Reliability of a System at time t (with life X)

$$R(t) = P(X > t).$$

Instantaneous Failure Rate of System at time t:

$$= \lim_{h \to 0} \frac{1}{h} P(\underbrace{t < X \le t+h}_{A} \mid \underbrace{X > t}_{B})$$

$$= \lim_{h \to 0} \frac{1}{h} \frac{P(t < X \le t+h)}{P(X > t)}$$

$$= \lim_{h \to 0} \frac{P(X \le t+h) - P(X \le t)}{h \quad P(X > t)}$$

$$= \lim_{h \to 0} \left\{ \frac{F_X(t+h) - F_X(t)}{h} \right\} \cdot \frac{1}{1 - F_X(t)}$$

$$= \frac{f_X(t)}{1 - F_X(t)} = \frac{f_X(t)}{R_X(t)} = H_X(t)$$

hazard rate at time t

We can now write

$$H_X(t) = -\frac{d}{dt} \log\left(1 - F_X(t)\right)$$

$$\Rightarrow \log\left(1 - F_X(t)\right) = -\int H_X(t)\,dt + c$$

$$\Rightarrow \quad 1 - F_X(t) = k\, e^{-\int H_X(t)\,dt}$$

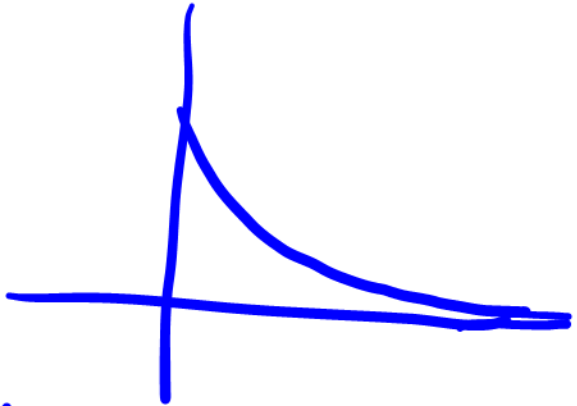
So for a continuous r.v. X describing life of a system, there is a one-to-one correspondence between the distribution (pdf or cdf) and its failure rate fn.

<u>For exponential distribution:</u>

$$f_X(t) = \begin{cases} \lambda e^{-\lambda t}, & t > 0 \\ 0, & \text{ew} \end{cases}$$

$$F_X(t) = 1 - e^{-\lambda t}$$

So $R_X(t) = e^{-\lambda t}$



$$H_X(t) = \frac{f_X(t)}{R_X(t)} = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda$$

So in the exponential dist^n, failure rate is constant.

Consider Weibull dist^n

$$f_X(t) = \alpha \beta t^{\beta-1} e^{-\alpha t^{\beta}}, \quad t > 0$$

$$F_X(t) = 1 - e^{-\alpha t^{\beta}}, \quad R_X(t) = e^{-\alpha t^{\beta}}$$

So for $\beta = 1$, it is exponential distⁿ

For $\beta > 1$, it goes to zero faster than exponential rate

For $\beta < 1$, it goes to zero slower than exponential rate.

$$H_X(t) = \frac{f_X(t)}{R_X(t)} = \frac{\alpha \beta t^{\beta-1} \cancel{e^{-\alpha t^{\beta}}}}{\cancel{e^{-\alpha t^{\beta}}}} = \alpha \beta t^{\beta-1}$$

For $\beta = 1$, $H_X(t) = \alpha$ constant

$\beta > 1$, $H_X(t)$ is increasing in t

So the lifetime has increasing failure rate (IFR)

$\beta < 1$, $H_X(t)$ is decreasing in t

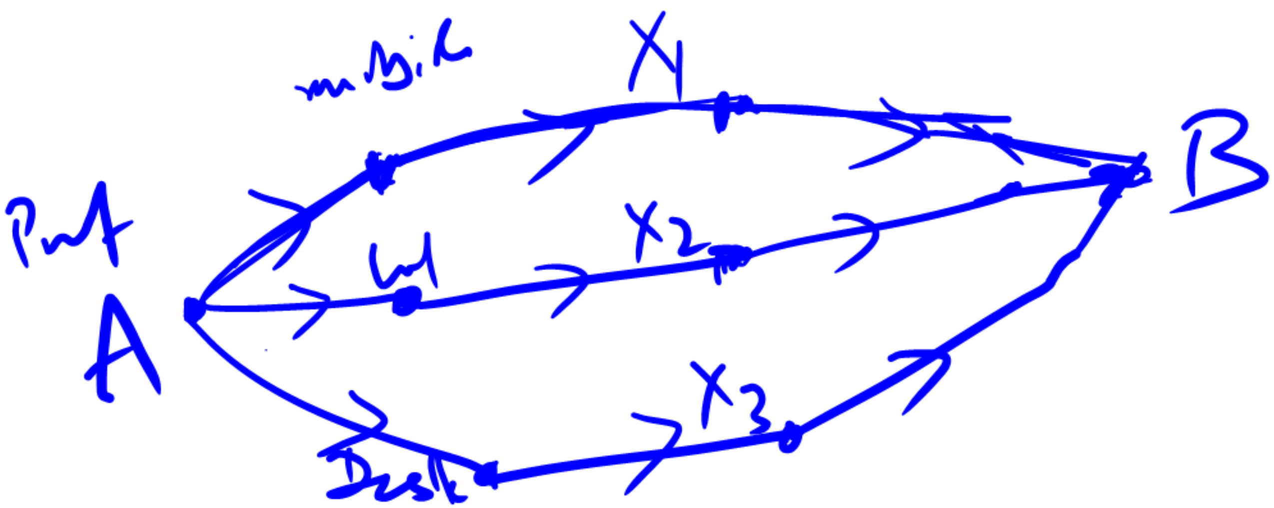
So the lifetime has decreasing failure rate (DFR)

Class: 1. Internet, 2. Device at the end of student → mobile/ laptop/ Desktop with all supporting equipment → Speaker, camera, mic

3. Professor → mobile/ laptop/ Desktop/ Speaker, camera, mic, wacom pen

4. Power

Series System, Parallel System



student

Reliability of a Series System

Suppose a system has k components

connected in a series. Let the system life be X and component lives be

$X_1, \ldots X_k$.

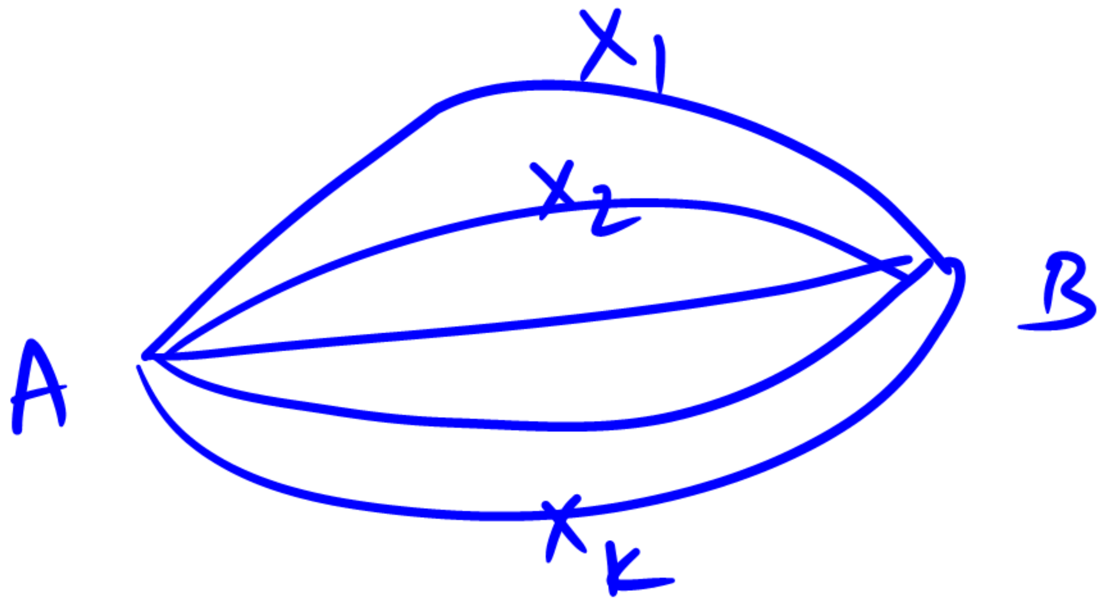
So the reliability of the system at time t

$$R_X(t) = P(X > t)$$

$$= P(X_1 > t, X_2 > t, \ldots, X_k > t)$$

If the component lives are assumed to be independent, then

$$R_X(t) = \prod_{i=1}^{k} P(X_i > t) = \prod_{i=1}^{k} R_{X_i}(t)$$



Reliability of a Parallel System

Suppose a system has k independent components connected in parallel.

System reliability

$$R_X(t) = P(X > t) = 1 - P(X \le t)$$

$$= 1 - P(X_1 \le t, X_2 \le t, \ldots, X_k \le t)$$

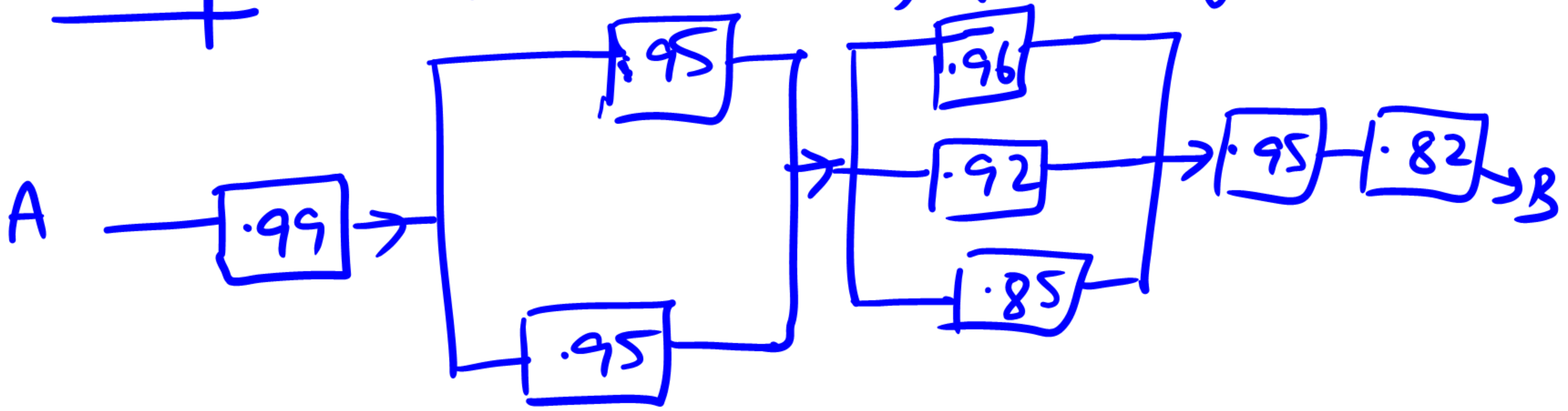
$$= 1 - \prod_{i=1}^{k} P(X_i \le t)$$

$$= 1 - \prod_{i=1}^{k} \left\{ 1 - R_{X_i}(t) \right\}$$

So in a series system the reliability decreases with addition of components.

In a parallel system the reliability increases with addition of components.

Example: Find reliability of this system



$$R_x(t) = (.99)\left\{1-(1-.95)^2\right\}\left\{1-(1-.96)(1-.92)(1-.85)\right\} \times .95 \times .82$$

$= (.99)(\underline{.9975})(\underline{.99952})(.95)(.82)$

$= 0.7689$

Example: Suppose a system has two independent components connected in a series. The life of first component is Weibull with $\alpha = 0.006$ & $\beta = 0.5$. The second has a life following exponential dist^n with mean (25000 hrs)

(i) What is the reliability of system at 2500 hrs?

(ii) What is the prob that the system will fail before 2000 hrs. ?

(iii) What is the system reliability if components are connected in parallel ?

$X_1 \to$ first component life

$$R_{X_1}(t) = e^{-\alpha t^{\beta}} = e^{-0.006\, t^{1/2}}$$

$X_2 \to$ second component life

$$R_{X_2}(t) = e^{-t/25000}$$

System reliability at time t

$$= e^{-0.006\sqrt{t}} \cdot e^{-t/25000}$$

$$R_X(2500) = e^{-0.006 \times 50} \; e^{-0.1}$$

$$= e^{-0.4} \cong 0.67$$

$$P(X < 2000) = 1 - R_X(2000)$$

$$= 1 - e^{-0.006 \times \sqrt{2000}} \; e^{-2/25} \approx \cdots$$

If the components are connected in parallel

$$R_X(2500) = 1 - (1 - e^{-0.006 \times 50})(1 - e^{-0.1})$$

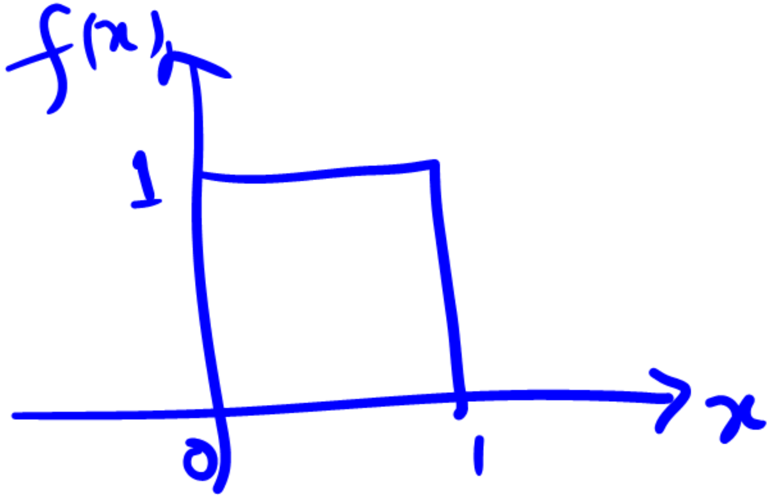
$$\cong 0.98$$

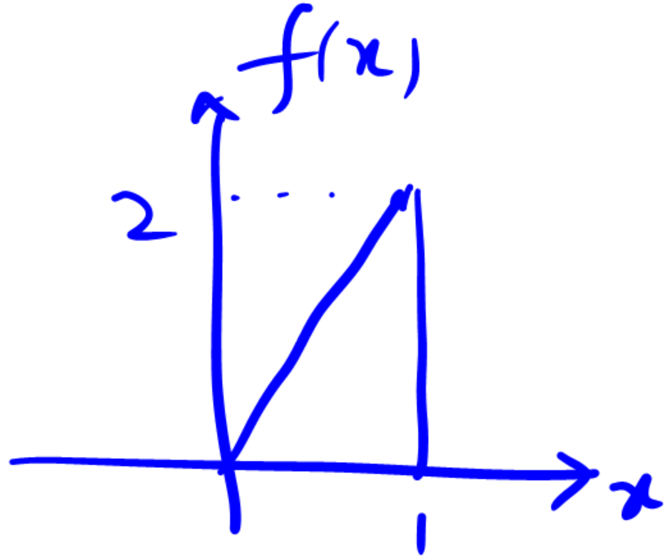
Beta Distribution : A r.v. X is said to have a Beta distⁿ with parameters α and β, (>0) if it has pdf

$$f_X(x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1} (1-x)^{\beta-1}, \quad 0<x<1$$

$$\alpha>0,\ \beta>0$$

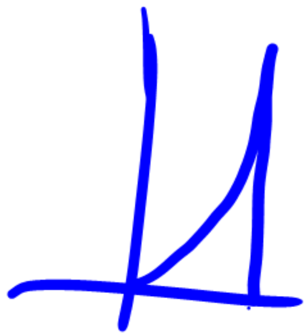
1. If $\alpha=\beta=1$, this a $U(0,1)$ distⁿ





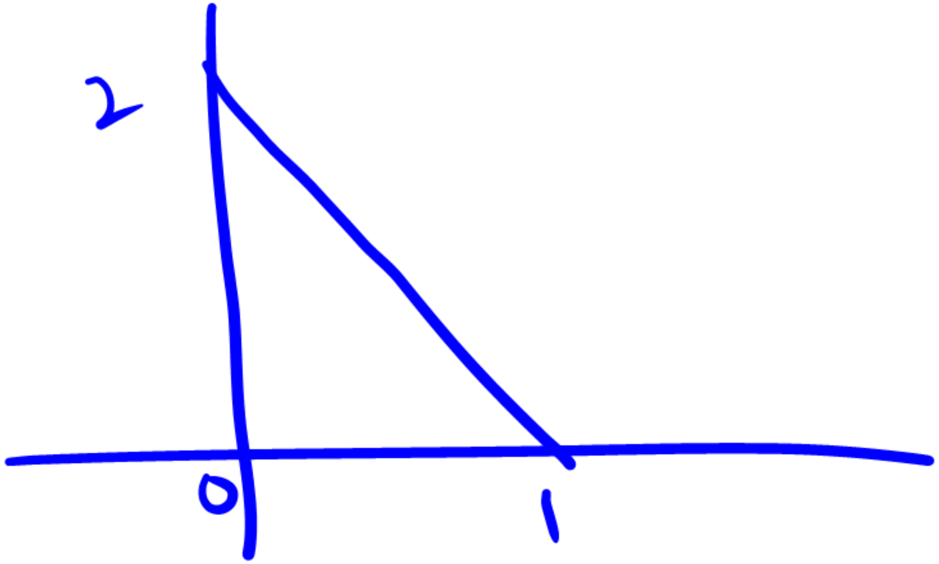
(ii) $\alpha = 2,\ \beta = 1,\qquad f_X(x) = 2x,\quad 0 < x < 1$

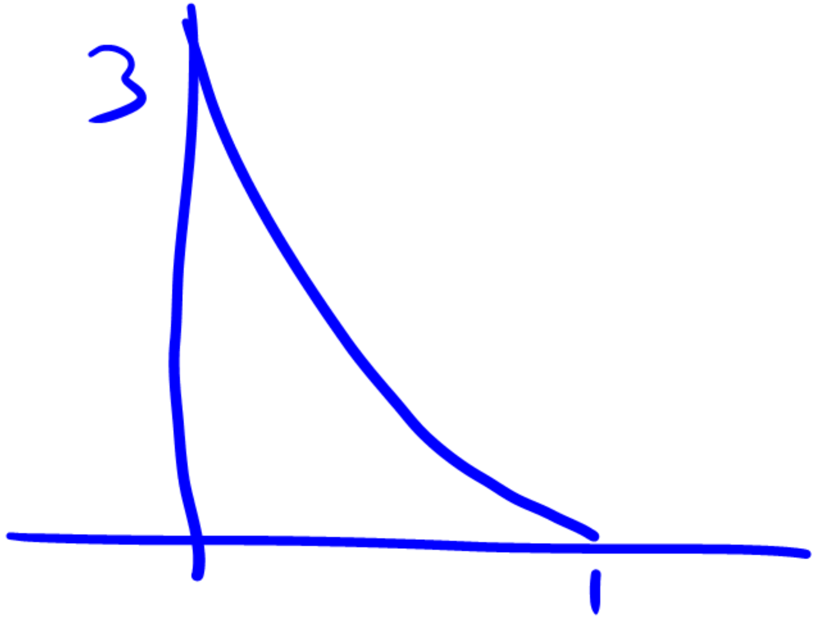
(iii) $\alpha = 3,\ \beta = 1\qquad f(x) = 3x^2,\quad 0 < x < 1$



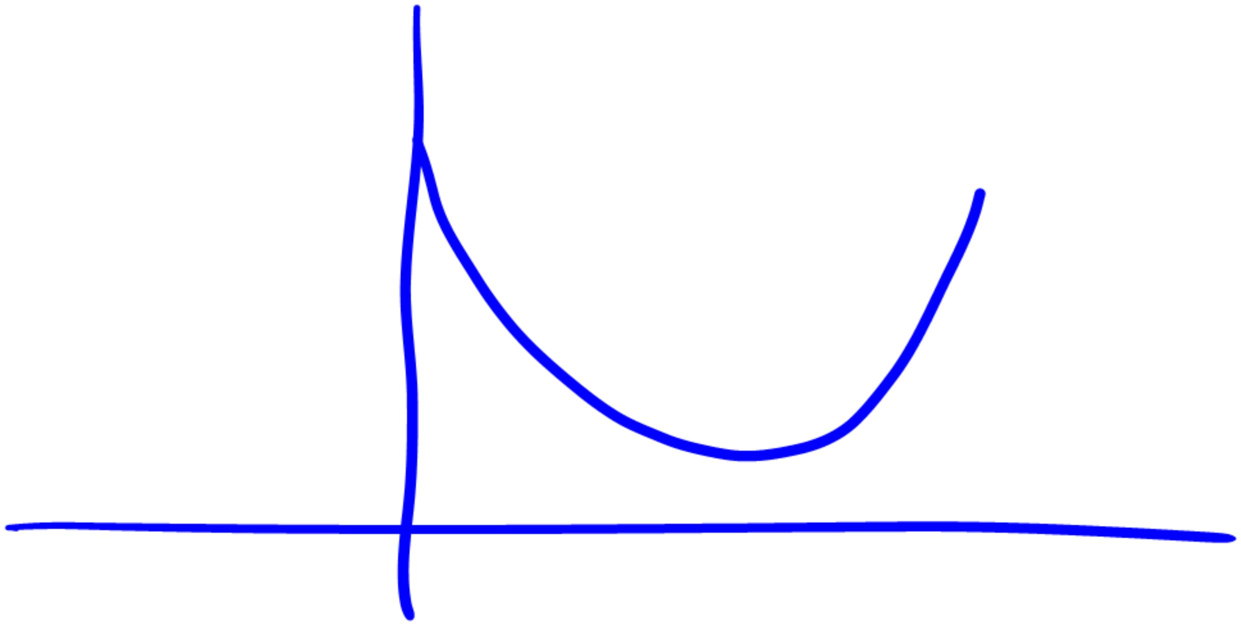
(iv) $\alpha = 1,\ \beta = 2,\qquad f_X(x) = 2(1-x),\quad 0 < x < 1$

(v) $\alpha = 1,\ \beta = 3\qquad f_X(x) = 3(1-x)^2,\quad 0 < x < 1$





(vi) $\alpha = \beta < \underline{1}$



(vii) $\alpha = \beta > 1$

So beta distⁿ can be used to model various kinds of datasets.

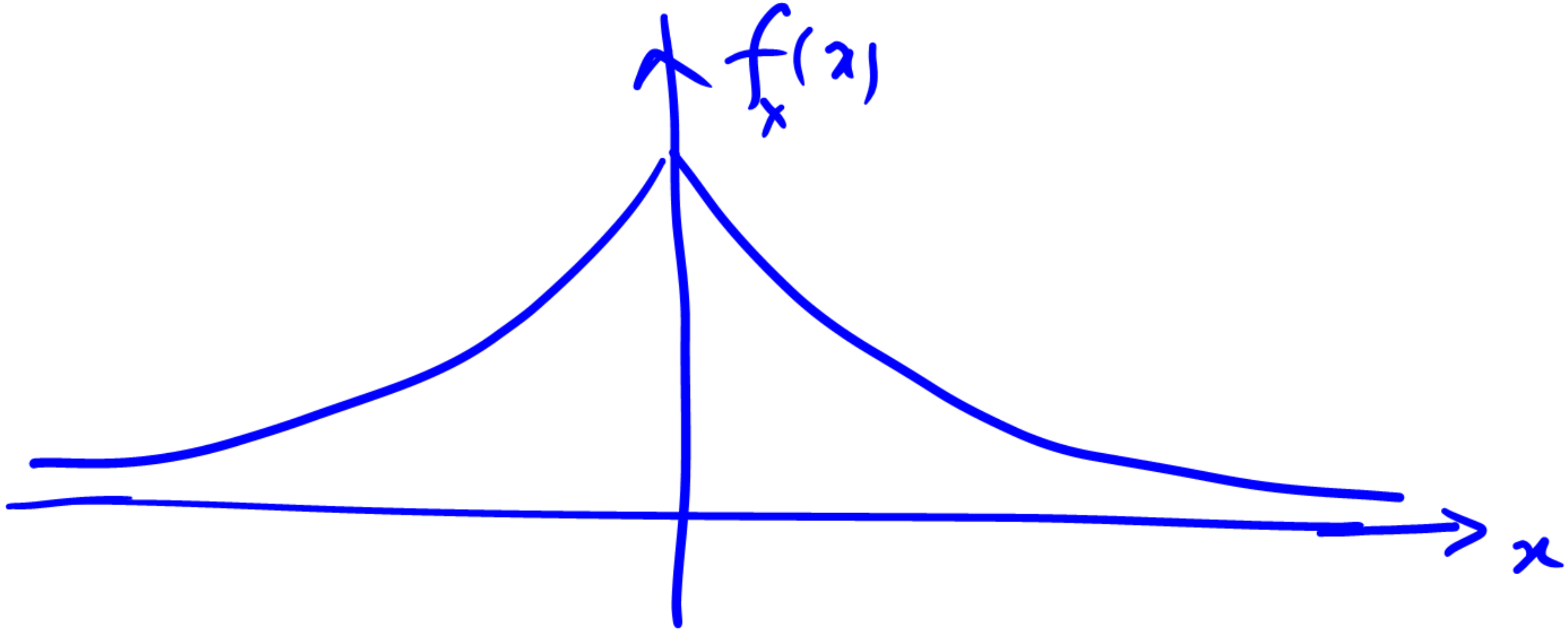
$$\mu'_k = E(X^k) = \int_0^1 \frac{1}{B(\alpha,\beta)} x^{\alpha+k-1} (1-x)^{\beta-1} dx$$

$$= \frac{B(\alpha+k, \beta)}{B(\alpha, \beta)}$$

$$\mu_1' = E(X) = \frac{\alpha}{\alpha+\beta}, \quad \mu_2' = \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}$$

$$\mu_2 = Var(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}.$$

Double Exponential or Laplace Distribution



$$f_X(x) = \frac{1}{2\sigma} e^{-\frac{|x-\mu|}{\sigma}}, \quad -\infty < x < \infty$$
$$-\infty < \mu < \infty$$
$$\sigma > 0$$

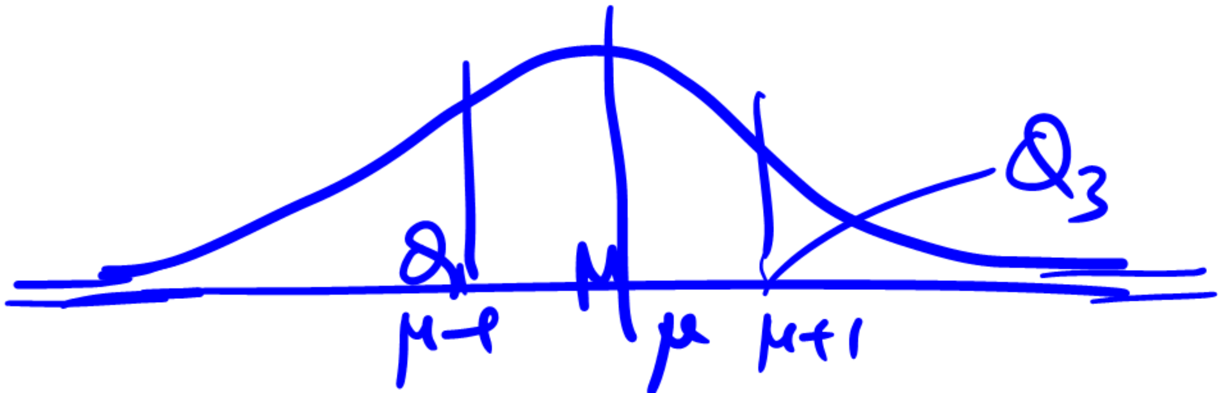
$E(X) = \mu$, $\text{Med}(X) = \mu$

$\text{Var}(X) = 2\sigma^2$

Find Q_1 and Q_3 also

Cauchy Distribution

$$f_X(x) = \frac{1}{\pi} \cdot \frac{1}{1 + (x-\mu)^2}, \quad x \in \mathbb{R}, \ \mu \in \mathbb{R}$$



Mean of this dist^n does not exist.

$$\text{Med}(X) = \mu.$$

$$F_X(x) = \frac{1}{\pi}\int_{-\infty}^{x} \frac{1}{1+(t-\mu)^2}\,dx$$

$$= \frac{1}{\pi}\left[\tan^{-1}(x-\mu) + \frac{\pi}{2}\right]$$

So $F_X(x) = \frac{1}{2}$ when $x = \mu$. So μ is median for Q_1:

$$\frac{1}{\pi}\left[\tan^{-1}(x-\mu)+\frac{\pi}{2}\right]=\frac{1}{4}$$

$$\Rightarrow \tan^{-1}(x-\mu) = -\frac{\pi}{4}$$

$$x-\mu = -1 \Rightarrow x = \mu-1 = Q_1$$

For Q_3:

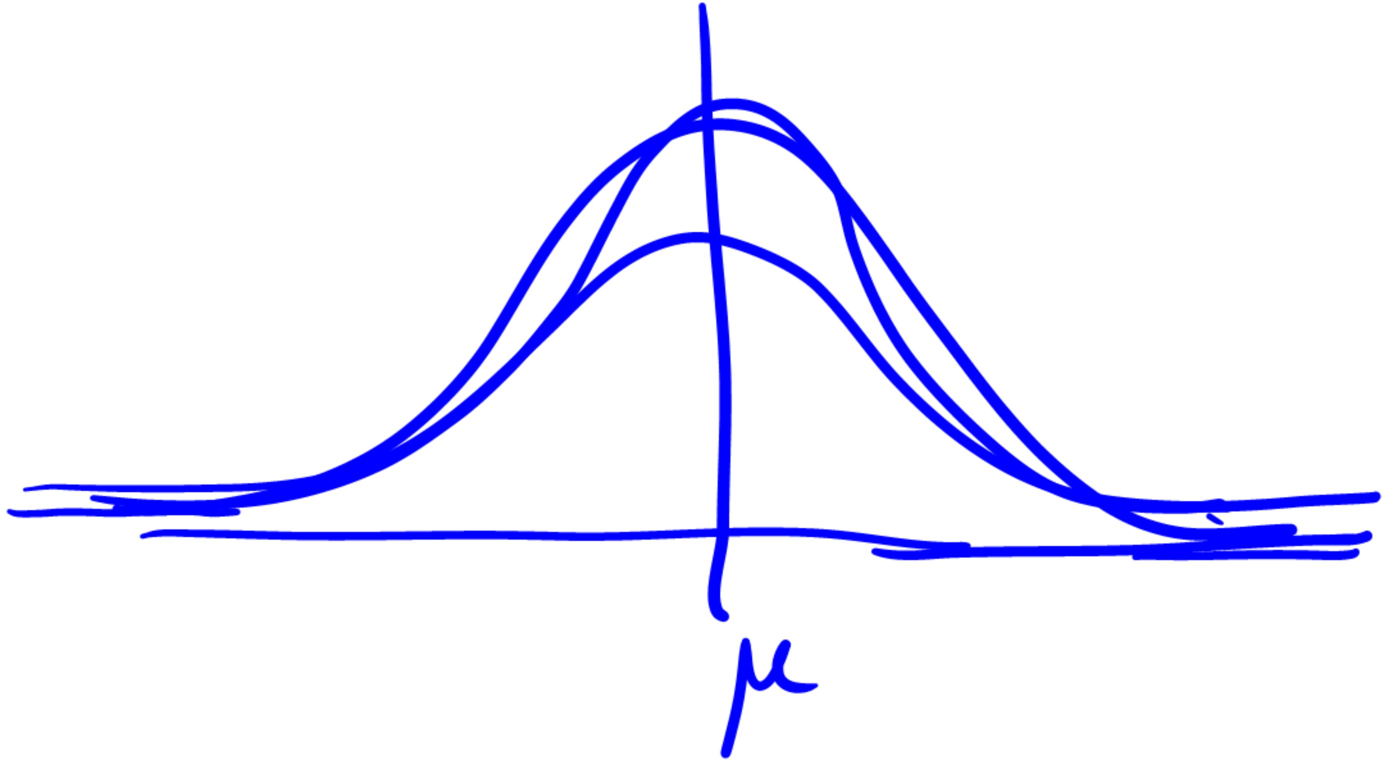
$$\frac{1}{\pi}\left[\tan^{-1}(x-\mu)+\frac{\pi}{2}\right]=\frac{3}{4}$$

$$\Rightarrow \tan^{-1}(x-\mu) = \frac{\pi}{4}$$

$$x-\mu = 1 \Rightarrow x = \mu+1 = Q_3$$

Normal Distribution : A continuous r.v. X is said to have a normal distribution with mean μ and variance σ^2 if its pdf is given by

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad \begin{array}{l} x \in \mathbb{R} \\ \mu \in \mathbb{R} \\ \sigma > 0 \end{array}$$



$$I = \int_{-\infty}^{\infty} f_X(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$z = \frac{x-\mu}{\sigma}, \quad dz = \frac{1}{\sigma} dx$$

$$I = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2}\, dz$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-z^2/2}\, dz \qquad t = \frac{z^2}{2}, \quad dz = \frac{1}{\sqrt{2t}}\, dt$$

$$I = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-t} \cdot \frac{1}{\sqrt{2t}}\, dt = \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} t^{-1/2} e^{-t}\, dt$$

$$= \frac{\Gamma(1/2)}{\sqrt{\pi}} = 1.$$

$$E\left(\frac{x-\mu}{\sigma}\right)^k = \int_{-\infty}^{\infty} \left(\frac{x-\mu}{\sigma}\right)^k \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \int_{-\infty}^{\infty} z^k \cdot \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

If $k = 2m+1$ (odd), then

$$\mu_k = 0.$$

So $E\left(\frac{X-\mu}{\sigma}\right) = 0 \Rightarrow E(X) = \mu$.

So μ represents the mean of normal distn.

So all odd ordered central moments of a normal distn vanish.

In particular $\mu_3 = 0$, so $\beta_1 = 0$

For $k = 2m$

$$\mu_{2m} = 2\sigma^{2m} \int_0^\infty z^{2m} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \, dz$$

$$= \frac{2\sigma^{2m}}{\sqrt{2\pi}} \int_0^\infty (2t)^m \, e^{-t} \cdot \frac{1}{\sqrt{2t}} \, dt$$

$$= \frac{2^m \sigma^{2m}}{\sqrt{\pi}} \int_0^\infty t^{m-\frac{1}{2}} \, e^{-t} \, dt$$

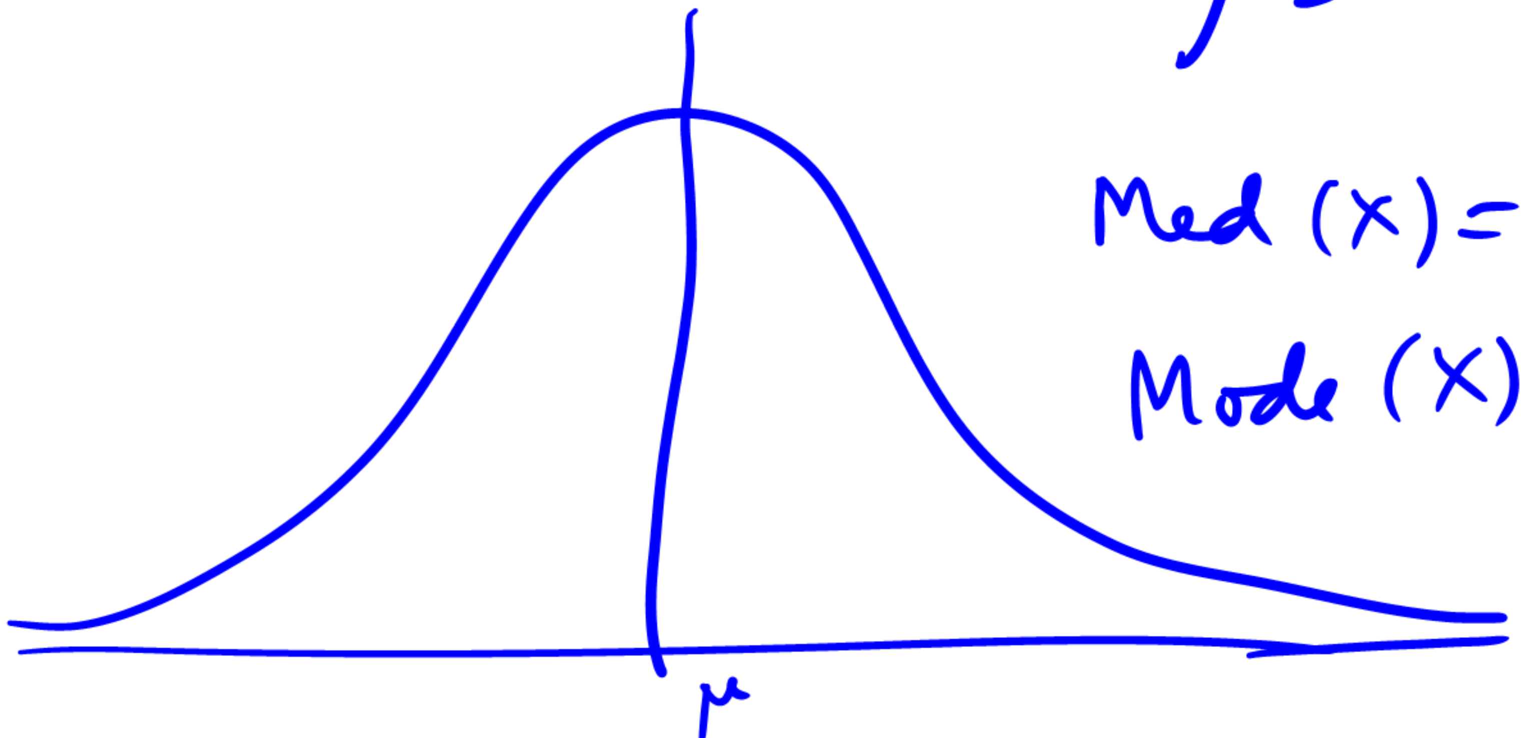
$$= \frac{2^m \sigma^{2m}}{\sqrt{\pi}} \Gamma\left(m+\frac{1}{2}\right) = \frac{2^m}{\cancel{\sqrt{\pi}}} \left(m-\frac{1}{2}\right)\left(m-\frac{3}{2}\right) \cdots \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cancel{\sqrt{\pi}} \, \sigma^{2m}$$

$$= (2m-1)(2m-3) \cdots \cdots 5 \cdot 3 \cdot 1 \cdot \sigma^{2m}$$

So $\mu_2 = \sigma^2$. ie σ^2 represents variance of a normal distn

$$\mu_4 = 3\sigma^4$$

Measure of Kurtosis $\beta_2 = \frac{\mu_4}{\mu_2^2} - 3 = 0$



$Med(X) = \mu$

$Mode(X) = \mu$