Assignment

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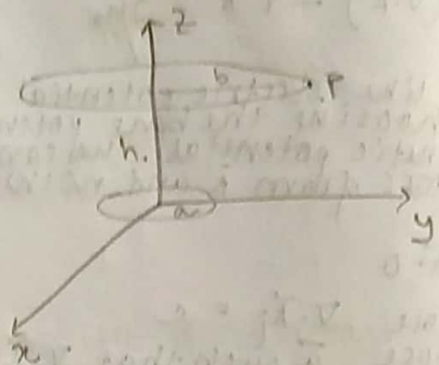
19EC10088

1. $a = 88$
$b = 2.5a$
$h = \frac{a+b}{2} = 1.75a$

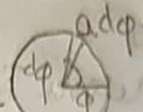
i) $\vec{A} = \oint \frac{\mu_0 I_1}{4\pi} \frac{\vec{dl}}{R}$



By symmetry, we expect the vector potential to be same everywhere on the loop. So we take point $P(0, b, h)$

$\vec{dl} = a\,d\varphi(-\sin\varphi\,\hat{a}_x + \cos\varphi\,\hat{a}_y)$
$\vec{R} = (-a\cos\theta,\, b - a\sin\theta,\, h)$

$$\vec{A} = \int_0^{2\pi} \frac{\mu_0 I_1}{4\pi} \frac{a\,d\varphi(-\sin\varphi\,\hat{a}_x + \cos\varphi\,\hat{a}_y)}{\sqrt{a^2\cos^2\varphi + (b - a\sin\varphi)^2 + h^2}}$$



$$\text{or } \vec{A} = \frac{\mu_0 I_1}{4\pi} a \int_0^{2\pi} \frac{(-\sin\varphi\,\hat{a}_x + \cos\varphi\,\hat{a}_y)\,d\varphi}{\sqrt{a^2\cos^2\varphi + h^2 + b^2 + a^2\sin^2\varphi - 2ab\sin\varphi}}$$

$$\text{or } \vec{A} = \frac{\mu_0 I_1}{4\pi} a \int_0^{2\pi} \frac{(-\sin\varphi\,\hat{a}_x + \cos\varphi\,\hat{a}_y)\,d\varphi}{\sqrt{a^2 + b^2 + h^2 - 2ab\sin\varphi}}$$

$$\text{or } \vec{A} = \frac{\mu_0 I_1}{4\pi} \int_0^{2\pi} \frac{(-\sin\varphi\,\hat{a}_x + \cos\varphi\,\hat{a}_y)\,d\varphi}{\sqrt{1 + \left(\frac{b}{a}\right)^2 + \left(\frac{h}{a}\right)^2 - 2\frac{b}{a}\sin\varphi}}$$

$$\text{or } \vec{A} = \frac{\mu_0 I_1}{4\pi} \int_0^{2\pi} \frac{(-\sin\varphi\,\hat{a}_x + \cos\varphi\,\hat{a}_y)\,d\varphi}{\sqrt{1 + 2.5^2 + 1.75^2 - 5\sin\varphi}}$$

$$\text{or } \vec{A} = \frac{\mu_0 I_1}{4\pi} \int_0^{2\pi} \frac{(-\sin\varphi\,\hat{a}_x + \cos\varphi\,\hat{a}_y)}{\sqrt{10.3125 - 5\sin\varphi}}\,d\varphi$$

The exact result was obtained using matlab and is presented below

$$\vec{A}(\vec{P}) = \frac{\mu_0 I_1}{4\pi}(0.2681969)(-\hat{a}_x) \qquad [\text{At } \vec{P} \;\; \hat{a}_\varphi = -\hat{a}_x]$$

Hence the general vector potential for any point on loop is

$$\vec{A} = \frac{\mu_0 I_1}{4\pi}(0.2681969)\,\hat{a}_\varphi$$

This holds good only when we need to find $\vec{A}$ on the wire. Although the magnitude of $\vec{A}$ remains constant on the surface of wire, it does change when we decide to find it at some other point. In other words magnitude of $\vec{A}$ depends on ρ and z. For calculating field anywhere with slight errors, we make the dipole assumption.

<u>Assumption</u>

The loop behaves as a dipole everywhere at far enough distances.

ii) using this assumption

$$\vec{A} = \frac{\mu_0 I a^2}{4r^2} \sin\theta \, \hat{a_\phi}$$

$$\vec{B} = \nabla \times \vec{A}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\phi)\,\hat{a_z} - \frac{\partial A_\phi}{\partial z}\,\hat{a_\rho}$$

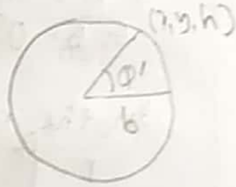
$$= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(r\sin\theta\, A_\phi) + \frac{1}{r}\cancel{\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi}}\left(-\frac{\partial}{\partial r}(rA_\phi)\right)$$

$$= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\mu_0 I a^2}{4r^2}\sin^2\theta\right) - \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\mu_0 I a^2}{4r}\sin\theta\right)$$

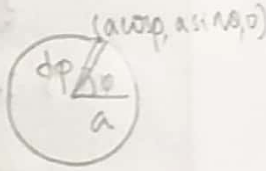
$$= \frac{\mu_0 I a^2}{4r^3}\left(2\cos\theta\,\hat{a_r} + \sin\theta\,\hat{a_\theta}\right)$$

without these assumptions

$$\vec{A} = \frac{\mu_0 I_1}{4\pi}\int_0^{2\pi}\frac{a\,d\phi\,(-\sin\phi\,\hat{a_x} + \cos\phi\,\hat{a_y})}{\sqrt{(x-a\cos\phi)^2 + (y-a\sin\phi)^2 + h^2}}$$



$$\text{or } \vec{A} = \frac{\mu_0 I_1}{4\pi} a \int_0^{2\pi}\frac{(-\sin\phi\,\hat{a_x} + \cos\phi\,\hat{a_y})\,d\phi}{\sqrt{x^2+y^2+h^2 - 2ax\cos\phi - 2ay\sin\phi}}$$



$$\text{or } \vec{A} = \frac{\mu_0 I_1}{4\pi} a \int_0^{2\pi}\frac{(-\sin\phi\,\hat{a_x} + \cos\phi\,\hat{a_y})\,d\phi}{\sqrt{x^2+y^2+h^2 - 2a(x\cos\phi + y\sin\phi)}}$$

$$\text{or } \vec{A} = \frac{\mu_0 I_1}{4\pi} a \int_0^{2\pi}\frac{(-\sin\phi\,\hat{a_x} + \cos\phi\,\hat{a_y})}{\sqrt{r^2 - 2ar\sin\theta\cos(\phi-\phi')}}\,d\phi$$

Clearly this integral is very tedious to compute. Hence the approximation was used. However numeric integration is possible, though it would not help in calculating the $\vec{B}$ field.

iii) $\Psi_{12} = \int_S \vec{B}\cdot d\vec{S}$

$= \int_S \nabla\times\vec{A}\cdot d\vec{S} = \oint \vec{A}\cdot d\vec{l_2}$

$= \int_0^{2\pi} A_\phi \hat{a_\phi} \cdot b\,d\phi\,\hat{a_\phi}$ [A_ϕ has already been computed $\vec{A} = A_\phi \hat{a_\phi}$]

$= bA_\phi \int_0^{2\pi} d\phi$

$= 2\pi b A_\phi$

$A_\phi = \frac{\mu_0 I_1}{4\pi}(0.2681969)$

$\Psi_{12} = \frac{\mu_0 I_1}{4\pi}(0.2681969) \times 2\pi b$

$= \frac{\mu_0}{2} I_1 (0.2682b) = (\mu_0\, 0.1341b)\, I_1$

$M = \frac{\Psi_{12}}{I_1} = \frac{\mu_0 b}{2}(0.2681969)$

$= 29.502\mu_0 = 2.95\ \mu H$

iv) The assumption taken was that the field due to loop acts as dipole everywhere. This is true only at very large distances from loop. So we can check the error in the approximation

$\vec{A} = \frac{\mu_0 I_1}{4\pi}\left(\frac{\pi a^2}{r^2}\sin\theta\right)\hat{a_\phi}$

or $\vec{A} = \frac{\mu_0 I_1}{4\pi}(0.276369)$

$\text{error} = 8.1721\times10^{-3}\ \frac{\mu_0 I_1}{4\pi}$

$\text{error}\ \% = \frac{8.1721\times10^{-3}}{0.2681969}\times100\%$

$= 3.047\%$

Thus even though $h \cancel{\gg} a$, the error with this approximation is just around 3%. The approximation thus holds good.

27) Using directly the formula for force between two loops

$$\vec{F} = \frac{\mu_0 I^2}{4\pi}\oint_{L_1}\oint_{L_2}\frac{\vec{dl_1}\times(\vec{dl_2}\times \hat{a}_{R21})}{R_{21}^2}$$

$P_1(a\cos\varphi_1,\ a\sin\varphi_1,\ 0)$

$P_2(b\cos\varphi_2,\ b\sin\varphi_2,\ h)$

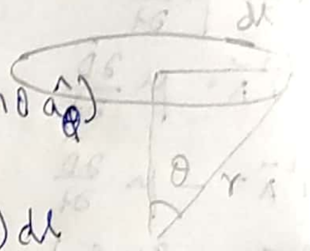
$R_{21}^2 = h^2 + a^2 + b^2 - 2ab\cos(\varphi_2 - \varphi_1)$

$\vec{dl_1}\times(\vec{dl_2}\times \vec{a}_{R21})$

$= -ab(a+b)(\cos\varphi_1\,\hat{a_x} + \sin\varphi_1\,\hat{a_y})$

This integral would be tedious to solve. So insted we use the $\vec{B}$ approximate on loop 2

$d\vec{F} = I\,\vec{dl}\times\vec{B}$



or $d\vec{F} = \frac{\mu_0 I^2 a^2}{4r^3}(dl\,\hat{a_\varphi})\times(2\cos\theta\,\hat{a_r} + \sin\theta\,\hat{a_\theta})$

or $d\vec{F} = \frac{\mu_0 I^2 a^2}{4r^3}(2\cos\theta\,\hat{a_\theta} - \sin\theta\,\hat{a_r})\,dl$

Now for different dl, the $\hat{a_r}$ and $\hat{a_\theta}$ directions are different. So we need to convert to $\hat{a_x}$, $\hat{a_y}$ and $\hat{a_z}$

$\hat{a_\theta} = \cos\theta(\cos\varphi\,\hat{a_x} + \sin\varphi\,\hat{a_y}) - \sin\theta\,\hat{a_z}$

$\hat{a_r} = \sin\theta(\cos\varphi\,\hat{a_x} + \sin\varphi\,\hat{a_y}) + \cos\theta\,\hat{a_z}$

$2\cos\theta\,\hat{a_\theta} - \sin\theta\,\hat{a_r}$

$= (2\cos^2\theta - \sin^2\theta)(\cos\varphi\,\hat{a_x} + \sin\varphi\,\hat{a_y}) - 3\sin\theta\cos\theta\,\hat{a_z}$

$\therefore \int d\vec{F} = \frac{\mu_0 I^2 a^2}{4r^3}\int_0^{2\pi}(3\cos^2\theta - 1)\cos\varphi\,\hat{a_x} + \sin\varphi\,\hat{a_y} - 3\sin\theta\cos\theta\,\hat{a_z}\,b$

clearly the x and y components vanish

$\vec{F} = \frac{\mu_0 I^2 a^2}{4r^3}\int_0^{2\pi}(-3\sin\theta\cos\theta)\,b\,d\varphi\,\hat{a_z}$

or $\vec{F} = \frac{\mu_0 I^2 a^2}{4r^3}(-1.5\sin 2\theta)\times 2\pi b\,\hat{a_z}$

or $\vec{F} = -\frac{3\mu_0 I^2\pi}{4}\left(\frac{a^2 b}{r^3}\right)\sin 2\theta\,\hat{a_z}$

or $\vec{F} = -0.2447\,\hat{a_z}$ mN

The following MATLAB code was used in order to determine the integral in computer.

```
a = 88;
b = 2.5 * a;
h = (a + b) / 2;

syms theta
deno = sqrt(h*h + a*a + b*b -2*a*b*cos(theta));
num = [-sin(theta); cos(theta)];
expr = num ./ deno;
result = vpaintegral(expr, theta, 0, 2*pi);
disp(a*result);

syms phi1 phi2;
deno = h*h + a*a + b*b -2*a*b*cos(phi1 - phi2);
num = [cos(phi1); sin(phi1)];
expr = num ./ deno;
result = vpaintegral(expr, phi1, [0 2*pi], phi2, [0 2*pi]);
disp(result);
```

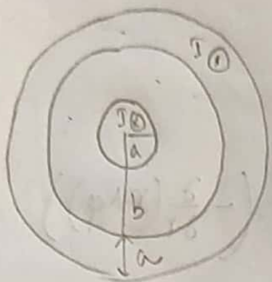
2. i) skin depth(δ)

$$= \frac{1}{\sqrt{\pi f \sigma \mu}} \qquad \rho = 1.7 \times 10^{-8}\ \Omega m$$
$$\mu = 1.256 \times 10^{-6}\ H/m$$

$$= \sqrt{\frac{\rho}{\pi f \mu}} \quad \left(\rho = \frac{1}{\sigma}\right)$$

For $f = 1GHz$ $\quad \delta = 2.075\ \mu m$
For $f = 10GHz$ $\quad \delta = 0.656\ \mu m$
For $f = 100GHz$ $\quad \delta = 0.207\ \mu m$

ii)



It is said that current is restricted between a depth 3δ.
We came same current I flows through both

$$J = J_s \exp\left[-\frac{(1+j)d}{\delta}\right]$$

In the region $0 \le \rho \le a-3d$

$$B * 2\pi\rho = 0$$
$$\text{or } B = 0$$

In the region $a-3\delta \le \rho \le \cancel{3\delta}\ a$

$$I = \int_{a-3\delta}^{a} J_s \exp\left[-\frac{(1+j)d}{\delta}\right] \cancel{\rho\, d\phi}\ 2\pi\rho\, d\rho$$

$$\text{or } I = 2\pi J_s \int_{a-3\delta}^{a} \exp\left[-\frac{(1+j)(a-\rho)}{\delta}\right] \rho\, d\rho$$

$$\text{or } I = 2\pi J_s\ \frac{j}{2} e^{-3(1+j)} \delta \left[\{e^{3(1+j)} - (4+3j)\}\delta - (1+j)\{e^{3(1+j)} - 1\} a\right]$$

$$I_{enc} = 2\pi J_s \left[\frac{j}{2}\delta\ \{\delta - (1+j)\rho\} e^{-\frac{(1+j)(a-\rho)}{\delta}} + \frac{1}{2} e^{-3(1+j)} \delta \{(3-4j)\delta - (1-j)a\}\right]$$

$$\frac{I_{enc}}{I} = \frac{j(\delta - (1+j)\rho)\exp\left[-\frac{(1+j)(a-\rho)}{\delta}\right] + e^{-3(1+j)}[(3-4j)\delta - (}{j e^{-3(1+j)}\left[\{-(4+3j) + e^{3(1+j)}\}\delta - (1+j)a\{e^{3(1+j)} - 1\}\right]}$$

$$\text{or } \frac{I_{enc}}{I} = \frac{[\delta-(1+j)\rho]\exp\left[-\frac{(1+j)(a-\rho)}{\delta}\right] + e^{-3(1+j)}[-(4+3j)\delta+(1+j)]}{e^{-3(1+j)}\left[\left\{e^{3(1+j)}-(4+3j)\right\}\delta - a(1+j)\left\{e^{3(1+j)}-1\right\}\right]}$$

$$\text{or } \frac{I_{enc}}{I} = \frac{e^{3(1+j)}[\delta-(1+j)\rho]\exp\left[\frac{-(1+j)(a-\rho)}{\delta}\right] + (1+j)-(4+3j)\delta}{\delta\left\{e^{3(1+j)}-(4+3j)\right\} - a(1+j)\left\{e^{3(1+j)}-1\right\}}$$

$$B \times 2\pi\rho = \mu_0 I_{enc}$$

$$\text{or } B = \frac{\mu_0 I}{2\pi\rho}\,\frac{e^{3(1+j)}[\delta-(1+j)\rho]\exp\left[-\frac{(1+j)(a-\rho)}{\delta}\right] + (1+j)-(4+3j)\delta}{\delta\left\{e^{3(1+j)}-(4+3j)\right\} - a(1+j)\left[e^{3(1+j)}-1\right]}$$

In the region $a \le \rho \le a+b-3\delta$

$$B \times 2\pi\rho = \mu_0 I$$

$$\text{or } B = \frac{\mu_0 I}{2\pi\rho}$$

In the region $a+b-3\delta \le \rho \le a+b$

$$B = \frac{\mu_0 I}{2\pi\rho}\left[1 - \frac{e^{3(1+j)}[\delta-(1+j)\rho]\exp\left[\frac{-(1+j)(a+b-\rho)}{\delta}\right] + (1+j)-(4+3j)\delta}{\delta\left\{e^{3(1+j)}-(4+3j)\right\} - a(1+j)\left\{e^{3(1+j)}-1\right\}(a+b)}\right]$$

In the region $a+b \le \rho < \infty$

$$B = 0$$

With changing frequency the skin depth would change. However the expressions would remain same. Clearly the decay would be faster with increasing frequency and magnetic field would be confined more and more towards the surface and inbetween conductors.

iii) $\Psi_{ext} = \int \vec{B}\cdot d\vec{s}$

$$\text{or } \Psi_{ext} = \frac{\mu_r\mu_0 I}{2\pi}\int_0^l\int_a^b \frac{1}{\rho} \times d\rho\, dz$$

$$\text{or } \Psi_{ext} = \frac{\mu_r\mu_0 I}{2\pi}\, l \ln\frac{b}{a}$$

$$\text{or } L_{ext} = \frac{\mu_r\mu_0 l}{2\pi}\ln\frac{b}{a} = \frac{\mu l}{2\pi}\ln\frac{b}{a}$$ for length l of conductor.

For internal inductance of second conductor

$$d\lambda = \frac{d\psi}{I} I_{enc}$$

$$\frac{I_{enc}}{I} \approx 1 - \frac{\rho \exp\left[-\frac{(1+j)(b+a-\rho)}{\delta}\right]}{(b+a)\left[1-e^{-3(1+j)}\right]}$$

$$B = \frac{\mu_0}{2\pi\rho} I_{enc}$$

$$\text{or } B = \frac{\mu_0 I}{2\pi\rho} \frac{I_{enc}}{I}$$

$$d\lambda = B\, d\rho\, dz \frac{I_{enc}}{I}$$

$$\text{or } d\lambda = \frac{\mu_r\mu_0 I}{2\pi\rho} \left(\frac{I_{enc}}{I}\right)^2 d\rho\, dz$$

$$\text{or } L_{int2} = \int_{a+b-3\delta}^{a+b} \frac{\mu_r\mu_0 l}{2\pi\rho} \left(\frac{I_{enc}}{I}\right)^2 d\rho$$

$$\text{or } L_{int2} = \frac{\mu_r\mu_0 l}{2\pi} \int_{a+b-3\delta}^{a+b} \frac{1}{\rho}\left(1 - \frac{\rho \exp\left[-\frac{(1+j)(a+b-\rho)}{\delta}\right]}{(a+b)\left[1-e^{-3(1+j)}\right]}\right)^2 d\rho$$

$$\text{or } L_{int2} = \frac{\mu_r\mu_0 l}{2\pi} \int_{a+b-3\delta}^{a+b} \frac{1}{\rho}\left[1 - \frac{2\rho \exp\left[-\frac{(1+j)(a+b-\rho)}{\delta}\right]}{(a+b)\left[1-e^{-3(1+j)}\right]} + \frac{\rho^2 \exp\left(-\frac{2(1+j)(a+b-\rho)}{\delta}\right)}{(a+b)^2\left(1-e^{-3(1+j)}\right)^2}\right]$$

$$\text{or } L_{int2} = \frac{\mu_r\mu_0 l}{2\pi} \int_{a+b-3\delta}^{a+b} \left[\frac{1}{\rho} - \frac{2\exp\left(-\frac{(1+j)(a+b-\rho)}{\delta}\right)}{(a+b)\left[1-e^{-3(1+j)}\right]} + \frac{\rho \exp\left(-\frac{2(1+j)(a+b-\rho)}{\delta}\right)}{(a+b)^2\left(1-e^{-3(1+j)}\right)^2}\right]$$

$$\text{or } L_{int2} = \frac{\mu_r\mu_0 l}{2\pi}\left[\ln\frac{a+b}{a+b-3\delta} - \frac{2(1-j)\left[1-e^{-3(1+j)}\right]\delta}{2(a+b)\left[1-e^{-3(1+j)}\right]} + \frac{j/8\,\delta\, e^{-6(1+j)}}{(a+b)^2\left(1-e^{-3(1+j)}\right)^2}\left[\left\{e^{6(1+j)} - (7+6j)\right\}\delta - 2(1+j)(a+b)\left(e^{6(1+j)}-1\right)\right]\right]$$

$$\text{or } L_{int2} \approx \frac{\mu_r\mu_0 l}{2\pi}\left[\ln\left(\frac{a+b}{a+b-3\delta}\right) - \frac{\delta(1-j)}{a+b} + \frac{\delta j\, e^{-6(1+j)}}{8(a+b)^2\left[1-e^{-3(1+j)}\right]^2} - 2(1+j)(a+b)\left(e^{6(1+j)}-1\right)\right]$$

$$\text{or } L_{int2} \approx \frac{\mu_r\mu_0 l}{2\pi}\left[-\ln\left(1-\frac{3\delta}{a+b}\right) + \frac{\delta(1-j)}{4(a+b)} \frac{1-e^{-6(1+j)}}{\left[1-e^{-3(1+j)}\right]^2}\right]$$

$$\text{or } L_{int2} \approx \frac{\mu_0 l}{2\pi} \ln\left(\frac{a+b}{a+b-3\delta}\right) \text{ or } -\frac{\mu_0 l}{2\pi}\ln\left(1-\frac{3\delta}{a+b}\right)$$

Scanned with CamScanner

For internal inductance of first conductor

$d\lambda = d\Psi \frac{I_{enc}}{I}$

Since the expression is very complicated we can make approximations

$\delta << \rho$

$a-\rho \approx \delta$

$$\frac{I_{enc}}{I} = \frac{e^{3(1+j)}[\delta-(1+j)\rho]\exp\left[\frac{-(1+j)(a-\rho)}{\delta}\right] + (1+j) - (4+3j)\delta}{\delta\{e^{3(j+1)}-(4+3j)\} - a(1+j)(e^{3(j+1)}-1)}$$

or $$\frac{I_{enc}}{I} \approx \frac{-e^{3(1+j)}(1+j)\rho\exp\left[\frac{-(1+j)(a-\rho)}{\delta}\right] + (1+j)}{-a(1+j)[e^{3(j+1)}-1]}$$

or $$\frac{I_{enc}}{I} \approx \frac{e^{+3(1+j)}\rho\exp\left[-\frac{(1+j)(a-\rho)}{\delta}\right] - 1}{a[e^{+3(j+1)}-1]}$$

or $$\frac{I_{enc}}{I} \approx \frac{\rho\exp\left[-\frac{(1+j)(a-\rho)}{\delta}\right] - 1}{a[1-e^{-3(1+j)}]} \approx \frac{\rho\exp\left[-\frac{(1+j)(a-\rho)}{\delta}\right]}{a[1-e^{-3(1+j)}]}$$

$$B \approx \frac{\mu_r\mu_0 I_{enc}}{2\pi\rho} \approx \frac{\mu_r\mu_0 I}{2\pi a} \frac{\exp\left[-\frac{(1+j)(a-\rho)}{\delta}\right] - 1/\rho}{1-e^{-3(1+j)}}$$

or $$B \approx \frac{\mu_r\mu_0 I}{2\pi a(1-e^{-3(1+j)})} \exp\left[-\frac{(1+j)(a-\rho)}{\delta}\right]$$

$d\lambda = B\,d\rho\,dz\,\frac{I_{enc}}{I}$

or $$d\lambda = \frac{\mu_r\mu_0 I\,d\rho\,dz}{2\pi a^2(1-e^{-3(1+j)})^2}\,\rho\exp\left[-\frac{2}{\delta}(1+j)(a-\rho)\right]$$

or $$\lambda = \frac{\mu_r\mu_0 I l}{2\pi a^2(1-e^{-3(1+j)})^2}\int_{a-3\delta}^{a}\rho\exp\left[-2\frac{(1+j)(a-\rho)}{\delta}\right]d\rho$$

or $L_{int1} = \frac{\mu_r \mu_0 l\, e^{-6(1+j)}\,\delta}{2\pi a^2 [1-e^{-3(1+j)}]^2} \left[\frac{j\delta}{8} \{ e^{6(1+j)} - (7+6j) \} \delta \right.$

$\left. - 2(1+j)\, a\, (e^{6(1+j)} - 1) \right]$

or $L_{int1} \approx \frac{\mu_r \mu_0 l}{2\pi a^2} \frac{e^{-6(1+j)}\,\delta}{[1-e^{-3(1+j)}]^2} \quad -2(1+j)\, a\, [e^{6(1+j)} - 1]$

or $L_{int1} \approx \frac{\mu_r \mu_0 l \delta}{2\pi a} [1 - e^{-6(1+j)}] (1+j)$

or $L_{int1} \approx 0$ (since $\delta \ll a$)

∴ Thus the net internal inductance is

$L_{int} = L_{int1} + L_{int2}$

or $L_{int} \approx \frac{\mu_0 l}{2\pi} \ln \left(\frac{a+b}{a+b-3\delta} \right)$

or $L_{int} \approx \frac{\mu l}{2\pi} \ln \left(\frac{a+b}{a+b-3\delta} \right)$

For different frequencies the values of inductances can be determined. We assume the wire thickness is a μm.

For $f = 1\,GHz$

$L_{int} = \frac{\mu l}{2\pi} \ln \left(\frac{3.5a}{3.5a-3\delta} \right) = \left(1.02 \frac{nH}{m} \right) l$

$L_{ext} = \frac{\mu l}{2\pi} \ln \frac{b}{a} = \frac{\mu l}{2\pi} \ln 2.5 = \left(0.183 \frac{\mu H}{m} \right) l$

For $f = 10\,GHz$

$L_{int} = \frac{\mu l}{2\pi} \ln \left(\frac{3.5a}{3.5a-3\delta} \right) = \left(0.32 \frac{nH}{m} \right) l$

$L_{ext} = \frac{\mu l}{2\pi} \ln 2.5 = \left(0.183 \frac{\mu H}{m} \right) l$

For $f = 100\,GHz$

$L_{int} = \frac{\mu l}{2\pi} \ln \left(\frac{3.5a}{3.5a-3\delta} \right) = \left(0.101 \frac{nH}{m} \right) l$

$L_{ext} = \frac{\mu l}{2\pi} \ln 2.5 = \left(0.183 \frac{\mu H}{m} \right) l$

3.i) The Maxwell's eq^n in differential form is

$$\nabla \cdot \vec{D} = \rho_v$$
$$\nabla \cdot \vec{B} = 0$$
$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

For deriving the phasor form, we make the following substitutions

$$\vec{D} = \vec{D_s} e^{j\omega t} \quad \vec{E} = \vec{E_s} e^{j\omega t} \quad \rho_v = \rho_{vs} e^{j\omega t}$$
$$\vec{B} = \vec{B_s} e^{j\omega t} \quad \vec{H} = \vec{H_s} e^{j\omega t} \quad \vec{J} = \vec{J_s} e^{j\omega t}$$

Hence the Maxwell's eq^n in differential form in phasor notation would be

$$\nabla \cdot \vec{D_s} = \rho_{vs}$$
$$\nabla \cdot \vec{B_s} = 0$$
$$\nabla \times \vec{E_s} = -j\omega \vec{B_s}$$
$$\nabla \times \vec{H_s} = \vec{J_s} + j\omega \vec{D_s}$$

The phasor notation is useful because:-

i. It converts time dependencies to simple algebraic product.

ii. Sinusoidal or ac currents can be easily handled.

iii. In case of varying electric and magnetic field, it provides a link how they vary with time in addition to phase.

(ii) $\nabla \times \vec{E}_s = -j\omega\mu\vec{H}$ $\quad\quad \vec{E} = \vec{E}_s e^{j\omega t}$

$\nabla \times \vec{H}_s = (\sigma + j\omega\epsilon)\vec{E}_s$ $\quad\quad \vec{H} = \vec{H}_s e^{j\omega t}$

The Poynting vector in phasor form is defined as $\vec{E} \times \vec{H}^*$

so first we can use the vector identity

$\nabla \cdot (\vec{H}^* \times \vec{E}) = \vec{E} \cdot (\nabla \times \vec{H}_s^*) - \vec{H}^* \cdot (\nabla \times \vec{E})$

or $\nabla \cdot (\vec{H}^* \times \vec{E}) = \vec{E} \cdot (\sigma - j\omega\epsilon)\vec{E}^* - \vec{H}^* \cdot (-j\omega\mu)\vec{H}$

or $\nabla \cdot (\vec{H}^* \times \vec{E}) = (\sigma - j\omega\epsilon) E^2 + j\omega\mu H^2$

or $\nabla \cdot (\vec{H}^* \times \vec{E}) = \sigma E^2 + j\omega (\mu H^2 - \epsilon E^2)$

or $\nabla \cdot (\vec{E} \times \vec{H}^*) = -\sigma E^2 - j\omega (\mu H^2 - \epsilon E^2)$

or $\iiint_V \nabla \cdot (\vec{E} \times \vec{H}^*) dv = -\iiint_V \sigma E^2 dv - j\omega \iint_V \mu H^2 - \epsilon E^2 dv$

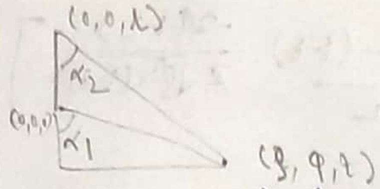
or $\oint_S (\vec{E} \times \vec{H}^*) \cdot d\vec{s} = -\iiint_V \sigma E^2 dv - j\omega \iiint_V \mu H^2 - \epsilon E^2 dv$

Here as usual the left side is the total power leaving the volume as given by Poynting vector. The first term in rhs is the ohmic loss whereas the second term shows how much energy decreases in electric and magnetic fields. This is just a statement of conservation of power or energy.

iii) $l = 1\,mm$ $\quad I = 3\cos(\omega t + \varphi)$

$f = 1\,GHz$

a.



$$\vec{H} = \int \frac{I_\rho\, d\vec{l} \times \hat{R}}{4\pi R^2} = \int_0^l \frac{I_\rho\, dz}{4\pi(\rho^2+z^2)^{3/2}}\, \hat{a}_\varphi$$

$$= \frac{I_\rho}{4\pi} \int_{\alpha_1}^{\alpha_2} \frac{-\rho\,\mathrm{cosec}^2\alpha\, d\alpha}{\rho^3 \mathrm{cosec}^3\alpha}\, \hat{a}_\varphi \qquad z = \rho \cot\alpha$$

$$= \frac{I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1)\, \hat{a}_\varphi$$

$$= \frac{I}{4\pi\rho}\left[\frac{z}{\sqrt{\rho^2+z^2}} + \frac{l-z}{\sqrt{\rho^2+(l-z)^2}}\right] \hat{a}_\varphi = H_\varphi\, \hat{a}_\varphi$$

b.

$$\nabla \times \vec{H} = \frac{1}{\rho}\begin{vmatrix} \hat{a}_\rho & \rho\hat{a}_\varphi & \hat{a}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ 0 & \rho H_\varphi & 0 \end{vmatrix} = \frac{1}{\rho}\left[\hat{a}_\rho\left(-\frac{\partial}{\partial z}(\rho H_\varphi)\right) + \hat{a}_z\left(\frac{\partial}{\partial\rho}\rho H_\varphi\right)\right]$$

$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho H_\varphi)\, \hat{a}_z - \frac{\partial}{\partial z} H_\varphi\, \hat{a}_\rho$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho H_\varphi)$$

$$= \frac{1}{\rho}\frac{I}{4\pi}\frac{\partial}{\partial\rho}\left[\frac{z}{\sqrt{\rho^2+z^2}} + \frac{l-z}{\sqrt{\rho^2+(l-z)^2}}\right]$$

$$= \frac{I}{4\pi\rho}\left[-\frac{1}{2}\frac{z \times 2\rho}{(\rho^2+z^2)^{3/2}} + \frac{(l-z)\left(-\frac{1}{2}\right) \times 2\rho}{[\rho^2+(l-z)^2]^{3/2}}\right]$$

$$= \frac{-I}{4\pi\rho}\left[\frac{\rho z}{(\rho^2+z^2)^{3/2}} + \frac{\rho(l-z)}{[\rho^2+(l-z)^2]^{3/2}}\right]$$

$$= -\frac{I}{4\pi}\left[\frac{z}{(\rho^2+z^2)^{3/2}} + \frac{l-z}{[\rho^2+(l-z)^2]^{3/2}}\right]$$

$$\frac{\partial}{\partial z} H_\varphi$$

$$= \frac{\partial}{\partial z}\frac{I}{4\pi\rho}\left[\frac{z}{\sqrt{\rho^2+z^2}} + \frac{l-z}{\sqrt{\rho^2+(l-z)^2}}\right]$$

$$= \frac{I}{4\pi\rho}\frac{\partial}{\partial z}\left[\frac{z}{(\rho^2+z^2)^{1/2}} + \frac{l-z}{[\rho^2+(l-z)^2]^{1/2}}\right]$$

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$$= \frac{I}{4\pi\rho}\frac{\partial}{\partial z}\left(\frac{z}{\sqrt{\rho^2+z^2}} + \frac{l-z}{\sqrt{\rho^2+(l-z)^2}}\right)$$

$$= \frac{I}{4\pi\rho}\left[\frac{\sqrt{\rho^2+z^2} - z\frac{2z}{2\sqrt{\rho^2+z^2}}}{\rho^2+z^2} + \frac{-\sqrt{\rho^2+(l-z)^2} - (l-z)\frac{-2(l-z)}{2\sqrt{\rho^2+(l-z)^2}}}{\rho^2+(l-z)^2}\right]$$

$$= \frac{I}{4\pi\rho}\left[\frac{\rho^2+z^2-z^2}{(\rho^2+z^2)^{3/2}} + \frac{(l-z)^2-\rho^2-(l-z)^2}{(\rho^2+(l-z)^2)^{3/2}}\right]$$

$$= \frac{I}{4\pi\rho}\left[\frac{\rho^2}{(\rho^2+z^2)^{3/2}} - \frac{\rho^2}{[\rho^2+(l-z)^2]^{3/2}}\right]$$

$$= \frac{I\rho}{4\pi}\left[\frac{1}{(\rho^2+z^2)^{3/2}} - \frac{1}{[\rho^2+(l-z)^2]^{3/2}}\right]$$

c. $\frac{\partial \vec{D}}{\partial t} = \nabla \times \vec{H}$

or $$\frac{\partial \vec{D}}{\partial t} = \frac{I\rho}{4\pi}\left[\frac{1}{(\rho^2+(l-z)^2)^{3/2}} - \frac{1}{(\rho^2+z^2)^{3/2}}\right]\hat{a}_\rho - \frac{I}{4\pi}\left[\frac{z}{(\rho^2+z^2)^{3/2}} + \frac{l-z}{[\rho^2+(l-z)^2]^{3/2}}\right]\hat{a}_z$$

or $$\frac{\partial \vec{D}}{\partial t} = \frac{I}{4\pi}\left[\left(\frac{\rho}{(\rho^2+(l-z)^2)^{3/2}} - \frac{\rho}{(\rho^2+z^2)^{3/2}}\right)\hat{a}_\rho - \left(\frac{z}{(\rho^2+z^2)^{3/2}} + \frac{l-z}{(\rho^2+(l-z)^2)^{3/2}}\right)\hat{a}_z\right]$$

or $$\vec{E} = \frac{\int I\,dt}{4\pi\epsilon}\left[\frac{\rho\hat{a}_\rho - (l-z)\hat{a}_z}{[\rho^2+(l-z)^2]^{3/2}} - \frac{\rho\hat{a}_\rho + z\hat{a}_z}{(\rho^2+z^2)^{3/2}}\right]$$

or $$\vec{E} = -\frac{3\sin(\omega_0 t+\varphi)}{4\pi\epsilon\omega_0}\left[\frac{\rho\hat{a}_\rho - (l-z)\hat{a}_z}{[\rho^2+(l-z)^2]^{3/2}} - \frac{\rho\hat{a}_\rho + z\hat{a}_z}{(\rho^2+z^2)^{3/2}}\right]$$

d. $\vec{P} = \vec{E}\times\vec{H}$

or $\vec{P} = (E_\rho\hat{a}_\rho + E_z\hat{a}_z)\times H_\varphi\hat{a}_\varphi$ $\qquad \vec{E} = E_\rho\hat{a}_\rho + E_z\hat{a}_z$, $\vec{H} = H_\varphi\hat{a}_\varphi$

or $\vec{P} = H_\varphi E_\rho\hat{a}_z - E_z H_\varphi\hat{a}_\rho$

$$H_\phi = \frac{3\omega\cos(\omega_0 t+\phi)}{4\pi\rho}\left[\frac{1}{\sqrt{\rho^2+z^2}} + \frac{l-z}{\sqrt{\rho^2+(l-z)^2}}\right]$$

$$E_\rho = -\frac{3\sin(\omega_0 t+\phi)\rho}{4\pi\epsilon\omega_0}\left[\frac{1}{(\rho^2+(l-z)^2)^{3/2}} - \frac{1}{(\rho^2+z^2)^{3/2}}\right]$$

$$E_z = \frac{3\sin(\omega_0 t+\phi)}{4\pi\epsilon\omega_0}\left[\frac{z}{(\rho^2+z^2)^{3/2}} + \frac{l-z}{(\rho^2+(l-z)^2)^{3/2}}\right]$$

$$\vec{P} = H_\phi\,(E_\rho\,\hat{a}_z - E_z\,\hat{a}_\rho)$$

e. $\nabla\times\vec{H} = \dfrac{\partial\vec{D}}{\partial t}$

or $\nabla\times\vec{B} = \mu\dfrac{\partial\vec{D}}{\partial t}$

or $-\nabla^2\vec{A} = \mu\dfrac{\partial\vec{D}}{\partial t}$

or $\nabla^2\vec{A} = -\mu\dfrac{\partial\vec{D}}{\partial t}$

or $$\nabla^2\vec{A} = \frac{3\mu\rho}{4\pi}\left[\frac{1}{(\rho^2+z^2)^{3/2}} - \frac{1}{(\rho^2+(l-z)^2)^{3/2}}\right]\hat{a}_\rho$$
$$\frac{I\mu}{4\pi}\left[\frac{z}{(\rho^2+z^2)^{3/2}} + \frac{l-z}{(\rho^2+(l-z)^2)^{3/2}}\right]\hat{a}_z$$

$$\therefore \nabla^2 A_x = \frac{\mu I x}{4\pi}\left[\frac{1}{(\rho^2+z^2)^{3/2}} - \frac{1}{(\rho^2+(l-z)^2)^{3/2}}\right]$$

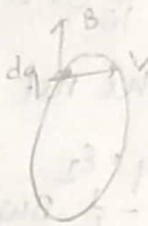
$$\nabla^2 A_y = \frac{\mu I y}{4\pi}\left[\frac{1}{(\rho^2+z^2)^{3/2}} - \frac{1}{(\rho^2+(l-z)^2)^{3/2}}\right]$$

$$\nabla^2 A_z = \frac{\mu I}{4\pi}\left[\frac{z}{(\rho^2+z^2)^{3/2}} + \frac{l-z}{(\rho^2+(l-z)^2)^{3/2}}\right]$$

The solutions to these non-homogenous Laplace eqn would give the three components of $\vec{A}$ field in cartesian coordinates (A_x, A_y, A_z)

4. Due to the time varying nature of field an emf is induced
i) given by Lenz's law

$$\mathcal{E} = -\frac{d\phi_B}{dt} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{s} = -\int \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s}$$



However since the loop is moving, there is another source of work.

A charge element dq experiences

$$d\vec{F} = dq\, \vec{v}\times\vec{B}$$

$$d^2W = d\vec{F}\cdot d\vec{l}$$

or $d^2W = dq\, \vec{v}\times\vec{B}\cdot d\vec{l}$

or $d\left(\frac{dW}{dq}\right) = \vec{v}\times\vec{B}\cdot d\vec{l}$

or $\frac{dW}{dq} = \oint (\vec{v}\times\vec{B})\cdot d\vec{l}$

or $\mathcal{E} = \oint (\vec{v}\times\vec{B})\cdot d\vec{l}$

Hence the total induced emf:

$$\mathcal{E} = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s} + \oint_L (\vec{v}\times\vec{B})\cdot d\vec{l}$$

ii) We construct a sinusoidal field $\vec{B} = B_0 \sin\omega_1 t\, \hat{a}_z$ in all of space. The loop moves in x-direction with velocity $v_0\hat{a}_x$

To make the $\vec{B}$-field space dependent we take $B_0 = B\cos\phi\, e^{-\phi}$ and $v_0 = v\cos(\omega_2 t)$

$$\vec{B} = B\cos\phi\, \sin\omega_1 t\, e^{-\phi}\hat{a}_z$$

$$\vec{v} = v\cos\omega_2 t\ \hat{a}_x$$

$$\mathcal{E}_1 = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s}$$

$$= -\int B\cos\phi\, \omega_1 \cos\omega_1 t\, ds\, e^{-\phi}$$

$$= -B\omega_1 \cos\omega_1 t \int_0^{r}\int_0^{2\pi} \cos\phi\, \rho\, d\phi\, d\rho\, e^{-\phi}$$

$$= -B\omega_1\cos\omega_1 t \times \left[\frac{\rho^2}{2}\right]_0^r \int e^{-\phi}\cos\phi\, d\rho$$

$$= -\frac{1}{2}B\omega_1 r^2 \cos(\omega_1 t)\,\frac{1}{2}\,(1^0 - e^{-2\pi})$$

$$= -\frac{1}{4}B\omega_1 r^2 (1-e^{-2\pi})\cos\omega_1 t$$

$$\mathcal{E}_2 = \oint_L \vec{v}\times\vec{B}\cdot d\vec{l}$$

$$= \int_0^{2\pi} -Bv\cos\phi\, e^{-\phi}\cos\omega_2 t\, \sin\omega_1 t \cdot r\cos\phi\, d\phi$$

$$= -\frac{Bv}{2}\sin(2\omega_1 t)\int_0^{2\pi} e^{-\phi}\cos^2\phi\, d\phi$$

$= -\frac{1}{2} Bv \sin(2\omega_1 t) \cdot \frac{3}{5}(1-e^{-2\pi})$

$= -\frac{3}{10} Bv \sin(2\omega_1 t)(1-e^{-2\pi})$

Thus the net induced emf is

$$\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2$$
$$= -\frac{1}{4} B\omega_1 r^2 (1-e^{-2\pi}) \cos\omega_1 t - \frac{3}{10} Bv \sin(2\omega_1 t)(1-e^{-2\pi})$$

$$= -B(1-e^{-2\pi})\left[\frac{\omega_1 r^2}{4}\cos\omega_1 t + \frac{3}{10} v \sin(2\omega_1 t)\right]$$

Thus using this scheme we can generate frequencies of ω_1 and $2\omega_1$. In general if the field is time dependent it introduces one frequency. If the velocity has a frequency dependente it would give two frequencies around the field frequency. If both frequencies are same we get one coincient frequency and an extra frequency

Velocity frequency ω_2
Field frequency ω_1

Frequencies available ω_1, $\omega_1 - \omega_2$ and $\omega_1 + \omega_2$ if $\omega_1 \neq \omega_2$
ω_1, $2\omega_1$ if $\omega_1 = \omega_2$

These extra frequencies are called heterodynes.

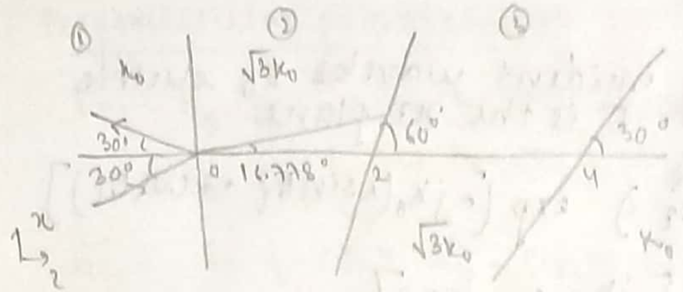
iii) Heterodynes are particularly useful when multiple signals share a common bandwith. Using the heterodynes the signal may be shifted along the spectrum. The field frequency would be the base or carrier frequency. While the velocity frequency would be modulation which gets shifted along the carrier frequency to

$\omega_1 - \omega_2$ or $\omega_1 + \omega_2$

The original frequency ω_1 is always aviable. Hence we can get 2-3 different frequencies.

Linear alternators are also possible with this scheme. The electric motor or generator if given a separate frequency of oscillation of velocity, will produce such signals with more than one frequency

5.



Region ① and ④ is free space

$n_2 = \frac{c}{v} = \frac{\sqrt{\mu\epsilon}}{\sqrt{\mu_0\epsilon_0}} = \sqrt{\mu_r\epsilon_r} = \sqrt{3}$

$n_3 = \sqrt{3}$

i) If the frequency of the wave is assumed to be ω

$k_0^2 = \omega^2\mu_0\epsilon_0$ since magnitude of k is not known (or ω is not known) we would take k_0 as the magnitude of the propagation vector

or $k_0 = \frac{\omega}{c}$

$\vec{k}_{inci} = k_0(\cos 30° \hat{a}_z + \sin 30° \hat{a}_x)$

$= \frac{k_0}{2}(\sqrt{3}\hat{a}_z + \hat{a}_x)$

$= \frac{k_0}{2}(\hat{a}_x + \sqrt{3}\hat{a}_z)$

$\vec{k}_{refl} = k_0(\sin 30° \hat{a}_x - \cos 30° \hat{a}_z)$

$= \frac{k_0}{2}(\hat{a}_x - \sqrt{3}\hat{a}_z)$

We can find angle of refraction by Snell's Law

$\sin\theta_i = n_2 \sin\theta_r$

or $\frac{1}{2} = \sqrt{3}\sin\theta_r$

or $\theta_r = \sin^{-1}\left(\frac{1}{2\sqrt{3}}\right) = 16.778°$

$\vec{k}_{tr2} = \sqrt{3}k_0(\sin\theta_r \hat{a}_x + \cos\theta_r \hat{a}_z)$

$= \sqrt{3}k_0\left(\frac{1}{2\sqrt{3}}\hat{a}_x + \frac{\sqrt{11}}{2\sqrt{3}}\hat{a}_z\right)$

$= \frac{\sqrt{3}k_0}{2\sqrt{3}}(\hat{a}_x + \sqrt{11}\hat{a}_z)$

$= \frac{\cancel{\sqrt{3}}}{2}k_0(\hat{a}_x + \sqrt{11}\hat{a}_z)$

$= k_0\left(0.5\hat{a}_x + \frac{\sqrt{11}}{2}\hat{a}_z\right)$

Parallel Polarization

we can assume that the incident wave has E_0 electric field and H_0 magnetic field. $\vec{E}$ is the xz plane

$$\vec{E}_{i1} = E_0(\cos\theta_i\,\hat{a}_x - \sin\theta_i\,\hat{a}_z)\ \exp[-jk_0(x\sin\theta_i + z\cos\theta_i)]$$

$$= \frac{E_0}{2}(\sqrt{3}\,\hat{a}_x - \hat{a}_z)\ \exp\left[\frac{-jk_0}{2}(x+\sqrt{3}z)\right]$$

$$\vec{H}_{i1} = \frac{E_0}{\eta_0}\ \exp\left[-\frac{jk_0}{2}(x+\sqrt{3}z)\right]$$

$$= E_0\ \exp\left[-\frac{jk_0}{2}(x+\sqrt{3}z)\right]$$

The reflected wave is

$$\vec{E}_{r1} = E_0R_{12}(\cos\theta_i\,\hat{a}_x + \sin\theta_i\,\hat{a}_z)\ \exp[-jk_0(x\sin\theta_i + z\cos\theta_i)]$$

$$= \frac{E_0R_{12}}{2}(\sqrt{3}\,\hat{a}_x + \hat{a}_z)\ \exp\left[\frac{-jk_0}{2}(x-\sqrt{3}z)\right]$$

$$\vec{H}_{r1} = -\frac{E_0R_{12}}{\eta_0}\cancel{\sqrt{3}}\ \exp\left[-j\frac{k_0}{2}(x-\sqrt{3}z)\right]\hat{a}_y$$

$$= -E_0R_{12}\ \exp\left[-\frac{jk_0}{2}(x-\sqrt{3}z)\right]\hat{a}_y$$

$$R_{12} = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i} = \frac{\frac{\eta_2}{\eta_1}\frac{\cos\theta_t}{\cos\theta_i} - 1}{\frac{\eta_2}{\eta_1}\frac{\cos\theta_t}{\cos\theta_i} + 1} = 0.3138$$

$$\cancel{T_{12} = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}}$$

The transmitted wave is

$$\vec{E}_{t2} = E_0T_{12}(\cos\theta_t\,\hat{a}_x - \sin\theta_t\,\hat{a}_z)\ \exp -j\sqrt{3}k_0(x\sin\theta_t + z\cos\theta_t)$$

$$= \frac{E_0T_{12}}{2\sqrt{3}}(\sqrt{11}\,\hat{a}_x - 1)\ \exp\left[-\frac{jk_0}{2}(x+\sqrt{11}z)\right]$$

$$\vec{H}_{t2} = \frac{E_0T_{12}}{\eta_2}\ \exp[-j\sqrt{3}k_0(x\sin\theta_t + z\cos\theta_t)]$$

$$= \frac{E_0T_{12}}{\sqrt{3}}\ \exp\left[-j\frac{k_0}{2}(x+\sqrt{11}z)\right]$$

$$T_{12} = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i} = \frac{\frac{2\eta_2}{\eta_1}\cos\theta_i}{\frac{\eta_2}{\eta_1}\cos\theta_t + \cos\theta_i} = 1.1884$$

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Perpendicular polarization

$$\vec{E_{i1}} = E_0 \exp\left[-jk_0(x\sin\theta_i + z\cos\theta_i)\right]\hat{a_y}$$
$$= E_0 \exp\left[-j\frac{k_0}{2}(x+\sqrt{3}z)\right]\hat{a_y}$$

$$\vec{H_{i1}} = \frac{E_0}{\eta_1}(-\cos\theta_i\,\hat{a_x} + \sin\theta_i\,\hat{a_z})\exp\left[-jk_0(x\sin\theta_i + z\cos\theta_i)\right]$$
$$= \frac{E_0}{2}(-\sqrt{3}\hat{a_x} + \hat{a_z})\exp\left[-j\frac{k_0}{2}(x+\sqrt{3}z)\right]$$

The reflected wave is

$$\vec{E_{r1}} = E_0R_{12}\exp\left[-jk_0(x\sin\theta_i - z\cos\theta_i)\right]\hat{a_y}$$
$$= E_0R_{12}\exp\left[-j\frac{k_0}{2}(x-\sqrt{3}z)\right]\hat{a_y}$$

$$\vec{H_{r1}} = \frac{E_0R_{12}}{\eta_1}\cancel{\exp}(\cos\theta_i\,\hat{a_x} + \sin\theta_i\,\hat{a_z})\exp\left[-j\frac{k_0}{2}(x-\sqrt{3}z)\right]$$
$$= \frac{E_0R_{12}}{2}(\sqrt{3}\hat{a_x} + \hat{a_z})\exp\left[-j\frac{k_0}{2}(x-\sqrt{3}z)\right]$$

$$R_{12} = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t} = \frac{\frac{\eta_2}{\eta_1}\frac{\cos\theta_i}{\cos\theta_t} - 1}{\frac{\eta_2}{\eta_1}\frac{\cos\theta_i}{\cos\theta_t} + 1} = 0.22$$

The transmitted wave is

$$\vec{E_{t2}} = E_0T_{12}\exp\left[-j\sqrt{3}k_0(x\sin\theta_t + z\cos\theta_t)\right]\hat{a_y}$$
$$= E_0T_{12}\exp\left[-j\frac{k_0}{2}(x+\sqrt{11}z)\right]\hat{a_y}$$

$$\vec{H_{t2}} = \frac{E_0T_{12}}{\eta_2}(-\cos\theta_t\,\hat{a_x} + \sin\theta_t\,\hat{a_z})\exp\left[-j\sqrt{3}k_0(x\sin\theta_t + z\cos\theta_t)\right]$$
$$= \frac{E_0T_{12}}{\sqrt{3}\times 2\sqrt{3}}(-\sqrt{11}\hat{a_x} + \hat{a_z})\exp\left[-j\frac{k_0}{2}(x+\sqrt{11}z)\right]$$

ii) since the refractive indices of the two medium are same, there can't be any reflection here.

$$\vec{K}_{inc2} = \vec{K}_{tr2}$$
$$= K_0\left(\frac{1}{2}\hat{a}_x + \frac{\sqrt{11}}{2}\hat{a}_z\right)$$

$$\vec{K}_{r2} = 0$$

$$\vec{K}_{t3} = \vec{K}_{inc2}$$
$$= K_0\left(\frac{1}{2}\hat{a}_x + \frac{\sqrt{11}}{2}\hat{a}_z\right)$$

The incident field may be both parallel or perpendicular polarization. The field incident is same as that of the transmitted field in regi

$$\vec{E}_{i2} = \vec{E}_{t2}$$
$$\vec{H}_{i2} = \vec{H}_{t2}$$

Irrespective of the nature of incident wave the ~~tra~~ reflected wave is 0

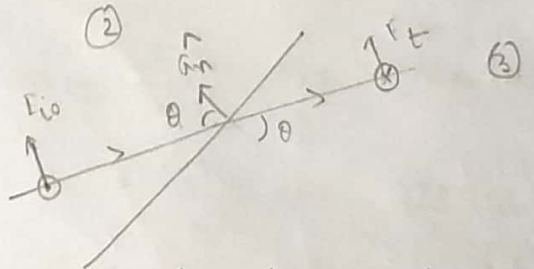
$$\vec{E}_{r2} = 0$$
$$\vec{H}_{r2} = 0$$

The transmitted wave would depend on whether the incident wave is parallel or perpendicularly polarized.

Parallel polarization

$$E_{i0} = E_0 T_{12p} = 1.1884 E_0$$
$$H_{i0} = \frac{E_{i0}}{\sqrt{3}} = 0.686 E_0$$

$$T_{23} = \frac{2n_1 \cos\theta_i}{n_2\cos\theta_t + n_1\cos\theta_i} = 1$$



For the boundary condition
$(\vec{E}_{i0} - \vec{E}_{t3}) \times \hat{a}_n = 0$ to be true
we must have

$$\vec{E}_{t3} = \vec{E}_{i2} = \frac{1.1884 E_0}{2\sqrt{3}} \exp\left[-j\frac{k_0}{2}(x + \sqrt{11}z)\right]\left(\hat{a}_z + \sqrt{11}\hat{a}_x\right)$$

Thus the electric field gets transmitted without any change

The other boundary condition to be satisfied is

$(\vec{B_1} - \vec{B_2}) \cdot \hat{a_n} = 0$

or $B_3 - B_2 \perp \hat{a_n}$

$$\vec{H_{t3}} = \frac{1.188 E_0}{\sqrt{3}} \exp\left[-j\frac{k_0}{2}(x \cdot \sqrt{11} z)\right]$$

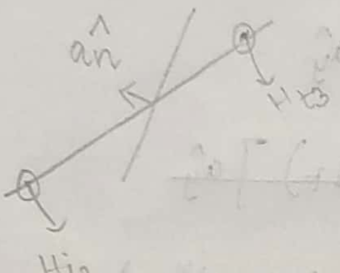
$\vec{H_{t3}} = \vec{H_{i2}} \sqrt{\frac{\mu_2 \epsilon_2}{\mu_3 \epsilon_3}}$

or $\vec{H_{t3}} = \frac{\vec{H_{i2}}}{\sqrt{3}}$ The magnetic field ~~gets scaled remains same~~ gets scaled

perpendicular polarization

$E_{i0} = 1.22 E_0$

$H_{i0} = \frac{E_{i0}}{\sqrt{3}}$



$(\vec{B_{i2}} - \vec{B_{t3}}) \cdot \hat{a_n} = 0$

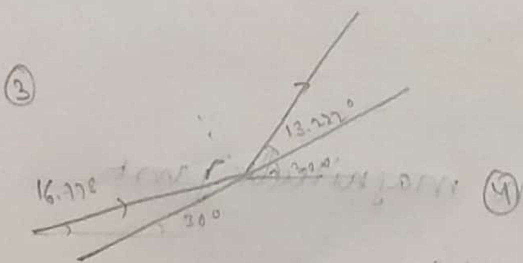
Here also the B incident and transmitted are parallel. So to satisfy boundary conditions

$\vec{H_{t3}} = \frac{\vec{H_{i2}}}{3} = \frac{\vec{H_{i2}}}{3}$

$\vec{E_{t3}} = \vec{E_{i2}}$

~~Thus irrespective of the polarization, magnetic field is scaled and electric field remains same.~~

iii)



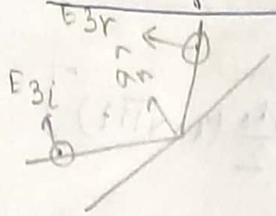
The angle of incidence is 76.778°
The critical angle is 35.26°
Thus there is total internal reflection

$\vec{k_{inc3}} = \vec{k_{ir2}}$

$\vec{k_{r3}} = k_0[(\cos 43.222°)\hat{a_z} + \sin 43.222° \hat{a_x}] = 0.728\hat{a_z} + 0.684\hat{a_x}$

$\vec{k_{t4}} = 0$

Parallel polarization



$(\vec{E_{3i}} - \vec{E_{3r}}) \times \hat{a}_n = 0$

This implies that both should have equal magnitude

$|\vec{E_{r3}}| = |\vec{E_{i3}}|$

$(\vec{B_{3i}} - \vec{B_{r3}}) \cdot \hat{a}_n = 0$

The B fields are perpendicular to the vector, so this is satisfied

$$\vec{H_{r3}} = \frac{|E_{r3}|}{n_1} \exp[-jk_0(x\sin\theta_i - z\cos\theta_i)]\,\hat{a_y}$$

$$\cancel{\vec{H_{r3}} = \frac{1.1884}{\sqrt{3}} \exp\left[jk_0(0.684x - 0.728z)\right]\hat{a_y}}$$

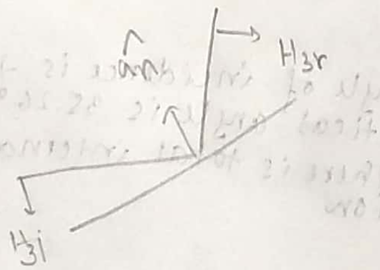
$$\cancel{\vec{E_{r3}} = \frac{1.1884E_0}{2\sqrt{3}} (-0.684\hat{a_z} + 0.728\hat{a_x}) \exp -jk_0 x}$$

$\theta_i = 76.778°$

$$\vec{H_{r3}} = \frac{1.1884E_0}{\sqrt{3}} \exp\left[jk_0(0.684x - 0.728z)\right]\hat{a_y}$$

$$\vec{E_{r3}} = 1.1884E_0 (-0.684\hat{a_z} + 0.728\hat{a_x}) \exp[-jk_0(0.728x - 0.684$$

Perpendicular polarization



Here H must be same in magnitude just rotated

$\vec{H_{3r}} = \vec{H_{3i}}\, e^{-j0.46}$ rotated by $26.44°$ $= 0.46$ radians

$\vec{E_{3r}} = 1.22E_0 \exp\left[-jk_0(0.684x + 0.728z)\right]$

(iv) The following assumptions are made

a. The wave impedance is small.

b. There is no loss in medium

c. Boundaries are smooth and abrupt.