmatrix}\sin^2\theta\cos\phi & \sin\theta\sin\phi & \cos\theta\\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta\\ -\sin\phi & \cos\phi & 0\end{bmatrix}\begin{bmatrix}\frac{y}{x^2+y^2}\\ \frac{-x}{x^2+y^2}\\ 10\end{bmatrix}$$

$$=\left(\frac{r\sin\theta\sin\phi}{r^2\sin^2\theta}\sin\theta\cos\phi-\frac{r\sin\theta\cos\phi}{r^2\sin^2\theta}\sin\theta\cos\phi\right.$$

$$\left.+10\cos\theta\right)\vec{a}_r+\left(\frac{r\sin\theta\sin\phi}{r^2\sin^2\theta}\cos\theta\cos\phi\right.$$

$$\left.-\frac{r\sin\theta\cos\phi}{r^2\sin^2\theta}\cos\theta\sin\phi-10\sin\theta\right)\vec{a}_\theta$$

$$+\left(-\frac{r\sin\theta\sin\phi}{r^2\sin^2\theta}\sin\phi-\frac{r\sin\theta\cos\phi}{r^2\sin^2\theta}\cos\phi\right)\vec{a}_\phi$$

$$\underline{\underline{\vec{S}(r,\theta,\phi)=10\cos\theta\,\vec{a}_r-10\sin\theta\,\vec{a}_\theta-\frac{1}{r\sin\theta}\vec{a}_\phi.}}$$

$\boxed{\text{Prob. 2.4}}$

(a) $\vec{A}=(x^2z^2\sin\phi+yz\sin\phi)\vec{a}_\rho+(yz\cos\phi-y^2z^2\cos\phi)\vec{a}_\phi$

$\quad+yx\,\vec{a}_z$

$\quad=(x^2z^2+yz)\sin\phi\,\vec{a}_\rho+yz(1-yz)\cos\phi\,\vec{a}_\phi$

$\quad+xy\,\vec{a}_z$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

$$A_x = A_\rho \cos\phi - A_\phi \sin\phi = x^2 z^2 \cos\phi \sin\phi + \cancel{yz \sin\phi \cos\phi}$$
$$- \cancel{yz \sin\phi \cos\phi} + y^2 z^2 \sin\phi \cos\phi$$
$$= (x^2+y^2) z^2 \cos\phi \sin\phi = (x^2+y^2) \cdot z^2 \cdot \frac{xy}{x^2+y^2} = xyz^2$$

$$A_y = A_\rho \sin\phi + A_\phi \cos\phi$$
$$= x^2 z^2 \sin^2\phi + yz \sin^2\phi + yz \cos^2\phi - y^2 z^2 \cos^2\phi$$
$$= yz + z^2 \left(x^2 \cdot \frac{\cancel{y^2}}{\cancel{x^2+y^2}} - y^2 \cdot \frac{\cancel{x^2}}{\cancel{x^2+y^2}} \right) = yz$$

$$\vec{A}(x,y,z) = \underline{\underline{xyz^2 \bar{a}_x + yz \bar{a}_y + xy \bar{a}_z}}.$$

(b) $\vec{B} = r^3 (-\sin\phi \vec{a}_x + \cos\phi \vec{a}_y)$

$$+ 4r \left[-\sin\theta \vec{a}_y + \cos\theta (\cos\phi \bar{a}_x + \sin\phi \vec{a}_y) \right] \cos\theta \sin\phi$$
$$+ 6r^2 \sin\theta \cos\phi \left[\cos\theta \bar{a}_z + \sin\theta (\cos\phi \bar{a}_x + \sin\phi \bar{a}_y) \right]$$

$$= \vec{a}_x \left(-r^3 \sin\phi + 4r \cos^2\theta \sin\phi \cos\phi + 6r^2 \sin^2\theta \cos^2\phi \right)$$
$$+ \bar{a}_y \left(r^3 \cos\phi + 4r \cos^2\theta \sin^2\phi + 6r^2 \sin^2\theta \sin\phi \cos\phi \right)$$
$$+ \bar{a}_z \left(-4r \sin\theta + 6r^2 \sin\theta \cos\theta \cos\phi \right)$$

But $r = \sqrt{x^2+y^2+z^2}$, $\tan\theta = \dfrac{\sqrt{x^2+y^2}}{z}$, $\tan\phi = \dfrac{y}{x}$

$$\sin\theta = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}, \quad \cos\theta = \frac{z}{\sqrt{x^2+y^2+z^2}},$$

$$\sin\phi = \frac{y}{\sqrt{x^2+y^2}}, \quad \cos\phi = \frac{x}{\sqrt{x^2+y^2}}.$$

$$\vec{B}(x,y,z) = \left[6x^2 + \frac{4xyz^2}{(x^2+y^2)(x^2+y^2+z^2)^{1/2}} - \frac{y(x^2+y^2+z^2)^{3/2}}{(x^2+y^2)^{1/2}}\right]\bar{a}_x$$

$$+ \left[6xyz + \frac{4y^2z^2}{(x^2+y^2)(x^2+y^2+z^2)^{1/2}} + \frac{x(x^2+y^2+z^2)^{3/2}}{(x^2+y^2)^{3/2}}\right]\bar{a}_y$$

$$+ \left[6xz - 4(x^2+y^2)^{1/2}\right]\bar{a}_z.$$

Prob. 2.5

(a) $\vec{a}_x \cdot \vec{a}_\rho = (\cos\phi\,\vec{a}_\rho - \sin\phi\,\vec{a}_\phi)\cdot\bar{a}_\rho = \cos\phi$

$\bar{a}_x \cdot \bar{a}_\phi = (\cos\phi\,\bar{a}_\rho - \sin\phi\,\bar{a}_\phi)\cdot\bar{a}_\phi = -\sin\phi$

$$\vec{a}_y \cdot \vec{a}_\rho = (\sin\phi\,\vec{a}_\rho + \cos\phi\,\vec{a}_\phi)\cdot\vec{a}_\rho = \sin\phi$$

$$\vec{a}_y \cdot \vec{a}_\phi = (\sin\phi\,\vec{a}_\rho + \cos\phi\,\vec{a}_\phi)\cdot\vec{a}_\phi = \cos\phi$$

(b) Since $\vec{a}_\rho$, $\vec{a}_\phi$, and $\vec{a}_z$ are mutually orthogonal,

$$\vec{a}_z \cdot \vec{a}_z = 1, \quad \vec{a}_z \cdot \vec{a}_\rho = 0 = \vec{a}_z \cdot \vec{a}_\phi.$$

Also, $\vec{a}_x \cdot \vec{a}_z = 0 = \vec{a}_y \cdot \vec{a}_z$. Hence

$$\begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \vec{a}_x\cdot\vec{a}_\rho & \vec{a}_x\cdot\vec{a}_\phi & \vec{a}_x\cdot\vec{a}_z \\ \vec{a}_y\cdot\vec{a}_\rho & \vec{a}_y\cdot\vec{a}_\phi & \vec{a}_y\cdot\vec{a}_z \\ \vec{a}_z\cdot\vec{a}_\rho & \vec{a}_z\cdot\vec{a}_\phi & \vec{a}_z\cdot\vec{a}_z \end{bmatrix}$$

(c) In spherical system,

$$\vec{a}_x = \sin\theta\cos\phi\,\vec{a}_r + \cos\theta\cos\phi\,\vec{a}_\theta - \sin\phi\,\vec{a}_\phi$$

$$\vec{a}_y = \sin\theta\sin\phi\,\vec{a}_r + \cos\theta\sin\phi\,\vec{a}_\theta + \cos\phi\,\vec{a}_\phi$$

$$\vec{a}_z = \cos\theta\,\vec{a}_r - \sin\theta\,\vec{a}_\theta.$$

Hence

$$\vec{a}_x \cdot \vec{a}_r = \sin\theta\cos\phi,$$

$$\vec{a}_x \cdot \vec{a}_\theta = \cos\theta\cos\phi,$$

$$\vec{a}_y \cdot \vec{a}_r = \sin\theta\sin\phi,$$

$$\vec{a}_y \cdot \vec{a}_\theta = \cos\theta\sin\phi,$$

$$\vec{a}_z \cdot \vec{a}_r = \cos\theta,$$

$$\vec{a}_z \cdot \vec{a}_\theta = -\sin\theta.$$

$\boxed{\text{Prob. 2.6}}$

(a) $r = \sqrt{x^2+y^2+z^2} = \sqrt{\rho^2+z^2}$

$\theta = \tan^{-1}\frac{\rho}{z}, \quad \phi = \phi$

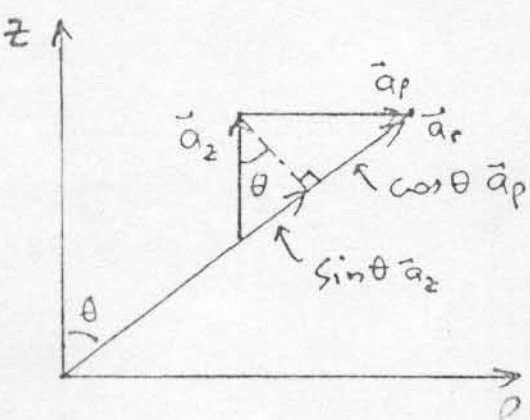
or

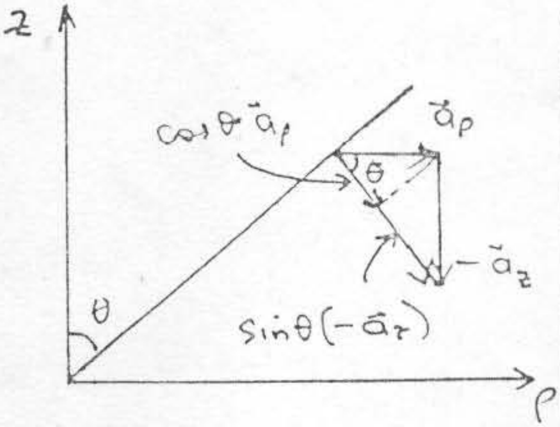
$\rho = \sqrt{x^2+y^2} = \sqrt{r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi}$

$= r\sin\theta,$

$z = r\cos\theta, \quad \phi = \phi.$

(b) from the figures below,





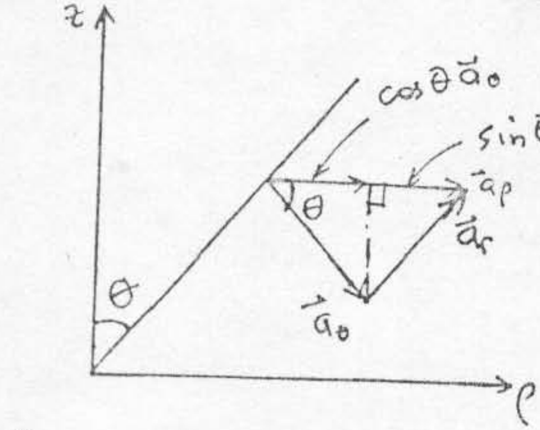
$\vec{a}_r = \sin\theta\,\vec{a}_z + \cos\theta\,\vec{a}_\rho, \quad \vec{a}_\theta = \cos\theta\,\vec{a}_\rho - \sin\theta\,\vec{a}_z, \quad \vec{a}_\phi = \vec{a}_\phi$

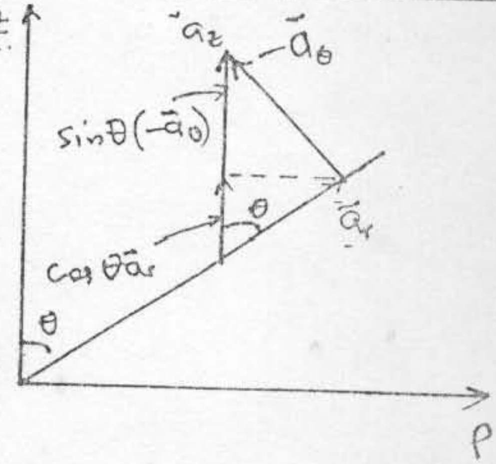
Hence

$$\begin{bmatrix}\vec{a}_r\\ \vec{a}_\theta\\ \vec{a}_\phi\end{bmatrix} = \begin{bmatrix}\sin\theta & 0 & \cos\theta\\ \cos\theta & 0 & -\sin\theta\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\vec{a}_\rho\\ \vec{a}_\phi\\ \vec{a}_z\end{bmatrix}$$

Or from the figures on the next page,

$\vec{a}_\rho = \cos\theta\,\vec{a}_\theta + \sin\theta\,\vec{a}_r, \quad \vec{a}_z = \cos\theta\,\vec{a}_r - \sin\theta\,\vec{a}_\theta, \quad \vec{a}_\phi = \vec{a}_\phi$





Hence

$$\begin{bmatrix} \vec{a}_\rho \\ \vec{a}_\phi \\ \vec{a}_z \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \vec{a}_r \\ \vec{a}_\theta \\ \vec{a}_\phi \end{bmatrix}.$$

Prob. 2.7

(a) Using the results in Prob. 2.6,

$$A_\rho = \rho z \sin\phi = r^2 \sin\theta \cos\theta \sin\phi,$$

$$A_\phi = 3\rho\cos\phi = 3r\sin\theta\cos\phi,$$

$$A_z = \rho\cos\phi\sin\phi = r\sin\theta\cos\phi\sin\phi.$$

Hence

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} r^2\sin\theta\cos\theta\sin\phi \\ 3r\sin\theta\cos\phi \\ r\sin\theta\cos\phi\sin\phi \end{bmatrix}$$

$$\vec{A}(r,\theta,\phi) = r\sin\theta\Big[\sin\phi\cos\theta\,(r\sin\theta + \cos\phi)\,\vec{a}_r + \sin\phi\,(r\cos^2\theta - \sin\theta\cos\phi)\,\vec{a}_\theta + 3\cos\phi\,\vec{a}_\phi\Big].$$

At $(10, \pi/2, 3\pi/4)$, $r = 10$, $\theta = \pi/2$, $\phi = 3\pi/4$,

$$\vec{A} = 10\left(0\vec{a}_r + \frac{1}{2}\vec{a}_\theta - \frac{3}{\sqrt{2}}\vec{a}_\phi\right) = \underline{\underline{5\vec{a}_\theta - 21.21\,\vec{a}_\phi}}.$$

(b) $\vec{B}_r = r^2 = (\rho^2 + z^2)$, $B_\theta = 0$, $B_\phi = \sin\theta = \dfrac{\rho}{\sqrt{\rho^2+z^2}}$

$$\begin{bmatrix} B_\rho \\ B_\phi \\ B_z \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} B_r \\ B_\theta \\ B_\phi \end{bmatrix}$$

$$\underline{\underline{\vec{B}(\rho, \phi, z) = \sqrt{\rho^2+z^2}\left(\rho\,\vec{a}_\rho + \frac{\rho}{\rho^2+z^2}\vec{a}_\phi + z\,\vec{a}_z\right)}}.$$

At $(2, \pi/6, 1)$, $\rho = 2$, $\phi = \pi/6$, $z = 1$,

$$\vec{B} = \sqrt{5}\left(2\vec{a}_\rho + \frac{2}{5}\vec{a}_\phi + \vec{a}_z\right)$$

$$= \underline{\underline{4.472\,\vec{a}_\rho + 0.8944\,\vec{a}_\phi + 2.236\,\vec{a}_z}}.$$

Prob. 2.8

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \underline{\underline{\begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{-y}{\sqrt{x^2+y^2}} & 0 \\ \frac{y}{\sqrt{x^2+y^2}} & \frac{x}{\sqrt{x^2+y^2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}}}$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

$$= \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2+z^2}} & \frac{xz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} & \frac{-y}{\sqrt{x^2+y^2}} \\ \frac{y}{\sqrt{x^2+y^2+z^2}} & \frac{yz}{\sqrt{x^2+y^2}\sqrt{x^2+y^2+z^2}} & \frac{x}{\sqrt{x^2+y^2}} \\ \frac{z}{\sqrt{x^2+y^2+z^2}} & \frac{-\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}} & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

Prob. 2.9

(a) $d = \sqrt{(6-2)^2 + (-1-1)^2 + (2-5)^2} = \sqrt{29} = \underline{\underline{5.385}}$.

(b) $d^2 = 3^2 + 5^2 - 2(3)(5)\cos\pi + (-1-5)^2 = 100$

$d = \sqrt{100} = \underline{\underline{10}}$.

(c) $d^2 = 10^2 + 5^2 - 2(10)(5)\cos\frac{\pi}{4}\cos\frac{\pi}{6}$

$-2(10)(5)\sin\frac{\pi}{4}\sin\frac{\pi}{6}\cos\left(\frac{7\pi}{4} - \frac{3\pi}{4}\right)$

$d = \sqrt{99.12} = \underline{\underline{9.956}}$.

Prob. 2.10

(a) $r = \sqrt{8^2 + (-15)^2 + 12^2} = 20.81$

$\theta = \tan^{-1}\sqrt{x^2+y^2}/z = \tan^{-1}\frac{17}{12} = 54.78°$

$$\phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{-15}{8} = 298^\circ$$

$$P(r,\theta,\phi) = \underline{\underline{P(20.81,\ 54.78^\circ,\ 298^\circ)}}.$$

(b) $$\begin{bmatrix} F_\rho \\ F_\phi \\ F_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2xy \\ -x^2 \\ 0 \end{bmatrix}$$

$F_\rho = 2xy\cos\phi - x^2\sin\phi$, $F_\phi = -2xy\sin\phi - x^2\cos\phi$,

$F_z = 0$.

But $x = \rho\cos\phi$, $y = \rho\sin\phi$,

$$F_\rho = \rho^2(2\cos^2\phi\sin\phi - \cos^2\phi\sin\phi) = \rho^2\cos^2\phi\sin\phi$$

$$F_\phi = -\rho^2(2\cos\phi\sin^2\phi + \cos^3\phi) = -\rho^2\cos\phi(1+\sin^2\phi)$$

$$\vec{F} = \underline{\underline{\rho^2\cos\phi\left[\cos\phi\sin\phi\,\vec{a}_\rho - (1+\sin^2\phi)\,\vec{a}_\phi\right]}}.$$

$\boxed{\text{Prob. 2.11}}$

(a) An infinite line parallel to the z-axis.

(b) Point $(2,-1,10)$.

(c) A circle of radius $r\sin\theta = 5$, i.e. the intersection of a cone and a sphere.

(d) An infinite line parallel to the z-axis.

(e) A semi-infinite line parallel to the x-y plane.

(f) A semi-circle of radius 5 in the x-y plane.

Prob. 2.12

At $T(2,3,-4)$,

$$\theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z} = \tan^{-1}\frac{\sqrt{13}}{-4} = 137.97^\circ$$

$$\cos\theta = -\frac{4}{\sqrt{29}} = -0.7428, \quad \sin\theta = \frac{\sqrt{13}}{\sqrt{29}} = 0.6695$$

$$\phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{3}{2} = 56.31^\circ$$

$$\cos\phi = \frac{2}{\sqrt{13}}, \quad \sin\phi = \frac{3}{\sqrt{13}}$$

$$\bar{a}_z = \cos\theta\,\bar{a}_r - \sin\theta\,\bar{a}_\theta = \underline{\underline{-0.7428\bar{a}_r - 0.6695\bar{a}_\theta}}$$

$$\bar{a}_r = \sin\theta\cos\phi\,\bar{a}_x + \sin\theta\sin\phi\,\bar{a}_y + \cos\theta\,\bar{a}_z$$

$$= \underline{\underline{0.3714\bar{a}_x + 0.5571\bar{a}_y - 0.7428\bar{a}_z}}$$

Prob. 2.13

At $P(0,2,-5)$, $\phi = 90^\circ$

$$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} B_\rho \\ B_\phi \\ B_z \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -5 \\ 1 \\ -3 \end{bmatrix}$$

$$\vec{B} = -\bar{a}_x - 5\bar{a}_y - 3\bar{a}_z$$

(a) $\vec{A} + \vec{B} = (2,4,10) + (-1,-5,-3) = \underline{\underline{\bar{a}_x - \bar{a}_y + 7\bar{a}_z}}$

(b) $\cos\theta_{AB} = \dfrac{\vec{A}\cdot\vec{B}}{AB} = \dfrac{-52}{\sqrt{4200}} \rightarrow \underline{\underline{\theta_{AB} = 143.26^\circ}}$

(c) $A_B = \vec{A} \cdot \vec{a}_B = \dfrac{\vec{A} \cdot \vec{B}}{B} = \dfrac{-52}{\sqrt{35}} = \underline{\underline{-8.789}}$

Prob. 2.14

At $P(8, 30^\circ, 60^\circ) = P(r, \theta, \phi)$,

$x = r \sin\theta \cos\phi = 8 \sin 30^\circ \cos 60^\circ = 2$

$y = r \sin\theta \sin\phi = 8 \sin 30^\circ \sin 60^\circ = 2\sqrt{3}$

$z = r\cos\theta = 4\sqrt{3}$

$\vec{G} = 14\,\vec{a}_x + 8\sqrt{3}\,\vec{a}_y + (48+24)\vec{a}_z = (14, 13.86, 72)$

$\vec{a}_\phi = -\sin\phi\,\vec{a}_x + \cos\phi\,\vec{a}_y = -\frac{\sqrt{3}}{2}\vec{a}_x + \frac{1}{2}\vec{a}_y$

$\vec{G}_\phi = (\vec{G} \cdot \vec{a}_\phi)\vec{a}_\phi = (-7\sqrt{3} + 4\sqrt{3})\frac{1}{2}(-\sqrt{3}\vec{a}_x + \vec{a}_y)$

$= \underline{\underline{4.5\,\vec{a}_x - 2.598\,\vec{a}_y}}$

Prob. 2.15

(a) $\vec{J}_z = (\vec{J} \cdot \vec{a}_z)\vec{a}_z$.

At $(2, \pi/2, 3\pi/2)$, $\vec{a}_z = \cos\theta\,\vec{a}_r - \sin\theta\,\vec{a}_\theta = -\vec{a}_\theta$

$\vec{J}_z = -\cos 2\theta \sin\phi\,\vec{a}_\theta = -\cos\pi \sin\frac{3\pi}{2}\,\vec{a}_\theta = \underline{\underline{-\vec{a}_\theta}}$

(b) $\vec{J}_\phi = \tan\frac{\theta}{2} \ln r\,\vec{a}_\phi = \tan\frac{\pi}{4}\ln 2\,\vec{a}_\phi = \ln 2\,\vec{a}_\phi = \underline{\underline{0.6931\,\vec{a}_\phi}}$

(c) $\vec{J}_t = \vec{J} - \vec{J}_n = \vec{J} - \vec{J}_r = -\vec{a}_\theta + \ln 2\,\vec{a}_\phi$

$= \underline{\underline{-\vec{a}_\theta + 0.6931\,\vec{a}_\phi}}$.

(d) $\vec{J}_P = (\vec{J} \cdot \vec{a}_x)\vec{a}_x$,

$\vec{a}_x = \sin\theta\cos\phi\,\vec{a}_r + \cos\theta\cos\phi\,\vec{a}_\theta - \sin\phi\,\vec{a}_\phi = \vec{a}_\phi$

at $(2, \pi/2, 3\pi/2)$,

$$\vec{J}_\rho = \ln 2\, \vec{a}_\phi = \underline{\underline{0.6931\, \vec{a}_\phi}}.$$

$\boxed{\text{Prob. 2.16}}$

At $P(3,-2,6)$, $x=3,\ y=-2,\ z=6$,

$$\rho = \sqrt{x^2+y^2} = \sqrt{13},\quad \phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{-2}{3}$$

$$r = \sqrt{x^2+y^2+z^2} = 7,\quad \theta = \tan^{-1}\frac{\rho}{z} = \tan^{-1}\frac{\sqrt{13}}{6}$$

(a) $$\vec{A} = \sqrt{13}\,\frac{3}{\sqrt{13}}\vec{a}_\rho + 6\left(\frac{-2}{\sqrt{13}}\right)\vec{a}_\phi - \sqrt{13}\,(36)\vec{a}_z$$

$$= \underline{\underline{3\vec{a}_\rho - 3.328\,\vec{a}_\phi - 129.8\,\vec{a}_z}}.$$

$$\vec{B} = 7\left(\frac{\sqrt{13}}{7}\right)\vec{a}_r + 49\left(\frac{6}{7}\right)\left(-\frac{2}{\sqrt{13}}\right)\vec{a}_\theta = \underline{\underline{3.606\,\vec{a}_r - 23.3\,\vec{a}_\theta}}.$$

(b) $$\vec{A}_B = (\vec{A}\cdot\vec{a}_B)\,\vec{a}_B = \frac{(\vec{A}\cdot\vec{B})\,\vec{B}}{B^2}$$

In spherical system.

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{13}}{17} & 0 & \frac{6}{7} \\ \frac{6}{7} & 0 & -\frac{\sqrt{13}}{7} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ \frac{-12}{\sqrt{13}} \\ -36\sqrt{13} \end{bmatrix} = \begin{bmatrix} \frac{3\sqrt{13}}{7} - 2\,\frac{16\sqrt{13}}{7} \\ \frac{18}{7} + \frac{36\times 13}{7} \\ -\frac{12}{\sqrt{13}} \end{bmatrix}$$

$$\vec{A} = -109.71\,\vec{a}_r + 69.43\,\vec{a}_\theta - 3.328\,\vec{a}_\phi.$$

Hence

$$\vec{A}_B = \frac{-2013.33}{555.9}(3.606\,\bar{a}_r - 23.3\,\bar{a}_\theta)$$

$$= \underline{\underline{-13.06\,\bar{a}_r + 84.39\,\bar{a}_\theta}}$$

(c) In cylindrical system,

$$\begin{bmatrix} B_\rho \\ B_\phi \\ B_z \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} B_r \\ B_\theta \\ B_\phi \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{13}}{7} & \frac{6}{7} & 0 \\ 0 & 0 & 1 \\ \frac{6}{7} & -\frac{\sqrt{13}}{7} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{13} \\ -\frac{12\times 49}{7\sqrt{13}} \\ 0 \end{bmatrix}$$

$$\vec{B} = \left(\frac{13}{7} - \frac{72}{\sqrt{13}}\right)\bar{a}_\rho + \left(\frac{6\sqrt{13}}{7} + 12\right)\bar{a}_z = -18.11\,\bar{a}_\rho + 15.09\,\bar{a}_z$$

$$\vec{A}\times\vec{B} = \begin{vmatrix} 3 & -3.328 & -129.8 \\ -18.11 & 0 & 15.09 \end{vmatrix}$$

$$= -52.92\,\bar{a}_\rho + 2396\,\bar{a}_\phi - 60.27\,\bar{a}_z$$

$$\hat{a}_{A\times B} = \underline{\pm\,(0.0221\,\bar{a}_\rho - 0.9994\,\bar{a}_\phi + 0.025\,\bar{a}_z)}.$$

$\boxed{\text{Prob. 2.17}}$

(a) $F_z = F\cos\gamma = F\cos\theta \longrightarrow \gamma = \theta,$

$F_x = F\sin\theta\cos\phi = F\cos\alpha \longrightarrow \cos\alpha = \sin\theta\cos\phi,$

$F_y = F\sin\theta\sin\phi = F\cos\beta \longrightarrow \cos\beta = \sin\theta\sin\phi,$

Hence,

$$\underline{\underline{\alpha = \cos^{-1}(\sin\theta\cos\phi),\ \beta = \cos^{-1}(\sin\theta\sin\phi),\ \gamma = \theta}}$$

(b) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi$
$\qquad + \cos^2\theta = \sin^2\theta + \cos^2\theta = 1$

Alternatively,

$$\frac{\vec{F}\cdot\vec{F}}{|\vec{F}|^2} = 1 \longrightarrow \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$\boxed{\text{Prob. 2.18}}$

(a) For $(x, y, z) = (-2, 3, 6)$,

$$r = \sqrt{x^2 + y^2 + z^2} = 7$$

$$x = r\cos\alpha \longrightarrow \cos\alpha = \frac{x}{r} = -\frac{2}{7} \rightarrow \alpha = 106.6^\circ$$

$$y = r\cos\beta \rightarrow \cos\beta = \frac{y}{r} = \frac{3}{7} \rightarrow \beta = 64.6^\circ$$

$$z = r\cos\gamma \rightarrow \cos\gamma = \frac{z}{r} = \frac{6}{7} \rightarrow \gamma = 31^\circ$$

Hence,

$$(r, \alpha, \beta, \gamma) = \underline{\underline{(7, 106.6^\circ, 64.6^\circ, 31^\circ)}}$$

(b) For $(\rho, \phi, z) = (4, 30^\circ, -3)$,

$$r = \sqrt{\rho^2 + z^2} = 5,$$

$$\cos\gamma = \frac{z}{r} = -\frac{3}{5} \longrightarrow \gamma = 126.9^\circ$$

$$\cos\alpha = \frac{x}{r} = \frac{\rho\cos\phi}{r} = \frac{4\cos 30^\circ}{5} \rightarrow \alpha = 46.15^\circ$$

$$\cos\beta = \frac{y}{r} = \frac{\rho\sin\phi}{r} = \frac{4}{5}\sin 30^\circ \longrightarrow \beta = 66.42^\circ$$

$$(r, \alpha, \beta, \gamma) = \underline{\underline{(5, 46.15^\circ, 66.42^\circ, 126.9^\circ)}}.$$

(c) For $(r, \theta, \phi) = (3, 30^\circ, 60^\circ)$,

$r = 3$, $\gamma = \theta = 30^\circ$,

$$\cos\alpha = \frac{x}{r} = \frac{r\sin\theta\cos\phi}{r} = \frac{1}{4} \longrightarrow \alpha = 75.52^\circ,$$

$$\cos\beta = \frac{y}{r} = \sin\theta\sin\phi = 0.433 \longrightarrow \beta = 64.34^\circ,$$

$$(r, \alpha, \beta, \gamma) = \underline{\underline{(3, 75.52^\circ, 64.34^\circ, 30^\circ)}}.$$

$\boxed{\text{Prob. 2.19}}$

$J_x = 3\cos\theta\sin\theta\cos\phi$, $J_y = 3\cos\theta\sin\theta\sin\phi$,

$J_z = 2 - 3\sin^2\theta\cos^2\phi - 3\sin^2\theta\sin^2\phi = 2 - 3\sin^2\theta$,

$$\begin{bmatrix} J_r \\ J_\theta \\ J_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} J_x \\ J_y \\ J_z \end{bmatrix}$$

$$J_r = 3\cos\theta\sin^2\theta\cos^2\phi + 3\cos\theta\sin^2\theta\sin^2\phi + 2\cos\theta - 3\cos\theta\sin^2\theta = 2\cos\theta$$

$$J_\theta = 3\cos^2\theta\sin\theta\cos^2\phi + 3\cos^2\theta\sin\theta\sin^2\phi - 2\sin\theta + 3\sin^3\theta = \sin\theta$$

$$J_\phi = -3\cos\theta\sin\theta\cos\phi\sin\phi + 3\cos\theta\sin\theta\sin\phi\cos\phi = 0$$

Hence $\vec{J} = \underline{2\cos\theta\,\vec{a}_r + \sin\theta\,\vec{a}_\theta}$.

CHAPTER 3

P.E.3.1

(a) $$DH = \int_{\phi=45^\circ}^{60^\circ} r \sin\phi \, d\phi \Big|_{r=3,\,\theta=90^\circ} = 3(1)\left(\frac{\pi}{4}-\frac{\pi}{3}\right) = -\frac{\pi}{4}$$

$$= \underline{\underline{0.7854}}.$$

(b) $$FG = \int_{\theta=60^\circ}^{90^\circ} r\, d\theta \Big|_{r=5} = 5\left(\frac{\pi}{3}-\frac{\pi}{2}\right) = \frac{5\pi}{6} = \underline{\underline{2.618}}$$

(c) $$AEHD = \int_{\theta=60^\circ}^{90^\circ}\int_{\phi=45^\circ}^{60^\circ} r^2 \sin\theta\, d\theta\, d\phi \Big|_{r=3} = 9(-\cos\theta)\Big|_{60^\circ}^{90^\circ} \phi\Big|_{45^\circ}$$

$$= 9\left(\frac{1}{2}\right)\left(\frac{\pi}{12}\right) = \frac{3\pi}{8} = \underline{\underline{1.178}}.$$

(d) $$ABCD = \int_{r=3}^{5}\int_{\theta=60^\circ}^{90^\circ} r\, d\theta\, dr = \frac{r^2}{2}\Big|_3^5 \left(\frac{\pi}{2}-\frac{\pi}{3}\right) = \frac{4\pi}{3}$$

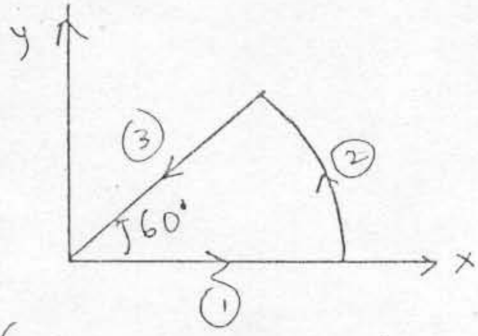
$$= \underline{\underline{4.189}}$$

(e) $$\text{volume} = \int_{r=3}^{5}\int_{\phi=45^\circ}^{60^\circ}\int_{\theta=60^\circ}^{90^\circ} r^2 \sin\theta\, d\theta\, d\phi$$

$$= \frac{r^3}{3}\Big|_3^5 (-\cos\theta)\Big|_{60^\circ}^{90^\circ} \phi\Big|_{45^\circ}^{60^\circ} = \frac{1}{3}(98)\left(\frac{1}{2}\right)\frac{\pi}{12}$$

$$= \frac{49\pi}{36} = \underline{\underline{4.276}}.$$

P.E.3.2



Let $\oint_L \vec{A}\cdot\vec{dl} = \left(\int_{①} + \int_{②} + \int_{③}\right)\vec{A}\cdot\vec{dl} = C_1 + C_2 + C_3$

Along ①, $C_1 = \int \vec{A}\cdot\vec{dl} = \int_0^2 \rho\cos\phi\, d\rho\Big|_{\phi=0} = \frac{\rho^2}{2}\Big|_0^2 = 2$

Along ②, $\vec{dl} = \rho\, d\phi\, \vec{a}_\phi$, $\vec{A}\cdot\vec{dl} = 0 \longrightarrow C_2 = 0$

Along ③,

$$C_3 = \int_2^0 \rho\cos\phi\, d\rho\Big|_{\phi=60^\circ} = \frac{\rho^2}{2}\Big|_0^2\left(\frac{1}{2}\right) = -1$$

$$\oint_L \vec{A}\cdot\vec{dl} = C_1 + C_2 + C_3 = 2 + 0 - 1 = \underline{\underline{1}}$$

P.E.3.3

(a) $\nabla U = \frac{\partial U}{\partial x}\vec{a}_x + \frac{\partial U}{\partial y}\vec{a}_y + \frac{\partial U}{\partial z}\vec{a}_z$

$= \underline{\underline{y(2x+z)\vec{a}_x + x(x+z)\vec{a}_y + xy\,\vec{a}_z}}$

(b) $\nabla V = \frac{\partial V}{\partial \rho}\vec{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\vec{a}_\phi + \frac{\partial V}{\partial z}\vec{a}_z$

$= (z\sin\phi + 2\rho)\vec{a}_\rho + \left(z\cos\phi - \frac{z^2}{\rho}\sin 2\phi\right)\vec{a}_\phi$

$$+ (\rho\cos\phi + 2z\cos^2\phi)\vec{a}_z.$$

(c) $$\nabla f = \frac{\partial f}{\partial r}\vec{a}_r + \frac{1}{r}\frac{\partial f}{\partial\theta}\vec{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\vec{a}_\phi$$

$$= \left(\frac{\cos\theta\sin\phi}{r} + 2r\phi\right)\vec{a}_r - \frac{\sin\theta\sin\phi}{r}\ln r\,\vec{a}_\theta$$

$$+ \left(\frac{\cot\theta\cos\phi}{r}\ln r + r\operatorname{cosec}\theta\right)\vec{a}_\phi.$$

$\boxed{\text{P.E. 3.4}}$

$$\nabla\Phi = (y+z)\vec{a}_x + (x+z)\vec{a}_y + (y+x)\vec{a}_z$$

At $(1,2,3)$, $\nabla\Phi = \underline{\underline{(5,4,3)}}$.

$$\nabla\Phi\cdot\vec{a}_l = (5,4,3)\cdot\frac{(2,2,1)}{3}, \text{ where } (2,2,1) = (3,4,4)-(1,2,3)$$

$$= \frac{21}{3} = \underline{\underline{7}}.$$

$\boxed{\text{P.E. 3.5}}$

Let $f = x^2y + z - 3$, $g = x\log z - y^2 + 4$,

$$\nabla f = 2xy\,\vec{a}_x + x^2\vec{a}_y + \vec{a}_z,$$

$$\nabla g = \log z\,\vec{a}_x - 2y\,\vec{a}_y + \frac{x}{z}\vec{a}_z.$$

At $P(-1,2,1)$,

$$\vec{n}_f = \pm\frac{\nabla f}{|\nabla f|} = -\frac{(-4\vec{a}_x + \vec{a}_y + \vec{a}_z)}{\sqrt{18}}.$$

$$\cos\theta = \vec{n}_f \cdot \vec{n}_g = \pm \frac{(-5)}{\sqrt{18 \times 17}} \longrightarrow \theta = \cos^{-1}\frac{5}{17.493}$$

$$\underline{\underline{\theta = 73.39^\circ}}.$$

$\boxed{\text{P.E. 3.6}}$

(a) $\nabla \cdot \vec{A} = \dfrac{\partial A_x}{\partial x} + \dfrac{\partial A_y}{\partial y} + \dfrac{\partial A_z}{\partial z} = 0 + 4x + 0 = \underline{\underline{4x}}.$

At $(1, -2, 3)$, $\nabla \cdot \vec{A} = \underline{\underline{4}}.$

(b) $\nabla \cdot \vec{B} = \dfrac{1}{\rho}\dfrac{\partial}{\partial \rho}(\rho B_\rho) + \dfrac{1}{\rho}\dfrac{\partial B_\phi}{\partial \phi} + \dfrac{\partial B_z}{\partial z}$

$= \dfrac{1}{\rho} 2\rho z \sin\phi - \dfrac{1}{\rho} 3\rho z^2 \sin\phi$

$+ 2z\sin\phi - 3z^2\sin\phi$

$= \underline{\underline{(2-3z) z \sin\phi}}.$

At $(5, \pi/2, 1)$, $\nabla \cdot \vec{B} = (2-3)(1) = \underline{\underline{-1}}$

(c) $\nabla \cdot \vec{C} = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r^2 A_r) + \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial \theta}(A_\theta \sin\theta)$

$+ \dfrac{1}{r\sin\theta}\dfrac{\partial A_\phi}{\partial \phi} = \dfrac{1}{r^2} 6r^2 \cos\theta \cos\phi$

$= \underline{\underline{6\cos\theta\cos\phi}}.$

At $(1, \pi/6, \pi/3)$, $\nabla \cdot \vec{C} = 6\cos\frac{\pi}{6}\cos\frac{\pi}{3} = \underline{\underline{2.598}}.$

$\boxed{\text{P.E. 3.7}}$

This is similar to Example 3.7.

$$\Psi = \oint_S \vec{A}\cdot d\vec{S} = \Psi_t + \Psi_b + \Psi_c$$

$\Psi_t = 0 = \Psi_b$ since $\vec{A}$ has no z-component.

$$\Psi_c = \iint \rho^2 \cos^2\phi \, \rho \, d\phi \, dz = \rho^3 \int_{\phi=0}^{2\pi} \cos^2\phi \, d\phi \int_{z=0}^{1} dz \Big|_{\rho=4}$$

$$= (4)^3 \pi (1) = 64\pi$$

$$\Psi = 0 + 0 + 64\pi = \underline{\underline{64\pi}}$$

By the divergence theorem,

$$\oint_S \vec{A}\cdot d\vec{S} = \int_V \nabla\cdot\vec{A} \, dv$$

$$\nabla\cdot\vec{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho^3\cos^2\phi) + \frac{1}{\rho}\frac{\partial}{\partial\phi} z\sin\phi + \frac{\partial \cancel{A_z}}{\partial z}^{\,0}$$

$$= 3\rho\cos^2\phi + \frac{1}{\rho}\cos\phi$$

$$\Psi = \int_V \nabla\cdot\vec{A} \, dv = \int_V \left(3\rho\cos^2\phi + \frac{1}{\rho}\cos\phi\right)\rho \, d\phi \, dz \, d\rho$$

$$= 3\int_0^4 \rho^2 d\rho \int_0^{2\pi} \cos^2\phi \, d\phi \int_0^1 dz + \int_0^4 d\rho \underbrace{\int_0^{2\pi} \cos\phi \, d\phi}_{0} \int_0^1 dz$$

$$= 3\frac{(4)^3}{3}\pi(1) = \underline{\underline{64\pi}}$$

P.E. 3.8

(a) $\nabla \times \vec{A} = \vec{a}_x (1-0) + \vec{a}_y (y-0) + \vec{a}_z (4y - z)$

$$= \underline{\vec{a}_x + y\,\vec{a}_y + (4y-z)\,\vec{a}_z}$$

At $(1,-2,3)$, $\nabla \times \vec{A} = \underline{\vec{a}_x - 2\vec{a}_y - 11\vec{a}_z}$

(b) $\nabla \times \vec{B} = \vec{a}_\rho (0 - 6\rho z \cos\phi) + \vec{a}_\phi (\rho \sin\phi - 0)$

$$+ \vec{a}_z \frac{1}{\rho} (6\rho z^2 \cos\phi - \rho z \cos\phi)$$

$$= \underline{-6\rho z \cos\phi\, \vec{a}_\rho + \rho \sin\phi\, \vec{a}_\phi + (6z-1) z \cos\phi\, \vec{a}_z}$$

At $(5, \pi/2, -1)$, $\nabla \times \vec{B} = \underline{5\,\vec{a}_\phi}$

(c) $\nabla \times \vec{C} = \vec{a}_r \dfrac{1}{r\sin\theta} (r^{1/2} \cos\theta - 0)$

$$+ \frac{\vec{a}_\theta}{r} \left(-\frac{2r\cos\theta \sin\phi}{\sin\theta} - \frac{3}{2} r^{1/2} \right)$$

$$+ \frac{\vec{a}_\phi}{r} (0 - 2r\sin\theta\cos\phi)$$

$$= r^{-1/2} \cot\theta\, \vec{a}_r - \left(2\cot\theta \sin\phi + \frac{3}{2} r^{-1/2}\right) \vec{a}_\theta$$

$$\underline{- 2\sin\theta\cos\phi\, \vec{a}_\phi}$$

At $(1, \pi/6, \pi/3)$,

$$\nabla \times \vec{C} = \underline{1.732\,\vec{a}_r - 4.5\,\vec{a}_\theta - 0.5\,\vec{a}_\phi}$$

$\boxed{\text{P.E. 3.9}}$

$$\oint_L \vec{A}\cdot d\vec{l} = \int_S (\nabla\times\vec{A})\cdot d\vec{S}$$

$$\nabla\times\vec{A} = \sin\phi\,\vec{a}_z + \frac{z\cos\phi}{\rho}\vec{a}_\rho, \quad d\vec{S} = \rho\, d\phi\, d\rho\, \vec{a}_z.$$

$$\int_S (\nabla\times\vec{A})\cdot d\vec{S} = \iint \rho\sin\phi\, d\phi\, d\rho = \frac{\rho^2}{2}\Big|_0^2 (-\cos\phi)\Big|_0^{60^\circ}$$

$$= 2\left(-\frac{1}{2}+1\right) = \underline{\underline{1}}.$$

$\boxed{\text{P.E. 3.10}}$

$$\nabla\times\nabla V = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ \frac{\partial V}{\partial x} & \frac{\partial V}{\partial y} & \frac{\partial V}{\partial z} \end{vmatrix} = \left(\frac{\partial^2 V}{\partial y\partial z} - \cancel{\frac{\partial^2 V}{\partial z\partial y}}\right)\vec{a}_x$$

$$+\left(\frac{\partial^2 V}{\partial x\partial z} - \cancel{\frac{\partial^2 V}{\partial z\partial x}}\right)\vec{a}_y + \left(\frac{\partial^2 V}{\partial x\partial y} - \cancel{\frac{\partial^2 V}{\partial y\partial x}}\right)\vec{a}_z = 0.$$

$\boxed{\text{P.E. 3.11}}$

(a) $$\nabla^2 U = \frac{\partial}{\partial x}(2xy+yz) + \cancel{\frac{\partial}{\partial y}(x^2+xz)}^{\,0} + \cancel{\frac{\partial}{\partial z}(xy)}^{\,0}$$

$$= \underline{\underline{2y}}.$$

(b) $$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\left(z\sin\phi + 2\rho\right) + \frac{1}{\rho^2}\Big(-\rho z\sin\phi$$

$$-2z^2\frac{\partial}{\partial\phi}\sin\phi\cos\phi\Big) + \frac{\partial}{\partial z}\left(\rho\sin\phi + 2z\cos^2\phi\right)$$

$$= \frac{1}{\rho}(z\sin\phi + 4\rho) - \frac{1}{\rho^2}(z\rho\sin\phi + 2z^2\cos 2\phi)$$

$$+ 2\cos^2\phi$$

$$= 4 + 2\cos^2\phi - \frac{2z^2}{\rho^2}\cos 2\phi .$$

(c) $$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left[\frac{1}{r^2}\,\frac{1}{r}\cos\theta\sin\phi + 2r^2\phi\right]$$

$$+ \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(-\sin^2\theta\sin\phi\ln r\right)$$

$$+ \frac{1}{r^2\sin^2\theta}\left(-\cos\theta\sin\theta\ln r\right)$$

$$= \frac{1}{r^2}\cos\theta\sin\phi\left(1 - 2\ln r - \operatorname{cosec}^2\theta\ln r\right) + 6\phi$$

$\boxed{\text{P.E. 3.12}}$

If $\vec{B}$ is conservative, $\nabla\times\vec{B} = 0$ must be satisfied.

$$\nabla\times\vec{B} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y + z\cos xz & x & x\cos xz \end{vmatrix}$$

$$= 0\,\vec{a}_x + (\cos xz - xz\sin xz - \cos xz + xz\sin xz)\,\vec{a}_y + (1-1)\,\vec{a}_z$$

$$= 0$$

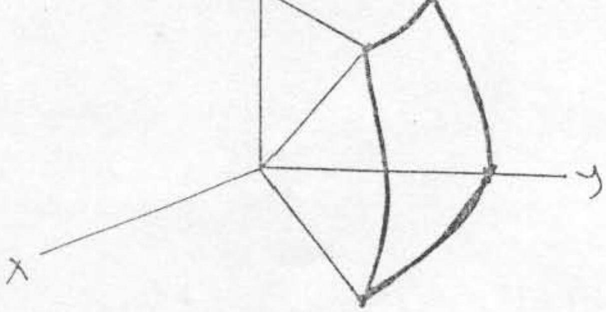
Hence $\vec{B}$ is a conservative field.

Prob. 3.1

(a) $L = \int dl = \int_{30^\circ}^{90^\circ} \rho\, d\phi \Big|_{\rho=10} = 10\left(\frac{\pi}{2} - \frac{\pi}{6}\right) = \frac{10\pi}{3} = \underline{\underline{10.47}}$

(b) $L = \int r\, d\theta = 5\left(\frac{3\pi}{4} - \frac{\pi}{4}\right) = \frac{5\pi}{2} = \underline{\underline{7.854}}$

(c)



$$L = 2\int_{\pi/6}^{\pi/2} r\, d\theta \Big|_{r=10} + r\left(\sin\pi/2 + \sin\pi/6\right)\int_{\pi/4}^{\pi/2} d\phi \Big|_{r=10}$$

$$= \frac{20\pi}{3} + \frac{15\pi}{4} = \frac{125\pi}{12} = \underline{\underline{32.72}}$$

Prob. 3.2

(a) $S = \int_{\rho=0}^{10}\int_{\phi=0}^{\pi} \rho\, d\phi\, d\rho = \pi \frac{\rho^2}{2}\Big|_0^{10} = 50\pi = \underline{\underline{157.1}}$

(b) $dS = \rho\, d\phi\, dz$, $S = \int_{z=0}^{4}\int_{\phi=\frac{\pi}{4}}^{\pi/2} \rho\, d\phi\, dz \Big|_{\rho=50}$

$$= 50(4)\left(\pi/4\right) = 50\pi = \underline{\underline{157.1}}$$

(c) $dS = r^2 \sin\theta\, d\theta\, d\phi$,

$$S = \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} r^2 \sin\theta \, d\theta \, d\phi \Big|_{r=5}$$

$$= 25(2\pi)(-\cos\theta)\Big|_0^{\pi/2} = 50\pi = \underline{\underline{157.1}}$$

(d) $dS = r\sin\theta \, dr \, d\phi,$

$$S = \int_{r=0}^{10} \int_{\phi=0}^{2\pi} r\sin\theta \, dr \, d\phi \Big|_{\theta=30^\circ}$$

$$= \sin 30^\circ (2\pi)\left(\frac{100}{2}\right) = 50\pi = \underline{\underline{157.1}}$$

(e) $dS = r^2 \sin\theta \, d\theta \, d\phi,$

$$S = (8.26)^2 \int_0^{2\pi} d\phi \int_{30^\circ}^{60^\circ} \sin\theta \, d\theta = (2\pi)^2 (2\pi)(-0.5 + 0.866)$$

$$= 50\pi = \underline{\underline{157.1}}$$

Prob. 3.3

(a) $dv = \rho \, d\rho \, d\phi \, dz,$

$$v = \int_1^2 \rho \, d\rho \int_{-1}^1 dz \int_{\pi/6}^{3\pi/4} d\phi = \frac{\rho^2}{2}\Big|_1^2 (2)\left(\frac{3\pi}{4} - \frac{\pi}{6}\right)$$

$$= \frac{7\pi}{4} = \underline{\underline{5.498}}$$

(b) $dv = r^2 \sin\theta \, dr \, d\theta \, d\phi,$

$$V = \int_3^5 r^2 dr \int_{\pi/3}^{\pi/2} \sin\theta\, d\theta \int_{5\pi/6}^{\pi} d\phi$$

$$= \frac{r^3}{3}\Big|_3^5 \left(-\cos\theta\Big|_{\pi/3}^{\pi/2}\right)\left(\pi - \frac{5\pi}{6}\right) = \frac{49\pi}{18} = \underline{\underline{8.552}}.$$

(c) $dV = dx\,dy\,dz$,

$$V = \int_{-4}^{4} dz \int_{x=0}^{2}\int_{y=0}^{x} dy\,dx = 8\int_{x=0}^{2} y\Big|_0^x dx$$

$$= 8\int_0^2 x\,dx = 8\frac{x^2}{2}\Big|_0^2 = \underline{\underline{16}}$$

$\boxed{\text{Prob. 3.4}}$

$$\oint_L \vec{F}\cdot d\vec{l} = \int (\nabla\times\vec{F})\cdot d\vec{S} \quad - \text{ stokes's theorem}$$

$$\nabla\times\vec{F} = -7\vec{a}_z, \quad d\vec{S} = \pm\, dx\,dy\,\vec{a}_z$$

(a) $\oint \vec{F}\cdot d\vec{l} = \pm\iint(-7)\,dx\,dy = \pm 7\pi r^2 = \underline{\underline{\pm 28\pi}}.$

(b) $\oint \vec{F}\cdot d\vec{l} = \pm 7\int_{-1}^{1}\int_{-1}^{1} dx\,dy = \pm 7(2)(2) = \underline{\underline{\pm 28}}.$

$\boxed{\text{Prob. 3.5}}$

(a) $$\int \vec{F}\cdot d\vec{l} = \int_{y=0}^{1}(x^2 - z^2)\,dy\Big|_{x=0,\, z=0} + \int_{x=0}^{2} 2xy\,dx\Big|_{y=1,\, z=0}$$

$$+ \int_{z=0}^{3}(-3xz^2)\,dz\Big|_{x=0,\, y=1}$$

$$= 0 + 2(1)\,\frac{x^2}{2}\Big|_0^2 - 3(2)\,\frac{z^3}{3}\Big|_0^3$$

$$= 0 + 4 - 54 = \underline{\underline{-50}}.$$

(b) Let $x = 2t,\ y = t,\ z = 3t,$

$dx = 2dt,\ dy = dt,\ dz = 3dt.$

$$\int \vec{F}\cdot d\vec{l} = \int_0^1 (8t^2 - 5t^2 - 162t^3)\,dt = -\frac{79}{2}$$

$$= \underline{\underline{-39.5}}.$$

$\boxed{\text{Prob. 3.6}}$

$$\int \vec{H}\cdot d\vec{l} = \int_{x=0}^{1} (x-y)\,dx\Big|_{y=0} + \int_{z=0}^{1} (x^2 + \underbrace{zy}_{0})\,dz\Big|_{y=0,\,z=0}$$

$$+ \int (x^2 + zy)\,dy + 5yz\,dz\Big|_{x=0,\ z=1-y/2}$$

$$= \frac{x^2}{2}\Big|_0^1 + 0 + \int_{y=0}^{2} y\left(1-\frac{y}{2}\right)dy + 5y\left(1-\frac{y}{2}\right)\left(-\frac{dy}{2}\right)$$

$$= -\frac{1}{2} + \int_0^2 \left(-\frac{3}{2}y + \frac{3y^2}{4}\right)dy$$

$$= \frac{1}{2} + \left(-\frac{3}{4}y^2 + \frac{y^4}{4}\right)\Big|_0^2 = -\frac{1}{2} - 3 + 2$$

$$= \underline{\underline{-1.5}}.$$

Prob. 3.7

Let the integrand be $\vec{F}$

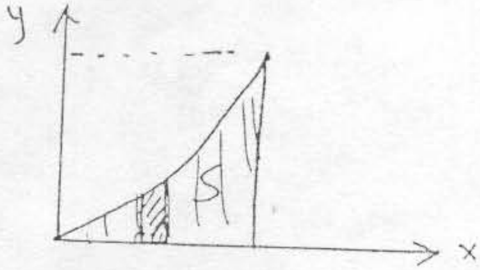
$$\vec{F}(x,y,z) \longrightarrow \vec{F}(\rho,\phi,z) = -\frac{\rho\sin\phi\,\vec{a}_x + \rho\cos\phi\,\vec{a}_y}{\rho}$$

$$\begin{bmatrix} F_\rho \\ F_\phi \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} -\sin\phi \\ \cos\phi \end{bmatrix}$$

$$\vec{F} = \vec{a}_\phi, \quad \vec{F}\cdot d\vec{l} = \rho\, d\phi$$

$$\int \vec{F}\cdot d\vec{l} = \int_0^{2\pi} \rho\, d\phi = \underline{\underline{2\pi a}}.$$

Prob. 3.8



$$\int \rho_s\, dS = \int_{x=0}^{1}\int_{y=0}^{x^2} (x^2 + xy)\, dy\, dx$$

$$= \int_0^1 \left(x^2 y + \frac{xy^2}{2} \Big|_0^{x^2} \right) dx = \int_0^1 \left(x^4 + \frac{x^5}{2}\right) dx$$

$$= \frac{1}{5} + \frac{1}{12} = \frac{17}{60} = \underline{\underline{0.2833}}$$

Prob. 3.9

$$\Psi = \oint_s \vec{A}\cdot d\vec{S} = \int_v \nabla\cdot\vec{A}\, dv = \iiint (4z - y)\, dx\, dy\, dz$$

$$= 4\int_0^1\int_0^1\int_0^1 z\,dx\,dy\,dz - \int_0^1\int_0^1\int_0^1 y\,dx\,dy\,dz$$

$$= 3(1)(1)\left(\tfrac{1}{2}\right) = \underline{\underline{1.5}}$$

Prob. 3.10

$$\oint_S \vec{A}\cdot\vec{dS} = \int_V \nabla\cdot\vec{A}\,dv$$

$$= \int\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + 0 + 0\right)dv$$

$$= \iiint\left(\frac{1}{r^2}\cdot 4r^3\right)r^2\sin\theta\,d\theta\,d\phi\,dr$$

$$= 4\int_0^5 r^3\,dr\int_0^{2\pi}d\phi\int_0^{\pi/2}\sin\theta\,d\theta$$

$$= (5)^4(2\pi)(1) = \underline{\underline{1250\pi}}$$

Prob. 3.11

$$V = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\alpha}\int_{r=0}^{a} r^2\sin\theta\,d\theta\,dr\,d\phi$$

$$= \frac{2\pi a^3}{3}(1-\cos\alpha)$$

$$V(\alpha=\pi/3) = \frac{2\pi a^3}{3}\left(1-\tfrac{1}{2}\right) = \underline{\underline{\pi a^3/3}}$$

$$V(\alpha=\pi/2) = \frac{2\pi a^3}{3}(1-0) = \underline{\underline{2\pi a^3/3}}$$

$\boxed{\text{Prob. 3.12}}$

$$\text{volume} = \int dv = \int_{\phi=0}^{2\pi}\int_{\rho=0}^{1}\int_{z=0}^{e^{-\rho^2}} \rho\, d\rho\, d\phi\, dz$$

$$= 2\pi \int_0^1 \rho e^{-\rho^2} d\rho = 2\pi \left(-\frac{1}{2} e^{-\rho^2}\right)\Big|_0^1$$

$$= \pi(1-e^{-1}) = \underline{1.986}$$

$\boxed{\text{Prob. 3.13}}$

$$\int g\, dv = \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi}\int_{r=0}^{1} r^4 \sin\theta\, dr\, d\phi\, d\theta$$

$$= 2\pi \frac{r^5}{5}\Big|_0^1 (-\cos\theta)\Big|_0^{\pi/2} = \frac{2\pi}{5} = \underline{\underline{1.257}}$$

$\boxed{\text{Prob. 3.14}}$

(a) $\nabla = \frac{\partial}{\partial x}\vec{a}_x + \frac{\partial}{\partial y}\vec{a}_y + \frac{\partial}{\partial z}\vec{a}_z$

$$= \left(\cos\phi \frac{\partial}{\partial\rho} - \frac{\sin\phi}{\rho}\frac{\partial}{\partial\phi}\right)\left(\cos\phi\, \vec{a}_\rho - \sin\phi\, \vec{a}_\phi\right)$$

$$+ \left(\sin\phi \frac{\partial}{\partial\rho} + \frac{\cos\phi}{\rho}\frac{\partial}{\partial\phi}\right)\left(\sin\phi\, \vec{a}_\rho + \cos\phi\, \vec{a}_\phi\right)$$

$$+ \frac{\partial}{\partial z}\vec{a}_z$$

$$= \frac{\partial}{\partial\rho}(\cos^2\phi + \sin^2\phi)\vec{a}_\rho + \frac{1}{\rho}\frac{\partial}{\partial\phi}(\sin^2\phi + \cos^2\phi)\vec{a}_\phi$$

$$+ \frac{\partial}{\partial z}\vec{a}_z = \frac{\partial}{\partial\rho}\vec{a}_\rho + \frac{1}{\rho}\frac{\partial}{\partial\phi}\vec{a}_\phi + \frac{\partial}{\partial z}\vec{a}_z$$

(b) $\nabla = \frac{\partial}{\partial x}\vec{a}_x + \frac{\partial}{\partial y}\vec{a}_y + \frac{\partial}{\partial y}\vec{a}_z$

$$= \left(\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\cos\phi\frac{\partial}{\partial\theta} - \frac{\sin\phi}{\rho}\frac{\partial}{\partial\phi}\right) \times$$

$$\left(\sin\theta\cos\phi\,\vec{a}_r + \cos\theta\cos\phi\,\vec{a}_r - \sin\phi\,\vec{a}_\phi\right)$$

$$+ \left(\sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{\rho}\frac{\partial}{\partial\phi}\right) \times$$

$$\left(\sin\theta\sin\phi\,\vec{a}_r + \cos\theta\sin\phi\,\vec{a}_\theta + \cos\phi\,\vec{a}_\phi\right)$$

$$+ \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta\,\vec{a}_r - \sin\theta\,\vec{a}_\theta\right)$$

$$= \vec{a}_r\left(\sin^2\theta\cos^2\phi\frac{\partial}{\partial r} + \cancel{\frac{\sin\theta\cos\theta}{r}\cos^2\phi\frac{\partial}{\partial\theta}}\right.$$

$$- \cancel{\frac{\sin\theta\sin\phi\cos\phi}{\rho}\frac{\partial}{\partial\phi}} + \frac{\sin^2\theta\sin^2\phi}{1}\frac{\partial}{\partial r}$$

$$+ \cancel{\frac{\sin\theta\cos\theta\sin^2\phi}{r}\frac{\partial}{\partial\theta}} + \cancel{\frac{\sin\theta\cos\phi\sin\phi}{\rho}\frac{\partial}{\partial\phi}}$$

$$\left. + \cos^2\theta\frac{\partial}{\partial r} - \frac{\sin\theta\cos\theta}{r}\frac{\partial}{\partial\theta}\right)$$

$$+ \vec{a}_\theta\left(\sin\theta\cos\theta\cos^2\phi\frac{\partial}{\partial r} + \cos^2\theta\cos^2\phi\frac{\partial}{\partial\theta}\right.$$

$$- \cancel{\frac{\cos\theta\sin\phi\cos\phi}{\rho}\frac{\partial}{\partial\phi}} + \cancel{\sin\theta\cos\theta\sin^2\phi\frac{\partial}{\partial r}}$$

$$+ \cos^2\theta\sin^2\phi\frac{\partial}{\partial\theta} + \cancel{\frac{\cos\theta\sin\phi\cos\phi}{\rho}\frac{\partial}{\partial\phi}}$$

$$-\sin\theta\cos\theta\frac{\partial}{\partial r} + \frac{\sin^2\theta}{r}\frac{\partial}{\partial\theta}\Big)$$

$$+\vec{a}_\phi\Big(-\sin\theta\cos\phi\sin\phi\frac{\partial}{\partial r} - \frac{\cos\theta\cos\phi\sin\phi}{r}\frac{\partial}{\partial\theta}$$

$$+\frac{\sin^2\phi}{\rho}\frac{\partial}{\partial\phi} + \sin\theta\sin\phi\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi\cos\phi}{r}\frac{\partial}{\partial\theta}$$

$$+\frac{\cos^2\phi}{\rho}\frac{\partial}{\partial\phi}\Big)$$

$$= \frac{\partial}{\partial r}\vec{a}_r + \frac{1}{r}\frac{\partial}{\partial\theta}\vec{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\vec{a}_\phi$$

$\boxed{\text{Prob. 3.15}}$

(a) $$\nabla U = \frac{\partial U}{\partial x}\vec{a}_x + \frac{\partial U}{\partial y}\vec{a}_y + \frac{\partial U}{\partial z}\vec{a}_z$$

$$= 4z^2\vec{a}_x + 3z\vec{a}_y + (8xz+3y)\vec{a}_z$$

(b) $$\nabla V = 2e^{(2x+3y)}\cos 5z\,\vec{a}_x + 3e^{(2x+3y)}\cos 5z\,\vec{a}_y - 5e^{(2x+3y)}\sin 5z\,\vec{a}_z$$

(c) $$\nabla T = \frac{\partial T}{\partial\rho}\vec{a}_\rho + \frac{1}{\rho}\frac{\partial T}{\partial\phi}\vec{a}_\phi + \frac{\partial T}{\partial z}\vec{a}_z$$

$$= 5e^{-2z}\sin\phi\,\vec{a}_\rho + 5e^{-2z}\cos\phi\,\vec{a}_\phi - 10\rho e^{-2z}\sin\phi\,\vec{a}_z$$

(d) $$\nabla W = 2(z^2+1)\cos\phi\,\vec{a}_\rho - 2(z^2+1)\sin\phi\,\vec{a}_\phi + 4\rho z\cos\phi\,\vec{a}_z$$

(e) $\nabla H = \frac{\partial H}{\partial r}\vec{a}_r + \frac{1}{r}\frac{\partial H}{\partial \theta}\vec{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial H}{\partial \phi}\vec{a}_\phi$

$$= \underline{\underline{2r\cos\theta\cos\phi\,\vec{a}_r - r\sin\theta\cos\phi\,\vec{a}_\theta - r\cot\theta\sin\phi\,\vec{a}_\phi}}$$

(f) $\nabla Q = \underline{\underline{\frac{-3}{r^4}\sin\theta\sin\phi\,\vec{a}_r + \frac{\cos\theta\sin\phi}{r^4}\vec{a}_\theta + \frac{\cos\phi}{r^4}\vec{a}_\phi}}$

Prob. 3.16

Let $f = xy + yz + zx - 5 = 0$

$$\nabla f = \frac{\partial f}{\partial x}\vec{a}_x + \frac{\partial f}{\partial y}\vec{a}_y + \frac{\partial f}{\partial z}\vec{a}_z$$

$$= (y+z)\vec{a}_x + (x+z)\vec{a}_y + (x+y)\vec{a}_z.$$

At $(-1,3,4)$, $\nabla f = 7\vec{a}_x + 3\vec{a}_y + 2\vec{a}_z$

$$\vec{a}_n = \pm\frac{\nabla f}{|\nabla f|} = \pm\frac{(7,3,2)}{\sqrt{62}}$$

$$= \underline{\pm(0.889\,\vec{a}_x + 0.381\,\vec{a}_y + 0.254\,\vec{a}_z)}$$

Prob. 3.17

$$\nabla T = 2x\,\vec{a}_x + 2y\,\vec{a}_y - \vec{a}_z$$

At $(1,1,2)$, $\nabla T = (2,2,-1)$. The mosquito should move in the direction of

$$\underline{\underline{2\vec{a}_x + 2\vec{a}_y - \vec{a}_z}}.$$

Prob. 3.18

(a) $\nabla \cdot \vec{A} = \underline{\underline{y e^{xy} + x\cos xy - 2x\cos zx \sin zx}}$

$$\nabla \times \vec{A} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{xy} & \sin xy & \cos^2 xz \end{vmatrix}$$

$$= (0-0)\bar{a}_x + (0 + 2z\cos xz \sin xz)\bar{a}_y + (y\cos xy - xe^{xy})\bar{a}_z$$

$$= \underline{\underline{z\sin 2xz\, \bar{a}_y + (y\cos xy - xe^{xy})\bar{a}_z}}$$

(b) $\nabla \cdot \vec{B} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho^2 z^2 \cos\phi) + 0 + \sin^2\phi$

$$= \underline{\underline{2z^2\cos\phi + \sin^2\phi}}$$

$$\nabla \times \vec{B} = \left(\frac{1}{\rho}\frac{\partial B_z}{\partial \phi} - 0\right)\bar{a}_\rho + \left(\frac{\partial B_\rho}{\partial z} - \frac{\partial B_z}{\partial \rho}\right)\bar{a}_\phi + \frac{1}{\rho}\left(0 - \frac{\partial B_\rho}{\partial \phi}\right)\bar{a}_z$$

$$= \underline{\underline{\frac{z\sin 2\phi}{\rho}\bar{a}_\rho + 2\rho z\cos\phi\, \bar{a}_\phi + z^2\sin\phi\, \bar{a}_z}}$$

(c) $\nabla \cdot \vec{C} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^3\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(-\frac{1}{r}\sin^2\theta\right) + 0$

$$= \underline{\underline{3\cos\theta - \frac{2\cos\theta}{r^2}}}$$

$$\nabla \times \vec{C} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(2r^2\sin^2\theta\right) - 0\right]\vec{a}_r$$
$$+ \frac{1}{r}\left[0 - \frac{\partial}{\partial r}\left(2r^3\sin\theta\right)\right]\vec{a}_\theta$$
$$+ \frac{1}{r}\left[\underbrace{\cancel{\frac{\partial}{\partial r}(-\sin\theta)}}_{0} + r\sin\theta\right]\vec{a}_\phi$$
$$= 4r\cos\theta\,\vec{a}_r - 6r\sin\theta\,\vec{a}_\theta + \sin\theta\,\vec{a}_\phi.$$

$\boxed{\text{Prob. 3.19}}$

(a) $$\nabla\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \underbrace{\cancel{\frac{\partial A_z}{\partial z}}}_{0} = 1 + yz + 2y$$
$$= 1 + 2y + yz$$

$$\nabla\times\vec{A} = (z-x)\vec{a}_x + (xy-2x)\vec{a}_y + (2x - z - xz)\vec{a}_z$$

(b) $$\nabla\cdot\vec{B} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho B_\rho) + \frac{1}{\rho}\frac{\partial B_\phi}{\partial\phi} + \underbrace{\cancel{\frac{\partial B_z}{\partial z}}}_{0} = 0 + \frac{1}{\rho}2\rho\cos\phi$$
$$= 2\cos\phi.$$

$$\nabla\times\vec{B} = \vec{a}_\rho\left(-\frac{1}{\rho}2\sin\phi\cos\phi - \rho\sin\phi\right) + \vec{a}_\phi\left(\frac{1}{\rho} - 0\right)$$
$$+ \frac{\vec{a}_z}{\rho}\left(\cos^2\phi - 0\right)$$
$$= -\vec{a}_\rho\left(\frac{\sin 2\phi}{\rho} - \rho\sin\phi\right) + \frac{\vec{a}_\phi}{\rho} + \vec{a}_z\frac{\cos^2\phi}{\rho}$$

(c) $\nabla\cdot\vec{C} = \frac{1}{r^2}\left(4r^3 + 2r\sin\theta\cos\phi\right)$

$+ \frac{1}{r\sin\theta}\left(2r\sin\theta\cos\theta\cos\phi\right) + \frac{1}{r\sin\theta}\cos\phi$

$= 4r + \frac{2\sin\theta\cos\phi}{r} + 2\cos\theta\cos\phi + \frac{\cos\phi}{r\sin\theta}$

$\nabla\times\vec{C} = \vec{a}_r \frac{1}{r\sin\theta}\Big[\frac{1}{r}\sec^2\theta\sin\theta + \frac{1}{r}\tan\theta\cos\theta$

$+ \sin\phi\cos\theta - r\sin\theta\sin\phi\Big]$

$+ \frac{\vec{a}_\theta}{r}\left[-\frac{1}{\sin\theta}\sin\theta\sin\phi - \sin\phi\right]$

$+ \frac{\vec{a}_\phi}{r}\left[2r\sin\theta\cos\phi - \cos\theta\cos\phi\right]$

$= \vec{a}_r\left[\frac{\sec^2\theta}{r^2} + \frac{1}{r^2} + \sin\phi\cot\theta - \sin\phi\right]$

$- \frac{\vec{a}_\theta}{r} 2\sin\phi + \vec{a}_\phi\left(2\sin\theta\cos\phi - \frac{\cos\theta\cos\phi}{r}\right]$

$\boxed{\text{Prob. 3.20}}$

(a) $\text{grad}(r) = \text{grad}\left(x^2+y^2+z^2\right)^{1/2}$

$= \left(x^2+y^2+z^2\right)^{-1/2}\left(\frac{1}{2}2x\,\vec{a}_x + \frac{1}{2}2y\,\vec{a}_y + \frac{1}{2}2z\,\vec{a}_z\right)$

$= \frac{\vec{r}}{r}$

(b) $\text{grad}\left(\frac{1}{r}\right) = \text{grad}\left(x^2+y^2+z^2\right)^{-1/2}$

$= \left(x^2+y^2+z^2\right)^{-3/2}\left(-\frac{1}{2}2x\,\vec{a}_x - \frac{1}{2}2y\,\vec{a}_y - \frac{1}{2}2z\,\vec{a}_z\right)$

$$= \frac{-\vec{r}}{r^3}.$$

(c) $\operatorname{grad}(\vec{A}\cdot\vec{r}) = \operatorname{grad}(A_x x + A_y y + A_z z)$

$$= A_x \vec{a}_x + A_y \vec{a}_y + A_z \vec{a}_z = \vec{A}.$$

since $\vec{A}$ is a uniform vector meaning that A_x, A_y, and A_z are independent of $x, y,$ and z.

(d) $(\vec{U}\cdot\operatorname{grad})\vec{r} = \left(U_x \frac{\partial}{\partial x} + U_y \frac{\partial}{\partial y} + U_z \frac{\partial}{\partial z}\right)\vec{r}$

$$= U_x \vec{a}_x + U_y \vec{a}_y + U_z \vec{a}_z = \vec{U}.$$

(e) $\operatorname{div}\vec{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1+1+1 = 3.$

(f) $\operatorname{div}(r^n\vec{r}) = \operatorname{div}\left[(x^2+y^2+z^2)^{n/2}(x\vec{a}_x + y\vec{a}_y + z\vec{a}_z)\right]$

$$= (x^2+y^2+z^2)^{n/2} + 2x^2 \cdot \frac{n}{2}(x^2+y^2+z^2)^{\frac{n-2}{2}}$$

$$+ (x^2+y^2+z^2)^{n/2} + 2y^2 \frac{n}{2}(x^2+y^2+z^2)^{\frac{n-2}{2}}$$

$$+ (x^2+y^2+z^2)^{n/2} + 2z^2 \frac{n}{2}(x^2+y^2+z^2)^{\frac{n-2}{2}}$$

$$= 3r^n + n(x^2+y^2+z^2)(x^2+y^2+z^2)^{\frac{n}{2}-1}$$

$$= 3r^n + nr^n = (n+3)r^n.$$

(g) $\operatorname{Curl}\vec{r} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0$

(h) Laplacian $\frac{1}{r} = \nabla\cdot\nabla\left(\frac{1}{r}\right)$

from (b), $\nabla\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$, hence

$$\nabla^2\left(\frac{1}{r}\right) = \text{div}\left(-r^3\,\vec{r}\right)$$

which is (A) if $n=3$. Hence

$$\nabla^2\left(\frac{1}{r}\right) = -\left(-3+3\right)r^{-3} = 0.$$

$\boxed{\text{Prob. 3.21}}$

(a) Let $\vec{F} = \vec{r}\,\Phi(r) = r\Phi\,\vec{a}_r$

since $\vec{r} = x\vec{a}_x + y\,\vec{a}_y + z\,\vec{a}_z = r\,\vec{a}_r$.

$$\nabla\cdot\vec{F} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 F_r\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^3\Phi\right)$$

$$= \frac{3r^2}{r^2}\Phi + \frac{r^3}{r^2}\Phi' = 3\Phi(r) + r\,\Phi'.$$

(b) Let $\vec{F} = \frac{\vec{r}}{r^3} = \frac{\vec{a}_r}{r^2}$

$$\nabla\cdot\vec{F} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 F_r\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(1) = 0.$$

(c) Let $V = r^n$

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) += \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 n\, r^{n-1}\right)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(n r^{n+1}\right) = n(n+1)r^{n-2}$$

(d) Let $\vec{A} = r^n\vec{r} = x r^n\vec{a}_x + y r^n\vec{a}_y + z r^n\vec{a}_z$

$$\nabla^2\vec{A} = \nabla^2(x r^n)\,\vec{a}_x + \nabla^2(y r^n)\,\vec{a}_y + \nabla^2(z r^n)\,\vec{a}_z$$

But $\nabla^2(xr^n) = \nabla^2\left[x(x^2+y^2+z^2)^{n/2}\right]$

$$= \frac{\partial^2}{\partial x^2}\left[x(x^2+y^2+z^2)^{n/2}\right]$$

$$+ \frac{\partial^2}{\partial y^2}\left[x(x^2+y^2+z^2)^{n/2}\right]$$

$$+ \frac{\partial^2}{\partial z^2}\left[x(x^2+y^2+z^2)^{n/2}\right]$$

$$= \frac{\partial}{\partial x}\left[(x^2+y^2+z^2)^{n/2} + \frac{n}{2}(2x)x(x^2+y^2+z^2)^{\frac{n}{2}-1}\right]$$

$$+ \frac{\partial}{\partial y}\left[\frac{n}{2}(2y)x(x^2+y^2+z^2)^{\frac{n}{2}-1}\right]$$

$$+ \frac{\partial}{\partial z}\left[\frac{n}{2}(2z)x(x^2+y^2+z^2)^{\frac{n}{2}-1}\right]$$

$$= \frac{n}{2}(2x)(x^2+y^2+z^2)^{\frac{n}{2}-1} + 2nx(x^2+y^2+z^2)^{\frac{n}{2}-}$$

$$+ nx^2(2x)\left(\frac{n}{2}-1\right)(x^2+y^2+z^2)^{n/2-2}.$$

$$+ nx(x^2+y^2+z^2)^{n/2-1} + nxy(2y)\left(\frac{n}{2}-1\right)(x^2+y^2+z^2)^{\frac{n}{2}-2}$$

$$+ nx(x^2+y^2+z^2)^{\frac{n}{2}-1} + nxz(2z)\left(\frac{n}{2}-1\right)(x^2+y^2+z^2)^{\frac{n}{2}-2}$$

$$= 5nx(x^2+y^2+z^2)^{n/2-1}$$

$$+ nx(x^2+y^2+z^2)(n-2)(x^2+y^2+z^2)^{\frac{n}{2}-2}$$

$$= 5nx\,(x^2+y^2+z^2)^{\frac{n}{2}-1} + n(n-2)\,x\,(x^2+y^2+z^2)^{\frac{n}{2}-1}$$

$$= \left(5nx - n^2x - 2nx\right)(x^2+y^2+z^2)^{\frac{n}{2}-1}$$

$$\nabla^2(x r^n) = n(n+3)\,x\,r^{n-2}$$

Similarly,

$$\nabla^2(y r^n) = n(n+3)\,y\,r^{n-2}$$

$$\nabla^2(z r^n) = n(n+3)\,z\,r^{n-2}$$

Hence,

$$\nabla^2 \vec{A} = n(n+3)\,r^{n-2}\,(x\,\vec{a}_x + y\,\vec{a}_y + z\,\vec{a}_z)$$

$$= n(n+3)\,r^{n-2}\,\vec{r}.$$

Prob. 3.22

$$\nabla\cdot\vec{H} = k\,\nabla\cdot\nabla T = k\,\nabla^2 T$$

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 50\sin\frac{\pi x}{2}\cosh\frac{\pi y}{2}\left(-\frac{\pi^2}{4}+\frac{\pi^2}{4}\right)$$

$$= 0$$

Hence $\nabla\cdot\vec{H} = 0$

Prob. 3.23

(a) $(\nabla\cdot\vec{r})\vec{T} = 3\vec{T} = \underline{\underline{6yz\,\vec{a}_x + 3xy^2\,\vec{a}_y + 3x^2yz\,\vec{a}_z}}$

(b) $(\vec{r}\cdot\nabla)\vec{T} = x\dfrac{\partial\vec{T}}{\partial x} + y\dfrac{\partial\vec{T}}{\partial y} + z\dfrac{\partial\vec{T}}{\partial z}$

$$= x\,(y^2\,\vec{a}_y + 2xyz\,\vec{a}_z)$$
$$+ y\,(2z\,\vec{a}_x + 2xy\,\vec{a}_y + x^2z\,\vec{a}_z)$$
$$+ z\,(2y\,\vec{a}_x + x^2y\,\vec{a}_z)$$

$$= \underline{\underline{4yz\,\vec{a}_x + 3xy^2\,\vec{a}_y + 4x^2yz\,\vec{a}_x}}.$$

(c) $\nabla\cdot\vec{r}\,(\vec{r}\cdot\vec{T}) = 3\,(2xyz + xy^3 + x^2yz^2)$

$$= \underline{\underline{6xyz + 3xy^3 + 3x^2yz^2}}$$

(d) $(\vec{r}\cdot\nabla)\vec{r} = \left(x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y} + z\dfrac{\partial}{\partial z}\right)(x^2+y^2+z^2)$

$$= x\cdot 2x + y\cdot 2y + z\cdot 2z$$
$$= \underline{\underline{2\,(x^2+y^2+z^2) = 2r^2}}$$

Prob. 3.24

(a) $\dfrac{\partial V_1}{\partial x} = -3x\,(x^2+y^2+z^2)^{-5/2}$

$$\frac{\partial^2 V_1}{\partial x^2} = 15x^2(x^2+y^2+z^2)^{-7/2} - 3\,(x^2+y^2+z^2)^{-5/2}$$

Hence,

$$\nabla^2 V_1 = 15(x^2+y^2+z^2)(x^2+y^2+z^2)^{-7/2} - 9(x^2+y^2+z^2)^{-5/2}$$

$$= \underline{\underline{6(x^2+y^2+z^2)^{-5/2}}}$$

(b) $\dfrac{\partial V_2}{\partial x} = ze^{-y} + 2x\ln y, \quad \dfrac{\partial^2 V_2}{\partial x^2} = 2\ln y$

$\dfrac{\partial V_2}{\partial y} = -xze^{-y} + \dfrac{2x}{y}, \quad \dfrac{\partial^2 V_2}{\partial y^2} = xze^{-y} - \dfrac{2x}{y^2}$

$$\nabla^2 V_2 = \underline{\underline{2\ln y + xze^{-y} - \frac{2x}{y^2}}}$$

(c) $\nabla^2 V_3 = \dfrac{1}{\rho}\dfrac{\partial}{\partial \rho}(20\rho^2 \sin 2\phi) - \dfrac{40\rho^2}{\rho^2}\sin 2\phi$

$$= 40\sin 2\phi - 40\sin 2\phi = \underline{\underline{0}}.$$

(d) $\nabla^2 V_4 = \dfrac{1}{\rho}\dfrac{\partial}{\partial \rho}(\rho z)(\cos\phi + \sin\phi) - \dfrac{\rho z}{\rho^2}(\cos\phi + \sin\phi)$

$$= \frac{z}{\rho}(\cos\phi + \sin\phi) - \frac{z}{\rho}(\cos\phi + \sin\phi)$$

$$= \underline{\underline{0}}.$$

(e) $\nabla^2 V_5 = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(10r^3)\sin\theta\cos\phi + \dfrac{5r^2\cos\phi}{r^2\sin\theta}\dfrac{\partial}{\partial\theta}(\sin\theta\cos\theta)$

$$- \frac{5r^2\sin\theta\cos\phi}{r^2\sin^2\theta}$$

$$= 30\sin\theta\cos\phi - 10\sin\theta\cos\phi$$

$$= \underline{\underline{20\sin\theta\cos\phi}}.$$

(f) $$\nabla^2 V_6 = \frac{\sin\theta}{r^2}\frac{\partial}{\partial r}\left(-\frac{2}{r}\right) + \frac{1}{r^4\sin\theta}\left(1-2\sin^2\theta\right)$$

$$= \underline{\underline{\frac{1}{r^4\sin\theta}}}.$$

$\boxed{\text{Prob. 3.25}}$

$$\nabla^2\vec{A} = \nabla(\nabla\cdot\vec{A}) - \nabla\times\nabla\times\vec{A}$$

(a) $\nabla\cdot\vec{A} = 4x$

$\nabla\times\vec{A} = \vec{a}_x + y\,\vec{a}_y + (4y - z)\,\vec{a}_z$

$\nabla^2\vec{A} = 4\,\vec{a}_x - \vec{a}_x(4 - 0) = \underline{\underline{0}}.$

(b) $\nabla\cdot\vec{B} = (2-3z)\sin\phi,$

$\nabla\times\vec{B} = -6\rho z\cos\phi\,\vec{a}_\rho + \rho\sin\phi\,\vec{a}_\phi + (6z-1)z\cos\phi\,\vec{a}_z$

$\nabla^2\vec{B} = 0\,\vec{a}_\rho + \frac{1}{\rho}(2-3z)z\cos\phi\,\vec{a}_\phi + (2-\cancel{6z})\sin\phi\,\vec{a}_z$

$- \vec{a}_\rho\left[-\frac{1}{\rho}(6z-1)z\sin\phi - 0\right] - \vec{a}_\phi(-6\rho\cos\phi)$

$- \frac{\vec{a}_z}{\rho}\left(2\rho\sin\phi - 6\rho z\sin\phi\right)$

$= \underline{\underline{\frac{1}{\rho}(6z-1)z\sin\phi\,\vec{a}_\rho + \left[(2-3z)\frac{z}{\rho} + 6\rho\right]\cos\phi\,\vec{a}_\phi}}$

At $(5, \pi/2, 1)$, $\nabla^2 \vec{B} = \underline{\underline{\vec{a}_\rho}}$

(c) $\nabla \cdot \vec{C} = 6 \cos\theta \cos\phi$,

$$\nabla(\nabla \cdot \vec{C}) = \frac{-6 \sin\theta \cos\phi}{r} \vec{a}_\theta - \frac{6 \cot\theta \sin\phi}{r} \vec{a}_\phi$$

$$\nabla \times \vec{C} = r^{-1/2} \cot\theta \, \vec{a}_r - \left(2 \cot\theta \sin\phi + \frac{3}{2} r^{-1/2}\right) \vec{a}_\theta + 2 \sin\theta \cos\phi \, \vec{a}_\phi$$

$$\nabla \times \nabla \times \vec{C} = \left[\frac{4 \cos\theta \cos\phi}{r} + \frac{2 \cos\theta \cos\phi}{r \sin^2\theta}\right] \vec{a}_r - \frac{2 \sin\theta \cos\phi}{r} \vec{a}_\theta + \left(r^{-3/2} \operatorname{cosec}^2\theta - \frac{2 \cos\theta \sin\phi}{r \sin\theta} - \frac{3}{4} r^{-3/2}\right) \vec{a}_\phi$$

$$\nabla^2 \vec{C} = \nabla(\nabla \cdot \vec{C}) - \nabla \times \nabla \times \vec{C}$$

$$= -\left[\frac{4 \cos\theta \cos\phi}{r} + \frac{2 \cos\theta \cos\phi}{r \sin^2\theta}\right] \vec{a}_r - \frac{4 \sin\theta \cos\phi}{r} \vec{a}_\theta - \left[\frac{1}{r^{3/2} \sin^2\theta} + \frac{4 \cos\theta \sin\phi}{r \sin\theta} - \frac{3}{4 r^{3/2}}\right] \vec{a}_\phi.$$

At $(1, \pi/6, \pi/3)$,

$$\nabla^2 \vec{C} = -(\sqrt{3} + 2\sqrt{3}) \vec{a}_r + (0.5 - 1.5) \vec{a}_\theta + (4 + 6 - 0.75) \vec{a}_\phi$$

$$\nabla^2 \vec{C} = \underline{\underline{5.196\,\vec{a}_r - \vec{a}_\theta - 9.25\,\vec{a}_\phi}}.$$

Prob. 3.26

(a) $$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial y^2}$$

$$= \underline{\underline{2(y^2 z^2 + x^2 z^2 + x^2 y^2)}}$$

(b) $$\nabla^2 \vec{A} = \nabla^2 A_x\, \vec{a}_x + \nabla^2 A_y\, \vec{a}_y + \nabla^2 A_z\, \vec{a}_z$$

$$= (2y + 0 + 0)\,\vec{a}_x + (0 + 0 + 6xz)\,\vec{a}_y + (0 - 2z^2 - 2y^2)\,\vec{a}_z$$

$$= \underline{\underline{2y\,\vec{a}_x + 6xz\,\vec{a}_y - 2(y^2 + z^2)\,\vec{a}_z}}$$

(c) $$\text{grad div } A = \nabla(\nabla \cdot \vec{A})$$

$$= \nabla(2xy + 0 - 2y^2 z)$$

$$= \underline{\underline{2y\,\vec{a}_x + 2(x - 2yz)\,\vec{a}_y - 2y^2\,\vec{a}_z}}$$

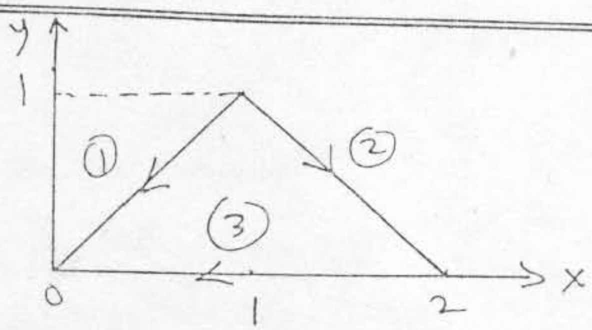
(d) $$\text{curl curl } \vec{A} = \nabla \times \nabla \times \vec{A}$$

$$= \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

From parts (b) and (c),

$$\nabla \times \nabla \times \vec{A} = \underline{\underline{2(x - 2yz - 3xz)\,\vec{a}_y + 2z^2\,\vec{a}_z}}$$

Prob. 3.27



(a) $\oint_L \vec{F}\cdot d\vec{l} = \left(\int_{①} + \int_{②} + \int_{③} \right) \vec{F}\cdot d\vec{l}$

for ①, $y = x \longrightarrow dy = dx$, $d\vec{l} = dx\,\vec{a}_x + dy\,\vec{a}_y$,

$$\int_{①} \vec{F}\cdot d\vec{l} = \int_0^1 x^3 dx - x\,dx = -\frac{1}{4}.$$

for ②, $y = -x+2$, $dy = -dx$, $d\vec{l} = dx\,\vec{a}_x + dy\,\vec{a}_y$,

$$\int_{②} \vec{F}\cdot d\vec{l} = \int_1^2 (-x^3 + 2x^2 - x + 2)\,dx = \frac{17}{12}.$$

$$\int_{③} \vec{F}\cdot d\vec{l} = \int_2^0 x^2 y\,dx \Big|_{y=0} = 0$$

$$\oint_L \vec{F}\cdot d\vec{l} = -\frac{1}{4} + \frac{17}{12} + 0 = \underline{\underline{\frac{7}{6}}}$$

(b) $\nabla\times\vec{F} = -x^2\vec{a}_z$, $d\vec{S} = dx\,dy\,(-\vec{a}_z)$

$$\int (\nabla\times\vec{F})\cdot d\vec{S} = -\iint (-x^2)\,dx\,dy$$

$$= \int_0^1\int_0^x x^2\,dy\,dx + \int_1^2\int_0^{-x+2} x^2\,dy\,dx$$

$$= \int_0^1 x^2 y \Big|_0^x dx + \int_1^2 x^2 y \Big|_0^{-x+2} dx$$

$$= \frac{x^4}{4}\Big|_0^1 + \int_1^2 x^2(-x+2)dx = \underline{\underline{\frac{7}{6}}}$$

(c) Yes.

Prob. 3.28

(a) $\nabla \times \vec{A} = 2\vec{a}_z$

$$\oint_L \vec{A}\cdot d\vec{l} = \int_S (\nabla \times \vec{A})\cdot d\vec{S} = \int 2\,dS = 2\int dS$$

$$= 2 \times \text{Area of } S.$$

(b) $\psi = \int \vec{r}\cdot d\vec{S} = \int \nabla\cdot\vec{r}\,dv = \int 3\,dv = 3v.$

Prob. 3.29

(a) for $x=0$, $\int \vec{F}\cdot d\vec{S} = -\int x^2\,dz\,dy\Big|_{x=0} = 0$

for $x=1$, $\int \vec{F}\cdot d\vec{S} = \int x^2\,dx\,dy\Big|_{x=1} = (1)(1) = 1.$

for $y=0$, $\int \vec{F}\cdot d\vec{S} = -\int y\,dx\,dz\Big|_{y=0} = 0.$

for $y=1$, $\int \vec{F}\cdot d\vec{S} = \int y\,dx\,dz\Big|_{y=1} = (1)(1) = 1.$

For $z=0$, $\int \vec{F}\cdot d\vec{S} = -\int z\,dx\,dy\Big|_{z=0} = 0$

for $z=1$, $\int \vec{F}\cdot d\vec{S} = \int z\,dx\,dy\Big|_{z=1} = (1)(1) = 1$

Hence,

$$\oint_S \vec{F}\cdot d\vec{S} = 1+0+1+0+1+0 = \underline{\underline{3}}.$$

(b) $\nabla\cdot\vec{F} = 2x+1+1 = 2(x+1)$,

$$\int_v \nabla\cdot\vec{F}\,dv = 2\int_0^1 (x+1)\,dx \int_0^1 dy \int_0^1 dz$$

$$= 2\left(\frac{x^2}{2}+1\right)\Big|_0^1 (1)(1) = \underline{\underline{3}}.$$

$\boxed{\text{Prob. 3.30}}$

(a) $$\oint_S \vec{E}\cdot d\vec{S} = \int \frac{1}{r^4}\sin^2\phi\; r^2\sin\theta\,d\theta\,d\phi\Big|_{r=4}$$

$$-\int \frac{1}{r^4}\sin^2\phi\; r^2\sin\theta\,d\theta\,d\phi\Big|_{r=2}$$

$$= \left(\frac{1}{4^2}-\frac{1}{2^2}\right)\int_{\phi=0}^{2\pi}\sin^2\phi\,d\phi\int_0^{\pi}\sin\theta\,d\theta$$

$$= \underline{\underline{-\frac{3\pi}{8}}}.$$

(b) $\nabla\cdot\vec{E} = -\frac{2}{r^5}\sin\phi$

$$\int \nabla\cdot\vec{E}\,dv = -2\int \frac{1}{r^3}\sin^2\phi\,\sin\theta\,d\theta\,dr\,d\phi$$

$$= -2\int_0^{2\pi}\sin^2\phi\,d\phi\int_0^{\pi}\sin\theta\,d\theta\int_2^4\frac{dr}{r^3}$$

$$= -2(\pi)(2)\left(\frac{1}{4^2}-\frac{1}{2^2}\right) = \underline{\underline{\frac{-3\pi}{8}}}$$

Prob. 3.31

$$\int \rho_v \, dv = \oint_S \vec{A}\cdot d\vec{s} \quad \text{— divergence theorem}$$

where $\rho_v = \nabla\cdot\vec{A} = x^2 + y^2$

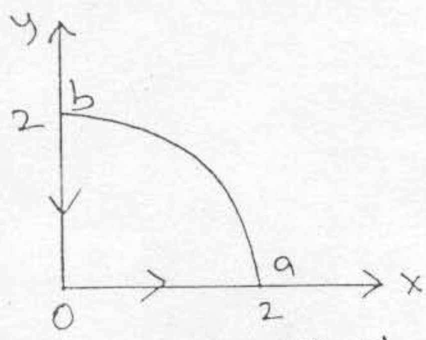
$$\frac{\partial A_x}{\partial x} = x^2 \longrightarrow A_x = \frac{x^3}{3} + c_1$$

$$\frac{\partial A_y}{\partial y} = y^2 \longrightarrow A_y = \frac{y^3}{3} + c_2$$

Hence,
$$\underline{\underline{\vec{A} = \left(\frac{x^3}{3} + c_1\right)\vec{a}_x + \left(\frac{y^3}{3} + c_2\right)\vec{a}_y}}$$

Prob. 3.32

(a)



$d\vec{l} = d\rho\,\vec{a}_\rho + \rho\, d\phi\,\vec{a}_\phi$, $\vec{A}\cdot d\vec{l} = \rho\sin\phi\, d\rho + \rho^3\, d\phi$.

Along Oa, $d\phi = 0$, $\phi = 0$, $\vec{A}\cdot d\vec{l} = 0$, $\int_0^a \vec{A}\cdot d\vec{l} = 0$

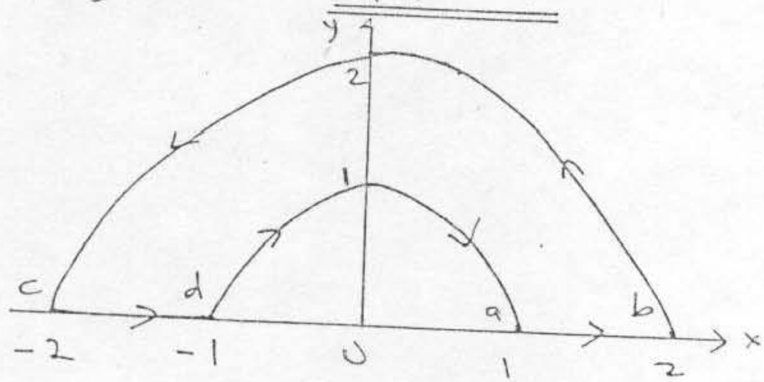
Along ab, $d\rho = 0$, $\rho = 2$, $\int_a^b \vec{A}\cdot d\vec{l} = \int_0^{\pi/2} 8\, d\phi = 4\pi$.

Along bo, $d\phi = 0$, $\phi = \pi/2$, $\int_b^0 \vec{A}\cdot d\vec{l} = \int_2^0 \rho\, d\rho = -2$

Hence,

$$\oint \vec{A}\cdot d\vec{l} = \underline{\underline{4\pi - 2}}.$$

(b)



Along ab, $d\phi = 0$, $\phi = 0$, $\vec{A}\cdot d\vec{l} = 0$, $\int_a^b \vec{A}\cdot d\vec{l} = 0$.

Along bc, $d\rho = 0$, $\vec{A}\cdot d\vec{l} = \rho^3 d\phi$,

$$\int_b^c \vec{A}\cdot d\vec{l} = \int_0^{\pi} \rho^3 d\phi = (2)^3(\pi - 0) = 8\pi$$

Along cd, $d\phi = 0$, $\phi = \pi$, $\vec{A}\cdot d\vec{l} = 0$, $\int_c^d \vec{A}\cdot d\vec{l} = 0$

Along da, $d\rho = 0$, $\vec{A}\cdot d\vec{l} = \rho^3 d\phi$,

$$\int_d^a \vec{A}\cdot d\vec{l} = \rho^3 \int_{\pi}^{0} d\phi = (1)^3(0 - \pi) = -\pi.$$

Hence,

$$\oint \vec{A}\cdot d\vec{l} = 0 + 8\pi + 0 - \pi = \underline{\underline{7\pi}}.$$

The result may be checked using Stokes's theorem.

$\boxed{\text{Prob. 3.33}}$

Let $\Psi = \oint \vec{F}\cdot d\vec{S} = \Psi_t + \Psi_b + \Psi_o + \Psi_i$

where Ψ_t, Ψ_b, Ψ_o, and Ψ_i are the fluxes through the top surface, bottom surface, outer surface ($\rho = 3$), and inner surface ($\rho = 2$) respectively (see Fig. 3.17).

For the top surface, $d\vec{S} = \rho\, d\phi\, d\rho\, a_z$, $z = 5$,

$$\vec{F}\cdot d\vec{S} = \rho^2 z\, d\phi\, dz. \quad \text{Hence}$$

$$\Psi_t = \int_{\rho=2}^{3} \int_{\phi=0}^{2\pi} \rho^2 z\, d\phi\, d\rho \Big|_{z=5} = \frac{190\pi}{3}.$$

For the bottom surface, $z = 0$, $d\vec{S} = \rho\, d\phi\, d\rho(-\bar{a}_z)$

$$\vec{F}\cdot d\vec{S} = \rho^2 \underbrace{z}_{0}\, d\phi\, d\rho = 0. \quad \text{Hence}$$

$$\Psi_b = 0$$

For the outer curved surface, $\rho = 3$, $d\vec{S} = \rho\, d\phi\, dz\, \bar{a}_\rho$

$$\vec{F}\cdot d\vec{S} = \rho^3 \sin\phi\, d\phi\, dz \quad \text{Hence}$$

$$\Psi_o = \int_{z=0}^{5} dz\ \rho^3 \underbrace{\int_{\phi=0}^{2\pi} \sin\phi\, d\phi}_{0} \Big|_{\rho=3} = 0$$

For the inner curved surface, $\rho = 2$, $d\vec{S} = \rho\, d\phi\, dz(-\bar{a}_\rho)$

$$\Psi_i = -\int_{z=0}^{5} dz\ \rho^3 \underbrace{\int_{\phi=0}^{2\pi} \sin\phi\, d\phi}_{0} \Big|_{\rho=2} = 0.$$

$$\Psi = \frac{190\pi}{3} + 0 + 0 + 0 = \underline{\underline{\frac{190\pi}{3}}}.$$

Using divergence theorem,

$$\Psi = \oint \vec{F} \cdot d\vec{s} = \int \nabla \cdot \vec{F}\, dv$$

$$\nabla \cdot \vec{F} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho^3 \sin\phi) + \frac{1}{\rho}\frac{\partial}{\partial \phi}(z\cos\phi) + \rho$$

$$= 3\rho\sin\phi - \frac{z}{\rho}\sin\phi + \rho$$

$$\int_v \nabla \cdot \vec{F}\, dv = \iiint \left(3\rho\sin\phi - \frac{z}{\rho}\sin\phi + \rho\right)\rho\, d\phi\, d\rho\, dz$$

$$= 0 + 0 + \int_0^5 dz \int_0^{2\pi} d\phi \int_2^3 \rho^2 d\rho$$

$$= \underline{\underline{\frac{190\pi}{3}}}.$$

Prob. 3.34

Let $\vec{B} = \nabla \times \vec{T}$

$$\Psi = \oint_s \vec{B} \cdot d\vec{s} = \int \nabla \cdot \vec{B}\, dv = \int \underbrace{\nabla \cdot \nabla \times \vec{T}}_{0}\, dv = 0$$

Prob. 3.35

$$\vec{Q} = \frac{r}{r\sin\theta}\, r\sin\theta \left[(\cos\phi - \sin\phi)\vec{a}_x + (\cos\phi + \sin\phi)\vec{a}_y\right]$$

$$= r(\cos\phi - \sin\phi)\vec{a}_x + r(\cos\phi + \sin\phi)\vec{a}_y$$

$$\begin{bmatrix} Q_r \\ Q_\theta \\ Q_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} Q_x \\ Q_y \\ Q_z \end{bmatrix}$$

$$\vec{Q} = r\sin\theta\,\vec{a}_r + r\cos\theta\,\vec{a}_\theta + r\,\vec{a}_\phi.$$

(a) $\vec{dl} = \rho\, d\phi\, \vec{a}_\phi$, $\rho = r\sin 30^\circ = 2\left(\frac{1}{2}\right) = 1$

$$z = r\cos 30^\circ = \sqrt{3}$$

$$Q_\phi = r = \sqrt{\rho^2 + z^2}$$

$$\oint \vec{Q}\cdot\vec{dl} = \int_0^{2\pi} \sqrt{\rho^2+z^2}\,\rho\, d\phi = 2(1)(2\pi) = \underline{\underline{4\pi}}$$

(b) $\nabla\times\vec{Q} = \cot\theta\,\vec{a}_r - 2\vec{a}_\theta + \cos\theta\,\vec{a}_\phi$

for S_1, $\vec{dS} = r^2\sin\theta\, d\theta\, d\phi\, \vec{a}_r$,

$$\int_{S_1} (\nabla\times\vec{Q})\cdot\vec{dS} = \int r^2 \sin\theta\cot\theta\, d\theta\, d\phi \Big|_{r=2}$$

$$= 4\int_0^{2\pi} d\phi \int_0^{30^\circ} \cos\theta\, d\theta = \underline{\underline{4\pi}}.$$

(c) for S_2, $\vec{dS} = r\sin\theta\, d\phi\, dr\, \vec{a}_\theta$,

$$\int_{S_2} (\nabla\times\vec{Q})\cdot\vec{dS} = -2\int r\sin\theta\, d\phi\, dr \Big|_{\theta=30^\circ}$$

$$= -2\sin 30^\circ \int_0^2 r\, dr \int_0^{2\pi} d\phi$$

$$= \underline{\underline{-4\pi}}.$$

(d) for S_1, $d\vec{S} = r^2 \sin\theta \, d\phi \, d\theta \, \vec{a}_r$,

$$\int_{S_1} \vec{Q} \cdot d\vec{S} = r^3 \int \sin^2\theta \, d\theta \, d\phi \Big|_{r=2}$$

$$= 8\int_0^{2\pi} d\phi \int_0^{30^\circ} \sin^2\theta \, d\theta$$

$$= 4\pi\left[\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right].$$

(e) for S_2, $d\vec{S} = r\sin\theta \, d\phi \, dr \, \vec{a}_\theta$

$$\int_{S_2} \vec{Q} \cdot d\vec{S} = \int r^2 \sin\theta \cos\theta \, d\phi \, dr \Big|_{\theta = 30^\circ}$$

$$= \frac{4\pi\sqrt{3}}{3}.$$

(f) $$\nabla \cdot \vec{Q} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^3 \sin\theta\right) + \frac{r}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\cos\theta\right) + 0$$

$$= 2\sin\theta + \cos\theta\cot\theta.$$

$$\int \nabla \cdot \vec{Q} \, dv = \int \left(2\sin\theta + \cos\theta\cot\theta\right) r^2 \sin\theta \, d\theta \, d\phi \, dr$$

$$= \frac{r^3}{3}\Big|_0^2 (2\pi) \int_0^{30^\circ} \left(1 + \sin^2\theta\right) d\theta$$

$$= \frac{4\pi}{3}\left(\pi - \frac{\sqrt{3}}{2}\right).$$

Check: $\int \nabla \cdot \vec{Q} \, dv = \left(\int_{S_1} + \int_{S_2}\right) (\nabla \times \vec{Q}) \cdot d\vec{S}$

$$= 4\pi \left[\frac{\pi}{3} - \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{3} \right]$$

$$= \frac{4\pi}{3} \left[\pi - \frac{\sqrt{3}}{2} \right].$$

It checks.

$\boxed{\text{Prob. 3.36}}$

Since $\vec{u} = \vec{w} \times \vec{r}$, $\nabla \times \vec{u} = \nabla \times (\vec{w} \times \vec{r})$

from Appendix A.10,

$$\nabla \times (\vec{A} \times \vec{B}) = \vec{A}(\nabla \cdot \vec{B}) - \vec{B}(\nabla \cdot \vec{A}) + (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B}$$

$$\nabla \times \vec{u} = \vec{w}\, \nabla \cdot \vec{r} - \vec{r}\, \underbrace{\nabla \cdot \vec{w}}_{0} + \underbrace{(\vec{r} \cdot \nabla)\, \vec{w}}_{0} - (\vec{w} \cdot \nabla)\, \vec{r}$$

$$= \vec{w}(3) - \vec{w} = 2\vec{w}$$

or $\quad \vec{w} = \frac{1}{2} \nabla \times \vec{u}.$

Alternatively, let $x = r\cos wt$, $y = r\sin wt$,

$$\vec{u} = \frac{\partial x}{\partial t}\vec{a}_x + \frac{\partial y}{\partial t}\vec{a}_y$$

$$= -wr\sin wt\, \vec{a}_x + wr\cos wt\, \vec{a}_y$$

$$= -wy\, \vec{a}_x + wx\, \vec{a}_y$$

$$\nabla \times \vec{u} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -wy & wx & 0 \end{vmatrix} = 2w\vec{a}_z = 2\vec{w}$$

i.e. $\quad \vec{w} = \frac{1}{2} \nabla \times \vec{u}.$

Prob. 3.37

(a) $\nabla (V\vec{A}) = \frac{\partial}{\partial x} VA_x + \frac{\partial}{\partial y} VA_y + \frac{\partial}{\partial z} VA_z$

$$= A_x \frac{\partial V}{\partial x} + V\frac{\partial A_x}{\partial x} + A_y \frac{\partial V}{\partial y} + V\frac{\partial A_y}{\partial y}$$

$$+ A_z \frac{\partial V}{\partial z} + V\frac{\partial A_z}{\partial z}$$

$$= \vec{A}\cdot\nabla V + V\nabla\cdot\vec{A}$$

(b) curl curl $\bar{A} = \nabla \times (\nabla \times \vec{A})$

$$= \nabla \times \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) & \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) & \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \end{vmatrix}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)\vec{a}_x + \frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)\vec{a}_y$$

$$+ \frac{\partial}{\partial z}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)\bar{a}_z$$

$$- \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(A_x \bar{a}_x + A_y \vec{a}_y + A_z \vec{a}_z\right)$$

$$= \nabla(\nabla\cdot\bar{A}) - \nabla^2\vec{A}.$$

$\boxed{\text{Prob. 3.38}}$

$$\nabla \times \vec{A} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 5x^2y & 3xy^2 & 0 \end{vmatrix} = (3y^2 - 5x^2)\,\vec{a}_z$$

$$\int \nabla \times \vec{A}\, dv = \iiint (3y^2 - 5x^2)\, dx\, dy\, dz\, \vec{a}_z$$

$$= \vec{a}_z \left[3\int_0^2 dx \int_{-1}^{1} y^2 dy \int_{-5}^{5} dz - 5\int_0^2 x^2 dx \int_{-1}^{1} dy \int_{-5}^{5} dz \right]$$

$$= \vec{a}_z \left[40 - \frac{800}{3} \right] = -226.67\, \vec{a}_z$$

$$\oint \vec{A} \times d\vec{S} = \int \vec{A} \times dy\,dz\,\vec{a}_x \Big|_{x=2} - \int \vec{A} \times dy\,dz\,\vec{a}_x \Big|_{x=0}$$

$$+ \int \vec{A} \times dx\,dz\,\vec{a}_y \Big|_{y=1} - \int \vec{A} \times dx\,dz\,\vec{a}_y \Big|_{y=-1}$$

$$+ \int \vec{A} \times dx\,dy\,\vec{a}_z \Big|_{z=5} - \int \vec{A} \times dx\,dy\,\vec{a}_z \Big|_{z=-5}$$

$$= -\iint 3xy^2\, dy\,dz \Big|_{x=2} \vec{a}_z + \iint 3\cancelto{0}{x}y^2\, dy\,dz \Big|_{x=0} \vec{a}_z$$

$$+ \iint 5x^2y\, dx\,dz \Big|_{y=1} \vec{a}_z - \iint 5x^2y\, dx\,dz \Big|_{y=-1} \vec{a}_z$$

$$+ \iint (\cancel{3xy^2\,\vec{a}_x} - 5x^2y\,\vec{a}_y)\, dx\,dy \Big|_{z=5}$$

$$-\iint (3xy^2 \vec{a}_x - 5x^2 y \vec{a}_y)\, dx\, dy \Big|_{z=-5}$$

$$= -6\int_{-1}^{1} y^2 dy \int_{-5}^{5} dz\, \vec{a}_z + 2(5)\int_{0}^{2} x^2 dx \int_{-5}^{5} dz\, \vec{a}_z$$

$$= \left(-40 + \frac{800}{3}\right)\vec{a}_z = 226.67\vec{a}_z.$$

Thus $\int \nabla \times \vec{A}\, dv = -\oint \vec{A} \times d\vec{S}$.

$\boxed{\text{Prob. 3.39}}$

$$\nabla \cdot \vec{R} = \frac{\partial R_x}{\partial x} + \frac{\partial R_y}{\partial y} + \frac{\partial R_z}{\partial z} = 4xy + 2xy + cxy + 0$$

$$0 = xy(6+c) \rightarrow \underline{\underline{c = -6}}.$$

$\boxed{\text{Prob. 3.40}}$

$$\nabla \times \vec{T} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \alpha x + \beta z^2 & 3x^2 - \gamma z & 3xz^2 - y \end{vmatrix}$$

$$= (-1+\gamma)\vec{a}_x + (3\beta z^2 - 3z^2)\vec{a}_y + (6x - \alpha x)\vec{a}_z$$

If $\vec{T}$ is irrotational, $\nabla \times \vec{T} = 0$, i.e.

$$\underline{\underline{\alpha = 1 = \beta = \gamma}}.$$

$$\nabla\cdot\vec{T} = \frac{\partial T_x}{\partial x} + \frac{\partial T_y}{\partial y} + \frac{\partial T_z}{\partial z} = \alpha y + 0 + 6xz$$

At $(2,-1,0)$, $\nabla\cdot\vec{T} = -1 + 0 = \underline{\underline{-1}}$.

Prob. 3.41

(a) Let $\vec{A} = (x-yz)\vec{a}_x + (y^2-xz)\vec{a}_y + (z^2-xy)\vec{a}_z$

$\nabla\times\vec{A} = 0$

Hence $\int \vec{A}\cdot d\vec{l}$ is independent of path taken.

Similarly, if $\vec{B} = z^2\vec{a}_x + 2y\vec{a}_y + 2xz\vec{a}_z$,

$\nabla\times\vec{B} = 0$. Hence $\int\vec{B}\cdot d\vec{l}$ is independent of path.

(b) Let $I_1 = \int \vec{A}\cdot d\vec{l}$, $I_2 = \int\vec{B}\cdot d\vec{l}$

(i) $x = y = z \rightarrow dx = dy = dz$

$$I_1 = \int_0^1 (x - x^2)\,dx = \frac{1}{2} - \frac{1}{3} = \underline{\underline{\frac{1}{6}}}$$

$$I_2 = \int_0^1 (2x + 3x^2)\,dx = 1 + 1 = \underline{\underline{2}}$$

(ii) $$I_1 = \int_0^1 x\,dx + \int_0^1 y^2\,dy + \int_0^1 (z^2-1)\,dz$$

$$= \frac{1}{2} + \frac{1}{3} + \frac{1}{3} - 1 = \underline{\underline{\frac{1}{6}}}$$

$$I_2 = \int_0^1 0\,dx + \int 2y\,dy + \int 2z\,dz = 1 + 1 = \underline{\underline{2}}.$$

CHAPTER 4

P.E. 4.1

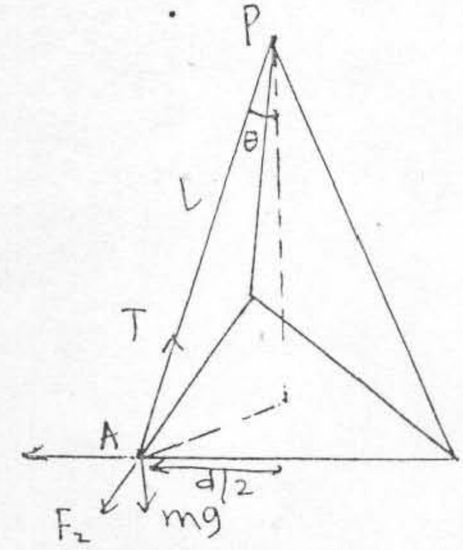
(a) $$\vec{F} = \frac{1\times10^{-9}}{4\pi \times \frac{10^{-9}}{36\pi}}\left[\frac{5\times10^{-9}\{(1,-3,7)-(2,0,4)\}}{|(1,-3,7)-(2,0,4)|^3} - \frac{2\times10^{-9}\{(1,-3,7)-(-3,0,5)\}}{|(1,-3,7)-(-3,0,5)|^3}\right]$$

$$= \left[\frac{45(-1,-3,3)}{19^{3/2}} - \frac{18(4,-3,2)}{29^{3/2}}\right] nN$$

$$= \underline{\underline{-1.004\,\vec{a}_x - 1.284\,\vec{a}_y + 1.4\,\vec{a}_z \ nN.}}$$

(b) $$\vec{E} = \frac{\vec{F}}{Q} = \underline{\underline{-1.004\,\vec{a}_x - 1.284\,\vec{a}_y + 1.4\,\vec{a}_z \ V/m.}}$$

P.E. 4.2



At point A,

$$T\sin\theta\cos 30^\circ = F_1 + F_2\cos 60^\circ$$

$$= \frac{q^2}{4\pi\epsilon_0 d^2} + \frac{q^2}{4\pi\epsilon_0 d^2}\cdot\frac{1}{2}$$

$$= \frac{3q^2}{8\pi\epsilon_0 d^2}$$

$$T\cos\theta = mg$$

Hence $$\tan\theta\cos 30^\circ = \frac{3q^2}{8\pi\epsilon_0 d^2 mg}$$

But $\sin\theta = \dfrac{h}{l} = \dfrac{d}{\sqrt{3}\,l} \longrightarrow \tan\theta = \dfrac{d/\sqrt{3}}{\sqrt{l^2 - d^2/3}}$

Thus

$$\frac{\frac{d}{\sqrt{3}}\cdot\frac{\sqrt{3}}{2}}{\sqrt{l^2 - d^2/3}} = \frac{3q^2}{8\pi\epsilon_0 d^2 mg}$$

or

$$q^2 = \frac{4\pi\epsilon_0 d^3 mg}{3\sqrt{l^2 - d^2/3}}.$$

But $q = Q/3 \longrightarrow q^2 = Q/9$. Hence

$$Q^2 = \frac{12\pi\epsilon_0 d^3 mg}{\sqrt{l^2 - d^2/3}}.$$

P.E. 4.3 $\qquad e\vec{E} = m\dfrac{d^2\vec{l}}{dt^2}$

$$eE_o\left(-2\vec{a}_x + \vec{a}_y\right) = m\left(\frac{d^2x}{dt^2}\vec{a}_x + \frac{d^2y}{dt^2}\vec{a}_y + \frac{d^2z}{dt^2}\vec{a}_z\right)$$

where $E_o = 200\ kV/m$.

$$\frac{d^2z}{dt^2} = 0 \longrightarrow z = c_1 t + c_2$$

$$m\frac{d^2z}{dt^2} = -2eE_o \longrightarrow x = -\frac{2eE_o t^2}{2m} + c_3 t + c_4$$

$$m\frac{d^2y}{dt^2} = eE_o \longrightarrow y = \frac{eE_o t^2}{2m} + c_5 t + c_6$$

At $t=0$, $(x,y,z) = (0,0,0) \longrightarrow c_2 = 0 = c_4 = c_6$.

Also, $\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = (0,0,0)$ at $t = 0 \Longrightarrow$

$$c_1 = 0 = c_3 = c_5.$$

Hence

$$(x,y) = \frac{eE_o t^2}{2m}(-2,1)$$

i.e. $2|y| = |x|$. Thus the largest horizontal distance is $80\text{cm} = \underline{\underline{0.8m}}$.

$\boxed{\text{P.E. 4.4}}$

(a) Consider an element of area dS of the disk. The contribution due to $dS = \rho\, d\phi\, d\rho$ is

$$dE = \frac{\rho_s\, dS}{4\pi\varepsilon_o r^2} = \frac{\rho_s\, dS}{4\pi\varepsilon_o(\rho^2 + h^2)}.$$

The sum of the contributions along ρ gives zero.

$$dE_z = dE\cos\alpha = \frac{\rho_s\, dS}{4\pi\epsilon_o(\rho^2+h^2)} \cdot \frac{h}{(\rho^2+h^2)^{1/2}}$$

$$E_z = \frac{\rho_s}{4\pi\epsilon_o}\int_{\rho=0}^{a}\int_{\phi=0}^{2\pi} \frac{z\,\rho\, d\rho\, d\phi}{(\rho^2+h^2)^{3/2}} = \frac{h\rho_s}{2\epsilon_o}\int_{\rho=0}^{a} \frac{\rho\, d\rho}{(\rho^2+h^2)^{3/2}}$$

$$= \frac{h\rho_s}{4\epsilon_o}\int_0^a (\rho^2+h^2)^{-3/2}\, d(\rho^2) = \frac{h\rho_s}{2\epsilon_o}\left(-2(\rho^2+h^2)^{-1/2}\right)\Big|_0^a$$

$$= \frac{\rho_s}{2\epsilon_o}\left[1 - \frac{h}{(h^2+a^2)^{1/2}}\right]$$

(b)

As $a \to \infty$, $\vec{E} = \dfrac{\rho_s}{2\epsilon_o}\vec{a}_z$.

$\boxed{\text{P.E. 4.5}}$

$$Q = \int \rho_s\, dS = \int_{-2}^{2}\int_{-2}^{2} 12|y|\, dx\, dy$$

$$= 12(4)\int_0^2 2y\, dy = \underline{\underline{192 \text{ mC}}}$$

$$\vec{E} = \int \frac{\rho_s\, dS}{4\pi\epsilon r^2}\vec{a}_r = \int \frac{\rho_s\, dS\, |\vec{r}-\vec{r}\,'|}{4\pi\epsilon_o|\vec{r}-\vec{r}\,'|^3}$$

where $\vec{r}-\vec{r}\,' = (0,0,10) - (x,y,z) = (-x,-y,10)$.

$$\vec{E} = \int_{x=-2}^{2} \int_{y=-2}^{2} \frac{12|y| \cdot 10^{-3} \, (-x, -y, 10)}{4\pi \cdot \frac{10^{-9}}{36\pi} \, (x^2+y^2+100)^{3/2}}$$

$$= 108 \times 10^{6} \Bigg[\cancelto{0}{\int_{-2}^{2} |y| \int_{-2}^{2} \frac{-x \, dx \, \vec{a}_x \, dy}{(x^2+y^2+100)^{3/2}}}$$

$$+ \cancelto{0}{\int_{-2}^{2} x \int_{-2}^{2} \frac{-y|y| \, dy \, \vec{a}_y \, dx}{(x^2+y^2+100)^{3/2}}}$$

$$+ 10 \, \vec{a}_z \int_{-2}^{2} \int_{-2}^{2} \frac{|y| \, dx \, dy}{(x^2+y^2+100)^{3/2}} \Bigg]$$

$$= 108 \, \vec{a}_z \cdot 10^{7} \int_{-2}^{2} \left[2 \int_{0}^{2} \frac{\frac{1}{2} d(y^2)}{(x^2+y^2+100)^{3/2}} \right] dx$$

$$= -216 \, \vec{a}_z \cdot 10^{7} \int_{-2}^{2} \left[\frac{1}{(x^2+104)^{1/2}} - \frac{1}{(x^2+100)^{1/2}} \right] dx$$

$$= -216 \, \vec{a}_z \cdot 10^{7} \ln \left| \frac{x + \sqrt{x^2+100}}{x + \sqrt{x^2+100}} \right| \Bigg|_{-2}^{2}$$

$$= -216 \, \vec{a}_z \cdot 10^{7} \left(\ln\left(\frac{2+\sqrt{108}}{2+\sqrt{104}}\right) - \ln\left(\frac{-2+\sqrt{108}}{-2+\sqrt{104}}\right) \right)$$

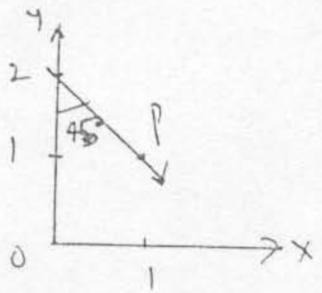
$$= -216 \, \vec{a}_z \cdot 10^{7} \left(-7.6202 \times 10^{-3}\right)$$

$$\underline{\underline{\vec{E} = 16.46 \, \vec{a}_z \ \text{MV/m}.}}$$

P.E. 4.6

$$\vec{E}_3 = \frac{\rho_L}{2\pi\epsilon_0 \rho}\,\vec{a}_\rho$$

To get $\vec{a}_\rho$, consider the $z = -1$ plane



$$\rho = \sqrt{2}$$

$$\hat{a}_\rho = \vec{a}_x \cos 45^\circ - \hat{a}_y \sin 45^\circ$$

$$= \frac{1}{\sqrt{2}}\left(\vec{a}_x - \hat{a}_y\right)$$

$$\vec{E}_3 = \frac{10 \times 10^{-9}}{2\pi \times \frac{10^{-9}}{36\pi}} \, \frac{1}{2}\left(\vec{a}_x - \vec{a}_y\right)$$

$$= 90\pi\left(\vec{a}_x - \vec{a}_y\right). \quad \text{Hence,}$$

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3$$

$$= -180\pi\vec{a}_x + 270\pi\vec{a}_y + 90\pi\vec{a}_x - 90\pi\vec{a}_y$$

$$= \underline{\underline{-282.7\,\vec{a}_x + 565.5\,\vec{a}_y \ \text{V/m}}}.$$

P.E. 4.7

from Example 4.7, $$E = \frac{2\rho_L \ell}{2\pi\epsilon_0 r\sqrt{\ell^2 + 4r^2}}$$

But $r = \ell/2$, $\rho_L = Q/\ell$. Hence

$$E = \frac{Q}{\pi\epsilon_0 \frac{\ell}{2}\sqrt{\ell^2 + \ell^2}} = \frac{\sqrt{2}\,Q}{\pi\epsilon_0 \ell^2}$$

$\boxed{\text{P.E. 4.8}}$

$$\vec{D} = \vec{D}_Q + \vec{D}_\rho = \frac{Q}{4\pi r^2}\vec{a}_r + \frac{\rho_s}{2}\vec{a}_n$$

$$= \frac{30\times10^{-9}}{4\pi(5)^2}\left[\frac{(0,4,3)-(0,0,0)}{5}\right] + \frac{10\times10^{-9}}{2}\vec{a}_y$$

$$= \frac{30}{500\pi}(0,4,3) + 5\vec{a}_y \ nC/m^2$$

$$= \underline{\underline{5.076\vec{a}_y + 0.0573\vec{a}_z \ nC/m^2}}.$$

$\boxed{\text{P.E. 4.9}}$

(a) $\rho_v = \nabla\cdot\vec{D} = 4x$

$\rho_v(-1,0,3) = \underline{\underline{-4 \ C/m^3}}.$

(b) $\Psi = Q = \int \rho_v \, dv = \int_0^1\int_0^1\int_0^1 4x \, dx\,dy\,dz$

$= 4(1)(1)(\tfrac{1}{2}) = \underline{\underline{2\ C}}.$

(c) $Q = \Psi = \underline{\underline{2C}}.$

$\boxed{\text{P.E. 4.10}}$

$$Q = \int \rho_v \, dv = \Psi = \oint \vec{D}\cdot d\vec{s}$$

for $0 \le r \le 10$,

$$D_r \cdot 4\pi r^2 = \iiint 2r \cdot r^2 \sin\theta \, d\theta \, dr \, d\phi$$

$$D_r \cdot 4\pi r^2 = 4\pi \cdot \frac{2r^4}{4}\Big|_0^r = 2\pi r^4$$

$$D_r = \frac{r^2}{2} \longrightarrow \vec{E} = \frac{r^2}{2\epsilon_0}\vec{a}_r \ \text{nV/m}$$

$$\vec{E}(r=2) = \frac{4 \times 10^{-9}}{2 \times \frac{10^{-9}}{36\pi}}\vec{a}_r = 72\pi\vec{a}_r = \underline{\underline{226\,\vec{a}_r \text{ V/m.}}}$$

for $r \geq 10$, $D_r \cdot 4\pi r^2 = 2\pi r_0^4$, $r_0 = 10$ m.

$$D_r = \frac{r_0^4}{2r^2} \longrightarrow \vec{E} = \frac{r_0^4}{2\epsilon_0 r^2}\vec{a}_r \ \text{nV/m}$$

$$\vec{E}(r=12) = \frac{10^4 \times 10^{-9}}{2 \times \frac{10^{-9}}{36\pi} \cdot 144}\vec{a}_r = 1250\pi\vec{a}_r$$

$$= \underline{\underline{3.927\,\vec{a}_r \text{ kV/m.}}}$$

$\boxed{\text{P.E. 4.11}}$

$$V(\vec{r}) = \sum_{k=1}^{3} \frac{Q_k}{4\pi\epsilon_0|\vec{r}-\vec{r}_k|} + C.$$

If $V(\infty) = 0$, $C = 0$

$$|\vec{r}-\vec{r}_1| = |(-1,5,2) - (2,-1,3)| = \sqrt{46}$$

$$|\vec{r}-\vec{r}_2| = |(-1,5,2) - (0,4,-2)| = \sqrt{18}$$

$$|\vec{r}-\vec{r}_3| = |(-1,5,2) - (0,0,0)| = \sqrt{30}$$

$$V(-1,5,2) = \frac{10^{-6}}{4\pi \times \frac{10^{-9}}{36\pi}}\left[\frac{-4}{\sqrt{46}} + \frac{5}{\sqrt{18}} + \frac{3}{\sqrt{30}}\right]$$

$$= \underline{\underline{10.3 \text{ kV}}}.$$

$\boxed{\text{P.E. 4.12}}$

$$V = \frac{Q}{4\pi\epsilon_0 r} + C$$

If $V(0,6,-8) = V(r=10) = 2$,

$$2 = \frac{5\times10^{-9}}{4\pi \times \frac{10^{-9}}{36\pi}} + C \longrightarrow C = -2.5$$

(a) $$V_A = \frac{5\times10^{-9}}{4\pi\times\frac{10^{-9}}{36\pi}\left|(-3,2,6)-(0,0,0)\right|} - 2.5$$

$$= \underline{\underline{3.929 \text{ V}}}$$

(b) $$V_B = \frac{45}{\sqrt{7^2+1^2+5^2}} - 2.5 = \underline{\underline{2.696 \text{ V}}}.$$

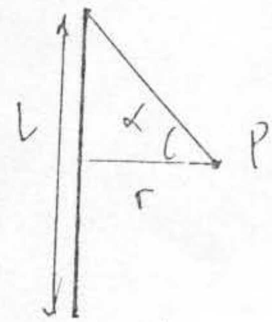
(c) $V_{AB} = V_B - V_A = 2.696 - 3.929 = \underline{\underline{-1.233 \text{ V}}}.$

$\boxed{\text{P.E. 4.13}}$

$$V = 4V_1$$

where V_1 is the potential due to one side.

from Example 4.13,



$$V_1 = V_P = \frac{\rho_L}{4\pi\varepsilon_0} \ln \frac{1+\sin\alpha}{1-\sin\alpha}$$

where $\tan\alpha = \frac{L/2}{r} \longrightarrow \sin\alpha = \frac{L/2}{\sqrt{r^2 + L^2/4}}$

$$V = 4V_1 = \frac{\rho_L}{\pi\varepsilon_0} \ln \frac{1+\sin 45^\circ}{1-\sin 45^\circ} = \frac{\rho_L}{\pi\varepsilon_0} \ln \left(\frac{1+\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}}\right)$$

$$= \frac{\rho_L}{\pi\varepsilon_0} \ln \left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)\left(\frac{\sqrt{2}+1}{\sqrt{2}+1}\right) = \frac{2\rho_L}{\pi\varepsilon_0} \ln(\sqrt{2}+1)$$

P.E. 4.14

(a) $-\frac{W}{Q} = \int \vec{E}\cdot d\vec{l} = \int (3x^2+y)\,dx + x\,dy$

$$= \int_0^2 (3x^2+y)\,dx \Big|_{y=5} + \int_5^{-1} x\,dy \Big|_{x=2}$$

$$= 18 - 12 = 6$$

$$W = -6Q = \underline{\underline{12 \text{ mJ}}}.$$

(b) $dy = -3\,dx$,

$$-\frac{W}{Q} = \int \vec{E}\cdot d\vec{l} = \int_0^2 (3x^2+5-3x)\,dx + x(-3)\,dx$$

$$= \int_0^2 (3x^2 - 6x + 5)\,dx = 8 - 12 + 10 = 6$$

$$W = \underline{\underline{12\,mJ.}}$$

P.E. 4.15

(a) $(0,0,10) \longrightarrow (r=10,\ \theta=0,\ \phi=0)$

$$V = \frac{100\cos 0}{4\pi\epsilon_o(10)}\cdot 10^{-12} = \frac{10^{-12}}{4\pi\times\frac{10^{-9}}{36\pi}} = \underline{\underline{9\,mV.}}$$

$$\vec{E} = \frac{100\times 10^{-12}}{4\pi\times\frac{10^{-9}}{36\pi}(10)^3}\left[2\cos 0\,\vec{a}_r + \sin 0\,\vec{a}_\theta\right]$$

$$= \underline{\underline{1.8\,\vec{a}_r\ mV/m.}}$$

(b) At $(1, \pi/3, \pi/2)$,

$$V = \frac{100\cos\frac{\pi}{3}\cdot 10^{-12}}{4\pi\times\frac{10^{-9}}{36\pi}(1)^2} = \underline{\underline{0.45\,V.}}$$

$$\vec{E} = \frac{100\times 10^{-12}}{4\pi\times\frac{10^{-9}}{36\pi}(1)^3}\left(2\cos\frac{\pi}{3}\,\vec{a}_r + \sin\frac{\pi}{3}\,\vec{a}_\theta\right)$$

$$= \underline{\underline{0.9\,\vec{a}_r + 0.7794\,\vec{a}_\theta\ V/m.}}$$

P.E. 4.16

(a) $$\frac{dy}{dx} = \frac{E_y}{E_x} = \frac{2xy}{y^2} = \frac{2x}{y}$$

$$\int y\,dy = \int 2x\,dx \longrightarrow \frac{y^2}{2} = x^2 + c$$

$$\underline{\underline{y^2 = 2x^2 + c.}}$$

which are hyperbolas. At $(2,-1,3)$,

$$x=2,\ y=-1 \longrightarrow 1 = 8 + c_0 \text{ or } c_0 = -7,$$

$$\underline{\underline{y^2 = 2x^2 - 7.}}$$

(b) $V = x^2 + y^2 = \text{constant}$

i.e. $x^2 + y^2 = \rho^2 = c_1$

which are cylinders of radius ρ.

$$\frac{dy}{dx} = -\frac{1}{E_y/E_x} = \frac{-y}{2x} \rightarrow \int \frac{dy}{y} = -\int \frac{dx}{x}$$

$$\ln y = -\tfrac{1}{2}\ln x + \ln c \rightarrow c = y\sqrt{x}$$

or $\underline{\underline{c_2 = xy^2.}}$

$\boxed{\text{P.E. 4.17}}$

After Q_1, $W_1 = 0$

After Q_2, $W_2 = Q_2 V_{21} = \dfrac{Q_2\,Q_1}{4\pi\epsilon_0\left|(1,0,0)-(0,0,0)\right|}$

$$= \frac{(1)(-2)\times 10^{-18}}{4\pi\times 10^{-9}/36\pi} = \underline{\underline{-18\,nJ.}}$$

After Q_3, $W_3 = Q_3(V_{31} + V_{32}) + Q_2 V_{21}$

$$= 3 \times 9 \times 10^{-9} \left(\frac{1}{|(0,0,-1)-(0,0,0)|} + \frac{-2}{|(0,0,-1)-(1,0,0)|} \right) - 18 \text{ nJ}$$

$$= 27\left(1 - \frac{2}{\sqrt{2}}\right) - 18 = \underline{\underline{-29.18 \text{ nJ}}}.$$

After Q_4,

$$W_4 = Q_4(V_{41} + V_{42} + V_{43}) + Q_3(V_{31} + V_{32}) + Q_2 V_{21}$$

$$= -4 \times 9 \times 10^{-9} \left[\frac{1}{|(0,0,1)-(0,0,0)|} + \frac{-2}{|(0,0,1)-(1,0,0)|} + \frac{3}{|(0,0,1)-(0,0,-1)|} \right] + W_3$$

$$= -36\left(1 - \frac{2}{\sqrt{2}} + \frac{3}{2}\right) + W_3$$

$$= -39.09 - 29.18 \text{ nJ} = \underline{\underline{-68.27 \text{ nJ}}}.$$

$\boxed{\text{P.E. 4.18}}$

$$\vec{E} = -\nabla V = -(y+1)\vec{a}_x + (1-x)\vec{a}_y - 2\vec{a}_z \text{ V/m}$$

At $(1,2,3)$, $\vec{E} = \underline{\underline{-3\vec{a}_x - 2\vec{a}_z \ \text{V/m}}}$.

$$W = \frac{1}{2}\epsilon_0 \int \vec{E}\cdot\vec{E}\, dv$$

$$= \frac{1}{2}\epsilon_0 \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} (x^2 + y^2 - 2x + 2y + 6)\, dx\, dy\, dz$$

$$= \frac{1}{2}\epsilon_0 \left[\int_{-1}^{1} x^2 dx \iint dy\, dz + \int_{-1}^{1} y^2 dy \iint dx\, dz \right.$$

$$\left. - 2\underbrace{\int_{-1}^{1} x\, dx}_{0} \iint dy\, dz + 2\underbrace{\int_{-1}^{1} y\, dy}_{0} \iint dx\, dz + 6(2)(2)(2) \right]$$

$$= \frac{1}{2}\epsilon_0 \left[2\,\frac{x^3}{3}\Big|_{-1}^{1}(2)(2) + 6(8) \right] = \frac{80\epsilon_0}{3}$$

$W = \underline{\underline{0.2358 \text{ nJ}}}$.

$\boxed{\text{Prob. 4.1}}$

(a) See Text.

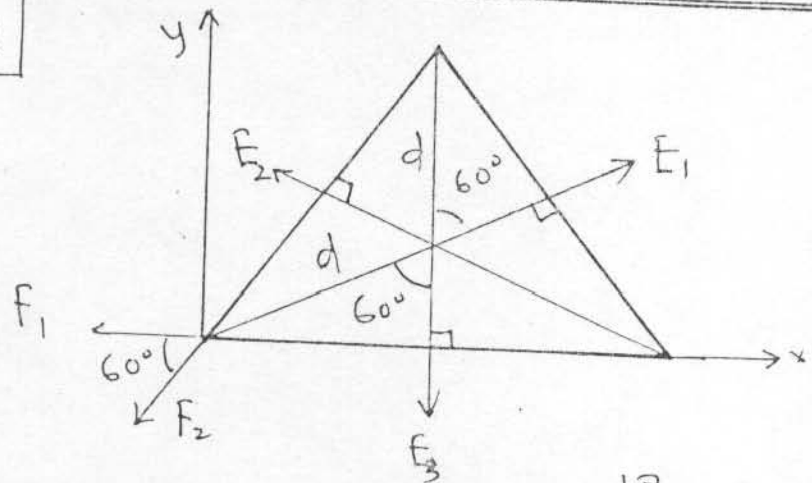
(b) $$\vec{F}_1 = \frac{Q_1 Q_2}{4\pi\epsilon_0} \frac{[(0,0,4) - (-2,6,1)]}{|(0,0,4) - (-2,6,1)|^3}$$

$$+ \frac{Q_1 Q_3}{4\pi\epsilon_0} \frac{[(0,0,4) - (3,-4,-8)]}{|(0,0,4) - (3,-4,-8)|^3}$$

$$= 9\left[\frac{(4,-12,6)}{7^3} + \frac{(9,-12,-36)}{13^3} \right] \text{kN}$$

$\vec{F}_1 = \underline{\underline{142\vec{a}_x - 364\vec{a}_y + 10\vec{a}_z \text{ N}}}$.

Prob. 4.2



(a) $|\vec{F_1}| = |\vec{F_2}| = \dfrac{Q_1 Q_2}{4\pi\epsilon_0 r^2} = \dfrac{100\times10^{-18}}{4\pi \times \frac{10^{-9}}{36\pi} \cdot 100\times10^{-4}}$

$= 9\times10^{-5}$ N

$F_x = F_1 + F_2 \cos 60° = \frac{3}{2} F_1$

$F_y = F_2 \sin 60° = \frac{\sqrt{3}}{2} F_1$

$F = \sqrt{F_x^2 + F_y^2} = \sqrt{3}\, F_1 = 9\sqrt{3}\cdot 10^{-5}$

$F = \underline{\underline{156\ \mu N}}$.

(b) $E_1 = E_2 = E_3 = \dfrac{Q}{4\pi\epsilon_0 d^2}$

$E_x = E_1 \sin 60° - E_2 \sin 60° = 0$

$E_y = E_1 \cos 60° + E_2 \cos 60° - E_3 = 0$

Hence $\underline{\underline{\vec{E} = 0}}$ at the center.

Prob. 4.3

$$\vec{F} = \frac{Q}{4\pi\epsilon_0}\left[\frac{Q_1\{(0,-3,4)-(0,-4,3)\}}{|(0,-3,4)-(0,-4,3)|^3} + Q_2\frac{\{(0,-3,4)-(0,1,1)\}}{|(0,-3,4)-(0,1,1)|^3}\right]$$

$$= \frac{Q}{4\pi\epsilon_0}\left[Q_1\frac{(0,1,1)}{2\sqrt{2}} + Q_2\frac{(0,-4,3)}{125}\right]$$

(a) If $F_z = 0$, then

$$\frac{Q_1}{2\sqrt{2}} + \frac{3Q_2}{125} = 0 \rightarrow Q_2 = -\frac{125}{3}\frac{Q_1}{2\sqrt{2}} = -\frac{125}{3\sqrt{2}}$$

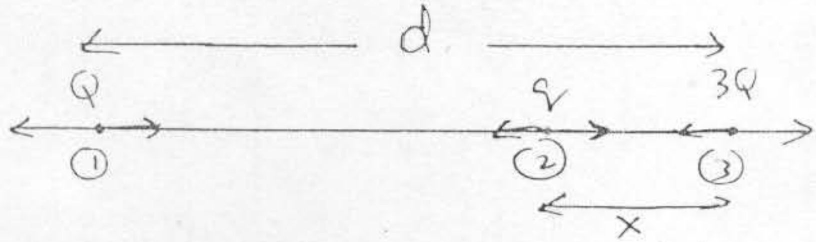
$$Q_2 = \underline{\underline{-29.46\text{ nC}}}.$$

(b) If $E_y = 0$, then

$$\frac{Q_1}{2\sqrt{2}} - \frac{4Q_2}{125} = 0 \rightarrow Q_2 = \frac{125}{4}\frac{Q_1}{2\sqrt{2}} = \frac{125}{4\sqrt{2}}\text{ nC}$$

$$Q_2 = \underline{\underline{22.1\text{ nC}}}.$$

Prob. 4.4



for the system to be in equilibrium, q must be negative and

$$\vec{F}_{12} = \vec{F}_{23} = \vec{F}_{13}$$

or

$$-\frac{Qq}{4\pi(d-x)^2} = -\frac{3Qq}{4\pi x^2} = \frac{4Q^2}{4\pi d^2}$$

i.e. $3(d-x)^2 = x^2 \longrightarrow 3d^2 - 6dx + 3x^2 = x^2$

$$2x^2 - 6dx + 3d^2 = 0$$

$$x = \frac{6d \pm \sqrt{36d^2 - 24d^2}}{4} = \frac{6d \pm d\sqrt{12}}{4}$$

$$x = 3 \pm \sqrt{3} = \underline{\underline{4.732\,m,\ 1.268\,m}}.$$

Also, $3qd^2 = -4Qx^2$

$$q = -\frac{4Qx^2}{d^2} = -Q(1.268)^2 \text{ or } -Q(4.732)^2$$

$$= \underline{\underline{-1.6078Q,\ -22.392Q}}.$$

$\boxed{\text{Prob. 4.5}}$

(a) $Q = \int \rho_L dl = \int_0^5 12x^2 dx = 4x^3\Big|_0^5 mC = \underline{\underline{0.5C}}.$

(b) $Q = \int \rho_s dS = \int_{z=0}^{4}\int_{\phi=0}^{2\pi} \rho z^2 \rho\, d\phi\, dz\Big|_{\rho=3} = 9(2\pi)\frac{z^3}{3}\Big|_0^4 nC$

$= \underline{\underline{1.206\,\mu C}}.$

(c) $Q = \int \rho_v \, dv = \iiint \frac{10}{r\sin\theta} r^2 \sin\theta \, d\theta \, d\phi \, dr$

$= 10\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \int_0^{4} r\,dr = 10(2\pi)(\pi)\frac{4^2}{2}$

$Q = \underline{\underline{157.91 \text{ C}}}$.

Prob. 4.6

$Q_A = \int_S \rho_s \, dS = \int_{x=0}^{2} \int_{y=0}^{x/2} 2\times10^{-3} \, dx\,dy$

$= 2\times10^{-2} \frac{x^2}{4}\Big|_0^2 = \underline{\underline{2 \text{ mC}}}$.

$Q_B = \int \rho_s \, dS = \int_{\rho=0}^{2} \int_{\phi=\frac{5\pi}{6}}^{\pi} 10^{-3} \rho \, d\rho \, d\phi$

$= \frac{\pi}{6} \cdot 10^{-3} \frac{\rho^2}{2}\Big|_0^2 = \underline{\underline{\frac{\pi}{3} \text{ mC}}}$

$Q_C = \int \rho_L \, dl = \int_{7\pi/6}^{3\pi/2} 10^{-6} \rho \, d\phi = \underline{\underline{\frac{2\pi}{3} \mu\text{C}}}$.

Prob. 4.7

(a) from Example 4.4,

$\vec{E} = \frac{\rho_L a h \, \bar{a}_x}{2\varepsilon_0 (h^2 + a^2)^{3/2}} = \frac{5\times10^{-9} \times 4 \times 3 \, \bar{a}_x}{2 \times \frac{10^{-9}}{36\pi} (5)^3}$

$= 18\pi(12)/25 \, \bar{a}_x = \underline{\underline{27.14 \, \bar{a}_x \text{ V/m}}}$.

(b) $\vec{E}$ due to the charges is

$$\vec{E}_Q = \frac{Q}{4\pi\epsilon_0}\left[\frac{(4,0,0)-(0,4,0)}{|(4,0,0)-(0,4,0)|^3}\right.$$

$$\left.+\frac{(4,0,0)-(0,-4,0)}{|(4,0,0)-(0,-4,0)|^3}\right] = \frac{2Q}{\pi\epsilon_0 32^{3/2}}\vec{a}_x$$

If $\vec{E}_Q + \vec{E}_l = 0 \longrightarrow Q = -\frac{32^{3/2}}{2}\pi\epsilon_0(27.14)$

$$Q = \underline{\underline{-68.23 \text{ nC}}}.$$

$\boxed{\text{Prob. 4.8}}$

Let $\rho_s = k\rho$

$$Q = \int k\rho \, dS = \int_{\rho=0}^{a}\int_{\phi=0}^{2\pi} k\rho \cdot \rho \, d\phi \, d\rho = \frac{2\pi k a^3}{3}.$$

Hence $k = \frac{3Q}{2\pi a^3}$ and $\rho_s = \frac{3Q\rho}{2\pi a^3}$

Due to symmetry, $\vec{E}$ has only z-component.

$$dE_z = \frac{\rho_s \, dS \cos\alpha}{4\pi\epsilon_0 r} = \frac{\rho_s \, dS}{4\pi\epsilon_0(\rho^2+h^2)} \cdot \frac{h}{(\rho^2+h^2)^{1/2}}$$

$$E_z = \frac{3Q}{2\pi a^3} \cdot \frac{1}{4\pi\epsilon_0}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{a} \frac{\rho h \cdot \rho \, d\phi \, d\rho}{(\rho^2+h^2)^{3/2}}$$

$$= \frac{3Qh}{4\pi\epsilon_0 a^3}\int_0^a \frac{\rho^2 \, d\rho}{(\rho^2+h^2)^{3/2}}.$$

Let $\rho = h\tan\theta$, $d\rho = h\sec^2\theta\, d\theta$.

$$\int_0^a \frac{\rho^2 d\rho}{(\rho^2+h^2)^{3/2}} = \int \frac{\tan^2\theta}{\sec\theta} d\theta = \int (\sec\theta - \cos\theta)\, d\theta$$

$$= \ln(\sec\theta + \tan\theta) - \sin\theta$$

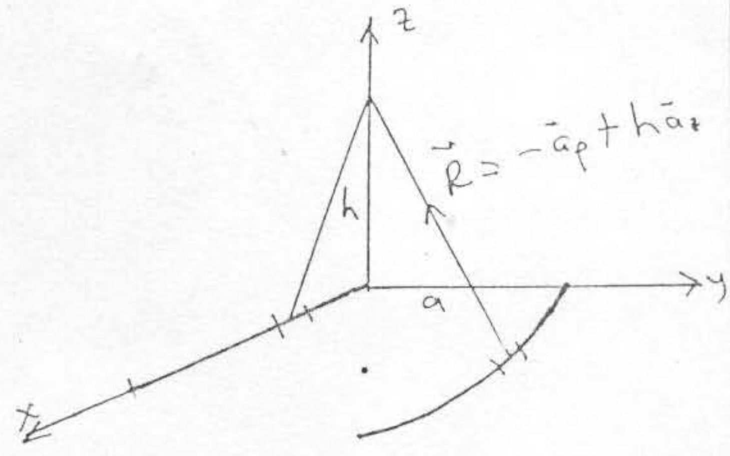
Hence,

$$E_z = \frac{3Qh}{4\pi\varepsilon_0 a^3} \left[\ln\left(\frac{\sqrt{\rho^2+h^2}+\rho}{h}\right) - \frac{\rho}{\sqrt{\rho^2+h^2}} \right]\Bigg|_0^a$$

or $$\vec{E} = \frac{3Qh}{4\pi\varepsilon_0 a^3} \left[\ln \frac{a+\sqrt{a^2+h^2}}{h} - \frac{a}{\sqrt{a^2+h^2}} \right] \vec{a}_z$$

as required.

$\boxed{\text{Prob. 4.9}}$



(a) $\rho_L = \frac{Q}{L} = \frac{2}{3}$ nC/m.

$$\vec{E} = \int \frac{\rho_L\, dl}{4\pi\varepsilon_0 R^2} \bar{a}_R = \frac{\frac{2}{3}\times 10^{-9}}{4\pi\times\frac{10^{-9}}{36\pi}} \int_0^3 \frac{(4\vec{a}_z - x\,\vec{a}_x)\,dx}{(x^2+16)^{3/2}}$$

$$= \frac{-6}{\sqrt{x^2+16}}\Bigg|_0^3 \bar{a}_x + \frac{24x}{\sqrt{x^2+16}}\Bigg|_0^3 \bar{a}_x$$

$$\vec{E} = -0.3\vec{a}_x + 0.9\vec{a}_z \quad V/m.$$

(b) $\vec{E} = -E_\rho \vec{a}_\rho + E_z \vec{a}_z$, $\rho_L = \frac{Q}{3\pi/4} = \frac{4Q}{3\pi}$

$$E_z = \frac{\rho_L a h}{4\pi\varepsilon_0 (h^2+a^2)^{3/2}} \int_{\phi=\pi/4}^{\pi/2} d\phi = \frac{\rho_L a h}{16\varepsilon_0 (a^2+h^2)^{3/2}}$$

$$= \frac{8}{3\pi} \cdot \frac{10^{-9}(3)(4)}{16 \times \frac{10^{-9}}{36\pi}(125)} = \frac{72}{125}$$

Since $\vec{a}_\rho = \cos\phi\, \vec{a}_x + \sin\phi\, \vec{a}_y$,

$$E_\rho = \frac{-\rho_L a^2}{4\pi\varepsilon_0 (h^2+a^2)^{3/2}} \int_{\pi/4}^{\pi/2} (\cos\phi\, \vec{a}_x + \sin\phi\, \vec{a}_y)\, d\phi$$

$$= \frac{-\rho_L a^2}{16\varepsilon_0 (h^2+a^2)^{3/2}} \left[\left(1 - \frac{1}{\sqrt{2}}\right)\vec{a}_x + \frac{1}{\sqrt{2}}\vec{a}_y \right]$$

$$= -0.1611\vec{a}_x - 0.3889\vec{a}_y$$

Thus $\vec{E} = \underline{-0.1611\vec{a}_x - 0.3889\vec{a}_y + 0.576\vec{a}_z \text{ V/m}}$

Prob. 4.10

Due to symmetry, $\vec{E}$ has only z-component given by

$$dE_z = dE \cos\alpha$$

$$dE_z = \frac{\rho_s\, dx\, dy}{4\pi\epsilon_0 (x^2+y^2+h^2)} \cdot \frac{h}{(x^2+y^2+h^2)^{1/2}}$$

$$E_z = \frac{\rho_s h}{4\pi\epsilon_0} \int_{-a}^{a}\int_{-b}^{b} \frac{dx\, dy}{(x^2+y^2+h^2)^{3/2}}$$

$$= \frac{\rho_s h}{\pi\epsilon_0} \int_0^a\int_0^b \frac{dx\, dy}{(x^2+y^2+h^2)^{3/2}}$$

$$= \frac{\rho_s h}{\pi\epsilon_0} \int_0^a \frac{y\, dx}{(x^2+h^2)(x^2+y^2+h^2)^{1/2}} \Bigg|_0^b$$

$$= \frac{\rho_s h}{\pi\epsilon_0} \int_0^a \frac{b\, dx}{(x^2+h^2)(x^2+b^2+h^2)^{1/2}}$$

By changing variables, we finally obtain

$$E_z = \frac{\rho_s}{\pi\epsilon_0} \tan^{-1}\left[\frac{ab}{h(a^2+b^2+h^2)^{1/2}}\right]$$

(b) $Q = \int \rho_s\, dS = \rho_s (2a)(2b) = 10^{-5}(4)(10) = \underline{\underline{0.4\ mC}}$

$$\vec{E} = \frac{10^{-5}}{\pi \times 10^{-9}/36\pi} \tan^{-1}\left[\frac{10}{10(4+25+100)^{1/2}}\right] \vec{a}_z$$

$$= 36\times10^{-3}\,(0.0878\ \text{radian})\,\vec{a}_z = \underline{\underline{31.61\,\vec{a}_z\ \mu V/m}}$$

Prob. 4.11

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3$$

$$= \frac{Q}{4\pi\epsilon_o r^2}\vec{a}_r + \frac{\rho_L}{2\pi\epsilon_o \rho}\vec{a}_\rho + \frac{\rho_S}{2\epsilon_o}\vec{a}_n$$

$$= \frac{100\times10^{-12}}{4\pi \times \frac{10^{-9}}{36\pi}} \frac{[(1,1,1)-(4,1,-3)]}{|(1,1,1)-(4,1,-3)|^3}$$

$$+ \frac{2\times10^{-9}}{2\pi \times \frac{10^{-9}}{36\pi}} \frac{[(1,1,1)-(1,0,0)]}{|(1,1,1)-(1,0,0)|^2} + \frac{5\times10^{-9}(-\vec{a}_z)}{2\times\frac{10^{-9}}{36\pi}}$$

$$= (-0.0216, 0, 0.0288) + (0, 18, 18) - 90\pi(0,0,1)$$

$$= \underline{\underline{-0.0216\vec{a}_x + 18\vec{a}_y - 264.7\vec{a}_z \text{ V/m}}}.$$

Prob. 4.12

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = \frac{\rho_L}{2\pi\epsilon_o\rho}\vec{a}_\rho + \frac{\rho_S}{2\epsilon_o}\cdot\vec{a}_n$$

$$= \frac{30\times10^{-9}}{2\pi\times\frac{10^{-9}}{36\pi}} \frac{[(0,0,0)-(0,1,-3)]}{|(0,0,0)-(0,1,-3)|^2} + \frac{20\times10^{-9}(-\vec{a}_x)}{2\times10^{-9}/36\pi}$$

$$= \underline{\underline{-1131\vec{a}_x - 54\vec{a}_y + 162\vec{a}_z \text{ V/m}}}.$$

Prob. 4.13

$$\vec{D} = \sum_{k=1}^{4} \frac{Q_k(\vec{r}-\vec{r}_k)}{4\pi|(\vec{r}-\vec{r}_k)|^3}$$

$$\vec{D} = \frac{Q}{4\pi}\left[\frac{2\{(0,0,6)-(2,2,0)\}}{|(0,0,6)-(2,2,0)|^3} - \frac{2\{(0,0,6)-(-2,-2,0)\}}{|(0,0,6)-(-2,-2,0)|^3} \right.$$

$$\left. + \frac{\{(0,0,6)-(-2,2,0)\}}{|(0,0,6)-(-2,2,0)|^3} - \frac{\{(0,0,6)-(2,-2,0)\}}{|(0,0,6)-(2,-2,0)|^3} \right]$$

$$= \frac{15}{4\pi}\left[\frac{2(-2,-2,6)}{44^{3/2}} - \frac{2(2,2,6)}{44^{3/2}} + \frac{(2,-2,6)}{44^{3/2}} - \frac{(-2,2,6)}{44^{3/2}} \right]$$

$$= \frac{15}{4\pi 44^{3/2}}(-4,-12,0) \ \mu C/m^2$$

$$= \underline{\underline{-16.36\,\vec{a}_x - 49.08\,\vec{a}_y}} \ nC/m^2.$$

Prob. 4.14

$$\vec{D} = \sum_{k=1}^{3} \frac{Q_k(\vec{r}-\vec{r}_k)}{4\pi|\vec{r}-\vec{r}_k|^3}$$

$$= \frac{1}{4\pi}\left[\frac{Q_1(0,0,-1)}{|(0,0,-1)|^3} + \frac{Q_2(6,-8,0)}{|(6,-8,0)|^3} + \frac{Q_3(0,-4,3)}{|(0,-4,3)|^3} \right]$$

$$\underline{\underline{\vec{D} = -2.39\,\vec{a}_x - 8.276\,\vec{a}_y - 314.5\,\vec{a}_z \ nC/m^2.}}$$

Prob. 4.15

(a) $D = \frac{\rho_L}{2\pi\rho}$, $\rho = \sqrt{6^2+8^2} = 10$

$D = \frac{\rho_L}{2\pi(10)} \longrightarrow \rho_L = 20\pi D = 60\pi \ nC/m$

$= \underline{188.5 \ nC/m}.$

(b) $\vec{D}(0,0,4) = \dfrac{\rho_L}{2\pi\rho}\vec{a}_z = \dfrac{60\pi}{2\pi(4)}\vec{a}_z = \underline{\underline{7.5\vec{a}_z \ nC/m^2}}.$

Prob. 4.16

Let Q_1 be located at the origin. At the spherical surface of radius r,

$$Q_1 = \oint \vec{D}\cdot d\vec{S} = \epsilon E_r \cdot 4\pi r^2$$

or
$$\vec{E} = \frac{Q_1}{4\pi\epsilon r^2}\vec{a}_r$$

by Gauss's law. If a second charge Q_2 is placed on the spherical surface, Q_2 experiences a force

$$\vec{F} = Q_2\vec{E} = \frac{Q_1 Q_2}{4\pi\epsilon r^2}\vec{a}_r$$

which is Coulomb's law.

Prob. 4.17

(a) $\rho_v = \nabla\cdot\vec{D} = \dfrac{\partial D_x}{\partial x} + \dfrac{\partial D_y}{\partial y} + \dfrac{\partial D_z}{\partial z} = 8y + 0 = \underline{\underline{8y \ C/m^3}}.$

(b) $\rho_v = \nabla\cdot\vec{D} = \dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}(\rho^2\sin\phi) + \dfrac{1}{\rho}\dfrac{\partial}{\partial\phi}(2\rho\cos\phi) + \dfrac{\partial}{\partial z}(2z^2)$

$= 2\sin\phi - 2\sin\phi + 4z = \underline{\underline{4z \ C/m^2}}.$

(c) $\rho_v = \nabla\cdot\vec{D} = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(\dfrac{2}{r}\cos\theta\right) + \dfrac{1}{r^4\sin\theta}\dfrac{\partial}{\partial\theta}(\sin^2\theta)$

$$\rho_v = \frac{-2}{r^3}\cos\theta + \frac{1}{r^4\sin\theta}\,2\sin\theta\cos\theta = \underline{\underline{0}}.$$

Prob. 4.18

for $r > a = 10\text{cm}$,

$$Q_T = \oint \vec{D}\cdot d\vec{S} = Q + \int \rho_v\, dv$$

If $\vec{D} = 0$ for $r > a$, then

$$Q = -\int \rho_v\, dv = -\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{0.1} 10^{-2} r^3 r^2 \sin\theta\, d\theta\, d\phi\, dr$$

$$= -4\pi\, 10^{-2} \frac{r^6}{6}\Big|_0^{0.1} = \underline{\underline{-21\ \text{nC}}}.$$

Prob. 4.19

(a) $\oint \vec{D}\cdot d\vec{S} = Q_T \longrightarrow \vec{E} = \frac{Q_T}{4\pi\epsilon_0 r^2}\vec{a}_r$

for $r = 4\text{cm}$,

$$Q_T = \int \rho_v\, dv = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\left[\int_{r=0}^{3}\rho_{v1} + \int_{r=3}^{4}\rho_{v2}\right] r^2\sin\theta\, d\theta\, d\phi\, dr$$

where $\rho_{v1} = 10$, $\rho_{v2} = 20$, $\rho_{v3} = 0$.

$$Q_T = \frac{4\pi}{3}\left[27\rho_{v1} + (64-27)\rho_{v2}\right]\cdot 10^{-6}$$

$$\vec{E}(r=4\text{cm}) = \frac{\frac{4\pi}{3}\left[270 + 37\times 20\right]\times 10^{-6}\times 10^{-9}\,\vec{a}_r}{4\pi\times\frac{10^{-9}}{36\pi}\times 16\times 10^{-4}}$$

$$= \underline{\underline{23.8\,\vec{a}_r\ \text{V/m}}}.$$

(b) for $r = 6$ cm,

$$Q_T = \frac{4\pi}{3}\left[\rho_{v1} r^3\Big|_0^3 + \rho_{v2} r^3\Big|_3^5 + \cancel{\rho_{v3}} r^3\Big|_5^6\right]$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad = 0$$

$$= \frac{4\pi}{3}\left[27\rho_{v1} + 98\rho_{v2}\right]$$

$$\vec{E}(r = 6\text{cm}) = \frac{\frac{4\pi}{3}(270 + 98 \times 20) \times 10^{-6} \times 10^{-9}\, \vec{a}_r}{4\pi \times \frac{10^{-9}}{36\pi} \times 36 \times 10^{-4}}$$

$$= \underline{\underline{23.35\, \vec{a}_r \text{ V/m}}}.$$

Prob. 4.20

From Gauss's law, $\oint \vec{D}\cdot d\vec{S} = Q = \Psi$. Since the faces are equidistant from the center, they must have the flux Ψ_1 thru them. Hence

$$6\Psi_1 = Q \longrightarrow \Psi_1 = \frac{Q}{6} = \underline{\underline{10 \text{ mC}}}.$$

Prob. 4.21

Let $Q_1 = 5\mu C$, $Q_2 = -3\mu C$, $Q_3 = 2\mu C$, and $Q_4 = 10\mu C$. The corresponding distances of the charge from the origin are:

$$r_1 = \sqrt{12^2 + 5^2} = 13\text{m},$$
$$r_2 = \sqrt{3^2 + 4^2} = 5\text{m},$$

$r_3 = \sqrt{2^2 + 6^2 + 3^2} = 7\text{m},$

$r_4 = \sqrt{3^2} = 3\text{m}.$

According to Gauss's law, $\Psi = Q_{enc}$.

(a) for $r = 1 < r_1, r_2, r_3, r_4$, $Q_{enc} = 0$

$\Psi = \underline{\underline{0}}.$

(b) for $r = 10 > r_2, r_3, r_4$,

$Q_{enc} = Q_2 + Q_3 + Q_4 = -3 + 2 + 10 = 9\,\mu C$

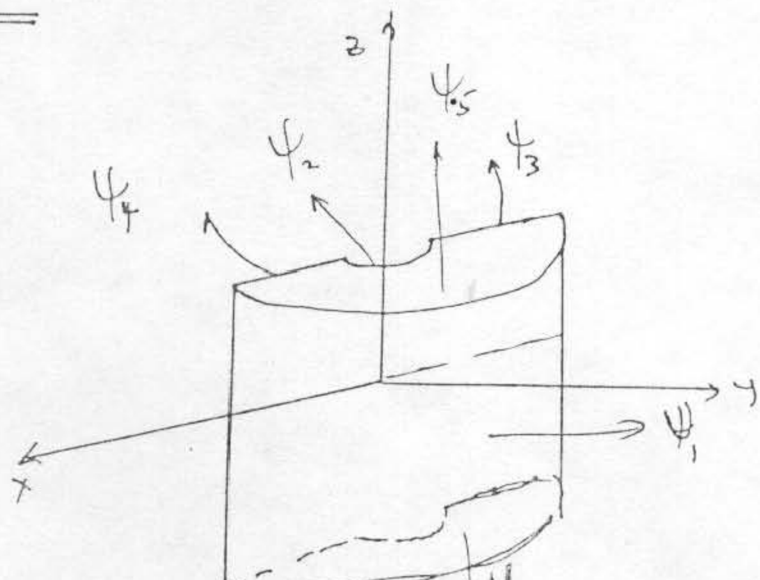
$\Psi = \underline{\underline{9\,\mu C}}.$

(c) for $r = 15 > r_1, r_2, r_3, r_4$

$Q_{enc} = Q_1 + Q_2 + Q_3 + Q_4 = 14\,\mu C$

$\Psi = \underline{\underline{14\,\mu C}}.$

Prob. 4.22



method 1

Let $\Psi = \Psi_1 + \Psi_2 + \Psi_3 + \Psi_4 + \Psi_5 + \Psi_6$

where Ψ_n's are shown on the previous page.

$$\Psi = \iint 2z^2 \sin\frac{\phi}{2}\, \rho\, d\phi\, dz \Big|_{\rho=4} - \iint 2z^2 \sin\frac{\phi}{2}\, \rho\, d\phi\, dz \Big|_{\rho=1}$$

$$+ \iint z^2 \cos\frac{\phi}{2}\, d\rho\, dz \Big|_{\phi=\pi} - \iint z^2 \cos\frac{\phi}{2}\, d\rho\, dz \Big|_{\phi=0}$$

$$+ \iint 4z\rho \sin\frac{\phi}{2}\, \rho\, d\phi\, d\rho \Big|_{z=1} - \iint 4z\rho \sin\frac{\phi}{2}\, \rho\, d\phi\, d\rho \Big|_{z=-2}$$

$$= 6\int_0^{\pi} \sin\frac{\phi}{2} \int_{-2}^{1} z^2 dz - \int_1^4 d\rho \int_{-2}^{1} z^2 dz + 12\int_{\phi}^{4} \rho^2 d\rho \int_0^{\pi} \sin\frac{\phi}{2} d\phi$$

$$= 4(9) - 9 + 8(63) = 531 \text{ mC}$$

$$\Psi = \underline{\underline{0.531\ C}}$$

Method 2

$$\nabla\cdot\vec{D} = \frac{2z^2 \sin\phi/2}{\rho} - \frac{z^2}{2\rho}\sin\frac{\phi}{2} + 4\rho \sin\frac{\phi}{2}$$

$$= \sin\frac{\phi}{2}\left(4\rho + \frac{3}{2}z^2\right)$$

$$\Psi = \int \nabla\cdot\vec{D}\, dv = \int_0^{\pi} \sin\frac{\phi}{2}\, d\phi \iint \left(\frac{3z^2}{2\rho} + 4\rho\right)\rho\, d\rho\, dz$$

$$= 2\left[4\int_{-2}^{1} dz \int_1^4 \rho^2 d\rho + \frac{3}{2}\int_1^4 d\rho \int_{-2}^{1} z^2 dz\right]$$

$$\Psi = 2\left[4(63) + \frac{3}{2}(9)\right] = 531 \text{ mC}$$

$$= \underline{\underline{0.531 \text{ C}}}$$

$\boxed{\text{Prob. 4.23}}$

(a) $$\Psi = \int \vec{D} \cdot d\vec{s} = \int_{\phi=0}^{2\pi} \int_{z=0}^{5} 2\rho z \cos^2\phi \, \rho \, d\phi \, dz \Big|_{\rho=3}$$

$$= 2(3)^2 \int_0^{2\pi} \cos^2\phi \, d\phi \int_0^5 z \, dz = 225\pi$$

$$\Psi = \underline{\underline{706.9 \text{ C}}}$$

(b) Since $d\vec{s}$ is normal to $-\vec{a}_z$ due to the fact that the surface is at the bottom of the cylinder,

$$\Psi = -\int_{\rho=0}^{3} \int_{\phi=0}^{2\pi} \rho^2 \cos^2\phi \, \rho \, d\phi \, d\rho$$

$$= -\int_0^{2\pi} \cos^2\phi \, d\phi \int_0^3 \rho^3 \, d\rho = -\frac{81\pi}{4}$$

$$\Psi = \underline{\underline{-63.6 \text{ C}}}$$

(c) Same as in (b) except that $d\vec{s}$ is along $+\vec{a}_z$ (top of the cylinder). Hence

$$\Psi = \underline{\underline{63.6 \text{ C}}}$$

(d) $$\Psi = \oint \vec{D} \cdot d\vec{s} = Q = 706.9 - 63.6 + 63.6$$

$$= 706.9 \text{ C}$$

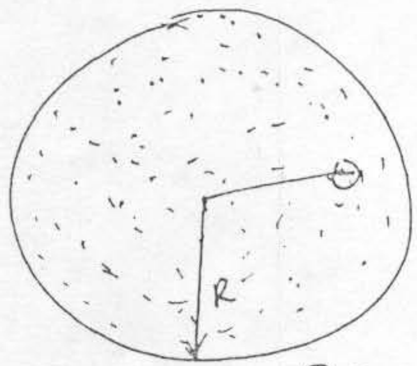
or $Q = \int \rho_v dv = \int \nabla \cdot \vec{D} dv$

$\nabla \cdot \vec{D} = 4z\cos^2\phi - z\cos 2\phi, \; dv = \rho d\phi d\rho dz$

$Q = 4\int_0^5 z\,dz \int_0^{2\pi} \cos^2\phi\, d\phi \int_0^3 \rho\, d\phi - 0 = 225\pi$

$Q = \underline{\underline{706.9\,C.}}$

$\boxed{\text{Prob. 4.24}}$



$F = eE, \quad \rho_0 = \dfrac{e}{\frac{4\pi R^3}{3}} = \dfrac{3e}{4\pi R^3}$

$\rho_v = \begin{cases} \rho_0, & 0 < r < R \\ 0, & \text{elsewhere} \end{cases}$

$\oint \vec{D} \cdot d\vec{S} = Q_{enc} = \int \rho_v dv = \dfrac{3e}{4\pi R^3} \dfrac{4\pi r^3}{3} = D_r 4\pi r^2$

$E_r = \dfrac{3er}{12\pi \epsilon_0 R^3}$

$F = eE = \dfrac{e^2 r}{4\pi \epsilon_0 R^3}$

$\boxed{\text{Prob. 4.25}}$

$\rho_v = \nabla \cdot \vec{D} = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r^2 D_r) = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(10r) = \dfrac{10}{r^2}\, nC/m^3$

$$Q = \int \rho_v\, dv = \iiint \frac{10}{r^2}\cdot r^2 \sin\theta\, d\theta\, dr\, d\phi$$

$$= 10\int_0^2 dr \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta = 10(2)(2\pi)(2)$$

$$= 80\pi = \underline{\underline{251.3 \text{ nC}}}$$

or $$Q = \oint \vec{D}\cdot d\vec{S} = \iint \frac{10}{r}\cdot r^2 \sin\theta\, d\theta\, d\phi \Big|_{r=2}$$

$$= 10(2)\int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi = 20(2)(2\pi) = 80\pi$$

$$= \underline{\underline{251.3 \text{ nC}}}$$

Prob. 4.26

Break up the path $P(1,2,-4) \rightarrow R(3,-5,6)$ as:

$$\begin{array}{ccc} P(1,2,-4) & & R(3,-5,6) \\ \downarrow & & \uparrow \\ P'(3,2,-4) & \longrightarrow & R'(3,-5,-4) \end{array}$$

$$-\frac{W}{Q} = \int \vec{E}\cdot d\vec{l} = \left(\int_P^{P'} + \int_{P'}^{R'} + \int_{R'}^{R}\right) \vec{E}\cdot d\vec{l}$$

$$= \int_{x=1}^{3} dx + \int_{y=2}^{-5} z^2 dy \Big|_{z=-4} + \int_{z=-4}^{6} 2yz\, dz \Big|_{y=-5}$$

$$= 2 + 16(-7) + 2(-5)\frac{z^2}{2}\Big|_{-4}^{6} = 2-112-100 = -210$$

$$W = 210Q = 210(5) = \underline{\underline{1050 \text{ J.}}}$$

Prob. 4.27

$$W = -Q\int \vec{E}\cdot d\vec{l}$$

(a) from A to B, $d\vec{l} = d\rho\,\vec{a}_\rho$, $z=1$, $\phi=0$.

$$W = -Q\int_0^2 20\rho z\cos\phi\, d\rho\Big|_{z=1,\phi=0} = \underline{\underline{-40\,\mu J}}.$$

(b) from B to C, $d\vec{l} = \rho\, d\phi\,\vec{a}_\phi$, $\rho = 2$, $z=1$

$$W = -Q\int_{\phi=0}^{\pi/2} (-10\rho z\sin\phi)\,\rho\, d\phi = \underline{\underline{20\,\mu J}}.$$

(c) from C to D, $d\vec{l} = dz\,\vec{a}_z$, $\rho=2$, $\phi = \pi/3$,

$$W = -Q\int_1^3 10\rho^2\cos\phi\, dz = \underline{\underline{-40\,\mu J}}.$$

(d) From A to D,

$$W = -40 + 20 - 40 = \underline{\underline{-60\,\mu J}}.$$

Prob. 4.28

(a) from A to B, $d\vec{l} = r\,d\theta\,\vec{a}_\theta$,

$$W_{AB} = -Q\int_{\theta=30^\circ}^{90^\circ} 10r\cos\theta\, r\, d\theta\Big|_{r=5} = \underline{\underline{-1250\,nJ}}.$$

(b) from A to C, $d\vec{l} = dr\,\vec{a}_r$,

$$W_{AC} = -Q\int_{r=5}^{10} 20r\sin\theta\, dr\Big|_{\theta=30^\circ} = \underline{\underline{-3750\,nJ}}.$$

(c) from A to D, $d\vec{l} = r\sin\theta\, d\phi\,\vec{a}_\phi$,

$$W_{AD} = -Q\int 0 \cdot r\sin\theta \, d\phi = \underline{\underline{0 \text{ J}}}.$$

(d) $W_{AE} = \underbrace{\cancel{W_{AD}}}_{0} + W_{DF} + W_{FE}$

where F is $(10, 30^\circ, 60^\circ)$. Hence

$$W_{AE} = -Q\left[\int_{r=5}^{10} 20r\sin\theta \, dr \bigg|_{\theta=30^\circ} + 10\int_{\theta=30^\circ}^{90^\circ} 10r\cos\theta r \, d\theta \bigg|_{r=10}\right]$$

$$= -100\left[\frac{75}{2} + \frac{100}{2}\right] \text{nJ} = \underline{\underline{-8750 \text{ nJ}}}$$

$\boxed{\text{Prob. 4.29}}$

$$V = V_1 + V_2 = \frac{Q_1}{4\pi\epsilon_0|\vec{r}-\vec{r}_1|} + \frac{Q_2}{4\pi\epsilon_0|\vec{r}-\vec{r}_2|}$$

(a) $|\vec{r}-\vec{r}_1| = |(0,1,0) - (0,0,0)| = 1,$

$|\vec{r}-\vec{r}_2| = |(0,1,0) - (0,0,1)| = \sqrt{2}$

$$V = \frac{10^{-9}}{4\pi \times \frac{10^{-9}}{36\pi}}\left[\frac{3}{1} - \frac{2}{\sqrt{2}}\right] = \underline{\underline{14.27 \text{ V}}}.$$

(b) $|\vec{r}-\vec{r}_1| = |(1,1,1) - (0,0,0)| = \sqrt{3}$

$|\vec{r}-\vec{r}_2| = |(1,1,1) - (0,0,1)| = \sqrt{2}$

$$V = \frac{10^{-9}}{4\pi \times \frac{10^{-9}}{36\pi}} \left[\frac{3}{\sqrt{3}} - \frac{2}{\sqrt{2}} \right] = \underline{\underline{2.86\ V}}$$

Prob. 4.30

$$V = -\int \vec{E} \cdot d\vec{l} = -\int \frac{\rho_L}{2\pi\epsilon_0 \rho} d\rho = -\frac{\rho_L}{2\pi\epsilon_0} \ln \rho + C$$

$$V_A - V_P = -\frac{\rho_L}{2\pi\epsilon_0} \ln \frac{\rho_A}{\rho_P}$$

where $\rho_A = \sqrt{x_A^2 + z_A^2} = 5$;

$\rho_P = \sqrt{x_P^2 + z_P^2} = 6$.

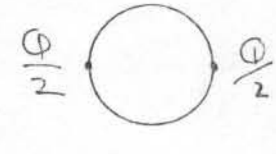
(a) $$V_A - 5 = -\frac{10 \times 10^{-9}}{2\pi \times \frac{10^{-9}}{36\pi}} \ln \frac{5}{6}$$

$$V_A = 32.82 + 5 = \underline{\underline{37.82\ V}}$$

(b) $$5 - V_P = -180 \ln 5/6$$

$$V_P = 5 - 32.82 = \underline{\underline{-27.82\ V}}$$

Prob. 4.31

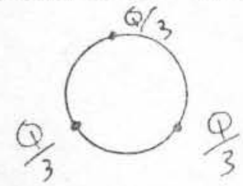


(a) $$V = \frac{2\left(\frac{Q}{2}\right)}{4\pi\epsilon_0 r} = \frac{Q}{4\pi\epsilon_0 r}$$

$$= \frac{60 \times 10^{-6}}{\frac{4\pi \times 10^{-9}}{36\pi} \cdot 4} \cdot = \underline{\underline{15\ kV}}$$

(b)

$$V = \frac{3\left(\frac{Q}{3}\right)}{4\pi\epsilon_0 r} = \underline{\underline{15\,kV}}.$$



(c) $$V = \int \frac{\rho_L\, dl}{4\pi\epsilon_0 r} = \frac{Q}{4\pi\epsilon_0 r} = \underline{\underline{15\,kV}}.$$

$\boxed{\text{Prob. 4.32}}$

$$V = \int \frac{\rho_s\, dS}{4\pi\epsilon_0 R} = \frac{\rho_s}{4\pi\epsilon_0} \int_{\phi=0}^{2\pi} \int_{\rho=0}^{a} \frac{\rho\, d\phi\, d\rho}{(h^2+\rho^2)^{\frac{1}{2}}}$$

$$= \frac{\rho_s}{4\pi\epsilon_0} (2\pi) \int_0^a (h^2+\rho^2)^{-\frac{1}{2}} \frac{1}{2} d(\rho^2)$$

$$= \frac{\rho_s}{4\epsilon_0}\, 2\, (h^2+\rho^2)^{\frac{1}{2}} \Big|_0^a$$

$$V = \frac{\rho_s}{2\epsilon_0} \left[\sqrt{h^2+a^2} - h \right].$$

$\boxed{\text{Prob. 4.33}}$

(a) $$\nabla \times \vec{E} = (-5\rho \sin\phi + 5\rho \sin\phi)\,\vec{a}_\rho$$

$$+ (10\rho\cos\phi - 10\rho\cos\phi)\,\vec{a}_\phi$$

$$+ \frac{1}{\rho}(-10\rho z \sin\phi + 10\rho z \sin\phi)\,\vec{a}_z = 0.$$

(b) $d\vec{l} = \rho\, d\phi\, \vec{a}_\phi$, $\rho = 1$, $z = 1$

$$\oint \vec{E}\cdot d\vec{l} = \int_0^{2\pi} (-5\rho z \sin\phi\, \rho\, d\phi) = -5\int_0^{2\pi} \sin\phi\, d\phi = 0.$$

Prob. 4.34

From Eq. (4.12),

$$\vec{E} = \frac{1}{4\pi\epsilon_o} \sum \frac{Q_k(\vec{r} - \vec{r}_k)}{|\vec{r} - \vec{r}_k|^3}.$$

Using $\nabla \times V\vec{A} = V\, \nabla \times \vec{A} + \nabla V \times \vec{A}$ where

$$V = \frac{1}{|\vec{r} - \vec{r}_k|^3}, \quad \vec{A} = \vec{r} - \vec{r}_k,$$

$$\nabla \times \frac{\vec{r} - \vec{r}_k}{|\vec{r} - \vec{r}_k|^3} = \frac{1}{|\vec{r} - \vec{r}_k|^3} \nabla \times (\vec{r} - \vec{r}_k) + \nabla \frac{1}{|\vec{r} - \vec{r}_k|^3} \times (\vec{r} - \vec{r}_k)$$

from Prob. 3.20(g), $\nabla \times (\vec{r} - \vec{r}_k) = 0$. It can also be shown that grad $r^n = n r^{n-2} \vec{r}$ so that

$$\nabla \frac{1}{|\vec{r} - \vec{r}_k|^3} = \frac{-3(\vec{r} - \vec{r}_k)}{|\vec{r} - \vec{r}_k|^5}.$$

Hence

$$\nabla \times \frac{\vec{r} - \vec{r}_k}{|\vec{r} - \vec{r}_k|} = 0 - \frac{3(\vec{r} - \vec{r}_k) \times (\vec{r} - \vec{r}_k)}{|\vec{r} - \vec{r}_k|^3} = 0$$

and thus $\nabla \times \vec{E} = 0$ as required.

Prob. 4.35

(a) $\vec{E} = -\nabla V = e^{-x}(-\vec{a}_x \sinh y \sin z + \vec{a}_y \cosh y \sin z + \vec{a}_z \sinh y \cos z)$ V/m.

(b) $\vec{E} = -\nabla V = -\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-3/2}\vec{a}_x + \cdots$

$$= \frac{3}{2}(x^2+y^2+z^2)^{-5/2}(2x)\vec{a}_x + \cdots$$

$$= \frac{3(x\vec{a}_x + y\vec{a}_y + z\vec{a}_z)}{(x^2+y^2+z^2)^{5/2}} \text{ V/m.}$$

(c) $\vec{E} = -\nabla V = e^{-z}(-\cos\phi\,\vec{a}_\rho + \sin\phi\,\vec{a}_\phi + \rho\cos\phi\,\vec{a}_z)$ V/m

(d) $\vec{E} = -\nabla V = \frac{1}{r^3}\left(2\cos\phi\sin\theta\,\vec{a}_r - \cos\theta\cos\phi\,\vec{a}_\theta + \sin\phi\,\vec{a}_\phi\right)$ V/m.

Prob. 4.36

(a) $V = -\int \vec{E}\cdot d\vec{l} = -\int \vec{E}\cdot dx\,\vec{a}_x = -\int 2xy\,dx$

$V = -x^2y + V_1(y,z)$ — (1)

or $V = -\int \vec{E}\cdot dy\,\vec{a}_y = -\int x^2\,dy = -x^2y + V_2(x,z)$ — (2)

or $V = -\int \vec{E}\cdot dz\,\vec{a}_z = \int dz = z + V_3(x,y)$ — (3)

from (1) to (3),

$$V = z - x^2y + C_1$$

(b) $V = -\int \vec{E}\cdot d\rho\,\vec{a}_\rho = -\rho e^{-z}\sin\phi + V_1(z,\phi)$ — (1)

or $V = -\int \vec{E}\cdot \rho\, d\phi\,\vec{a}_\phi = -\rho e^{-z}\sin\phi + V_2(\rho,z)$ — (2)

or $V = -\int \vec{E}\cdot dz\,\vec{a}_z = -\rho e^{-z}\sin\phi + V_3(\rho,\phi)$ —(3)

From (1) to (3),

$$\underline{\underline{V = c_2 - \rho e^{-z}\sin\phi}}$$

(c) $V = -\int \vec{E}\cdot dr\,\vec{a}_r = -\ln r\sin\theta + V_1(\phi,\theta)$ —(1)

or $V = -\int \vec{E}\cdot r\,d\theta\,\vec{a}_\theta = -\ln r\sin\theta + V_2(r;\phi)$ —(2)

from (1) and (2),

$$\underline{\underline{V = c_3 - \ln r\sin\theta}}$$

Prob. 4.37

(a) $$m\frac{d^2y}{dt^2} = eE \longrightarrow u\frac{dy}{dt} = \frac{eEt}{m} + c_0$$

$$y = \frac{eEt^2}{2m} + c_0t + c_1$$

"From rest" implies $c_1 = 0 = c_0$.

At $t = t_0$, $y = d$, $E = \frac{V}{d}$ or $V = Ed$.

$$t^2 = \frac{2md}{eE}.$$

Hence $$u = \frac{eE}{m}\sqrt{\frac{2md}{eE}} = \sqrt{\frac{2eV}{m}}$$

i.e. $u \propto \sqrt{V}$

or $u = k\sqrt{V}$

(b) $k = \sqrt{\frac{2e}{m}} = \sqrt{\frac{2 \times 1.603 \times 10^{-19}}{9.1066 \times 10^{-31}}}$

$= \underline{\underline{5.933 \times 10^{5}}}.$

(c) $V = \frac{u^2 m}{2e} = \frac{9 \times 10^{16} \times \frac{1}{100}}{2 \times 1.76 \times 10^{11}} = 2557$

$V = \underline{\underline{2.557 \text{ kV}}}.$

$\boxed{\text{Prob. 4.38}}$

(a) This is similar to Example 4.3.

$u_y = \frac{eEt}{m}, \quad u_x = u_o$

$y = \frac{eEt^2}{2m}, \quad x = u_o t$

$t = \frac{x}{u_o} = \frac{10 \times 10^{-2}}{10^7} = 10 \text{ ns}.$

Since $x = 10\text{cm}$ when $y = 1\text{cm}$,

$E = \frac{2my}{et^2} = \frac{2 \times 10^{-2}}{1.76 \times 10^{11} \times 10^{-16}} = 1.136 \text{ kV/m}$

$\vec{E} = \underline{\underline{-1.136 \vec{a}_y \text{ kV/m}}}.$

(b) $u_x = u_o = 10^7, \quad u_y = \frac{2000}{1.76} \times 1.76 \times 10^{11} \times 10^{-8} = 2 \times 10^6$

$\vec{u} = \underline{\underline{(\vec{a}_x + 0.2 \vec{a}_y) \times 10^7 \text{ m/s}}}.$

Prob. 4.39

(a) Applying Gauss's law,

$$\oint \vec{E} \cdot d\vec{s} = \frac{Q_{enc}}{\epsilon_0}$$

$$E_r \cdot 4\pi r^2 = \frac{Q_{enc}}{\epsilon_0}$$

$$Q_{enc} = \int_0^a 4\pi r^2 \rho_v \, dv = 4\pi \rho_0 \int_0^a \left(1 - \frac{r^2}{a^2}\right) r^2 dr$$

$$= \frac{8\pi a^3 \rho_0}{15} = E_r \cdot 4\pi r^2 \epsilon_0$$

$$\vec{E} = \frac{2 a^3 \rho_0}{15 \epsilon_0 r^2} \vec{a}_r , \quad r \geq a$$

$$V = -\int E_r \, dr = \frac{2 a^3 \rho_0}{15 \epsilon_0 r} + c_1$$

But $V(r \to \infty) = 0$, hence $c_1 = 0$ and

$$V = \frac{2 a^3 \rho_0}{15 \epsilon_0 r} , \quad r \geq a$$

(b) When $r < a$, i.e. inside the sphere,

$$Q_{enc} = \int_0^r \rho_0 \left(1 - \frac{r^2}{a^2}\right) 4\pi r^2 dr$$

$$= 4\pi \rho_0 \left(\frac{r^3}{3} - \frac{r^5}{15 a^2}\right) = E_r \cdot 4\pi r^2 \epsilon_0$$

$$\vec{E} = \frac{\rho_0}{\epsilon_0}\left[\frac{r}{3} - \frac{r^3}{5a^2}\right]\vec{a}_r \ \ V/m, \ r \le a.$$

$$V = -\int E_r dr = -\frac{\rho_0}{\epsilon_0}\left(\frac{r^2}{6} - \frac{r^4}{20a^2}\right) + C_2$$

From (a), $V(r=a) = \dfrac{2a^2\rho_0}{15\epsilon_0}$, hence

$$\frac{2a^2\rho_0}{15\epsilon_0} = \frac{\rho_0}{\epsilon_0}\left(\frac{a^2}{20} - \frac{a^2}{6}\right) + C_2$$

i.e. $C_2 = \dfrac{a^2\rho_0}{4\epsilon_0}$

Thus

$$V = \frac{\rho_0}{\epsilon_0}\left[\frac{a^2}{4} + \frac{r^4}{20a^2} - \frac{r^2}{6}\right], \ r \le a$$

(c) $\vec{E}$ is maximum for $r \le a$.

$$\frac{dE_r}{dr} = \frac{\rho_0}{\epsilon_0}\left(\frac{1}{3} - \frac{3r^2}{5a^2}\right) = 0 \longrightarrow r^2 = \frac{5a^2}{9}$$

$$r = a\sqrt{5/9} = \underline{0.745a}$$

(d) V is maximum for $r \le a$ where

$$\frac{dV}{dr} = 0 \longrightarrow \frac{4r^3}{20a^2} - \frac{2r}{6} = 0$$

i.e. $r = 0, \ a\sqrt{5/3}$

$V(r=0) = \dfrac{\rho_0 a^2}{4\epsilon_0}$, $V\left(r = a\sqrt{5/3}\right) = \dfrac{a^2\rho_0}{9\epsilon_0}$

Hence
$$V_{max} = \frac{\rho_o a^2}{4\epsilon_o} \quad \text{at } r = 0.$$

Prob. 4.40

$$V = \frac{p\cos\theta}{4\pi\epsilon_o r} = \frac{k\cos\theta}{r}.$$

At $(0, 1nm)$, $\theta = 0$, $r = 1nm$, $V = 9$; i.e.

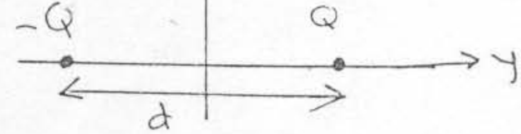
$$9 = \frac{k \cdot 1}{1\times 10^{-18}} \longrightarrow k = 9\times 10^{-18}$$

$$V = 9\times 10^{-18}\frac{\cos\theta}{r}$$

At $(1,1)nm$, $r = \sqrt{2}\ nm$, $\theta = 45°$,

$$V = \frac{9\times 10^{-18}\cos 45°}{10^{-18}\sqrt{2}} = \frac{9}{2\sqrt{2}} = \underline{\underline{3.182\ V.}}$$

Prob. 4.41



The dipole is oriented along y-axis.

$$V = \frac{\vec{p}\cdot\vec{r}}{4\pi\epsilon_o r^2}, \quad \vec{p}\cdot\vec{r} = Qd\,\vec{a}_y\cdot\vec{a}_r$$
$$= Qd\sin\theta\sin\phi$$

$$V = \frac{Qd\sin\theta\sin\phi}{4\pi\epsilon_o r^2}$$

$$\vec{E} = -\nabla V = -\frac{\partial V}{\partial r}\vec{a}_r - \frac{1}{r}\frac{\partial V}{\partial \theta}\vec{a}_\theta - \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\vec{a}_\phi$$

$$= \frac{Qd}{4\pi\epsilon_0}\left[\frac{2\sin\theta\sin\phi}{r^3}\vec{a}_r - \frac{\cos\theta\sin\phi}{r^3}\vec{a}_\theta - \frac{\cos\phi}{r^3}\vec{a}_\phi\right]$$

$$\underline{\underline{\vec{E} = \frac{Qd}{4\pi\epsilon_0 r^3}\left(2\sin\theta\sin\phi\,\vec{a}_r - \cos\theta\sin\phi\,\vec{a}_\theta - \cos\phi\,\vec{a}_\phi\right)}}$$

Prob. 4.42

(a) $\frac{dr}{E_r} = \frac{r\,d\theta}{E_\theta} = \frac{r\sin\theta\,d\phi}{E_\phi}$. Hence

$$\frac{dr}{2\cos\theta} = \frac{r\,d\theta}{\sin\theta} \longrightarrow \frac{dr}{r} = \frac{2\cos\theta}{\sin\theta}d\theta$$

Integrating both sides,

$$\ln r = 2\int\frac{d(\sin\theta)}{\sin\theta} = 2\ln\sin\theta + \ln c$$

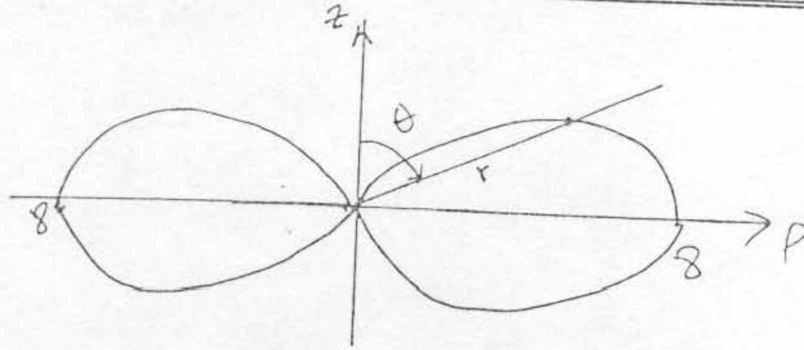
or $\quad r = c\sin^2\theta$

At $(4, \pi/4, \pi/2)$, $r=1$, $\theta = \pi/4$,

$$4 = c\cdot\tfrac{1}{2} \longrightarrow c = 8$$

$$r = 8\sin^2\theta.$$

The flux line is as sketched on the next page.



Since $\vec{E}$ is tangential to the line,

$$\vec{E}(1, \pi/4, \pi/2) = \frac{(2\vec{a}_r + \vec{a}_\theta)}{4^3\sqrt{2}}$$

A unit vector along $\vec{E}$ is

$$\vec{a} = \frac{1}{\sqrt{5}}(2\vec{a}_r + \vec{a}_\theta) = \underline{\underline{0.8944\vec{a}_r + 0.4472\vec{a}_\theta}}$$

(b) The equipotential line is $V=0$, i.e.

$$0 = V = V_1 + V_2 = \frac{Q_1}{4\pi\epsilon_0 r_1} + \frac{Q_2}{4\pi\epsilon_0 r_2}$$

At an arbitrary point $(x, y, 0)$,

$$0 = \frac{10^{-9}}{4\pi\epsilon_0}\left[\frac{-3}{|(x,y)-(1,0)|} + \frac{4}{|(x,y)-(-1,0)|}\right]$$

or

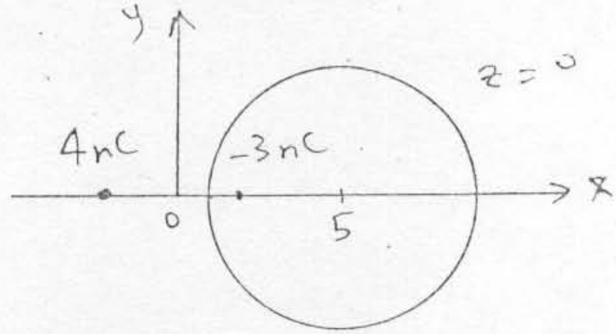
$$\frac{3}{\sqrt{(x-1)^2 + y^2}} = \frac{4}{\sqrt{(x+1)^2 + y^2}}$$

$$16(x-1)^2 + 16y^2 = 9(x+1)^2 + 9y^2$$

or

$$(x-5)^2 + y^2 = (4.86)^2$$

i.e. a circle of radius 4.86 centered at (5, 0). Thus the equipotential line is sketched below



Prob. 4.43

(a) $W = Q_3 (V_{31} + V_{32}) = Q_3 \left(\frac{Q_1}{4\pi\epsilon_0 r_1} + \frac{Q_2}{4\pi\epsilon_0 r_2} \right)$

$$= \frac{2\times10^{-18}}{4\pi \times \frac{10^{-9}}{36\pi}} \left[\frac{-3}{|(0,0,0)-(1,0,0)|} + \frac{4}{|(0,0,0)-(-1,0,0)|} \right]$$

$W = \underline{\underline{18 \text{ nJ}}}.$

(b) $W = W_1 + W_2 + W_3 = 0 + Q_2 V_{21} + Q_3 (V_{31} + V_{32})$

$$= \frac{1}{4\pi\epsilon_0} \left[\frac{Q_2 Q_1}{|\vec{r}_2 - \vec{r}_1|} + \frac{Q_3 Q_1}{|\vec{r}_3 - \vec{r}_1|} + \frac{Q_3 Q_2}{|\vec{r}_3 - \vec{r}_2|} \right]$$

$$= 9\times10^{9} \times 10^{-18} \left[-\frac{12}{2} - \frac{6}{1} + \frac{8}{1} \right]$$

$= \underline{\underline{-36 \text{ nJ}}}.$

$\boxed{\text{Prob. 4.44}}$

$$W = \frac{1}{2}\int \vec{D}\cdot\vec{E}\,dv = \frac{\epsilon_0}{2}\int |\vec{E}|\,dv, \quad \vec{E} = \frac{Q\,\vec{a}_r}{4\pi\epsilon_0 r^2}$$

$$W = \frac{\epsilon_0}{2}\iiint \frac{Q^2}{16\pi^2\epsilon_0^2 r^4}\cdot r^4 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \frac{Q^2}{32\pi^2\epsilon_0}\cdot 4\pi\int_a^\infty \frac{1}{r^2}\,dr = \underline{\underline{\frac{Q^2}{8\pi\epsilon_0 a}}}$$

$\boxed{\text{Prob. 4.45}}$

$$W = \frac{1}{2}\int \vec{D}\cdot\vec{E}\,dv = \frac{1}{2\epsilon_0}\int \vec{D}\cdot\vec{D}\,dv$$

$$= \frac{10^{-18}}{2\epsilon_0}\iiint (y^2z^2 + x^2z^2 + x^2y^2)\,dx\,dy\,dz$$

$$= \frac{3\times10^{-18}}{2\times\frac{10^{-9}}{36\pi}}\left[\int_0^1 dx\int_0^1 y^2\,dy\int_0^1 z^2\,dz\right]$$

$$= 6\pi \text{ nJ} = \underline{\underline{18.85 \text{ nJ}}}.$$

$\boxed{\text{Prob. 4.46}}$

$$W = \frac{1}{2}\epsilon_0\int |\vec{E}|^2 dv = \frac{1}{2}\epsilon_0\iiint (4r^2\sin^2\theta\cos^2\phi$$
$$+ r^2\cos^2\theta\cos^2\phi + r^2\sin^2\phi)\, r^2 \sin\theta\, d\theta\, d\phi\, dr$$

$$= \frac{1}{2}\epsilon_o \int_0^2 r^4 dr \left[4\int_0^{2\pi} \cos^2\phi \, d\phi \int_0^{\pi} \sin^3\theta d\theta \right.$$

$$\left. + \int_0^{2\pi} \cos^2\phi \, d\phi \int_0^{\pi} \cos^2\theta \sin\theta d\theta + \int_0^{2\pi} \sin^2\phi \, d\phi \int_0^{\pi} \sin\theta d\theta \right]$$

$$= \frac{1}{2}\epsilon_o \frac{r^5}{5}\Big|_0^2 \left[4 \ \frac{1}{2}(2\pi) \int_0^{\pi} (1-\cos^2\theta)\, d(-\cos\theta) \right.$$

$$\left. + \frac{1}{2}(2\pi)\int_0^{\pi} \cos^2\theta \, d(-\cos\theta) + \frac{1}{2}(2\pi)\ (-\cos\theta)\Big|_0^{\pi} \right]$$

$$= 3.2\epsilon_o \left[4\pi \left(\frac{\cos^3\theta}{3} - \cos\theta \right)\Big|_0^{\pi} + \pi\left(-\frac{\cos^3\theta}{3}\right)\Big|_0^{\pi} + 2\pi \right]$$

$$= 3.2\,\epsilon_o (8\pi) = 25.6\pi \times \frac{10^{-9}}{36\pi} = \underline{\underline{0.7111 \ nJ}}.$$

$\boxed{\text{Prob. 4.47}}$ $\vec{E} = -\nabla V = -\left(\frac{\partial V}{\partial \rho}\vec{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\vec{a}_\phi + \frac{\partial V}{\partial z}\vec{a}_z \right)$

$$= -\left(2\rho z \sin\phi \,\vec{a}_\rho + \rho z \cos\phi\, \vec{a}_\phi + \rho^2 \sin\phi \,\vec{a}_z\right)$$

$$W = \frac{1}{2}\epsilon_o \int |\vec{E}|^2 dv = \iiint \left(4\rho^2 z^2 \sin^2\phi + \rho^2 z^2 \cos^2\phi + \rho^4 \sin^2\phi\right) \rho \, d\phi \, dz \, d\rho$$

$$\frac{2W}{\epsilon_o} = 4\int_1^4 \rho^3 dz \int_{-2}^{2} z^2 dz \int_0^{\pi/3} \sin^2\phi \, d\phi + \int_1^4 \rho^3 d\rho \int_{-2}^{2} z^2 dz \int_0^{\pi/3} \cos^2\phi \, d\phi$$

$$+ \int_1^4 \rho^5 d\rho \int_{-2}^{2} dz \int_0^{\pi/3} \sin^2\phi \, d\phi$$

But $\int_0^{\pi/3} \cos^2\phi \, d\phi = \frac{1}{2}\int_0^{\pi/3} [1+\cos 2\phi] d\phi = \frac{\pi}{6} + \frac{1}{4}\sin\frac{2\pi}{3}$

$$= 0.7401$$

$$\int_0^{\pi/2} \sin^2\phi\, d\phi = \frac{1}{2}\int_0^{\pi/2}(1-\cos 2\phi)\,d\phi = \frac{\pi}{6} - \frac{1}{4}\sin\frac{2\pi}{3}$$

$$= 0.3071$$

$$\frac{2W}{\epsilon_0} = \frac{4}{4}\rho^4\Big|_0^1 \frac{2z^3}{3}\Big|_0^2 (0.3071) + \frac{\rho^4}{4}\Big|_0^1 \frac{2z^3}{3}\Big|_0^2 (0.7401)$$

$$+ \frac{\rho^6}{6}\Big|_1^4 (4)(0.3071)$$

$$= 255 \times \frac{16}{3} \times 0.3071 + \frac{225}{4} \times \frac{16}{3} \times 0.7041 + \frac{4096}{6}(0.3071)$$

$$= 417.67 + 239.394 + 838.59 = 1495.6$$

$$W = \frac{1495.6}{2} \times \frac{10^{-9}}{36\pi} = \underline{\underline{6.612\ nJ}}.$$

CHAPTER 5

P.E. 5.1

$$d\vec{S} = \rho\, d\phi\, dz\, \vec{a}_\rho$$

$$I = \int \vec{J}\cdot d\vec{S} = \int_{\phi=0}^{2\pi}\int_{z=1}^{5} 10z\sin^2\phi\ \rho\, dz\, d\phi \Big|_{\rho=2}$$

$$= 10(2)\ \frac{z^2}{2}\Big|_1^5 \int_0^{2\pi} \frac{1}{2}(1-\cos 2\phi)\,d\phi = 240\pi$$

$$I = \underline{\underline{754\ A}}.$$

P.E. 5.2

$$I = \rho_s w u = 0.5 \times 10^{-6} \times 0.1 \times 10 = 0.5\,\mu A$$

$$V = iR = 10^{14} \times 0.5 \times 10^{-6} = \underline{\underline{50 \text{ mV}}}.$$

$\boxed{\text{P.E. 5.3}}$

$$\sigma = 5.8 \times 10^{7} \text{ ℧/m}$$

$$J = \sigma E \longrightarrow E = \frac{J}{\sigma} = \frac{8 \times 10^{6}}{5.8 \times 10^{7}}$$

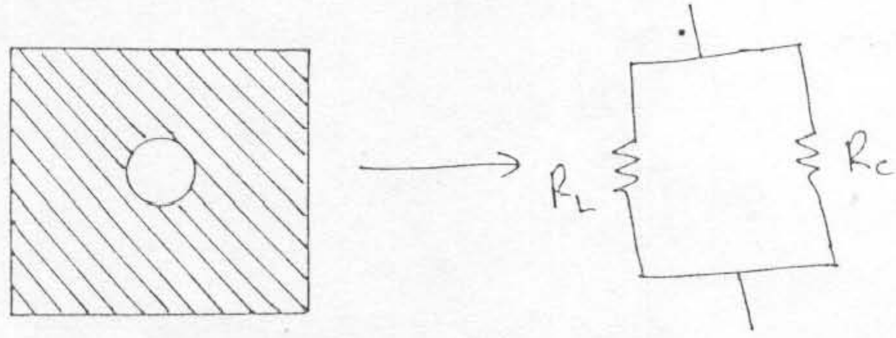
$$= \underline{\underline{0.138 \text{ V/m}}}.$$

$$J = \rho_v u \longrightarrow u = \frac{J}{\rho_v} = \frac{8 \times 10^{6}}{1.81 \times 10^{10}}$$

$$= \underline{\underline{4.42 \times 10^{-4} \text{ m/s}}}.$$

$\boxed{\text{P.E. 5.4}}$

The composite bar can be modeled by a parallel combination of resistors as shown below.



for the lead, $R_L = \dfrac{l}{\sigma_L S_L}$, $S_L = d^2 - \pi r^2 = 9 - \frac{\pi}{4}$ cm²

$$R_L = 974 \mu\Omega = 0.974 \text{ m}\Omega.$$

for copper, $R_c = \dfrac{l}{\sigma_c S_c}$, $S_c = \pi r^2 = \dfrac{\pi}{4}\ \text{cm}^2$.

$$R_c = \frac{4}{5.8\times10^6 \times \frac{\pi}{4}\times10^{-4}} = 8.781\ \text{m}\Omega$$

$$R = \frac{R_L R_c}{R_L + R_c} = \frac{0.974 \times 8.781}{0.974 + 8.781}$$

$$= 0.8767\ \text{m}\Omega = \underline{\underline{876.7\ \mu\Omega}}.$$

$\boxed{\text{P.E.5.5}}$

$$\rho_{ps} = P_x = \vec{P}\cdot\vec{a}_x = ax^2 + b$$

$$\rho_{ps}\big|_{x=0} = \vec{P}\cdot(-\vec{a}_x)\big|_{x=0} = \underline{\underline{-b}}$$

$$\rho_{ps}\big|_{x=L} = \vec{P}\cdot\vec{a}_x\big|_{x=L} = \underline{\underline{aL^2+b}}$$

$$Q_s = \int \rho_{ps}\, dS = -b\cdot A + (aL^2+b)\cdot A = AaL^2$$

$$\rho_{pv} = -\nabla\cdot\vec{P} = -\frac{d}{dx}(ax^2+b) = -2ax$$

$$\rho_{pv}\big|_{x=0} = \underline{\underline{0}}, \quad \rho_{pv}\big|_{x=L} = \underline{\underline{-2aL}}$$

$$Q_v = \int \rho_{pv}\, dv = \int_0^L (-2ax)\, A\, dx = -AaL^2$$

Hence $Q_T = Q_v + Q_s = -AaL^2 + AaL^2 = 0$

P.E. 5.6

$$\vec{E} = \frac{V}{d}\vec{a}_x = \frac{1\times10^3}{2\times10^{-3}}\vec{a}_x = \underline{\underline{500\,\vec{a}_x\ kV/m.}}$$

$$\vec{P} = \chi_e\epsilon_o\vec{E} = (2.25-1)\times\frac{10^{-9}}{36\pi}\times0.5\times10^6$$

$$= \underline{\underline{6.853\,\vec{a}_x\ \mu C/m^2.}}$$

$$\rho_{ps} = \vec{P}\cdot\vec{a}_x = \underline{\underline{6.853\ \mu C/m^2}}$$

P.E. 5.7

(a) Since $\vec{P} = \epsilon_o\chi_e\vec{E}$, $P_x = \chi_e\epsilon_o E_x$

$$\chi_e = \frac{P_x}{\epsilon_o E_x} = \frac{3\times10^{-9}}{10\pi}\cdot\frac{1}{5}\times36\pi\times10^9 = \underline{\underline{2.16}}$$

(b) $$\vec{E} = \frac{\vec{P}}{\chi_e\epsilon_o} = \frac{36\pi\times10^9}{2.16}\cdot\frac{1}{10\pi}(3,-1,4)\cdot10^{-9}$$

$$= \underline{\underline{5\vec{a}_x - 1.67\vec{a}_y + 6.67\vec{a}_z\ V/m.}}$$

(c) $$\vec{D} = \epsilon_o\epsilon_r\vec{E} = \frac{\epsilon_r\vec{P}}{\chi_e} = \frac{3.16}{2.16}\left(\frac{1}{10\pi}\right)(3,-1,4)\ nC/m^2$$

$$= \underline{\underline{139.7\vec{a}_x - 46.6\vec{a}_y + 186.3\vec{a}_z\ pC/m^2.}}$$

P.E. 5.8

from Example 5.8,

$$F = \frac{\rho_s^2 S}{2\epsilon_o}$$

i.e. $$\rho_s^2 = \frac{2\epsilon_0 F}{S}$$

But $$\rho_s = \epsilon_0 E = \epsilon_0 \frac{V_d}{d}.$$ Hence

$$\rho_s^2 = \frac{\epsilon_0^2 V_d^2}{d^2} = \frac{2\epsilon_0 F}{S} \longrightarrow V_d^2 = \frac{2Fd^2}{\epsilon_0 S}$$

i.e.

$$V_d = V_1 - V_2 = \sqrt{\frac{2Fd^2}{\epsilon_0 S}} \quad \text{as required.}$$

$\boxed{\text{P.E. 5.9}}$ (a) Since $\vec{a}_n = \vec{a}_x$,

$$\vec{D}_{1n} = 12\vec{a}_x, \quad \vec{D}_{1t} = -10\vec{a}_y + 4\vec{a}_z$$

$$\vec{D}_{2n} = \vec{D}_{1n} = 12\vec{a}_x$$

$$\vec{E}_{2t} = \vec{E}_{1t} \longrightarrow \vec{D}_{2t} = \frac{\epsilon_2 \vec{D}_{1t}}{\epsilon_1} = \frac{1}{2.5}(-10\vec{a}_y + 4\vec{a}_z)$$

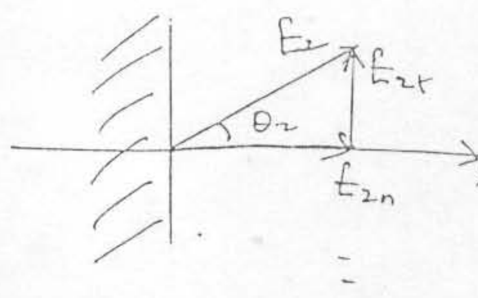
$$= -4\vec{a}_y + 1.6\vec{a}_z$$

$$\vec{D}_2 = \vec{D}_{2n} + \vec{D}_{2t} = \underline{-12\vec{a}_x - 4\vec{a}_y + 1.6\vec{a}_z \ \text{nC/m}^2}$$

(b) $$\tan\theta_2 = \frac{D_{2t}}{D_{2n}} = \frac{\sqrt{(-4)^2 + (1.6)^2}}{12} = 0.359$$

$$\rightarrow \underline{\underline{\theta_2 = 19.75^\circ}}$$

(b)



$$E_{1t} = E_{2t}$$
$$= E_2 \sin\theta_2 = 12\sin 60^\circ$$
$$= 10.392$$

$$E_{1n} = \frac{\epsilon_{r2}}{\epsilon_{r1}} E_{2n} = \frac{1}{2 \cdot 5}\ 12 \cos 60^\circ = 2 \cdot 4.157$$

$$E_1 = \sqrt{E_{1t}^2 + E_{1n}^2} \qquad = \underline{\underline{10 \cdot 67}}$$

$$\tan\theta_1 = \frac{\epsilon_{r1}}{\epsilon_{r2}} \tan\theta_2 = \frac{2 \cdot 5}{1} \tan 60^\circ = 4.33$$

$$\rightarrow \theta_1 = \underline{77^\circ}.$$

Note that $\theta_1 > \theta_2$.

$\boxed{\text{P.E. 5.10}}$

$$\vec{D} = \epsilon_0 \vec{E} = \frac{10^{-9}}{36\pi} (60, 20, -30) \cdot 10^{-3}$$

$$= \underline{\underline{0.531\, \vec{a}_x + 0.177\, \vec{a}_y - 0.265\, \vec{a}_z\ p\, C/m^2}}.$$

$$\rho = D_n = |\vec{D}| = \frac{10^{-9}}{36\pi} (10) \sqrt{36 + 4 + 9} \cdot 10^{-3}$$

$$= \underline{\underline{0.619\ p\, C/m^2}}.$$

$\boxed{\text{Prob. 5.1}}$

$$I = \int \vec{J} \cdot d\vec{S} = \int_{\rho=0}^{a} \int_{\phi=0}^{2\pi} \frac{500}{\rho} \rho\, d\phi\, d\rho = 500\,(2\pi a)$$

$$= 1000\pi \times 1.6 \times 10^{-3} = 1.6\pi$$

$$= \underline{\underline{5.026\ A}}$$

Prob. 5.2

$$I = \int \vec{J}\cdot d\vec{S} = 10\int_{\rho=0}^{a}\int_{\phi=0}^{2\pi} e^{-(1-\rho/a)}\,\rho\, d\phi\, d\rho$$

$$= 20\pi \int_0^a \rho e^{-(1-\rho/a)}\, d\rho$$

But $\int x e^{ax} = \frac{e^{ax}}{a^2}(ax-1)$,

$$I = 20\pi e^{-1} a^2 \left(\frac{\rho}{a}-1\right) e^{\rho/a}\Big|_0^a = \frac{20\pi a^2}{e}(1+0)$$

$$= \underline{\underline{23.11a^2 \quad A}}.$$

Prob. 5.3

$$I = \frac{dQ}{dt} = -3\times10^{-4} e^{-3t}$$

$$I(t=0) = \underline{\underline{-0.3\, mA}}$$

$$I(t=2.5) = -0.3e^{-7.5} = \underline{\underline{-166\, nA}}$$

Prob. 5.4

$$Q = It = \underline{\underline{12C}}$$

$$n = \frac{Q}{e} = \frac{12}{1.6\times10^{-19}} = \underline{\underline{7.5\times10^{19} \text{ electrons}}}$$

Prob. 5.5

(a) $\frac{Q}{t} = I = \underline{\underline{10\,\mu C/s}}.$

(b) $$\rho_s = \frac{I}{wu} = \frac{10\times10^{-6}}{25\times0.5} = \underline{\underline{0.8\ \mu C/m^2}}.$$

$\boxed{\text{Prob. 5.6}}$

(a) $$R = \frac{\rho l}{S} \rightarrow \rho = \frac{RS}{l} = \frac{4.04}{10^3}\cdot\frac{\pi d^2}{4} = 2.855\times10^{-8}$$

$$\sigma = \frac{1}{\rho} = \underline{\underline{3.5\times10^7\ \text{mhos/m}\quad(\text{Aluminium})}}.$$

(b) $$J = \frac{I}{S} = \frac{40}{\frac{\pi}{4}\times90\times10^{-6}} = 5.66\times10^6\ A/m^2$$

or $$J = \sigma E = 3.5\times0.1616\times10^7$$
$$= \underline{\underline{5.66\times10^6\ A/m^2}}.$$

$\boxed{\text{Prob. 5.7}}$

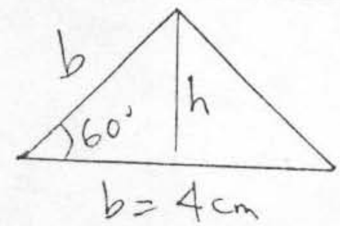
$$P = I^2R = I^2\frac{\rho l}{S} = \left(\frac{I}{S}\right)^2\rho l S = J^2\rho l S$$

But $\rho_m = \frac{M}{V} = \frac{M}{lS}$, hence

$$P = J^2\rho\frac{M}{\rho_m} = \frac{(0.8\times10^6)^2\times0.176\times10^{-6}\times1}{8.9\times10^{-3}\times10^6}$$

$$= \underline{\underline{1.26\ W}}$$

Prob. 5.8



$h = b\sin 60°, \quad S = \frac{1}{2}hb = \frac{1}{2}b^2 \sin 60°$

$$R = \frac{l}{\sigma S} = \frac{2l}{\sigma b^2 \sin 60°} = \frac{2 \times 3}{10^{-15} \times 16 \times 10^{-4} \sin 60°}$$

$$= \underline{\underline{4.33 \times 10^{18}\ \Omega}}$$

Prob. 5.9

(a) $R = \frac{\rho_c l}{S}, \quad S_i = \pi r_i^2 = \pi (1.5)^2 \times 10^{-4}$

$= 2.25\pi \times 10^{-4}$

$S_o = \pi (r_o^2 - r_i^2) = \pi (4 - 2.25) \times 10^{-4}$

$= 1.75\pi \times 10^{-4}$

$$R = R_i \| R_o = \frac{R_i R_o}{R_i + R_o} = \left(\frac{\frac{\rho_i}{S_i} \cdot \frac{\rho_o}{S_o}}{\frac{\rho_i}{S_i} + \frac{\rho_o}{S_o}} \right) l$$

$$= 10 \frac{\left(\frac{1.77 \times 11.8 \times 10^{-16}}{2.25\pi \times 1.75\pi \times 10^{-8}} \right)}{\frac{1.77 \times 10^{-8}}{1.75\pi \times 10^{-4}} + \frac{11.8 \times 10^{-8}}{2.25\pi \times 10^{-4}}} = 27 \times 10^{-4}$$

$= \underline{\underline{0.27\ m\Omega}}.$

(b) $V = I_i R_i = I_o R_o \longrightarrow \dfrac{I_i}{I_o} = \dfrac{R_o}{R_i} = \dfrac{0.3219}{1.669} = 0.1929$

$I_i + I_o = 1.1929\, I_o = 60\,A$

$I_o = \underline{\underline{50.3\ A}}$ (copper)

$I_i = \underline{\underline{9.7\,A}}$ (steel)

(c) $R = \dfrac{10 \times 1.77 \times 10^{-8}}{1.75\pi \times 10^{-4}} = \underline{\underline{0.322\ m\Omega}}$

Prob. 5.10

$$R = \frac{l}{\sigma S} = \frac{h}{\sigma\pi(b^2-a^2)} = \frac{2}{10^5\pi\ (25-9)\times 10^{-4}}$$

$= \underline{\underline{4\ m\Omega}}.$

Prob. 5.11

$|\vec{P}| = n|\vec{p}| = nQd = 2ned \quad \cdot (Q = 2e)$

$= \chi_e \epsilon_o E$

$$\chi_e = \frac{2ned}{\epsilon_o E} = \frac{2\times 5\times 10^{25} \times 1.602\times 10^{-19}\times 10^{-18}}{\frac{10^{-9}}{36\pi}\times 10^4}$$

$= 0.000182$

$\epsilon_r = 1 + \chi_e = \underline{\underline{1.000182}}$

$\boxed{\text{Prob. 5.12}}$

$$\vec{P} = \frac{\sum_{i=1}^{N} q_i \vec{d}_i}{V} = \frac{\sum_{i=1}^{N} \vec{p}_i}{V}$$

$$|\vec{P}| = \frac{N}{V}|\vec{p}| = 2\times10^{19} \times 1.8\times10^{-27} = 3.6\times10^{-8}$$

$$\vec{P} = |\vec{P}|\vec{a}_x = \underline{\underline{3.6\times10^{-18}\ \vec{a}_x\ \ \text{C/m}^2}}.$$

But $\vec{P} = \chi_e \epsilon_o \vec{E}$

or $$\chi_e = \frac{P}{\epsilon_o E} = \frac{3.6\times 36\pi\times10^{9}\times10^{-18}}{10^5}$$

$$= 0.0407$$

$$\epsilon_r = 1+\chi_e = \underline{\underline{1.0407}}$$

$\boxed{\text{Prob. 5.13}}$

(a) $$\vec{E} = -\nabla V = -\frac{dV}{dz}\vec{a}_z = 600z\ \vec{a}_z$$

$$\vec{D} = \epsilon_o\epsilon_r\vec{E} = \frac{10^{-9}}{36\pi}(2.4)\,600z\ \vec{a}_z$$

$$= \underline{\underline{12.73z\ \vec{a}_z\ \ \text{nC/m}^2}}.$$

$$\rho_v = \nabla\cdot\vec{D} = \frac{\partial D_z}{\partial z} = \underline{\underline{12.73\ \ \text{nC/m}^3}}$$

(b) $\chi_e = \epsilon_r - 1 = 1.4$

$$\vec{P} = \epsilon_o\chi_e\vec{E} = \frac{\chi_e D}{\epsilon_r}$$

$$\vec{P} = \frac{1.4}{2.4}(12.73z)\vec{a}_z = \underline{\underline{7.427z\,\vec{a}_z \ nC/m^2}}.$$

$$\rho_{pv} = -\nabla\cdot\vec{P} = \underline{\underline{-7.427 \quad nC/m^3}}.$$

Prob. 5.14

(a) $\vec{P} = \chi_e \epsilon_o \vec{E} = (\epsilon_r - 1)\epsilon_o \vec{E} = (\epsilon - \epsilon_o)\vec{E}$

$$\vec{D} = \epsilon\vec{E} = \frac{\epsilon}{\epsilon - \epsilon_o}\vec{P} = \frac{\epsilon_r}{\epsilon_r - 1}\vec{P}.$$

(b) $\epsilon_r = 1 + \chi_e = \underline{\underline{3.4}}$

$$E = \frac{D}{\epsilon_o \epsilon_r} = \frac{300\times10^{-6}}{3.4}\,36\pi\times10^{9}$$

$$= \underline{\underline{9.979 \ MV/m}}.$$

$$P = \epsilon_o \chi_e E = \epsilon_o \chi_e \frac{D}{\epsilon_o \epsilon_r} = \frac{\chi_e}{\epsilon_r} D$$

$$= \frac{2.4}{3.4}(300) = \underline{\underline{211.8 \ \mu C/m^2}}.$$

Prob. 5.15

(a) Applying coulomb's law,

$$E_r = \begin{cases} \dfrac{D_r}{\epsilon_o} = \dfrac{Q}{4\pi\epsilon_o r^2}, & b < r < a \\[2ex] \dfrac{D_r}{\epsilon} = \dfrac{Q}{4\pi\epsilon r^2}, & a < r < b \end{cases}$$

From the previous problem,

$$\vec{P} = \frac{\epsilon_r - 1}{\epsilon_r}\vec{D} \quad (= \vec{D} - \epsilon\vec{E}) \quad \text{Hence}$$

$$P_r = \frac{\epsilon_r - 1}{\epsilon_r} \cdot \frac{Q}{4\pi r^2}, \quad a < r < b.$$

(b) $$\rho_{pv} = -\nabla \cdot \vec{P} = -\frac{1}{r^2}\frac{d}{dr}(r^2 P_r) = \underline{\underline{0}}.$$

(c) $$\rho_{ps} = \vec{P} \cdot (-\vec{a}_r) = -\frac{Q}{4\pi a^2}\left(\frac{\epsilon_r - 1}{\epsilon_r}\right), \quad r = a$$

$$\rho_{ps} = \vec{P} \cdot \vec{a}_r = \frac{Q}{4\pi b^2}\left(\frac{\epsilon_r - 1}{\epsilon_r}\right), \quad r = b.$$

Prob. 5.16

(a) $$\vec{J} = 10^4(9+16)\vec{a}_z = \underline{\underline{250\vec{a}_z \text{ kA/m}^2}}.$$

(b) $$\frac{\partial \rho_v}{\partial t} = -\nabla \cdot \vec{J} = \underline{\underline{0}}.$$

(c) $$I = \int \vec{J} \cdot d\vec{S} = \int_{\phi=0}^{2\pi} \int_{\rho=0}^{0.005} 10^4 \rho^2 \rho \, d\phi \, d\rho$$

$$= 10^4 (2\pi) \frac{\rho^4}{4}\Big|_0^{0.005}$$

$$= \underline{\underline{9.817 \mu A}}.$$

Prob. 5.17

(a) $I = \int \vec{J}\cdot d\vec{S} = \int_{\phi=0}^{2\pi}\int_{z=0}^{3} \frac{e^{-10^3 t}}{\rho^2}\,\rho\, d\phi\, dz$

$= (2\pi)(3)\frac{e^{-10}}{2} = \underline{\underline{428\ \mu A}}.$

(b) $\frac{\partial \rho_v}{\partial t} = -\nabla\cdot\vec{J} = -\frac{e^{-10^3 t}}{\rho}\frac{d}{d\rho}\left(\frac{1}{\rho}\right) = \frac{e^{10^3 t}}{\rho^3}$

$\rho_v = \frac{-10^{-3} e^{-10^3 t}}{\rho^3} + c(t)$

If we let $\rho_v \to 0$ as $t \to \infty$, $c(t) = 0$.

$\rho_v(\rho=2,\ t=10\,ms) = -\frac{10^{-3} e^{-10}}{8} = \underline{\underline{-5.67\ nC/m^3}}.$

Prob. 5.18

(a) $T = \frac{\epsilon_0 \epsilon_r}{\sigma} = \frac{2.55 \times 10^{-9}}{10^{-16} \times 36\pi} = 2.255 \times 10^5\ s$

$= 62.64 \text{ hours} = \underline{\underline{2.61 \text{ days}}}.$

(b) $T = \frac{\epsilon_0 \epsilon_r}{\sigma} = \frac{20 \times 10^{-9}}{10^{-4} \times 36\pi} = \underline{\underline{1.76\ \mu s}}$

(c) $T = \frac{10^{-9}}{1.6 \times 10^7 \times 36\pi} = \underline{\underline{5.53 \times 10^{-19}\ s}}.$

Prob. 5.19

$$T = \frac{\epsilon}{\sigma} \rightarrow \sigma = \frac{\epsilon}{T} = \frac{6\times10^{-9}}{53\times36\pi} = \underline{\underline{10^{-12}\ \mho/m.}}$$

$\underline{\underline{\text{Glass or Porcelain.}}}$

Prob. 5.20

$$T_r = \frac{\epsilon}{\sigma} = \frac{2.5\times10^{-9}}{5\times10^{-6}\times36\pi} = 4.42\ \mu s$$

$$\rho_{vo} = \frac{Q}{V} = \frac{10\times10^{-6}}{\frac{4\pi}{3}\times10^{-6}\times8} = \underline{\underline{0.2984\ C/m^3.}}$$

$$\rho_v = \rho_{vo}e^{-t/T_r} = 0.2984e^{-2/4.42} = \underline{\underline{0.1898\ C/m^3}}$$

Prob. 5.21

(a) $\vec{E}_{2t} = \vec{E}_{1t} = -300\vec{a}_x + 50\vec{a}_y$, $\vec{E}_{1n} = 70\vec{a}_z$

$\vec{D}_{2n} = \vec{D}_{1n} \rightarrow \epsilon_2\vec{E}_{2n} = \epsilon_1\vec{E}_{1n}$

$$\vec{E}_{2n} = \frac{\epsilon_1}{\epsilon_2}\vec{E}_{1n} = \frac{2.5}{4}(70\vec{a}_z) = 43.75\vec{a}_z$$

$\vec{E}_2 = -30\vec{a}_x + 50\vec{a}_y + 43.75\vec{a}_z$.

$$\vec{D}_2 = \epsilon_0\epsilon_{r2}\vec{E}_2 = \frac{4\times10^{-9}}{36\pi}(-30, 50, 43.75)$$

$$= \underline{\underline{-1.061\vec{a}_x + 1.768\vec{a}_y + 1.547\vec{a}_z\ nC/m^2.}}$$

(b) $\vec{P}_2 = \epsilon_o \chi_{e2} \vec{E}_2 = \frac{3 \times 10^{-9}}{36\pi}(-30, 50, 43.75)$

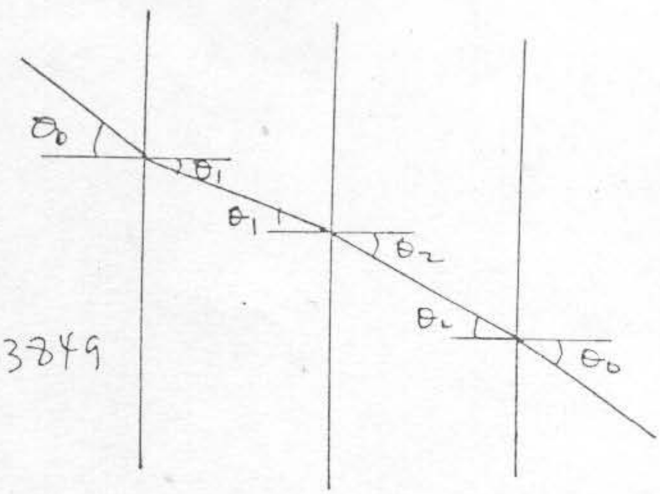
$= \underline{0.7958\,\vec{a}_x + 1.326\,\vec{a}_y + 1.161\,\vec{a}_z \text{ nC/m}^2}.$

(c) $\vec{E}_1 \cdot \vec{a}_z = E_1 \cos\theta_n$

$\cos\theta_n = \frac{70}{\sqrt{30^2 + 50^2 + 70^2}} = 0.7683$

$\theta_n = \underline{39.79^\circ}.$

Prob. 5.22



(b)

$\frac{\tan\theta_0}{\tan\theta_1} = \frac{\epsilon_0}{\epsilon_1} = \frac{1}{1.5}$

$\tan\theta_0 = \frac{\tan 30^\circ}{1.5} = 0.3849$

$\theta_0 = \underline{21.1^\circ}$

(a) $\frac{\tan\theta_2}{\tan\theta_1} = \frac{\epsilon_{r2}}{\epsilon_{r1}} = \frac{3}{1.5} = 2,$

$\tan\theta_2 = 2\tan 30^\circ = 1.155$

$\theta_2 = \underline{49.1^\circ}.$

Prob. 5.23

(a) $\vec{P}_1 = \chi_e \epsilon_o \vec{E}_1 = \frac{1.5 \times 10^{-9}}{36\pi}(2, 5, -4)\cdot 10^3$

$= \underline{26.53\,\vec{a}_\rho + 66.31\,\vec{a}_\phi - 53.05\,\vec{a}_z \text{ nC/m}^2}$

$$\rho_{pv_1} = -\nabla\cdot\vec{P}_1 = -\frac{1}{\rho}\frac{\partial}{\partial\rho}(26.53\rho)$$

$$= -\frac{26.53}{\rho}\ nC/m^3.$$

(b) $\vec{E}_{2t} = \vec{E}_{1t} = 5\vec{a}_\phi - 4\vec{a}_z$

$$D_{2n} = D_{1n} \longrightarrow E_{2n} = \frac{\epsilon_1}{\epsilon_2}E_{1n}$$

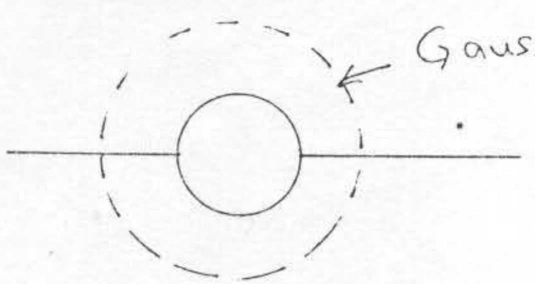
$$E_{2n} = \frac{2.5}{1.0}(2) = 5$$

$$\vec{E}_2 = 5\vec{a}_\rho + 5\vec{a}_\phi - 4\vec{a}_z\ kV/m.$$

$$\vec{D}_2 = \epsilon_0\epsilon_r\vec{E} = 2.5\frac{\times 10^{-9}}{36\pi}(5,5,-4)\cdot 10^3$$

$$= 110.52\vec{a}_\rho + 110.52\vec{a}_\phi - 88.42\vec{a}_z\ nC/m^2.$$

Prob. 5.24



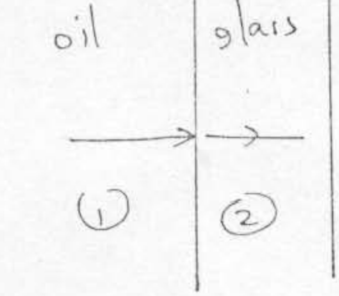
$$Q = \int\vec{D}\cdot d\vec{s} = \epsilon_1 E_r\cdot\frac{4\pi r^2}{2} + \epsilon_2 E_r\cdot\frac{4\pi r^2}{2}$$

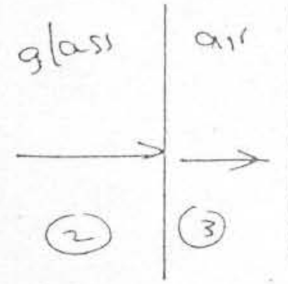
$$= 2\pi r^2(\epsilon_1+\epsilon_2)E_r.$$

$$E_r = \begin{cases} \dfrac{Q}{2\pi(\epsilon_1+\epsilon_2)}, & r > a \\ 0, & r < a. \end{cases}$$

Prob. 5.25

(a).





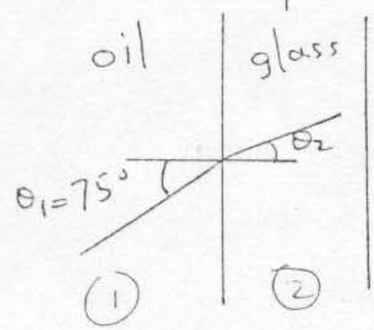
$E_{1n} = 2000, \; E_{1t} = 0 = E_{2t} = E_{3t}$

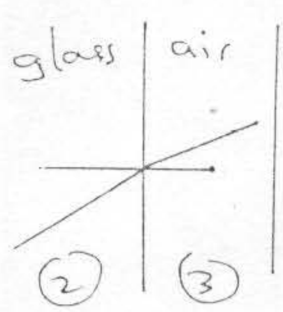
$D_{1n} = D_{2n} = D_{3n} \longrightarrow \epsilon_1 E_{1n} = \epsilon_2 E_{2n} = \epsilon_3 E_{3n}$

$\rightarrow E_{2n} = \frac{\epsilon_1}{\epsilon_2} E_{1n} = \frac{3.0}{8.5}(2000) = \underline{\underline{705.9 \text{ V/m}}}, \theta_2 = 0$

$E_{3n} = \frac{\epsilon_1}{\epsilon_3} E_{1n} = \frac{3.0}{1.0}(2000) = \underline{\underline{6000 \text{ V/m}, \; \theta_3 = 0^\circ}}.$

(b)





$E_{1n} = 2000 \cos 75^\circ = 517.63$

$E_{1t} = 2000 \sin 75^\circ = E_{2t} = E_{3t} = 1931.85$

$E_{2n} = \frac{\epsilon_1}{\epsilon_2} E_{1n} = \frac{3}{8.5}(517.63) = 182.7$

$E_{3n} = \frac{\epsilon_1}{\epsilon_3} E_{1n} = \frac{3}{1}(517.63) = 1552.89$

$E_2 = \sqrt{E_{2n}^2 + E_{2t}^2} = \underline{\underline{1940.5}}, \; \theta_2 = \tan^{-1} \frac{E_{2t}}{E_{2n}} = \underline{\underline{84.6^\circ}}$

$E_3 = \sqrt{E_{3n}^2 + E_{3t}^2} = \underline{\underline{2478.6}}, \; \theta_3 = \tan^{-1} E_{3t}/E_{3n} = \underline{\underline{51.2^\circ}}$

Prob. 5.26

(a) $\rho_s = D_n = \epsilon_0 E_n = \dfrac{10^{-9}}{36\pi}\sqrt{15^2+8^2} = \dfrac{17}{36\pi}\ nC/m^2$

$= \underline{\underline{0.1503\ nC/m^2}}$

(b) $D_n = \rho_n = -20\ nC$

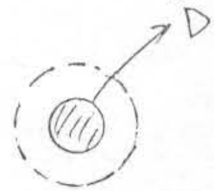
$\vec{D} = \vec{D}_n \vec{a}_n = (-20nC)(-\vec{a}_y) = \underline{\underline{20\,\vec{a}_y\ nC/m^2}}.$

Prob. 5.27

(a) $D_n = \rho_s = \dfrac{Q}{4\pi a^2} = \dfrac{12\times10^{-9}}{4\pi\times25\times10^{-4}} = \dfrac{1200}{\pi}\ nC/m^2$

$|\vec{D}| = \underline{\underline{381.97\ nC/m^2.}}$

(b) Using Gauss's law,



$D_r \cdot 4\pi r^2 = Q \longrightarrow D_r = \dfrac{Q}{4\pi r^2}$

$\vec{D} = \dfrac{Q}{4\pi r^2}\vec{a}_r = \dfrac{12}{4\pi r^2}\vec{a}_r\ nC/m^2 = \underline{\underline{\dfrac{0.955}{r^2}\vec{a}_r\ nC/m^2}}$

(c) $W = \dfrac{1}{2\epsilon_0}\int |\vec{D}|\,dv = \dfrac{Q^2}{2\epsilon_0 \cdot 16\pi^2}\iiint \dfrac{1}{r^4}\cdot r^2\sin\theta\, d\theta\, d\phi\, dr$

$= \dfrac{Q^2}{32\pi^2\epsilon_0}\cdot 4\pi\displaystyle\int_a^\infty \dfrac{dr}{r^2} = \dfrac{Q^2}{8\pi\epsilon_0 a}$

$= \dfrac{144\times10^{-18}}{8\pi\times\frac{10^{-9}}{36\pi}\times5\times10^{-2}} = \underline{\underline{12.96\ \mu J}}.$

Prob. 5.28

(a) $\vec{J} = \sigma \vec{E} = \underline{\underline{0}}$ since $\vec{E} = 0$.

(b) $\rho_s = D_n = \frac{Q}{S} = \frac{Q}{4\pi r^2} = \frac{50 \times 10^{-9}}{4\pi \times 1600 \times 10^{-4}}$

$= \underline{\underline{24.87 \ nC/m^2}}$.

Prob. 5.29

For steady state current,

$$\nabla \cdot \vec{J} = 0 \longrightarrow \oint \vec{J} \cdot d\vec{S} = 0.$$

Applying the integral form to a thin disc at the interface,

$$J_{1n} = J_{2n}.$$

Also, at the interface,

$$E_{1t} = E_{2t} \longrightarrow \sigma_1 \frac{J_{1t}}{\sigma_1} = \frac{J_{2t}}{\sigma_2}$$

CHAPTER 6

P.E. 6.1

$$\nabla^2 V = -\frac{\rho}{\varepsilon} \longrightarrow \frac{d^2V}{dx^2} = -\frac{\rho_0 x}{\varepsilon a}$$

$$V = -\frac{\rho_0 x^3}{6\varepsilon a} + Ax + B$$

$$\vec{E} = -\frac{dV}{dx}\vec{a}_x = \left(\frac{\rho_0 x^2}{2\varepsilon a} - A\right)\vec{a}_x$$

If $\vec{E}=0$ at $x=0$, then

$$0 = 0 - A \longrightarrow A = 0$$

If $V=0$ at $x=a$, then

$$0 = -\frac{\rho_o a^3}{6\epsilon a} + B \longrightarrow B = \frac{\rho_o a^2}{6\epsilon}$$

Thus

$$V = \frac{\rho_o}{6\epsilon a}\left(a^3 - x^3\right), \quad \vec{E} = \frac{\rho x^2}{2\epsilon a}$$

P.E.6.2

$$V_1 = A_1 x + B_1, \quad V_2 = A_2 x + B_2$$

$$V_1(x=d) = V_o = A_1 d + B_1 \longrightarrow B_1 = V_o - A_1 d$$

$$V_2(x=0) = 0 = 0 + B_2 \longrightarrow B_2 = 0$$

$$V_1(x=a) = V_2(x=a) \longrightarrow A_1 a + B_1 = A_2 a$$

$$D_{1n} = D_{2n} \longrightarrow \epsilon_1 A_1 = \epsilon_2 A_2 \longrightarrow A_2 = \frac{\epsilon_1}{\epsilon_2} A_1$$

$$A_1 a + V_o - A_1 d = \frac{\epsilon_1}{\epsilon_2} a A_1$$

$$V_o = A_1\left(-a + d + \frac{\epsilon_1}{\epsilon_2} a\right)$$

or

$$A_1 = \frac{V_o}{d - a + \frac{\epsilon_1}{\epsilon_2} a}, \quad A_2 = \frac{\epsilon_1}{\epsilon_2} A_1 = \frac{\epsilon_1 V_o}{\epsilon_2 d - \epsilon_2 a + \epsilon_1 a}$$

Hence

$$\vec{E}_1 = -A_1 \vec{a}_x = \frac{-V_o \vec{a}_x}{d - a + \frac{\epsilon_1}{\epsilon_2} a}, \quad \vec{E}_2 = -A_2 \vec{a}_x = \frac{-V_o \vec{a}_x}{a + \frac{\epsilon_2}{\epsilon_1} d - \frac{\epsilon_2}{\epsilon_1} a}$$

P.E. 6.3

from Example 6.3,

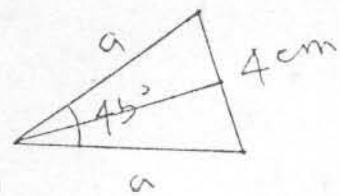
$$\vec{E} = -\frac{V_o}{\rho\phi_o}\vec{a}_\phi, \quad \vec{D} = \epsilon_o \vec{E}$$

$$\rho_s = D_n(\phi = 0) = -\frac{V_o \epsilon}{\rho \phi_o}$$

The charge on the plate $\phi = 0$ is

$$Q = \int \rho_s \, dS = -\frac{V_o \epsilon}{\phi_o} \int_{z=0}^{L} \int_{\rho=a}^{b} \frac{1}{\rho} dz \, d\rho = -\frac{V_o \epsilon L}{\phi_o} \ln \frac{b}{a}$$

$$C = \frac{|Q|}{V_o} = \frac{\epsilon L}{\phi_o} \ln \frac{b}{a}$$



$$a \sin \frac{45^\circ}{2} = 2 \longrightarrow a = \frac{2}{\sin 22.5} = 5.226 \text{ mm}$$

$$C = \frac{1.5 \times 10^{-9}}{36\pi \times \frac{\pi}{4}} \, 5 \ln \frac{1000}{5.226} = 444 \text{ pF}$$

$$Q = CV_o = 444 \times 10^{-12} \times 50 = \underline{\underline{22.2 \, nC}}$$

P.E. 6.4 from Example 6.4, $V_o = 50$, $\theta_1 = \frac{\pi}{2}$

$\theta_2 = 45^\circ$, $r = \sqrt{3^2 + 4^2 + 2^2} = \sqrt{29}$, $\theta = \tan^{-1} \frac{\rho}{z} = \frac{5}{2} \rightarrow \theta = 68.2^\circ$

$$V = \frac{50 \ln(\tan 34.1^\circ)}{\tan(22.5^\circ)} = \underline{\underline{22.13 \, V}}$$

$$\vec{E} = \frac{-50 \, \vec{a}_\theta}{\sqrt{29} \sin 68.2 \ln(\tan 22.5^\circ)} = \underline{\underline{11.364 \, \vec{a}_\theta \, V/m}}$$

P.E.6.5

$$\vec{E} = -\nabla V = -\frac{\partial V}{\partial x}\vec{a}_x - \frac{\partial V}{\partial y}\vec{a}_y$$

$$= -\frac{4V_o}{b}\sum_{n=odd}^{\infty}\frac{1}{\sinh\frac{n\pi a}{b}}\left[\cos\frac{n\pi x}{b}\sinh\frac{n\pi y}{b}\vec{a}_x + \sin\frac{n\pi x}{b}\cosh\frac{n\pi y}{b}\vec{a}_y\right]$$

(a) At $(x,y) = (a, a/2)$,

$$V = \frac{400}{\pi}(0.3775 - 0.0313 + 0.00394 - 0.000585 + \ldots)$$

$$= \underline{\underline{44.51\,V}}$$

$$\vec{E} = 0\vec{a}_x + (-115.118 + 19.127 - 3.9431 + 0.8192 - 0.1703 + 0.035 - 0.0094 + \ldots)\vec{a}_y$$

$$= \underline{\underline{-99.25\,\vec{a}_y \text{ V/m.}}}$$

(b) At $(x,y) = (3a/2, a/4)$,

$$V = \frac{400}{\pi}(0.1238 + 0.00626 - 0.00383 - 0.000264 + \ldots)$$

$$= \underline{\underline{16.50\,V}}$$

$$\vec{E} = (24.757 - 3.7358 - 0.3834 + 0.0369 + 0.00351 - 0.00033 + \ldots)\vec{a}_x + (-66.25 - 4.518 + 0.3988 + 0.03722 - 0.00352 - 0.000333 + \ldots)\vec{a}_y$$

$$= \underline{\underline{20.68\,\vec{a}_x - 70.34\,\vec{a}_y \text{ V/m.}}}$$

P.E. 6.6

$$V(y=a) = V_o \sin\frac{7\pi x}{b} = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{b} \sinh\frac{n\pi a}{b}$$

By equating coefficients, we notice that

$$c_n = 0 \quad \text{for} \quad n \neq 7.$$

for $n=7$;

$$V_o \sin\frac{7\pi x}{b} = c_7 \sin\frac{7\pi x}{b} \sinh\frac{7\pi a}{b}$$

or $$c_7 = V_o \Big/ \sinh\frac{7\pi a}{b}.$$

Hence

$$V(x,y) = \frac{V_o \sin\frac{7\pi x}{b} \sinh\frac{7\pi y}{b}}{\sinh\frac{7\pi a}{b}}$$

P.E. 6.7

Let $V(r,\theta,\phi) = R(r)F(\theta)\Phi(\phi)$,

substituting this in Laplace's equation gives

$$\frac{\Phi F}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{R\Phi}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dF}{d\theta}\right)$$

$$+ \frac{RF}{r^2\sin^2\theta}\frac{d^2\Phi}{d\phi^2} = 0$$

Dividing thru by $RF\Phi / r^2\sin^2\theta$ gives

$$\frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2R'\right) + \frac{\sin\theta}{F}\frac{d}{d\theta}\left(\sin\theta F'\right) = -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = \lambda^2$$

$$\Phi'' + \lambda^2 \Phi = 0.$$

$$\frac{1}{R}\frac{d}{dr}(r^2 R') + \frac{1}{F\sin\theta}\frac{d}{d\theta}(\sin\theta F') = \frac{\lambda^2}{\sin^2\theta}$$

$$\frac{1}{R}\frac{d}{dr}(r^2 R') = \frac{\lambda^2}{\sin^2\theta} - \frac{1}{F\sin\theta}\frac{d}{d\theta}(\sin\theta F') = \mu^2$$

$$2R' + r^2 R'' = \mu^2 R$$

or

$$R'' + \frac{2}{r}R' - \frac{\mu^2}{r^2}R = 0.$$

$$\frac{\sin\theta}{F}\frac{d}{d\theta}(\sin\theta F') - \lambda^2 + \mu^2\sin^2\theta = 0$$

or

$$F'' + \cot\theta F' + (\mu^2 - \lambda^2 \operatorname{cosec}^2\theta) F = 0.$$

P.E. 6.8

(a) This is similar to Example 6.8 (a) except that here, $0 < \phi < 2\pi$ instead of $0 < \phi < \pi/2$. Hence

$$I = \frac{2\pi t V_o \sigma}{\ln\frac{b}{a}}$$

and

$$R = \frac{V_o}{I} = \frac{\ln\frac{b}{a}}{2\pi t \sigma}$$

(b) This is similar to Example 6.8 (b) except that here $0 < \phi < 2\pi$. Hence

$$I = \frac{V_o\sigma}{t}\int_a^b\int_0^{2\pi} \rho\, d\rho\, d\phi = \frac{V_o\sigma\pi\,(b^2 - a^2)}{t}$$

and $R = \frac{V_o}{I} = \frac{t}{\sigma\pi(b^2 - a^2)}$

P.E. 6.9

from Example 6.9,

$$J_1 = \frac{\sigma_1 V_o}{\rho \ln\frac{b}{a}}, \quad J_2 = \frac{\sigma_2 V_o}{\rho \ln\frac{b}{a}}$$

$$I = \int \vec{J}\cdot d\vec{S} = \int_{z=0}^{l} \left[\int_{\phi=0}^{\pi} J_1 \rho \, d\phi + \int_{\phi=\pi}^{2\pi} J_2 \rho \, d\phi \right] dz$$

$$= \frac{V_o l}{\ln\frac{b}{a}} \left[\pi\sigma_1 + \pi\sigma_2 \right]$$

$$R = \frac{V_o}{I} = \frac{\ln\frac{b}{a}}{\pi l(\sigma_1 + \sigma_2)}$$

P.E. 6.10

(a) $C = \frac{4\pi\epsilon}{\frac{1}{a} - \frac{1}{b}}$, C_1 and C_2 are in series.

$$C_1 = 4\pi \times \frac{10^{-9}}{36\pi}\left(\frac{2.5}{\frac{10^3}{2} - \frac{10^3}{3}}\right) = \frac{5}{3} \text{ pF}$$

$$C_2 = 4\pi \times \frac{10^{-9}}{36\pi}\left(\frac{3.5}{\frac{10^3}{1} - \frac{10^3}{2}}\right) = \frac{7}{9} \text{ pF}$$

$$C = \frac{C_1 C_2}{C_1 + C_2} = \frac{5/3 \left(\frac{7}{9}\right)}{5/3 + 7/9} = \frac{35}{36} \text{ pF} = \underline{0.53 \text{ pF}}$$

(b) $C = \dfrac{2\pi\epsilon}{\frac{1}{a}-\frac{1}{b}}$, C_1 and C_2 are in parallel.

$$C_1 = \frac{2\pi\times10^{-9}}{36\pi}\left(\frac{2\cdot5}{\frac{10^3}{1}-\frac{10^3}{3}}\right) = \frac{5}{24}\ pF,$$

$$C_2 = \frac{2\pi\times10^{-9}}{36\pi}\left(\frac{3\cdot5}{\frac{10^3}{1}-\frac{10^3}{3}}\right) = \frac{7}{24}\ pF,$$

$$C = C_1 + C_2 = \frac{12}{24}\ pF = \underline{\underline{0\cdot5\ pF}}$$

$\boxed{P.E.\ 6\cdot11}$

As in Example 6·8, assuming $V(\rho=a)=0$, $V(\rho=b)=V_0$,

$$V = V_0\frac{\ln \rho/a}{\ln b/a},\quad \vec{E} = -\nabla V = \frac{-V_0\,\bar{a}_\rho}{\rho\ln b/a}$$

$$Q = \int\epsilon\vec{E}\cdot d\vec{S} = \frac{V_0\epsilon}{\ln\frac{b}{a}}\int_{z=0}^{L}\int_{\phi=0}^{2\pi}\frac{1}{\rho}\,dz\,\rho\,d\phi = V_0\cdot\frac{2\pi\epsilon L}{\ln\frac{b}{a}}$$

$$C = \frac{Q}{V_0} = \underline{\underline{\frac{2\pi\epsilon L}{\ln\frac{b}{a}}}}$$

$\boxed{P.E.\ 6\cdot12}$

(a) C_1 and C_2 are in series

$$C_1 = \frac{2\pi\epsilon_{r1}\epsilon_0}{\ln\frac{c}{a}} = \frac{2\pi\times2\cdot5\times10^{-9}}{\ln\frac{2}{1}\times36\pi} = 200\ pF/m$$

$$C_2 = \frac{2\pi\epsilon_{r2}\epsilon_0}{\ln\frac{b}{c}} = \frac{2\pi\times3\cdot5\times10^{-9}}{\ln 3/2} = 480\ pF/m$$

$$C = \frac{C_1 C_2}{C_1 + C_2} = \frac{200 \times 480}{680} = 141.12 \text{ pF/m}.$$

$$C_T = Cl = \underline{\underline{1.41 \text{ nF}}}.$$

(b) C_1 and C_2 are in parallel.

$$C = C_1 + C_2 = \frac{\pi \epsilon_0 \epsilon_{r1}}{\ln \frac{b}{a}} + \frac{\pi \epsilon_0 \epsilon_{r2}}{\ln \frac{b}{a}} = \frac{\pi \epsilon_0 (\epsilon_{r1} + \epsilon_{r2})}{\ln \frac{b}{a}}$$

$$= \frac{6\pi \times \frac{10^{-9}}{36\pi}}{\ln \frac{3}{1}} = 151.7 \text{ pF/m}$$

$$C = Cl = \underline{\underline{1.52 \text{ nF}}}.$$

P.E. 6.13

Instead of eq. (6.31), we now have

$$V = -\int_b^a \frac{Q\,dr}{4\pi\epsilon r^2} = -\int_b^a \frac{Q}{4\pi \frac{10\epsilon_0}{r} r^2} dr$$

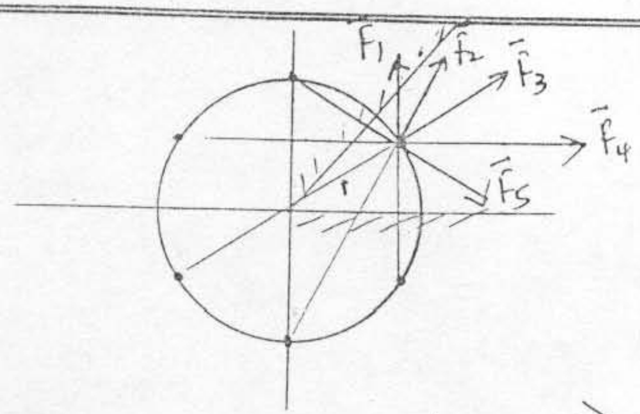
$$= \frac{-Q}{40\pi\epsilon_0} \ln \frac{b}{a}.$$

$$C = \frac{Q}{|V|} = \frac{40\pi \cdot \frac{10^{-9}}{36\pi}}{\ln \frac{4}{1.5}} = \underline{\underline{1.13 \text{ nF}}}.$$

P.E. 6.14

Let $\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4 + \vec{F}_5$

where $\vec{F}_i$, $i = 1, \cdots 5$ are as shown on the next page.



$$\vec{F} = \frac{-Q^2 \vec{a}_y}{4\pi\epsilon_0 r^2} + \frac{Q^2 (\vec{a}_x \sin 30^\circ + \vec{a}_y \cos 30^\circ)}{4\pi\epsilon_0 (r\cos 30^\circ)^2}$$

$$- \frac{Q^2}{4\pi\epsilon(2r)^2}(\vec{a}_x \cos 30^\circ + \vec{a}_y \sin 30^\circ) + \frac{Q^2 \vec{a}_x}{4\pi\epsilon_0 (r\cos 30^\circ)^2}$$

$$- \frac{Q^2}{4\pi\epsilon_0 r^2}(\vec{a}_x \cos 30^\circ - \vec{a}_y \sin 30^\circ)$$

$$= \frac{Q^2}{4\pi\epsilon_0 r^2}\Big[-\vec{a}_y + \frac{4}{3}\Big(\frac{\vec{a}_x}{2} + \vec{a}_y \frac{\sqrt{3}}{2}\Big) - \frac{1}{4}\Big(\vec{a}_x\frac{\sqrt{3}}{2} + \frac{\vec{a}_y}{2}\Big)$$

$$+ \frac{4}{3}\vec{a}_x - \vec{a}_x\frac{\sqrt{3}}{2} + \frac{\vec{a}_y}{2}\Big]$$

$$= 9\times10^{-5}\Big[\vec{a}_x\Big(2 - \frac{5\sqrt{3}}{8}\Big) + \vec{a}_y\Big(\frac{4\sqrt{3}-5}{8}\Big)\Big]$$

$$= 82.57\,\vec{a}_x + 21.69\,\vec{a}_y \ \mu N$$

$$|\vec{F}| = \underline{\underline{85.37\ \mu N}}.$$

$\boxed{\text{Prob. 6.1}}$ If $V'' = f$

$$V' = \int_0^x f(x)\,dx + c_1$$

$$V = \int_0^x \int_0^\lambda f(\mu)\, d\mu\, d\lambda + c_1 x + c_2$$

$$V(x=0) = V_1 = c_2 \longrightarrow c_2 = V_1$$

$$V(x=L) = V_2 = \int_0^L \int_0^\lambda f(\mu)\, d\mu\, d\lambda + c_1 L + c_2$$

$$c_1 = \frac{1}{L}\left[V_2 - V_1 - \int_0^L \int_0^\lambda f(\mu)\, d\mu\, d\lambda\right]$$

Thus,

$$V = \frac{x}{L}\left[V_2 - V_1 - \int_0^L \int_0^\lambda f(\mu)\, d\mu\, d\lambda\right] + V_1$$

$$+ \int_0^x \int_0^\lambda f(\mu)\, d\mu\, d\lambda$$

Prob. 6.2

$$\nabla^2 V = \frac{d^2V}{dy^2} = -\frac{\rho_v}{\epsilon} = \frac{-50(1-y^2)}{\epsilon} \cdot 10^{-6}$$

$$= -k(1-y^2)$$

where $k = \dfrac{50 \times 10^{-6}}{3 \times 10^{-9}/36\pi} = 600\pi \cdot 10^3$

$$\frac{dV}{dy} = -k\left(y - \frac{y^3}{3}\right) + A$$

$$V = -k\left(\frac{y^2}{2} - \frac{y^4}{12}\right) + Ay + B$$

$$= \frac{ky^4}{12} - \frac{ky^2}{2} + Ay + B$$

$$V = 50\pi \cdot 10^3 y^4 - 300\pi \cdot 10^3 y^2 + Ay + B$$

When $y = 2\text{cm}$, $V = 30 \times 10^3$,

$$30 \times 10^3 = 50\pi \times 10^3 \times 16 \times 10^{-6} - 30\pi \cdot 10^3 \times 4 \times 10^{-4} + Ay + B$$

or

$$30,374.5 = 0.02A + B \quad -(1)$$

When $y = -2\text{cm}$, $V = 30 \times 10^3$,

$$30,374.5 = -0.02A + B \quad -(2)$$

from (1) and (2), $A = 0$, $B = 30,374.5$. Thus

$$\underline{\underline{V = 157.08y^4 - 942.5y^2 + 30.374 \text{ kV}}}$$

Prob. 6.3

(a) $\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) = -\frac{\rho_v}{\varepsilon} = \frac{-10 \times 10^{-12}}{\rho \times \frac{10^{-9}}{36\pi}}$

$$\frac{d}{d\rho}\left(\rho \frac{dV}{d\rho}\right) = -1.131$$

$$\rho \frac{dV}{d\rho} = -1.131\rho + A \longrightarrow \frac{dV}{d\rho} = -1.131 + \frac{A}{\rho}$$

$$V = -1.131\rho + A\ln\rho + B$$

If $V = 0$ at $\rho = 1$, $0 = -1.131 + B \rightarrow B = 1.131$

If $V = 100$ at $\rho = 4$, $100 = -1.131 \times 4 + A\ln 4 + 1.131$

$$\longrightarrow A = 74.58$$

Hence,

$$V = -1.131\rho + 74.58\ln\rho + 1.131$$

$$V(\rho = 3) = \underline{\underline{79.67}} \text{ V}$$

(b) $\vec{E} = -\nabla V = -\frac{\partial V}{\partial \rho}\vec{a}_\rho = \left(1.131 - \frac{A}{\rho}\right)\vec{a}_\rho$

$\vec{E}(\rho=2) = \underline{\underline{-36.14\,\vec{a}_\rho \text{ V/m.}}}$

$\boxed{\text{Prob. 6.4}}$

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = -\frac{\rho_v}{\epsilon} = -\frac{\rho_o a^4}{\epsilon_o r^4}$$

$$r^2 \frac{dV}{dr} = \frac{\rho_o a^4}{\epsilon_o r} + A$$

$$V = -\frac{\rho_o a^4}{2\epsilon_o r^2} - \frac{A}{r} + B$$

$V(r \to \infty) = 0 \longrightarrow B = 0$

$V(r=a) = 1.5 \times 10^6 = V_o$

$$V_o = \frac{-\rho_o a^2}{2\epsilon_o} - \frac{A}{a} \Longrightarrow A = -a\left(V_o + \frac{\rho_o a^2}{2\epsilon_o}\right)$$

$$\vec{E} = -\nabla V = -\left(\frac{\rho_o a^4}{\epsilon_o r^3} + \frac{A}{r^2}\right)\vec{a}_r$$

$$\vec{E}(r=a) = \left(\frac{V_o}{a} - \frac{\rho_o a}{2\epsilon_o}\right)\vec{a}_r$$

$$= \left[\frac{1.5 \times 10^6}{0.1} - \frac{1.2 \times 10^{-3} \times 0.1}{2 \times 10^{-9}/36\pi}\right]\vec{a}_r$$

$$= \underline{8.214\,\vec{a}_r \text{ MV/m.}}$$

With $\rho_o = a$, $\vec{E} = 15$ MV/m which is greater. $\underline{\underline{\text{No!}}}$

Prob. 6.5

from eq. (6.3), $\nabla \cdot (\epsilon \nabla V) = -\rho_v$.

Let $f = \epsilon$ and $\vec{A} = \nabla V$. Using the vector identity

$$\nabla \cdot (f\vec{A}) = f \nabla \cdot \vec{A} + (\vec{A} \cdot \nabla) f$$

gives

$$\nabla \cdot (\epsilon \nabla V) = \epsilon \nabla \cdot \nabla V + \nabla V \cdot \nabla \epsilon$$
$$= \epsilon \nabla^2 V + \nabla V \cdot \nabla \epsilon$$

Hence

$$\epsilon \nabla^2 V + \nabla V \cdot \nabla \epsilon = -\rho_v.$$

Prob. 6.6

(a) $\nabla^2 V = V_o\left(-\frac{n^2\pi^2}{a^2} - \frac{n^2\pi^2}{a^2}\right) \sin\left(\frac{n\pi x}{a}\right) \cos\left(\frac{n\pi y}{a}\right)$

$= -\frac{2\pi^2 n^2}{a^2} V \neq 0$

(b) $\nabla^2 V_o = V_o\left(-\frac{n^2\pi^2}{a^2} + \frac{n^2\pi^2}{a^2}\right) \sin\left(\frac{n\pi x}{a}\right) \cosh\left(\frac{n\pi y}{a}\right) = 0$

(c) $\nabla^2 V = -\frac{2\pi^2 n^2}{a^2} V \neq 0$

(d) $V = (x+y) r^{-3/2}$

$\frac{\partial V}{\partial x} = r^{-3/2} - \frac{3}{2}(x+y) r^{-5/2}(2x) = r^{-3/2} - 3x(x+y) r^{-5/2}$

$\frac{\partial^2 V}{\partial x^2} = -\frac{3}{2} r^{-5/2}(2x) - 3(2x+y) r^{-5/2} + \frac{15}{2}(x^2+xy) r^{-7/2}(2x)$

$$\frac{\partial^2 V}{\partial x^2} = -3(3x+y)r^{-5/2} + 15(x^3+x^2y)r^{-7/2}$$

Similarly,

$$\frac{\partial^2 V}{\partial y^2} = -3(x+3y)r^{-5/2} + 15(y^3+y^2x)r^{-7/2}$$

$$\frac{\partial V}{\partial z} = -\frac{3}{2}(x+y)r^{-5/2}(2z) = -3z(x+y)r^{-5/2}$$

$$\frac{\partial^2 V}{\partial z^2} = -3(x+y)r^{-5/2} - 3z(x+y)\left(-\tfrac{5}{2}\right)r^{-7/2}(2z)$$

$$= -3(x+y)r^{-5/2} + 15z^2(x+y)r^{-7/2}$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

$$= -3(5x+5y)r^{-5/2} + 15(x+y)(x^2+y^2+z^2)r^{-7/2}$$

$$= -15(x+y)r^{-5/2} + 15(x+y)r^{-5/2} = 0$$

Or expressing V in spherical system,

$$V = \frac{\sin\theta}{r^2}(\cos\phi + \sin\phi)$$

$$\nabla^2 V = \frac{\sin\theta}{r^2}\frac{\partial}{\partial r}\left[r^2\cdot\left(-\frac{2}{r^3}\right)\right](\cos\phi+\sin\phi)$$

$$+ \frac{(\cos\phi+\sin\phi)}{r^4\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\cos\theta)$$

$$- \frac{1}{r^4\sin\theta}(\cos\phi+\sin\phi)$$

$$\nabla^2 V = \frac{(\cos\phi + \sin\phi)}{r^4}\left[2\sin\theta + \frac{1}{\sin\theta}(-\sin^2\theta + \cos^2\theta) - \frac{1}{\sin\theta}\right]$$

$$= \frac{(\cos\phi + \sin\phi)}{r^4}\left(\cancel{2\sin\theta} + \cancel{\frac{1}{\sin\theta}} - \cancel{2\sin\theta} - \cancel{\frac{1}{\sin\theta}}\right) = 0$$

Thus,
(a) and (c) do not satisfy Laplace's equation while (b) and (d) satisfy it.

$\boxed{\text{Prob. 6.7}}$

(a) $\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 30yz - 30yz + 0 = 0$

(b) $\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2} + \underbrace{\frac{\partial^2 V}{\partial z^2}}_{0}$

$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(-\frac{\cos\phi}{\rho}\right) + \frac{1}{\rho^2}\left(-\frac{\cos\phi}{\rho}\right)$$

$$= \frac{\cos\phi}{\rho^3} - \frac{\cos\phi}{\rho^3} = 0.$$

(c) $\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}$

$$= \frac{10\sin\theta\sin\phi}{r^2}\frac{\partial}{\partial r}\left(-\frac{2}{r}\right) + \frac{10\sin\phi}{r^4\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\cos\theta) - \frac{10}{r^4\sin^2\theta}\sin\theta\sin\phi$$

$$= \frac{20\sin\theta\sin\phi}{r^4} + \frac{10\sin\phi(1-2\sin^2\theta)}{r^4\sin\theta} - \frac{10\sin\phi}{r^4\sin\theta} = 0$$

$\boxed{\text{Prob. 6.8}}$

If $\nabla^2 V = \dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial z^2} = 0$, then

$$0 = -\frac{\partial}{\partial x}\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}\right)$$

$$= \frac{\partial^2}{\partial x^2}\left(-\frac{\partial V}{\partial x}\right) + \frac{\partial^2}{\partial y^2}\left(-\frac{\partial V}{\partial x}\right) + \frac{\partial^2}{\partial z^2}\left(-\frac{\partial V}{\partial z}\right)$$

$$= \frac{\partial^2}{\partial x^2} E_x + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial y^2} = \nabla^2 E_x$$

i.e. $\nabla^2 E_x = 0$.

The same holds for E_y and E_z.

$\boxed{\text{Prob. 6.9}}$

$$\nabla^2 V = \frac{d^2 V}{dz^2} = 0 \longrightarrow V = Az + B$$

when $z = 0$, $V = 0 \longrightarrow B = 0$

when $z = d$, $V = V_o \longrightarrow V_o = Ad$ or $A = \dfrac{V_o}{d}$.

Hence $\quad V = \dfrac{V_o}{d} z$

$$\vec{E} = -\nabla V = -\frac{dV}{dz}\vec{a}_z = -\frac{V_o}{d}\vec{a}_z$$

$$\vec{D} = \epsilon \vec{E} = -\epsilon_o \epsilon_r \frac{V_o}{d}\vec{a}_z.$$

Since $V_o = 50\text{V}$ and $d = 2\text{mm}$,

$$V = \underline{\underline{252\ \text{kV}}}, \quad \vec{E} = \underline{\underline{-25\vec{a}_z\ \text{kV/m}}},$$

$$\vec{D} = -\frac{10^{-9}}{36\pi}(1.5)\,25\times10^{3}\,\vec{a}_z = \underline{\underline{-332\,\vec{a}_z\ nC/m^2}},$$

$$\rho_s = D_n = \underline{\underline{\pm\,332\ nC/m^2}}.$$

The surface charge density is positive on the plate at $z = d$ and negative on the plate at $z = 0$.

$\boxed{\text{Prob } 6.10}$

From Example 6.8, solving $\nabla^2 V = 0$ when $V = V(\rho)$ leads to

$$V = \frac{V_o \ln \rho/a}{\ln b/a}$$

$$\vec{E} = -\nabla V = -\frac{V_o}{\rho \ln b/a}\,\vec{a}_\rho$$

$$\vec{D} = \epsilon\vec{E} = -\frac{V_o\,\epsilon_o\,\epsilon_r}{\rho \ln b/a}\,\vec{a}_\rho \quad .$$

$$\rho_s = D_n = \pm\left.\frac{V_o\,\epsilon_o\,\epsilon_r}{\rho \ln b/a}\right|_{\rho = a, b}$$

In this case, $V_o = 100V$, $b = 5mm$, $a = 15m$, $\epsilon_r = 2.0$.
Hence at $\rho = 10mm$,

$$V = \frac{100 \ln \frac{10}{5}}{\ln \frac{15}{5}} = \underline{\underline{36.91\ V}}.$$

$$\vec{E} = \frac{100}{10\times10^{-3}\ln 3}\vec{a}_\rho = \underline{\underline{9.102\,\vec{a}_\rho}}\ \text{kV/m}.$$

$$\vec{D} = 9.102\times10^{3}\times\frac{10^{-9}}{36\pi}\times 2\vec{a}_\rho = \underline{\underline{161\,\vec{a}_\rho\ \text{nC/m}^2}}.$$

$$\rho_s(\rho=5\text{mm}) = \frac{10^{-9}}{36\pi}(2)\frac{10^{5}}{5\ln 3} = \underline{\underline{322\ \text{nC/m}^2}}$$

$$\rho_s(\rho=15\text{mm}) = -\frac{10^{-9}}{36\pi}(2)\frac{10^{5}}{15\ln 3} = \underline{\underline{-107.33\ \text{nC/m}^2}}.$$

$\boxed{\text{Prob. 6.11}}$

$$\nabla^2 V = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = 0 \longrightarrow V = -\frac{A}{r} + B.$$

$$V(r=0.1) = 0 \longrightarrow 0 = -\frac{A}{0.1} + B \rightarrow B = 10A$$

$$V(r=2) = 100 \longrightarrow 100 = -\frac{A}{2} + B = \frac{19}{2}A$$

$$\text{or}\quad A = \frac{200}{19} \qquad B = \frac{2000}{19}$$

$$\underline{\underline{V = -\frac{10.53}{r} + 105.3\ \text{V}.}}$$

$$\vec{E} = -\nabla V = -\frac{dV}{dr}\vec{a}_r = \underline{\underline{-\frac{10.53}{r^2}\vec{a}_r}}\ \text{V/m}.$$

$$\vec{D} = \epsilon_0\vec{E} = \underline{\underline{-\frac{9.32\times10^{-11}}{r^2}\vec{a}_r}}\ \text{C/m}^2$$

Prob. 6.12

$$\nabla^2 V = 0 \longrightarrow V = -\frac{A}{r} + B$$

At $r = 0.5$, $V = -50 \longrightarrow -50 = -\frac{A}{0.5} + B$

or $$-50 = -2A + B \quad —(1)$$

At $r = 1$, $V = 50 \longrightarrow 50 = -A + B \quad —(2)$

From (1) and (2), $A = 100$, $B = 150$. Hence

$$V = -\frac{100}{r} + 150$$

$$\vec{E} = -\nabla V = -\frac{A}{r^2}\vec{a}_r = \frac{-100}{r^2}\vec{a}_r \text{ V/m.}$$

Prob. 6.13

From Example 6.4,

$$V = \frac{V_o \ln\left(\dfrac{\tan\theta/2}{\tan\theta_1/2}\right)}{\ln\left(\dfrac{\tan\theta_2/2}{\tan\theta_1/2}\right)}$$

$V_o = 100$, $\theta_1 = 30°$, $\theta_2 = 120°$, $r = \sqrt{3^2 + 0^2 + 4^2} = 5$.

$$\theta = \tan^{-1}\frac{\rho}{z} = \tan^{-1}\frac{3}{4} = 36.87°$$

$$V = 100\ln\left(\frac{\tan 18.435°}{\tan 15°}\right) \Big/ \ln\left(\frac{\tan 60°}{\tan 15°}\right) = \underline{11.7\text{ V}}$$

$$\vec{E} = \frac{-V_o \vec{a}_\theta}{r\sin\theta \ln\left(\frac{\tan\theta_2/2}{\tan\theta_1/2}\right)} = \frac{-100\,\vec{a}_\theta}{5\sin 36.87^\circ \ln 6.464}$$

$$= \underline{\underline{-17.86\,\vec{a}_\theta \ \ V/m.}}$$

$\boxed{\text{Prob. } 6.14}$

(a) $\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) = 0 \longrightarrow V = A\ln\rho + B$

$V(\rho=b) = 0 \longrightarrow 0 = A\ln b + B \rightarrow B = -A\ln b.$

$V(\rho=a) = V_o \longrightarrow V_o = A\ln\frac{a}{b} \longrightarrow A = -\frac{V_o}{\ln\frac{b}{a}}$

$$V = -\frac{V_o}{\ln\frac{b}{a}}\ln\frac{\rho}{b} = \frac{V_o \ln\frac{b}{\rho}}{\ln\frac{b}{a}}$$

$$V(\rho = 15\,mm) = \frac{70\ln 2}{\ln 50} = \underline{\underline{12.4\ V}}.$$

(b) As the electron decelerates, potential energy gained = K.E. loss

$$e[70-12.4] = \frac{1}{2}m\left[(10^7)^2 - u^2\right]$$

$$10^{14} - u^2 = \frac{2e}{m}\times 57.6$$

$$u^2 = 10^{14} - \frac{2\times1.6\times10^{-19}}{9.1\times10^{-31}}\times 57.6 = 10^{12}(100-20.25)$$

$$u = \underline{\underline{8.93\times10^6\ m/s.}}$$

Prob. 6.15

(a) for the parallel-plate capacitor,

$$\vec{E} = -\frac{V_0}{d}\vec{a}_x$$

from example 6.11.

$$C = \frac{1}{V_0^2}\int \epsilon|\vec{E}|^2 dv = \frac{1}{V_0^2}\int \epsilon\frac{V_0^2}{d^2}dv$$

$$= \frac{\epsilon}{d^2}\cdot Sd = \frac{\epsilon S}{d}.$$

(b) for the cylindrical capacitor,

$$\vec{E} = -\frac{V_0}{\rho\ln\frac{b}{a}}\vec{a}_\rho$$

from example 6.8.

$$C = \frac{1}{V_0^2}\iiint\frac{\epsilon V_0^2}{\rho^2\left(\ln\frac{b}{a}\right)^2}\rho\, d\rho\, d\phi\, dz$$

$$= \frac{2\pi L\epsilon}{\left(\ln\frac{b}{a}\right)^2}\int_a^b\frac{d\rho}{\rho} = \frac{2\pi\epsilon L}{\ln\frac{b}{a}}$$

(c) For the spherical capacitor,

$$\vec{E} = \frac{V_0}{r^2\left(\frac{1}{a}-\frac{1}{b}\right)}\vec{a}_r$$

from example 6.10.

$$C = \frac{1}{V_0^2}\iiint\frac{\epsilon V_0^2}{r^4\left(\frac{1}{a}-\frac{1}{b}\right)^2}r^2\sin\theta\, d\theta\, dr\, d\phi$$

$$C = \frac{\epsilon}{\left(\frac{1}{a}-\frac{1}{b}\right)^2} \cdot 4\pi \int_a^b \frac{dr}{r^2} = \frac{4\pi\epsilon}{\frac{1}{a}-\frac{1}{b}}.$$

Prob. 6.16

This is similar to case 1 in example 6.5.

$$X = c_1 x + c_2, \quad Y = c_3 y + c_4$$

But $X(0) = 0 \longrightarrow 0 = c_2$

$Y(0) = 0 \longrightarrow 0 = c_4$

Hence, $V(x,y) = XY = a_0 xy, \quad a_0 = c_1 c_3$

Also, $V(xy=4) = 20 \longrightarrow 20 = a_0 4 \longrightarrow a_0 = 5.$

Thus $V(x,y) = 5xy$ and $\vec{E} = -\nabla V = -5y\,\bar{a}_x - 5x\,\bar{a}_y$

At $(x,y) = (1,2)$,

$V = \underline{\underline{10V}}, \quad \vec{E} = \underline{\underline{-10\bar{a}_x - 5\bar{a}_y \text{ V/m}}}.$

Prob. 6.17

(a) As in example 6.5, $X(x) = A \sin \frac{n\pi x}{b}$

for Y,

$$Y(y) = c_1 \cosh \frac{n\pi y}{b} + c_2 \sinh \frac{n\pi y}{b}.$$

$$Y(a) = 0 \longrightarrow 0 = c_1 \cosh \frac{n\pi a}{b} + c_2 \sinh \frac{n\pi a}{b}$$

or $c_1 = -c_2 \tanh \frac{n\pi a}{b}$

$$V = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi y}{b} \left(\sinh \frac{n\pi y}{b} - \tanh \frac{n\pi a}{b} \cosh \frac{n\pi y}{b} \right)$$

$$V(x, y=0) = V_o = -\sum_{n=1}^{\infty} a_n \tanh \frac{n\pi a}{b} \sin \frac{n\pi x}{\cancel{b}}.$$

$$-a_n \tanh \frac{n\pi a}{b} = \frac{2}{b} \int_0^b V_o \sin \frac{n\pi y}{b}\, dy = \begin{cases} \frac{4V_o}{n\pi}, & n = \text{odd} \\ 0, & n = \text{even} \end{cases}$$

Hence,

$$V = -\frac{4V_o}{\pi} \sum_{n=\text{odd}}^{\infty} \sin \frac{n\pi x}{b} \left[\frac{\sin \frac{n\pi y}{b}}{n \tanh \frac{n\pi a}{b}} - \frac{\cosh \frac{n\pi y}{b}}{n} \right]$$

$$= -\frac{4V_o}{\pi} \sum \frac{\sin \frac{n\pi x}{b}}{n \sinh \frac{n\pi a}{b}} \left[\sin \frac{n\pi y}{b} \cosh \frac{n\pi a}{b} - \cosh \frac{n\pi y}{b} \sinh \frac{n\pi a}{b} \right]$$

$$= \frac{4V_o}{\pi} \sum_{n=\text{odd}}^{\infty} \frac{\sin \frac{n\pi x}{a} \sinh \frac{n\pi}{b}(a-y)}{n \sinh \frac{n\pi a}{b}}$$

Alternatively, for Y,

$$Y(y) = c_1 \sinh \frac{n\pi}{b}(y - c_2)$$

$$Y(a) = 0 \rightarrow 0 = c_1 \sinh \frac{n\pi}{b}(a - c_2) \rightarrow c_2 = a$$

$$V = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{b} \sinh \frac{n\pi}{b}(y-a)$$

where

$$b_n = \begin{cases} -\dfrac{4V_0}{n\pi \sinh \frac{n\pi a}{b}}, & n = odd \\ 0, & n = even. \end{cases}$$

(b) This is the same as example 6.5 except that we exchange y and x. Hence

$$V(x,y) = \frac{4V_0}{\pi} \sum_{n=odd}^{\infty} \frac{\sin \frac{n\pi y}{a} \sinh \frac{n\pi x}{a}}{n \sinh \frac{n\pi b}{a}}.$$

(c) This is the same as part (a) except that we must exchange x and y. Hence

$$V(x,y) = \frac{4V_0}{\pi} \sum_{n=odd}^{\infty} \frac{\sin \frac{n\pi y}{b} \sinh \frac{n\pi}{b}(a-x)}{n \sinh \frac{n\pi a}{b}}.$$

Prob. 6.18

(a) $X(x)$ is the same as in example 6.5. Hence

$$V(x,y) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{b} \left(a_n \sinh \frac{n\pi y}{b} + b_n \cosh \frac{n\pi y}{b} \right)$$

At $y = 0$, $V = V_1$

$$V_1 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{b} \longrightarrow b_1 = \begin{cases} \frac{4V_1}{n\pi}, & n = odd \\ 0, & n = even. \end{cases}$$

At $y = b$, $V = V_2$,

$$V_2 = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{a} \left[a_n \sinh \frac{n\pi a}{b} + b_n \cosh \frac{n\pi a}{b} \right]$$

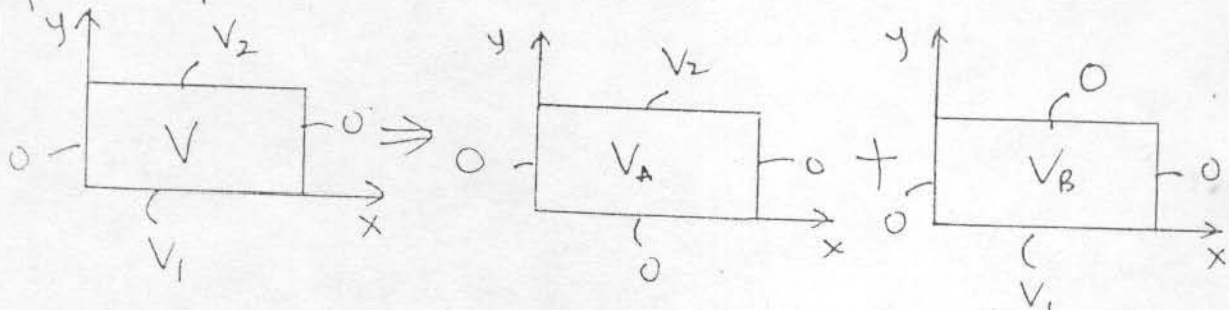
$\Longrightarrow$

$$a_n \sinh \frac{n\pi a}{b} + b_n \cosh \frac{n\pi a}{b} = \begin{cases} \frac{4V_2}{n\pi}, & n = \text{odd} \\ 0, & n = \text{even.} \end{cases}$$

or

$$a_n = \begin{cases} \dfrac{4}{n\pi \sinh \frac{n\pi a}{b}} \left(V_2 - V_1 \cosh \frac{n\pi a}{b} \right), & n = \text{odd} \\ 0, & n = \text{even} \end{cases}$$

Alternatively, we may apply superposition principle.



i.e. $V = V_A + V_B$.

V_A is exactly the same as example 6.5 with $V_0 = V_2$ while V_B is exactly the same as Prob. 6.17(a). Hence,

$$V = \frac{4}{\pi} \sum_{n=\text{odd}}^{\infty} \frac{\sin \frac{n\pi x}{b}}{n \sinh \frac{n\pi a}{b}} \left[V_1 \sinh \frac{n\pi}{b}(a-y) + V_2 \sinh \frac{n\pi y}{b} \right]$$

(b) $V(x,y) = \left(a_1 e^{-\alpha x} + a_2 e^{+\alpha x}\right)\left(a_3 \sin\alpha y + a_4 \cos\alpha y\right)$

$\lim_{x \to \infty} V(x,y) = 0 \longrightarrow a_2 = 0$

$$V(x, y=0) = 0 \longrightarrow a_4 = 0$$

$$V(x, y=a) = 0 \longrightarrow \alpha = \frac{n\pi}{a}, \quad n=1,2,3.$$

Hence,

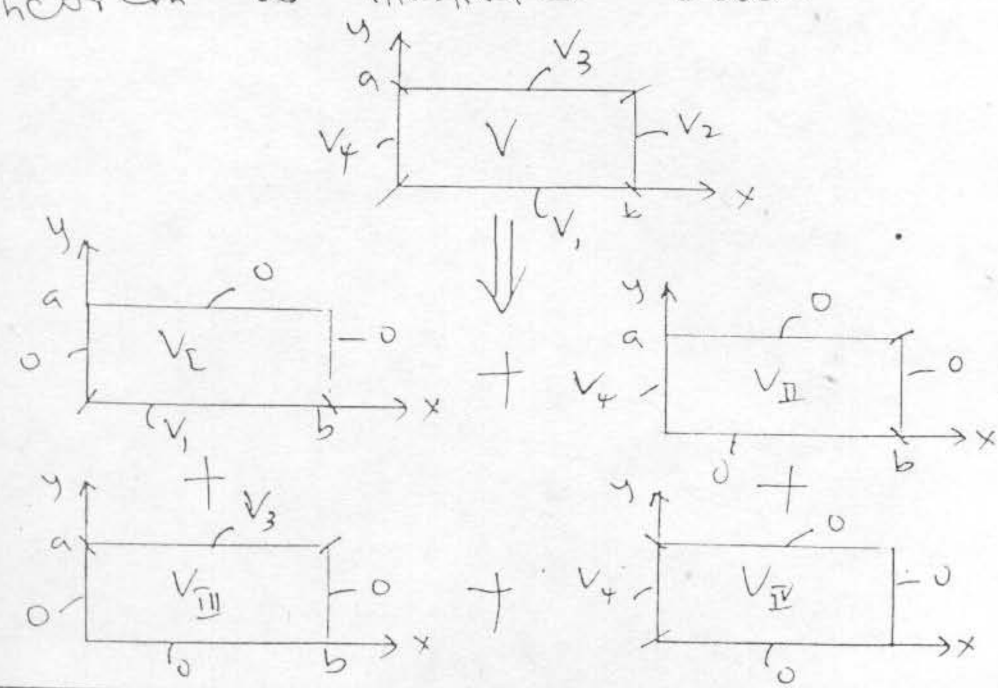
$$V(x,y) = \sum_{n=1}^{\infty} a_n e^{-\frac{n\pi x}{a}} \sin \frac{n\pi y}{a}.$$

$$V(x=0, y) = V_0 = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi y}{a}$$

$$\Longrightarrow \quad a_n = \begin{cases} \dfrac{4V_0}{n\pi}, & n = \text{odd} \\ 0, & n = \text{even}. \end{cases}$$

$$V(x,y) = \frac{4V_0}{\pi} \sum_{n=\text{odd}}^{\infty} \frac{1}{n} \sin \frac{n\pi y}{a} \exp\left(-\frac{n\pi x}{a}\right).$$

(c) The problem is easily solved using superposition theorem as illustrated below.



Therefore,

$$V = V_{I} + V_{II} + V_{III} + V_{IV}$$

where

$$V_{I} = \frac{4V_1}{\pi} \sum_{n=odd}^{\infty} \frac{\sin \frac{n\pi x}{b} \sinh \frac{n\pi}{b}(a-y)}{n \sinh \frac{n\pi a}{b}},$$

$$V_{II} = \frac{4V_2}{\pi} \sum_{n=odd}^{\infty} \frac{\sinh \frac{n\pi x}{a} \sin \frac{n\pi y}{a}}{n \sinh \frac{n\pi b}{a}},$$

$$V_{III} = \frac{4V_3}{\pi} \sum_{n=odd}^{\infty} \frac{\sin \frac{n\pi x}{b} \sinh \frac{n\pi y}{b}}{n \sinh \frac{n\pi a}{b}},$$

$$V_{IV} = \frac{4V_4}{\pi} \sum_{n=odd}^{\infty} \frac{\sin \frac{n\pi y}{a} \sinh \frac{n\pi}{a}(b-x)}{n \sinh \frac{n\pi b}{a}}$$

Prob. 6.19

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial\phi^2} = 0$$

If we let $V(\rho,\phi) = R(\rho)\,\Phi$.

$$\frac{\Phi}{\rho}\frac{\partial}{\partial\rho}(\rho R') + \frac{1}{\rho^2} R\,\Phi'' = 0$$

or $$\frac{\rho}{R}\frac{d}{d\rho}(\rho R') = -\frac{\Phi''}{\Phi} = \lambda$$

Hence $$\underline{\Phi'' + \lambda\Phi = 0}$$

and $$\frac{d}{d\rho}(\rho R') - \frac{\lambda R}{\rho} = 0$$

or $$R'' + \frac{R'}{\rho} - \frac{\lambda R}{\rho^2} = 0 .$$

Prob. 6.20

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) = 0.$$

If $V(r,\theta) = R(r)\,F(\theta)$, $r \neq 0$,

$$F\frac{d}{dr}(r^2 R') + \frac{R}{\sin\theta}\frac{d}{d\theta}(\sin\theta\, F') = 0$$

Dividing thru by RF gives

$$\frac{1}{R}\frac{d}{dr}(r^2 R') = -\frac{1}{F\sin\theta}\frac{d}{d\theta}(\sin\theta\, F') = \lambda$$

Hence,

$$\sin\theta\, F' + \cos\theta\, F' + \lambda F \sin\theta = 0$$

or $$F'' + \cot\theta\, F' + \lambda F = 0 .$$

Also, $$\frac{d}{dr}(r^2 R') - \lambda R = 0$$

or $$R'' + \frac{2R'}{r} - \frac{\lambda}{r^2} R = 0 .$$

Prob. 6.21

If the ends at $\phi = 0$ and $\phi = \pi/2$ are maintained at a potential difference of V_o, from example 6.3,

$$E_\phi = \frac{2V_o}{\pi\rho}, \quad J = \sigma E.$$

Hence,

$$I = \int \vec{J}\cdot d\vec{S} = \frac{2V_o\sigma}{\pi}\int_{\rho=a}^{b}\int_{z=0}^{t} \frac{1}{\rho}\, d\rho\, dz = \frac{2V_o\sigma t}{\pi} \ln\frac{b}{a}$$

and

$$R = \frac{V_o}{I} = \frac{\pi}{2\sigma t \ln\frac{b}{a}}.$$

$\boxed{\text{Prob. 6.22}}$

If $V(r=a)=0$, $V(r=b)=V_o$, from example 6.9,

$$E = \frac{V_o}{r^2\left(\frac{1}{a}-\frac{1}{b}\right)}, \quad J = \sigma E.$$

Hence,

$$I = \int \vec{J}\cdot d\vec{S} = \frac{\sigma V_o}{\frac{1}{a}-\frac{1}{b}}\int_{\theta=0}^{\alpha}\int_{\phi=0}^{2\pi} \frac{1}{r^2}\, r^2 \sin\theta\, d\theta\, d\phi$$

$$= \frac{\sigma V_o 2\pi}{\frac{1}{a}-\frac{1}{b}}\left(-\cos\theta\right)\Big|_0^{\alpha}$$

$$R = \frac{V_o}{I} = \frac{\frac{1}{a}-\frac{1}{b}}{2\pi\sigma(1-\cos\alpha)}$$

$\boxed{\text{Prob. 6.23}}$

For a spherical capacitor, from eq. (6.38),

$$R = \frac{\frac{1}{a} - \frac{1}{b}}{4\pi\sigma}.$$

For the hemisphere, $R' = 2R$. Since the sphere consists of two hemispheres in parallel. As $b \to \infty$,

$$R' = \lim_{b \to \infty} \frac{\frac{1}{a} - \frac{1}{b}}{2\pi\sigma} = \frac{1}{2\pi a\sigma}$$

$$G = \frac{1}{R'} = 2\pi a\sigma$$

Alternatively, for an isolated sphere,

$$C = 4\pi\epsilon a$$

But $RC = \frac{\epsilon}{\sigma} \longrightarrow R = \frac{1}{4\pi a\sigma}$

$$R' = 2R = \frac{1}{2\pi a\sigma} \quad \text{or} \quad G = 2\pi a\sigma$$

$\boxed{\text{Prob. 6.24}}$

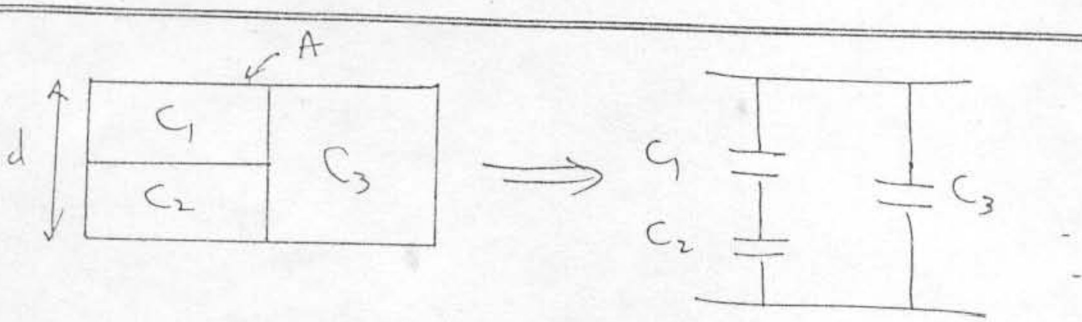
$$C = \frac{\epsilon S}{d} \longrightarrow S = \frac{Cd}{\epsilon_0\epsilon_r} = \frac{2\times10^{-9} \times 10^{-6}}{4 \times \frac{10^{-9}}{36\pi}} \text{ m}^2$$

$$S = 0.18\pi \text{ cm}^2 = \underline{\underline{0.5655 \text{ cm}^2}}.$$

$\boxed{\text{Prob. 6.25}}$

From the figure on the next page,

$$C = \frac{C_1 C_2}{C_1 + C_2} + C_3, \quad \text{where}$$



$$C_1 = \frac{\epsilon_0 A/2}{d/2} = \frac{\epsilon_0 A}{d}, \quad C_2 = \frac{\epsilon_0 \epsilon_r A}{d}, \quad C_3 = \frac{\epsilon_0 A}{2d}$$

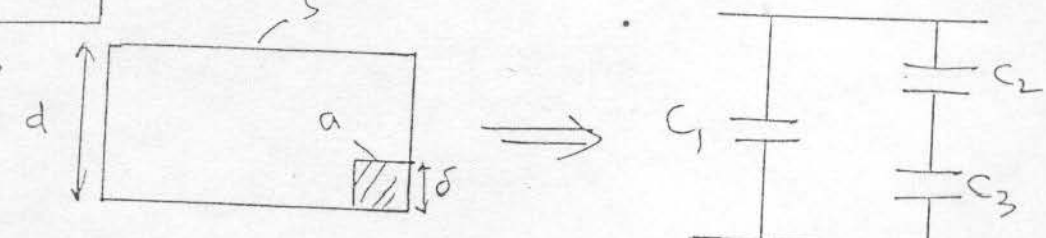
Hence,
$$C = \frac{\epsilon_0 \epsilon_0 \epsilon_r A^2/d^2}{\epsilon_0(\epsilon_r+1)A/d} + \frac{\epsilon_0 A}{2d}$$

$$= \frac{\epsilon_0 A}{d}\left(\frac{1}{2} + \frac{\epsilon_r}{\epsilon_r+1}\right)$$

$$= \frac{10^{-9}}{36\pi} \cdot \frac{10\times10^{-4}}{2\times10^{-3}}\left(\frac{1}{2}+\frac{6}{7}\right) = 5.99\times10^{-12}$$

$$\underline{\underline{C \simeq 6\,pF}}.$$

$\boxed{\text{Prob. 6.26}}$



from the figure above,

$$C = C_1 + \frac{C_2 C_3}{C_2 + C_3}$$

where $C_1 = \frac{\epsilon_0(S-a)}{d}$, $C_2 = \frac{\epsilon_0 a}{d-\delta}$, $C_3 = \frac{\epsilon_0 \epsilon_r a}{\delta}$

$$C_1 = \frac{10^{-9}}{36\pi} \frac{(2000-9)\times 10^{-6}}{5\times 10^{-3}} = 3.52\,pF.$$

$$C_2 = \frac{10^{-9}}{36\pi} \frac{9\times 10^{-6}}{2\times 10^{-3}} = 0.04\,pF,$$

$$C_3 = \frac{10^{-9}}{36\pi} \frac{4.6\times 9\times 10^{-6}}{3\times 10^{-3}} = 0.122\,pF.$$

Hence,

$$C = 3.52 + \frac{0.04\times 0.122}{0.162} = \underline{\underline{3.55\,pF}}$$

Prob. 6.27

$$F dx = dW_E \longrightarrow F = \frac{dW_E}{dx}$$

$$W_E = \int \frac{1}{2}\epsilon|\vec{E}|^2 dv = \frac{1}{2}\epsilon_o\epsilon_r E^2 xad + \frac{1}{2}\epsilon_o E^2 da(l-x)$$

where $E = V_o/d$

$$\frac{dW_E}{dx} = \frac{1}{2}\epsilon_o\frac{V_o^2}{d^2}(\epsilon_r-1)da \rightarrow F = \frac{\epsilon_o(\epsilon_r-1)V_o^2 a}{2d}$$

Alternatively,

$$W_E = \frac{1}{2}CV_o^2$$

where

$$C = C_1 + C_2 = \frac{\epsilon_o\epsilon_r ax}{d} + \frac{\epsilon_o\epsilon_r(l-x)}{d}$$

$$\frac{dW_E}{dx} = \frac{1}{2}V_o^2\frac{\epsilon_o a}{d}(\epsilon_r-1)$$

$$\underline{\underline{F = \frac{\epsilon_o(\epsilon_r-1)V_o^2 a}{d}}}$$

$\boxed{\text{Prob. 6.28}}$

(a) $C = \dfrac{\epsilon_0 S}{d} = \dfrac{10^{-9}}{36\pi} \cdot \dfrac{200 \times 10^{-4}}{3 \times 10^{-3}} = \underline{\underline{59 \text{ pF}}}.$

(b) $\rho_s = D_n = 10^{-6} \text{ nC/m}^2.$

But $D_n = \epsilon E_n = \dfrac{\epsilon_0 V_0}{d} = \rho_s$

or $V_0 = \dfrac{\rho_s d}{\epsilon_0} = 10^{-6} \times 3 \times 10^{-3} \times 36\pi \times 10^{9}$

$= \underline{\underline{339.3 \text{ V}}}.$

(c) $F = \dfrac{Q^2}{2S\epsilon_0} = \dfrac{\rho_s^2 S}{2\epsilon_0} = \dfrac{10^{-12} \times 200 \times 10^{-4} \times 36\pi \times 10^{9}}{2}$

$F = 36\pi \times 10^{-5} = \underline{\underline{1.131 \text{ mN}}}.$

$\boxed{\text{Prob. 6.29}}$

(a) $\vec{E}$ can be found in 2 ways,

<u>method 1:</u> $\vec{E} = \dfrac{\rho_s}{\epsilon}(-\vec{a}_x)$, where ρ_s is to be determined.

$V_0 = -\int \vec{E} \cdot d\vec{l} = -\int_0^d -\dfrac{\rho_s}{\epsilon}\, dx$

$= \rho_s \int_0^d \left(\dfrac{x+d}{2d\epsilon_0}\right) dx = \dfrac{3\rho_s d}{4\epsilon_0} \longrightarrow \rho_s = \dfrac{4\epsilon_0 V_0}{3d}$

$\vec{E} = -\dfrac{4V_0\epsilon_0}{3d} \dfrac{x+d}{2d\epsilon_0} \vec{a}_x = \underline{\underline{\dfrac{-2V_0(x+d)}{3d^2} \vec{a}_x}}$

Method 2: We solve Laplace's equation for inhomogeneous media.

$$\nabla\cdot(\epsilon\nabla V) = \frac{d}{dx}\left(\epsilon\frac{dV}{dx}\right) = 0$$

$$\epsilon\frac{dV}{dx} = A$$

$$\frac{dV}{dx} = \frac{A}{\epsilon} = \frac{A}{2d\epsilon_0}(x+d) = c_1(x+d)$$

$$V = c_1\left(\frac{x^2}{2} + dx\right) + c_2$$

$V(x=0) = 0 \longrightarrow c_2 = 0$

$V(x=d) = V_0 \longrightarrow c_1 = \dfrac{V_0}{d^2/2 + d^2} = \dfrac{2V_0}{3d^2}$

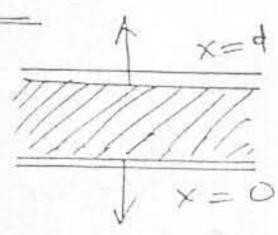
$$V = \frac{2V_0}{3d^2}\left(\frac{x^2}{2} + dx\right)$$

$$\vec{E} = -\nabla V = -\frac{2V_0}{3d^2}(x+d)\,\vec{a}_x$$

(b) $$\vec{P} = (\epsilon_r - 1)\epsilon_0\vec{E} = -\frac{2V_0}{3d^2}(x+d)\left(\frac{2d}{x+d} - 1\right)\epsilon_0\,\vec{a}_x$$

$$= \frac{-2V_0}{3d^2}\epsilon_0(d-x)\,\vec{a}_x$$

(c) $$\rho_{ps}\Big|_{x=0} = \vec{P}\cdot(-\vec{a}_x)\Big|_{x=0} = \frac{2V_0\epsilon_0}{3d}$$



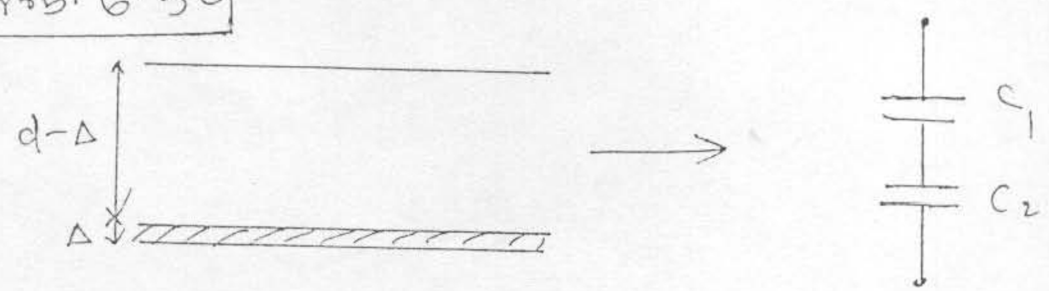
$$\rho_{ps}\Big|_{x=d} = \vec{P}\cdot\vec{a}_x\Big|_{x=d} = \underline{\underline{0}}.$$

(d) $$Q = \int \rho_s\, dS = \rho_s S = \frac{4V_o\epsilon_o S}{3d}$$

$$C = \frac{Q}{V_o} = \frac{4\epsilon_o S}{3d} = \frac{4}{3}\cdot\frac{10^{-9}}{36\pi}\cdot\frac{200\times10^{-4}}{2.5\times10^{-3}}$$

$$= \frac{8}{27\pi}\ \text{nF} = \underline{\underline{94.31\ \text{pF}}}$$

$\boxed{\text{Prob. 6.30}}$



From the figure shown above,

$$C = \frac{C_1 C_2}{C_1 + C_2}, \text{ where } C_1 = \frac{\epsilon_o S}{d-\Delta}, \quad C_2 = \frac{\epsilon_o \epsilon_r S}{\Delta}$$

$$C = \epsilon_o S \frac{\left(\frac{\epsilon_r}{\Delta}\cdot\frac{1}{d-\Delta}\right)}{\frac{\epsilon_r}{\Delta} + \frac{1}{d-\Delta}} = \frac{\epsilon_o \epsilon_r S}{\epsilon_r(d-\Delta) + \Delta}$$

$$= \frac{\epsilon_r d C_o}{\epsilon_r d + (1-\epsilon_r)\Delta} = \frac{\epsilon_r d C}{\epsilon_r d - \chi_e \Delta}$$

Prob. 6.31

$$C = \frac{2\pi\epsilon_o\epsilon_r L}{\ln\frac{b}{a}} = \frac{2\pi \times 3.5 \times 10^{-9} \times 10^{3}}{36\pi \ln\frac{2}{1}} = \frac{3.5 \times 10^{-6}}{18 \ln 2}$$

$$= \underline{\underline{280.5 \text{ nF/km.}}}$$

Prob. 6.32

$$C = \frac{4\pi\epsilon}{\frac{1}{a} - \frac{1}{b}}$$

Since $b \to \infty$, $C = 4\pi a \epsilon_o \epsilon_r$

$$= 4\pi \times 5 \times 10^{-2} \times 80 \times \frac{10^{-9}}{36\pi}$$

$$= \underline{\underline{444 \text{ pF}}}.$$

Prob. 6.33

$$C = \frac{4\pi\epsilon}{\frac{1}{a} - \frac{1}{b}} = \frac{4\pi \times \frac{10^{-9}}{36\pi} \times 5.9}{\left(\frac{1}{2} - \frac{1}{5}\right) \cdot 10^{2}} = \frac{5.9}{2.7} \times 10^{-11}.$$

$$= \underline{\underline{21.85 \text{ pF}}}$$

Prob. 6.34

$$C = \frac{2\pi\epsilon_o L}{\ln\frac{b}{a}} = 2\pi \times \frac{10^{-9}}{36\pi} \times \frac{100 \times 10^{-6}}{\ln\frac{600}{20}} = 1.633 \times 10^{-15}$$

$$V = \frac{Q}{C} = \frac{50 \times 10^{-15}}{1.633 \times 10^{-15}} = \underline{\underline{30.62 \text{ V}}}$$

Prob. 6.35

$$V = V_o e^{-t/T_r}$$

where $T_r = RC = 10\times10^{-6} \times 100\times10^{6} = 1000$

$$50 = 100 e^{-t/T_r} \rightarrow 2 = e^{t/T_r}$$

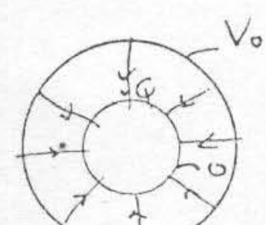
$$t = 1000 \ln 2 = \underline{\underline{693.1 s}}.$$

Prob. 6.36

$$RC = \frac{C}{G} = \frac{\epsilon}{\sigma} \rightarrow G = \frac{C\sigma}{\epsilon}$$

$$G = \underline{\underline{\frac{\pi\sigma}{\cosh^{-1}(d/2a)}}}$$

Prob. 6.37



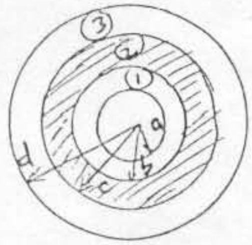
$$\vec{E} = \frac{Q}{4\pi\epsilon r^2}\vec{a}_r$$

$$W = \frac{1}{2}\int \epsilon |\vec{E}|^2 dv = \iiint \frac{Q^2}{32\pi^2\epsilon^2 r^2}\epsilon r^2 \sin\theta \, d\theta \, d\phi \, dr$$

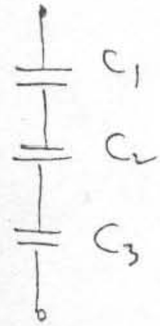
$$= \frac{Q^2}{32\pi^2\epsilon}(2\pi)(2)\int_b^c \frac{dr}{r^2} = \frac{Q^2}{8\pi\epsilon}\left(\frac{1}{c} - \frac{1}{b}\right)$$

$$W = \frac{Q^2(b-c)}{8\pi\epsilon bc}.$$

Prob. 6.38



$$\longrightarrow$$



$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$$

$$C_1 = \frac{4\pi\epsilon_o}{\frac{1}{a} - \frac{1}{b}}, \quad C_2 = \frac{4\pi\epsilon_o\epsilon_r}{\frac{1}{b} - \frac{1}{c}}, \quad C_3 = \frac{4\pi\epsilon_o}{\frac{1}{c} - \frac{1}{d}}$$

$$\frac{4\pi\epsilon_o}{C} = \frac{1}{a} - \frac{1}{b} + \frac{1}{\epsilon_r}\left(\frac{1}{b} - \frac{1}{c}\right) + \frac{1}{c} - \frac{1}{d}$$

$$\frac{4\pi\epsilon_o}{C} = \frac{1}{a} - \frac{1}{d} + \frac{\epsilon_r - 1}{\epsilon_r}\left(\frac{1}{c} - \frac{1}{b}\right)$$

$$C = \frac{4\pi\epsilon_o}{\frac{1}{a} - \frac{1}{d} + \frac{\epsilon_r - 1}{\epsilon_r}\left(\frac{1}{c} - \frac{1}{b}\right)}$$

Prob. 6.39

$$C = 4\pi\epsilon_o a = 4\pi \times \frac{10^{-9}}{36\pi} \times 6.37 \times 10^6 = \frac{6.37}{9}\,\text{mF}$$

$$= \underline{\underline{0.7078\,\text{mF}}}$$

Prob. 6.40

(a)

$$V = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{|(6,3,2)|} - \frac{1}{|(6,3,8)|}\right]$$

$$= \frac{10\times10^{-9}}{4\pi \times \frac{10^{-9}}{36\pi}}\left[\frac{1}{7} - \frac{1}{\sqrt{109}}\right] = \underline{\underline{4.237\,V}}.$$

$$\vec{E} = \frac{10\times10^{-9}}{4\pi\times\frac{10^{-9}}{36\pi}}\left[\frac{(6,3,2)}{7^3} - \frac{(6,3,8)}{109^{3/2}}\right]$$

$$= \underline{\underline{1.1\,\vec{a}_x + 0.55\,\vec{a}_y - 0.108\,\vec{a}_z\ V/m}}.$$

(b)
$$\vec{F} = \frac{Q_1 Q_2}{4\pi\epsilon_0 r^2}\vec{a}_r = \frac{-10\times10\times10^{-18}\left[(0,0,-3)-(0,0,3)\right]}{4\pi\times\frac{10^{-9}}{36\pi}\left|(0,0,-3)-(0,0,3)\right|^3}$$

$$= -900\times10^{-9}\,\frac{(0,0,-6)}{6^3} = \underline{\underline{-25\,\vec{a}_z\ N}}.$$

Prob. 6.41

(a) $Q_i = -(3nC - 4nC) = \underline{\underline{1nC}}$.

(b) The force of attraction between the

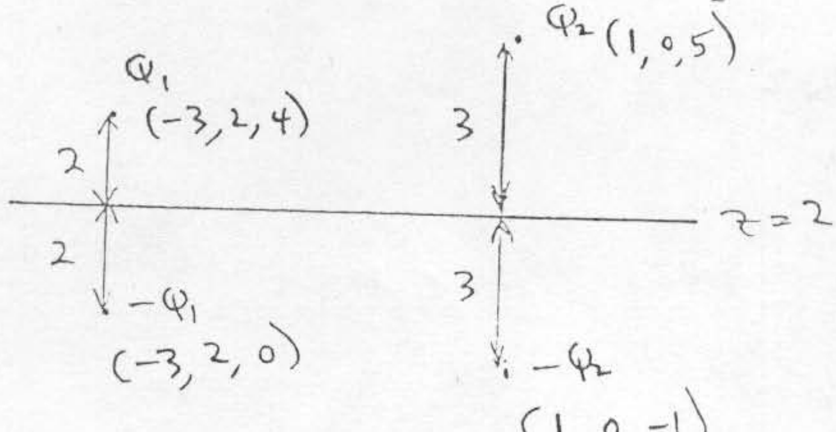
charges and the plates is

$$\vec{F} = \vec{F}_{13} + \vec{F}_{14} + \vec{F}_{23} + \vec{F}_{24}$$

$$|\vec{F}| = \frac{10^{-18}}{4\pi \times \frac{10^{-9}}{36\pi}} \left[\frac{9}{2^2} - \frac{2(12)}{3^2} + \frac{16}{4^2} \right]$$

$$= \frac{81}{4} - 24 + 9 \text{ nN} = \underline{\underline{5.25 \text{ nN}}}.$$

$\boxed{\text{Prob. 6.42}}$



$$\vec{D}(x,y,z) = \frac{Q_1}{4\pi}\left[\frac{(x,y,z)-(-3,2,4)}{|\quad\downarrow\quad|^3} - \frac{(x,y,z)-(-3,2,0)}{|\quad\downarrow\quad|^3} \right]$$

$$+ \frac{Q_2}{4\pi}\left[\frac{(x,y,z)-(1,0,5)}{|\quad\downarrow\quad|^3} - \frac{(x,y,z)-(1,0,-1)}{|\quad\downarrow\quad|^3} \right]$$

$$= \frac{50}{4\pi}\left[\frac{(x+3,\ y-2,\ z-4)}{|(x+3)^2+(y-2)^2+(z-4)|^{3/2}} - \frac{(x+3,\ y-2,\ z)}{|(x+3)^2+(y-2)^2+z^2|^{3/2}} \right]$$

$$-\frac{20}{4\pi}\left[\frac{(x-1,\ y,\ z-5)}{|(x-1)^2+y^2+(z-5)^2|^{3/2}} - \frac{(x-1,\ y,\ z+1)}{|(x-1)^2+y^2+(z+1)^2|^{3/2}} \right]$$

(a) At $(x,y,z) = (7,-2,2)$,

$$\rho_s = D_z\Big|_{z=2} = \frac{50}{4\pi}\left[\frac{2-4}{(10^2+4^2+2^2)^{3/2}} - \frac{2}{(10^2+4^2+2^2)^{3/2}}\right]$$

$$-\frac{20}{4\pi}\left[\frac{-3}{(6^2+4^2+3^2)^{3/2}} - \frac{3}{(6^2+4^2+3^2)^{3/2}}\right]$$

$$= \frac{-200}{4\pi\,(120)^{3/2}} + \frac{120}{4\pi\,(61)^{3/2}}$$

$$= -0.01211 + 0.02004 \ \text{nC/m}^2$$

$$\rho_s = \underline{\underline{7.934 \ \text{pC/m}^2}}.$$

(b) At $(3,4,8)$,

$$\vec{D} = \frac{50}{4\pi}\left[\frac{(6,2,4)}{(36+4+16)^{3/2}} - \frac{(6,2,8)}{(36+4+64)^{3/2}}\right]$$

$$-\frac{20}{4\pi}\left[\frac{(2,4,3)}{(4+16+9)^{3/2}} - \frac{(2,4,9)}{(4+16+81)^{3/2}}\right] \text{nC/m}^2$$

$$= \frac{25}{\pi}\left[\frac{(3,1,2)}{56^{3/2}} - \frac{(3,1,4)}{104^{3/2}}\right] - \frac{5}{\pi}\left[\frac{(2,4,3)}{29^{3/2}} - \frac{(2,4,9)}{101^{3/2}}\right]$$

$$= (34.46, 18.24, 7.974) - (17.25, 34.53, 16.46) \ \text{pC/m}^2$$

$$= \underline{\underline{17.21\,\vec{a}_x - 16.29\,\vec{a}_y - 8.486\,\vec{a}_z \ \text{pC/m}^2}}.$$

(c) Since $(1,1,1)$ is below the ground plane,

$$\vec{D} = \underline{\underline{0}}$$

Prob. 6.43

We have 7 images as follows: $-Q$ at $(-1,1,1)$, $-Q$ at $(1,-1,1)$, $-Q$ at $(1,1,-1)$, $-Q$ at $(-1,-1,-1)$, Q at $(1,-1,-1)$, Q at $(-1,-1,1)$, and Q at $(-1,1,-1)$.

$$\vec{F} = \frac{Q}{4\pi\epsilon_o}\left[-\frac{2\vec{a}_x}{2^3} - \frac{2\vec{a}_y}{2^3} - \frac{2\vec{a}_z}{2^3} - \frac{(2\vec{a}_x + 2\vec{a}_y + 2\vec{a}_z)}{12^{3/2}} + \frac{(2\vec{a}_y + 2\vec{a}_z)}{8^{3/2}} + \frac{(2\vec{a}_x + 2\vec{a}_z)}{8^{3/2}} + \frac{(2\vec{a}_x + 2\vec{a}_z)}{8^{3/2}} \right]$$

$$= 0.9(\vec{a}_x + \vec{a}_y + \vec{a}_z)\left(-\frac{1}{4} - \frac{1}{12\sqrt{3}} + \frac{1}{4\sqrt{2}}\right)$$

$$= \underline{\underline{-0.1891\,(\vec{a}_x + \vec{a}_y + \vec{a}_z)\text{ N}}}.$$

Prob. 6.44

(a) $$\vec{E} = \vec{E}_+ + \vec{E}_- = \frac{\rho_L}{2\pi\epsilon_o}\left(\frac{\vec{a}_{\rho_1}}{\rho_1} - \frac{\vec{a}_{\rho_2}}{\rho_2}\right)$$

$$= \frac{20\times10^{-9}}{2\pi \times \frac{10^{-9}}{36\pi}}\left[\frac{(1,2,3)-(1,1,5)}{|\ \ |^2} - \frac{(1,2,3)-(1,1,-3)}{|\ \ |^2}\right]$$

$$= 360\left[\frac{(0,1,-2)}{5} - \frac{(0,1,8)}{65}\right] = \underline{\underline{66.46\,\vec{a}_y - 188.3\,\vec{a}_z\text{ V/m}}}.$$

(b) Since $\vec{E}$ does not exist below plane $z=0$,

$$\vec{E} = \underline{\underline{0}}.$$

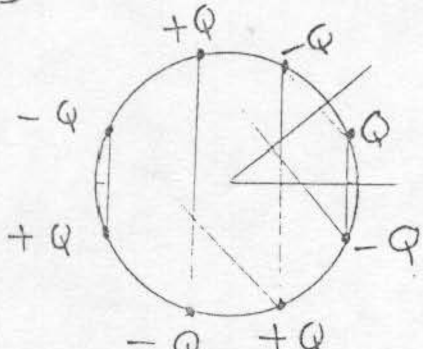
Prob. 6.45

$$\vec{E} = \sum \frac{\rho_s}{2\varepsilon_0}\vec{a}_n = \frac{10^{-9}}{2\times\frac{10^{-9}}{36\pi}}\left[-5\vec{a}_x + 10\vec{a}_x - 10\vec{a}_x - 5\vec{a}_x\right]$$

$$= -180\pi\,\vec{a}_x = \underline{\underline{-565.5\,\vec{a}_x}}\ \mathrm{V/m}.$$

Prob. 6.46

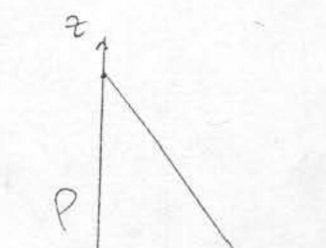
$$N = \left(\frac{360^\circ}{45^\circ} - 1\right) = 8 - 1 = \underline{\underline{7}}.$$



CHAPTER 7

P.E. 7.1

$\rho = 5,\ \cos\alpha_1 = 0,\ \cos\alpha_2 = \sqrt{\frac{2}{27}}$



$$\vec{a}_\phi = \vec{a}_l \times \vec{a}_\rho$$

$$= \left(\frac{-\vec{a}_x - \vec{a}_y}{\sqrt{2}}\right)\times\vec{a}_z = \frac{-\vec{a}_x + \vec{a}_y}{\sqrt{2}}$$

$$\vec{H}_3 = \frac{10}{4\pi(5)}\left(\sqrt{\frac{2}{27}} - 0\right)\left(\frac{-\vec{a}_x + \vec{a}_y}{2}\right) = \underline{\underline{-30.63\vec{a}_x + 30.63\vec{a}_y\ \mathrm{mA/m}}}$$

P.E. 7.2

(a) $\vec{H} = \frac{2}{4\pi(2)}\left(1+\frac{3}{\sqrt{13}}\right)\vec{a}_z = \underline{\underline{0.1458\,\vec{a}_z \text{ A/m.}}}$

(b) $\rho = \sqrt{3^2+4^2} = 5,\ \alpha_2 = 0,\ \cos\alpha_1 = -\frac{12}{13},$

$\vec{a}_\phi = -\vec{a}_y \times \left(\frac{3\vec{a}_x - 4\vec{a}_z}{5}\right) = \frac{4\vec{a}_x + 3\vec{a}_z}{5}$

$\vec{H} = \frac{2}{4\pi(5)}\left(1+\frac{12}{13}\right)\left(\frac{4\vec{a}_x + 3\vec{a}_z}{5}\right) = \frac{1}{26\pi}(4\vec{a}_x + 3\vec{a}_z)$

$= \underline{\underline{48.97\,\vec{a}_x + 36.73\,\vec{a}_z \text{ mA/m.}}}$

P.E. 7.3

(a) from Example 7.3,

$$\vec{H} = \frac{I a^2}{2(a^2+z^2)^{3/2}}\vec{a}_z$$

At $(0,0,-1)$, $z = 2$ cm,

$$\vec{H} = \frac{50\times10^{-3}\times 25\times10^{-4}}{2(5^2+2^2)^{3/2}\times10^{-6}}\vec{a}_z \text{ A/m}$$

$= \underline{\underline{400.2\,\vec{a}_z \text{ mA/m.}}}$

(b) At $(0, 0, 10\text{cm})$, $z = 9$cm,

$$\vec{H} = \frac{50\times10^{-3}\times25\times10^{-4}\,\vec{a}_z}{2(5^2+9^2)^{3/2}\times10^{-6}} \text{ A/m}$$

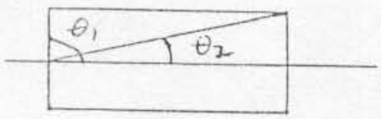
$= \underline{\underline{57.3\,\vec{a}_z \text{ mA/m.}}}$

$\boxed{P.E.7.4}$

$$\vec{H} = \frac{NI}{2L}(\cos\theta_2 - \cos\theta_1)\,\bar{a}_z = \frac{2\times10^3 \times 50\times10^{-3}}{2\times 0.75}(\;)\,\bar{a}_z$$

$$= \frac{100}{1.5}(\cos\theta_2 - \cos\theta_1)\,\bar{a}_z.$$

(a) At $(0,0,0)$, $\theta_1 = 90°$, $\cos\theta_2 = \dfrac{0.75}{\sqrt{0.75^2+0.05^2}}$

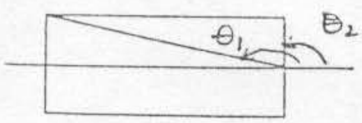


$$= 0.9978$$

$$\vec{H} = \frac{100}{1.5}(0.9978 - 1)\,\bar{a}_z$$

$$= \underline{\underline{66.52\,\bar{a}_z \;\; A/m.}}$$

(b) At $(0,0,0.75)$, $\theta_2 = 90°$, $\cos\theta_1 = -0.9978$

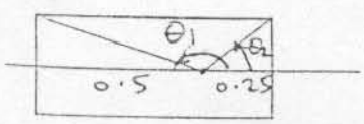


$$\vec{H} = \frac{100}{1.5}(0 + 0.9978)\,\bar{a}_z$$

$$= \underline{\underline{66.52\,\bar{a}_z \;\; A/m.}}$$

(c) At $(0,0,0.5)$, $\cos\theta_1 = \dfrac{-0.5}{\sqrt{0.5^2+0.05^2}} = -0.995$

$$\cos\theta_2 = \frac{0.25}{\sqrt{0.25^2+0.05^2}} = 0.9806$$



$$\hat{H} = \frac{100}{1.5}(0.9806 + 0.995)\,\bar{a}_z$$

$$= \underline{\underline{131.7\,\bar{a}_z \;\; A/m.}}$$

P.E. 7.5

$$\vec{H} = \frac{1}{2}\vec{K} \times \vec{a}_n$$

(a) $\vec{H}(0,0,0) = \frac{1}{2}\, 50\,\vec{a}_z \times (-\vec{a}_y) = \underline{\underline{25\,\vec{a}_x \text{ mA/m}}}$

(b) $\vec{H}(1,5,-3) = \frac{1}{2}\, 50\,\vec{a}_z \times \vec{a}_y = \underline{\underline{-25\,\vec{a}_x \text{ mA/m}}}$.

P.E. 7.6

$$|\vec{H}| = \begin{cases} \dfrac{NI}{2\pi\rho}, & \rho - a < \rho < \rho + a = 9 < \rho < 11 \\ 0, & \text{otherwise} \end{cases}$$

(a) At $(3,-4,0)$, $\rho = \sqrt{3^2+4^2} = 5 \text{ cm} < 9 \text{ cm}$

$$|\vec{H}| = \underline{\underline{0}}.$$

(b) At $(6,9,0)$, $\rho = \sqrt{6^2+9^2} = \sqrt{117} < 11$

$$\vec{H} = \frac{10^3 \times 100 \times 10^{-3}}{2\pi\sqrt{117} \times 10^{-2}} = \underline{\underline{147.1 \text{ A/m}}}$$

P.E. 7.7

(a) $\vec{B} = \nabla \times \vec{A} = (-4xz - 0)\vec{a}_x + (0 + 4yz)\vec{a}_y + (y^2 - x^2)\vec{a}_z$

$\vec{B}(-1,2,5) = \underline{\underline{20\,\vec{a}_x + 40\,\vec{a}_y + 3\,\vec{a}_z \text{ Wb/m}^2}}$.

(b) $\psi = \int \vec{B}\cdot d\vec{S} = \int_{y=-1}^{4}\int_{x=0}^{1} (y^2 - x^2)\,dx\,dy = \int_{-1}^{4} y^2\,dy - 5\int_0^1 x^2\,dx$

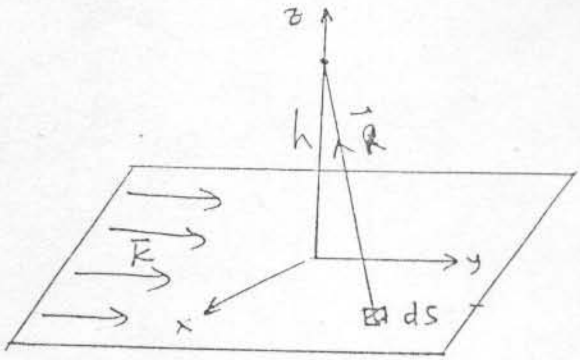
$$= \frac{1}{3}(64+1) - \frac{5}{3} = \underline{\underline{20 \text{ Wb}}}$$

Alternatively,

$$\Psi = \oint \vec{A}\cdot d\vec{l} = \int_0^1 x^2(-1)\,dx + \int_{-1}^{4} y^2(1)\,dy + \int_1^0 x^2(4)\,dx + 0$$

$$= -\frac{5}{3} + \frac{65}{3} = \underline{\underline{20 \text{ Wb}}}.$$

$\boxed{\text{P.E. 7.8}}$



$$\vec{H} = \int \frac{\vec{K}\,ds \times \vec{R}}{4\pi R^3},$$

$ds = dx\,dy$, $\vec{K} = k_y \vec{a}_y$,

$\vec{R} = (-x, -y, h)$,

$\vec{K} \times \vec{R} = (h\vec{a}_x + x\vec{a}_z)k_y$

$$\vec{H} = \int \frac{k_y(h\vec{a}_x + x\vec{a}_z)\,dx\,dy}{4\pi(x^2+y^2+h^2)^{3/2}}$$

$$= \frac{k_y h \vec{a}_x}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dx\,dy}{(x^2+y^2+h^2)^{3/2}} + \underbrace{\frac{k_y \vec{a}_z}{4\pi}\cancel{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{x\,dx\,dy}{(x^2+y^2+h^2)^{3/2}}}}_{0,\ \text{odd function of } x}$$

$$= \frac{k_y h \vec{a}_x}{4\pi}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{\infty} \frac{\rho\,d\phi\,d\rho}{(\rho^2+h^2)^{3/2}} = \frac{k_y h\, 2\pi \vec{a}_x}{4\pi}\int_0^{\infty} (\rho^2+h^2)^{-3/2}\,\frac{d(\rho^2)}{2}$$

$$= \frac{k_y h}{2}\vec{a}_z\left(\frac{-1}{(\rho^2+h^2)^{1/2}}\right)\Bigg|_0^{\infty} = \frac{k_y}{2}\vec{a}_x$$

Similarly, for point $(0,0,-h)$, $\vec{H} = -\frac{1}{2}k_y\vec{a}_x$

Hence,
$$\vec{H} = \begin{cases} \frac{1}{2}k_y\vec{a}_x, & z > 0 \\ -\frac{1}{2}k_y\vec{a}_x, & z < 0 \end{cases}$$

Prob. 7.1

Since the element is so small, from Biot-Savart law,

$$\vec{H} = \frac{I d\vec{l} \times \vec{R}}{4\pi R^3},$$

where $\vec{R} = (3,-6,2) - (0,0,0) = (3,-6,2)$,

$R = |\vec{R}| = 7, \quad d\vec{l} = dy\,\vec{a}_y$

$$d\vec{l} \times \vec{R} = \begin{vmatrix} 0 & dy & 0 \\ 3 & -6 & 2 \end{vmatrix} = 2dy\,\vec{a}_x - 3dy\,\vec{a}_z$$

$$\vec{H} = \frac{16\cdot 10^{-3}(2\vec{a}_x - 3\vec{a}_z)\, 2\times 10^{-3}}{4\pi \times 7^3} \text{ A/m}$$

$$= \underline{\underline{14.85\,\vec{a}_x - 22.27\,\vec{a}_z \text{ nA/m}}}$$

Prob. 7.2

(a) $\vec{H} = \frac{I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1)\vec{a}_\phi = \frac{2}{4\pi(5)}\left(\frac{10}{5\sqrt{2}} - 0\right)\vec{a}_y$

$= \underline{\underline{28.47\,\vec{a}_y \text{ mA/m}}}$.

(b) $\vec{H} = \frac{2}{4\pi 5\sqrt{2}}\left(\frac{10}{5\sqrt{6}} - 0\right)\vec{a}_\phi$, where $\vec{a}_\phi = \vec{a}_z \times \left(\frac{\vec{a}_x + \vec{a}_y}{\sqrt{2}}\right)$

$= \frac{1}{5\pi\sqrt{12}}\left(\frac{-\vec{a}_x + \vec{a}_y}{\sqrt{2}}\right) = \underline{\underline{-13\,\vec{a}_x + 13\,\vec{a}_y \text{ mA/m}}}$.

(c) $\vec{H} = \frac{2}{4\pi(5)\sqrt{10}}\left(\frac{10}{5\sqrt{14}} - 0\right)\vec{a}_\phi, \quad \vec{a}_\phi = \vec{a}_z \times \left(\frac{\vec{a}_x + 3\vec{a}_y}{\sqrt{10}}\right)$

$= \frac{1}{50\pi\sqrt{14}}(-3\vec{a}_x + \vec{a}_y) = \underline{\underline{-5.1\,\vec{a}_x + 1.7\,\vec{a}_y \text{ mA/m}}}$.

(d) $\vec{H} = \underline{\underline{5.1\vec{a}_x + 1.7\vec{a}_y \ mA/m^2.}}$

Prob. 7.3

Let $\vec{H} = \vec{H}_z + \vec{H}_x$

For $\vec{H}_z = \dfrac{I}{2\pi\rho}\vec{a}_\phi, \quad \rho = \sqrt{6^2+8^2} = 10,$

$$\vec{a}_\phi = \vec{a}_z \times \left(\frac{6\vec{a}_y + 8\vec{a}_y}{10}\right)$$

$$H_z = \frac{20}{2\pi(100)}\left(-8\vec{a}_x + 6\vec{a}_y\right) = \frac{1}{\pi}\left(-0.8\vec{a}_x + 0.6\vec{a}_y\right)$$

for $H_x = \dfrac{I}{2\pi\rho}\vec{a}_\phi, \quad \rho = \sqrt{8^2+6^2} = 10$

$$\vec{a}_\phi = \vec{a}_x \times \left(\frac{8\vec{a}_y - 6\vec{a}_z}{10}\right)$$

$$\vec{H}_x = \frac{30}{2\pi(100)}\left(6\vec{a}_y + 8\vec{a}_z\right) = \frac{1}{\pi}\left(0.9\vec{a}_y + 1.2\vec{a}_z\right)$$

$$\vec{H} = \vec{H}_x + \vec{H}_z = \frac{1}{\pi}\left(-0.8\vec{a}_x + 1.5\vec{a}_y + 1.2\vec{a}_z\right)$$

$$= \underline{\underline{-0.255\vec{a}_x + 0.4775\vec{a}_y + 0.382\vec{a}_z \ A/m.}}$$

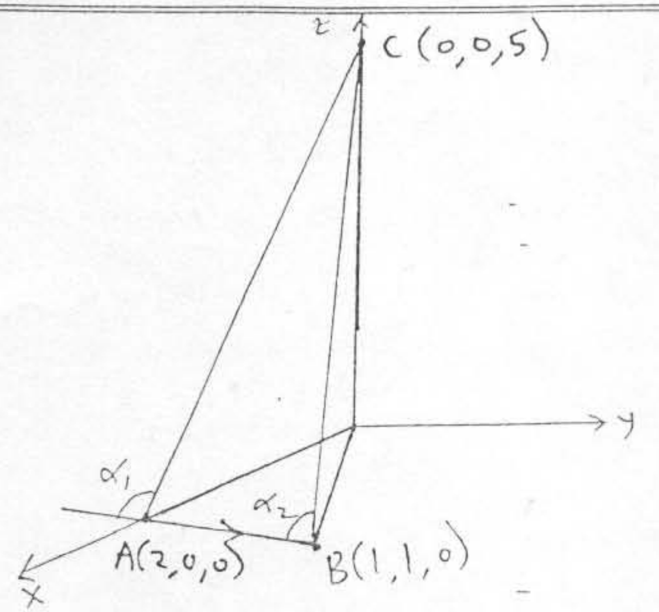
Prob. 7.4

(a) Consider the figure on the next page.

$\vec{AB} = (1,1,0) - (2,0,0) = (-1,1,0)$

$\vec{AC} = (0,0,5) - (2,0,0) = (-2,0,5)$

$\vec{AB} \cdot \vec{AC} = 2$, i.e $\vec{AB}$ and $\vec{AC}$ are not



perpendicular.

$$\cos(180-\alpha_1) = \frac{\vec{AB}\cdot\vec{AC}}{|\vec{AB}||\vec{AC}|} = \frac{2}{\sqrt{2}\sqrt{29}} \rightarrow \cos\alpha_1 = -\sqrt{\frac{2}{29}}$$

$$\vec{BC} = (0,0,5) - (1,1,0) = (-1,-1,5)$$

$$\vec{BA} = (1,-1,0)$$

$$\cos\alpha_2 = \frac{\vec{B}C\cdot B\vec{A}}{|\vec{BC}|\,|\vec{BA}|} = \frac{-1+1}{|\vec{BC}|\,|\vec{BA}|} = 0$$

i.e. $\vec{BC} = \vec{\rho} = (-1,-1,5),\quad \rho = \sqrt{27}$

$$\vec{a}_\phi = \vec{a}_l \times \vec{a}_\rho = \frac{(-1,1,0)}{\sqrt{2}} \times \left(\frac{-1,-1,5)}{\sqrt{27}}\right. = \frac{(5,5,2)}{\sqrt{54}}$$

$$\vec{H}_2 = \frac{10}{4\pi\sqrt{27}}\left(0+\sqrt{\frac{2}{29}}\right)\frac{(5,5,2)}{\sqrt{27}} = \frac{5}{2\pi\sqrt{29}}\cdot\frac{(5,5,2)}{27}\ \text{A/m}$$

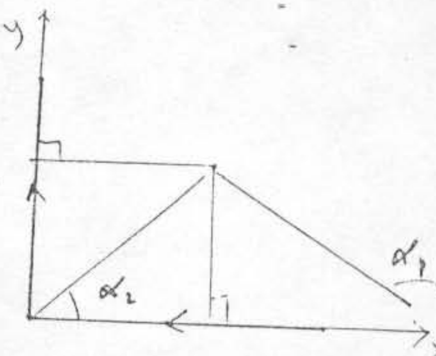
$$= \underline{\underline{27.37\,\vec{a}_x + 27.37\,\vec{a}_y + 10.95\,\vec{a}_z\ \text{mA/m}}}.$$

(b) $\vec{H} = \vec{H}_1 + \vec{H}_2 + \vec{H}_3 = (0,-59.1,0) +$

$$(27.37, 27.37, 10.95) + (-30.63, 30.63, 0)$$

$$= \underline{\underline{-3.26\,\bar{a}_x - 1.1\,\bar{a}_y + 10.95\,\bar{a}_z \text{ mA/m}}}.$$

Prob. 7.5



(a) Let $\vec{H} = \vec{H}_x + \vec{H}_y = 2\vec{H}_x$

$$\vec{H}_x = \frac{I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1)\,\bar{a}_\phi,$$

where $\bar{a}_\phi = -\bar{a}_x \times \bar{a}_y = -\bar{a}_z$, $\alpha_1 = 180°$, $\alpha_2 = 45°$.

$$\vec{H}_x = \frac{5}{4\pi(2)}(\cos 45° - \cos 180°)(-\bar{a}_z)$$

$$= \underline{\underline{-0.6792\,\bar{a}_z \text{ A/m}}}$$

(b) $\vec{H} = \vec{H}_x + \vec{H}_y$

where $\vec{H}_x = \dfrac{5}{4\pi(2)}(1-0)\,\bar{a}_\phi$, $\bar{a}_\phi = -\bar{a}_x \times -\bar{a}_y = \bar{a}_z$

$$= 198.9\,\bar{a}_z \text{ mA/m}.$$

$\vec{H}_y = 0$ since $\alpha_1 = \alpha_2 = 0$.

$$\vec{H} = \underline{\underline{0.1989\,\bar{a}_z \text{ A/m}}}$$

(c) $\vec{H} = \vec{H}_x + \vec{H}_y$

where $\vec{H}_x = \dfrac{5}{4\pi(2)}(1-0)(-\bar{a}_x \times \bar{a}_z) = 198.9\,\bar{a}_y \text{ mA/m}$

$$\vec{H}_y = \frac{5}{4\pi(2)}(1-0)(\bar{a}_y \times \bar{a}_z) = 198.9\,\bar{a}_x \text{ mA/m}$$

$\vec{H} = \underline{0.1989\,\vec{a}_x + 0.1989\,\vec{a}_y \quad A/m.}$

Prob. 7.6

For the side of the loop along y-axis,

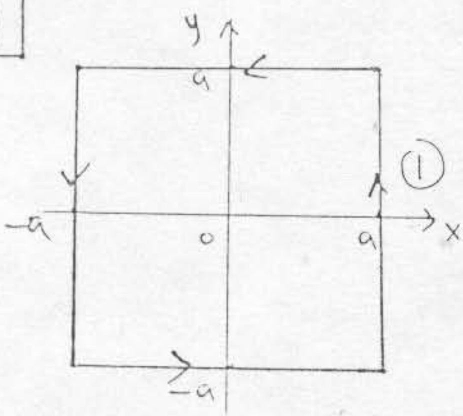
$$\vec{H}_1 = \frac{I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1)\,\vec{a}_\phi,$$

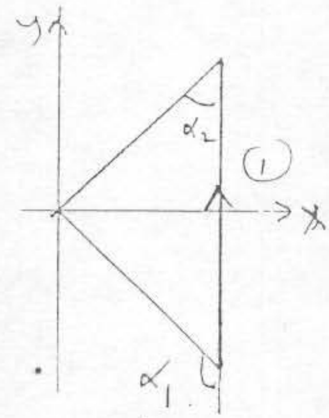
where $\vec{a}_\phi = -\vec{a}_x$, $\rho = 2\tan 30^\circ = \frac{2}{\sqrt{3}}$, $\alpha_2 = 30^\circ$, $\alpha_1 = 150^\circ$

$$\vec{H}_1 = \frac{5}{4\pi}\,\frac{\sqrt{3}}{2}\,(\cos 30^\circ - \cos 150^\circ)(-\vec{a}_x) = -\frac{15}{8\pi}\,\vec{a}_x$$

$$\vec{H} = 3\vec{H}_1 = \underline{-1.79\,\vec{a}_x \quad A/m.}$$

Prob. 7.7





$\vec{H} = 4\vec{H}_1$, where $\vec{H}_1$ is due to side 1.

$$\vec{H}_1 = \frac{I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1)\,\vec{a}_\phi,$$

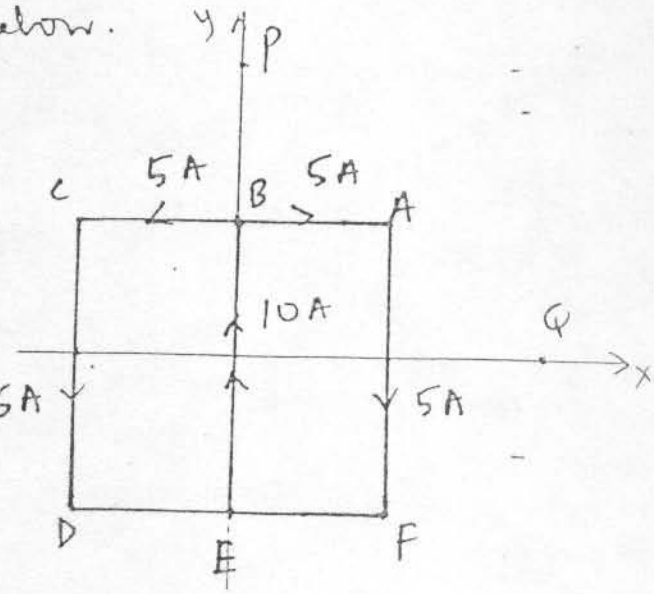
$\rho = a$, $\alpha_2 = 45^\circ$, $\alpha_1 = 135^\circ$, $\vec{a}_\phi = \vec{a}_y \times -\vec{a}_x = \vec{a}_z$

$$\vec{H}_1 = \frac{I}{4\pi\rho}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)\vec{a}_z = \frac{2I}{4\pi a\sqrt{2}}\,\vec{a}_z$$

$$\vec{H} = 4\vec{H}_1 = \frac{\sqrt{2}\,I}{\pi a}\,\vec{a}_z.$$

Prob. 7.8

Consider the figure below.



(a) At P, due to symmetry, $\vec{B} = \underline{\underline{0}}$.

(b) At Q, the contribution by side AB cancels that due to side FF. Similarly, the contribution due to side BC cancels that due to side DE. Thus, at Q,

$$\vec{B} = \vec{B}_{AF} + \vec{B}_{BE} + \vec{B}_{CD}$$

We now apply

$$\vec{B} = \mu \vec{H} = \frac{\mu I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1)\vec{a}_\phi.$$

$$\vec{B}_{AF} = \frac{5\mu_0}{4\pi(1)}\left(2\frac{1}{\sqrt{2}}\right)(-\vec{a}_y \times \vec{a}_x) = \frac{10\mu_0}{4\pi\sqrt{2}}\vec{a}_z$$

$$\vec{B}_{BE} = \frac{10\mu_0}{4\pi(2)}\left(2\frac{1}{\sqrt{5}}\right)(\vec{a}_y \times \vec{a}_z) = -\frac{10\mu_0}{4\pi\sqrt{5}}\vec{a}_z.$$

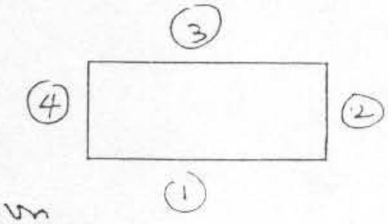
$$\vec{B}_{CD} = \frac{5\mu_o}{4\pi(3)}\left(2\frac{1}{\sqrt{10}}\right)(-\vec{a}_y \times \vec{a}_x) = \frac{10\mu_o}{4\pi 3\sqrt{10}}\vec{a}_z$$

$$\vec{B} = \frac{10\mu_o}{4\pi}\vec{a}_z\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{5}} + \frac{1}{3\sqrt{10}}\right)$$

$$= \frac{10\times 4\pi\times 10^{-7}}{4\pi}\vec{a}_z\,(0.707 - 0.4472 + 0.1054)\,\text{Wb/m}^2$$

$$= \underline{\underline{0.3652\,\vec{a}_z\ \mu\text{Wb/m}^2}}.$$

$\boxed{\text{Prob. 7.9}}$



Let $\vec{H} = \vec{H}_1 + \vec{H}_2 + \vec{H}_3 + \vec{H}_4$

where $\vec{H}_n$ is the contribution by side n.

(a) $\vec{H} = 2\vec{H}_1 + \vec{H}_2 + \vec{H}_4$ since $\vec{H}_1 = \vec{H}_3$.

$$\vec{H}_1 = \frac{I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1)\vec{a}_\phi = \frac{10}{4\pi(2)}\left(\frac{6}{\sqrt{40}} + \frac{1}{\sqrt{2}}\right)\vec{a}_z$$

$$\vec{H}_2 = \frac{10}{4\pi(6)}\left(2\times\frac{2}{\sqrt{40}}\right)\vec{a}_z,\quad \vec{H}_4 = \frac{10}{4\pi(2)}\left(2\cdot\frac{1}{\sqrt{2}}\right)\vec{a}_z.$$

$$\vec{H} = \left[\frac{5}{2\pi}\left(\frac{3}{\sqrt{10}} + \frac{1}{\sqrt{2}}\right) + \frac{5}{6\pi\sqrt{10}} + \frac{5}{2\pi\sqrt{2}}\right]\vec{a}_z = \underline{\underline{1.964\,\vec{a}_z\ \text{A/m}}}.$$

(b) At $(4, 2, 0)$, $\vec{H} = 2(\vec{H}_1 + \vec{H}_4)$

$$\vec{H}_1 = \frac{10}{4\pi(2)}\frac{8}{\sqrt{20}}\vec{a}_z,\quad \vec{H}_4 = \frac{10}{4\pi(4)}\frac{4}{\sqrt{20}}\vec{a}_z,$$

$$\vec{H} = \frac{2\sqrt{5}}{\pi}\left(1+\frac{1}{4}\right)\vec{a}_z = \underline{1.78\,\vec{a}_z \text{ A/m.}}$$

(c) At $(4,8,0)$, $\vec{H} = \vec{H}_1 + 2\vec{H}_2 + \vec{H}_3$

$$\vec{H}_1 = \frac{10}{4\pi(8)}\left(2\cdot\frac{4}{4\sqrt{5}}\right)\vec{a}_z, \quad \vec{H}_2 = \frac{10}{4\pi(4)}\left(\frac{8}{4\sqrt{5}} - \frac{1}{\sqrt{2}}\right)\vec{a}_z$$

$$\vec{H}_3 = \frac{10}{4\pi(4)}\left(\frac{2}{\sqrt{2}}\right)(-\vec{a}_z)$$

$$\vec{H} = \frac{5}{8\pi}(\vec{a}_z)\left(\frac{1}{\sqrt{5}} + \frac{4}{\sqrt{5}} - \frac{4}{\sqrt{2}}\right) = \underline{-0.1178\,\vec{a}_z \text{ A/m.}}$$

(d) At $(0,0,2)$,

$$\vec{H}_1 = \frac{10}{4\pi(2)}\left(\frac{8}{\sqrt{64}} - 0\right)(\vec{a}_x \times \vec{a}_z) = -\frac{10}{\pi\sqrt{68}}\vec{a}_y,$$

$$\vec{H}_2 = \frac{10}{4\pi\sqrt{64}}\left(\frac{4}{\sqrt{84}} - 0\right)\vec{a}_y \times \left(\frac{2\vec{a}_z - 8\vec{a}_x}{\sqrt{68}}\right) = \frac{5(\vec{a}_x + 4\vec{a}_y)}{17\pi\sqrt{84}}$$

$$\vec{H}_3 = \frac{10}{4\pi\sqrt{20}}\left(-\frac{8}{\sqrt{84}} - 0\right)\vec{a}_x \times \left(\frac{2\vec{a}_x - 4\vec{a}_y}{\sqrt{20}}\right) = \frac{\vec{a}_y + 2\vec{a}_z}{\pi\sqrt{21}}$$

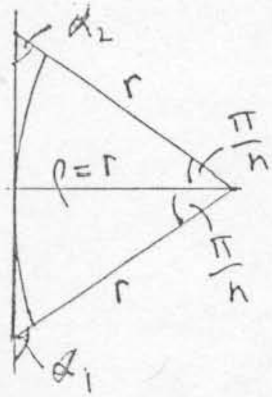
$$\vec{H}_4 = \frac{10}{4\pi(2)}\left(0 + \frac{4}{\sqrt{20}}\right)(-\vec{a}_y \times \vec{a}_z) = -\frac{5\vec{a}_x}{\pi\sqrt{20}}$$

$$\vec{H} = \left(\frac{5}{34\pi\sqrt{21}} - \frac{5}{\pi\sqrt{20}}\right)\vec{a}_x + \left(\frac{1}{\pi\sqrt{21}} - \frac{10}{\pi\sqrt{68}}\right)\vec{a}_x$$

$$+ \left(\frac{20}{34\pi\sqrt{21}} - \frac{2}{\pi\sqrt{21}}\right)\vec{a}_z$$

$$= \underline{-0.3457\,\vec{a}_x - 0.3165\,\vec{a}_y + 0.1798\,\vec{a}_z \text{ A/m}}$$

Prob. 7.10



(a) Consider one side of the polygon as shown. The angle subtended by the side at the center of the circle is

$$\frac{360^\circ}{n} = \frac{2\pi}{n}.$$

The field due to this side is

$$H_1 = \frac{I}{4\pi\rho}\left(\cos\alpha_2 - \cos\alpha_1\right)$$

where $\rho = r$, $\cos\alpha_2 = \cos\left(90 - \frac{\pi}{n}\right) = \sin\frac{\pi}{n}$

$$\cos\alpha_1 = -\sin\frac{\pi}{n}$$

$$H_1 = \frac{I}{4\pi r}\, 2\sin\frac{\pi}{n}$$

$$H = nH_1 = \frac{nI}{2\pi r}\sin\frac{\pi}{n}$$

(b) for $n = 3$, $H = \dfrac{3I}{2\pi r}\sin\pi/3$

$r\cot 30^\circ = 2 \longrightarrow r = \dfrac{2}{\sqrt{3}}$

$$H = \frac{3\times 5}{2\pi\, \frac{2}{\sqrt{3}}}\cdot\frac{\sqrt{3}}{2} = \frac{45}{8\pi} = \underline{\underline{1.78\ A/m.}}$$

for $n = 4$, $H = \dfrac{4I}{2\pi r}\sin\dfrac{\pi}{4} = \dfrac{4\times 5}{2\pi(2)}\dfrac{1}{\sqrt{2}}$

$$= \underline{\underline{1.125\ A/m}}$$

(i) As $n \to \infty$,

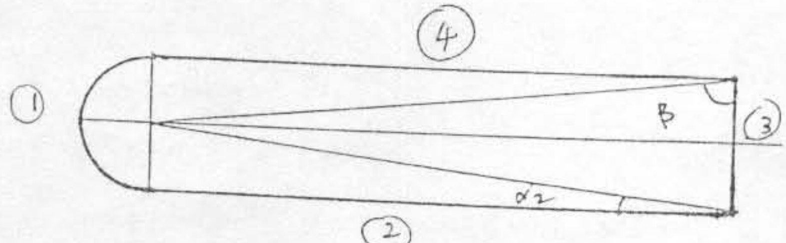
$$H = \lim_{n\to\infty} \frac{nI}{2\pi r} \sin\frac{\pi}{n} = \frac{nI}{2\pi r}\cdot\frac{\pi}{n} = \underline{\underline{\frac{I}{2r}}}.$$

From Example 7-3, when $h = 0$,

$$H = \frac{I}{2r}$$

which agrees.

$\boxed{\text{Prob. 7.11}}$



Let $\vec{H} = \vec{H}_1 + \vec{H}_2 + \vec{H}_3 + \vec{H}_4$

$$\vec{H}_1 = \frac{I}{4a}\vec{a}_z = \frac{10}{4\times4\times10^{-2}}\vec{a}_z = 62.5\,\vec{a}_z$$

$$\vec{H}_2 = \vec{H}_4 = \frac{I}{4\pi\times4\times10^{-2}}(\cos\alpha_2 - \cos 90^\circ)\vec{a}_z, \quad \alpha_2 = \tan^{-1}\frac{4}{100} = 2.29^\circ$$

$$= 19.88\,\vec{a}_z$$

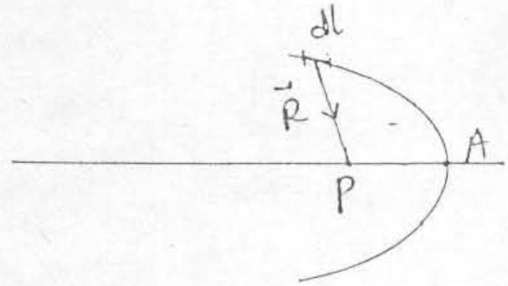
$$\vec{H}_3 = \frac{I}{4\pi(1)}\, 2\cos\beta\,\vec{a}_z, \quad \beta = \tan^{-1}\frac{100}{4} = 87.7^\circ$$

$$= \frac{10}{4\pi}\,2\cos 87.7^\circ\,\vec{a}_z = 0.06361\,\vec{a}_z$$

$$\vec{H} = (62.5 + 2\times19.88 + 0.06361)\vec{a}_z$$

$$= \underline{102.32\,\vec{a}_z \text{ A/m}}.$$

Prob. 7.12



$$\vec{H} = \int \frac{I\vec{dl} \times \vec{R}}{4\pi R^3}$$

$$\left| \frac{\vec{R} \times \vec{dl}}{R^3} \right| = \frac{1}{r^2} r d\theta = \frac{d\theta}{r}$$

$$H = \frac{I}{4\pi} \int \frac{d\theta}{r}$$

If P is the focus of the parabola and the apex A is the point $r = d$, $\theta = 0$, then $r(1+\cos\theta) = 2d$. Hence at P,

$$H = \frac{I}{4\pi} \int_{-\pi}^{\pi} \frac{(1+\cos\theta)}{2d} d\theta = \frac{I}{8\pi d} (\theta - \sin\theta)\Big|_{-\pi}^{\pi}$$

$$= \frac{I}{4d} = \frac{3}{0.1} = \underline{\underline{30 \text{ A/m}}}.$$

Prob. 7.13

From Example 7.3, $\vec{H}$ due to circular loop is

$$\vec{H} = \frac{I\rho^2}{2(\rho^2 + z^2)} \vec{a}_z.$$

(a) $$\vec{H}(0,0,0) = \frac{5 \times 2^2}{2(2^2+0^2)^{3/2}} \vec{a}_z + \frac{5 \times 2^2}{2(2^2+4^2)^{3/2}} \vec{a}_z$$

$$= \underline{\underline{1.36 \vec{a}_z \text{ A/m}}}.$$

(b) $\vec{H}(0,0,2) = 2\,\dfrac{5\times 2^2}{2(2^2+2^2)^{3/2}}\,\vec{a}_z$

$= \underline{\underline{0.884\,\vec{a}_z\ \text{A/m}}}$.

Prob. 7.14

Let $\vec{H} = \vec{H}_L + \vec{H}_s$

where $\vec{H}_L = \dfrac{I}{2\pi\rho}\,\vec{a}_\phi$, $I = 50\pi$ mA,

$\rho = |(3,4,5)-(3,1,4)| = \sqrt{10}$

$\vec{a}_\phi = \vec{a}_l \times \vec{a}_\rho = \vec{a}_x \times \left(\dfrac{3\vec{a}_y+\vec{a}_z}{\sqrt{10}}\right) = \dfrac{-\vec{a}_y+3\vec{a}_z}{\sqrt{10}}$

$\vec{H}_L = \dfrac{50\pi}{2\pi\times 10}\left(-\vec{a}_y+3\vec{a}_z\right) = -2.5\vec{a}_y + 7.5\,\vec{a}_z$ mA/m

$\vec{H}_s = \frac{1}{2}\left(20\,\vec{a}_x \times \vec{a}_z\right) = -10\vec{a}_y$ mA/m.

Thus $\vec{H} = \vec{H}_L + \vec{H}_s = \underline{\underline{-12.5\,\vec{a}_y + 7.5\,\vec{a}_z\ \text{mA/m}}}$.

Prob. 7.15

(a) See text.

(b) for $0<\rho<a$, $\oint \vec{H}\cdot d\vec{l} = H_\phi \cdot 2\pi\rho = I\cdot\dfrac{\pi\rho^2}{\pi a^2}$

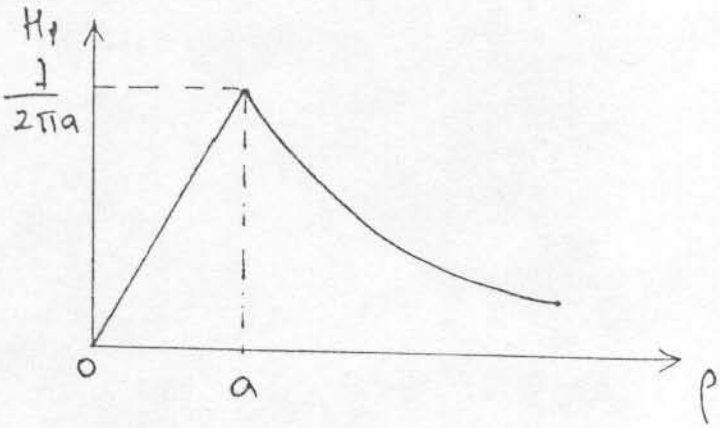
$H_\phi = \dfrac{I\rho}{2\pi a^2}$.

for $\rho>a$, $\oint \vec{H}\cdot d\vec{l} = H_\phi \cdot 2\pi\rho = I$

or $H_\phi = \dfrac{I}{2\pi\rho}$

Thus $\vec{H} = \begin{cases} \dfrac{I\rho}{2\pi a^2}\,\vec{a}_\phi, & 0 < \rho < a \\ \dfrac{I}{2\pi\rho}\,\vec{a}_\phi, & \rho > a \end{cases}$

$|\vec{H}|$ is sketched below.



Prob. 7.16

Applying Ampere's law,

$$\oint \vec{H}\cdot d\vec{l} = I = \int \vec{J}\cdot d\vec{S}$$

For $\rho < a$, $H_\phi \cdot 2\pi\rho = J_o \cdot \dfrac{2\pi\rho^3}{3} \rightarrow H_\phi = J_o \dfrac{\rho^2}{3}$

for $\rho > a$, $H_\phi \cdot 2\pi\rho = J_o \cdot \dfrac{2\pi a^3}{3} \rightarrow H_\phi = J_o \dfrac{a^3}{\rho}$

Thus, $H_\phi = \begin{cases} J_o \rho^2/3, & \rho < a \\ \dfrac{J_o a^3}{3\rho}, & \rho > a. \end{cases}$

$\boxed{\text{Prob. 7.17}}$

(a) Applying Ampere's law,

$$H_\phi \cdot 2\pi\rho = I \cdot \frac{\pi\rho^2}{\pi a^2} \rightarrow H_\phi = \frac{I\rho}{2\pi a^2}$$

i.e. $$\vec{H} = \frac{I\rho}{2\pi a^2}\vec{a}_\phi .$$

$$\vec{J} = \nabla \times \vec{H} = -\underbrace{\cancel{\frac{\partial H_\phi}{\partial z}}}_{0}\vec{a}_\rho + \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho H_\phi)\, a_z$$

$$= \frac{1}{\rho}\frac{I}{2\pi a^2}\cdot 2\rho\vec{a}_z = \frac{I}{\pi a^2}\vec{a}_z .$$

(b) from Prob. 7.15,

$$H_\phi = \begin{cases} \dfrac{I\rho}{2\pi a^2}, & \rho < a \\ \dfrac{I}{2\pi\rho}, & \rho > a \end{cases}$$

At (0, 1cm, 0),

$$H_\phi = \frac{3 \times 1 \times 10^{-2}}{2\pi \times 4 \times 10^{-4}} = \frac{300}{8\pi}$$

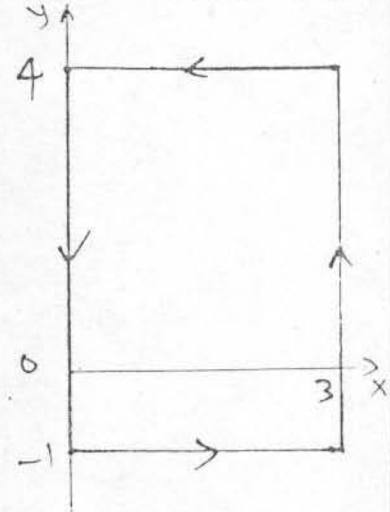
$$\vec{H} = 11.94\,\vec{a}_\phi \text{ A/m.}$$

At (0, 4cm, 0),

$$H_\phi = \frac{3}{2\pi \times 4 \times 10^{-2}} = \frac{300}{8\pi}$$

$$\vec{H} = 11.94\,\vec{a}_\phi \text{ A/m}$$

$\boxed{\text{Prob. 7.18}}$



(a) $\vec{J} = \nabla \times \vec{H} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix}$

$\vec{J} = \underline{\underline{-2\vec{a}_z \ A/m^2}}$.

(b) $\oint \vec{H}\cdot d\vec{l} = I_{enc}$

$I_{enc} = \int \vec{J}\cdot d\vec{S} = \int_{x=0}^{3}\int_{y=-1}^{4} (-2)\,dx\,dy = (-2)(3)(5) = -30\,A$

$\oint \vec{H}\cdot d\vec{l} = \int_0^3 y\,dx \Big|_{y=-1} + \int_{y=-1}^{4} (-x)\,dy \Big|_{x=3} + \int_3^0 y\,dx \Big|_{y=4}$

$+ \int_4^{-1} (-x)\,dy \Big|_{x=0} = (-1)(3) + (-3)(5) + (4)(-3)$

$= -30\,A$

Thus $\oint \vec{H}\cdot d\vec{l} = I_{enc} = \underline{\underline{-30\,A}}$.

$\boxed{\text{Prob. 7.19}}$

(a) $\vec{J} = \nabla \times \vec{H} = -2y(x+1)\vec{a}_x + (y^2+z^2)\vec{a}_y$

$\vec{J}(1,0,-3) = \underline{\underline{9\vec{a}_y \ A/m^2}}$.

(b) $I = \int \vec{J}\cdot d\vec{S} = \int_{x=0}^{1}\int_{z=0}^{1} (1+z^2)\,dx\,dz = (1)\left(z+\frac{z^3}{3}\right)\Big|_0^1 = \frac{4}{3}$

$I = \underline{\underline{1.333\ A}}$

Prob. 7.20

(a) $\vec{B} = \dfrac{\mu_o I}{2\pi\rho}\,\vec{a}_\phi$. At $(-3, 4, 5)$, $\rho = 5$

$$\vec{B} = \frac{4\pi \times 10^{-7} \times 2}{2\pi(5)}\,\vec{a}_\phi = \underline{\underline{80\,\vec{a}_\phi \ nWb/m^2}}$$

(b) $$\Phi = \int \vec{B}\cdot d\vec{S} = \frac{\mu_o I}{2\pi}\iint \frac{d\rho\, dz}{\rho}$$

$$= \frac{4\pi \times 10^{-7} \times 2}{2\pi}\ \ln\rho\,\Big|_2^6\ z\,\Big|_0^4 = 16 \times 10^{-7}\ln 3$$

$$= \underline{\underline{1.756\ \mu Wb}}.$$

Prob. 7.21

$$\psi = \int \vec{B}\cdot d\vec{S} = \mu_o \int_{\phi=0}^{\pi/2}\int_{z=0}^{2} \frac{10^6}{\rho}\cos\phi\ \rho\, d\phi\, dz$$

$$= 4\pi \times 10^{-7} \times 10^6 \int_0^2 dz \int_0^{\pi/2} \cos\phi\, d\phi = 0.4\pi(2)(1)$$

$$\psi = \underline{\underline{2.513\ Wb}}.$$

Prob. 7.22

$$\psi = \int \vec{B}\cdot d\vec{S} = \mu_o \int_{z=0}^{0.2}\int_{\phi=0}^{50^\circ} \frac{10^6}{\rho}\sin 2\phi\ \rho\, d\phi\, dz$$

$$\Psi = 4\pi \times 10^{-7} \times 10^{6} (0.2) \left(-\frac{\cos 2\phi}{2}\right)\Big|_0^{50°}$$

$$= 0.04\pi (1 - \cos 100°) = \underline{\underline{0.1475 \text{ Wb.}}}$$

Prob 7.23

Let $\vec{H} = \vec{H}_1 + \vec{H}_2$

where $\vec{H}_1$ and $\vec{H}_2$ are due to the wires centered at $x=0$ and $x=10$ cm respectively.

(a) for $\vec{H}_1$, $\rho = 5$ cm, $\vec{a}_\phi = \vec{a}_l \times \vec{a}_\rho = \vec{a}_z \times \vec{a}_x = \vec{a}_y$

$$\vec{H}_1 = \frac{5}{2\pi(5 \times 10^{-2})}\vec{a}_y = \frac{50}{\pi}\vec{a}_y$$

for $\vec{H}_2$, $\rho = 5$ cm, $\vec{a}_\phi = -\vec{a}_z \times -\vec{a}_x = \vec{a}_y$, $\vec{H}_2 = \vec{H}_1$

$$\vec{H} = 2\vec{H}_1 = \frac{100}{\pi}\vec{a}_y = \underline{\underline{31.43\,\vec{a}_y \text{ A/m.}}}$$

(b) for $\vec{H}_1$, $\vec{a}_\phi = \vec{a}_z \times \left(\frac{2\vec{a}_x + \vec{a}_y}{\sqrt{5}}\right) = \frac{2\vec{a}_y - \vec{a}_x}{\sqrt{5}}$

$$\vec{H}_1 = \frac{5}{2\pi 5\sqrt{5} \times 10^{-2}}\left(\frac{-\vec{a}_x + 2\vec{a}_y}{\sqrt{5}}\right) = -3.183\,\vec{a}_x + 6.366\,\vec{a}_y$$

for $\vec{H}_2$, $\vec{a}_\phi = -\vec{a}_z \times \vec{a}_y = \vec{a}_x$.

$$\vec{H}_2 = \frac{5}{2\pi(5)}\vec{a}_x = 15.924\,\vec{a}_x$$

$$\vec{H} = \vec{H}_1 + \vec{H}_2 = \underline{\underline{12.79\,\vec{a}_x + 6.366\,\vec{a}_y \text{ A/m.}}}$$

Prob. 7.24

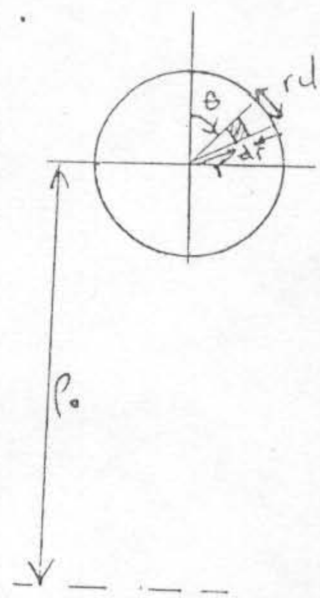
The stoke of lightning may be regarded as a semi-infinite filamentary current.

$$B = \frac{\mu_o I}{4\pi\rho} = \frac{4\pi \times 10^{-7} \times 50\times 10^{3}}{4\pi\,(100)}$$

$$= \underline{\underline{50\ \mu Wb/m^2}}$$

Prob. 7.25

(a) $$\vec{B} = \frac{\mu_o N I}{2\pi\rho}, \quad \rho_o - a < \rho < \rho_o + a.$$



from the figure,

$$\rho = \rho_o + r\cos\theta$$

$$ds = r\,d\theta\,dr$$

$$\Psi = \int \vec{B}\cdot d\vec{s} = \int_{r=0}^{a}\int_{\theta=0}^{2\pi} \frac{\mu_o N I\, r\, d\theta\, dr}{2\pi\,(\rho_o + r\cos\theta)}.$$

$$= \frac{\mu_o N I}{2\pi}\int_0^a dr \int_0^{2\pi} \frac{r\,d\theta}{(\rho_o + r\cos\theta)}$$

$$= \frac{\mu_o N I}{2\pi}\int_0^a \frac{2\pi}{\sqrt{\rho_o^2 - r^2}}\, r\,dr = \mu_o N I\left[-(\rho^2 - r^2)\right]\Big|_0^a$$

$$\Psi = \mu_o N I\left[\rho_o - (\rho_o^2 - a^2)^{\frac{1}{2}}\right]$$ as required.

If $\rho_o >> a$, by binomial expansion

$$\rho_o - (\rho_o^2 - a^2)^{1/2} = \rho_o - \left(\rho_o - \frac{1}{2}\frac{a^2}{\rho_o} + \cdots\right) = \frac{a^2}{2\rho_o}.$$

Hence
$$\psi = \underline{\underline{\frac{\mu_o N I a^2}{2\rho_o}}}$$

(b)
$$\psi = \int \bar{B}\cdot d\vec{S} = \frac{\mu_o N I}{2\pi}\int_{z=0}^{5}\int_{\rho=\rho_o - a/2}^{\rho_o + \frac{a}{2}} \frac{d\rho\, dz}{\rho}$$

$$= \frac{\mu_o N I a}{2\pi}\ln\frac{\rho_o + \frac{1}{2}a}{\rho_o - \frac{1}{2}a} = \underline{\underline{\frac{\mu_o N I a}{2\pi}\ln\frac{2\rho_o + a}{2\rho_o - a}}}$$

$\boxed{\text{Prob. 7.26}}$

On the slant side of the ring, $z = \frac{h}{b}(\rho - a)$

$$\psi = \int \bar{B}\cdot d\vec{S} = \int \frac{\mu_o I}{2\pi\rho}\, d\rho\, dz$$

$$= \frac{\mu_o I}{2\pi}\int_{\rho=a}^{a+b}\int_{z=0}^{\frac{h}{b}(\rho-a)} \frac{dz\, d\rho}{\rho} = \frac{\mu_o I h}{2\pi b}\int_a^{a+b}\left(1 - \frac{a}{\rho}\right)d\rho$$

$$= \frac{\mu_o I h}{2\pi b}\left(b - a\ln\frac{a+b}{a}\right)$$ as required.

If $a = 30\text{cm}$, $b = 10\text{cm}$, $h = 5\text{cm}$, $I = 10\text{A}$.

$$\psi = \frac{4\pi\times10^{-7}\times10\times0.05}{2\pi\times0.1}\left(0.1 - 0.3\ln\frac{4}{3}\right)$$

$$= \underline{\underline{1.37\times10^{-8}\ \text{Wb}}}$$

Prob. 7.27

(a) $\nabla \cdot \vec{A} = 0$, $\nabla \times \vec{A} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & -y & 0 \end{vmatrix} = 0$

i.e. $\vec{A}$ is $\underline{\underline{\vec{E}\text{-field in a charge-free region.}}}$

(b) $\nabla \cdot \vec{B} = \frac{1}{(x^2+y^2)^2}(-2xy + 2xy) = 0$, $\nabla \times \vec{B} \neq 0$

i.e. $\vec{B}$ is $\underline{\underline{\vec{H}\text{-field.}}}$

(c) $\nabla \cdot \vec{C} = e^{-y}(-\sin x + \sin x) = 0$

$\nabla \times \vec{C} = e^{-y}(-\cos x + \cos x) = 0$

i.e. $\vec{C}$ is $\underline{\underline{\vec{E}\text{-field in a charge-free region.}}}$

(d) $\nabla \cdot \vec{D} = 0$, $\nabla \times \vec{D} = -10\rho e^{-2z} \neq 0$

i.e. $\vec{D}$ is $\underline{\underline{\vec{H}\text{-field.}}}$

(e) $\nabla \cdot \vec{E} = \frac{4\cos\theta}{r^2} \neq 0$, $\nabla \times \vec{E} = \frac{2\sin\theta}{r^2}\vec{a}_\phi \neq 0$

i.e. $\vec{E}$ is $\underline{\underline{\text{neither electric nor magnetic.}}}$

Prob. 7.28

from eq. (7.44), $\vec{B} = \frac{\mu_o}{4\pi}\int_L \frac{I\,d\vec{l}' \times \vec{R}}{R^3}$

Using this leads to eq. (7.47), namely

$$\vec{B} = -\frac{\mu_o}{4\pi}\int I\vec{dl}' \times \nabla\left(\frac{1}{R}\right)$$

$$\nabla\cdot\vec{B} = -\frac{\mu_o I}{4\pi}\nabla\cdot\left[\int \vec{dl}' \times \nabla\left(\frac{1}{R}\right)\right]$$

Since $\int dl'$ operates on (x', y', z') and ∇ operates on (x,y,z), the two operators can be reversed in order, i.e.

$$\nabla\cdot\vec{B} = -\frac{\mu_o I}{4\pi}\int \nabla\cdot\left[\vec{dl}' \times \nabla\left(\frac{1}{R}\right)\right]$$

But $\nabla\cdot(\vec{A}\times\vec{C}) = \vec{C}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{C})$.

Taking $\vec{A} = \vec{dl}'$ and $\vec{C} = \nabla\left(\frac{1}{R}\right)$,

$$\nabla\cdot\vec{B} = -\frac{\mu_o I}{4\pi}\int\left[\nabla\left(\frac{1}{R}\right)\cdot\underbrace{\left(\nabla\times\vec{dl}'\right)}_{0} - \vec{dl}'\cdot\underbrace{\left(\nabla\times\nabla\left(\frac{1}{R}\right)\right)}_{0}\right] = 0$$

Since ∇ operates on (x,y,z) and $\vec{dl}'$ depends on (x', y', z'), $\nabla\times\vec{dl}' = 0$. Also, since $\nabla\times\nabla V = 0$, $\nabla\times\nabla\left(\frac{1}{R}\right) = 0$. Thus

$$\nabla\cdot\vec{B} = 0$$

Prob. 7.29

(a) $\vec{B} = \nabla\times\vec{A} = \underline{-\left(e^{-z}\cos y + \cos x\right)\vec{a}_y + e^{-z}\sin y\,\vec{a}_z \text{ Wb/m}^2}$.

(b) $\psi = \int_0^\pi\int_0^\pi e^{-z}\sin y\,dx\,dy\Big|_{z=0} = (1)(\pi)(-\cos\pi + \cos 0)$
$= 2\pi = \underline{\underline{6.283 \text{ Wb}}}$.

Prob. 7.30

$$\vec{B} = \nabla \times \vec{A} = \frac{1}{\rho}\frac{\partial A_z}{\partial \phi}\vec{a}_\rho - \frac{\partial A_z}{\partial \rho}\vec{a}_\phi$$

$$= \frac{15}{\rho}e^{-\rho}\cos\phi\,\vec{a}_\rho + 15e^{-\rho}\sin\phi\,\vec{a}_\phi$$

$$\vec{B}(3, \tfrac{\pi}{4}, -10) = 5e^{-3}\frac{1}{\sqrt{2}}\vec{a}_\rho + 15e^{-3}\frac{1}{\sqrt{2}}\vec{a}_\phi$$

$$\vec{H} = \frac{\vec{B}}{\mu_o} = \frac{10^7}{4\pi}\frac{15}{\sqrt{2}}e^{-3}\left(\frac{1}{3}\vec{a}_\rho + \vec{a}_\phi\right)$$

$$\vec{H} = \underline{\underline{(14\,\vec{a}_\rho + 42\,\vec{a}_\phi)\cdot 10^4}} \text{ A/m.}$$

$$\psi = \int \vec{B}\cdot d\vec{S} = \iint \frac{15}{\rho}e^{-\rho}\cos\phi\,\rho\,d\phi\,dz$$

$$= 15z\Big|_0^{10}(-\sin\phi)\Big|_0^{\pi/2}e^{-5} = -150e^{-5}$$

$$\psi = \underline{\underline{-1.011 \text{ Wb.}}}$$

Prob. 7.31

$$\vec{A} = \frac{\mu_o I}{4\pi}\int\frac{d\vec{l}}{R},$$

$\vec{R} = (x, y, z) - (0, 0, z')$, $R = \sqrt{x^2 + y^2 + (z - z')^2}$. At a distant point, $z >> z'$ so that

$$R \simeq \sqrt{x^2 + y^2 + z^2} = r.$$

Hence, $\vec{A} = \dfrac{\mu_o I}{4\pi r}\displaystyle\int_{-L}^{L} dz'\,\vec{a}_z = \dfrac{\mu_o I L}{2\pi r}\,\vec{a}_z$

$$\vec{A} = \frac{\mu_o I L \,\vec{a}_z}{2\pi (x^2+y^2+z^2)^{1/2}}$$

$$\vec{B} = \nabla \times \vec{A} = \frac{\mu_o I L}{2\pi} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & \frac{1}{r} \end{vmatrix}$$

$$\frac{\partial}{\partial y}\left(\frac{1}{r}\right) = \frac{-y}{r^3}, \quad \frac{\partial}{\partial x}\left(\frac{1}{r}\right) = -\frac{x}{r^3}$$

$$\vec{B} = \frac{\mu_o I L}{2\pi r^3}(-y\,\vec{a}_x + x\,\vec{a}_y) = \frac{\mu_o I L(-y\,\vec{a}_x + x\,\vec{a}_y)}{2\pi (x^2+y^2+z^2)^{3/2}}$$

$\boxed{\text{Prob. 7.32}}$

$$\vec{H} = \frac{\nabla \times \vec{A}}{\mu_o} = \frac{I}{2\pi\rho}\,\vec{a}_\rho$$

$$\nabla \times \vec{A} = \frac{\mu_o I}{2\pi\rho}\,\vec{a}_\phi = \Bigg[\underbrace{\frac{\partial A_\rho}{\partial z}}_{0} - \frac{\partial A_z}{\partial \rho}\Bigg]\vec{a}_\phi$$

$$\frac{dA_z}{d\rho} = -\frac{\mu_o I}{2\pi\rho} \longrightarrow A_z = -\frac{\mu_o I}{2\pi}\ln\rho$$

$$\vec{A} = -\frac{\mu_o I}{2\pi}\ln\rho\,\vec{a}_z.$$

$\boxed{\text{Prob. 7.33}}$

$$\vec{A} = -\frac{I_o\mu_o}{4\pi a^2}(x^2+y^2)\,\vec{a}_z = -\frac{I_o\mu_o\rho^2\,\vec{a}_z}{4\pi a^2}$$

$$\vec{B} = \nabla \times \vec{A} = \frac{I_o\mu_o\rho}{2\pi a^2}\,\vec{a}_\phi = \mu_o\vec{H}$$

i.e. $$\vec{H} = \frac{I_o\rho}{2\pi a^2}\,\vec{a}_\phi = \frac{I_o\sqrt{x^2+y^2}}{2\pi a^2}\,\vec{a}_\phi.$$

By Ampere's law, $\oint \vec{H}\cdot d\vec{l} = I_{enc}$

$$H_\phi \cdot 2\pi\rho = I_o \cdot \frac{\rho^2}{a^2}$$

or $$\vec{H} = \frac{I_o \rho}{2\pi a^2}\,\vec{a}_\phi \,.$$

$\boxed{\text{Prob. 7.34}}$

$$\vec{J} = \nabla\times\vec{H} = \nabla\times\frac{\nabla\times\vec{A}}{\mu_o} = \frac{1}{\mu_o}\nabla\times\nabla\times\vec{A}$$

$$\nabla\times\vec{A} = -\frac{\partial A_z}{\partial\rho}\vec{a}_\phi = +\frac{20}{\rho^3}\vec{a}_\phi$$

$$\nabla\times\nabla\times\vec{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\phi)\,\vec{a}_z = -\frac{40}{\rho^4}\vec{a}_z$$

$$\vec{J} = -\frac{40}{\mu_o\rho^4}\vec{a}_z \ \text{A/m}^2$$

or $\nabla^2\vec{A} = -\mu_o\vec{J}$

or $$\vec{J} = -\frac{1}{\mu_o}\nabla^2\vec{A} = -\frac{1}{\mu_o}\nabla^2 A_z\,\vec{a}_z$$

$$= -\frac{1}{\mu_o}\vec{a}_z\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial A_z}{\partial\rho}\right) + \frac{1}{\rho^2}\underbrace{\frac{\partial^2 A_z}{\partial\phi^2}}_{0} + \underbrace{\frac{\partial^2 A_z}{\partial z^2}}_{0}\right]$$

$$= \frac{1}{\mu_o}\vec{a}_z\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(-\frac{20}{\rho^2}\right) = -\frac{40\,\vec{a}_z}{\mu_o\rho^4} \ \text{A/m}^2.$$

$\boxed{\text{Prob. 7.35}}$

From Fig. 7.5, $$d\vec{A} = \frac{\mu_o I\,d\vec{l}}{4\pi R}$$

where $\vec{dl} = dz\,\vec{a}_z$, $\vec{R} = \sqrt{\rho^2 + z^2}$.

$$d\vec{A} = \frac{\mu_o I\,dz\,\vec{a}_z}{4\pi(\rho^2 + z^2)^{1/2}}$$

$$d\vec{B} = \nabla \times d\vec{A} = -\frac{\partial}{\partial\rho}\,dA_z\,\vec{a}_\phi = \frac{\mu_o I\,dz}{4\pi}\,\frac{\rho}{(\rho^2+z^2)^{3/2}}\,\vec{a}_\phi$$

$$\vec{B} = \frac{\mu_o I\,\vec{a}_\phi}{4\pi}\int \frac{\rho\,dz}{(\rho^2+z^2)^{3/2}}.$$

Let $z = \rho\cot\alpha$, $dz = \rho\,\mathrm{cosec}^2\alpha\,d\alpha$

$$\vec{H} = \frac{\vec{B}}{\mu} = \frac{I\,\vec{a}_\phi}{4\pi}\int_{\alpha_1}^{\alpha_2} \frac{\rho^2\,\mathrm{cosec}^2\alpha\,d\alpha}{\rho^3\,\mathrm{cosec}^3\alpha}$$

$$= \frac{I}{4\pi}(\cos\alpha_2 - \cos\alpha_1)\,\vec{a}_\phi.$$

$\boxed{\text{Prob. 7.36}}$

$$\vec{H} = -\nabla V_m \;\rightarrow\; V_m = -\int \vec{H}\cdot d\vec{l}$$

from Example 7.3, $\vec{H} = \dfrac{Ia^2}{2(z^2+a^2)^{3/2}}\,\vec{a}_z$

$$V_m = -\frac{Ia^2}{2}\int (z^2+a^2)^{-3/2}\,dz = \frac{-Iz}{2(z^2+a^2)^{1/2}} + c$$

As $z \rightarrow \infty$, $V_m = 0$, i.e.

$$0 = -\frac{I}{2} + c \;\rightarrow\; c = \frac{I}{2}.$$

Hence,

$$V_m = \frac{I}{2}\left[1 - \frac{z}{\sqrt{z^2+a^2}}\right].$$

Prob 7.37

for the outer conductor,

$$J_z = \frac{-I}{\pi(c^2-b^2)} = \frac{-I}{\pi(16-9)a^2} = \frac{-I}{7\pi a^2}$$

Let $\vec{A} = A_z \vec{a}_z$. Using Poisson's equation,

$$\nabla^2 A_z = -\mu_o J_z$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial A_z}{\partial\rho}\right) = \frac{\mu_o I}{7a^2\pi}$$

or $$\frac{d}{d\rho}\left(\rho\frac{dA_z}{d\rho}\right) = \frac{\mu_o I\rho}{7\pi a^2}$$

Integrating once,

$$\rho\frac{dA_z}{d\rho} = \frac{\mu_o I\rho^2}{14\pi a^2} + c_1$$

or $$\frac{dA_z}{d\rho} = \frac{\mu_o I\rho}{14\pi a^2} + \frac{c_1}{\rho}$$

Integrating again,

$$A_z = \frac{\mu_o I\rho^2}{28\pi a^2} + c_1\ln\rho + c_2$$

But $A_z = 0$ when $\rho = 3a$.

$$0 = \frac{9}{28\pi}\mu_o I + c_1\ln 3a + c_2$$

$$c_2 = -c_1\ln 3a - \frac{9}{28\pi}\mu_o I$$

i.e. $$A_z = \frac{\mu_o I}{28\pi}\left(\frac{\rho^2}{a^2} - 9\right) + c_1\ln\frac{\rho}{3a}$$

But $\nabla \times \vec{A} = \vec{B} = \mu_o \vec{H}$

$$\nabla \times \vec{A} = -\frac{\partial A_z}{\partial \rho}\vec{a}_\phi = -\left(\frac{\mu_o I \rho}{14\pi a^2} + \frac{c_1}{\rho}\right)\vec{a}_\phi$$

At $\rho = 3a$, $\oint \vec{H}\cdot d\vec{l} = I \longrightarrow 2\pi(3a)\,H_\phi = I$

or $$H_\phi = \frac{I}{6\pi a}$$

Thus $\nabla \times \vec{A}\Big|_{\rho=3a} = \mu_o \vec{H}\,(\rho = 3a)$ implies that

$$-\left(\frac{3\mu_o I}{14\pi a} + \frac{c_1}{3a}\right) = \frac{\mu_o I}{6\pi a}$$

or $$c_1 = -\frac{I\mu_o}{2\pi} - \frac{9\mu_o I}{14\pi} = -\frac{16\mu_o I}{14\pi}$$

Thus,
$$\underline{\underline{A_z = \frac{\mu_o I}{28\pi}\left(\frac{\rho^2}{a^2} - 9\right) - \frac{8\mu_o I}{7\pi}\ln\frac{\rho}{3a}}}.$$

Prob. 7.38

For an infinite line current, $\vec{H} = \frac{I}{2\pi\rho}\vec{a}_\phi$.

But $\vec{H} = -\nabla V_m \quad (\vec{J} = 0)$

$$\frac{I}{2\pi\rho}\vec{a}_\phi = -\frac{1}{\rho}\frac{\partial V_m}{\partial \phi}\vec{a}_\phi \longrightarrow V_m = -\frac{I}{2\pi}\phi + C$$

(a) If $V_m = 0$ at $\phi = 0 \rightarrow C = 0$

$$V(6, \pi/4, 0) = -\frac{20}{2\pi}\cdot\frac{\pi}{4} = \underline{\underline{-2.5\ A}}.$$

$$+\frac{1}{\rho}\left(\frac{\partial^2 V}{\partial\rho\partial\phi}-\frac{\partial^2 V}{\partial\phi\partial\rho}\right)\vec{a}_z=0$$

(b) $$\nabla\cdot(\nabla\times\vec{A})=\nabla\cdot\left[\left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi}-\frac{\partial A_\phi}{\partial z}\right)\vec{a}_\rho\right.$$

$$\left.+\left(\frac{\partial A_\rho}{\partial z}-\frac{\partial A_z}{\partial\rho}\right)\vec{a}_\phi+\frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_\phi)-\frac{\partial A_\rho}{\partial\phi}\right)\vec{a}_z\right]$$

$$=\frac{1}{\rho}\frac{\partial^2 A_z}{\partial\rho\partial\phi}-\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial A_\phi}{\partial z}\right)+\frac{1}{\rho}\frac{\partial^2 A_\rho}{\partial\phi\partial z}-\frac{1}{\rho}\frac{\partial^2 A_z}{\partial\phi\partial\rho}$$

$$+\frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\phi)\right)-\frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial A_\rho}{\partial\phi}\right)$$

$$=-\frac{\partial^2 A_\phi}{\partial\rho\partial z}-\frac{1}{\rho}\frac{\partial A_\phi}{\partial z}+\frac{\partial^2 A_\phi}{\partial z\partial\rho}+\frac{1}{\rho}\frac{\partial A_\phi}{\partial z}=0$$

$\boxed{\text{Prob. 7.41}}$

(a) $$\nabla\times\psi\vec{F}=\left(\frac{\partial}{\partial y}(\psi F_z)-\frac{\partial}{\partial z}(\psi F_y)\right)\vec{a}_x$$

$$+\left(\frac{\partial}{\partial z}(\psi F_x)-\frac{\partial}{\partial x}(\psi F_z)\right)\vec{a}_y$$

$$+\left(\frac{\partial}{\partial x}(\psi F_y)-\frac{\partial}{\partial y}(\psi F_x)\right)\vec{a}_z$$

$$=\psi\left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right)\vec{a}_x+\psi\left(\frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x}\right)\vec{a}_y$$

$$+\psi\left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)\vec{a}_z+\left(F_z\frac{\partial\psi}{\partial y}-F_y\frac{\partial\psi}{\partial z}\right)\vec{a}_x$$

$$+\frac{1}{\rho}\left(\cancel{\frac{\partial^2 V}{\partial\rho\partial\phi}}-\frac{\partial^2 V}{\partial\phi\partial\rho}\right)\vec{a}_z=0$$

(b) $$\nabla\cdot(\nabla\times\vec{A})=\nabla\cdot\left[\left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi}-\frac{\partial A_\phi}{\partial z}\right)\vec{a}_\rho\right.$$

$$\left.+\left(\frac{\partial A_\rho}{\partial z}-\frac{\partial A_z}{\partial\rho}\right)\vec{a}_\phi+\frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_\phi)-\frac{\partial A_\rho}{\partial\phi}\right)\vec{a}_z\right]$$

$$=\frac{1}{\rho}\cancel{\frac{\partial^2 A_z}{\partial\rho\partial\phi}}-\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial A_\phi}{\partial z}\right)+\frac{1}{\rho}\cancel{\frac{\partial^2 A_\rho}{\partial\phi\partial z}}-\frac{1}{\rho}\cancel{\frac{\partial^2 A_z}{\partial\phi\partial\rho}}$$

$$+\frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\phi)\right)-\frac{\partial}{\partial z}\left(\frac{1}{\rho}\cancel{\frac{\partial A_\rho}{\partial\phi}}\right)$$

$$=-\cancel{\frac{\partial^2 A_\phi}{\partial\rho\partial z}}-\frac{1}{\rho}\cancel{\frac{\partial A_\phi}{\partial z}}+\cancel{\frac{\partial^2 A_\phi}{\partial z\partial\rho}}+\frac{1}{\rho}\cancel{\frac{\partial A_\phi}{\partial z}}=0$$

$\boxed{\text{Prob. 7.41}}$

(a) $$\nabla\times\psi\vec{F}=\left(\frac{\partial}{\partial y}(\psi F_z)-\frac{\partial}{\partial z}(\psi F_y)\right)\vec{a}_x$$

$$+\left(\frac{\partial}{\partial z}(\psi F_x)-\frac{\partial}{\partial x}(\psi F_z)\right)\vec{a}_y$$

$$+\left(\frac{\partial}{\partial x}(\psi F_y)-\frac{\partial}{\partial y}(\psi F_x)\right)\vec{a}_z$$

$$=\psi\left(\frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z}\right)\vec{a}_x+\psi\left(\frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x}\right)\vec{a}_y$$

$$+\psi\left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)\vec{a}_z+\left(F_z\frac{\partial\psi}{\partial y}-F_y\frac{\partial\psi}{\partial z}\right)\vec{a}_x$$

$$+\left(F_x \frac{\partial \psi}{\partial z} - F_z \frac{\partial \psi}{\partial x}\right)\vec{a}_y + \left(F_y \frac{\partial \psi}{\partial x} - F_x \frac{\partial \psi}{\partial y}\right)\vec{a}_z$$

$$= \underline{\psi \nabla \times \vec{F} + \nabla \psi \times \vec{F}}$$

(b) $$\nabla \times \nabla \times \vec{A} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) & \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) & \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \end{vmatrix}$$

$$= \left(\frac{\partial^2 A_y}{\partial x \partial y} + \frac{\partial^2 A_z}{\partial x \partial z} - \frac{\partial^2 A_x}{\partial z^2} - \frac{\partial^2 A_x}{\partial^2 y^2}\right)\vec{a}_x$$

$$-\left(\frac{\partial^2 A_y}{\partial z^2} + \frac{\partial^2 A_y}{\partial x^2} - \frac{\partial^2 A_z}{\partial y \partial z} - \frac{\partial^2 A_x}{\partial x \partial y}\right)\vec{a}_y$$

$$-\left(\frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial x^2} - \frac{\partial^2 A_y}{\partial z \partial y} - \frac{\partial^2 A_z}{\partial x \partial z}\right)\vec{a}_z$$

$$\nabla(\nabla \cdot \vec{A}) = \left(\frac{\partial}{\partial x}\vec{a}_x + \frac{\partial}{\partial y}\vec{a}_y + \frac{\partial}{\partial z}\vec{a}_z\right)\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)$$

$$= \left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_y}{\partial x \partial y} + \frac{\partial^2 A_z}{\partial x \partial z}\right)\vec{a}_x$$

$$+ \left(\frac{\partial^2 A_x}{\partial y \partial x} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_z}{\partial y \partial z}\right)\vec{a}_y$$

$$+ \left(\frac{\partial^2 A_x}{\partial z \partial x} + \frac{\partial^2 A_y}{\partial z \partial y} + \frac{\partial^2 A_z}{\partial z^2}\right)\vec{a}_z$$

$$\nabla^2 \vec{A} = \nabla^2 A_x \vec{a}_x + \nabla^2 A_y \vec{a}_y + \nabla^2 A_z \vec{a}_z$$

$$= \left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2}\right)\vec{a}_x +$$

$$\left(\frac{\partial^2 A_y}{\partial x^2}+\frac{\partial^2 A_y}{\partial y^2}+\frac{\partial^2 A_y}{\partial z^2}\right)\vec{a}_y+\left(\frac{\partial^2 A_z}{\partial x^2}+\frac{\partial^2 A_z}{\partial y^2}+\frac{\partial^2 A_z}{\partial z^2}\right)\vec{a}_z$$

$$\nabla(\nabla\cdot\vec{A})-\nabla^2\vec{A}=\left(\frac{\partial^2 A_y}{\partial x\partial y}+\frac{\partial^2 A_z}{\partial x\partial z}-\frac{\partial^2 A_x}{\partial y^2}-\frac{\partial^2 A_z}{\partial z^2}\right)\vec{a}_x$$

$$+\left(\frac{\partial^2 A_x}{\partial x\partial y}+\frac{\partial^2 A_z}{\partial z\partial y}-\frac{\partial^2 A_y}{\partial y^2}-\frac{\partial^2 A_y}{\partial z^2}\right)\vec{a}_z$$

$$=\nabla\times\nabla\times\vec{A}.$$

Alternatively, we may use the bac-cab rule;

$$\vec{A}\times(\vec{B}\times\vec{C})=\vec{B}\,(\vec{A}\cdot\vec{C})-\vec{C}\,(\vec{A}\cdot\vec{B})$$

and let $\vec{A}\rightarrow\nabla$, $\vec{B}\rightarrow\nabla$, $\vec{C}\rightarrow\vec{A}$, we obtain

$$\nabla\times\nabla\times\vec{A}=\nabla(\nabla\cdot\vec{A})-\vec{A}\,(\nabla\cdot\nabla)$$

$$=\nabla(\nabla\cdot\vec{A})-\nabla^2\vec{A}.$$

$\boxed{\text{Prob. 7.42}}$

$$R=|\vec{r}-\vec{r}'|=\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{1/2}$$

$$\nabla\frac{1}{R}=\left(\frac{\partial}{\partial x}\vec{a}_x+\frac{\partial}{\partial y}\vec{a}_y+\frac{\partial}{\partial z}\vec{a}_z\right)\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-1/2}$$

$$=-\frac{1}{2}\,2(x-x')\vec{a}_x\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-3/2}$$

$$+\ \dots\ +\ \dots$$

$$=-\left[(x-x')\vec{a}_x+(y-y')\vec{a}_y+(z-z')\vec{a}_z\right]\Big/R^3=-\frac{\vec{R}}{R^3}.$$

$$\nabla' \frac{1}{R} = \left(\frac{\partial}{\partial x'}\vec{a}_x + \frac{\partial}{\partial y'}\vec{a}_y + \frac{\partial}{\partial z'}\vec{a}_z\right)\left[(x-x')^2+(y-y')^2 + (z-z')^2\right]^{-1/2}$$

$$= \left(-\tfrac{1}{2}\right)(-2)(x-x')\vec{a}_x\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-3/2}$$

$$= \frac{\vec{R}}{R^3}.$$

CHAPTER 8

P.E. 8.1

(a) $F = m\dfrac{d\vec{u}}{dt} = Q\vec{E} = \underline{\underline{6\,\vec{a}_z\ N}}$

(b) $\dfrac{d\vec{u}}{dt} = 6\,\vec{a}_z = \dfrac{d}{dt}(u_x, u_y, u_z) \Longrightarrow$

$$\frac{du_x}{dt} = 0 \longrightarrow u_x = A$$

$$\frac{du_y}{dt} = 0 \longrightarrow u_y = B.$$

$$\frac{du_z}{dt} = 6 \longrightarrow u_z = 6t + C$$

Since $\vec{u}(t=0) = 0$, $A = B = C = 0$

$$u_x = 0 = u_y, \quad u_z = 6t$$

$$u_x = \frac{dx}{dt} = 0 \longrightarrow x = A_1$$

$$u_y = \frac{dy}{dt} = 0 \longrightarrow y = B_1$$

$$u_z = \frac{dz}{dt} = 6t \longrightarrow z = 3t^2 + c_1$$

At $t=0$, $(x,y,z) = (0,0,0) \longrightarrow A_1 = 0 = B_1 = c_1$

Hence $(x,y,z) = (0,0,3t^2)$, $\vec{u} = 6t\,\vec{a}_z$ at any time. At $P(0,0,12)$, $z = 12 = 3t^2 \rightarrow t = 2s$

$$\underline{\underline{t = 2s}}$$

(c) $\vec{u} = 6t\,\vec{a}_z = \underline{\underline{12\vec{a}_z \text{ m/s}}}$.

$$\vec{a} = \frac{d\vec{u}}{dt} = \underline{\underline{6\vec{a}_z \text{ m/s}^2}}.$$

(d) $K \cdot E = \frac{1}{2} m|\vec{u}|^2 = \frac{1}{2}(1)(144) = \underline{\underline{72\ J}}$.

$\boxed{\text{P.E. 8.2}}$

(a) $m\vec{a} = e\vec{u} \times \vec{B} = (eB_o u_y, -eB_o u_x, 0)$

$$\frac{d^2x}{dt^2} = \frac{eB_o}{m}\frac{dy}{dt} = w\frac{dy}{dt} \quad — (1)$$

$$\frac{d^2y}{dt^2} = -\frac{eB_o}{m}\frac{dx}{dt} = -w\frac{dx}{dt} \quad — (2)$$

$$\frac{d^2z}{dt^2} = 0 \longrightarrow \frac{dz}{dt} = c_1 \quad — (3)$$

from (1) and (2),

$$\frac{d^3x}{dt^3} = w\frac{d^2y}{dt^2} = -w^2\frac{dx}{dt}$$

$$(D^3 + w^2 D)X = 0 \longrightarrow DX = (0, \pm jw)x$$

$x = c_2 + c_3 \cos wt + c_4 \sin wt$

$\frac{dx}{dt} = -c_3 w \sin wt + c_4 w \cos wt$

$\frac{dy}{dt} = \frac{1}{w}\frac{d^2x}{dt^2} = -c_3 w \cos wt - c_4 w \sin wt$

At $t=0$, $\vec{u} = (\alpha, 0, \beta)$. Hence

$c_1 = \beta, \quad c_3 = 0, \quad c_4 = \frac{\alpha}{w}$

$$\underline{\frac{dx}{dt} = \alpha \cos wt, \quad \frac{dy}{dt} = -\alpha \sin wt, \quad \frac{dz}{dt} = \beta.}$$

(b) Solving these yields

$x = \frac{\alpha}{w}\sin wt, \quad y = \frac{\alpha}{w}\cos wt, \quad z = \beta t.$

(c) $x^2 + y^2 = \frac{\alpha^2}{w^2}, \quad z = \beta t$

showing that the particles move along a helix of radius α/w placed along the z-axis.

$\boxed{\text{P.E. 8.3}}$

(a) from Example 8.3, $QuB = QE$ regardless of the sign of the charge.

$E = uB = 8 \times 10^6 \times 0.5 \times 10^{-3} = \underline{\underline{4 \text{ kV/m}}}.$

(b) Yes, since $QuB = QE$ holds for any Q and m.

P.E. 8.4

By Newton's 3rd law, $\vec{F}_{12} = -F_{21}$, the force on the infinitely long wire is

$$\vec{F}_l = -\vec{F} = \frac{\mu_o I_1 I_2 b}{2\pi}\left(\frac{1}{\rho_o} - \frac{1}{\rho_o + a}\right)\vec{a}_\rho$$

$$= \frac{4\pi \times 10^{-7} \times 50 \times 3}{2\pi}\left(\frac{1}{2} - \frac{1}{3}\right)\vec{a}_\rho = \underline{\underline{5\vec{a}_\rho \ \mu N.}}$$

P.E. 8.5

$$\vec{m} = IS\vec{a}_n = 10 \times 10^{-4} \times 50 \ \frac{(2, 6, -3)}{7}$$

$$= 7.143 \times 10^{-3} \ (2, 6, -3)$$

$$= \underline{\underline{(1.429\vec{a}_x + 4.286\vec{a}_y - 2.143\vec{a}_z) \times 10^{-2} \ A\text{-}m^2.}}$$

P.E. 8.6

(a) $$\vec{T} = \vec{m} \times \vec{B} = \frac{10 \times 10^{-4} \times 50}{7 \times 10}\begin{vmatrix} 2 & 6 & -3 \\ 6 & 4 & 5 \end{vmatrix}$$

$$= \underline{\underline{0.03\vec{a}_x - 0.02\vec{a}_y - 0.02\vec{a}_z \ N.m.}}$$

(b) $|\vec{T}| = ISB\sin\theta \longrightarrow |\vec{T}|_{max} = ISB$

$$|\vec{T}|_{max} = \frac{50 \times 10^{-2}}{10}\left|6\vec{a}_x + 4\vec{a}_y + 5\vec{a}_z\right| = 0.4387$$

or $$|\vec{T}|_{max} = |\vec{m} \times \vec{B}| = |-0.3055\vec{a}_x + 0.076\vec{a}_y + 0.3055\vec{a}_z|$$

$$= \underline{\underline{0.4387 \ N.m.}}$$

P.E. 8.7

(a) $\mu_r = \frac{\mu}{\mu_o} = 4.6$, $\chi_m = \mu_r - 1 = \underline{\underline{3.6}}$

(b) $\vec{H} = \frac{\vec{B}}{\mu} = \frac{10 \times 10^{-3} e^{-y}}{4\pi \times 10^{-7} \times 4.6} \vec{a}_z \text{ A/m} = \underline{\underline{1730 e^{-y} \vec{a}_z \text{ A/m}}}$

(c) $\vec{M} = \chi_m \vec{H} = \underline{\underline{6228 e^{-y} \hat{a}_z \text{ A/m}}}$.

P.E. 8.8

$$\vec{a}_n = \frac{3\vec{a}_x + 4\vec{a}_y}{5}$$

$$\vec{B}_{1n} = (\vec{B}_1 \cdot \vec{a}_n)\vec{a}_n = \frac{(6+32)(6\vec{a}_x + 8\vec{a}_y)}{1000}$$

$$= 0.228\vec{a}_x + 0.304\vec{a}_y = \vec{B}_{2n}$$

$$\vec{B}_{1t} = \vec{B}_1 - \vec{B}_{1n} = -0.128\vec{a}_x + 0.096\vec{a}_y + 0.2\vec{a}_z$$

$$\vec{B}_{2t} = \frac{\mu_2}{\mu_1}\vec{B}_{1t} = 10\vec{B}_{1t} = -1.28\vec{a}_x + 0.96\vec{a}_y + 2\vec{a}_z$$

$$\vec{B}_2 = \vec{B}_{2n} + \vec{B}_{2t} = \underline{\underline{-1.052\vec{a}_x + 1.264\vec{a}_y + 2\vec{a}_z \text{ Wb/m}^2}}$$

P.E. 8.9

(a) $\vec{B}_{1n} = \vec{B}_{2n} \longrightarrow \mu_1 \vec{H}_{1n} = \mu_2 \vec{H}_{2n}$

or $\mu_1 \vec{H}_1 \cdot \vec{a}_{n21} = \mu_2 \vec{H}_2 \cdot \vec{a}_{n21}$

$$\mu_o \frac{(60+2-36)}{7} = 2\mu_o \frac{(6H_{2x} - 10 - 12)}{7}$$

$$35 = 6H_{2x}$$

$$H_{2x} = 5.833$$

(b) $\vec{K} = (\vec{H}_1 - \vec{H}_2) \times \vec{a}_{n12} = \vec{a}_{n21} \times (\vec{H}_1 - \vec{H}_2)$

$$= \vec{a}_{n21} \times \left[(10,1,12) - \left(\frac{35}{6}, -5, 4\right) \right]$$

$$= \frac{1}{7} \begin{vmatrix} 6 & 2 & -3 \\ \frac{25}{6} & 6 & 8 \end{vmatrix}$$

$$\vec{K} = 4.86\,\vec{a}_x - 8.64\,\vec{a}_y + 3.95\,\vec{a}_z \text{ A/m.}$$

(c) Since $\vec{B} = \mu \vec{H}$, $\vec{B}_1$ and $\vec{H}_1$ are parallel, i.e. they make the same angle with the normal to the interface.

$$\cos\theta_1 = \frac{\vec{H}_1 \cdot \vec{a}_{n21}}{|\vec{H}_1|} = \frac{26}{7\sqrt{100+1+144}} = 0.2373$$

$$\theta_1 = 76.27^\circ$$

$$\cos\theta_2 = \frac{\vec{H}_2 \cdot \vec{a}_{n21}}{|\vec{H}_2|} = \frac{13}{7\sqrt{(5.833)^2+25+16}} = 0.2144$$

$$\theta_2 = 77.62^\circ.$$

P.E. 8.10

(a) $L' = \mu_0 \mu_r n^2 S = 4\pi \times 10^{-7} \times 1000 \times 16 \times 10^6 \times 4 \times 10^{-4}$

$$= 8.042 \text{ H/m}$$

(b) $W_m' = \frac{1}{2} L' I^2 = \frac{1}{2}(8.042)(0.5)^2 = 1.005 \text{ J/m.}$

P.E. 8.11

from Example 8.11,

$$L_{in} = \frac{8l}{8\pi}$$

$$L_{ext} = \frac{2W_m}{I^2} = \frac{1}{I^2}\iiint \frac{\mu I^2}{4\pi^2\rho^2}\,\rho\, d\rho\, d\phi\, dz$$

$$= \frac{1}{4\pi^2}\int_0^l dz \int_0^{2\pi} d\phi \int_a^b \frac{2\mu_0}{(1+\rho)\rho}\, d\rho$$

$$= \frac{2\mu_0}{4\pi^2}\cdot 2\pi l \int_a^b \left[\frac{1}{\rho} - \frac{1}{1+\rho}\right] d\rho$$

$$= \frac{\mu_0 l}{\pi}\left[\ln\frac{b}{a} - \ln\frac{1+b}{1+a}\right]$$

$$L = L_{in} + L_{ext} = \frac{\mu_0 l}{8\pi} + \frac{\mu_0 l}{\pi}\left[\ln\frac{b}{a} - \ln\frac{(1+b)}{(1+a)}\right]$$

P.E. 8.12

(a) $L'_{in} = \frac{\mu_0}{8\pi} = \frac{4\pi\times 10^{-7}}{8\pi} = \underline{\underline{0.05\ \mu H/m.}}$

$L'_{ext} = L' - L'_{in} = 1.2 - 0.05 = \underline{\underline{1.15\ \mu H/m.}}$

(b) $L' = \frac{\mu_0}{2\pi}\left[\frac{1}{4} + \ln\frac{d-a}{a}\right]$

$$\ln\frac{d-a}{a} = \frac{2\pi l'}{\mu_0} - 0.25 = \frac{2\pi\times 1.2\times 10^{-6}}{4\pi\times 10^{-7}} - 0.25$$

$$= 6 - 0.25 = 5.75$$

$$\frac{d-a}{a} = e^{5.75} = 314.19$$

$$d-a = 314.19a = 314.19 \times \frac{2.588\times10^{-3}}{2} = 406.6\,mm$$

$$d = 407.9\,mm = \underline{\underline{40.79\,cm.}}$$

P.E. 8.13

This is similar to Example 8.13. In this case, however, $h=0$ so that

$$\vec{A}_1 = \frac{\mu_0 I_1 a^2 b}{4b^3}\,\vec{a}_\phi$$

$$\Phi_{12} = \frac{\mu_0 I_1 a^2}{4b^2}\cdot 2\pi b = \frac{\mu_0 \pi I_1 a^2}{2b}$$

$$M_{12} = \frac{\Phi_{12}}{I_1} = \frac{\mu_0 \pi a^2}{b} = \frac{4\pi\times10^{-7}\times\pi\times4}{2\times3}$$

$$= \underline{\underline{2.632\ \mu H.}}$$

P.E. 8.14

$$L_{in} = \frac{\mu_0 l}{8\pi} = \frac{\mu_0 2\pi\rho_0}{8\pi} = \frac{4\pi\times10^{-7}\times10\times10^{-2}}{4}$$

$$= \underline{\underline{31.42\ nH.}}$$

P.E. 8.15

(a) From Example 7.6, $B_{ave} = \frac{\mu_0 N I}{l} = \frac{\mu_0 N I}{2\pi\rho_0}$

$$\Phi = B_{ave} \cdot S = \frac{\mu_o N I}{2\pi\rho_o} \cdot \pi a^2$$

or $$I = \frac{2\rho_o \Phi}{\mu a^2 N} = \frac{2 \times 10 \times 10^{-2} \times 0.5 \times 10^{-3}}{4\pi \times 10^{-7} \times 10^{-4} \times 10^3}$$

$$= \underline{\underline{795.77\ A}}$$

Alternatively, using circuit approach

$$R = \frac{l}{\mu S} = \frac{2\pi\rho_o}{\mu_o S} = \frac{2\pi\rho_o}{\mu_o \cdot \pi a^2}$$

$$\mathcal{F} = NI = \Phi R \longrightarrow I = \frac{\Phi R}{N} = \frac{2\rho_o \Phi}{\mu a^2 N}$$

as obtain before.

$$R = \frac{2\rho_o}{\mu_o a^2} = \frac{2 \times 10 \times 10^{-2}}{4\pi \times 10^{-7} \times 10^{-4}} = 1.591 \times 10^9$$

$$\mathcal{F} = \Phi R = 0.5 \times 10^{-3} \times 1.591 \times 10^9 = 7.955 \times 10^5$$

$$I = \frac{\mathcal{F}}{N} = 795A$$ as obtained before.

(b) If $\mu = 500\mu_o$, $I = \frac{795.77}{500} = \underline{\underline{1.592\ A}}$.

P.E. 8.16

$$f = \frac{B_a^2 S}{2\mu_o} = \frac{(1.5)^2 \times 10 \times 10^{-4}}{2 \times 4\pi \times 10^{-7}} = \frac{22500}{8\pi}$$

$$= \underline{\underline{895.25}}\ N.$$

$\boxed{\text{Prob. 8.1}}$

(a) $\vec{F}_e = Q\vec{E} = \underline{\underline{10\bar{a}_x - 30\bar{a}_y + 80\bar{a}_z \text{ N.}}}$

(b) $\vec{F}_m = Q\vec{u} \times \vec{B} = 10\begin{vmatrix} 2 & 0 & -4 \\ 0.3 & 0.1 & 0 \end{vmatrix} = \underline{\underline{4\bar{a}_x - 12\bar{a}_y + 2\bar{a}_z \text{ N.}}}$

(c) $\vec{F} = \vec{F}_e + \vec{F}_m = \underline{\underline{14\bar{a}_x - 42\bar{a}_y + 82\bar{a}_z \text{ N.}}}$

$\boxed{\text{Prob. 8.2}}$

from Lorentz force equation,

$$\vec{F} = Q\vec{u} \times \vec{B}$$

$$dW = \vec{F}\cdot d\vec{l} = Q\left(\frac{d\vec{l}}{dt} \times \vec{B}\right)\cdot d\vec{l} = 0$$

since $d\vec{l} \times \vec{B}$ is perpendicular to $d\vec{l}$. Thus a magnetic field does not contribute to the work done on a charged particle undergoing displacement $d\vec{l}$.

$\boxed{\text{Prob. 8.3}}$

(a) $\vec{a} = \dfrac{Q\vec{E}}{m} = \dfrac{3}{2}(12\bar{a}_x + 10\bar{a}_y) = \underline{\underline{18\bar{a}_x + 15\bar{a}_y \text{ m/s}^2}}$

(b) $\vec{a} = \left(\dfrac{du_x}{dt}, \dfrac{du_y}{dt}, \dfrac{du_z}{dt}\right) = 18\bar{a}_x + 15\bar{a}_y$

$$\frac{du_x}{dt} = 18 \longrightarrow u_x = 18t + A$$

$$\frac{du_y}{dt} = 15 \longrightarrow u_y = 15t + B$$

$$\frac{du_z}{dt} = 0 \longrightarrow u_z = C.$$

At $t=0$, $\vec{u} = (4,0,3)$, hence $A=4$, $B=0$, $C=3$

and $\vec{u} = (18t+4, 15t, 3)$

$$\vec{u}(t=1s) = \underline{\underline{22\vec{a}_x + 15\vec{a}_y + 3\vec{a}_z}} \text{ m/s.}$$

(c) $K.E. = \frac{1}{2}m|\vec{u}|^2 = \frac{1}{2}(2)(22^2+15^2+3^2) = \underline{\underline{718\text{ J.}}}$

(d) $\frac{dx}{dt} = u_x = 18t+4 \longrightarrow x = 9t^2 + 4t + A_1$

$$\frac{dy}{dt} = u_y = 15t \longrightarrow y = 7.5t^2 + B_1$$

$$\frac{dz}{dt} = u_z = 3 \longrightarrow z = 3t + C_1$$

At $t=0$, $(x,y,z) = (1,-2,0)$. Hence

$$A_1 = 1, \quad B_1 = -2, \quad C_1 = 0.$$

$$(x,y,z) = (9t^2+4t+1,\ 7.5t^2-2,\ 3t)$$

At $t=1$, $(x,y,z) = \underline{\underline{(14,\ 5.5,\ 3)}}.$

Prob. 8.4

(a) $\vec{F} = m\vec{a} = Q(\vec{E} + \vec{u}\times\vec{B})$

$$\frac{d}{dt}(u_x, u_y, u_z) = 2\left[-4\vec{a}_y + \begin{vmatrix} u_x & u_y & u_z \\ 5 & 0 & 0 \end{vmatrix}\right]$$

$$= -8\vec{a}_y + 10u_z\vec{a}_y - 10u_y\vec{a}_z$$

i.e.

$$\frac{dU_x}{dt} = 0 \longrightarrow U_x = A_1 \quad — (1)$$

$$\frac{dU_y}{dt} = -8 + 10U_z \quad — (2)$$

$$\frac{dU_z}{dt} = -10U_y \quad — (3)$$

$$\frac{d^2U_y}{dt^2} = 0 + 10\frac{dU_z}{dt} = -100U_y$$

$$\ddot{U}_y + 100U_y = 0 \longrightarrow U_y = B_1\cos 10t + B_2\sin 10t$$

from (2),

$$10U_z = 8 + \dot{U}_y = 8 - 10B_1\sin 10t + 10B_2\cos 10t$$

$$U_z = 0.8 - B_1\sin 10t + B_2\cos 10t$$

At $t=0$, $\vec{u}=0 \longrightarrow A_1 = 0,\ B_1 = 0,\ B_2 = -0.8$

Hence,

$$\vec{u} = (0, -0.8\sin 10t, 0.8 - 0.8\cos 10t) — (4)$$

$$U_x = \frac{dx}{dt} = 0 \longrightarrow x = C_1$$

$$U_y = \frac{dy}{dt} = -0.8\sin 10t \longrightarrow y = 0.08\cos 10t + C_2$$

$$U_z = \frac{dz}{dt} = 0.8 - 0.8\cos 10t \longrightarrow z = 0.8t + C_3 - 0.08\sin 10t$$

At $t=0$, $(x, y, z) = (2, 3, -4) \Longrightarrow$

$C_1 = 2,\ C_2 = 2.92,\ C_3 = -4.$

Hence $(x, y, z) = (2, 2 + 0.08\cos 10t, 0.8t - 0.08\sin 10t - 4)$

At $t = 1$,

$$(x, y, z) = \underline{\underline{(2, 1.933, -3.156)}}.$$

(b) from (4), at $t = 1$, $\vec{u} = (0, 0.435, 1.471)$ m/s

$$K.E = \frac{1}{2} m |\vec{u}|^2 = \frac{1}{2}(1)(0.435^2 + 1.471^2)$$

$$= \underline{\underline{1.177 \text{ J}}}.$$

$\boxed{\text{Prob. 8.5}}$

$$m\vec{a} = Q\vec{u} \times \vec{B}$$

$$10^{-3}\vec{a} = -2 \times 10^{-3} \begin{vmatrix} u_x & u_y & u_z \\ 0 & 6 & 0 \end{vmatrix}$$

$$\frac{d}{dt}(u_x, u_y, u_z) = (12u_z, 0, -12u_x)$$

i.e. $\frac{du_x}{dt} = 12u_z$ —— (1)

$\frac{du_y}{dt} = 0 \longrightarrow u_y = A_1$ — (2)

$\frac{du_z}{dt} = -12u_x$ — (3)

from (1) and (2),

$$\ddot{u}_x = -12\dot{u}_z = -144u_x$$

or $\ddot{u}_x + 144u_x = 0 \rightarrow u_x = c_1\cos 12t + c_2\sin 12t$

from (1), $u_z = -c_1\sin 12t + c_2\cos 12t$

At $t=0$, $u_x = 2$, $u_y = 0$, $u_z = 0 \rightarrow A_1 = 0 = C_2$,

$C_1 = 5$

Hence $\vec{u} = (5\cos 12t,\ 0,\ -5\sin 12t)$

$\vec{u}(t=10s) = (5\cos 120,\ 0,\ -5\sin 120)$

$= \underline{\underline{4.071\,\vec{a}_x - 2.903\,\vec{a}_z}}$ m/s.

$u_x = \frac{dx}{dt} = 5\cos 12t \longrightarrow x = \frac{5}{12}\sin 12t + B_1$

$u_y = \frac{dy}{dt} = 0 \longrightarrow y = B_2$

$u_z = \frac{dz}{dt} = -5\sin 12t \rightarrow z = \frac{5}{12}\cos 12t + B_3$

At $t=0$, $(x,y,z) = (0,1,2) \rightarrow B_1 = 0,\ B_2 = 1,$

$B_3 = \frac{19}{12}$

$$(x,y,z) = \left(\frac{5}{12}\sin 12t,\ 1,\ \frac{5}{12}\cos 12t + \frac{19}{12}\right) \quad -(4)$$

At $t = 10s$, $(x,y,z) = \left(\frac{5}{12}\sin 120,\ 1,\ \frac{5}{12}\cos 120 + \frac{19}{12}\right)$

$= \underline{\underline{(0.2419,\ 1,\ 1.923)}}$

By eliminating t from (4),

$$x^2 + \left(z - \frac{19}{12}\right) = \left(\frac{5}{12}\right)^2,\quad y = 1$$

which is a helix with axis on line

$y = 1$, $z = \frac{19}{12}$.

Prob. 8.6

(a) $m\vec{a} = e(\vec{u} \times \vec{B})$

$$\frac{m}{e}\frac{d}{dt}(u_x, u_y, u_z) = \begin{vmatrix} u_x & u_y & u_z \\ 0 & 0 & B_z \end{vmatrix}$$

$$= u_y B_o \vec{a}_x - B_o u_x \vec{a}_y$$

$$\frac{du_z}{dt} = 0 \longrightarrow u_z = c = 0$$

$$\frac{du_x}{dt} = u_y \frac{B_o e}{m} = u_y w, \text{ where } w = \frac{B_o e}{m}$$

$$\frac{du_y}{dt} = -u_x w$$

Hence $\ddot{u}_x = w\dot{u}_y = -w^2 u_x$

or $\ddot{u}_x + w^2 u_x = 0 \longrightarrow u_x = A\cos wt + B\sin wt$

$$u_y = \frac{\dot{u}_x}{w} = -A\sin wt + B\cos wt$$

At $t=0$, $u_x = u_o$, $u_y = 0 \longrightarrow A = u_o$, $B = 0$.

Hence

$$u_x = u_o \cos wt = \frac{dx}{dt} \longrightarrow x = \frac{-u_o}{w}\sin wt + c_1$$

$$u_y = -u_o \sin wt = \frac{dy}{dt} \rightarrow y = \frac{-u_o}{w}\cos wt + c_2$$

At $t=0$, $x = 0 = y \longrightarrow c_1 = 0$, $c_2 = \frac{u_o}{w}$. Hence

$$x = -\frac{u_o}{w}\sin wt, \quad y = \frac{u_o}{w}(1-\cos wt)$$

$$\frac{u_o^2}{w^2}(\cos^2 wt + \sin^2 wt) = \left(\frac{u_o}{w}\right)^2 = x^2 + \left(y - \frac{u_o}{w}\right)^2$$

showing that the electron would move in a circle centered at $(0, \frac{u_0}{w})$. But since the field does not exist throughout the circular region, the electron passes through a semi-circle and leaves the field horizontally.

(b) $d =$ twice the radius of the semi-circle

$$= \frac{2u_0}{w} = \frac{2u_0 m}{B_0 e}.$$

Alternatively, due to the centrifugal force,

$$F = e u_0 B = \frac{m u_0^2}{r} \longrightarrow r = \frac{m u_0}{B_0 e}$$

$$d = 2r = \frac{2 m u_0}{B_0 e}.$$

Prob. 8.7

(a) Let $\vec{F}_2$ be the force on 2 due to the field produced by 1,

$$d\vec{F}_2 = I_2 d\vec{l}_2 \times \vec{B}_1 = I_2\, dz\,(-\vec{a}_z) \times \frac{\mu_0 I_1}{2\pi\rho}\vec{a}_\phi.$$

The force per unit length on 2 is

$$\vec{f}_2 = \frac{d\vec{F}_2}{dz} = \frac{\mu_0 I_1 I_2}{2\pi\rho}\vec{a}_\rho \quad \text{(repulsive)}$$

(b) $f = \dfrac{4\pi\times10^{-7}\times5\times10}{2\pi\times 2} = 5\ \mu N/m.$

(i) $\underline{\underline{5\ \mu N/m\text{, attractive}}}$, (ii) $\underline{\underline{5\ \mu N/m\text{, repulsive}}}$.

Prob. 8.8

$\vec{F} = I\vec{L}\times\vec{B} \longrightarrow \vec{f} = \dfrac{\vec{F}}{L} = I_1\vec{a}_l\times\vec{B}_2 = \dfrac{\mu_0 I_1 I_2\ \vec{a}_l\times\vec{a}_\rho}{2\pi\rho}$

(a) $\vec{f}_{21} = \dfrac{\vec{a}_z\times(-\vec{a}_x)}{2\pi}\ \dfrac{4\pi\times10^{-7}\times(-2\times10^4)}{4}$

$= \underline{\underline{\vec{a}_y\ mN/m}}$ (repulsive).

(b) $\vec{f}_{12} = -\vec{f}_{21} = \underline{\underline{-\vec{a}_y\ mN/m}}$ (repulsive)

(c) $\vec{a}_l\times\vec{a}_\rho = \vec{a}_z\times\left(-\frac{4}{5}\vec{a}_x+\frac{3}{5}\vec{a}_y\right) = -\frac{3}{5}\vec{a}_x-\frac{4}{5}\vec{a}_y,\ \rho=5$

$\vec{f}_{31} = \dfrac{4\pi\times10^{-7}}{2\pi(5)}(-3\times10^4)\left(-\frac{3}{5}\vec{a}_x-\frac{4}{5}\vec{a}_y\right)$

$= \underline{\underline{0.72\vec{a}_x+0.96\vec{a}_y}}\ mN/m$ (repulsive).

(d) $\vec{f}_3 = \vec{f}_{31}+\vec{f}_{32}$

$\vec{f}_{32} = \dfrac{4\pi\times10^{-7}\times6\times10^4}{2\pi(3)}\vec{a}_z\times\vec{a}_y = -4\vec{a}_x\ mN/m$ (attractive)

$\vec{f}_3 = \underline{\underline{-3.28\vec{a}_x+0.96\vec{a}_y\ mN/m}}.$

(attractive due to L_2 and repulsive due to L_1).

Prob. 8.9

$$W = -\int \vec{F}\cdot d\vec{l}, \quad \vec{F} = \int I d\vec{l} \times \vec{B} = 3(2\vec{a}_z) \times \cos\frac{\phi}{3}\,\vec{a}_\rho$$

$$= 6\cos\frac{\phi}{3}\,\vec{a}_\phi \text{ mN}.$$

$$W = -\int_0^{2\pi} 6\cos\frac{\phi}{3}\,\rho_o\,d\phi = -6\times 3\sin\frac{\phi}{3}\Big|_0^{2\pi} \text{ mJ}$$

$$= -18\sin\frac{2\pi}{3} = \underline{\underline{-15.59 \text{ mJ}}}.$$

Prob. 8.10

(a) $$\vec{F}_1 = \int_{\rho=2}^{4} \frac{\mu_o I_1 I_2}{2\pi\rho}\,d\rho\,\vec{a}_\rho \times \vec{a}_\phi = \frac{4\pi\times10^{-7}}{2\pi}(2)(5)\ln\frac{4}{2}\,\vec{a}_z$$

$$= 2\ln 2\,\vec{a}_z \;\mu\text{N} = \underline{\underline{1.3863\,\vec{a}_z \;\mu\text{N}}}.$$

(b) $$\vec{F}_2 = \int I_2 d\vec{l}_2 \times \vec{B}_1$$

$$= \frac{I_1 I_2 \mu_o}{2\pi}\int \frac{1}{\rho}\left[d\rho\,\vec{a}_\rho + dz\,\vec{a}_z\right] \times \vec{a}_\phi$$

$$= \frac{\mu_o I_1 I_2}{2\pi}\int \frac{1}{\rho}\left[d\rho\,\vec{a}_z - dz\,\vec{a}_\rho\right]$$

But $\rho = z + 2, \; dz = d\rho$.

$$\vec{F}_2 = \frac{4\pi\times10^{-7}}{2\pi}(5)(2)\int_{\rho=4}^{2} \frac{1}{\rho}\left[d\rho\,\vec{a}_z - d\rho\,\vec{a}_\rho\right]$$

$$= 2\ln\frac{2}{4}\left(\vec{a}_z - \vec{a}_\rho\right)\mu\text{N} = 1.386\,\vec{a}_\rho - 1.386\,\vec{a}_z \;\mu\text{N}$$

$$\vec{F}_3 = \frac{\mu_o I_1 I_2}{2\pi} \int \frac{1}{\rho} \left[d\rho\, \vec{a}_z - dz\, \vec{a}_\rho \right]$$

But $z = -\rho + 6$, $dz = -d\rho$

$$\vec{F}_3 = \frac{4\pi \times 10^{-7} (5)(2)}{2\pi} \int_{\rho=6}^{4} \frac{1}{\rho} \left[d\rho\, \vec{a}_z + d\rho\, \vec{a}_\rho \right]$$

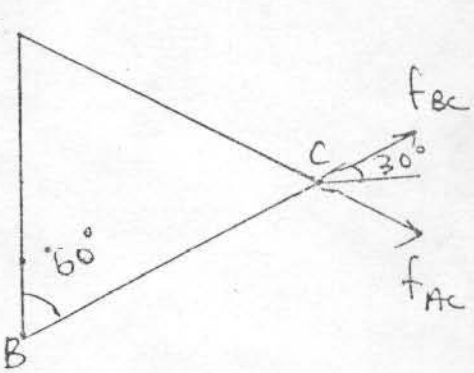
$$= 2 \ln \frac{4}{6} (\vec{a}_\rho + \vec{a}_z)\ \mu N = -0.8109\, \vec{a}_\rho - 0.8109\, \vec{a}_z\ \mu N$$

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3$$

$$= 1.3863\, \vec{a}_z + 1.386\, \vec{a}_\rho - 1.386\, \vec{a}_z - 0.8109\, \vec{a}_\rho - 0.8109\, \vec{a}_z$$

$$= \underline{\underline{0.5751\, \vec{a}_\rho - 0.8109\, \vec{a}_z\ \mu N}}.$$

Prob. 8.11



From Prob. 8.7,

$$f = \frac{\mu_o I_1 I_2}{2\pi\rho}\, \vec{a}_\rho \quad N/m$$

$$\vec{f} = \vec{f}_{AC} + \vec{f}_{BC}$$

$$f_{AC} = f_{BC} = \frac{4\pi \times 10^{-7} \times 75 \times 150}{2\pi \times 2} = 1.125 \times 10^{-3}$$

$$\vec{f} = 2 \times 1.125 \cos 30^\circ\, \vec{a}_x \quad mN/m$$

$$= \underline{\underline{1.949\, \vec{a}_x\ \ mN/m}}$$

Prob. 8.12

$$\vec{F} = \int I\,d\vec{l} \times \vec{B} = \int \vec{J}\,dv \times \vec{B}$$

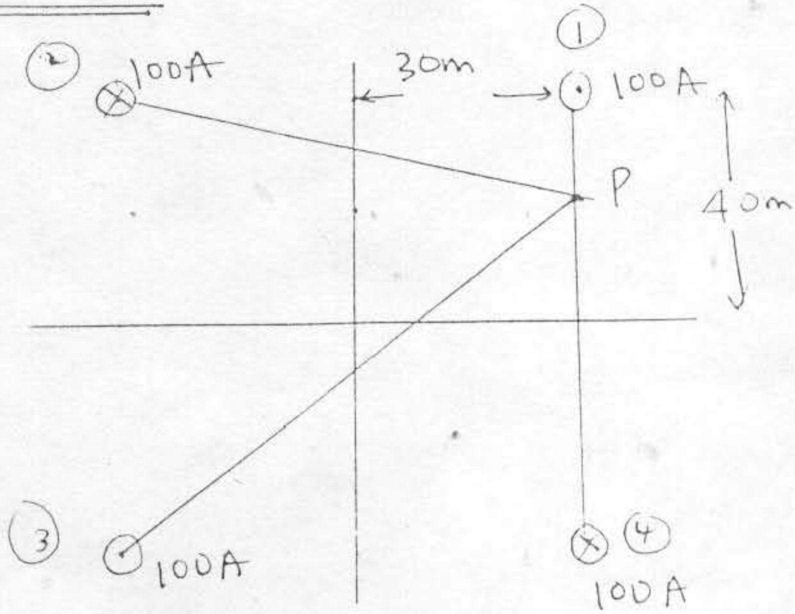
$$\vec{J} = \frac{I}{\pi(b^2-a^2)}\,\vec{a}_z, \quad \vec{B} = B_o\,\vec{a}_\rho$$

$$\vec{F} = \frac{I}{\pi(b^2-a^2)} \int \hat{a}_z\,dv \times B_o\,\hat{a}_\rho = \frac{I B_o\,\vec{a}_\phi}{\pi(b^2-a^2)} \int dv$$

$$= \frac{I B_o}{\pi(b^2-a^2)}\,\pi(a^2-b^2)L$$

$$\vec{f} = \frac{\vec{F}}{L} = \underline{\underline{I B_o\,\vec{a}_\phi.}}$$

Prob. 8.13



Let $\vec{B} = \vec{B}_1 + \vec{B}_2 + \vec{B}_3 + \vec{B}_4$

where $\vec{B}_n = \dfrac{\mu_o \mu_r I}{2\pi\rho}\,\vec{a}_\phi$.

for ①, $\vec{a}_\phi = \vec{a}_l \times \vec{a}_\rho = \vec{a}_z \times (-\vec{a}_y) = \vec{a}_x$

$$\vec{B}_1 = \frac{4\pi \times 10^{-7} \times 2000 \times 100}{2\pi \times 20 \times 10^{-3}} \vec{a}_x = 2\vec{a}_x$$

for ②, $\vec{\rho} = 6\vec{a}_x - 2\vec{a}_y$

$$\vec{a}_\phi = -\vec{a}_z \times \frac{(6\vec{a}_x - 2\vec{a}_y)}{\sqrt{40}} = \frac{(-2\vec{a}_x - 6\vec{a}_y)}{\sqrt{40}}$$

$$\vec{B}_2 = \frac{4\pi \times 10^{-7} \times 2000 \times 100}{2\pi \times 400 \times 10^{-3}} (-2\vec{a}_x - 6\vec{a}_y)$$

$$= -0.2\,\vec{a}_x - 0.6\vec{a}_y$$

for ③, $\vec{\rho} = 6\vec{a}_x + 6\vec{a}_y$

$$\vec{a}_\phi = \vec{a}_z \times \frac{(6\vec{a}_x + 6\vec{a}_y)}{\sqrt{72}} = \frac{-6\vec{a}_x + 6\vec{a}_y}{\sqrt{72}}$$

$$\vec{B}_3 = \frac{4\pi \times 10^{-7} \times 2000 \times 100}{2\pi \times 720 \times 10^{-3}} (-6\vec{a}_x + 6\vec{a}_y)$$

$$= -0.3333\,\vec{a}_x + 0.3333\,\vec{a}_y$$

for ④, $\vec{a}_\phi = -\vec{a}_z \times \vec{a}_y = \vec{a}_x$

$$\vec{B}_4 = \frac{4\pi \times 10^{-7} \times 2000 \times 100}{2\pi \times 60 \times 10^{-3}} \vec{a}_x = 0.6667\,\vec{a}_x$$

$$\vec{B} = \left(2 + \frac{2}{3} - \frac{1}{5} - \frac{1}{3}\right)\vec{a}_x + \left(-\frac{3}{5} + \frac{1}{3}\right)\vec{a}_y$$

$$= \underline{\underline{2.133\,\vec{a}_x - 0.2667\,\vec{a}_y \ \text{Wb/m}^2.}}$$

Prob. 8.14

$$\vec{m} = m\vec{a}_z = m(\cos\theta\,\vec{a}_r - \sin\theta\,\vec{a}_\theta)$$

Hence

$$\frac{\vec{m}}{r^3} = \frac{m}{r^3}(\cos\theta\,\vec{a}_r - \sin\theta\,\vec{a}_\theta)$$

$$\frac{3(\vec{m}\cdot\vec{r})\vec{r}}{r^5} = \frac{3mr\cos\theta\,(r\,\vec{a}_r)}{r^5} = \frac{3m\cos\theta}{r^3}\vec{a}_r$$

Thus

$$\frac{3(\vec{m}\cdot\vec{r})\vec{r}}{r^5} - \frac{\vec{m}}{r^3} = \frac{2m\cos\theta\,\vec{a}_r + m\sin\theta\,\vec{a}_\theta}{r^3}$$

and

$$\vec{B} = \frac{\mu_0}{4\pi}\left[3\frac{(\vec{m}\cdot\vec{r})\vec{r}}{r^5} - \frac{\vec{m}}{r^3}\right] = \frac{\mu_0 m}{4\pi r^3}(2\cos\theta\,\vec{a}_r + \sin\theta\,\vec{a}_\theta)$$

Prob. 8.15

$$T = mB\sin\theta \longrightarrow T_{max} = mB$$

or $$B = \frac{T_{max}}{ISN} = \frac{50\times10^{-6}}{10\times10^{-3}\times800\times0.6\times10^{-4}}$$

$$= \underline{\underline{104.2\ \text{mWb/m}^2}}.$$

Prob. 8.16

$$T = mB = NISB = 1000\times2\times10^{-3}\times300\times10^{-6}\times0.4$$

$$= \underline{\underline{240\ \mu\text{N.m}}}.$$

Prob. 8.17

(a) $\vec{m} = IS\vec{a}_n = 1(\pi)(0.1)^2\vec{a}_z = \underline{0.03142\vec{a}_z \text{ A}\cdot\text{m}}$

(b) $$\vec{H} = \frac{m}{4\pi r^3}\left(2\cos\theta\,\vec{a}_r + \sin\theta\,\vec{a}_\theta\right)$$

where $r = \sqrt{4+4+4} = 2\sqrt{3}$, $\tan\theta = \frac{\rho}{z} = \frac{2\sqrt{2}}{2} = \sqrt{2}$

i.e. $\sin\theta = \sqrt{\frac{2}{3}}$, $\cos\theta = \frac{1}{\sqrt{3}}$. Hence

$$\vec{H} = \frac{0.01\pi}{4\pi(24\sqrt{3})}\left(\frac{2}{\sqrt{3}}\vec{a}_r + \sqrt{\frac{2}{3}}\,\vec{a}_\theta\right)$$

$$= \underline{69.44\vec{a}_r + 49.1\vec{a}_\theta \ \mu\text{A/m}}.$$

(c) $$\vec{B} = \frac{\mu_o m}{4\pi r^3}\left(2\cos\theta\,\vec{a}_r + \sin\theta\,\vec{a}_\theta\right)$$

where $r = \sqrt{36+64+100} = 10\sqrt{2}$, $\tan\theta = \frac{10}{10} = 1$. Thus

$$\vec{B} = \frac{4\pi\times10^{-7}\times0.01\pi}{4\pi(2000\sqrt{2})}\left(\frac{2}{\sqrt{2}}\vec{a}_r + \frac{1}{\sqrt{2}}\vec{a}_\theta\right)$$

$$= \underline{1.571\vec{a}_r + 0.7853\vec{a}_\theta \ \text{pWb/m}^2}.$$

Prob. 8.18

$$\vec{B} = \frac{4\pi\times10^{-7}\times10^{-2}}{288}\left(2\vec{a}_r + \sqrt{2}\,\vec{a}_\theta\right)$$

$$\vec{m} = 4\vec{a}_z = 4(\cos\theta\,\vec{a}_r - \sin\theta\,\vec{a}_\theta) = 4\left(\frac{\vec{a}_r}{\sqrt{3}} - \sqrt{\frac{2}{3}}\,\vec{a}_\theta\right)$$

$$\vec{T} = \vec{m} \times \vec{B} = \frac{4\times10^{-2} \times 4\pi\times10^{-7}}{288} \begin{vmatrix} \vec{a}_r & \vec{a}_\theta & \vec{a}_\phi \\ \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} & 0 \\ 2 & \sqrt{2} & 0 \end{vmatrix}$$

$$= -\sqrt{\tfrac{2}{3}} \cdot \frac{\pi}{8} \cdot 10^{-9}\, \vec{a}_\phi = \underline{\underline{-1.425 \times 10^{-10}\, \vec{a}_\phi \text{ N·m}}}$$

$\boxed{\text{Prob. 8.19}}$

$$\vec{B} = \frac{k}{r^3}\left(2\cos\theta\, \vec{a}_r + \sin\theta\, \vec{a}_\theta\right)$$

At $(10, 0, 0)$, $r = 10$, $\theta = \pi/2$, $\vec{a}_r = \vec{a}_x$, $\vec{a}_\theta = -\vec{a}_z$

$$-0.5\times10^{-3}\, \vec{a}_z = \frac{k}{10^3}\left(0 - \vec{a}_z\right) \longrightarrow k = 0.5$$

Thus

$$\vec{B} = \frac{0.5}{r^3}\left(2\cos\theta\, \vec{a}_r + \sin\theta\, \vec{a}_\theta\right).$$

(a) At $(0,3,0)$, $r = 3$, $\theta = \pi/2$, $\vec{a}_r = \vec{a}_y$, $\vec{a}_\theta = -\vec{a}_z$

$$\vec{B} = \frac{0.5}{27}\left(0 - \vec{a}_z\right) = \underline{\underline{-18.52\, \vec{a}_z \text{ mWb/m}^2}}.$$

(b) At $(3,4,0)$, $r = 5$, $\theta = \pi/2$, $\vec{a}_\theta = -\vec{a}_z$.

$$\vec{B} = \frac{0.5}{125}\left(0 - \vec{a}_z\right) = \underline{\underline{-4\, \vec{a}_z \text{ mWb/m}^2}}.$$

(c) At $(1,-1,1)$, $r = \sqrt{3}$, $\tan\theta = \frac{\rho}{z} = \frac{\sqrt{2}}{-1}$, i.e.

$$\sin\theta = \sqrt{\tfrac{2}{3}}, \quad \cos\theta = -\frac{1}{\sqrt{3}}$$

$$\vec{B} = \frac{0.5}{3\sqrt{3}}\left(-\frac{2}{\sqrt{3}}\vec{a}_r + \sqrt{\tfrac{2}{3}}\, \vec{a}_\theta\right) = \underline{\underline{-111\, \vec{a}_r + 78.6\, \vec{a}_\theta \text{ mWb/m}^2}}$$

Prob. 8.20

$$\vec{T}_2 = \vec{m}_2 \times \vec{B}_1$$

where $\vec{m}_2 = 3\vec{a}_y$, $m_1 = 5\vec{a}_x$. From Prob. 8.14,

$$\vec{B}_1 = \frac{\mu_0}{4\pi}\left[\frac{3(\vec{m}_1\cdot\vec{r})\vec{r}}{r^5} - \frac{\vec{m}_1}{r^3}\right]$$

$$= \frac{4\pi\times10^{-7}}{4\pi}\left[\frac{3(20)(4,-3,10)}{(5\sqrt{5})^3} - \frac{(5,0,0)}{(5\sqrt{5})^3}\right]$$

$$= (-2.204, -1.03, 3.434)\cdot10^{-10}$$

Hence

$$\vec{T}_2 = 3\vec{a}_y \times \vec{B}_1$$

$$= \underline{\underline{(10.302\,\vec{a}_x + 6.612\,\vec{a}_z)\cdot10^{-10}}} \text{ N·m}$$

Prob. 8.21

(a) $\chi_m = \mu_r - 1 = \underline{\underline{3}}$.

(b) $\vec{H} = \dfrac{\vec{B}}{\mu} = \dfrac{(2,-5,4)\times10^{-3}}{4\pi\times10^{-7}(4)}$

$$= \underline{\underline{398\,\vec{a}_x - 995\,\vec{a}_y + 796\,\vec{a}_z \text{ A/m}}}.$$

(c) $\vec{M} = \chi_m\vec{H} = 3\vec{H} = \underline{\underline{1.19\vec{a}_x - 2.98\vec{a}_y + 2.39\vec{a}_z \text{ kA/m}}}.$

(d) $w_m = \frac{1}{2}\vec{B}\cdot\vec{H} = \dfrac{|\vec{B}|^2}{2\mu}$

$$W_m = \frac{(4+25+16)\times 10^{-6}}{2\times 4\pi\times 10^{-7}\times 4} = \underline{\underline{4.475\ \text{J/m}^3}}.$$

$\boxed{\text{Prob. 8.22}}$

(a) $\vec{M} = \chi_m \vec{H} = \underline{\underline{4z\,\vec{a}_y\ \text{A/m}}}.$

(b) $\vec{J} = \nabla\times\vec{H} = -\frac{\partial H_y}{\partial z}\vec{a}_x = \underline{\underline{-\vec{a}_x\ \text{A/m}}}.$

(c) $\vec{J}_b = \chi_m \vec{J} = \underline{\underline{-4\,\vec{a}_x\ \text{A/m}}}.$

$\boxed{\text{Prob. 8.23}}$

(a) for an infinitely long wire,

$$H_\phi = \begin{cases} \frac{I\rho}{2\pi a^2}, & \rho < a = 2\,\text{m} \\ \frac{I}{2\pi\rho}, & \rho > a \end{cases}$$

$\vec{m} = \chi_m \vec{H}$, $\vec{J}_b = \nabla\times\vec{m}$, $\vec{J}_f = \nabla\times\vec{H}$, $\vec{J} = \vec{J}_b + \vec{J}_f$
$= 5\,\nabla\times\vec{H}$

where $\nabla\times\vec{H} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho H_\phi)\,\vec{a}_z$

At $\rho = 1\,\text{mm} < a$, $\nabla\times\vec{H} = \frac{I}{2\pi a^2}\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho^2)\vec{a}_z = \frac{I}{\pi a^2}\vec{a}_z$

$$\vec{J}_b = 4\,\nabla\times\vec{H} = \frac{4\times 4\,\vec{a}_z}{\pi\times 4\times 10^{-6}} = \underline{\underline{1.273\,\vec{a}_z\ \text{MA/m}^2}}$$

$$\vec{J} = 5\,\nabla\times\vec{H} = \underline{\underline{1.592\,\vec{a}_z\ \text{MA/m}^2}}$$

$$\vec{M} = \frac{4 \times 1 \times 10^{-3}}{2\pi \times 4 \times 10^{-6}} \vec{a}_\phi = \underline{\underline{159.15\, \vec{a}_\phi \ \ A/m}}$$

At $\rho = 3\text{mm} > a$,

$$\underline{\underline{\vec{M} = 0 = \vec{J}_b = \vec{J}}}$$

(b) $\vec{K}_b = \vec{M} \times \vec{a}_n = 4\, H_\phi\, \vec{a}_\phi \times \vec{a}_\rho \Big|_{\rho = 2\text{mm}}$

$$= \frac{4 \times 4\,(-\vec{a}_z)}{2\pi \times 2 \times 10^{-3}} = \underline{\underline{-1.273\, \vec{a}_z \ \ kA/m^2}}.$$

$\boxed{\text{Prob. 8.24}}$

(a) from $H_{1t} - H_{2t} = K$ and $M = \chi_m H$, we obtain

$$\underline{\underline{\frac{M_{1t}}{\chi_{m1}} - \frac{M_{2t}}{\chi_{m2}} = K}}$$

Also from $B_{1n} = B_{2n}$ and $B = \mu H = \frac{\mu}{\chi_m} M$, we get

$$\underline{\underline{\frac{\mu_1 M_{1n}}{\chi_{m1}} = \frac{\mu_2 M_{2n}}{\chi_{m2}}}}$$

(b) from $B_1 \cos\theta_1 = B_{1n} = B_{2n} = B_2 \cos\theta_2 \quad — (1)$

and $\dfrac{B_1}{\mu_1} \sin\theta_1 = H_{2t} = K + H_{2t}$

$$= K + \frac{B_2}{\mu_2} \sin\theta_2 \quad — (2),$$

dividing (2) by (1) gives

$$\frac{\tan\theta_1}{\mu_1} = \frac{K}{B_2 \cos\theta_2} + \frac{\tan\theta_2}{\mu_2} = \frac{\tan\theta_2}{\mu_2}\left(1 + \frac{K \mu_2}{B_2 \sin\theta_2}\right)$$

i.e. $$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\mu_1}{\mu_2}\left(1 + \frac{K\mu_2}{B_2 \sin\theta_2}\right).$$

$\boxed{\text{Prob. 8.25}}$

(a) $\vec{H}_{1t} = \vec{H}_{2t} = -6\,\vec{a}_\phi + 7\,\vec{a}_z$

$\vec{B}_{1n} = \vec{B}_{2n} \longrightarrow \mu_o \vec{H}_{1n} = 3\mu_o H_{2n} = 3\mu_o(4\,\vec{a}_\rho)$

$$\vec{H}_{1n} = 12\,\vec{a}_\rho$$

Hence $\vec{H}_1 = \vec{H}_{1n} + \vec{H}_{1t} = \underline{\underline{12\,\vec{a}_\rho - 6\,\vec{a}_\phi + 7\,\vec{a}_z \text{ kA/m}}}$

(b) $\vec{B}_2 = \mu_2 \vec{H}_2 = 3(4\pi\times10^{-7})(4,-6,7)\cdot10^3$

$= \underline{15.1\,\vec{a}_\rho - 22.6\,\vec{a}_\phi + 26.4\,\vec{a}_z \text{ mWb/m}^2}.$

(c) $\vec{H}_2\cdot\vec{a}_\rho = H_2\cos\theta_2 \rightarrow \cos\theta_2 = \dfrac{4}{\sqrt{4^2+6^2+7^2}}$

$= 0.398$

or $\theta_2 = \underline{\underline{66.55^\circ}}$.

$\vec{H}_1\cdot\vec{a}_\rho = H_1\cos\theta_1 \rightarrow \cos\theta_1 = \dfrac{12}{\sqrt{12^2+6^2+7^2}} = 0.793$

or $\theta_1 = \underline{\underline{37.53^\circ}}$.

$\boxed{\text{Prob. 8.26}}$

(a) $\vec{B}_{1n} = \vec{B}_{2n} = 50\,\vec{a}_y$

$$\vec{H}_{1t} = \vec{H}_{2t} \longrightarrow \vec{B}_{2t} = \frac{\mu_2}{\mu_1}\vec{B}_{1t} = \frac{1}{2}(40\vec{a}_x - 30\vec{a}_z)$$
$$= 20\vec{a}_x - 15\vec{a}_z$$

$$\vec{B}_2 = \vec{B}_{2n} + \vec{B}_{2t} = \underline{\underline{20\vec{a}_x + 50\vec{a}_y - 15\vec{a}_z}} \text{ mWb/m}^2.$$

(b) $$\frac{H_1}{H_2} = \frac{B_1}{B_2}\cdot\frac{\mu_{r2}}{\mu_{r1}} = \frac{10\sqrt{4^2+5^2+3^2}}{10\sqrt{2^2+5^2+1.5^2}}\cdot\frac{1}{2}$$
$$= \underline{\underline{0.6324}}$$

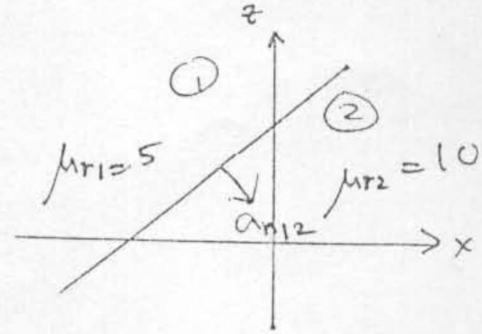
(c) $$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\mu_1}{\mu_2} = 2$$

or $$\tan\theta_1 = \frac{\sqrt{40^2+30^2}}{50} = 1 \longrightarrow \theta_1 = 45^\circ$$

$$\tan\theta_2 = \frac{\sqrt{20^2+15^2}}{50} = 0.5 \longrightarrow \theta_2 = 26.56^\circ$$

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{1}{0.5} = \underline{\underline{2}}.$$

Prob. 8.27



Let $\vec{H}_2 = (H_x, H_y, H_z)$

$$(\vec{H}_1 - \vec{H}_2)\times\vec{a}_{n12} = \vec{K}$$

where $f(x,z) = 5z - 4x = 0$ and

$$a_{n12} = -\frac{\nabla f}{|\nabla f|} = \frac{4\vec{a}_x - 5\vec{a}_z}{\sqrt{41}}$$

$$(\vec{H}_1-\vec{H}_2)\times\vec{a}_{n12}=\frac{1}{\sqrt{41}}\begin{vmatrix}25-H_x & -30-H_y & 45-H_z\\ 4 & 0 & -5\end{vmatrix}$$

$$=\frac{1}{\sqrt{41}}\Big[150+5H_y,\ 180-4H_z+125-5H_x,$$
$$120+4H_y\Big]=\vec{K}=35\vec{a}_y.$$

Equating components,

$\vec{a}_x$: $150+5H_y=0 \longrightarrow H_y=-30$

$\vec{a}_y$: $305-4H_z-5H_x=35 \longrightarrow 4H_z+5H_x=270$

$\vec{a}_z$: $120+4H_y=0 \longrightarrow H_y=-30.$

Also, $\vec{B}_{1n}=\vec{B}_{2n} \longrightarrow \mu_1\vec{H}_{1n}=\mu_2\vec{H}_{2n}$

$$5\mu_0(25,-30,45)\cdot\frac{(4,0,5)}{\sqrt{41}}=10\mu_0(H_x,H_y,H_z)\cdot\frac{(4,0,5)}{\sqrt{41}}$$

$$100-225=8H_x-10H_z$$

or $125=10H_z-8H_x$

$=10H_z-8(54-0.8H_z)$

$\longrightarrow H_z=33.96$

and $H_x=54-0.8H_z=26.83$

Thus $\vec{H}_2=\underline{\underline{26.83\vec{a}_x-30\vec{a}_y+33.96\vec{a}_z\ A/m.}}$

Prob. 8.28

$\vec{H}_{1n} = -3\vec{a}_z$, $\vec{H}_{1t} = 10\vec{a}_x + 15\vec{a}_y$

$\vec{H}_{2t} = \vec{H}_{1t} = 10\vec{a}_x + 15\vec{a}_y$,

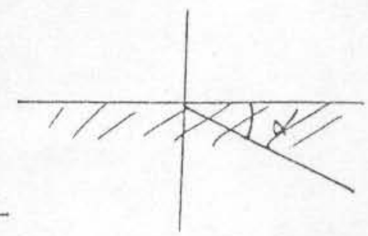
$\vec{H}_{2n} = \frac{\mu_1}{\mu_2} H_{1n} = \frac{1}{200}(-3\vec{a}_z) = -0.015\vec{a}_z$

$\vec{H}_2 = 10\vec{a}_x + 15\vec{a}_y - 0.015\vec{a}_z$

$\vec{B}_2 = \mu_2 \vec{H}_2 = 200 \times 4\pi \times 10^{-7}(10, 15, -0.015)$

$\vec{B}_2 = \underline{\underline{2.51\vec{a}_x + 3.77\vec{a}_y - 0.0037\vec{a}_z}}$ mWb/m^2

$\tan\alpha = \frac{B_{2n}}{B_{2t}}$



or $\alpha = \tan^{-1} \frac{0.0037}{\sqrt{2.51^2 + 3.77^2}}$

$= \underline{\underline{0.047^\circ}}$.

Prob. 8.29

(a) $\vec{H} = \frac{1}{2}\vec{K} \times \vec{a}_n = \frac{1}{2}(30-40)\,\vec{a}_x \times (-\vec{a}_z) = \underline{\underline{-5\vec{a}_y \text{ A/m}}}$

$\vec{B} = \mu_0 \vec{H} = 4\pi \times 10^{-7}(-5\vec{a}_y) = \underline{\underline{-6.283\vec{a}_y \ \mu\text{Wb/m}^2}}$

(b) $\vec{H} = \frac{1}{2}(-30-40)\vec{a}_y = \underline{\underline{-35\vec{a}_y \text{ A/m}}}$.

$\vec{B} = \mu_0\mu_r \vec{H} = 4\pi \times 10^{-7}(-35\vec{a}_y) = \underline{\underline{-110\vec{a}_y \ \mu\text{Wb/m}^2}}$.

(c) $\vec{H} = \frac{1}{2}(-30+40)\vec{a}_y = \underline{\underline{5\vec{a}_y \text{ A/m}}}.$

$\vec{B} = \mu_o \vec{H} = \underline{\underline{6.283\,\vec{a}_y \ \mu\text{Wb/m}^2}}.$

Prob. 8.30

(a) $B = 70 + (210)^2 = 44.17 \text{ mWb/m}^2.$

$$\mu_r = \frac{B}{\mu_o H} = \frac{44.17 \times 10^{-3}}{4\pi \times 10^{-7} \times 210} = \underline{\underline{167.4}}$$

(b) $$W_m = \int_0^{H_o} H\,dB = \int_0^{H_o} H\left(\frac{1}{3} + 2H\right)dH$$

$$= \frac{H_o^2}{6} + \frac{2}{3}H_o^3 = 7350 + 6174000$$

$$= \underline{\underline{6181.35 \text{ kJ/m}^3}}.$$

Prob. 8.31

$$L' = \frac{L}{l} = \mu_o n^2 S = \frac{\mu_o N^2 S}{l^2} = \frac{\mu_o N^2 \pi d^2}{4 l^2}$$

$$L = \frac{4\pi \times 10^{-7} \times 10^6 \pi \times 100 \times 10^{-4}}{4(0.4)} = \underline{\underline{24.674 \text{ mH}}}.$$

Prob. 8.32

(a) $$L = \frac{\lambda}{I} = \frac{N\Psi}{I} = \frac{\mu_o N^2 I a}{2\pi} \ln\left(\frac{2\rho_o + a}{2\rho_o - a}\right)$$

(b) $$L = \frac{N\Psi}{I} = \mu_o N^2 \left[\rho_o - (\rho_o^2 - a^2)^{\frac{1}{2}}\right]$$

When $\rho_o >> a$, binomial series expansion gives

$$L = \frac{\mu_o N^2 a^2}{2\rho_o}$$

or from Example 8.10,

$$L = L'l = \frac{\mu_o N^2 l S}{l^2} = \frac{\mu_o N^2 \pi a^2}{2\pi \rho_o} = \frac{\mu_o N^2 a^2}{2\rho_o}.$$

Prob. 8.33

(a) $L' = \frac{L}{l} = \frac{\mu_o \mu_r}{8\pi} = \frac{5 \times 4\pi \times 10^{-7}}{8\pi} = \underline{\underline{250 \text{ nH/m}}}.$

(b) $W_H' = \frac{W_H}{l} = \frac{1}{2} L' I^2 = \frac{1}{2} \times 250 \times 10^{-9} \times 16$

$= \underline{\underline{2 \ \mu J/m^3}}.$

Prob. 8.34

for $d >> a$,

$$L' = \frac{L}{l} = \frac{\mu_o}{\pi} \ln \frac{d}{a} = \frac{4\pi \times 10^{-7}}{\pi} \ln \frac{d}{a} = 2.5 \times 10^{-6}$$

or $\ln \frac{d}{a} = 6.25 \longrightarrow \frac{d}{a} = e^{6.25} = 518.01$

$$a = \frac{3}{518.01} = 5.78 \text{mm}$$

$D = 2a = \underline{\underline{11.58 \text{ mm}}}.$

Prob. 8.35

$$L = \frac{N^2 \mu S}{l} \longrightarrow N^2 = \frac{Ll}{\mu S} = \frac{L\, 2\pi\rho_o}{\mu_o \mu_r S}$$

$$= \frac{2.5 \times 2\pi \times 0.5}{4\pi \times 10^{-7} \times 200 \times 12 \times 10^{-4}}$$

$$= \frac{25}{96} \times 10^{8}$$

$$N = \underline{\underline{5103 \text{ turns}}}.$$

Prob. 8.36

$$\Psi_{12} = \int \vec{B}_1 \cdot d\vec{S} = \int_{\rho=\rho_o}^{\rho_o+a} \int_{z=0}^{b} \frac{\mu_o I}{2\pi\rho}\, dz\, d\rho = \frac{\mu_o I b}{2\pi} \ln \frac{a+\rho_o}{\rho_o}$$

for $N=1$,

$$M_{12} = \frac{N\Psi_{12}}{I_1} = \frac{\mu_o b}{2\pi} \ln \frac{a+\rho_o}{\rho_o}$$

$$= \frac{4\pi \times 10^{-7} (1) \ln 2}{2\pi} = \underline{\underline{0.1386 \ \mu H}}.$$

Prob. 8.37

from Prob. 7.25,

$$\lambda = \Psi N = \frac{\mu_o N I}{2\pi} \int_0^{2\pi} \int_{\rho=0}^{a} \frac{r\, d\theta\, dr}{\rho_o + r\cos\theta}$$

$$= \mu_o N I \left[\rho_o - (\rho_o^2 - a^2)^{\frac{1}{2}}\right]$$

$$M = \frac{\lambda}{I} = \mu_o N \left[\rho_o - (\rho_o^2 - a^2)^{1/2} \right].$$

Prob. 8.38

We may approximate the longer solenoid as infinite so that $B_1 = \mu_o \frac{N_1 I_1}{l_1}$. The flux linking the second solenoid is

$$\Psi_2 = N_2 B_1 S_1 = \mu_o \frac{N_1 I_1}{l_1} \pi r_1^2$$

$$M = \frac{\Psi_2}{I_1} = \frac{\mu_o N_1 N_2}{l_1} \cdot \pi r_1^2$$

Prob 8.39

(a) $\Psi = B \cdot S = \mu H S$

$$\rightarrow \mu = \frac{\Psi}{HS} = \frac{0.8 \times 10^{-3}}{700 \times 12 \times 10^{-4}} = \frac{8}{8400}$$

$$R = \frac{l}{\mu S} = \frac{0.6 \times 8400}{8 \times 12 \times 10^{-4}} = 525{,}000$$

or $R = \frac{l}{S} \cdot \frac{HS}{\Psi} = \frac{lH}{\Psi} = \frac{0.6 \times 700}{0.8 \times 10^{-3}}$

$= \underline{5.25 \times 10^5 \text{ A-t/Wb}}.$

(b) $NI = \Psi R = Hl = 700 \times 0.6 = 420$ A-t or $NI = \Psi R = 0.8 \times 5.25 \times 10^{5} \times 10^{-3}$

$= \underline{\underline{420 \ A \cdot t}}$

$\boxed{\text{Prob. 8.40}}$

$\mathcal{F} = NI = 400 \times 0.5 = 200 \ A \cdot t.$

$$\mathcal{F}_a = \frac{R_a}{R_a + R_3 + R_1 \| R_2} = \frac{796 \times 10^{3} \times 200}{(796 + 383) \times 10^{3}} = \underline{\underline{190.8 \ A \cdot t}}$$

$$H_a = \frac{\mathcal{F}_a}{l_a} = \frac{190.8}{1 \times 10^{-2}} = \underline{\underline{19{,}080 \ A/m.}}$$

$\boxed{\text{Prob. 8.41}}$

Total $\mathcal{F} = NI = 2000 \times 10 = 20{,}000 \ A \cdot t.$

$$R_c = \frac{l_c}{\mu_0 \mu_r S} = \frac{(24 + 20 - 0.6) \times 10^{-2}}{4\pi \times 10^{-7} \times 1500 \times 2 \times 10^{-4}} = \underline{\underline{0.115 \times 10^{7} \ A \cdot t/m}}$$

$$R_a = \frac{l_a}{\mu_0 \mu_r S} = \frac{0.6 \times 10^{-2}}{4\pi \times 10^{-7} (1) \times 2 \times 10^{-4}} = \underline{\underline{2.387 \times 10^{7} \ A \cdot t/m}}$$

$R = R_a + R_c = 2.502 \times 10^{7} \ A \cdot t/m$

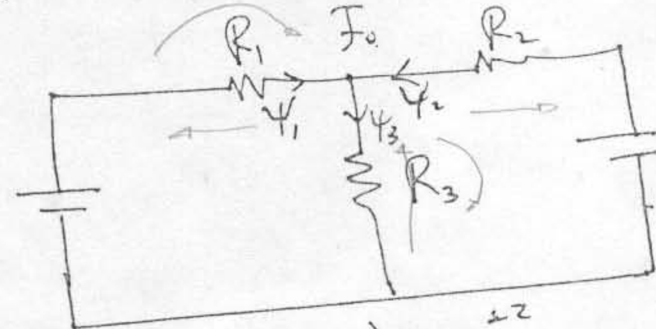
$$\Psi = \frac{\mathcal{F}}{R} = \Psi_a = \Psi_c = \frac{20{,}000}{2.502 \times 10^{7}}$$

$= \underline{\underline{8 \times 10^{-4} \ Wb/m^2.}}$

$$\mathcal{F}_a = \frac{R_a}{R_a + R_c}\mathcal{F} = \frac{2.387 \times 20{,}000}{2.502} = \underline{\underline{19{,}081 \text{ A·t}}}$$

$$\mathcal{F}_c = \frac{R_c}{R_a + R_c}\cdot\mathcal{F} = \frac{0.115}{2.502} \times 20{,}000 = \underline{\underline{919 \text{ A·t}}}.$$

$\boxed{\text{Prob. 8.42}}$



$$R_1 = R_2 = \frac{l}{\mu S} = \frac{(6+6+8)\times 10^{-2}}{4\pi \times 10^{-7} \times 1000 \times 4 \times 10^{-4}} = 3.98 \times 10^5$$

$$R_3 = \frac{8 \times 10^{-2}}{4\pi \times 10^{-7} \times 1000 \times 6 \times 10^{-4}} = 1.061 \times 10^5$$

$$\Psi_1 + \Psi_2 = \Psi_3 \longrightarrow \frac{\mathcal{F}_0 - 3000}{R_1} + \frac{\mathcal{F}_0 - 1000}{R_2} = \frac{0 - \mathcal{F}_0}{R_3}$$

Since $R_1 = R_2$,

$$-\mathcal{F}_0 = \frac{R_3}{R_1}(2\mathcal{F}_0 - 4000) = \frac{1.061}{3.98}(2\mathcal{F}_0 - 4000)$$

$$\mathcal{F}_0 = 695.5 \text{ A·t}$$

$$B_1 = \frac{\Psi_1}{S_1} = \frac{\mathcal{F}_0 - 3000}{R_1 S_1} = \frac{695.5 - 3000}{3.98 \times 10^5 \times 4 \times 10^{-4}} = -14.475$$

$$B_2 = \frac{\Psi_2}{S_2} = \frac{F_0 - 1000}{R_2 S_2} = \frac{695.5 - 1000}{3.98\times10^5 \times 4\times10^{-4}} = -1.913$$

$$B_3 = \frac{\Psi_3}{S_3} = \frac{0 - F_0}{R_3 S_3} = \frac{-695.5}{1.061\times10^5} = -10.925$$

Thus

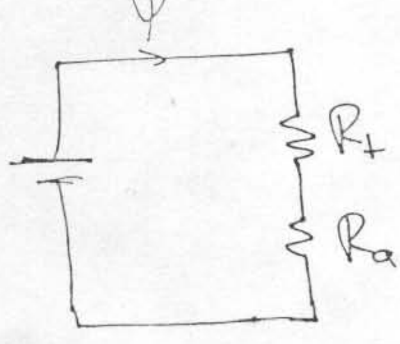
$$\underline{\underline{B_1 = 14.475 \text{ Wb/m}^2, \quad B_2 = 1.913 \text{ Wb/m}^2, \quad B_3 = 10.925 \text{ Wb/m}^2.}}$$

$\boxed{\text{Prob. 8.43}}$

$$F = \frac{B^2 S}{2\mu_o} = \frac{\Psi^2}{2\mu_o S} = \frac{4\times10^{-6}}{2\times4\pi\times10^{-7}\times0.3\times10^{-4}} = \underline{\underline{53.05 \text{ kN}}}$$

$\boxed{\text{Prob. 8.44}}$

(a)



$$F = NI = 200\times10^{-3}\times750$$
$$= 150 \text{ A-t}$$

$$R_a = \frac{l_a}{\mu_o S} = \frac{10^{-3}}{25\times10^{-6}\mu_o} = 3.183\times10^7$$

$$R_t = \frac{l_t}{\mu_o\mu_r S} = \frac{2\pi\times0.1}{\mu_o\times300\times25\times10^{-6}} = 20\times10^7$$

$$\Psi = \frac{F}{R_a + R_t} = \frac{150}{10^7(3.183+20)} = 6.47\times10^{-7}$$

$$F = \frac{B^2 S}{2\mu_0} = \frac{\Psi^2}{2\mu_0 S} = \frac{41.861 \times 10^{-14}}{2 \times 4\pi \times 10^{-7} \times 25 \times 10^{-6}}$$

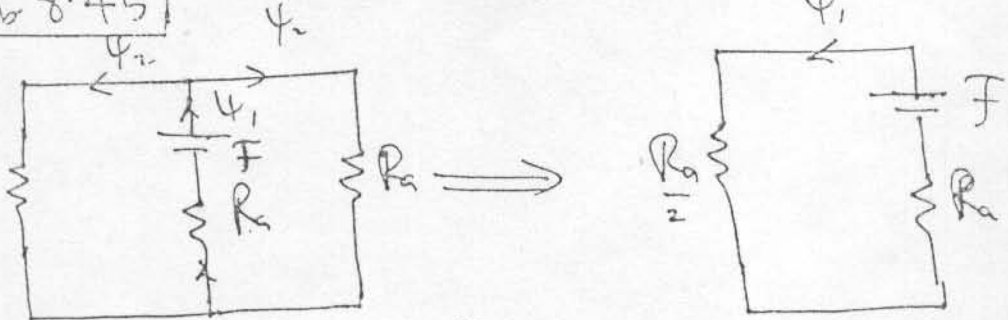
$$= \underline{\underline{6.66\ mN}}.$$

(b) If $\mu_t \to \infty$, $\mathcal{R}_t = 0$, $\Psi = \frac{\mathcal{F}}{\mathcal{R}_a} = \frac{150}{3.183 \times 10^7}$

$$F_2 = I_2 dl_2 \cdot B_1 = I_2 dl_2 \frac{\Psi_1}{S} = \frac{2 \times 10^{-3} \times 5 \times 10^{-3} \times 150}{3.183 \times 10^7 \times 25 \times 10^{-6}}$$

$$F_2 = \underline{\underline{1.885\ nN}}$$

$\boxed{\text{Prob 8.45}}$



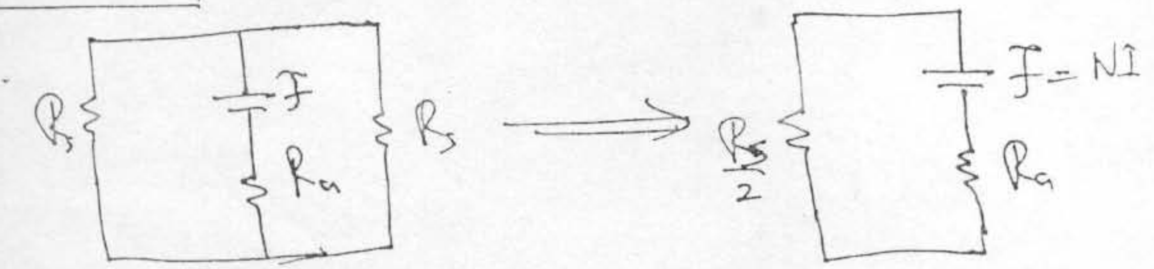
$$\Psi_1 = 2\Psi_2, \quad \Psi_1 = \frac{\mathcal{F}}{\frac{3}{2}\mathcal{R}_a} = \frac{2\mathcal{F}}{3\mathcal{R}_a} \to \Psi_2 = \frac{\mathcal{F}}{3\mathcal{R}_a}$$

$$F = 2\left(\frac{\Psi_2^2}{2\mu_0 S}\right) + \frac{\Psi_1^2}{2\mu_0 S} = \frac{3\Psi_1^2}{\mu_0 S} = \frac{\mathcal{F}^2}{3\mathcal{R}_a^2 \mu_0^2}$$

$$= \frac{\mu_0 S \mathcal{F}^2}{3 l_a^2} = \frac{4\pi \times 10^{-7} \times 210 \times 10^{-4} \times 9 \times 10^6}{3 \times 10^{-6}}$$

$$= 24\pi \times 10^3 = mg \longrightarrow m = \frac{24\pi \times 10^3}{9.8} = \underline{7694\ kg}.$$

Prob. 8.46



Since $\mu \to \infty$ for the core, $\mathcal{R}_c = 0$.

$$\mathcal{F} = NI = \Psi\left(\mathcal{R}_a + \frac{\mathcal{R}_s}{2}\right) = \frac{\Psi\,(a/2 + x)}{\mu_o S}$$

$$= \frac{\Psi(2x + a)}{2\mu_o S}$$

$$F = \frac{B^2 S}{2\mu_o} = \frac{\Psi^2}{2\mu_o S} = \frac{1}{2\mu_o S} \cdot \frac{N^2 I^2 4\mu_o^2 S^2}{(a+2x)^2}$$

$$= \frac{2N^2 I^2 \mu_o S}{(a+2x)^2}$$

$\vec{F} = -F\vec{a}_x$ since the force is attractive, i.e.

$$\vec{F} = \frac{-2N^2 I^2 \mu_o S\, \vec{a}_x}{(a+2x)^2}.$$

$$\frac{B^2 S}{2N_o} = \Psi\frac{(a+2x)}{2N_oS} = F$$

$$\frac{B^2 S^2}{\Psi(a+2x)}$$

CHAPTER 9

P.E. 9.1

(a) $V_{emf} = \int (\vec{u} \times \vec{B}) \cdot d\vec{l} = uBl = 8(0.5)(0.1) = \underline{\underline{0.4 \text{ V}}}$

(b) $I = \dfrac{V_{emf}}{R} = \dfrac{0.4}{20} = \underline{\underline{20 \text{ mA}}}.$

(c) $\vec{F}_m = I\vec{l} \times \vec{B} = 0.2\left(0.1\,\vec{a}_y \times -0.5\,\vec{a}_z\right) = \underline{\underline{-\vec{a}_x \text{ mN}}}$

(d) $P = Fu = I^2R = 8 \text{ mW}$

or $P = \dfrac{V_{emf}^2}{R} = \dfrac{(0.4)^2}{20} = \underline{\underline{8 \text{ mW}}}.$

P.E 9.2

(a) $V_{emf} = \int (\vec{u} \times \vec{B}) \cdot d\vec{l}$

where $\vec{B} = B_o \vec{a}_y = B_o(\sin\phi\, \vec{a}_\rho + \cos\phi\, \vec{a}_\phi)$, $B_o = 0.05$

$(\vec{u} \times \vec{B}) \cdot d\vec{l} = -\rho\omega B_o \sin\phi\, dz = -0.2\pi \sin(\omega t + \pi/2)\, dz$

$V_{emf} = \int_0^{0.03} (\vec{u} \times \vec{B}) \cdot d\vec{l} = -6\pi \cos 100\pi t \text{ mV}$

At $t = 1$ ms, $V_{emf} = -6\pi \cos 0.1\pi = \underline{\underline{-17.93 \text{ mV}}}$

$i = \dfrac{V_{emf}}{R} = -60\pi \cos 100\pi t \text{ mA}$

At $t = 3$ ms, $i = -60\pi \cos 0.3\pi = \underline{\underline{-0.1108 \text{ A}}}.$

(b) Method 1:

$$\Psi = \int \vec{B}\cdot d\vec{S} = \int B_o t\,(\cos\phi\,\vec{a}_\rho - \sin\phi\,\vec{a}_\phi)\cdot d\rho\, dz\, \vec{a}_\phi$$

$$= -\int_0^{\rho_o}\int_0^{z_o} Bt \sin\phi \, d\rho\, dz = -B_o\rho_o z_o t \sin\phi$$

where $B_o = 0.02$, $\rho_o = 0.04$, $z_o = 0.03$,

$$\phi = wt + \pi/2 .$$

$$\Psi = -B_o\rho_o z_o t \cos wt$$

$$V_{emf} = -\frac{\partial\Psi}{\partial t} = B_o\rho_o z_o \cos wt - B_o\rho_o z_o t w \sin wt$$

$$= (0.02)(0.04)(0.03)[\cos wt - wt\sin wt]$$

$$= 24\,[\cos wt - wt \sin wt]\ \mu V$$

Method 2:

$$V_{emf} = -\int \frac{\partial \vec{B}}{\partial t}\cdot d\vec{S} + \int (\vec{u}\times\vec{B})\cdot d\vec{l}$$

$$\vec{B} = B_o t\,\vec{a}_x = B_o t\,(\cos\phi\,\vec{a}_\rho - \sin\phi\,\vec{a}_\phi),\ \phi = wt + \pi/2$$

$$\frac{\partial \vec{B}}{\partial t} = B_o\,(\cos\phi\,\vec{a}_\rho - \sin\phi\,\vec{a}_\phi)$$

Note that only explicit dependence of $\vec{B}$ on time is accounted for, i.e. we make ϕ = constant because it is transformer (stationary) emf. Thus,

$$V_{emf} = -B_o \int_0^{\rho_o}\int_0^{z_o} (\cos\phi\, \vec{a}_\rho - \sin\phi\, \vec{a}_\phi)\cdot d\rho\, dz\, \vec{a}_\phi$$

$$+ \int_{z_o}^{0} -\rho_o w B_o t \cos\phi\, dz$$

$$= B_o \rho_o z_o (\sin\phi + wt\cos\phi), \quad \phi = wt + \pi/2$$

$$= B_o \rho_o z_o (\cos wt - wt \sin wt)$$

as obtained earlier.

At $t = 1ms$,

$$V_{emf} = 24 \left[\cos 18^\circ - 100\pi \times 10^{-3} \sin 18^\circ\right] \mu V$$

$$= \underline{\underline{20.5 \ \mu V}}.$$

At $t = 3ms$,

$$i = 240 \left[\cos 54^\circ - 0.3\pi \sin 54^\circ\right] mA$$

$$= \underline{\underline{-41.92 \ mA}}.$$

$\boxed{P.E. 9.3}$

$$V_1 = -N_1 \frac{d\psi}{dt}, \quad V_2 = -N_2 \frac{d\psi}{dt}$$

$$\frac{V_2}{V_1} = \frac{N_2}{N_1} \longrightarrow V_2 = \frac{N_2}{N_1} V_1 = \frac{300 \times 120}{500} = \underline{\underline{72 V}}$$

$\boxed{P.E. 9.4}$

(a) $\vec{J}_d = \frac{\partial \vec{D}}{\partial t} = \underline{\underline{-20 w \varepsilon_o \sin(wt - 50x)\, \vec{a}_y \quad A/m^2}}$

(b) $\nabla \times \vec{H} = \vec{J}_d \rightarrow -\dfrac{\partial H_z}{\partial x}\vec{a}_y = -20\,w\epsilon_o \sin(wt - 50x)\,\vec{a}_y$

or $\vec{H} = \dfrac{20\,w\epsilon_o}{50} \cos(wt - 50x)\,\vec{a}_z$

$= \underline{\underline{0.4\,w\epsilon_o \cos(wt - 50x)\,\vec{a}_z}}$ A/m.

(c) $\nabla \times \vec{E} = -\mu_o \dfrac{\partial \vec{H}}{\partial t} \rightarrow \dfrac{\partial E_y}{\partial x}\vec{a}_z = 0.4\,\mu_o w \epsilon_o \sin(wt - 50x)\,\vec{a}_z$

$1000 = 0.4\,\mu_o \epsilon_o w^2 = 0.4\,\dfrac{w^2}{c^2}$

or $w = \underline{\underline{1.5 \times 10^{10}\ \text{rad/s}}}$.

$\boxed{\text{P.E. 9.5}}$

(a) $j^3\left(\dfrac{1+j}{2-j}\right)^2 = -j\left[\dfrac{\sqrt{2}\,\angle 45^\circ}{\sqrt{5}\,\angle -26.56^\circ}\right]^2 = -j\left(\dfrac{2}{5}\angle 143.13^\circ\right)$

$= \underline{\underline{0.24 + j0.32}}$

(b) $6\angle 30^\circ + j5 - 3 + e^{j45^\circ} = 5.196 + j3 + j5 - 3 + 0.7071(1+j)$

$= \underline{\underline{2.903 + j8.707}}$.

$\boxed{\text{P.E. 9.6}}$

$\vec{P} = 2\sin(10t + x - \pi/4)\,\vec{a}_y$

$= 2\cos(10t + x - \pi/4 - \pi/2)\,\vec{a}_y, \quad w = 10$

$= \text{Re}\left[2e^{j(x - 3\pi/4)}\,\vec{a}_y\, e^{jwt}\right] = \text{Re}\left[\check{P}_s e^{jwt}\right]$

i.e. $\check{P}_s = \underline{2e^{j(x-3\pi/4)}\,\vec{a}_y}$

$$\tilde{Q} = Re\left[\vec{Q}_s e^{jwt}\right] = Re\left[e^{j(x+wt)}(\vec{a}_x - \vec{a}_z)\right]\sin\pi y$$

$$= \underline{\underline{\sin\pi y\cos(wt+x)(\vec{a}_x - \vec{a}_z)}}.$$

$\boxed{P.E\cdot 9.7}$

$$-\mu\frac{\partial\vec{H}}{\partial t} = \nabla\times\vec{E} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(E_\phi\sin\theta)\vec{a}_r - \frac{1}{r}\frac{\partial}{\partial r}(rE_\phi)\vec{a}_\theta$$

$$= \frac{2\cos\theta}{r^2}\cos(wt-\beta r)\vec{a}_r - \frac{\beta}{r}\sin\theta\sin(wt-\beta r)\vec{a}_\theta$$

$$\vec{H} = \frac{2\cos\theta}{wr^2}\sin(wt-\beta r)\vec{a}_r + \frac{\beta}{wr}\sin\theta\cos(wt-\beta r)\vec{a}_\theta.$$

$$\beta = \frac{w}{c} = \frac{6\times10^7}{3\times10^8} = \underline{\underline{0.2\ rad/m.}}$$

$$\vec{H} = \frac{10^{-7}}{3r^2}\cos\theta\sin(6\times10^7t-0.2r)\vec{a}_r$$

$$+\frac{10^{-8}}{3r}\sin\theta\cos(6\times10^7t-0.2r)\vec{a}_\theta.$$

$\boxed{P.E\cdot 9.8}$

$$w = \frac{3}{\sqrt{\mu\epsilon}} = \frac{3c}{\sqrt{\epsilon_r\mu_r}} = \frac{9\times10^8}{\sqrt{10}} = \underline{\underline{2.846\times10^8\ rad/s.}}$$

$$\vec{E} = \frac{1}{\epsilon}\int\nabla\times\vec{H}\,dt = -\frac{6}{w\epsilon}\cos(wt-3y)\vec{a}_x$$

$$= \frac{-6}{\frac{9\times10^8}{\sqrt{10}}\cdot\frac{10^{-9}}{36\pi}(5)}\cos(wt-3y)\vec{a}_x$$

$$\vec{E} = \underline{\underline{-476.8\cos(2.846\times10^8t-3y)\vec{a}_x\ V/m.}}$$

Prob. 9.1

$$V = -\frac{\partial \Psi}{\partial t} = -\frac{\partial}{\partial t}\int \vec{B}\cdot d\vec{s} = -\frac{\partial \vec{B}}{\partial t}\cdot \vec{S}$$

$$= 3770 \sin 377t \quad \pi(0.2)^2 \quad \times 10^{-3}$$

$$= \underline{\underline{0.4738 \sin 377t \text{ V}}}.$$

Prob. 9.2

$$V_{emf} = -\frac{\partial \lambda}{\partial t} = -N\frac{\partial}{\partial t}\int \vec{B}\cdot d\vec{s} = -NBS\frac{d\phi}{dt}$$

$$= -NBS\omega = -50 \times 0.06 \times 0.3 \times 0.4$$

$$= \underline{\underline{-54 \text{ V}}}.$$

Prob 9.3

(a) $$I = \frac{V}{R} = \frac{1}{R}\frac{\partial \Psi}{\partial t} = \frac{1}{R} S \frac{\partial B}{\partial t}$$

$$= \frac{0.3 \times 0.5 \times 0.05}{10} = \underline{\underline{0.75 \text{ mA}}}$$

(b) $$\Psi = \int \vec{B}\cdot d\vec{s} = \int_{y=0}^{0.3}\int_{x=0}^{0.5} 5 \sin(120\pi t - 2y) \times 10^{-3}\, dx\, dy$$

$$= \frac{2.5}{2}\cos(120\pi t - 2y)\Big|_0^{0.3} \text{ mWb}$$

$$= 1.25\left[\cos(120\pi t - 0.6) - \cos 120\pi t\right] \text{ mWb}$$

$$I = \frac{1}{R}\frac{\partial \Psi}{\partial t} = \frac{1}{10} \times 1.25(-120\pi) \times 10^{-3}\left[\sin(120\pi t - 0.6) - \sin 120\pi t\right]$$

$$I = -15\pi\left[\sin(120\pi t - 0.6) - \sin 120\pi t\right] \text{ mA}$$

$$= -15\pi(2)\cos(\omega t - 0.3)\sin(-0.3)$$

$$= \underline{\underline{27.85\cos(120\pi t - 0.3) \text{ mA}}}.$$

Prob. 9.4

$$\vec{B} = \frac{\mu_o I}{2\pi y}(-\hat{a}_x)$$

$$\Psi = \int \vec{B}\cdot d\vec{s} = \frac{\mu_o I}{2\pi}\int_{z=0}^{a}\int_{y=\rho}^{\rho+a}\frac{dz\,dy}{y} = \frac{\mu_o I a}{2\pi}\ln\frac{\rho+a}{\rho}$$

$$V_{emf} = -\frac{\partial\Psi}{\partial t} = -\frac{\partial\Psi}{\partial\rho}\cdot\frac{\partial\rho}{\partial t} = -\frac{\mu_o I a}{2\pi}u_o\frac{d}{d\rho}\left[\ln(\rho+a) - \ln\rho\right]$$

$$= -\frac{\mu_o I a u_o}{2\pi}\left[\frac{1}{\rho+a} - \frac{1}{\rho}\right] = \underline{\underline{\frac{\mu_o a^2 I u_o}{2\pi\rho(\rho+a)}}}$$

Prob 9.5

This is similar to Prob. 9.4. Assume loop is of width z.

$$\Psi = \frac{\mu_o I z}{2\pi}\ln\frac{\rho+a}{\rho}$$

$$V_{emf} = -\frac{\partial\Psi}{\partial t} = -\frac{d\Psi}{dz}\cdot\frac{dz}{dt} = -\frac{\mu_o I}{2\pi}\ln\frac{\rho+a}{\rho}\cdot u$$

$$= -\frac{4\pi\times10^{-7}}{2\pi}\times15\times3\ln\frac{60}{20} = -9.888\,\mu V$$

Thus the induced emf = 9.888 mV, point A at higher potential.

$\boxed{\text{Prob 9.6}}$

$$V_{emf} = -\int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} + \int (\vec{u} \times \vec{B}) \cdot d\vec{l}$$

where $\vec{B} = B_o \cos wt\, \vec{a}_x$, $\vec{u} = u_o \cos wt\, \vec{a}_y$, $d\vec{l} = dz\, \vec{a}_z$.

$$V_{emf} = \int_{z=0}^{l} \int_{y=-a}^{y} B_o w \sin wt \, dy\, dz - \int_0^l B_o u_o \cos^2 wt \, dz$$

$$= B_o w l (y+a) \sin wt - B_o u_o l \cos^2 wt$$

Alternatively,

$$\Psi = \int \vec{B} \cdot d\vec{S} = \int_{z=0}^{l} \int_{y=-a}^{y} B_o \cos wt\, \vec{a}_x \cdot dy\, dz\, \vec{a}_x$$

$$= B_o (y+a) l \cos wt$$

$$V_{emf} = -\frac{\partial \Psi}{\partial t} = B_o (y+a) l w \sin wt - B_o \frac{dy}{dt} l \cos wt$$

But $\frac{dy}{dt} = u = u_o \cos wt \longrightarrow y = \frac{u_o}{w} \sin wt$

$$V_{emf} = B_o w l (y+a) \sin wt - B_o u_o l \cos^2 wt$$

$$= B_o u_o l \sin^2 wt + B_o w a l \sin wt - B_o u l \cos^2 wt$$

$$= -B_o u_o l \cos 2wt + B_o w a l \sin wt$$

$$= 6 \times 10^{-3} \times 5 \left[10 \times 10 \sin 10t - 2 \cos 20t \right]$$

$$V_{emf} = \underline{\underline{3\sin 10t - 0.06\cos 20t \ V.}}$$

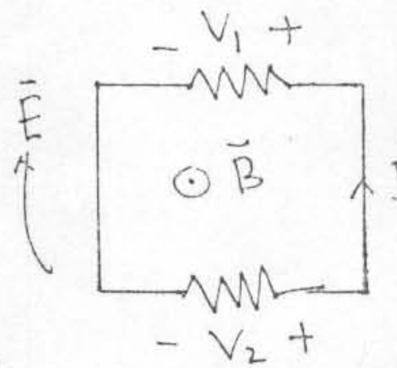
$\boxed{\text{Prob. 9.7}}$

$$V_{emf} = \int (\vec{u} \times \vec{B}) \cdot d\vec{l} = uBl$$

$$= \frac{80 \times 10^3 \times 70 \times 10^{-6} \times 2.5}{3600}$$

$$= \underline{\underline{3.89 \text{ mV}.}}$$

$\boxed{\text{Prob. 9.8}}$



$$\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt}\int \vec{B} \cdot d\vec{S}$$

$$= I(R_1 + R_2)$$

$$\frac{dB}{dt} \cdot S = I(R_1 + R_2) \quad \text{--- (1)}$$

Also, $\oint \vec{E} \cdot d\vec{l} = V_1 - V_2 = -\frac{dB}{dt} \cdot S \quad \text{--- (2)}$

Hence, $V_1 = IR_1 = -\frac{SR_1}{R_1 + R_2}\frac{dB}{dt}$

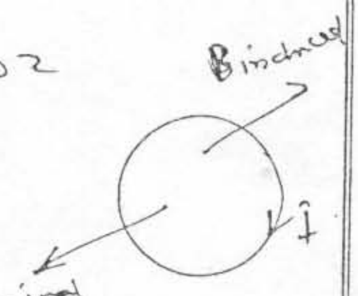
$$V_2 = -IR_2 = \frac{SR_2}{R_1 + R_2}\frac{dB}{dt}$$

$$V_1 = \frac{10 \times 10^{-4} \times 10}{15} \times 0.2 \times 150\pi \sin 150\pi t = \underline{\underline{0.0628 \sin 150\pi t \ V}}$$

$$V_2 = \frac{-10 \times 10^{-4} \times 5}{15} \times 0.2 \times 150\pi \sin 150\pi t = \underline{\underline{-0.0314 \sin 150\pi t \ V}}$$

Prob. 9.9

$d\Psi = 0.63 - 0.45 = 0.18, \quad dt = 0.02$



$$V_{emf} = N\frac{d\Psi}{dt} = 10\left(\frac{0.18}{0.02}\right) = 90V$$

$$I = \frac{V_{emf}}{R} = \frac{90}{15} = \underline{\underline{6\ A}}$$

Using Lenz's law, the direction of the induced current is $\underline{\underline{\text{counterclockwise}}}$

Prob. 9.10

$$\mathcal{R} = \mathcal{R}_c + \mathcal{R}_a = \frac{1}{\mu_0 S}\left(\frac{l_c}{\mu_r} + \frac{l_a}{1}\right)$$

$$= \frac{1}{4\pi\times10^{-7}\times4\times10^{-4}}\left[6\times10^{-3} + \frac{(40-0.6)\cdot10^{-2}}{1000}\right]$$

$$= 12.72\times10^{6}$$

$$\Psi = \frac{\mathcal{F}}{\mathcal{R}} = \frac{N_1 I_1}{\mathcal{R}}, \quad V_2 = -N_2\frac{\partial\Psi}{\partial t} = -\frac{N_1 N_2}{\mathcal{R}}\frac{dI_1}{dt}$$

$$V_2 = \frac{500(300)\,1200\pi\sin120\pi t}{12.72\times100}$$

$$= \underline{\underline{44.46\sin120\pi t\ V.}}$$

Prob. 9.11

$V = \int(\vec{u}\times\vec{B})\cdot d\vec{l}$, where $\vec{u} = \rho\omega\,\vec{a}_\phi$, $\vec{B} = B_0\,\vec{a}_z$

$$V = \int_{\rho_1}^{\rho_2}\rho\omega B_0\,d\rho = \frac{\omega B_0}{2}\left(\rho_2^2 - \rho_1^2\right)$$

$$V = \frac{60 \times 5}{2} \cdot 10^{-3}(100-4) \cdot 10^{-4} = \underline{\underline{4.32 \text{ mV}}}.$$

$\boxed{\text{Prob. 9.12}}$

$$J_{ds} = j\omega D_s \longrightarrow |J_{ds}|_{max} = \omega \epsilon E_s = \omega \epsilon \frac{V_s}{d}$$

$$= \frac{10^{-9}}{36\pi} \times \frac{2\pi \times 20 \times 10^6 \times 50}{0.2 \times 10^{-3}}$$

$$= \underline{\underline{277.8 \text{ A/m}^2}}$$

$$I_{ds} = J_{ds} \cdot S = \frac{1000}{36} \times 2.8 \times 10^{-4} = \underline{\underline{77.78 \text{ mA}}}$$

$\boxed{\text{Prob. 9.13}}$

$$J = \sigma E = \underline{\underline{4 \cos 10^7 t \text{ nA/m}^2}}$$

$$J_d = \frac{\partial D}{\partial t} = \epsilon_r \epsilon_o \frac{\partial E}{\partial t} = -20 \times 10^7 \times 10^{-6} \times \frac{10^{-9}}{36\pi} \times 5 \sin 10^7 t$$

$$= \underline{\underline{-8.842 \sin 10^7 t \text{ nA/m}^2}}.$$

$\boxed{\text{Prob. 9.14}}$

$$J_c = \frac{I_c}{S} = \sigma E \longrightarrow E = \frac{I_c}{\sigma S}$$

$$J_d = j\omega\epsilon E \longrightarrow |J_d| = \omega \epsilon E = \frac{\omega \epsilon I_c}{\sigma S}$$

$$|J_d| = \frac{10^9 \times 4.6 \times 10^{-9}/36\pi \times 0.2 \times 10^{-3}}{2.5 \times 10^6 \times 10 \times 10^{-4}} \text{ A/m} = \underline{\underline{3.254 \text{ nA/m}^2}}$$

$\boxed{\text{Prob. 9.15}}$

If $|\vec{J}_d| = |\vec{J}|$, then $\epsilon\omega|\vec{E}| = \sigma|\vec{E}|$ or

$$2\pi f \epsilon_o \epsilon_r = \sigma \longrightarrow f = \frac{\sigma}{2\pi\epsilon_o\epsilon_r} = \frac{4\times10^{-3}}{2\pi\times\frac{10^{-9}}{36\pi}\times 80}$$

$$= \underline{\underline{900\ \text{kHz}}}.$$

$\boxed{\text{Prob. 9.16}}$

(a) $\nabla\cdot\vec{E}_s = \rho_s/\epsilon, \quad \nabla\cdot\vec{H}_s = 0$

$\underline{\underline{\nabla\times\vec{E}_s = j\omega\mu\vec{H}_s, \quad \nabla\times\vec{H}_s = (\sigma - j\omega\epsilon)\vec{E}_s}}.$

(b) $\nabla\cdot\vec{D} = \rho_v \longrightarrow \dfrac{\partial D_x}{\partial x} + \dfrac{\partial D_y}{\partial y} + \dfrac{\partial D_z}{\partial z} = \rho_v \quad -(1)$

$\nabla\cdot\vec{B} = 0 \longrightarrow \dfrac{\partial B_x}{\partial x} + \dfrac{\partial B_y}{\partial y} + \dfrac{\partial B_z}{\partial z} = 0 \quad -(2)$

$\nabla\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t} \rightarrow \dfrac{\partial E_z}{\partial y} - \dfrac{\partial E_y}{\partial z} = -\dfrac{\partial B_x}{\partial t} \quad -(3)$

$\dfrac{\partial E_x}{\partial z} - \dfrac{\partial E_z}{\partial x} = -\dfrac{\partial B_y}{\partial t} \quad -(4)$

$\dfrac{\partial E_y}{\partial x} - \dfrac{\partial E_x}{\partial y} = -\dfrac{\partial B_z}{\partial t} \quad -(5)$

$\nabla\times\vec{H} = \vec{J} + \dfrac{\partial\vec{D}}{\partial t} \rightarrow \dfrac{\partial H_z}{\partial y} - \dfrac{\partial H_y}{\partial z} = J_x + \dfrac{\partial D_x}{\partial t} \quad -(6)$

$\dfrac{\partial H_x}{\partial z} - \dfrac{\partial H_z}{\partial x} = J_y + \dfrac{\partial D_y}{\partial t} \quad -(7)$

$\underline{\underline{\dfrac{\partial H_y}{\partial x} - \dfrac{\partial H_x}{\partial y} = J_z + \dfrac{\partial D_z}{\partial t} \quad -(8)}}$

Prob.9.19

If $\rho_v = 0$, $\nabla\cdot\vec{E} = 0$ —(1)

$$\vec{H} = -\frac{1}{\mu_o}\int \nabla\times\vec{E}\,dt$$

But $\nabla\times\vec{E} = -30k\sin 2x\cos(kz-wt)\vec{a}_x + 60\cos 2x\sin(kz-wt)\vec{a}_z$

$$\vec{H} = -\frac{30k}{\mu_o w}\sin 2x\sin(kz-wt)\vec{a}_x - \frac{60}{\mu_o w}\cos 2x\cos(kz-wt)\vec{a}_z \text{ A/m}$$

$\nabla\cdot\vec{H} = 0$ — (2)

$$\vec{J}_d = \epsilon_o\frac{\partial\vec{E}}{\partial t} = \underline{\underline{-30w\epsilon_o\sin 2x\cos(kz-wt)\vec{a}_y \text{ A/m}^2}}$$

$$\nabla\times\vec{H} = \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\vec{a}_y$$

$$= -\frac{30\epsilon_o}{w}\sin 2x\cos(kz-wt)\left[\frac{k^2}{\mu_o\epsilon_o} + \frac{4}{\mu_o\epsilon_o}\right]\vec{a}_y$$

$$= -30\epsilon_o w\sin 2x\cos(kz-wt)\vec{a}_y = \vec{J}_d \quad —(3)$$

since $\frac{k^2}{\mu_o\epsilon_o} + \frac{4}{\mu_o\epsilon_o} = w^2$. Thus the four Maxwell's equations are satisfied by $\vec{E}$ showing that $\vec{E}$ is a genuine EM field.

Prob. 9.20

$$\epsilon \frac{\partial \vec{E}}{\partial t} = \nabla \times \vec{H} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & H_z \end{vmatrix} = -\frac{\partial H_z}{\partial x} \vec{a}_y$$

$$= 0.6\beta \sin\beta x \cos wt \, \vec{a}_y$$

$$\vec{E} = \frac{1}{\epsilon} \int \nabla \times \vec{H} \, \partial t = \frac{0.6\beta}{w\epsilon} \sin\beta x \sin wt \, \vec{a}_y$$

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_y & 0 \end{vmatrix} = \frac{\partial E_y}{\partial x} \vec{a}_z$$

$$= \frac{0.6\beta^2}{w\epsilon} \cos\beta x \sin wt \, \vec{a}_z$$

$$\vec{H} = -\frac{1}{\mu} \int \nabla \times \vec{E} \, \partial t = \frac{0.6\beta^2}{w^2 \mu\epsilon} \cos\beta x \cos wt \, \vec{a}_z$$

Thus $\beta = w\sqrt{\mu\epsilon} = \frac{w}{c}\sqrt{\mu_r \epsilon_r} = \frac{10^8 (2.25)}{3 \times 10^8}$

$$= \underline{\underline{0.8333 \text{ rad/m.}}}$$

$$E_0 = \frac{0.6\beta}{w\epsilon} = \frac{0.6 w\sqrt{\mu\epsilon}}{w} = 0.6\sqrt{\frac{\mu}{\epsilon}} = 0.6(377)\sqrt{\frac{\mu_r}{\epsilon_r}}$$

$$= \frac{0.6 \times 377}{2.25} = 100.5$$

$$\vec{E} = \underline{\underline{100.5 \sin\beta x \sin wt \, \vec{a}_y \text{ V/m.}}}$$

Prob. 9.21

from Maxwell's equations,

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad -(1)$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad -(2)$$

Dotting both sides of (2) with $\vec{E}$ gives

$$\vec{E} \cdot (\nabla \times \vec{H}) = \vec{E} \cdot \vec{J} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \quad -(3)$$

But for any arbitrary vectors $\vec{A}$ and $\vec{B}$,

$$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$$

Applying this on the left-hand side of (3) we get by letting $\vec{A} \equiv \vec{B}$ and $\vec{B} \equiv \vec{E}$, we get

$$\vec{H} \cdot (\nabla \times \vec{E}) + \nabla \cdot (\vec{H} \times \vec{E}) = \vec{E} \cdot \vec{J} + \frac{1}{2} \frac{\partial}{\partial t} (\vec{D} \cdot \vec{E}) \quad -(4)$$

from (1),

$$\vec{H} \cdot (\nabla \times \vec{E}) = \vec{H} \cdot \left(-\frac{\partial \vec{B}}{\partial t}\right) = -\frac{1}{2} \frac{\partial}{\partial t} (\vec{B} \cdot \vec{H})$$

Substituting this in (4) gives

$$-\frac{1}{2} \frac{\partial}{\partial t} (\vec{B} \cdot \vec{H}) - \nabla \cdot (\vec{E} \times \vec{H}) = \vec{J} \cdot \vec{E} + \frac{1}{2} \frac{\partial}{\partial t} (\vec{D} \cdot \vec{E})$$

Rearranging terms and then taking the volume integral of both sides

$$\int_v \nabla \cdot (\vec{E} \times \vec{H}) dv = -\frac{\partial}{\partial t} \frac{1}{2} \int_v (\vec{E} \cdot \vec{D} + \vec{H} \cdot \vec{B}) dv - \int_v \vec{J} \cdot \vec{E} \, dv$$

$$\oint_s (\vec{E} \times \vec{H}) \cdot d\vec{s} = -\frac{\partial W}{\partial t} - \int_v \vec{J} \cdot \vec{E} \, dv$$

or

$$\frac{\partial W}{\partial t} = -\oint_s (\vec{E} \times \vec{H}) \cdot d\vec{S} - \int \vec{E} \cdot \vec{J}\, dv$$

as required.

$\boxed{\text{Prob. 9.22}}$

In air, $\sigma = 0$, $\epsilon = \epsilon_0$, $\mu = \mu_0$

$$\nabla \times \vec{H} = \vec{J}_d = -\frac{\partial H_\phi}{\partial z} \vec{a}_\rho$$

$$\vec{J}_d = \underline{\underline{\frac{10}{3\rho} \cos(10^8 t - \tfrac{z}{3}) \vec{a}_\rho \ \text{A/m}^2.}}$$

$$\vec{J}_d = \frac{\partial \vec{D}}{\partial t} \rightarrow \quad \vec{E} = \frac{1}{\epsilon_0} \int \vec{J}\, dt$$

$$= \frac{10}{3\rho} \cdot \frac{1}{10^8 \epsilon_0} \sin(10^8 t - z/3)\, \vec{a}_\rho$$

$$= \underline{\underline{\frac{1200\pi}{\rho} \sin(10^8 t - z/3) \vec{a}_\rho \ \text{V/m}.}}$$

$\boxed{\text{Prob 9.23}}$

$$\nabla \times \vec{E} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho E_\phi) \vec{a}_z = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho^2 t e^{-\rho - t}\right) \vec{a}_z$$

$$= (2 - \rho) t e^{-\rho - t} \vec{a}_z$$

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times \vec{E} \longrightarrow \vec{B} = -\int (\nabla \times \vec{E})\, dt$$

$$= \int \underbrace{(\rho - 2) t}_{v} \underbrace{e^{-\rho - t} dt}_{du} \vec{a}_z$$

Integrating by parts yields

$$\vec{B} = \left[-(\rho-2)te^{-\rho-t} + \int(\rho-2)e^{-\rho-t}\,dt\right]\vec{a}_z$$

$$= \underline{\underline{(2-\rho)(1+t)e^{-\rho-t}\,\vec{a}_z \quad Wb/m^2.}}$$

$$\vec{J} = \nabla\times\vec{H} = \nabla\times\frac{\vec{B}}{\mu_0} = -\frac{1}{\mu_0}\frac{\partial B_z}{\partial\rho}\vec{a}_\phi$$

$$= -\frac{1}{\mu_0}(1+t)(-1-2+\rho)e^{-\rho-t}\,\vec{a}_\phi$$

$$\underline{\underline{\vec{J} = \frac{(1+t)(3-\rho)10^7e^{-\rho-t}}{4\pi}\,\vec{a}_\phi \quad A/m^2}}$$

Prob. 9.24

$$\nabla^2\vec{A} = \nabla(\nabla\cdot\vec{A}) - \nabla\times\nabla\times\vec{A} \quad \text{— (1)}$$

$$\nabla\cdot\vec{A} = -\frac{\mu}{4\pi}e^{j\omega(t-r/c)}\cos\theta\left(\frac{1}{r^2}+\frac{j\omega}{cr}\right)$$

$$\nabla(\nabla\cdot\vec{A}) = \frac{\mu}{4\pi}e^{j\omega(t-r/c)}\left[\cos\theta\left(\frac{2j\omega}{cr^2}-\frac{\omega^2}{c^2r}+\frac{2}{r^3}\right)\vec{a}_r\right.$$

$$\left.+\sin\theta\left(\frac{1}{r^3}+\frac{j\omega}{cr^2}\right)\vec{a}_\theta\right] \quad \text{— (2)}$$

$$\nabla\times\vec{A} = \frac{\mu}{4\pi}e^{j\omega(t-r/c)}\sin\theta\left(\frac{j\omega}{cr}+\frac{1}{r^2}\right)\vec{a}_\phi$$

$$\nabla\times\nabla\times\vec{A} = \frac{\mu}{4\pi}e^{j\omega(t-r/c)}\left[\cos\theta\left(\frac{2j\omega}{cr^2}+\frac{2}{r^3}\right)\vec{a}_r\right.$$

$$\left.+\sin\theta\left(-\frac{\omega^2}{c^2r}+\frac{j\omega}{cr^2}+\frac{1}{r^3}\right)\vec{a}_\theta\right] \quad \text{— (3)}$$

Substituting (2) and (3) into (1) give

$$\nabla^2 \vec{A} = -\frac{w^2 \mu}{4\pi c^2 r}\left(\cos\theta\, \vec{a}_r - \sin\theta\, \vec{a}_\theta\right) e^{jw(t-r/c)} \quad — (4)$$

$$\mu\epsilon \frac{\partial^2 \vec{A}}{\partial t^2} = -\frac{w^2}{c^2}\vec{A} = -\frac{w^2 \mu}{4\pi c^2 r}\left(\cos\theta\cdot \vec{a}_r - \sin\theta\, \vec{a}_\theta\right) e^{jw(t-r/c)} \quad — (5)$$

Thus

$$\nabla^2 \vec{A} - \mu\epsilon \frac{\partial^2 \vec{A}}{\partial t^2} = 0$$

But $\nabla\cdot\vec{A} = -\mu\epsilon\frac{\partial V}{\partial t} = -\frac{1}{c^2}\frac{\partial V}{\partial t}$

or $V = -c^2 \int \nabla\cdot\vec{A}\, dt$

$$= \frac{\mu_0 \cos\theta}{4\pi c} e^{jw(t-r/c)}\left(\frac{1}{cr} + \frac{1}{jwr^2}\right)$$

$\boxed{\text{Prob. 9.25}}$

$$\nabla^2(\nabla\cdot\vec{A}) = \nabla\cdot\nabla^2\vec{A}$$

But $\nabla^2\vec{A} = -\mu\vec{J}$ and $\nabla\cdot\vec{A} = -\mu\epsilon\frac{\partial V}{\partial t}$

$$-\mu\epsilon\frac{\partial}{\partial t}\nabla^2 V = -\mu\nabla\cdot\vec{J}$$

Also $\nabla^2 V = -\frac{\rho_v}{\epsilon}$, $-\frac{\partial \rho_v}{\partial t} = \nabla\cdot\vec{J}$

which is the continuity equation.

Prob. 9.26

(a) $\frac{3-j2}{4-j} - \frac{2}{1+j} + \frac{(2+j)^{\frac{1}{2}}}{(-3+j5)^*} = (0.8235 - j0.2941)$

$-(1-j) + (-0.1791 + j0.1837)$

$= \underline{\underline{-0.3554 + j0.8896}}.$

(b) $-10\angle 30^\circ + (20\angle 120^\circ)^* = -8.66 - j5 - 10 - j17.32$

$= \underline{\underline{-18.66 - j22.32}}.$

(c) $10\ln(3-j4) = 10\ln 5 + 10\ln e^{-j53.1^\circ}$

$= \underline{\underline{16.09 + j(20n - 2.95)\pi}}$

where n = integer.

Prob. 9.27

(a) $\frac{2+j3}{j(-10+j2)} = \frac{3.606\angle 56.31^\circ}{10.198\angle 258.69^\circ} = \underline{\underline{0.354\angle -202.38^\circ}}$

(b) $\frac{8\angle 30^\circ - 5\angle 60^\circ}{1+j} = \frac{4.4282 - j0.33}{1+j} = \frac{4.44\angle -4.26^\circ}{1.414\angle 45^\circ}$

$= \underline{\underline{3.14\angle -49.26^\circ}}$

(c) $\frac{(1+j3)(4-j2)}{(1+j3)+(4-j2)} = \frac{10+j10}{5+j} = \underline{\underline{2.77\angle 33.7^\circ}}$

(d) $\left[\frac{(15-j7)(3+j2)^*}{(4+j6)^*(3\angle 70^\circ)}\right]^* = \left(\frac{16.555\angle -25.02^\circ}{7.211\angle -56.31^\circ} \cdot \frac{3.606\angle -33.69^\circ}{3\angle 70^\circ}\right)^*$

$$= (2.7596\angle{-92.4})^{*} = \underline{\underline{2.76\angle{72.4}^{\circ}}}.$$

(e) $$\left[(-10\angle{30^{\circ}})(4e^{j\pi/4})\right]^{1/2} = \left[40\angle{210^{\circ}+45^{\circ}}\right]^{1/2}$$

$$= \underline{\underline{6.325\angle{127.5}^{\circ}}}.$$

Prob. 9.28

(a) $$\vec{A} = Re\left[5e^{j\omega t}e^{-j2z}e^{-j\pi/2}\vec{a}_x\right] = Re\left[\vec{A}_s e^{j\omega t}\right]$$

$$\longrightarrow \quad \underline{\underline{\vec{A}_s = -je^{-j2z}\vec{a}_x}}.$$

(b) $$\vec{B} = Re\left[15e^{-2y}e^{j\omega t}e^{-jy}\vec{a}_z\right]$$

$$\vec{B}_s = 15e^{-2y}e^{-jy}\vec{a}_z = \underline{\underline{15e^{-y(2+j)}\vec{a}_z}}.$$

(c) $$\vec{C} = 5\cos\omega t\,\vec{a}_y - 8\cos(\omega t - x - \pi/2)\,\vec{a}_z$$

$$\vec{C}_s = 5\vec{a}_y - 8e^{-jx}e^{-j\pi/2}\vec{a}_z = \underline{\underline{5\vec{a}_y + j8e^{-jx}\vec{a}_z}}.$$

(d) $$\vec{D} = Re\left[\frac{10}{\rho}e^{j\omega t}e^{-j3z}\vec{a}_\phi\right].$$

$$\underline{\underline{\vec{D}_s = \frac{10}{\rho}e^{-j3z}\vec{a}_\phi}}.$$

Prob. 9.29

(a) $$\vec{A} = Re\left(\vec{A}_s e^{j\omega t}\right) = Re\left[5e^{j90^{\circ}}e^{-j20^{\circ}}e^{j\omega t}\vec{a}_x - 5xe^{j\alpha}e^{j\omega t}\vec{a}_y\right]$$

where $\alpha = \tan^{-1}\frac{4}{3} = 53.13^{\circ}$

$$\underline{\underline{\vec{A} = 5\cos(\omega t + 70^{\circ})\vec{a}_x - 5x\cos(\omega t + 53.13^{\circ})\vec{a}_y}}.$$

(b) $\vec{B} = Re[10 e^{-jkz} e^{j\omega t} \vec{a}_x + 5 e^{j\pi/2} e^{jkz + \pi/4} e^{j\omega t} \vec{a}_y]$

$= 10\cos(\omega t - kz)\vec{a}_x + 5\cos(\omega t + kz + \pi/2 + \pi/4)\vec{a}_y$

$= \underline{\underline{10\cos(\omega t - kz)\vec{a}_x - 5\sin(\omega t + kz + \pi/4)\vec{a}_y}}.$

(c) $C = Re[C_s e^{j\omega t}]$

$= Re[2 e^{-j90^\circ} e^{-j3x} e^{j\omega t} \sin x + e^{3x - j4x + j\omega t}]$

$= 2\sin x \cos(\omega t - 3x - 90^\circ) + e^{3x}\cos(\omega t - 4x)$

$= \underline{\underline{2\sin x \sin(\omega t - 3x) + e^{3x}\cos(\omega t - 4x)}}.$

Prob. 9.30

for time factor $e^{-j\omega t}$, replace every j by $-j$ and obtain

$$\vec{B}_s = \nabla \times \vec{A}_s$$
$$\vec{E}_s = -\nabla V_s - j\omega \vec{A}_s$$
$$\nabla \cdot \vec{A}_s = -j\omega\mu\epsilon V_s$$
$$\nabla^2 V_s + \omega^2 \mu\epsilon V_s = -\rho_s/\epsilon$$
$$\nabla^2 \vec{A}_s + \omega^2 \mu\epsilon \vec{A}_s = -\mu \vec{J}_s.$$

Prob. 9.31

$\vec{E} = Re[\vec{E}_s e^{j\omega t}] = \underline{\underline{20\sin(k_x x)\sin(k_y y)\cos\omega t\, \vec{a}_z \text{ V/m}}}.$

$$-\frac{\partial \vec{B}}{\partial t} = \nabla \times \vec{E} = \frac{\partial E_z}{\partial y}\vec{a}_x - \frac{\partial E_z}{\partial x}\vec{a}_y$$

$$= 20k_y \sin(k_x x)\cos(k_y y)\cos wt\,\bar{a}_x$$
$$-20k_x \cos(k_x x)\sin(k_y y)\cos wt\,\hat{a}_y$$

$$\vec{B} = -\int \nabla \times \vec{E}\,dt$$
$$= -\frac{20}{w}\left[k_y \sin(k_x x)\cos(k_y y)\sin wt\,\bar{a}_x - k_x\cos(k_x x)\sin(k_y y)\sin wt\,\hat{a}_y\right] \text{ Wb/m}^2.$$

CHAPTER 10

P.E. 10.1

(a) $T = \frac{2\pi}{w} = \frac{2\pi}{2\times10^8} = 10\pi \text{ ns} = \underline{31.42 \text{ ns}}$.

$\lambda = uT = 3\times10^8 \times \frac{\pi}{10^8} = \underline{9.425 \text{ m}}$.

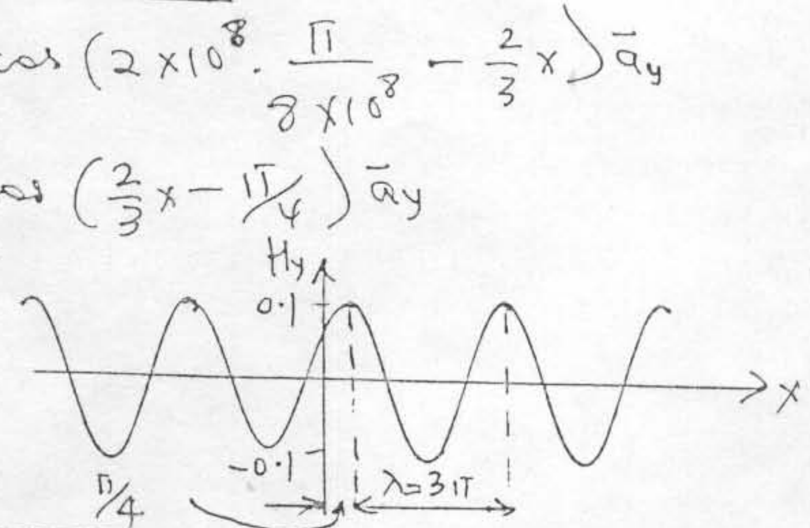
$k = \beta = \frac{2\pi}{\lambda} = \frac{2}{3} = \underline{0.667 \text{ rad/m}}$

(b) $t_1 = \frac{T}{8} = \underline{3.927 \text{ ns}}$.

(c) $\vec{H}(t=t_1) = 0.1\cos\left(2\times10^8 \cdot \frac{\pi}{8\times10^8} - \frac{2}{3}x\right)\bar{a}_y$

$$= 0.1\cos\left(\frac{2}{3}x - \pi/4\right)\bar{a}_y$$

as sketched below.



i.e. $\alpha^2+\beta^2 = w\mu\sqrt{(\sigma^2+w^2\epsilon^2)}$ — (1)

$Re\,\gamma^2 = \alpha^2-\gamma^2 = -w^2\mu\epsilon$

or $\beta^2-\alpha^2 = w^2\mu\epsilon$ — (2)

Substracting and adding (1) and (2) lead respectively to

$$\alpha = w\sqrt{\frac{\mu\epsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{w\epsilon}\right)^2}-1\right]}$$

$$\beta = w\sqrt{\frac{\mu\epsilon}{2}\left[\sqrt{1+\left(\frac{\sigma}{w\epsilon}\right)^2}+1\right]}$$

(b) from Eq. (10.29), $\vec{E}_s(z) = E_o e^{-\gamma z}\vec{a}_x$.

$\nabla\times\vec{E} = -jw\mu\vec{H}_s \longrightarrow \vec{H}_s = \frac{j}{w\mu}\nabla\times\vec{E}_s$

$$= \frac{j}{\mu w}\left(-\gamma E_o e^{-\gamma z}\vec{a}_y\right)$$

But $\vec{H}_s = H_o e^{-\gamma z}\vec{a}_y$, hence $H_o = \frac{E_o}{\eta} = -\frac{j\gamma E_o}{w\mu}$

$$\eta = \frac{jw\mu}{\gamma}$$

(c) From (b),

$$\eta = \frac{jw\mu}{\sqrt{jw\mu(\sigma+jw\epsilon)}} = \sqrt{\frac{jw\mu}{\sigma+jw\epsilon}} = \frac{\sqrt{\mu/\epsilon}}{\sqrt{1-j\frac{\sigma}{w\epsilon}}}$$

$$|\eta| = \frac{\sqrt{\mu/\epsilon}}{\sqrt[4]{1+\left(\frac{\sigma}{w\epsilon}\right)^2}},\quad \tan 2\theta_\eta = \left(\frac{w\epsilon}{\sigma}\right)^{-1} = \frac{\sigma}{w\epsilon}.$$

Prob. 10.3

(a) Along the <u>positive x direction.</u>

(b) $\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{6} = \underline{\underline{1.0472\,m}}.$

(c) $w = 10^8,\ \beta = 6,\ u = \frac{w}{\beta} = \frac{10^8}{6} = \underline{\underline{1.667\times10^7\ m/s}}.$

Prob. 10.4

$$(\nabla^2 - \gamma^2)(E_{xs}, E_{ys}, E_{zs}) = (0,0,0)$$

i.e.
$$\frac{\partial^2 E_{xs}}{\partial x^2} + \frac{\partial^2 E_{xs}}{\partial y^2} + \frac{\partial^2 E_{xs}}{\partial z^2} - \gamma^2 E_{xs} = 0$$

$$\frac{\partial^2 E_{ys}}{\partial x^2} + \frac{\partial^2 E_{ys}}{\partial y^2} + \frac{\partial^2 E_{ys}}{\partial z^2} - \gamma^2 E_{ys} = 0$$

$$\frac{\partial^2 E_{zs}}{\partial x^2} + \frac{\partial^2 E_{zs}}{\partial y^2} + \frac{\partial^2 E_{zs}}{\partial z^2} - \gamma^2 E_{zs} = 0.$$

Prob. 10.5

(a) Let $u = \frac{\sigma}{w\epsilon}$ = loss tangent

$$\beta = w\sqrt{\frac{\mu\epsilon}{2}\left[\sqrt{1+u^2}+1\right]} = \frac{w}{c}\sqrt{\frac{5\times2}{2}}\sqrt{\sqrt{1+u^2}+1}$$

$$10 = \frac{w}{c}\sqrt{5}\sqrt{\sqrt{1+u^2}+1}$$

or $$\sqrt{\sqrt{1+u^2}+1} = \frac{10c}{w\sqrt{5}} = \frac{10\times3\times10^8}{2\pi\times5\times10^6\sqrt{5}} = \frac{300}{\pi\sqrt{5}}$$

$$\sqrt{1+u^2} = 1823 \longrightarrow u = \frac{\sigma}{w\epsilon} = \underline{1823}.$$

(b) $\sigma = \omega \epsilon u = 2\pi \times 5 \times 10^{6} \times 1823 \times \frac{10^{-9}}{36\pi}$

$= \underline{1.013 \ \mho/m.}$

(c) $\epsilon_c = \epsilon' - j\epsilon'' = \epsilon - j\frac{\sigma}{\omega} = \frac{2 \times 10^{-9}}{36\pi} - j\frac{1.013}{2\pi \times 5 \times 10^{6}}$

$= \underline{1.768 \times 10^{-11} - j3.224 \times 10^{-8} \ F/m.}$

(d) $\frac{\alpha}{\beta} = \frac{\sqrt{\sqrt{1+u^2} - 1}}{\sqrt{\sqrt{1+u^2} + 1}} = \sqrt{\frac{1822}{1824}}$

$\alpha = \underline{9.995 \ Np/m.}$

(e) $|\eta| = \frac{\sqrt{\mu/\epsilon}}{\sqrt[4]{1+u^2}} = \frac{120\pi\sqrt{\epsilon/2}}{\sqrt{1823}} = 13.96 \ \Omega$

$\tan 2\theta_\eta = u = 1823 \longrightarrow \theta_\eta = 44.98°$

$\eta = \underline{13.96 \angle 44.98° \ \Omega}$

$\boxed{\text{Prob. 10.6}}$

(a) $\frac{\sigma}{\omega\epsilon} = \tan 2\theta_\eta = \tan 60° = \underline{1.732}$

(b) $|\eta| = 240 = \frac{\frac{120\pi}{\sqrt{\epsilon_r}}}{\sqrt[4]{1+3}} = \frac{120\pi}{\sqrt{2\epsilon_r}}$

$\epsilon_r = \frac{\pi}{8} = \underline{1.234}.$

(c) $\epsilon_c = \epsilon(1 - j\sigma/\omega\epsilon) = 1.234 \times \frac{10^{-9}}{36\pi}(1 - j1.732)$

$\epsilon_c = \underline{\underline{(1.091 - j1.89) \times 10^{-11} \text{ F/m}}}.$

(d) $\alpha = \frac{w}{c} \sqrt{\frac{\mu_r \epsilon_r}{2} \left[\sqrt{1 + \left(\frac{\sigma}{w\epsilon}\right)^2} - 1 \right]}$

$= \frac{2\pi \times 10^6}{3 \times 10^8} \sqrt{\frac{1}{2} \cdot \frac{\pi^2}{8} \left[\sqrt{1+3} - 1 \right]} = \frac{\pi^2}{600}$

$= \underline{0.0164 \text{ Np/m}}.$

$\boxed{\text{Prob. 10.7}}$

$\tan 2\theta_\eta = \tan 2 \times 35.26° = 2.827 = \frac{\sigma}{w\epsilon}$

loss tangent $= \underline{\underline{2.827}}$

loss angle $= \theta = 2\theta_\eta = \underline{\underline{70.52°}}$.

$\frac{\alpha}{\beta} = \sqrt{\frac{\sqrt{1 + (\sigma/w\epsilon)^2} - 1}{\sqrt{1 + (\sigma/w\epsilon)^2} + 1}} = \sqrt{\frac{\sqrt{9} - 1}{\sqrt{9} + 1}} = \frac{1}{\sqrt{2}}$

$\beta = \alpha\sqrt{2} = 0.1\sqrt{2}, \quad \lambda = \frac{2\pi}{\beta} = \frac{2\pi}{0.1\sqrt{2}} = \underline{\underline{44.43 \text{ m}}}.$

$\boxed{\text{Prob. 10.8}}$

$\frac{\sigma}{w\epsilon} = \frac{4}{2\pi \times 10^5 \times 81 \times 10^{-9}/36\pi} = \frac{80,000}{9} \gg 1$

$\alpha = \beta = \sqrt{\frac{w\mu\sigma}{2}} = \sqrt{\frac{2\pi \times 10^5 \times 4\pi \times 10^{-7} \times 4}{2}} = 0.4\pi$

(a) $u = \frac{w}{\beta} = \frac{2\pi \times 10^5}{0.4\pi} = \underline{\underline{5 \times 10^5 \text{ m/s}}}.$

(b) $\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{0.4\pi} = \underline{\underline{5 \text{ m}}}.$

(c) $\delta = \frac{1}{\alpha} = \frac{1}{0.4\pi} = \underline{\underline{0.796}}$.

Prob. 10.9

(a) Along $\underline{\underline{\vec{a}_z}}$.

(b) $\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{1} = \underline{\underline{6.283\text{ m}}}$.

$T = \frac{2\pi}{w} = \frac{2\pi}{3\times10^8} = \underline{\underline{20.94\text{ ms}}}$.

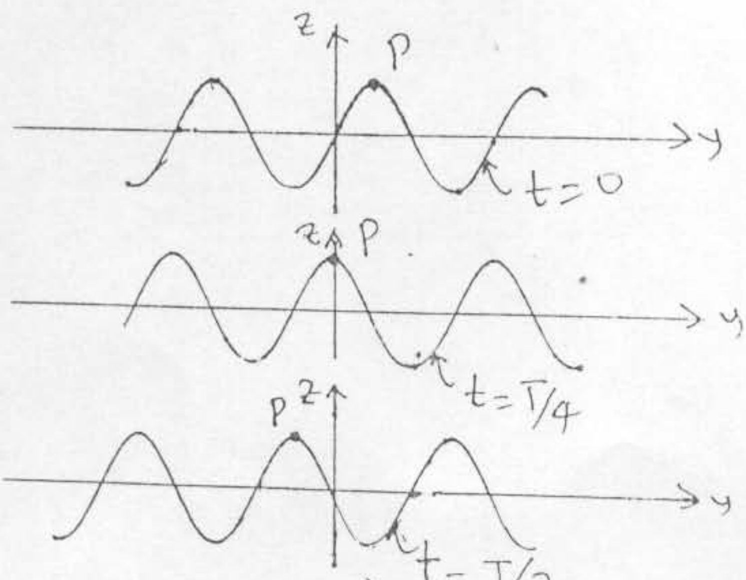
$u = c = \underline{\underline{3\times10^8\text{ m/s}}}$.

(c) At $t=0$, $\vec{E} = 5\sin y\,\vec{a}_z$ V/m

At $t = T/4$, $\vec{E} = 5\sin\left(w\cdot\frac{2\pi}{4w} + y\right)\vec{a}_z = 5\cos y\,\vec{a}_z$

At $t = T/2$, $\vec{E} = 5\sin\left(y + w\cdot\frac{2\pi}{2w}\right)\vec{a}_z = -5\sin y\,\vec{a}_z$

The wave is sketched below.



(d) Let $\vec{H} = H_o\sin(wt+y)\,\vec{a}_H$,

$H_o = \frac{E_o}{|\eta|} = \frac{5}{377} = 0.01326$, $a_z \times a_H = -\vec{a}_y \rightarrow \vec{a}_H = -\vec{a}_x$

$\vec{H} = -13.26 \sin\left(3\times10^{8} t + y\right) \vec{a}_x$ mA/m.

Prob. 10.10

(a) $\beta = \frac{w}{c} = \frac{2\pi\times10^{6}}{3\times10^{8}} = \frac{2\pi}{300} = 0.02094$ rad/m.

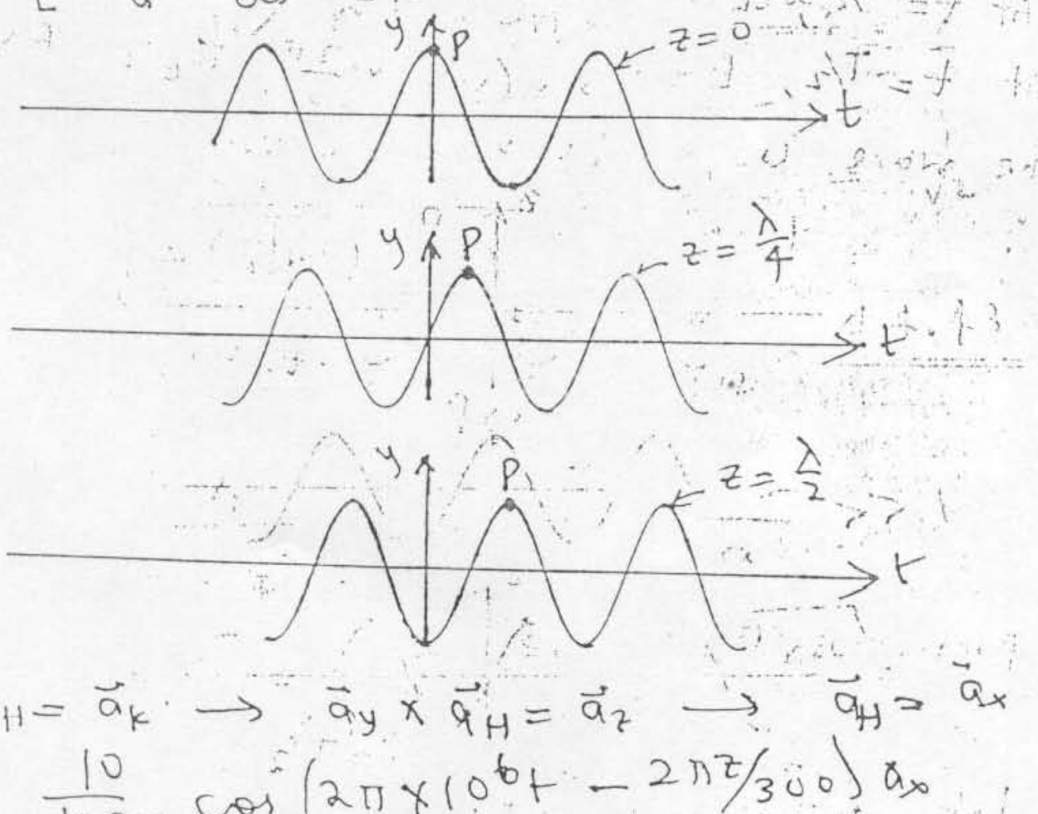
$\lambda = \frac{2\pi}{\beta} = 300\text{m}$.

(b) When $z=0$, $E_y = 10\cos wt$

$z=\lambda/4$, $E_y = 10\cos\left(wt - \frac{2\pi}{\lambda}\cdot\frac{\lambda}{4}\right) = 10\sin wt$

$z=\frac{\lambda}{2}$, $E_y = 10\cos(wt - \pi) = -10\cos wt$

Thus E is as sketched below:



(c) $\vec{a}_E \times \vec{a}_H = \vec{a}_k \longrightarrow \vec{a}_y \times \vec{a}_H = \vec{a}_z \longrightarrow \vec{a}_H = \vec{a}_x$

$\vec{H} = \frac{10}{120\pi}\cos\left(2\pi\times10^{6}t - 2\pi z/300\right)a_x$

$= 26.53\cos\left(2\pi\times10^{6}t - 0.02094z\right)\vec{a}_x$ mA/m.

Prob. 10.11

$$\frac{\sigma}{w\epsilon} = \frac{5\times10^{-9}}{2\pi\times10^{9}\times 9\times\frac{10^{-9}}{36\pi}} = 10^{-8} \simeq 0$$

$$\alpha \simeq 0, \quad \beta \simeq w\sqrt{\mu\epsilon} = \frac{2\pi f}{c}\sqrt{\mu_r\epsilon_r} = \frac{2\pi\times10^{9}\sqrt{9}}{3\times10^{8}} = 20\pi$$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{20\pi} = \underline{\underline{0.1\ m}}$$

It is a $\underline{\underline{\text{lossless medium}}}$ since $\frac{\sigma}{w\epsilon} \simeq 0$.

Prob. 10.12

$$\beta = \frac{2\pi}{\lambda} = \frac{w}{c} = \frac{4\pi\times10^{6}}{3\times10^{8}} = \frac{4\pi}{300} = \underline{\underline{0.04189\ \text{rad/m}}}.$$

$$\vec{a}_k = \underline{\underline{\vec{a}_z}}.$$

$$E_0 = \eta_0 H_0 = 120\pi(0.3) = 36\pi$$

$$-\vec{a}_E = \vec{a}_k \times \vec{a}_H = \vec{a}_z \times (-\vec{a}_x) \longrightarrow \vec{a}_E = \vec{a}_y$$

$$\vec{E} = (36\pi\vec{a}_y)\sin(4\pi\times10^{8}t - kz) \longrightarrow \vec{A} = \underline{\underline{36\pi\vec{a}_y}}.$$

Prob. 10.13

If $\vec{A}$ is a uniform vector and $\phi(r)$ is a scalar,

$$\nabla\times(\phi\vec{A}) = \nabla\phi\times\vec{A} + \phi\,\underbrace{(\nabla\times\vec{A})}_{0} = \nabla\phi\times\vec{A}$$

$$\nabla\times\vec{E} = \left(\frac{\partial}{\partial x}\vec{a}_x + \frac{\partial}{\partial y}\vec{a}_y + \frac{\partial}{\partial z}\vec{a}_z\right)\times\vec{E}_0\, e^{j(k_x x + k_y y + k_z z - wt)}$$

$$= j(k_x \vec{a}_x + k_y \vec{a}_y + k_z \vec{a}_z)\, e^{j(\ )} \times \vec{E}_o$$

$$= j\vec{k} \times \vec{E}_o e^{j(\ )} = j\vec{k} \times \vec{E}$$

Also $-\dfrac{\partial \vec{B}}{\partial t} = j\omega\mu \vec{H}$. Hence $\nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$ becomes

$$\vec{k} \times \vec{E} = \omega\mu \vec{H}.$$

from this, $\vec{a}_k \times \vec{a}_E = \vec{a}_H$.

$\boxed{\text{Prob. 10.14}}$

$$\nabla \cdot \vec{E} = \left(\frac{\partial}{\partial x}\vec{a}_x + \frac{\partial}{\partial y}\vec{a}_y + \frac{\partial}{\partial z}\vec{a}_z\right) \cdot \vec{E}_o e^{j(k_x x + k_y y + k_z z - \omega t)}$$

$$= j(k_x \vec{a}_x + k_y \vec{a}_y + k_z \vec{a}_z)\, e^{j(\ )} \cdot \vec{E}_o$$

$$= j\vec{k} \cdot \vec{E}_o e^{j(\ )} = j\vec{k} \cdot \vec{E} = 0 \rightarrow \vec{k} \cdot \vec{E} = 0.$$

Similarly, $\nabla \cdot \vec{H} = j\vec{k} \cdot \vec{H} = 0 \longrightarrow \vec{k} \cdot \vec{H} = 0$

It has been shown in Prob. 10.13 that

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \longrightarrow \vec{k} \times \vec{E} = \mu\omega \vec{H}.$$

Similarly, $\nabla \times \vec{H} = \dfrac{\partial \vec{D}}{\partial t} \rightarrow \vec{k} \times \vec{H} = -\epsilon\omega \vec{E}$.

From $\vec{k} \times \vec{E} = \mu\omega \vec{H}$, $\quad \vec{a}_k \times \vec{a}_E = \vec{a}_H$

and from $\vec{k} \times \vec{H} = -\epsilon\omega \vec{E}$, $\quad \vec{a}_k \times \vec{a}_H = -\vec{a}_E$.

Prob. 10.15

(a) $\beta = \frac{w}{c}\sqrt{\epsilon_r} \longrightarrow \sqrt{\epsilon_r} = \frac{\beta c}{w} = \frac{5 \times 3 \times 10^8}{2\pi \times 10^8} = \frac{15}{2\pi}$

$$\epsilon_r = \underline{\underline{5.6993}}.$$

(b) $\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{5} = \underline{\underline{1.2566\ m}}.$

$u = \frac{c}{\sqrt{\mu_r \epsilon_r}} = \frac{3 \times 10^8}{15/2\pi} = \underline{\underline{1.257 \times 10^8\ m/s}}.$

(c) $\eta = \eta_o \sqrt{\frac{\mu_r}{\epsilon_r}} = \frac{120\pi}{15/(2\pi)} = 16\pi^2 = \underline{\underline{157.91\ \Omega}}.$

(d) $\vec{a}_E \times \vec{a}_H = \vec{a}_z \longrightarrow \vec{a}_E \times \vec{a}_z = \vec{a}_x \rightarrow \vec{a}_E = \underline{\underline{\vec{a}_y}}.$

(e) $\vec{E} = 30 \times 10^{-3}\,(157.91)\sin(wt - \beta x)\,\vec{a}_E$

$= \underline{\underline{4.737 \sin(2\pi \times 10^8 t - 5x)\,\vec{a}_y\ V/m}}.$

(f) $\vec{J}_d = \frac{\partial \vec{D}}{\partial t} = \nabla \times \vec{H} = \underline{\underline{0.15 \cos(2\pi \times 10^8 t - 5x)\,\vec{a}_y\ A/m}}.$

Prob. 10.16

$\beta = w\sqrt{\mu\epsilon} = \frac{w}{c}\sqrt{\mu_r\epsilon_r} = \frac{2\pi \times 10^7}{3 \times 10^8}(10) = \frac{2\pi}{3}$

$= \underline{\underline{2.0943\ rad/m}}.$

$\vec{H} = -\frac{1}{\mu}\int \nabla \times \vec{E}\,dt$

$\nabla \times \vec{E} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_y(x) & E_z(x) \end{vmatrix} = -\frac{\partial E_z}{\partial x}\hat{a}_x + \frac{\partial E_y}{\partial x}\vec{a}_z$

$= -10\beta \sin(wt - \beta x)(\vec{a}_y - \vec{a}_z)$

$$\vec{H} = \frac{-10\beta}{w\mu}\cos(wt-\beta x)(\vec{a}_y - \vec{a}_z)$$

$$= \frac{10 \times \frac{2\pi}{3}}{2\pi \times 10^7 \times 50 \times 4\pi \times 10^{-7}} \cos(wt - \beta x)(-\vec{a}_y + \vec{a}_z)$$

$$\vec{H} = \underline{5.305 \cos(2\pi \times 10^7 t - 2.0943x)(-\vec{a}_y + \vec{a}_z)\ mA/m.}$$

Prob. 10.17

(a) $\frac{\sigma}{w\varepsilon} = \frac{4}{2\pi \times 10^6 \times \frac{10^{-9}}{36\pi} \times 80} = \underline{\underline{900}}$.

(b) $\frac{\sigma}{w\varepsilon} = 18000 \times \frac{10^{-3}}{80} = \underline{\underline{0.225}}$.

(c) $\frac{\sigma}{w\varepsilon} = 18000 \times \frac{10^{-3}}{10} = \underline{\underline{1.8}}$

(d) $\frac{\sigma}{w\varepsilon} = 18000 \times \frac{10^{-5}}{3} = \underline{\underline{0.06}}$

Prob. 10.18

$$\delta = \frac{1}{\alpha} = \frac{1}{\beta} = \frac{1}{\sqrt{\pi f \mu \sigma}}$$

or $f = \frac{1}{\delta^2 \pi \mu_0 \sigma} = \frac{1}{4 \times 10^{-6} \times \pi \times 4\pi \times 10^{-7} \times 6.1 \times 10^7}$

$= \underline{1.038\ kHz}$

$$\lambda = \frac{2\pi}{\beta} = 2\pi\delta = 4\pi = \underline{\underline{12.566\ mm}}$$

$$u = \frac{w}{\beta} = \frac{2}{\delta \mu_o \sigma} = \frac{2}{2\times10^{-3} \times 4\pi\times10^{-7} \times 6.1\times10^{7}}$$

$$= \underline{\underline{13.04 \text{ m/s}}}.$$

$\boxed{\text{Prob. 10.19}}$

$$\alpha = w\sqrt{\frac{\mu\epsilon}{2}\left[\sqrt{1+(\sigma/w\epsilon)^2} - 1\right]}$$

$$= \frac{2\pi f}{c}\sqrt{\frac{\mu_r\epsilon_r}{2}\left[\sqrt{1.0049} - 1\right]}$$

$$= \frac{2\pi\times6\times10^{6}}{3\times10^{8}}\sqrt{\frac{4}{2}\times2.447\times10^{-3}} = 8.791\times10^{-3}$$

$$\delta = \frac{1}{\alpha} = \underline{\underline{113.75 \text{ m}}}.$$

$$\beta = \frac{4\pi}{100}\sqrt{\frac{4}{2}\left[\sqrt{1.0049} + 1\right]} = 0.2515$$

$$u = \frac{w}{\beta} = \frac{2\pi\times6\times10^{6}}{0.2515} = \underline{\underline{1.5\times10^{8} \text{ m/s}}}.$$

$\boxed{\text{Prob. 10:20}}$

(a) $0.4E_o = E_o e^{-\alpha z} \longrightarrow \frac{1}{0.4} = e^{2\alpha}$

or $\alpha = \frac{1}{2}\ln\frac{1}{0.4} = 0.4581$

$$\delta = 1/\alpha = \underline{\underline{2.183 \text{ m}}}.$$

(b) $\lambda = 2\pi/\beta = 2\pi/1.6$

$$u = f\lambda = 10^{7}\times\frac{2\pi}{1.6} = \underline{\underline{3.927\times10^{7} \text{ m/s}}}.$$

Prob. 10.21

(a) $R_{d.c} = \frac{l}{\sigma S} = \frac{l}{\sigma \pi a^2} = \frac{600}{5.8 \times 10^7 \times \pi \times (1.2)^2 \times 10^{-6}}$

$= \underline{\underline{2.287\ \Omega}}$

(b) $R_{ac} = \frac{l}{\sigma\, 2\pi a \delta}$

At 100 MHz, $\delta = 6.6 \times 10^{-3}$ mm for copper (see Table 10.2).

$R_{ac} = \frac{600}{5.8 \times 10^7 \times 2\pi \times 1.2 \times 6.6 \times 10^{-3} \times 10^{-6}}$

$= \underline{\underline{207.88\ \Omega}}$

(c) $\frac{R_{ac}}{R_{dc}} = \frac{a}{2\delta} = 1 \longrightarrow \delta = \frac{a}{2} = \frac{66.1 \times 10^{-3}}{\sqrt{f}}$

$\sqrt{f} = \frac{66.1 \times 2 \times 10^{-3}}{a} = \frac{66.1 \times 2}{1.2}$

$\longrightarrow f = \underline{\underline{12.137\ \text{kHz}}}$

Prob. 10.22

$w = 10^6 \pi = 2\pi f \longrightarrow f = 0.5 \times 10^6$

$\delta = \frac{1}{\sqrt{\pi f \sigma \mu}} = \frac{1}{(\pi \times 0.5 \times 10^6 \times 3.5 \times 10^7 \times 4\pi \times 10^{-7})^{1/2}}$

$= \underline{\underline{0.1203\ \text{mm}}}$

$$R_{ac} = \frac{l}{\sigma \delta w}$$

since δ is very small, $w = 2\pi \rho_{outer}$

$$R_{ac} = \frac{l}{\sigma\delta\, 2\pi \rho_{outer}} = \frac{40}{3.5\times10^{7}\times 0.1203\times 2\pi \times 12\times10^{-6}}$$

$$= \underline{\underline{0.126\ \Omega}}.$$

Prob. 10.23

$$P_{ave} = \frac{E_o^2}{2\eta_o} = \frac{1}{2}\eta_o H_o^2 = \frac{1}{2}(120\pi)(0.1)^2 = \underline{\underline{1.885\ W/m^2}}.$$

$\vec{P}$ is along $\underline{\underline{-\vec{a}_y}}$.

Prob. 10.24

(a) $\vec{E} = Re(\vec{E}_s e^{j\omega t}) = \underline{\underline{40\cos(\omega t - 8y + 20^\circ)\vec{a}_z}}.$

(b) $\eta = 377\sqrt{\frac{\mu_r}{\epsilon_r}} = 377\sqrt{2.25} = 565.5\ \Omega.$

$\vec{a}_E \times \vec{a}_H = \vec{a}_k \longrightarrow \vec{a}_z \times \vec{a}_H = \vec{a}_y \longleftrightarrow \vec{a}_H = \vec{a}_x.$

$$\vec{H} = \frac{40}{565.5}\cos(\omega t - 8y + 20^\circ)$$

$$= \underline{\underline{0.0707\cos(\omega t - 8y + 20^\circ)\vec{a}_x\ A/m}}.$$

(c) $\lambda = \frac{2\pi}{\beta} = \frac{\pi}{4} = \underline{\underline{0.7854\ m}}$, $\eta = \underline{\underline{565.5\ \Omega}}.$

(d) $\vec{P}_{ave} = \frac{E_o^2}{2\eta}\vec{a}_y = \frac{(40)^2}{2(565.5)}\vec{a}_y = \underline{\underline{1.415\,\vec{a}_y\ W/m^2}}.$

Prob. 10.25

(a) $\vec{P} = \vec{E} \times \vec{H} = \frac{E_o^2}{\eta_o} \cos^2(wt - \beta z)\,\vec{a}_z = \frac{100}{120\pi} \cos^2(wt - \beta z)\,\vec{a}_z$

$= \underline{0.2652 \cos(2\pi \times 10^6 t - \beta z)\,\vec{a}_z}$ W/m^2.

(b) $\vec{P} = \vec{E} \times \vec{H} = \frac{E_o^2}{|\eta|} e^{-2z/3} \sin(wt - \beta z) \sin(wt - \beta z - \theta_n)\,\vec{a}_z$

$= \frac{(0.5)^2}{177.72} e^{-2\alpha z} \sin(wt - \beta z) \sin(wt - \beta z - \theta_n)\,\vec{a}_z$

$= \underline{1.41\, e^{-2z/3} \sin(10^8 t - \beta z) \sin(10^8 t - \beta z - 13.63^\circ)\,\vec{a}_z}$ mW/m^2.

Prob 10.26

Let $\vec{E}_s = \vec{E}_r + j\vec{E}_i$ and $\vec{H}_s = \vec{H}_r + j\vec{H}_i$

$\vec{E} = Re(\vec{E}_s e^{jwt}) = \vec{E}_r \cos wt - \vec{E}_i \sin wt$

similarly, $\vec{H} = \vec{H}_r \cos wt - \vec{H}_i \sin wt$

$\vec{P} = \vec{E} \times \vec{H} = \vec{E}_r \times \vec{H}_r \cos^2 wt + \vec{E}_i \times \vec{H}_i \sin^2 wt$

$- \frac{1}{2}(\vec{E}_r \times \vec{H}_i + \vec{E}_i \times \vec{H}_r) \sin 2wt$

$\vec{P}_{ave} = \frac{1}{T}\int_0^T \vec{P}\,dt = \frac{1}{T}\int_0^T \cos^2 wt\,dt\,(\vec{E}_r \times \vec{H}_r)$

$+ \frac{1}{T}\int_0^T \sin^2 wt\,dt\,(\vec{E}_i \times \vec{H}_i) - \frac{1}{2T}\underbrace{\int_0^T \sin 2wt\,dt}_{0}\,(\vec{E}_i \times \vec{H}_i + \vec{E}_i \times \vec{H}_r)$

$$= \frac{1}{2}(\vec{E}_r \times \vec{H}_r + \vec{E}_i \times \vec{H}_i)$$

$$= \frac{1}{2} Re\left[(\vec{E}_r + j\vec{E}_i) \times (\vec{H}_r - j\vec{H}_i)\right]$$

$$\vec{P}_{ave} = \frac{1}{2} Re(\vec{E}_s \times \vec{H}_s^*) \quad \text{as required.}$$

Prob. 10.27

(a) $\beta = \frac{w}{c} = \frac{10^7}{3 \times 10^8} = \underline{\underline{0.0333 \text{ rad/m.}}}$

$E_o = \eta_o H_o = 377, \quad -\vec{a}_E = \vec{a}_k \times \vec{a}_H = \vec{a}_r \times \vec{a}_\theta \rightarrow \vec{a}_E = -\vec{a}_\phi$

$$\underline{\underline{\vec{E} = -\frac{377 \sin\theta}{r} \cos(10^7 t - 0.0333 r)\, \vec{a}_\phi \text{ V/m.}}}$$

(c) $P_t = \int \vec{P}_{ave} \cdot d\vec{S} = \int_{\theta=0}^{\pi/3} \int_{\phi=0}^{2\pi} \frac{60\pi \sin^2\theta}{r^2} r^2 \sin\theta \, d\theta \, d\phi$

$$= 60\pi (2\pi) \int_0^{\pi/3} (1 - \cos^2\theta)\, d(-\cos\theta)$$

$$= 120\pi^2 \left(\frac{5}{24}\right) = \underline{\underline{246.74 \text{ W.}}}$$

(b) $\vec{P} = \vec{E} \times \vec{H} = 377 \frac{\sin^2\theta}{r^2} \cos^2(10^7 t - \beta r)\, \vec{a}_r$

$$\underline{\underline{= \frac{188.5 \sin^2\theta}{r^2} \vec{a}_r \text{ W/m}^2}}$$

Prob. 10.28

$$\vec{H}_i(0) + \vec{H}_r(0) = \vec{H}_t(0) \longrightarrow \quad H_{io} + H_{ro} = H_{to} \qquad -(1)$$

$$\vec{E}_i(0) + \vec{E}_r(0) = \vec{E}_t(0) \longrightarrow \eta_1 (H_{io} - H_{ro}) = \eta_2 H_{to} \qquad -(2)$$

$$(1) \times \eta_2 - (2) \Longrightarrow \quad H_{io}(\eta_2 - \eta_1) + H_{ro}(\eta_2 + \eta_1) = 0$$

$$\text{or} \quad -\frac{H_{ro}}{H_{io}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \Gamma$$

$$(1) \times \eta_1 + (2) \Longrightarrow \quad 2\eta_1 H_{io} = (\eta_1 + \eta_2) H_{to}$$

$$\text{or} \quad \frac{H_{to}}{H_{io}} = \frac{2\eta_1}{\eta_1 + \eta_2} = \frac{\eta_1}{\eta_2}\tau$$

Prob 10.29

$$P_{i\,ave} = \frac{E_{io}^2}{2\eta_1}, \quad P_{r\,ave} = \frac{E_{ro}^2}{2\eta_1}, \quad P_{t\,ave} = \frac{E_{to}^2}{2\eta_2}$$

$$R = \frac{P_{r\,ave}}{P_{i\,ave}} = \frac{E_{ro}^2}{E_{io}^2} = \Gamma^2 = \left(\frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}\right)^2$$

$$T = \frac{P_{t\,ave}}{P_{i\,ave}} = \frac{\eta_1}{\eta_2}\frac{E_{to}^2}{E_{io}^2} = \frac{\eta_1}{\eta_2}\tau^2 = \frac{4\eta_1\eta_2}{(\eta_1 + \eta_2)^2}$$

$$R + T = \frac{\eta_2^2 - 2\eta_1\eta_2 + \eta_1^2 + 4\eta_1\eta_2}{(\eta_1 + \eta_2)^2} = 1$$

Alternatively,

$$\mathcal{P}_{i\,ave} = \mathcal{P}_{r\,ave} + \mathcal{P}_{t\,ave}$$

$$1 = \frac{\mathcal{P}_{r\,ave}}{\mathcal{P}_{i\,ave}} + \frac{\mathcal{P}_{t\,ave}}{\mathcal{P}_{i\,ave}} = R + T.$$

$\boxed{\text{Prob. 10.30}}$

$$\eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}} = \frac{\eta_o}{2}, \quad \eta_2 = \eta_o$$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{1}{3}, \quad \tau = \frac{4}{3}$$

$$E_{or} = \Gamma E_{io} = \left(\frac{1}{3}\right)(5) = \frac{5}{3}, \quad E_{ot} = \tau E_{io} = \frac{20}{3}.$$

$$\beta = \frac{w}{c}\sqrt{\mu_r \epsilon_r} = \frac{10^8}{3\times10^8}\sqrt{4} = \frac{2}{3}$$

(a) $E_r = \frac{5}{3}\cos\left(10^8 t - \frac{2}{3}y\right)\vec{a}_z$

$$\vec{E}_1 = \vec{E}_i + \vec{E}_r$$

$$= \underline{\underline{5\cos\left(10^8 t + \frac{2}{3}y\right)\vec{a}_z + \frac{5}{3}\cos\left(10^8 t - \frac{2}{3}y\right)\vec{a}_z \text{ V/m}}}.$$

(b) $$\vec{\mathcal{P}}_{ave\,1} = \frac{E_{io}^2}{2\eta_1}(-\vec{a}_y) + \frac{E_{ro}^2}{2\eta_1}(\vec{a}_y)$$

$$= \frac{25}{2(60\pi)}\left(1 - \frac{1}{9}\right)(-\vec{a}_y) = \underline{\underline{-0.0589\,\vec{a}_y \text{ W/m}^2}}.$$

(c) $$\vec{\mathcal{P}}_{ave\,2} = \frac{E_{to}^2}{2\eta_2}(-\vec{a}_y) = \frac{400(-\vec{a}_y)}{9(2)(120\pi)} = \underline{\underline{-0.0589\,\vec{a}_y \text{ W/m}^2}}.$$

Prob. 10.31

(a) $\beta = 1 = \frac{w}{u} = \frac{w}{c}\sqrt{\mu_r \epsilon_r} \longrightarrow w = \frac{c}{\sqrt{\mu_r \epsilon_r}}$

$$w = \frac{3\times 10^8}{\sqrt{3\times 12}} = \underline{\underline{0.5 \times 10^8 \text{ rad/s}}}$$

(b) $\eta_1 = \eta_o$, $\eta_2 = \eta_o\sqrt{\frac{\mu_r}{\epsilon_r}} = \eta_o\sqrt{\frac{3}{12}} = \frac{\eta_o}{2}$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{\frac{1}{2} - 1}{\frac{1}{2} + 1} = -\frac{1}{3}, \quad \tau = 1 + \Gamma = \frac{2}{3}$$

$$s = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + \frac{1}{3}}{1 - \frac{1}{3}} = \underline{\underline{2}}.$$

(c) Let $\vec{H}_r = H_{or}\cos(wt + z)\,\vec{a}_H$

where $\vec{E}_r = -\frac{1}{3}(30)\cos(wt + z)\vec{a}_y = -10\cos(wt + z)\vec{a}_y$,

$$H_{or} = \frac{10}{\eta_o} = \frac{10}{120\pi}$$

$\vec{a}_E \times \vec{a}_H = \vec{a}_k \longrightarrow -\vec{a}_y \times \vec{a}_H = -\vec{a}_z \longrightarrow \vec{a}_H = -\vec{a}_x$

$$\vec{H}_r = -\frac{10}{120\pi}\cos(0.5\times 10^8 t + z)\,\vec{a}_x \text{ A/m}$$

$$= \underline{\underline{-26.53\cos(0.5\times 10^8 t + z)\,\vec{a}_x \text{ mA/m}}}.$$

Prob. 10.32

(a) $\vec{a}_E \times \vec{a}_H = \vec{a}_k \longrightarrow \vec{a}_E \times \vec{a}_z = \vec{a}_x \longrightarrow \vec{a}_E = -\vec{a}_y$.

i.e. polarization is along the y-axis.

(b) $\beta = w\sqrt{\mu\epsilon} = \frac{2\pi f}{c}\sqrt{\mu_r\epsilon_r} = \frac{2\pi \times 30\times10^6}{3\times10^8}\sqrt{4\times9}$

$\beta = 3.77$ rad/m.

(c) $\vec{J}_d = \nabla\times\vec{H} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & H_z(x,t) \end{vmatrix} = -\frac{\partial H_z}{\partial x}\vec{a}_y$

$= -10\beta\cos(wt+\beta x)\vec{a}_y, \quad \beta = \frac{6\pi}{5}$

$= -37.6\cos(wt+\beta x)\vec{a}_y$ mA/m.

(d) $\eta_2 = \eta_0, \quad \eta_1 = \eta_0\sqrt{\frac{4}{9}} = \frac{2}{3}\eta_0$

$\Gamma = \frac{\eta_2-\eta_1}{\eta_2+\eta_1} = \frac{1-\frac{2}{3}}{1+\frac{2}{3}} = \frac{1}{5}, \quad \tau = 1+\Gamma = \frac{6}{5}$

$\vec{E}_i = 10\eta_1\sin(wt+\beta x)\vec{a}_E$ mV/m, $\vec{a}_E = -\vec{a}_y$

$\vec{E}_r = \Gamma 10\eta_1\sin(wt-\beta x)(-\vec{a}_y)$ mV/m

$a_E\times a_H = a_k \longrightarrow -\vec{a}_y\times a_H = +a_x \longrightarrow \vec{a}_H = -\vec{a}_z$

$\vec{H}_r = \Gamma 10\sin(wt-\beta x)(-\vec{a}_z)$ mA/m

$= -2\sin(wt-\beta x)\vec{a}_z$ mA/m.

$\vec{E}_t = \tau 10\eta_1\sin(wt+\beta x)(-\vec{a}_y)$ mV/m

$\vec{a}_E\times\vec{a}_H = \vec{a}_k \longrightarrow -\vec{a}_y\times\vec{a}_H = -\vec{a}_x \longrightarrow \vec{a}_H = \vec{a}_z$

$$\vec{H}_r = \frac{6}{5}(10)\frac{\eta_1}{\eta_2} \sin(\omega t + \beta x)\, \vec{a}_z \quad mA/m$$

$$= \underline{\underline{8 \sin(\omega t + \beta x)\, \vec{a}_z \quad mA/m.}}$$

(e) $$\vec{P}_{1ave} = \frac{E_{io}^2}{2\eta_1}(-\vec{a}_x) + \frac{E_{ro}^2}{2\eta_1}\vec{a}_x = -\frac{E_{io}^2}{2\eta_1}(1-\Gamma^2)\vec{a}_x$$

$$= -\frac{\eta_1^2 H_{io}^2}{2\eta_1}(1-\Gamma^2)\vec{a}_x = -\frac{1}{3}\eta_o 100\left(1-\frac{1}{25}\right)\vec{a}_x$$

$$= -32\eta_o \vec{a}_x \ \mu W/m^2 = \underline{\underline{-0.012064\,\vec{a}_x \quad W/m^2.}}$$

$$\vec{P}_{2ave} = \frac{E_{ot}^2}{2\eta_2}(-\vec{a}_x), \quad E_{ot} = \tau E_{oi} = \tau \eta_1 H_{io}$$

$$= -\frac{\tau^2 \eta_1^2 H_{io}^2}{2\eta_2}\vec{a}_x = 32\eta_o(-\vec{a}_x) \quad \mu W/m^2$$

$$= \underline{\underline{-0.012064\,\vec{a}_x \quad W/m^2.}}$$

$\boxed{\text{Prob. 10.33}}$

(a) In air, $\beta_1 = 1$, $\lambda_1 = \frac{2\pi}{\beta_1} = 2\pi = \underline{6.283\ m.}$

$$\omega = \beta_1 c = \underline{3 \times 10^8 \ rad/s.}$$

In the dielectric medium, ω is the same

$$\omega = \underline{3 \times 10^8 \ rad/s}$$

$$\beta_2 = \frac{\omega}{c}\sqrt{\epsilon_{r2}} = \beta_1\sqrt{\epsilon_{r2}} = 1\sqrt{3}$$

$$\lambda_2 = \frac{2\pi}{\beta_2} = \frac{2\pi}{\sqrt{3}} = \underline{\underline{3.6276\ m.}}$$

(b) $H_o = \dfrac{E_o}{\eta_o} = \dfrac{10}{120\pi} = 0.0265$

$$\vec{a}_H = \vec{a}_k \times \vec{a}_E = \vec{a}_z \times \vec{a}_y = \vec{a}_x$$

$$\vec{H}_i = \underline{\underline{-26.5 \cos(\omega t - z)\,\vec{a}_x \ mA/m}}$$

(c) $\eta_1 = \eta_o$, $\eta_2 = \dfrac{\eta_o}{\sqrt{3}}$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{\frac{1}{\sqrt{3}} - 1}{\frac{1}{\sqrt{3}} + 1} = \frac{1-\sqrt{3}}{1+\sqrt{3}} = \underline{\underline{-0.268}},\quad \tau = 1 + \Gamma = \underline{\underline{0.732}}$$

(d) $E_{to} = \tau E_{io} = 7.32$, $E_{ro} = \Gamma E_{io} = -2.68$

$$\vec{E}_1 = \vec{E}_i + \vec{E}_r = \underline{\underline{10\cos(\omega t - z)\,\vec{a}_y - 2.68\cos(\omega t + z)\,\vec{a}_y \ V/m.}}$$

$$\vec{E}_2 = \vec{E}_t = \underline{7.32\cos(\omega t - z)\,\vec{a}_y \ V/m.}$$

$$\vec{P}_{1ave} = \frac{1}{2\eta_1}\vec{a}_z\left(E_{io}^2 - E_{ro}^2\right) = \frac{1}{2(120\pi)}\vec{a}_z\left(10^2 - 2.68^2\right)$$

$$= \underline{\underline{0.1231\,\vec{a}_z \ W/m^2}}.$$

$$\vec{P}_{2ave} = \frac{E_{ot}^2}{2\eta_2}\vec{a}_z = \frac{\sqrt{3}}{2\times 120\pi}(7.32)^2\vec{a}_z = \underline{\underline{0.1231\,\vec{a}_z \ W/m^2}}$$

$\boxed{\text{Prob. 10.34}}$

(a) $\omega = \beta c = 3 \times 3 \times 10^8 = \underline{\underline{9 \times 10^8 \ rad/s.}}$

(b) $\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{3} = \underline{\underline{2.094 \text{ m}}}$

(c) $\frac{\sigma}{\omega\epsilon} = \frac{4}{9\times10^{8}\times 80 \times \frac{10^{-9}}{36\pi}} = 2\pi = \underline{\underline{6.288}}$

$\tan 2\theta_\eta = \frac{\sigma}{\omega\epsilon} = 6.288 \longrightarrow \theta_\eta = 40.47^\circ$

$$|\eta_2| = \frac{\sqrt{\mu_2/\epsilon_2}}{\sqrt[4]{1+\left(\frac{\sigma_2}{\omega\epsilon_2}\right)^2}} = \frac{377/\sqrt{80}}{\sqrt[4]{1+4\pi^2}} = 16.71$$

$\eta_2 = \underline{\underline{16.71 \angle 40.47^\circ \ \Omega}}.$

(d) $$\Gamma = \frac{\eta_2-\eta_1}{\eta_2+\eta_1} = \frac{16.71\angle 40.47^\circ - 377}{16.71\angle 40.47^\circ + 377} = 0.935 \angle 176.7^\circ$$

$E_{or} = \Gamma E_{oi} = 9.35 \angle 176.7^\circ$

$\vec{E}_r = \underline{\underline{9.35 \sin(\omega t - 3z + 176.7^\circ)\, \bar{a}_x \ \text{V/m}}}.$

$$\alpha_2 = \frac{\omega}{c}\sqrt{\frac{\mu_{r2}\epsilon_{r2}}{2}\left[\sqrt{1+\left(\frac{\sigma_2}{\epsilon_2\omega}\right)^2} - 1\right]}.$$

$$= \frac{9\times10^{8}}{3\times10^{8}}\sqrt{40}\sqrt{\sqrt{1+4\pi^2} - 1} = 43.94 \text{ Np/m}$$

$$\beta_2 = \frac{9\times10^{8}}{3\times10^{8}}\sqrt{40}\sqrt{\sqrt{1+4\pi^2} + 1} = 51.48 \text{ rad/m}$$

$$\tau = \frac{2\eta_2}{\eta_1+\eta_2} = \frac{33.42\angle 40.47^\circ}{377+16.71\angle 40.47^\circ} = 0.0857 \angle 38.89^\circ$$

$$E_{ot} = \tau E_o = 0.857 \angle 38.59^\circ$$

$$\underline{\underline{E_t = 0.857 e^{43.94z} \sin(9\times10^8 t + 51.48z + 38.89^\circ)\ \text{V/m}.}}$$

$\boxed{\text{Prob. 10.35}}$

Both media are lossless, hence Γ is real.

$$\Gamma = \frac{s-1}{s+1} = \frac{1.2}{3.2} = \frac{3}{8}, \quad \tau = 1 + \Gamma = \frac{11}{8}$$

from Prob. 10.29, $P_t = T P_i = \frac{\eta_1}{\eta_2} \tau^2 P_i$.

But $\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{1-x}{1+x} = \frac{3}{8}$, where $x = \frac{\eta_1}{\eta_2}$.

$$3x + 3 = 8 - 8x \longrightarrow x = \frac{5}{11} = \frac{\eta_1}{\eta_2}$$

$$P_t = \frac{5}{11}\left(\frac{11}{8}\right)^2 (3) = \underline{\underline{2.578\ \text{W/m}^2}}.$$

$\boxed{\text{Prob. 10.36}}$

$$\eta_1 = \eta_o, \quad \eta_2 = \sqrt{\frac{\mu}{\epsilon}} = 3\eta_o$$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{1}{2}, \quad \tau = 1 + \Gamma = \frac{3}{2}, \quad E_{to} = \tau E_{io} = \frac{3}{2}(2) = 2$$

$$P_{t\,ave} = \frac{E_{to}^2}{2\eta_2} = \frac{\frac{1}{2}(3)^2}{3(120\pi)} = \frac{1}{80\pi} = \underline{\underline{3.98\ \text{mW/m}^2}}.$$

$\boxed{\text{Prob. 10.37}}$

$$\eta_1 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377$$

$$\tan 2\theta_{\eta_2} = \frac{\sigma_2}{\omega\epsilon_2} = \frac{0.02}{10^9 \times \frac{10^{-9}}{36\pi}} = 2.262 \rightarrow \theta_{\eta_2} = 33.08^\circ$$

$$|\eta_2| = \frac{377}{\sqrt[4]{1+(2.262)^2}} = 239.72$$

$$\eta_2 = 239.72 \angle 33.08^\circ$$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{239.72 \angle 33.08^\circ - 377}{239.72 \angle 33.08^\circ + 377}$$

$$= 0.371 \angle 130.61^\circ$$

$$E_{ro} = \Gamma E_{io}, \quad E_{rs} = 0.371 \angle 130.61^\circ \left(40 e^{-j\beta_1 z} \vec{a}_x + 30 e^{-j(\beta_1 z - 90^\circ)} \vec{a}_y\right)$$

$$\vec{E}_r = 14.84 \cos(10^9 t + 3.33z + 130.61^\circ)\vec{a}_x$$
$$+ 11.13 \sin(10^9 t + 3.33z + 130.61^\circ)\vec{a}_y \ \text{V/m}$$

$$a_H = \vec{a}_k \times \vec{a}_E \longrightarrow \vec{a}_{H_1} = -\vec{a}_z \times \vec{a}_x = -\vec{a}_y$$
$$\vec{a}_{H_2} = -\vec{a}_z \times \vec{a}_y = \vec{a}_x$$

$$\vec{H}_r = \frac{-14.84}{377} \cos(\omega t + 3.33z + 130.61^\circ)\vec{a}_y$$
$$+ \frac{11.13}{377} \sin(\omega t + 3.33z + 130.61^\circ)\vec{a}_x$$

$$\vec{H}_r = 29.52 \sin(10^9 t + 3.33z + 130.61^\circ)\vec{a}_x$$

$$-39.36\cos(10^9t+3.33z+130.61^\circ)\vec{a}_y \text{ mA/m.}$$

$$\tau = 1+\Gamma = 1+0.371\underline{/130.61^\circ} = 0.81\underline{/20.3^\circ},$$

$E_{to} = \tau E_{io}$,

$$\vec{E}_{ts} = 0.81\underline{/20.3^\circ}\left[40e^{-j\beta_2 z}\vec{a}_x + 30e^{-j(\beta_2 z-90^\circ)}\vec{a}_y\right]$$

But $\beta_2 = 10^9\sqrt{\frac{\mu_o\varepsilon_o}{2}\left[\sqrt{1+(2.262)^2}+1\right]} = 4.393$

$$\vec{E}_t = 32.4\cos(10^9t-4.393z+20.3^\circ)\vec{a}_x$$
$$+24.3\sin(10^9t-4.393z+20.3^\circ)\vec{a}_y \text{ V/m.}$$

Since $\vec{a}_{H1} = \vec{a}_z\times\vec{a}_x = \vec{a}_y$, $\vec{a}_{H2} = \vec{a}_z\times\vec{a}_y = -\vec{a}_x$

and $\eta_2 = 239.72\underline{/33.08^\circ}$,

$$\vec{H}_t = \frac{32.4}{239.72}\cos(10^9t-4.393z+20.3^\circ-33.08^\circ)\vec{a}_y$$
$$-\frac{24.3}{239.72}\sin(10^9t-4.393z+20.3^\circ-33.08^\circ)\vec{a}_x$$

or $$\vec{H}_t = -101.6\sin(10^9t-4.393z-12.78^\circ)\vec{a}_x$$
$$+135.5\cos(10^9t-4.393z-12.78^\circ)\vec{a}_y \text{ mA/m.}$$

Prob-10.38

(a) $\frac{\lambda}{2} = 4\text{m}$ or $\lambda = 8\text{m}$, $f = \frac{c}{\lambda} = \frac{3\times10^8}{8} = 37.5 \text{ MHz}$.

(b) $|\Gamma| = \frac{s-1}{s+1} = \frac{1}{3}$, $\theta_r = 0$, $\Gamma = 0.3333$.

$\boxed{\text{Prob. 10-39}}$

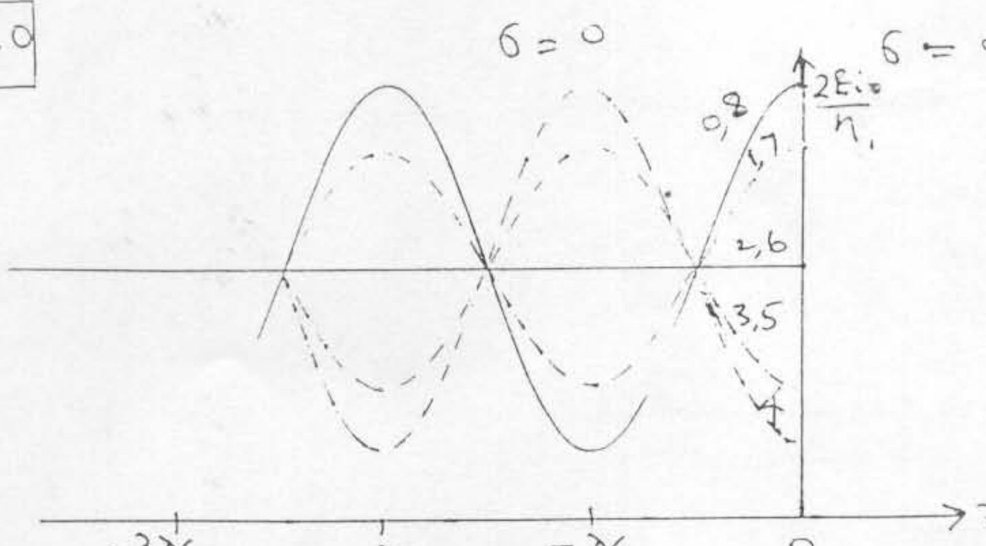
$$\vec{E} = \vec{E}_i + \vec{E}_r$$
$$= -40[\cos(wt+\beta z) - \cos(wt-\beta z)]\,\vec{a}_x$$
$$-30[\sin(wt+\beta z) - \sin(wt-\beta z)]\,\vec{a}_y$$
$$= 80\sin wt \sin\beta z\,\vec{a}_x - 60\cos wt \sin\beta z\,\vec{a}_y \text{ V/m}$$

which is a standing wave

$$\vec{H} = \vec{H}_i + \hat{E}_r$$
$$= \frac{1}{3\pi}[\cos(wt+\beta z) + \cos(wt-\beta z)]\,\vec{a}_y$$
$$-\frac{1}{4\pi}[\sin(wt+\beta z) + \sin(wt-\beta z)]\,\vec{a}_x$$
$$= \frac{2}{3\pi}\cos wt \cos\beta z\,\vec{a}_y - \frac{2}{4\pi}\sin wt \cos\beta z\,\vec{a}_x \text{ A/m}$$

which is a standing wave.

$\boxed{\text{Prob. 10.40}}$



Curve 0 is at $t = 0$, curve 1 is at $t = T/8$, curve 2 is at $t = \frac{T}{4}$, curve 3 is at $t = 3T/8$, etc.

$\boxed{\text{Prob. 10.41}}$

Since $\mu_o = \mu_1 = \mu_2$,

$$\sin\theta_{t1} = \sin\theta_i \sqrt{\frac{\epsilon_o}{\epsilon_1}} = \frac{\sin 45^\circ}{\sqrt{4.5}} = 0.3333 \longrightarrow \theta_{t1} = \underline{\underline{19.47^\circ}}$$

$$\sin\theta_{t2} = \sin\theta_{t1}\sqrt{\frac{\epsilon_2}{\epsilon_1}} = \frac{1}{3}\sqrt{\frac{2.25}{4.5}} = 0.2357$$

$$\rightarrow \theta_{t2} = \underline{\underline{13.63^\circ}}.$$

$\boxed{\text{Prob. 10.42}}$

$$\vec{E}_s = 20\frac{(e^{jk_x x} - e^{-jk_x x})}{2j}\frac{(e^{jk_y y} + e^{-jk_y y})}{2}\vec{a}_z$$

$$= -j5\left[e^{j(k_x x + k_y y)} + e^{j(k_x x - k_y y)} - e^{-j(k_x x - k_y y)} - e^{-j(k_x x + k_y y)}\right]\vec{a}_z$$

which consists of four plane waves.

$$\nabla \times \vec{E}_s = -j\omega\mu_o \vec{H}_s \longrightarrow \vec{H}_s = \frac{j}{\omega\mu_o}\nabla\times\vec{E}_s$$

$$= \frac{j}{\omega\mu_o}\left(\frac{\partial E_z}{\partial y}\vec{a}_x - \frac{\partial E_z}{\partial x}\vec{a}_y\right)$$

$$\vec{H}_s = -\frac{j20}{\omega\mu_o}\left[k_y \sin(k_x x)\sin(k_y y)\vec{a}_x + k_x\cos(k_x x)\cos(k_y y)\vec{a}_y\right].$$

$\boxed{\text{Prob. 10.43}}$

If $\mu_1 = \mu_2 = \mu_o$, $\eta_1 = \frac{\eta_o}{\sqrt{\epsilon_{r1}}}$, $\eta_2 = \frac{\eta_o}{\sqrt{\epsilon_{r2}}}$

$$\Gamma_{11} = \frac{\frac{1}{\sqrt{\epsilon_{r2}}}\cos\theta_t - \frac{1}{\sqrt{\epsilon_{r1}}}\cos\theta_i}{\frac{1}{\sqrt{\epsilon_{r2}}}\cos\theta_t + \frac{1}{\sqrt{\epsilon_{r1}}}\cos\theta_i}$$

$$\sqrt{\epsilon_{r1}}\sin\theta_i = \sqrt{\epsilon_{r2}}\sin\theta_t \longrightarrow \frac{\sqrt{\epsilon_{r2}}}{\sqrt{\epsilon_{r1}}} = \frac{\sin\theta_i}{\sin\theta_t},$$

$$\Gamma_{11} = \frac{\cos\theta_t - \frac{\sin\theta_i}{\sin\theta_t}\cos\theta_i}{\cos\theta_t + \frac{\sin\theta_i}{\sin\theta_t}\cos\theta_i} = \frac{\sin\theta_t\cos\theta_t - \sin\theta_i\cos\theta_i}{\sin\theta_t\cos\theta_t + \sin\theta_i\cos\theta_i}$$

Dividing both numerator and denominator by $\cos\theta_i\cos\theta_t$ gives

$$\Gamma_{11} = \frac{\tan\theta_t - \tan\theta_i}{\tan\theta_t + \tan\theta_i} = \frac{\frac{\tan\theta_t - \tan\theta_i}{1+\tan\theta_i\tan\theta_t}}{\frac{\tan\theta_t - \tan\theta_i}{1+\tan\theta_i\tan\theta_t}}$$

$$= \frac{\tan(\theta_t - \theta_i)}{\tan(\theta_t + \theta_i)}.$$

Similarly,

$$\tau_{11} = \frac{\frac{2}{\sqrt{\epsilon_{r2}}}\cos\theta_i}{\frac{1}{\sqrt{\epsilon_{r2}}}\cos\theta_t + \frac{1}{\sqrt{\epsilon_{r1}}}\cos\theta_i} = \frac{2\cos\theta_i}{\cos\theta_t + \frac{\sin\theta_i}{\sin\theta_t}\cos\theta_i}$$

$$= \frac{2\cos\theta_i\sin\theta_t}{\sin\theta_t\cos\theta_t(\sin^2\theta_i + \cos^2\theta_i) + \sin\theta_i\cos\theta_i(\sin^2\theta_t + \cos^2\theta_i)}$$

$$\tau_{\parallel} = \frac{2\cos\theta_i \sin\theta_t}{(\sin\theta_i\cos\theta_t + \sin\theta_t\cos\theta_i)(\cos\theta_i\cos\theta_t + \sin\theta_i\sin\theta_t)}$$

$$= \frac{2\cos\theta_i\sin\theta_t}{\sin(\theta_t+\theta_i)\cos(\theta_t-\theta_i)}$$

$$\Gamma_{\perp} = \frac{\frac{1}{\sqrt{\epsilon_{r2}}}\cos\theta_i - \frac{1}{\sqrt{\epsilon_{r1}}}\cos\theta_t}{\frac{1}{\sqrt{\epsilon_{r2}}}\cos\theta_i + \frac{1}{\sqrt{\epsilon_{r1}}}\cos\theta_t} = \frac{\cos\theta_i - \frac{\sin\theta_i\cos\theta_t}{\sin\theta_t}}{\cos\theta_i + \frac{\sin\theta_i\cos\theta_t}{\sin\theta_t}}$$

$$= \frac{\sin(\theta_t-\theta_i)}{\sin(\theta_t+\theta_i)}$$

$$\tau_{\perp} = \frac{\frac{2}{\sqrt{\epsilon_{r2}}}\cos\theta_i}{\frac{1}{\sqrt{\epsilon_{r2}}}\cos\theta_i + \frac{1}{\sqrt{\epsilon_{r1}}}\cos\theta_t} = \frac{2\cos\theta_i}{\cos\theta_i + \frac{\sin\theta_i}{\sin\theta_t}\cos\theta_t}$$

$$= \frac{2\cos\theta_i\sin\theta_t}{\sin(\theta_t+\theta_i)}$$

Prob. 10.44

$$\vec{H} = H_o\vec{a}_H = \frac{E_o}{\eta}\vec{a}_k \times \vec{a}_E = \frac{50}{120\pi}\vec{a}_k \times \vec{a}_E$$

where $\vec{a}_k = \sin 45^\circ \vec{a}_x + \cos 45^\circ \vec{a}_z$

$$\vec{a}_k \times \vec{a}_E = -\cos 45^\circ \vec{a}_x + \sin 45^\circ \vec{a}_z$$

$$\vec{H} = \frac{50}{120\pi}(-\cos 45^\circ\vec{a}_x + \sin 45^\circ a_z)\cos(wt - \ldots)$$

$$\vec{H} = 93.78(-\vec{a}_x + \vec{a}_z)\cos(\omega t - \beta_1 x\sin 45^\circ - \beta_1 z\cos 45^\circ)\ mA/m$$

$$\tau_\perp = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_i + \eta_1\cos\theta_t} = \frac{2}{1 + \frac{\eta_1}{\eta_2}\frac{\cos\theta_t}{\cos\theta_i}},$$

$$\cos\theta_i = \vec{a}_n \cdot \vec{a}_k = \cos 45^\circ \longrightarrow \theta_i = 45^\circ$$

$$k_i\sin\theta_i = k_t\sin\theta_t \longrightarrow \omega\sqrt{\mu_o\epsilon_o}\sin 45^\circ = \omega\sqrt{\mu_o\epsilon_o(2.25)}\sin\theta_t$$

$$\sin\theta_t = \frac{\sin 45^\circ}{1.5} \longrightarrow \theta_t = 28.125^\circ, \frac{\eta_1}{\eta_2} = 1.5$$

$$\tau_\perp = \frac{2}{1 + 1.5\frac{\cos 28.125^\circ}{\cos 45^\circ}} = \underline{\underline{0.6967}}.$$

$\boxed{\text{Prob. 10.45}}$

(a) $\tan\theta_i = \frac{k_{ix}}{k_{iz}} = \frac{1}{\sqrt{8}} \longrightarrow \theta_i = \theta_r = \underline{\underline{19.47^\circ}}.$

$\sin\theta_t = \sin\theta_i\sqrt{\frac{\epsilon_{r1}}{\epsilon_{r2}}} = \frac{1}{3}(3) = 1 \longrightarrow \theta_t = \underline{\underline{90^\circ}}.$

(b) $\beta_1 = \frac{\omega}{c}\sqrt{\epsilon_{r1}} = \frac{10^9}{3\times10^8}\times 3 = 10 = k\sqrt{1+8} = 3k$

$\longrightarrow k = \frac{10}{3} = \underline{\underline{3.333}}.$

(c) $\lambda = \frac{2\pi}{\beta}, \quad \lambda_1 = \frac{2\pi}{\beta_1} = \frac{2\pi}{10} = \underline{\underline{0.6283\ m}}.$

$\beta_2 = \frac{\omega}{c} = \frac{10}{3}, \quad \lambda_2 = \frac{2\pi}{\beta_2} = \frac{2\pi\times3}{10} = \underline{\underline{1.885\ m}}.$

(d) $\vec{E}_i = \eta_1 \vec{a}_k \times \vec{H}_i = 40\pi \dfrac{(\vec{a}_x + \sqrt{8}\,\vec{a}_z)}{3} \times 0.2 \cos(\omega t - \vec{k}\cdot\vec{r})\, \vec{a}_y$

$= \underline{\underline{(-213.3\,\vec{a}_x + 75.4\,\vec{a}_z) \cdot \cos(10^9 t - kx - k\sqrt{8}z)\ V/m}}$

(e) $\tau_{\parallel} = \dfrac{2\cos\theta_i \sin\theta_t}{\sin(\theta_i + \theta_t)\cos(\theta_t - \theta_i)}$

$= \dfrac{2\cos 19.47^\circ \sin 90^\circ}{\cos 19.47^\circ \sin 19.47^\circ} = 6$

$\Gamma_{\parallel} = -\dfrac{\cot 19.47^\circ}{\cot 19.47^\circ} = -1$

Let $\vec{E}_t = -E_{to}(\cos\theta_t\, \vec{a}_x - \sin\theta_t\, \vec{a}_z)\cos(10^9 t - \beta_2 x \sin\theta_t - \beta_2 z \cos\theta_t)$

where $\vec{E}_i = -E_{io}(\cos\theta_i\, \vec{a}_x - \sin\theta_i\, \vec{a}_z)\cos(10^9 t - \beta_1 x \sin\theta_i - \beta_1 z \cos\theta_i)$

$\sin\theta_t = 1,\ \cos\theta_t = 0,\ \beta_2 \sin\theta_t = \dfrac{10}{3},$

$E_{to}\sin\theta_t = \tau_{\parallel} E_{io} = 6(24\pi)(3)(1) = 1357.2$

Hence

$\vec{E}_t = \underline{\underline{1357\cos(10^9 t - 3.333x)\,\vec{a}_z\ V/m.}}$

Since $\Gamma = -1$, $\theta_r = \theta_i$

$\vec{E}_r = \underline{\underline{(213.3\,\vec{a}_x + 75.4\,\vec{a}_z)\cos(10^9 t - kx + k\sqrt{8}z)\ V/m.}}$

(f) $\tan\theta_{B\parallel} = \sqrt{\frac{\epsilon_2}{\epsilon_1}} = \sqrt{\frac{\epsilon_0}{9\epsilon_0}} = \frac{1}{3} \longrightarrow \theta_{B\parallel} = \underline{\underline{18.43^\circ}}$.

Prob. 10.46

Total transmission occurs at Brewster angle. Since $\mu_1 = \mu_2 = \mu_0$, this can only occur for parallel polarization.

$\eta_1 = 1, \quad \eta_2 = \sqrt{\epsilon_{r2}} = 2$

$\theta_{B\parallel} = \theta_i = \tan^{-1}\frac{n_2}{n_1} = \tan^{-1}\frac{2}{1} \longrightarrow \theta_i = 63.43^\circ$

from Snell's law, $\sin\theta_t = \frac{n_1}{n_2}\sin\theta_i = \frac{\sin 63.43^\circ}{2}$

$\longrightarrow \quad \theta_t = \underline{\underline{26.56^\circ}}$

Prob. 10.47

$\beta_1 = \sqrt{3^2 + 4^2} = 5 = \frac{w}{c} \rightarrow w = \beta_1 c = \underline{\underline{15 \times 10^8 \text{ rad/s}}}$.

Let $\vec{E}_r = (E_{ox}, E_{oy}, E_{oz})\sin(wt + 3x + 4y)$. In order for $\nabla \cdot \vec{E}_r = 0$, $\quad 3E_{ox} + 4E_{oy} = 0 \quad$ —(1)

Also at $y = 0$, $\vec{E}_{1tan} = E_{2tan} = 0$

$\vec{E}_{1tan} = 0 \longrightarrow 8\bar{a}_x + 5\bar{a}_z + E_{ox}\bar{a}_x + E_{oz}\bar{a}_z = 0$

Equating components, $E_{ox} = -8, \quad E_{oz} = -5$

from (1), $4E_{oy} = -3E_{ox} = 24 \longrightarrow \quad E_{oy} = 6$

Hence $\vec{E}_r = \underline{\underline{(-8\bar{a}_x + 6\bar{a}_y - 5\bar{a}_z)\sin(15\times10^8 t + 3x + 4y) \text{ V/m}}}$

CHAPTER 11

P.E. 11.1

Since Z_o is real and $\alpha \neq 0$, this is a distortionless line.

$$Z_o = \sqrt{\frac{R}{G}} \quad -(1)$$

or

$$\frac{L}{R} = \frac{C}{G} \quad -(2)$$

$$\alpha = \sqrt{RG} \quad -(3)$$

$$\beta = wL\sqrt{\frac{G}{R}} = \frac{wL}{Z} \quad -(4)$$

$$(1)\times(3) \longrightarrow \quad R = \alpha Z_o = 0.04 \times 80 = \underline{\underline{3.2\ \Omega/m}}.$$

$$(3)\div(1) \longrightarrow \quad G = \frac{\alpha}{Z_o} = \frac{0.04}{80} = \underline{\underline{5\times10^{-4}\ \mho/m}}.$$

$$L = \frac{\beta Z_o}{w} = \frac{1.5\times80}{2\pi\times5\times10^{8}} = \underline{\underline{38.2\ nH/m}}.$$

$$C = \frac{LG}{R} = \frac{12}{\pi}\cdot10^{-8}\times\frac{0.04}{80}\times\frac{1}{0.04\times80} = \underline{\underline{5.97\ pF/m}}.$$

P.E. 11.2

(a)

$$Z_o = \sqrt{\frac{R+jwL}{G+jwC}} = \sqrt{\frac{0.03+j2\pi\times10^{3}\times0.1\times10^{-3}}{0+j2\pi\times10^{3}\times0.02\times10^{-6}}}$$

$$= 70.73 - j1.688 = \underline{\underline{70.75\ \angle{-1.367^\circ}\ \Omega}}.$$

(b)

$$\gamma = \sqrt{(R+jwL)(G+jwC)} = \sqrt{(0.03+j0.2\pi)(j0.4\times10^{-4}\pi)}$$

$$= \underline{\underline{2.121\times10^{-4} + j8.888\times10^{-3}\ /m}}.$$

(c) $u = \frac{w}{\beta} = \frac{2\pi \times 10^3}{8.888 \times 10^{-3}} = \underline{\underline{7.069 \times 10^5 \text{ m/s}}}$.

P.E. 11.3

(a) $Z_o = Z_L \longrightarrow Z_{in} = Z_o = \underline{\underline{30 + j60 \ \Omega}}$

(b) $V_{in} = V_o = \frac{Z_{in}}{Z_{in} + Z_g} V_g = \frac{V_g}{2} = \underline{\underline{7.5 \angle 0° \ V_{rms}}}$

$$I_{in} = I_o = \frac{V_g}{Z_g + Z_{in}} = \frac{V_g}{2Z_o} = \frac{15 \angle 0°}{2(30 + j60)}$$

$$= \underline{\underline{0.05 \angle -63.43° \ A}}$$

(c) Since $Z_o = Z_L$, $\Gamma = 0 \longrightarrow V_o^- = 0$, $V_o^+ = V_o$.

The load voltage is

$$V_L = V_s(z = l) = V_o^+ e^{-\gamma l}$$

$$e^{\gamma l} = \frac{V_o^+}{V_L} = \frac{7.5 \angle 0°}{5 \angle -48°} = 1.5 \angle 48°$$

$$e^{\alpha l} e^{j\beta l} = 1.5 \angle 48°$$

$$e^{\alpha l} = 1.5 \longrightarrow \alpha = \frac{1}{l} \ln 1.5 = \frac{1}{40} \ln 1.5 = 0.0101$$

$$e^{j\beta l} = e^{j48°} \longrightarrow \beta = \frac{1}{l} \frac{48°}{180} \pi \text{ rad} = 0.02094$$

$$\gamma = \underline{\underline{0.0101 + j0.2094 \ /m}}$$