Probability and Stochastic Process

(MA20106)

Assignment Solution - Probability.

1. $P(A) = 0.5$, $P(B) = 0.4$, $P(A \cap B^c) = 0.2$

$P(B^c \mid A \cup B) = ?$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= P(A \cap B^c) + P(B) = 0.6.$$

Now, $$P(B^c \mid A \cup B) = \frac{P(B^c \cap (A \cup B))}{P(A \cup B)}$$

$$= \frac{P(B^c \cap A)}{P(A \cup B)} = \frac{0.2}{0.6} = \frac{1}{3} \text{ Ans.}$$

---

2. Let X_i = Number of defective components in Box i, $i = 1, 2$.

$X_i \sim Bin(2, \frac{1}{4})$

P[Exactly one box contains exactly one defective component.] = ?

$$P(X_1 = 1, X_2 = 0 \text{ or } 2) + P(X_2 = 1, X_1 = 0 \text{ or } 2)$$

$= 2 \cdot P(X_1 = 1, X_2 = 0 \text{ or } 2)$ $\{\because X_1 \,\&\, X_2$ are identical$\}$.

$= 2 \cdot P(X_1 = 1) \cdot P(X_2 = 0 \text{ or } 2)$ $\{\because X_1 \,\&\, X_2$ are independent.$\}$.

$= 2 \cdot P(X_1 = 1) \{P(X_2 = 0) + P(X_2 = 2)\}$.

$$= 2 \binom{2}{1} \left(\frac{1}{4}\right) \left(\frac{3}{4}\right) \cdot \left\{ \binom{2}{0} \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^2 + \binom{2}{2} \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^0 \right\}$$

$$= \frac{15}{32} \text{ Ans.}$$

3. Each cashew nut has prob $\frac{1}{1000}$ to be present in a biscuit.
Let X = No. of cashew nuts in a biscuit.

$\Rightarrow X \sim Bin\left(2000, \frac{1}{1000}\right)$.

Since n is very large, and $p \to 0$, $np = 2000 \times \frac{1}{1000} = 2$ (finite).

$\Rightarrow X \sim PP(2)$.

$$\text{Prob.}(X=0) = \frac{e^{-2}\, 2^{0}}{0!} = e^{-2} = 0.13533 \text{ Ans.}$$

---

4. Prob (system works) = Prob(at least one of the components works.).

$$= 1 - (1-0.8)^n$$

$$= 1 - (0.2)^n$$

Now:

$$0.97 \le 1 - (0.2)^n$$

$$\Rightarrow (0.2)^n \le 0.03$$

$\Rightarrow$ minimum value of n is 3, which satisfy this. Ans

---

5. Prob. (at least one of n components functions) $= \frac{63}{64}$

$\Rightarrow 1 - P(\text{no component is functioning}) = \frac{63}{64}$

$$\Rightarrow 1 - \left(1 - \frac{3}{4}\right)^n = \frac{63}{64}$$

$$\Rightarrow \left(\frac{1}{4}\right)^n = \frac{1}{64}$$

$$\Rightarrow \boxed{n=3} \text{ Ans}$$

6. Let E_i = Event that 6 comes at the ith roll of a fair die. $i=1,2,--$

$\Rightarrow P(E_i) = \frac{1}{6}$, $P(E_i^c) = \frac{5}{6}$, $i=1,2,\ldots$

Probability that S wins is given by.

$P[(E_1^c \cap E_2^c \cap E_3^c \cap E_4) \cup (\cap_{i=1}^{7} E_i^c \cap E_8) \cup (\cap_{i=1}^{11} E_i^c \cap E_{12}) \cup \ldots]$

$= P[(E_1^c \cap E_2^c \cap E_3^c \cap E_4)] + P(\cap_{i=1}^{7} E_i^c \cap E_8) + P(\cap_{i=1}^{11} E_i^c \cap E_{12}) + \ldots$

$= \left(\prod_{i=1}^{3} P(E_i^c)\right) \cdot P(E_4) + \left(\prod_{i=1}^{7} P(E_i^c)\right) \cdot P(E_8) + \ldots$

$= \left(\frac{5}{6}\right)^3 \cdot \frac{1}{6} + \left(\frac{5}{6}\right)^7 \cdot \frac{1}{6} + \left(\frac{5}{6}\right)^{11} \frac{1}{6} + \ldots$

$= \left(\frac{5}{6}\right)^3 \cdot \frac{1}{6} \left\{1 + \left(\frac{5}{6}\right)^4 + \left(\frac{5}{6}\right)^8 + \ldots\right\}. = \underline{\frac{125}{671}}$ Ans.

---

7. X = No. of white balls transferred to U_2.

pmf of X is, $P(X=i) = \dfrac{\binom{4}{i}\binom{4}{4-i}}{\binom{8}{4}}$, $i=0,1,2,3,4$.

P(Ball drawn from U_2 is white)

$= \sum_{i=0}^{4} P(\text{Ball drawn from } U_2 \text{ is white} / X=i) \cdot P(X=i)$

$= \sum_{i=0}^{4} \frac{i}{4} \times \dfrac{\binom{4}{i}\binom{4}{4-i}}{\binom{8}{4}}$

$= \underline{\frac{1}{2}}$ Ans.

8. we have

$P(E|F) + P(F|E) = 1$, $P(E \cap F) = \frac{2}{9}$, $P(F) < P(E)$.

$P(E) = ?$

let $P(E) = a$, $P(F) = b$, so $b < a$.

we have $P(E|F) = P(E) = a$

$P(F|E) = P(F) = b$ $\because$ E & F are ind. events.

$P(E \cap F) = P(E) \cdot P(F) = a \cdot b$.

$\Rightarrow a + b = 1$, $ab = \frac{2}{9}$

Now $a - b = \sqrt{(a+b)^2 - 4ab} = \sqrt{1 - \frac{8}{9}} = \frac{1}{3}$

$\Rightarrow a = \frac{2}{3}$, $b = \frac{1}{3}$ $\Rightarrow$ $\boxed{P(E) = \frac{2}{3}}$ Ans.

---

9. X = No. of children in a randomly chosen family.

$P(M \geq 1, F \geq 1) = 1 - P(M = 0 \text{ or } F = 0)$

$= 1 - [P(M=0) + P(F=0)]$

$= 1 - 2 \cdot P(M=0)$

$= 1 - 2 \sum_{k=1}^{\infty} P(M=0 / X=k) \cdot P(X=k)$

$= 1 - 2 \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k \cdot (0.5)^k$

$= 1 - 2 \sum_{k=1}^{\infty} \left(\frac{1}{4}\right)^k$

$= \frac{1}{3}$ Ans.

(5)

10. X = No. of heads observed.

Y = No. appread on the upper face of the die.

$$P(X=x) = \binom{4}{x}\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{4-x}, \quad x=0,1,2,3,4.$$

$$P(Y=y) = \frac{1}{6}, \quad y=1,2,3,4,5,6.$$

$$P(X>Y) = \sum_{y=1}^{6} P(X>Y \mid Y=y)\cdot P(Y=y)$$

$$= \sum_{y=1}^{3} P(X>y)\cdot P(Y=y)$$

$$= \frac{1}{6}\left[P(X>1) + P(X>2) + P(X>3)\right].$$

$$= \frac{1}{6}\left[P(X=2) + 2\cdot P(X=3) + 3\cdot P(X=4)\right]. = \underline{\frac{17}{96}} \text{ Ans.}$$

---

11. $E \cap F = \phi$, $P(E \cup F) = 1$, $P(E \cap G) = \frac{1}{4}$, $P(G) = \frac{7}{12}$

$P(F \cap G) = ?$

we have

$$P(E \cap G) + P(F \cap G) = P(G)$$

$$\Rightarrow P(F \cap G) = P(G) - P(E \cap G)$$

$$= \underline{\frac{1}{3}} \text{ Ans.}$$

12. Let P_i = Prob that U_i is selected $\Rightarrow P_i = \frac{1}{5}$, $\forall i = 1, 2 \ldots 5$.

$$\text{Prob}(R \mid U_i) = \frac{5}{8}, \quad i = 1, 2, 3, 4$$

$$\text{Prob}(R \mid U_5) = \frac{4}{8} = \frac{1}{2}.$$

$$\text{Prob}(U_5 \mid R) = \frac{P(R \mid U_5) \cdot P_5}{\sum_{i=1}^{5} P(R \mid U_i) \cdot P_i}$$

$$= \frac{\cancel{4} \times \frac{1}{2} \times \frac{1}{5}}{\left(4 \times \frac{5}{8} + \frac{1}{2}\right) \times \frac{1}{5}} = \underline{\frac{1}{6}} \text{ Ans.}$$

---

13. $E \cap F = \phi \Rightarrow E \cap F \cap G = \phi$

$p = P(E) + P(F)$, $q = P(G)$.

$$P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(E \cap F) - P(F \cap G) - P(E \cap G) + P(E \cap F \cap G).$$

$$= p + q - 0 - P(F) \cdot P(G) - P(E) \cdot P(G) + 0$$

$$= \underline{p + q - pq} \text{ Ans}$$

---

14. $P(A|B) = 0.3$, $P(A|B^c) = 0.4$

$P(B|A) = ?$

$P(B^c|A^c) = ?$

we have

$$P(A|B) = \frac{P(A\cap B)}{P(B)} \Rightarrow P(A\cap B) = 0.3\,P(B). \quad \text{---} (1)$$

and

$$P(A|B^c) = \frac{P(A\cap B^c)}{P(B^c)} \Rightarrow P(A\cap B^c) = 0.4\,P(B^c)$$

$$\Rightarrow P(A) - P(A\cap B) = 0.4\,(1 - P(B))$$

$$\Rightarrow P(A) = 0.4 - 0.1\,P(B). \quad \text{---} (2)$$

Now

$$P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{0.3\,P(B)}{0.4 - 0.1\,P(B)} = \underline{\frac{3\,P(B)}{4 - P(B)}}$$

$$P(B^c|A^c) = \frac{P(A^c\cap B^c)}{P(A^c)} = \frac{P[(A\cup B)^c]}{P(A^c)} = \frac{1 - [P(A) + P(B) - P(A\cap B)]}{1 - P(A)}$$

using (1) & (2).

$$= \underline{\frac{6 - 6\,P(B)}{6 + P(B)}}$$

Given that

$$\frac{1}{4} \le P(B|A) \le \frac{1}{3} \quad \text{and} \quad \frac{1}{4} \le P(B^c|A^c) \le \frac{9}{16}$$

$$\Rightarrow \frac{1}{4} \le \frac{3\,P(B)}{4 - P(B)} \le \frac{1}{3} \quad \text{and} \quad \frac{1}{4} \le \frac{6 - 6\,P(B)}{6 + P(B)} \le \frac{9}{16}$$

$$\Rightarrow \frac{4}{13} \le P(B) \le \frac{2}{5} \quad \text{and} \quad \frac{2}{5} \le P(B) \le \frac{18}{25}$$

$$\Rightarrow \boxed{P(B) = \frac{2}{5}}$$

15. Let A_n denotes the event that 2 appears on the nth trial.
& B_n denotes the event that 1, 4 or 6 appears on the nth trial.

$P(A_n) = \frac{1}{6}$, $P(B_n) = \frac{3}{6} = \frac{1}{2}$

Prob that 2 appears before 3 or 5

$$= P(A_1) + P(B_1 \cap A_2) + P(B_1 \cap B_2 \cap A_3) + \cdots$$
$$= P(A_1) + P(B_1)P(A_2) + P(B_1) \cdot P(B_2) P(A_3) + \cdots$$
$$= \frac{1}{6} + \frac{1}{2} \times \frac{1}{6} + \left(\frac{1}{2}\right)^2 \times \frac{1}{6} + \cdots$$
$$= \frac{1}{6}\left\{1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots\right\} = \frac{1}{3} \text{ Ans.}$$

---

16. Let i and j elements have been selected from B to form nonempty sets P and Q, resp.
Total no. of ways to do so is
$$\sum_{i=1}^{n} \sum_{j=1}^{n} \binom{n}{i}\binom{n}{j}$$
Now, we want that P and Q do not have any common element. So, we select i elements from n and then j elements from the remaining $(n-i)$.
$i \in \{1, 2, \ldots, n-1\}$ and $j \in \{1, 2, \ldots n-i\}$.
So, the no. of favourable cases is
$$\sum_{i=1}^{n-1} \sum_{j=1}^{n-i} \binom{n}{i}\binom{n-i}{j}$$

and the req. prob

$$= \frac{\sum_{i=1}^{n-1} \sum_{j=1}^{n-i} \binom{n}{i}\binom{n-i}{j}}{\sum_{i=1}^{n} \sum_{j=1}^{n} \binom{n}{i}\binom{n}{j}} \text{ Ans}$$

17. 9.

(a) Let C_i be the event that coin i is selected, $i=1,2$.

$P(H/C_1) = \frac{1}{3}$, $P(H/C_2) = \frac{1}{2}$

$P(C_1) = P(C_2) = \frac{1}{2}$

$P(C_2/H) = ?$

$$P(C_2/H) = \frac{P(H/C_2) \cdot P(C_2)}{P(H/C_1)\cdot P(C_1) + P(H/C_2)\cdot P(C_1)} = \underline{\frac{3}{5}}$$

(b) Let D_0 : The weather is dry today.

D_n : " " will be dry n days later.

$P_n = P(D_n / D_0)$.

Given that $P_1 = P(D_1/D_0) = p$

$P(D_n / D_{n-1}) = p$, $P(D_n / D_{n-1}^c) = 1-p$, $\forall n \geq 1$.

<u>Now</u>

$$
\begin{aligned}
P_n &= P(D_n / D_0) \\
&= P(D_n \cap D_{n-1} / D_0) + P(D_n \cap D_{n-1}^c / D_0) \\
&= P(D_n / D_{n-1}) \cdot P(D_{n-1}/D_0) + P(D_n / D_{n-1}^c)\cdot P(D_{n-1}^c / D_0). \\
&= p P_{n-1} + (1-p)(1-P_{n-1}) \\
&= \underline{(2p-1) P_{n-1} + (1-p), \quad n\geq 1}
\end{aligned}
$$

Setting $p = \frac{3}{4}$, we obtain

$$P_{50} = \frac{3}{2^{51}} + \frac{1}{2^{50}} + \frac{1}{2^{49}} + \cdots + \frac{1}{2^2}$$

$$P_{50} = \underline{\frac{1+2^{50}}{2^{51}}}$$

18. Let events Y and D denote, resp, that the test is positive and the dease is present.

$P(Y|D) = 0.99$, $P(Y^c|D) = 0.01$, $P(Y|D^c) = 0.01$.

$P(Y^c|D^c) = 0.99$, $P(D) = 0.01$, $P(D^c) = 0.99$.

$P(D/Y) = ?$

$$P(D/Y) = \frac{P(Y|D).P(D)}{P(Y|D).P(D) + P(Y|D^c).P(D^c)}$$

$$= \underline{0.5}$$

---

19. Let E_k : The event that the urn U_k is selected.

$P(E_k) \propto k+1$

$\Rightarrow P(E_k) = c.(k+1)$

Now $\sum_{k=1}^{n} P(E_k) = 1$

$\Rightarrow c \sum_{k=1}^{n} (k+1) = 1 \Rightarrow \boxed{c = \frac{2}{n(n+3)}}$

$\Rightarrow P(E_k) = \frac{2(k+1)}{n(n+3)}$, $k = 1, 2, \ldots, n$.

we have $P(B/E_k) = \frac{k^2}{k+k^2} = \frac{k}{1+k}$

(a) $P(B) = \sum_{k=1}^{n} P(B|E_k).P(E_k) = \underline{\frac{n+1}{n+3}}$

(b) $P(E_n|B^c) = \frac{P(B^c|E_n).P(E_n)}{P(B^c)} = \frac{\{1-P(B|E_n)\}.P(E_n)}{1-P(B)}$

$$= \underline{\frac{1}{n}}$$

20. $P(E)>0$, $P(F|E)=0.3$, $P(E\cap F^c)=0.2$.

$P(E)=?$

$$P(E) = P(E\cap F) + P(E\cap F^c) \qquad \left\{\because P(F/E) = \frac{P(E\cap F)}{P(E)}\right\}.$$

$$= 0.3 \cdot P(E) + 0.2$$

$$\Rightarrow P(E) = \frac{2}{7}$$

---

21. P(both coins show up tails) = $(1-u)(1-v)$

P(" " " " heads) = uv

Given

P(both coins show up tails) = P(both coins show up heads).

$$1-(u+v)+uv = uv$$

$$\Rightarrow \boxed{(u+v)=1}$$

---

22. Let E_1, E_2, E_3, E_4 denote the events that a student chosen at random is an Asian, American, European and African, resp

Let E be the event that a student chosen at random is a girl.

$P(E_1)=0.30$, $P(E_2)=0.40$, $P(E_3)=0.20$, $P(E_4)=0.10$

$P(E|E_1)=0.40$, $P(E|E_2)=0.50$, $P(E|E_3)=0.60$, $P(E|E_4)=0.20$.

(a) $$P(E_1|E) = \frac{P(E|E_1)\cdot P(E_1)}{\sum_{i=1}^{4} P(E|E_i)\cdot P(E_i)} = \frac{6}{23}$$

22\. (b) $P(A_i) = \frac{1}{2}$, $i = 1,2,3$.

A_3 and $A_1 \cup A_2$ are ind.

$\Rightarrow P(A_3 \cap (A_1 \cup A_2)) = P(A_3) \cdot P(A_1 \cup A_2)$. ——— (1).

L.H.S. $= P(A_3 \cap (A_1 \cup A_2))$

$= P(A_3) \cdot P(A_1) + P(A_3) \cdot P(A_2) - P(A_1 \cap A_2 \cap A_3)$.

$= \frac{1}{2} - P(A_1 \cap A_2 \cap A_3)$.

R.H.S. $= P(A_3) \cdot P(A_1 \cup A_2)$

$= P(A_3) \cdot \{P(A_1) + P(A_2) - P(A_1) \cdot P(A_2)\}$

$= \frac{3}{8}$

from (1) we have.

$\frac{1}{2} - P(A_1 \cap A_2 \cap A_3) = \frac{3}{8}$

$\Rightarrow P(A_1 \cap A_2 \cap A_3) = \frac{1}{8}$.

23\. X = no. of attemps made to destroy the target.

$P(X=x) = (0.2)^{x-1} \cdot (0.8)$, if $x = 1,2,\ldots$

$P(X=8 \,/\, \text{first five attamp fail})$

$= P(X=3)$

$= (0.2)^2 \times 0.8$

$= 0.032$.

24.

$P(E) = 0.7, \quad P(F) = 0.4\cancel{8}, \quad P(E \cap F^c) = 0.4$

$$P(F \mid E \cup F^c) = \frac{P(F \cap (E \cup F^c))}{P(E \cup F^c)}$$

$$= \frac{P(F \cap E)}{P(E \cup F^c)}$$

$$= \frac{P(E) - P(E \cap F^c)}{P(E) + P(F^c) - P(E \cap F^c)}$$

$$= \frac{0.7 - 0.4}{0.7 + 0.6 - 0.4} = \underline{\frac{1}{3}}.$$