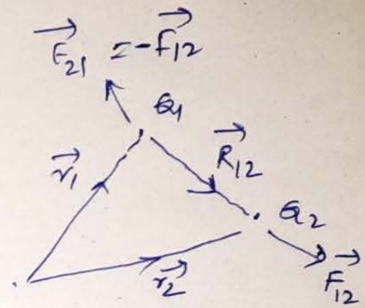
Electrostatics



- Coulomb's Law (1785) — $\vec{F}_{12} = \dfrac{Q_1 Q_2}{4\pi \varepsilon_0 R^2} \hat{a}_{R_{12}}$
- Gauss's Law.

Q: Coloumb.
F: newton
R: meters

$\vec{R}_{12} = \vec{r}_2 - \vec{r}_1$

$R = |\vec{R}_{12}|$

$\hat{a}_{R_{12}} = \dfrac{\vec{R}_{12}}{R}$

ε_0: Permittivity in free space

$= 8.854 \times 10^{-12} \simeq \dfrac{10^{-9}}{36\pi}$ F/m

$\dfrac{1}{4\pi\varepsilon_0} \simeq 9 \times 10^9$ m/F.

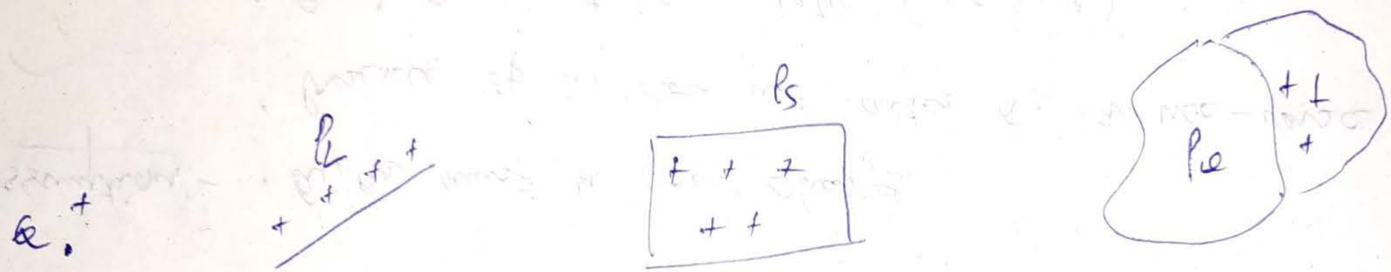
1 coloumb charge $\simeq$ 6×10^{18} electrons

Assumptions:-
- Q_1, Q_2 must be point charges.
- Presence of Q_1 does not alter Q_2 or vice-versa
- Q_1 & Q_2 must be static (at rest)

Principle of Superposition: For N charges $Q_1, Q_2, \ldots, Q_N$ located respectively at position vectors $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N$. The resultant $\vec{F}$ on charge Q located at point $\vec{r}$ is $\vec{F} = \dfrac{Q}{4\pi\varepsilon_0} \sum_{k=1}^{N} \dfrac{Q_k(\vec{r} - \vec{r}_k)}{|\vec{r} - \vec{r}_k|^3}$

Electric Field Strength $\vec{E}$: Force/charge when placed in an electric field.

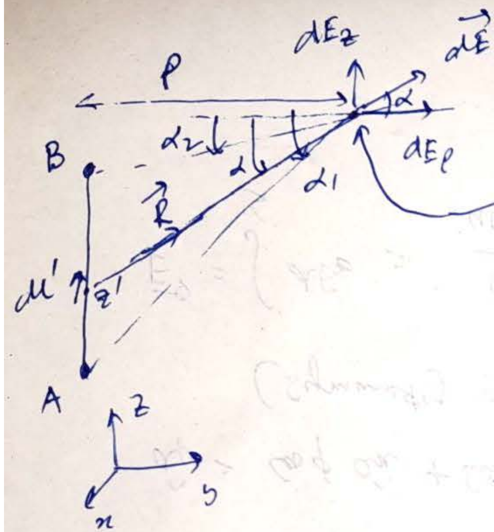
$$\vec{F} = \lim_{Q \to 0} \frac{\vec{F}}{Q} \quad (N/m) \text{ or } (V/m).$$



Point charge	Line charge	Surface charge	Volume charge.
C	C/m	C/m^2	C/m^3.

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\hat{a}_R = \int_L \frac{\rho_L \, dl}{4\pi\varepsilon_0 R^2}\hat{a}_R = \iint_S \frac{\rho_s \, ds}{4\pi\varepsilon_0 R^2}\hat{a}_R = \iiint \frac{\rho_v \, dv}{4\pi\varepsilon_0 R^2}\hat{a}_R$$

Line charge.



uniform charge density ρ_L.

$P(x, y, z)$.

dl' at $z = z'$, $\vec{R} = x\,\hat{a}_x + y\,\hat{a}_y + (z-z')\,\hat{a}_z$

$= \rho\hat{a}_\rho + (z-z')\,\hat{a}_z$

$$\frac{\vec{R}}{|\vec{R}|^3} = \frac{\rho\,\hat{a}_\rho + (z-z')\,\hat{a}_z}{\left[\rho^2 + (z-z')^2\right]^{3/2}}, \quad R = |\vec{R}| = \left[\rho^2 + (z-z')^2\right]^{1/2}$$

$= \rho \sec\alpha$

$$\vec{E} = \frac{\rho_L}{4\pi\varepsilon_0}\int_A^B \frac{\vec{R}}{|\vec{R}|^3}\,dz'$$

$z' = z - \rho\tan\alpha$

$dz' = -\rho\sec^2\alpha\,d\alpha$

$$= -\frac{\rho_L}{4\pi\varepsilon_0}\int_{\alpha_1}^{\alpha_2} \frac{\rho\sec^2\alpha\left[\cos\alpha\,\hat{a}_\rho + \sin\alpha\,\hat{a}_z\right]}{\rho^2\sec^2\alpha}\,d\alpha = \frac{\rho_L}{4\pi\varepsilon_0\rho}\Big[-(\sin\alpha_2 - \sin\alpha_1)\,\hat{a}_\rho + (\cos\alpha_2 - \cos\alpha_1)\,\hat{a}_z\Big].$$

For infinite line charge, $B(0,0,\infty)$, $A(0,0,-\infty)$, $\vec{E} = \dfrac{\rho_L}{2\pi\varepsilon_0\rho}\,\hat{a}_\rho$

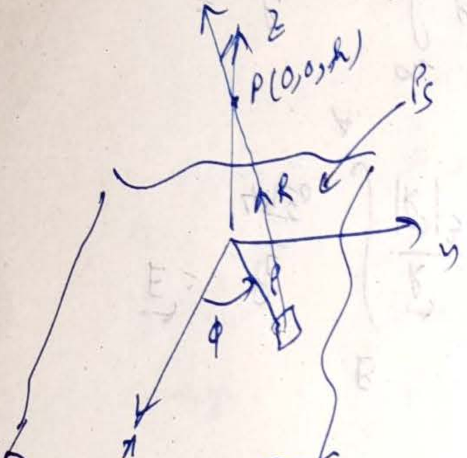
$\alpha_1 = \pi/2$, $\alpha_2 = -\pi/2$

Surface charge

infinite sheet of charge with uniform charge density ρ_s.

$$d\vec{E} = \frac{\rho_s\, ds}{4\pi\varepsilon_0 R^2}\,\hat{a}_R\,, \qquad \vec{R} = \rho(-\hat{a}_\rho) + h\,\hat{a}_z$$

$$R = |\vec{R}| = \left[\rho^2 + h^2\right]^{1/2}$$



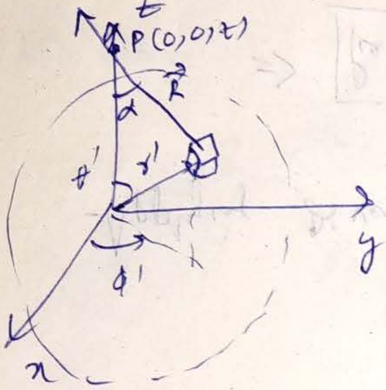
$$\hat{a}_R = \frac{\vec{R}}{R}, \qquad dQ = \rho_s\, ds = \rho_s(\rho\, d\phi)\, d\rho$$

$$d\vec{E} = \frac{\rho_s \rho\, d\phi\, d\rho\left[-\rho\hat{a}_\rho + h\hat{a}_z\right]}{4\pi\varepsilon_0\left[\rho^2+h^2\right]^{3/2}}$$

$\hat{a}_\rho = \cos\phi\,\hat{a}_x + \sin\phi\,\hat{a}_y$, Integration of $\cos\phi$ or $\sin\phi$ $0<\phi<2\pi = 0$, so $\hat{a}_\rho$ component vanishes.
(Symmetry arguement – physical picture).

$$\vec{E} = \int_S dE_z = \frac{\rho_s}{4\pi\varepsilon_0}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{\infty}\frac{h\,\rho\, d\rho\, d\phi}{\left[\rho^2+h^2\right]^{3/2}}\,\hat{a}_z = \frac{\rho_s}{2\varepsilon_0}\,\hat{a}_z$$

Volume charge



A Sphere of radius 'a' with uniform volume-charge density ρ_v (C/m³)

$$d\vec{E} = \frac{\rho_v \, dv}{4\pi\varepsilon_0 R^2}\,\hat{a}_R \qquad \hat{a}_R = \cos\alpha\,\hat{a}_z + \sin\alpha\,\hat{a}_\rho$$

$$dv = r'^2 \sin\theta'\, dr'\, d\theta'\, d\phi' \quad , \quad 0<\theta'<\pi,\ 0<\phi<2\pi,\ 0<r'<a$$

$$R^2 = z^2 + r'^2 - 2zr'\cos\theta' \quad , \quad r'^2 = z^2 + R^2 - 2zR\cos\alpha$$

$$\cos\alpha = \frac{z^2+R^2-r'^2}{2zR} \ , \quad \cos\theta' = \frac{z^2+r'^2-R^2}{2zr'} \ , \quad \sin\theta'\,d\theta' = \frac{R\,dR}{z\,r'}$$

As θ' varies from 0 to π, R varies from $(z-r')$ to $(z+r')$

$$E_z = \frac{\rho_v}{4\pi\varepsilon_0}\int_{\phi'=0}^{2\pi} d\phi' \int_{r'=0}^{a}\int_{R=z-r'}^{z+r'} r'^2\,\frac{R\,dR}{z\,r'}\,dr'\,\frac{z^2+R^2-r'^2}{2zR}\,\frac{1}{R^2}$$

$$= \left(\frac{4}{3}\pi a^3 \rho_v\right)\frac{1}{4\pi\varepsilon_0 z^2}\,\hat{a}_z$$

$E_\rho = 0$ (by symmetry arguement).

Electric flux Density ($\vec{D}$)

$\vec{D} = \varepsilon_0 \vec{E}$ (in free space) , $\vec{D} = \varepsilon \vec{E}$ (ε = permittivity)

ψ = Electric flux $= \iint_S \vec{D} \cdot \vec{ds}$

Gauss's Law :- The total electric flux ψ through any closed surface is equal to the total charge enclosed by that surface.

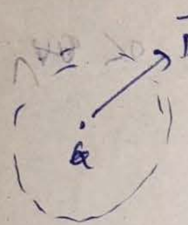
$$\psi = Q_{enc.} = \iiint_V \rho_v \, dv$$
$$= \oint_S d\psi = \oint_S \vec{D} \cdot \vec{ds}$$

Applying Divergen Th., $\oint_S \vec{D} \cdot \vec{ds} = \iiint_V (\nabla \cdot \vec{D}) \, dv$

$$\Rightarrow \boxed{\rho_v = \nabla \cdot \vec{D}}$$

It provides an easy means of finding $\vec{E}$ for symmetrical charge distⁿ. However when the charge distⁿ. is not symmetric, we must use Coulomb's Law.

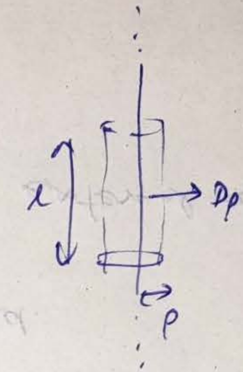
Pt. charge



$$Q = \oint_S \vec{D}\cdot\vec{ds} = D_r\, 4\pi r^2$$

$$\vec{D} = \frac{Q}{4\pi r^2}\,\hat{a}_R$$

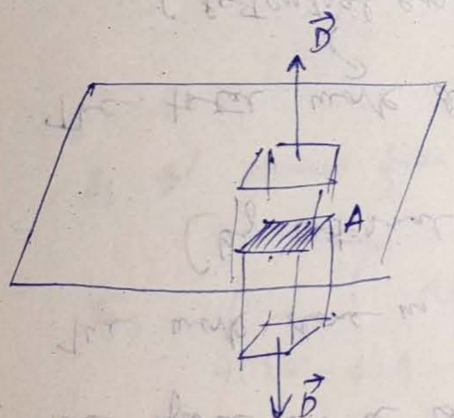
Infinite line charge



$$Q = \rho_L l = D_\rho\, 2\pi\rho l$$

$$\Rightarrow \vec{D} = \frac{\rho_L}{2\pi\rho}\,\hat{a}_\rho$$

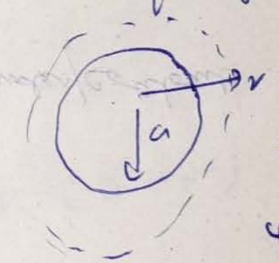
Infinite sheet



$$\rho_S \int_S ds = D_z \int_{top} ds + D_z \int_{bottom} ds$$

$$\rho_S A = D_z 2A \Rightarrow \vec{D_z} = \frac{\rho_S}{2}\,\hat{a}_z$$

Uniformly charged sphere



$r \geq a$

$$Q = \rho_v \frac{4}{3}\pi a^3$$

$$\psi = D_r\, 4\pi r^2$$

$$\vec{D} = \rho_v \frac{a^3}{3r^2}\,\hat{a}_r\,,$$

$$Q = \rho_v \frac{4}{3}\pi r^3$$

$$\psi = D_r\, 4\pi r^2$$

$$\vec{D} = \frac{r}{3}\rho_v\,\hat{a}_r$$

Electric Potential

Suppose we move a pt. charge Q from A to B in an electric field $\vec{E}$.

The force on Q is $\vec{F} = Q\vec{E}$

The work done in displacing the charge by $\vec{dl}$ is $dw = -\vec{F}\cdot\vec{dl}$

$= -Q\vec{E}\cdot\vec{dl}$

(by external agent)

The total work done, $W = -Q\int_A^B \vec{E}\cdot\vec{dl}$

(potential energy)

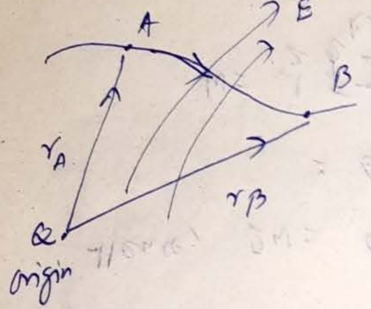
Potential difference : $V_{AB} = W/Q = -\int_A^B \vec{E}\cdot\vec{dl}$ (Joules/coulomb or volts V)

$V_{AB} < 0 \Rightarrow$ loss in PE in moving Q from A to B.

work is being done by the field.

$V_{AB} > 0 \Rightarrow$ Gain in PE $\Rightarrow$ work done by external agent.

V_{AB} is independent of the path taken



$\vec{E}$ field due to a point charge Q located at origin,

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\,\hat{a}_r$$

$$V_{AB} = -\int_{r_A}^{r_B} \frac{Q}{4\pi\varepsilon_0 r^2}\,\hat{a}_r \cdot dr\,\hat{a}_r = \frac{Q}{4\pi\varepsilon_0}\left[\underbrace{\frac{1}{r_B}}_{V_B} - \underbrace{\frac{1}{r_A}}_{V_A}\right]$$

$$V_B - V_A.$$

Absolute potentials

$$V_{BA} = -V_{AB}$$

For pt. charges, choose infinity as reference.

Potential at infinity is zero.

If $r_A \to \infty$, $V_A \to 0$, then potential at any pt. 'r' is $V = \frac{Q}{4\pi\varepsilon_0 r}$

$\left[V = \frac{Q}{4\pi\varepsilon_0 r} + C \right.$ For any other reference pt. $\left.\right]$

if charge is located at $\vec{r}'$,

$$V = -\int_{\infty}^{r} \vec{E}\cdot\vec{dl} \;;\quad \oint_L \vec{E}\cdot\vec{dl} = 0$$

$$V = \frac{Q}{4\pi\varepsilon_0 |\vec{r}-\vec{r}'|}$$

Superposition Th.

Line integral does not depend on the path of integration, $\vec{E}$: Conservative field.

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{k=1}^{n}\frac{Q_k}{|\vec{r}-\vec{r}_k|} \quad \text{(pt. charges)}$$

$$= \frac{1}{4\pi\varepsilon_0}\int_L \frac{\rho_L(\vec{r}')\,dl'}{|\vec{r}-\vec{r}'|} \quad \text{(line charge)}$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\iiint_V \frac{\rho_v(\vec{r}')\,dv'}{|\vec{r}-\vec{r}'|} \quad \text{(volume charge)}$$

$$= \frac{1}{4\pi\varepsilon_0}\iint_S \frac{\rho_S(\vec{r}')\,ds'}{|\vec{r}-\vec{r}'|} \quad \text{(Surface charge)}$$

$\oint_L \vec{E}\cdot d\vec{l} = 0$ (no net work in moving a charge along a closed path in an electrostatic field)

$= \iint_S (\nabla\times\vec{E})\cdot d\vec{s}$ - true for all 's'. $\Rightarrow$ $\boxed{\nabla\times\vec{E} = 0}$ irrotational.

potential $V = -\int \vec{E}\cdot d\vec{l}$ $\Rightarrow$ $\boxed{\vec{E} = -\nabla V}$

$\hookrightarrow$ -ve sign $\Rightarrow$ direction of $\vec{E}$ is opposite to the direction of increase in V.

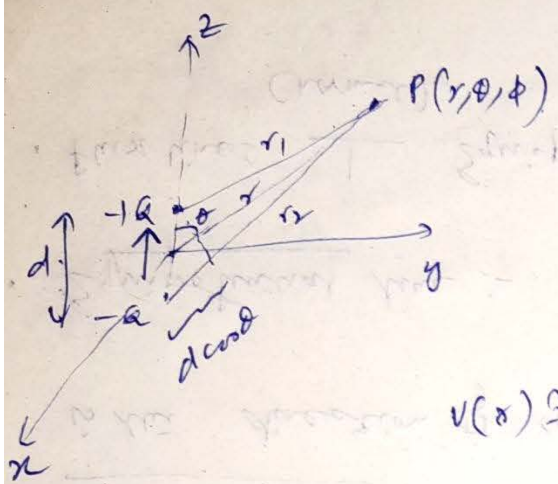
i.e. $\vec{E}$ is directed from high to low potential.

. Electric flux Line is a line drawn in such a way that its direction at any pt. is the direction of the electric field at that point.

. Equipotential line :- Any surface on which the potential is same throughout.

. flux lines $\perp$ Equipotential Surfaces (normal).

Electric Dipole



$$V = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_1} - \frac{1}{r_2}\right] = \frac{Q}{4\pi\varepsilon_0}\,\frac{r_2 - r_1}{r_1 r_2}$$

if $r >> d$, $r_2 - r_1 \simeq d\cos\theta$, $r_1 r_2 \simeq r^2$

$$V(r) \simeq \frac{Q}{4\pi\varepsilon_0}\,\frac{d\cos\theta}{r^2} = \frac{\vec{p}\cdot\hat{a}_r}{4\pi\varepsilon_0 r^2},$$

$\vec{p} = Q\vec{d}$ (Dipole moment) directed from $-Q$ to $+Q$

$\vec{d}\cdot\hat{a}_r = d\cos\theta$

$\vec{d} = d\,\hat{a}_z$

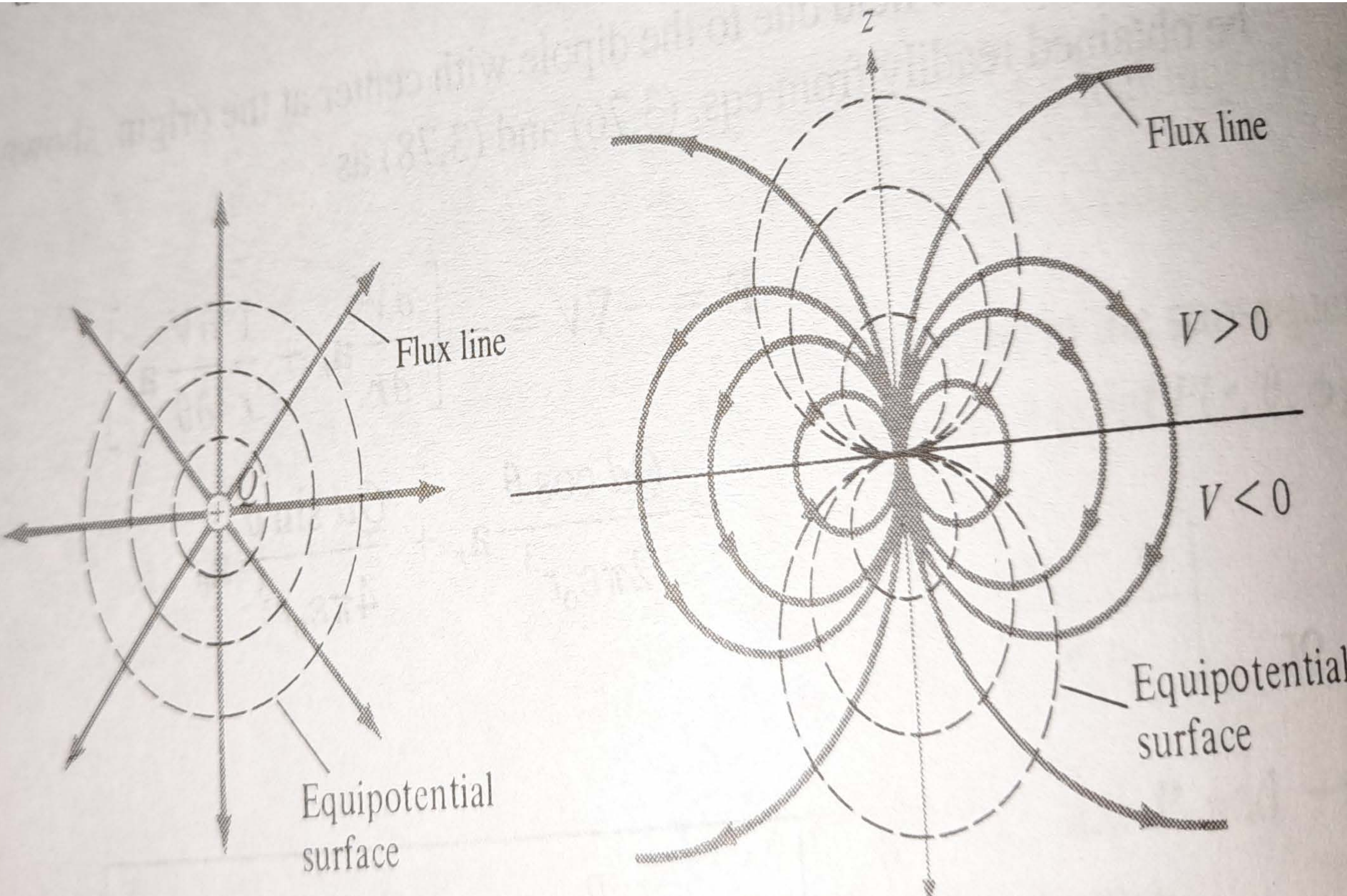
$$\vec{E} = -\nabla V = -\left[\frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial\theta}\hat{a}_\theta\right] = \frac{|\vec{p}|}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{a}_r + \sin\theta\,\hat{a}_\theta\right)$$

- A pt. charge (Monopole) ~ field varies as $\frac{1}{r^2}$, potential " $\frac{1}{r}$

A dipole ~ field varies as $\frac{1}{r^3}$, Potential as $\frac{1}{r^2}$

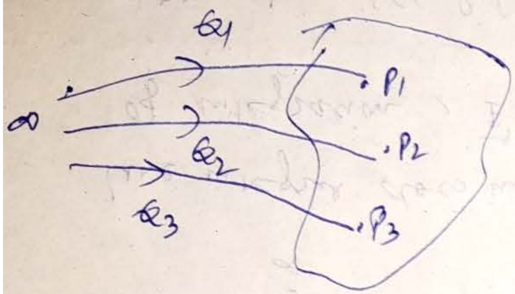
Quadrapole (2 dipoles) ~ $\frac{1}{r^4}$ (Fields)

Octupole (2 quad.) ~ $\frac{1}{r^5}$ "



Flux line
Q
Equipotential surface
z
Flux line
V > 0
V < 0
Equipotential surface

Energy Density



Suppose we wish to position 3 pt. charges Q_1, Q_2, Q_3.

No work (W_1) is reqd. to transfer Q_1, from ∞ to P_1

work reqd. . . . Q_2, ... P_2 is W_2

Q_3, ... P_3 is W_3

Total work $W = W_1 + W_2 + W_3$

$= 0 + Q_2 V_{21} + Q_3(V_{31} + V_{32})$

If charges were positioned in reverse order, i.e. first Q_3, then Q_2, finally Q_1.

$W = W_3 + W_2 + W_1$

$= 0 + Q_2(V_{23}) + Q_1(V_{12} + V_{13})$

Hence, $2W = Q_1(V_{12}+V_{13}) + Q_2(V_{21} + V_{23}) + Q_3(V_{31} + V_{32})$

$= Q_1 V_1 + Q_2 V_2 + Q_3 V_3$

$$W = \frac{1}{2} \sum_{k=1}^{n} Q_k V_k$$

(Joules)

$$W = \frac{1}{2}\int_L \rho_L V\, dl = \frac{1}{2}\iint_S \rho_S V\, ds = \frac{1}{2}\iiint_V \rho_V V\, dv.$$

$$W = \frac{1}{2} \iiint_V \rho_v V \, dv \quad , \quad \nabla \cdot \vec{D} = \rho_v$$

$$= \frac{1}{2} \iiint_V (\nabla \cdot \vec{D}) V \, dv$$

Using $\nabla \cdot V\vec{A} = \vec{A} \cdot \nabla V + V(\nabla \cdot \vec{A})$

we have, $(\nabla \cdot \vec{A}) V = \nabla \cdot V\vec{A} - \vec{A} \cdot \nabla V$

$$= \frac{1}{2} \iiint_V (\nabla \cdot V\vec{D}) dv - \frac{1}{2} \iiint_V (\vec{D} \cdot \nabla V) dv$$

$$= \frac{1}{2} \underbrace{\oint_S (V\vec{D}) \cdot \vec{ds}}_{\to 0} - \frac{1}{2} \iiint_V (\vec{D} \cdot \nabla V) dv$$

at $S \to \infty$ (using Divergence Th).

, $V \sim \frac{1}{r}$, $\vec{D} \sim \frac{1}{r^2}$ (pt. charges).

$V \sim \frac{1}{r^2}$, $\vec{D} \sim \frac{1}{r^3}$ (dipoles).

$\vdots$

$V\vec{D} \sim \frac{1}{r^3}, \frac{1}{r^5}, \dots$

$ds \sim r^2$

$$\Rightarrow W = \frac{1}{2} \iiint_V \vec{D} \cdot \vec{E} \, dv$$

$$= \frac{1}{2} \iiint_V \varepsilon_0 E^2 \, dv$$

Electrostatic energy density $(\omega_E) = \dfrac{dW}{dv} = \dfrac{1}{2} \vec{D} \cdot \vec{E} = \dfrac{1}{2} \varepsilon_0 E^2$, $W = \iiint_V \omega_E \, dv$

J/m^3