

MA 20104 Probability and Statistics
Hints/Solutions to Assignment No. 2

1. $c = \frac{2}{3}$.

$$F(x) = \begin{cases} 0, & x \leq 0 \\ \frac{x^3}{3}, & 0 < x \leq 1. \\ 1 - \frac{2}{3x}, & x > 1 \end{cases}$$

$E(X)$ and $Var(X)$ do not exist.

$$Med(X) = \frac{4}{3}, \quad P(0.5 < X < 2) = \frac{5}{8}, \quad P(X > 3) = \frac{2}{9}.$$

2. $P\left(\frac{1}{2} < X < \frac{5}{2}\right) = F\left(\frac{5}{2}-\right) - F\left(\frac{1}{2}\right) = \frac{11}{12} - \frac{1}{8} = \frac{19}{24}.$

$$P(1 < X < 3) = F(3-) - F(1) = \frac{11}{12} - \frac{1}{2} = \frac{5}{12}.$$

The random variable X is continuous in the intervals $(0, 1)$ and $(1, 2)$ with the uniform density $f(x) = \frac{1}{4}$, and discrete at points 1, 2 and 3 with probabilities $1/4$, $1/6$ and $1/12$ respectively. So

$$E(X) = \int_0^1 \frac{x}{4} dx + \int_1^2 \frac{x}{4} dx + 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{12} = \frac{4}{3}.$$

$$E(X^2) = \int_0^1 \frac{x^2}{4} dx + \int_1^2 \frac{x^2}{4} dx + 1^2 \cdot \frac{1}{4} + 2^2 \cdot \frac{1}{6} + 3^2 \cdot \frac{1}{12} = \frac{7}{3}.$$

$$V(X) = \frac{5}{9}, \quad Median(X) = 1.$$

3. Let S denote a survival and D denote a death of a guinea pig during the trial, then the sample space for this experiment can be described by

$$\Omega = \{SS, SDS, SDD, DSS, DSD, DDSS, DDSD, DDDS, DDDD\}$$

and the probabilities associated with these sample points are given by $4/9$, $4/27$, $2/27$, $4/27$, $2/27$, $4/81$, $2/81$, $2/81$, $1/81$ respectively.

Let X = the number of survivors, Y = number of deaths.

Then the pmf of X is

$$P(X = 2) = P(\{SS, SDS, DSS, DDSS\}) = 64/81,$$

$$P(X = 1) = P(\{SDD, DSD, DDSD, DDDS\}) = 16/81,$$

$$P(X = 0) = P(\{DD\}) = 1/81.$$

The pmf of Y is

$$\begin{aligned} P(Y = 0) &= P(\{SS\}) = 4/9, P(Y = 1) = P(\{SDS, DSS\}) = 8/27, \\ P(Y = 2) &= P(\{SDD, DSD, DDSS\}) = 16/81, \\ P(Y = 3) &= P(\{DDSD, DDDS\}) = 4/81, P(Y = 4) = P(\{DDDD\}) = 1/81. \end{aligned}$$

4. Let X = the number of second generation particles,

Let Y = the number of third generation particles,

Then $X \rightarrow 1, 2, 3; Y \rightarrow 1, 2, \dots, 9$.

$$P(Y = 1) = 1/9, P(Y = 2) = 4/27, P(Y = 3) = 16/81,$$

$$P(Y = 4) = 4/27 = P(Y = 5),$$

$$P(Y = 6) = 10/81, P(Y = 7) = 2/27,$$

$$P(Y = 8) = 1/27, P(Y = 9) = 1/81.$$

5. Let X denote the scores on IQ test.

$$P(X < 52 \text{ or } X > 148) = P(|X - 100| > 48) \leq \frac{V(X)}{(48)^2} = \frac{1}{9}.$$

6. The cdf is given by

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{x^2}{4}, & 0 \leq x \leq 1 \\ \frac{2x-1}{4}, & 1 < x \leq 2. \\ \frac{6x-x^2-5}{4}, & 2 < x \leq 3 \\ 1, & x \geq 3 \end{cases}$$

$$E(X) = 3/2, \text{Median}(X) = 3/2, \text{Var}(X) = 5/12.$$

7. Note that $\int_{-\infty}^{\infty} f(x) dx = 1$ for all $k \in \mathbb{R}$. To have $f(x) \geq 0 \quad \forall x \in \mathbb{R}$, we must have

$0 \leq k \leq 1$. The cdf is given by

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{kx}{4}, & 0 \leq x \leq 1 \\ \frac{k+x^2-1}{4}, & 1 < x \leq 2 \\ \frac{(3k+1)+(1-k)x}{4}, & 2 < x < 3 \\ 1, & x \geq 3 \end{cases}$$

In order to determine the median we note that the maximum value of $F(x)$ for

$0 \leq x \leq 1$ is $\frac{k}{4}$ and will be less than $\frac{1}{2}$. In the range $1 < x \leq 2$, it will be between

$\frac{k}{4}$ and $\frac{k+3}{4}$ and so median will lie in this range. Equating $\frac{k+M^2-1}{4}$ to $\frac{1}{2}$ we get $M = \sqrt{3-k}$ and so $\sqrt{2} \leq M \leq \sqrt{3}$.

8. Let X be the number of tests required. Then X is either 1 or 11.

$$P(X = 1) = P(\text{none has the disease}) = (0.99)^{10},$$

$$P(X = 11) = P(\text{at least one has disease}) = 1 - (0.99)^{10},$$

$$E(X) = 11 - 10(0.99)^{10}.$$

9. $P(X = i) = \frac{n_i}{m}$, $i = 1, \dots, r$, $E(X) = \sum_{i=1}^r \frac{in_i}{m}$.

$$P(Y = i) = \frac{in_i}{\sum_{i=1}^r in_i}, \quad i = 1, \dots, r, \quad E(Y) = \frac{\sum_{i=1}^r i^2 n_i}{\sum_{i=1}^r in_i}.$$

The proof of $E(Y) \geq E(X)$ follows by noticing $E(U^2) \geq \{E(U)\}^2$.

10. $0 \leq p_X(i) \leq 1$, $i = 1, \dots, 4$ yields $-\frac{1}{3} \leq d \leq \frac{1}{4}$.

$$E(X) = \frac{10-9d}{4}, \quad E(X^2) = \frac{30-47d}{4}, \quad V(X) = \frac{20-8d-81d^2}{16}.$$

$$V(X) \text{ is minimized at } d = \frac{1}{4}.$$

11. $\sum_{x=0}^{\infty} p_X(x) = k$, so $k = 1$.

$$\begin{aligned} F(x) &= 0, & \text{if } x < 0, \\ &= 1/2, & \text{if } 0 \leq x < 1, \\ &= 2/3, & \text{if } 1 \leq x < 2, \\ &\vdots \\ &= n/(n+1), & \text{if } n-1 \leq x < n, \\ &\vdots \end{aligned}$$

$E(X)$ does not exist. Any M between 0 and 1 is a median.

$$12. \quad P(40 < X < 60) = P(|X - \mu| < 10) \geq 1 - \frac{\sigma^2}{100} = 0.75.$$

$$13. \quad X \sim \text{Bin}(4, 0.4). \text{ So } E(X) = 1.6, V(X) = \frac{24}{25}, P(X \leq 1) = \frac{297}{625}.$$

As $p < 0.5$, the distribution is positively skewed.

14. In order to have two consecutive heads ending on n^{th} trial we must have sequence of the type $\underset{n-1}{HTHT} \underset{n}{HH}$. The probability of this is $\left(\frac{1}{2}\right)^n$.

Similarly in order to have two consecutive tails ending on n^{th} trial we must have sequence of the type $\underset{n-1}{THTH} \underset{n}{TT}$. The probability of this is also $\left(\frac{1}{2}\right)^n$.

$$\text{So } P(X = n) = \frac{1}{2^{n-1}}, \quad n = 2, 3, 4, \dots$$

$P(\text{even number of tosses is required to end the experiment})$

$$= \sum_{m=1}^{\infty} P(X = 2m) = \sum_{m=1}^{\infty} \left(\frac{1}{2}\right)^{2m-1} = \left(\frac{2}{3}\right).$$

$$P(X = 6 | X = \text{even}) = \frac{P(X = 6)}{P(X = \text{even})} = \frac{3}{16}.$$

15. Let U = upper face on dice of player A and V = upper face on dice of player B.

$$\text{Then } P(U = i) = \frac{i}{21}, \quad i = 1, 2, \dots, 6 \text{ and } P(V = j) = \frac{j^2}{91}, \quad j = 1, 2, \dots, 6.$$

$X = |U - V|$. So X can take values 0, 1, ..., 5.

$$P(X = 0) = \sum_{i=1}^6 P(U = i, V = i) = \frac{441}{1911}.$$

$$P(X = 1) = \sum_{i=1}^5 P(U = i, V = i+1) + \sum_{i=1}^5 P(U = i+1, V = i) = \frac{630}{1911}.$$

$$P(X = 2) = \sum_{i=1}^4 P(U = i, V = i+2) + \sum_{i=1}^4 P(U = i+2, V = i) = \frac{420}{1911}.$$

$$P(X = 3) = \sum_{i=1}^3 P(U = i, V = i+3) + \sum_{i=1}^3 P(U = i+3, V = i) = \frac{252}{1911}.$$

$$P(X = 4) = \sum_{i=1}^2 P(U = i, V = i+4) + \sum_{i=1}^2 P(U = i+4, V = i) = \frac{126}{1911}.$$

$$P(X = 5) = P(U = 1, V = 6) + P(U = 6, V = 1) = \frac{42}{1911}.$$