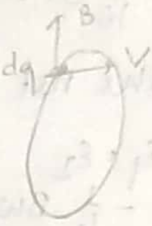
4. Due to the time varying nature of field an emf is induced

i) given by Lenz's law

$$\mathcal{E} = -\frac{d\phi_B}{dt} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{s} = -\int \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s}$$

However since the loop is moving, there is another source of work.



A charge element dq experiences

$$d\vec{F} = dq\, \vec{v}\times\vec{B}$$

$$d^2W = d\vec{F}\cdot d\vec{l}$$

or $d^2W = dq\, \vec{v}\times\vec{B}\cdot d\vec{l}$

or $d\left(\frac{dW}{dq}\right) = \vec{v}\times\vec{B}\cdot d\vec{l}$

or $\frac{dW}{dq} = \oint (\vec{v}\times\vec{B})\cdot d\vec{l}$

or $\mathcal{E} = \oint (\vec{v}\times\vec{B})\cdot d\vec{l}$

Hence the total induced emf:

$$\mathcal{E} = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s} + \oint_L (\vec{v}\times\vec{B})\cdot d\vec{l}$$

ii) We construct a sinusoidal field $\vec{B} = B_0 \sin\omega_1 t\, \hat{a}_z$ in all of space. The loop moves in x-direction with velocity $v_0\hat{a}_x$. To make the B-field space dependent we take $B_0 = B\cos\phi\, e^{-\phi}$ and $v_0 = v\cos(\omega_2 t)$

$$\vec{B} = B\cos\phi \sin\omega_1 t\, e^{-\phi}\hat{a}_z$$

$$\vec{v} = v\cos\omega_2 t\, \hat{a}_x$$

$$\mathcal{E}_1 = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s}$$

$$= -\int_{\text{u.r}} B\cos\phi\, \omega_1 \cos\omega_1 t\, ds\, e^{-\phi}$$

$$= -B\omega_1 \cos\omega_1 t \int_0^r \int_0^{2\pi} \cos\phi\, \rho\, d\phi\, d\rho\, e^{-\phi}$$

$$= -B\omega_1\cos\omega_1 t \times \left[\frac{\rho^2}{2}\right]_0^r \int e^{-\phi}\cos\phi\, d\phi$$

$$= -\frac{1}{2} B\omega_1 r^2 \cos(\omega_1 t)\, \frac{1}{2}(1 - e^{-2\pi})$$

$$= -\frac{1}{4} B\omega_1 r^2 (1 - e^{-2\pi})\cos\omega_1 t$$

$$\mathcal{E}_2 = \oint_L \vec{v}\times\vec{B}\cdot d\vec{l}$$

$$= \int_0^{2\pi} -Bv\cos\phi\, e^{-\phi}\cos\omega_1 t \sin\omega_1 t \cdot r\cos\phi\, d\phi$$

$$= -\frac{Bv}{2}\sin(2\omega_1 t)\int_0^{2\pi} e^{-\phi}\cos^2\phi\, d\phi$$

$= -\frac{1}{2} Bv \sin(2\omega_1 t) \cdot \frac{3}{5}(1-e^{-2\pi})$

$= -\frac{3}{10} Bv \sin(2\omega_1 t)(1-e^{-2\pi})$

Thus the net induced emf is

$$\varepsilon = \varepsilon_1 + \varepsilon_2$$

$$= -\frac{1}{4} B\omega_1 r^2 (1-e^{-2\pi}) \cos\omega_1 t - \frac{3}{10} Bv \sin(2\omega_1 t)(1-e^{-2\pi})$$

$$= -B(1-e^{-2\pi}) \left[\frac{\omega_1 r^2}{4}\cos\omega_1 t + \frac{3}{10} v \sin(2\omega_1 t)\right]$$

Thus using this scheme we can generate frequencies of ω_1 and $2\omega_1$. In general if the field is time dependent it introduces one frequency. If the velocity has a frequency dependence it would give two frequencies around the field frequency. If both frequencies are same we get one coincident frequency and an extra frequency

Velocity frequency ω_2
Field frequency ω_1
Frequencies available ω_1, $\omega_1-\omega_2$ and $\omega_1+\omega_2$ if $\omega_1 \neq \omega_2$
ω_1, $2\omega_1$ if $\omega_1 = \omega_2$

These extra frequencies are called heterodynes.

iii) Heterodynes are particularly useful when multiple signals share a common bandwith. Using the heterodynes the signal may be shifted along the spectrum. The field frequency would be the base or carrier frequency. While the velocity frequency would be modulation which gets shifted along the carrier frequency to

$\omega_1 - \omega_2$ or $\omega_1 + \omega_2$

The original frequency ω_1 is always aviable. Hence we can get 2-3 different frequencies.

Linear alternators are also possible with this scheme. The electric motor or generator if given a separate frequency of oscillation of velocity, will produce such signals with more than one frequency