Examples: 1. Find the Corr(X,Y) if (X,Y) is jointly continuous with pdf

$$f(x,y) = \begin{cases} x+y, & 0<x<1, \quad 0<y<1 \\ 0, & \text{ew} \end{cases}$$

$$E(XY) = \int_0^1\int_0^1 xy\,(x+y)\,dx\,dy = \frac{1}{6}+\frac{1}{6} = \frac{1}{3}$$

The marginal pdf of X is

$$f_X(x) = \int_0^1 (x+y)\,dy = x+\frac{1}{2}, \quad 0<x<1$$

The marginal pdf of Y is

$$f_Y(y) = \int_0^1 (x+y)\,dx = y+\frac{1}{2}, \quad 0<y<1$$

$$E(X) = \int_0^1 x\left(x+\frac{1}{2}\right)dx = \frac{1}{3}+\frac{1}{4} = \frac{7}{12}$$

$E(Y) = 7/12$. $\quad E(X^2) = \int_0^1 x^2\left(x+\frac{1}{2}\right)dx = \frac{1}{4}+\frac{1}{6} = \frac{5}{12}$

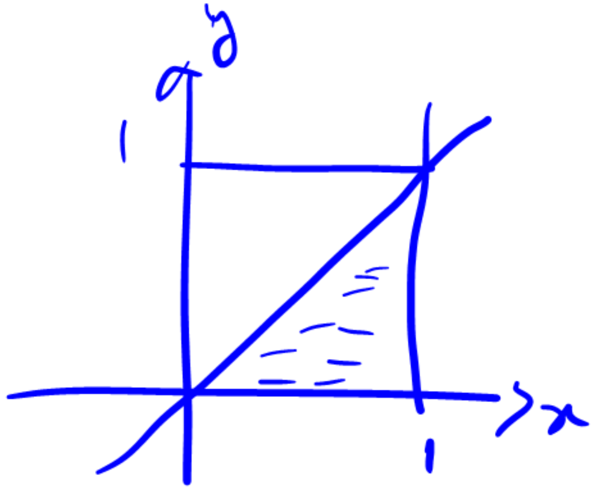
$$V(X) = \frac{5}{12} - \frac{49}{144} = \frac{11}{144} = V(Y)$$

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)\,E(Y) = \frac{1}{3} - \frac{49}{144} = -\frac{1}{144}$$

$$\rho_{X,Y} = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}} = -\frac{1}{11}.$$

2. Let (X,Y) be jointly distributed continuous r.v.'s with pdf

$$f(x,y) = \begin{cases} 2, & 0<y<x<1 \\ 0, & \text{ew} \end{cases}$$



$$f_X(x) = \int_0^x 2\,dy = \begin{cases} 2x, & 0<x<1 \\ 0, & \text{ew} \end{cases}$$

$$f_Y(y) = \int_y^1 2\,dx = \begin{cases} 2(1-y), & 0<y<1 \\ 0, & \text{ew} \end{cases}$$

$$E(X) = \int_0^1 2x^2\,dx = \frac{2}{3}, \quad E(X^2) = \int_0^1 2x^3\,dx = \frac{1}{2}$$

$$V(X) = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$$

$$E(Y) = \int_0^1 2y(1-y)\,dy = 1 - \frac{2}{3} = \frac{1}{3}$$

$$E(Y^2) = \int_0^1 2y^2(1-y)\,dy = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$$

$$V(Y) = \frac{1}{6} - \frac{1}{9} = \frac{1}{18}$$

$$E(XY) = \int_0^1 \int_0^x 2xy\,dy\,dx = \frac{1}{4}$$

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y) = \frac{1}{4} - \frac{2}{3} \cdot \frac{1}{3} = \frac{1}{36}$$

$$\rho_{X,Y} = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}} = \frac{1/36}{1/18} = \frac{1}{2}$$

The joint mgf of X and Y is

$$M_{X,Y}(s,t) = E\left(e^{sX+tY}\right)$$

provided it exists in a neighbourhood of $(0,0)$.

<u>Theorem</u>: X and Y are independent

$$\iff M_{X,Y}(s,t) = M_X(s)\,M_Y(t) \quad \forall (s,t) \in \mathbb{R}^2$$

<u>Theorem</u>: If X and Y are independent, then

$$M_{X+Y}(t) = M_X(t)\,M_Y(t)$$

Pf. Let X and Y be independent. Then

$$M_{X+Y}(t) = E\left\{e^{t(X+Y)}\right\} = E\left(e^{tX} \cdot e^{tY}\right)$$

$$= E\left(e^{tX}\right) E\left(e^{tY}\right) = M_X(t) M_Y(t)$$

Bivariate Normal Distribution

A continuous jointly distributed r.v. (X, Y) is said to have a bivariate normal distribution if it has pdf given by

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{(1-\rho^2)}} e^{-\frac{1}{2}Q},$$

where $$Q = \frac{1}{(1-\rho^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 + \left(\frac{y-\mu_2}{\sigma_2}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right)\right]$$

$(x,y) \in \mathbb{R}^2$, $\mu_1 \in \mathbb{R}$, $\mu_2 \in \mathbb{R}$, $\sigma_1 > 0$, $\sigma_2 > 0$

$-1 < \rho < 1$

Now we can write

$$Q = \left(\frac{x-\mu_1}{\sigma_1}\right)^2 + \frac{1}{(1-\rho^2)}\left\{\frac{y-\mu_2}{\sigma_2} - \rho\left(\frac{x-\mu_1}{\sigma_1}\right)\right\}^2$$

Then we can express

$$f_{X,Y}(x,y) = \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}$$

$$\frac{1}{\sigma_2\sqrt{1-\rho^2}\,\sqrt{2\pi}} e^{-\frac{1}{2\underbrace{\sigma_2^2(1-\rho^2)}}\left[y-\underbrace{\left\{\mu_2+\rho\sigma_2\left(\frac{x-\mu_1}{\sigma_1}\right)\right\}}\right]^2}$$

So the marginal pdf of X is

$$f_X(x) = \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}, \quad x \in \mathbb{R}$$

So $X \sim N(\mu_1, \sigma_1^2)$

Also we can obtain the conditional pdf

of Y given $X=x$ as

$$f_{Y|X=x}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

$$= \frac{1}{\sigma_2\sqrt{1-\rho^2}\sqrt{2\pi}} e^{-\frac{1}{2\sigma_2^2(1-\rho^2)}\left[y-\left\{\mu_2+\rho\sigma_2\left(\frac{x-\mu_1}{\sigma_1}\right)\right\}\right]^2}$$

So

$$Y|_{X=x} \sim N\left(\mu_2+\rho\sigma_2\left(\frac{x-\mu_1}{\sigma_1}\right),\ \sigma_2^2(1-\rho^2)\right)$$

We can also write

$$Q = \left(\frac{y-\mu_2}{\sigma_2}\right)^2 + \frac{1}{1-\rho^2}\left\{\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - \rho\left(\frac{y-\mu_2}{\sigma_2}\right)\right\}^2$$

So the joint pdf of X and Y can be expressed as

$$f_{X,Y}(x,y) = \frac{1}{\sigma_2\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{y-\mu_2}{\sigma_2}\right)^2}$$

$$\frac{1}{\sigma_1\sqrt{1-\rho^2}\,\sqrt{2\pi}}\, e^{-\frac{1}{2\sigma_1^2(1-\rho^2)}\left[x-\left\{\mu_1+\rho\sigma_1\left(\frac{y-\mu_2}{\sigma_2}\right)\right\}\right]^2}$$

So the marginal pdf of Y is obtained as

$$f_Y(y) = \frac{1}{\sigma_2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\mu_2}{\sigma_2}\right)^2}, \quad y \in \mathbb{R}$$

So $Y \sim N(\mu_2, \sigma_2^2)$

We get the conditional pdf of X given $Y=y$ as

$$f_{X|Y=y}(x|y) = \frac{1}{\sigma_1\sqrt{1-\rho^2}\sqrt{2\pi}} e^{-\frac{1}{2\sigma_1^2(1-\rho^2)}\left[x-\left\{\mu_1+\rho\sigma_1\left(\frac{y-\mu_2}{\sigma_2}\right)\right\}\right]^2}, \quad x \in \mathbb{R}$$

So

$$X|_{Y=y} \sim N\left(\mu_1+\rho\sigma_1\left(\frac{y-\mu_2}{\sigma_2}\right), \; \sigma_1^2(1-\rho^2)\right)$$

Theorem: (X,Y) have bivariate normal distribution iff the marginal distributions of X and Y and the conditional distributions of X given $Y=y$ & Y given $X=x$ are univariate normal.

$$(X,Y) \sim BVN(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$$

Examples 1. $(X,Y) \sim BVN(6, 4, 1, 0.25, 0.1)$

$X \sim N(6, 1) \qquad Y \sim N(4, 0.25)$

$$Y|_{X=x} \sim N\left(4 + (0.1)(0.5)\left(\frac{x-6}{1}\right),\ 0.25(1-0.01)\right)$$

$$\mu_2 + \rho\sigma_2\left(\frac{x-\mu_1}{\sigma_1}\right),\ \sigma_2^2(1-\rho^2)$$

$$X|_{Y=y} \sim N\left(6+(0.1)(1)\left(\frac{y-4}{0.5}\right),\ (1-0.01)\right)$$

$$P(X \le 5) = P\left(Z \le \frac{5-6}{1}\right) = \Phi(-1) = 0.1587$$

$$P(Y \le 5 | X=5) = P\left(Z \le \frac{5-3.975}{\sqrt{0.2475}}\right) = \Phi(2.06) \cong 0.98$$

$$Y|_{X=5} \sim N(3.975,\ 0.2475)$$

2. $(X,Y) \sim BVN(2000,\ 0.1,\ 2500,\ 0.01,\ 0.87)$

$X \sim N(2000,\ 2500)$, $Y \sim N(0.1,\ 0.01)$

$$P(X > 1950 | Y = 0.098) = P\left(Z > \frac{1950 - 2000.87}{\sqrt{607.25}}\right)$$

$$X\Big|_{Y=0.098} \sim N\left(2000 + 0.87 \times 50\left(\frac{0.098 - 0.1}{0.1}\right),\right.$$

$$\left. 2500\left(1 - (0.87)^2\right)\right)$$

$$\equiv N\left(2000.87,\ \ 607.25\right)$$

The Reqd prob $= \quad P(Z > -2.06)$

$$= \Phi(2.06) \approx 0.98$$

Suppose (X, Y) are jointly distributed

$$E\left(g(X,Y)\right) = \int\int g(x,y)\, f_{X,Y}(x,y)\, dx\, dy$$

$$= \int \left(\int g(x,y) \frac{f_{X,Y}(x,y)}{f_Y(y)} dx \right) f_Y(y)\, dy$$

$$= \int E\{g(X,Y) \mid Y=y\}\, f_Y(y)\, dy$$

$$= E^{Y} E^{X|Y}\{g(X,Y) \mid Y\}$$

Let (X,Y) be jointly distributed r.v's and $g: \mathbb{R}^2 \to \mathbb{R}$ be a measureable function.

Then $E\{g(X,Y)\} = E^{Y} E(g(X,Y)|Y)$

$$= E^{X} E(g(X,Y)|X)$$

provided expectation exists.

Moments / Product Moments of Bivariate Normal Distribution:

Let $(X,Y) \sim BVN(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$

So $X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$

$E(X) = \mu_1$, $V(X) = \sigma_1^2$; $E(Y) = \mu_2$, $V(Y) = \sigma_2^2$

$$X\big|_{Y=y} \sim N\left(\mu_1 + \rho\sigma_1\left(\frac{y-\mu_2}{\sigma_2}\right), \ \sigma_1^2(1-\rho^2)\right)$$

$$E(X|Y=y) = \mu_1 + \rho\sigma_1\left(\frac{y-\mu_2}{\sigma_2}\right)$$

$$V(X|Y=y) = \sigma_1^2(1-\rho^2)$$

$$Y\big|_{X=x} \sim N\left(\mu_2 + \rho\sigma_2\left(\frac{x-\mu_1}{\sigma_1}\right), \ \sigma_2^2(1-\rho^2)\right)$$

$$E(Y \mid X=x) = \mu_2 + \rho\,\sigma_2\left(\frac{x-\mu_1}{\sigma_1}\right)$$

$$V(Y \mid X=x) = \sigma_2^2(1-\rho^2)$$

$$\mathrm{Cov}(X,Y) = E\{(X-\mu_1)(Y-\mu_2)\}$$

$$= E^X\left[E\{(X-\mu_1)(Y-\mu_2) \mid X\}\right]$$

$$= E^X\left[(X-\mu_1)\quad E\{(Y-\mu_2) \mid X\}\right] = E\left[(X-\mu_1)\,\rho\,\sigma_2\left(\frac{X-\mu_1}{\sigma_1}\right)\right]$$

$$= \frac{\rho\sigma_2}{\sigma_1} E(X-\mu_1)^2 = \frac{\rho\sigma_2}{\sigma_1}\sigma_1^2 = \rho\sigma_1\sigma_2$$

So $$\text{Corr}(X,Y) = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)}\sqrt{\text{Var}(Y)}} = \frac{\rho\sigma_1\sigma_2}{\sigma_1\sigma_2}$$

$$= \rho$$

The joint mgf of (X,Y)

$$M_{X,Y}(s,t) = E\left(e^{sX+tY}\right)$$

$$= E^{Y}\left[E\left\{ e^{sX+tY} \middle| Y \right\} \right]$$

$$= E^{Y}\left[e^{tY} E\left(e^{sX} \middle| Y \right) \right] = E\left[e^{tY} M_{X|Y}(s) \right]$$

$$= E\left[e^{tY} \cdot e^{\left\{ \mu_1 + \rho\sigma_1 \left(\frac{Y-\mu_2}{\sigma_2} \right) \right\} s + \frac{1}{2}\sigma_1^2 (1-\rho^2) s^2} \right]$$

$$= e^{\left\{ \mu_1 s - \frac{\rho\sigma_1\mu_2}{\sigma_2} s + \frac{1}{2}\sigma_1^2 (1-\rho^2) s^2 \right\}} E\left[e^{\left(t + \frac{\rho\sigma_1}{\sigma_2} s \right) Y} \right]$$

$$= e^{\mu_1 s - \frac{\rho\sigma_1\mu_2 s}{\sigma_2} + \frac{1}{2}\sigma_1^2 s^2 - \frac{1}{2}\sigma_1^2\rho^2 s^2} \, e^{\mu_2\left(t + \frac{\rho\sigma_1}{\sigma_2}s\right) + \frac{1}{2}\sigma_2^2\left(t + \frac{\rho\sigma_1}{\sigma_2}s\right)^2}$$

$$= e^{\left[\mu_1 s + \mu_2 t + \frac{1}{2}\sigma_1^2 s^2 + \frac{1}{2}\sigma_2^2 t^2 + \rho\sigma_1\sigma_2 st\right]}$$

<u>Theorem</u>: Let $(X, Y) \sim BVN(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$.

Then X and Y are independent $\Longleftrightarrow \rho = 0$

<u>Proof</u> : X and Y are independent

$$\iff M_{X,Y}(s,t) = M_X(s)\, M_Y(t) \quad \forall (s,t)$$

$$\iff e^{\mu_1 s + \mu_2 t + \frac{1}{2}\sigma_1^2 s^2 + \frac{1}{2}\sigma_2^2 t^2 + \rho\sigma_1\sigma_2 st}$$

$$= e^{\mu_1 s + \frac{1}{2}\sigma_1^2 s^2} \cdot e^{\mu_2 t + \frac{1}{2}\sigma_2^2 t^2} \quad \forall (s,t)$$

$$\iff \rho = 0$$

<u>Theorem</u> : $(X,Y) \sim BVN(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$

$$\Leftrightarrow\quad aX+bY \sim N(a\mu_1+b\mu_2,\ a^2\sigma_1^2+b^2\sigma_2^2+2ab\rho\sigma_1\sigma_2)$$

for all $a, b \in \mathbb{R}$

Pf. ⊛ Ex

Random Vectors :

$$\underline{X} = (X_1, \ldots, X_k) : \Omega \to \mathbb{R}^k \quad \text{(measurable)}$$

The joint cdf of $\underline{X}$ is $\qquad \underline{x} = (x_1, \ldots x_k)$

$$F_{\underline{X}}(\underline{x}) = P(X_1 \le x_1, X_2 \le x_2, \ldots, X_k \le x_k)$$

Properties of Joint CDF :

1. In order to get joint cdf of a subset $(X_{i_1}, \ldots, X_{i_r})$, $1 \le r < k$ we take limit $x_j \to \infty$ for $j \neq i_1, \ldots, i_r$

2. $\lim_{x_i \to -\infty} F_{\underline{X}}(\underline{x}) = 0 \quad \forall \; i = 1, \ldots, k$

3. F is non-decreasing in each of its arguments

4. F is continuous from right in each of its arguments.

In case $(X_1, \ldots, X_k)$ is jointly discrete, we have joint pmf $p_{\underline{X}}(\underline{x})$ satisfying

(i) $0 \leq p_{\underline{X}}(\underline{x}) \leq 1 \quad \forall \quad \underline{x} \in \mathbb{R}^k$

(ii) $$\sum \sum \cdots \sum_{\underline{x} \in \mathbb{R}^k} p_{\underline{X}}(\underline{x}) = 1$$

(iii) $p_{\underline{X}}(\underline{x}) = P(X_1 = x_1, \ldots, X_k = x_k)$

The marginals & conditional pmfs can be evaluated from the joint pmf

Let $(X_1, \ldots X_k)$ be jointly continuous with pdf $f_{\underline{X}}(x)$. Then $f_{\underline{X}}(\underline{x})$ satisfies

(i) $f_{\underline{X}}(\underline{x}) \geq 0 \quad \forall \; \underline{x} \in R^k$

(ii) $\int \cdots \int f_{\underline{X}}(\underline{x}) \, dx_1 \cdots dx_k = 1$

(iii) $P(\underline{X} \in A) = \int \cdots \int_A f_{\underline{X}}(\underline{x})\, dx_1 \cdots dx_k$

for any $A \subset \mathbb{R}^k$.

The joint mgf of $\underline{X} = (X_1, \ldots, X_k)$ is

$$M_{\underline{X}}(\underline{t}) = E\left[e^{(t_1 X_1 + \cdots + t_k X_k)}\right]$$

$\underline{t} = (t_1 \ldots t_k)$

If $X_1, \ldots X_k$ are independently distributed then $$M_Y(t) = \prod_{i=1}^{k} M_{X_i}(t)$$

where $Y = \sum_{i=1}^{k} X_i$.

Additive Properties of Some distributions

1. Let $X_1, \ldots, X_k$ be i.i.d. (independent and identically distributed) r.v. with

$$X_i \sim Bin(n_i, p), \quad i = 1 \ldots k$$

Then $Y = \sum_{i=1}^{k} X_i \sim Bin\left(\sum_{i=1}^{k} n_i, p\right)$

Pf. $M_Y(t) = \prod_{i=1}^{k} M_{X_i}(t) = \prod_{i=1}^{k} (q + pe^t)^{n_i}$

$= (q + pe^t)^{\sum_{i=1}^{k} n_i}$ which is mgf of $Bin\left(\sum_{i=1}^{k} n_i, p\right)$

2. $X_1, \ldots X_k$ are i.i.d. Poisson r.v's

with $X_i \sim \mathcal{P}(\lambda_i)$, $i = 1 \ldots k$

$Y = \sum_{i=1}^{k} X_i$. Then $Y \sim \mathcal{P}\left(\sum_{i=1}^{k} \lambda_i\right)$

Pf. $M_Y(t) = \prod_{i=1}^{k} M_{X_i}(t)$

$$= \prod_{i=1}^{k} \left[e^{\lambda_i (e^t - 1)} \right]$$

$$= e^{\left(\sum_{i=1}^{k} \lambda_i\right)(e^t - 1)}$$

which is mgf of $P(\sum \lambda_i)$ dist^n

3. Let $X_1 \dots X_k$ be i.i.d. $Geo(p)$

Then $Y = \sum_{i=1}^{k} X_i \sim$ Neg. Bin (k, p)

Pf. $\circledast$ Ex

4. Let $X_1 \dots X_k$ be i.i.d. $Exp(\lambda)$

Then $Y = \sum_{i=1}^{k} X_i \sim$ Gamma (k, λ).

Linearity Property of Normal Distribution

Let $X_1 \ldots X_k$ be independent normal r.v.'s with $X_i \sim N(\mu_i, \sigma_i^2)$, $i=1, \ldots, k$

Let $Y = \sum_{i=1}^{k} (a_i X_i + b_i)$

Then $Y \sim N\left(\sum (a_i \mu_i + b_i), \sum a_i^2 \sigma_i^2\right)$

Pf ⊛ Ex