Principles of Programming Languages

Module M04: λ-Calculus: Semantics

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Source:

- λ- Calculus Overview
- Lambda calculus, Wikipedia
- Operational Semantics of Pure Functional Languages

Programming Languages

Three main characteristics of programming languages:

- **Syntax**: What is the appearance and structure of its programs?
  - *Lexical syntax* for defining the rules for basic symbols involving identifiers, literals, punctuators and operators.
  - *Concrete syntax* specifies the real representation of the programs with the help of lexical symbols like its alphabet.
  - *Abstract syntax* conveys only the vital program information.
- **Semantics**: What is the meaning of programs?
  - *Static Semantics*: Semantic rules that can be checked prior to execution (that is, which are type correct)
  - *Dynamic Semantics*: Semantic rules that describe the effect of executing programs (how to interpret the meaning of valid programs)
- **Pragmatics**: What is the usability of the language?
  - How easy is it to implement? What kinds of applications does it suit?

Semantics of Programming Languages

- The meaning of **semantics** is [Essential Meaning of semantics]
  - *the meanings of words and phrases in a particular context*,
  - *the study of the meanings of words and phrases in language*, etc.
- While semantics relates to any language including Natural Languages, here we concern ourselves with *Programming Languages* only. Hence, **semantics** is
  - **the field concerned with the rigorous mathematical study of the meaning of programming languages** [Semantics (computer science)]
- Consider various semnatics for an expression `x + y` where `x` and `y` are variables of type `T` (= `int` or `double` or `string` or `Complex`, etc.):
  - `operator+`: `int` $\times$ `int` $\rightarrow$ `int`, `x + y` $\Rightarrow$ *integer addition*
  - `operator+`: `double` $\times$ `double` $\rightarrow$ `double`, `x + y` $\Rightarrow$ *floating point addition*
  - `operator+`: `string` $\times$ `string` $\rightarrow$ `string`, `x + y` $\Rightarrow$ *concatenation of strings*
  - `operator+`: `Complex` $\times$ `Complex` $\rightarrow$ `Complex`, `x + y` $\Rightarrow$ *addition of complex numbers (user-defined in* `Complex` *UDT)*
- **Same syntax** may be associated with **multiple semantics** based on the *context* or *type*

Semantics of Programming Languages

- Semantics of `x + y` for `int` (`operator+`: `int` × `int` → `int`)
  - In **Simple English**: *Integer addition* of `x` and `y`
  - In **C++03 Standard (English)**: *The additive operators + (and −) group left-to-right. The usual arithmetic conversions are performed for operands of arithmetic or enumeration type. For addition, either both operands shall have arithmetic or enumeration type, or one operand shall be a pointer to a completely-defined effective object type and the other shall have integral or enumeration type.*
  - In **x86 Assembly**:

```
mov eax, DWORD PTR _x$[ebp] // _x$[ebp] represents x as offset _x$ from [ebp]
add eax, DWORD PTR _y$[ebp] // eax represents the resulting expression
```

  - In **recursive definition** (using `succ(x) = ++x` & `pred(x) = --x`):

```
+(x, y) = succ(+(x, pred(y)), y > 0
        = pred(+(x, succ(y)), y < 0
        = x, y = 0
```

- Similar semantics may be defined for `double` or `string` or `Complex`, etc.

Semantics of Programming Languages

- To express semantics of a syntactically correct string in a language we need a *vehicular language*. *This leads to circularity since the expression in vehicular language further needs to be interpreted*. Here are important characteristics of vehicular languages:
  - **Natural Languages**: Like *Simple English & C++03 Standard (English)*
    - Is imprecise, ambiguous and verbose
    - Is most common to put one's early thoughts about the semantics
    - Needs re-interpretation in other languages
  - **Near-machine Languages**: Like *x86 Assembly, Bytecode & Binary Code*
    - Is cryptic, machine-dependent, difficult to understand
    - Is executable
    - May need re-interpretation in other languages (Assembly to binary, Bytecode to VM)
  - ...
  - **Mathematical Languages**: Like *Recursive Definition*
    - Is precise, unambiguous, crisp, universal, easy to understand
    - Is provable, easy to reason with
    - Does not need re-interpretation
- Hence, **semantics deals with mathematical study of the meaning of languages**

Semantics of λ-Expressions

- A λ **expression** has as its meaning the λ **expression** that results after all its function applications (combinations) are carried out
- The λ **Semantics** has several advantages including:
  - λ **expression** itself works as the *Vehicular Language* for its semantics – no other language is needed
  - It is *formal*, *mathematical* and *simple*
  - It can be *untyped* as well as *typed* (Module 05)
  - It is the basis for *Denotational Semantics* (Module 08) that can define formal semantics for programming languages
  - It can also represent other semantics styles like *Operational* and *Axiomatic*
  - It is the basis for *Functional Programming*, and can be used for *Imperative Paradigm* as well
- *Evaluating a λ expression is called* **reduction**
  - The basic *reduction rule* involves *substituting* expressions for *free variables* in a manner similar to the way that the parameters in a function definition are passed as arguments in a function call

Free and Bound Variables

Free and Bound Variables

- An occurrence of a variable x is said to be *bound* when it occurs in the body M of an abstraction $\lambda x.\ M$
- We say that λx is a *binder* whose scope is M
- An occurrence of x is *free* if it appears in a position where it is *not bound* by an enclosing abstraction on x
- For example,
  - Occurrences of x in xy and $\lambda y.xy$ are *free*
  - Occurrences of x in $\lambda x.x$ and $\lambda z.\lambda x.\lambda y.x(yz)$ are *bound*
  - In $(\lambda x.x)x$ the first occurrence of x is *bound* and the second is *free*
- In a loose parallel to C functions, consider the *bound* variables as *local* (including *parameters*) and *free* variables as *global* or *non-local*

Free and Bound Variables

- In an abstraction, the variable named is referred to as the *bound* variable and the associated λ-expression is the *body* of the abstraction
- In an expression of the form:

$$\lambda v.\ e$$

  occurrences of variable v in expression e are *bound*
- All occurrences of other variables are *free*
- Example:

$$((\lambda x.\ \lambda y.\ (xy))(yw))$$

  - x, and y are *bound* in first part
  - y, and w are *free* in second part

Free and Bound Variables: Non-λ Contexts

- *Binder*: *Definite Integral*
  - $\int_0^1 x^2 dx = \int_0^1 y^2 dy = \frac{1}{3}$; $\int_0^1 A * x^2 dx = \int_0^1 A * y^2 dy = \frac{A}{3}$: x, y are *bound*, A is *free*
- *Binder*: *Definite Summation*
  - $\sum_{x=1}^{10} \frac{1}{x} = \sum_{y=1}^{10} \frac{1}{y}$; $\sum_{x=1}^{10} K * \frac{1}{x} = \sum_{y=1}^{10} K * \frac{1}{y}$: x, y are *bound*, K is *free*
- *Binder*: *Limit*
  - $\lim_{x\to\infty} e^{-x} = \lim_{y\to\infty} e^{-y}$; $\lim_{x\to\infty}(M + e^{-x}) = \lim_{y\to\infty}(M + e^{-y})$: x, y are *bound*, M is *free*
- *Binder*: *Function Scope*
  - `int N; int f(int x) { return x + N; }` $\equiv$ `int f(int y) { return y + N; }`: x, y are *bound*, N is *free*
- *Binder*: *Universal Quantifier*
  - $\forall x \in \mathbb{R}, x > 1 \Rightarrow \frac{1}{x} < 1 \equiv \forall y \in \mathbb{R}, y > 1 \Rightarrow \frac{1}{y} < 1$: x, y are *bound*
- *Bound variables* can be renamed (α reduction)

Free and Bound Variables

- **Definition**: An occurrence of a variable v in a λ-expression is called *bound* if it is within the scope of a λv; otherwise it is called *free*
  - A variable may occur both bound and free in the same λ-expression – for example, in $\lambda x.\ y\ \lambda y.\ y\ x$ the first occurrence of y is free and the other two are bound

Module M04

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Semantics of λ-Expressions

Free and Bound Variables

Substitution

Reduction

α-Reduction

β-Reduction

η-Reduction

δ-Reduction

Order of Evaluation

Normal and Applicative Order

Set of Free Variables

- **Definition**: The **set of free variables** in an expression E, denoted by $FV(E)$, is defined as follows:

  [1] $FV(c) = \Phi$ for any constant c

  [2] $FV(x) = \{x\}$ for any variable x

  [3] $FV(E1\ E2) = FV(E1) \cup FV(E2)$

  [4] $FV(\lambda x.\ E) = FV(E) - \{x\}$

- A λ-expression E with *no free variables* ($FV(E) = \Phi$) is called **closed**

Substitution

Substitution

- The notation $E[v \to E1]$ refers to the λ-expression obtained by replacing each free occurrence of the variable v in E by the λ-expression $E1$
- **Naive Rules of Substitution**

  [1] $v[v \to E_1] = E_1$ for any variable v
  [2] $x[v \to E_1] = x$ for any variable $x \neq v$
  [3] $(\lambda v.\ E)[v \to E_1] = \lambda v.\ (E[v \to E_1])$
  [4] $(E_{rator}\ \ E_{rand})[v \to E_1] = ((E_{rator}[v \to E_1])(E_{rand}[v \to E_1]))$
- Does it work?

$$(\lambda y.x)[x \to (\lambda z.zw)] = \lambda y.\lambda z.zw$$

  YES!

Unsafe Substitution: Example

- Consider:

$$(\lambda x.\ x)[x \rightarrow y] = \lambda x.\ (x[x \rightarrow y]) = \lambda x.\ y$$

  conflicts with a basic understanding that the names of bound variables (that is, parameters) do not matter.
- The identity function is the same whether we write it as $\lambda x.x$ or $\lambda z.z$ or $\lambda fred.fred$.
- If these do not behave the same way under substitution they would not behave the same way under evaluation and that seems wrong
- The mistake is that the substitution should only apply to **free** variables and **not bound** ones
- Here x is bound in the term so we should not substitute it
- That seems to give us what we want:

$$(\lambda x.x)[x \rightarrow y] = \lambda x.x$$

Unsafe Substitution: Example

- Again, the naive substitution

$$(\lambda x.\ (mul\ \ y\ \ x))[y \rightarrow x] \Rightarrow (\lambda x.\ (mul\ \ x\ \ x))$$

  is unsafe since the result represents a squaring operation whereas the original lambda expression does not
  - A substitution is **valid** or **safe** if no free variable in $E1$ becomes bound as a result of the substitution $E[v \rightarrow E1]$
  - An invalid substitution involves a **variable capture** or **name clash**
- Correct way would be:

$$(\lambda x.\ (mul\ \ y\ \ x))[y \rightarrow x] \Rightarrow (\lambda z.\ (mul\ \ y\ \ z))[y \rightarrow x]$$

$$(\lambda z.\ (mul\ \ y\ \ z))[y \rightarrow x] \Rightarrow (\lambda z.\ (mul\ \ x\ \ z))$$

- Unsafe substitutions change in semantics!

Substitution

- **Definition**: The **substitution** of an expression for a (*free*) variable in a λ-expression is denoted by $E[v \to E_1]$ and is defined as follows:
  [1] $v[v \to E_1] = E_1$ for any variable v
  [2] $x[v \to E_1] = x$ for any variable $x \neq v$
  [3] $c[v \to E_1] = c$ for any constant c
  [4] $(E_{rator}\ E_{rand})[v \to E_1] = ((E_{rator}[v \to E_1])(E_{rand}[v \to E_1]))$
  [5] $(\lambda v.\ E)[v \to E_1] = (\lambda v.\ E)$ when v is not free in E, that is, $v \notin FV(E)$
  [6] $(\lambda x.\ E)[v \to E_1] = \lambda x.\ (E[v \to E_1])$ when $x \neq v$ and $x \notin FV(E_1)$
  [7] $(\lambda x.\ E)[v \to E_1] = \lambda z.\ (E[x \to z][v \to E_1])$ when $x \neq v$ and $x \in FV(E_1)$, where $z \neq v$ and $z \notin FV(E\ E_1)$
- In part ([7]), the first substitution $E[x \to z]$ replaces the bound variable x that will capture the free x's in E_1 by an entirely new bound variable z. Then the intended substitution can be performed safely.

Substitution Example

$(\lambda y.\ (\lambda f.\ f\ x)\ y)\ [x \rightarrow f\ y]$	$\Rightarrow_\alpha$	
$\lambda z.\ ((\lambda f.\ f\ x)\ z)\ [x \rightarrow f\ y]$	$\Rightarrow$	by [7]) since $y \in FV(f\ y)$
$\lambda z.\ ((\lambda f.\ f\ x)\ [x \rightarrow f\ y]\ z[x \rightarrow f\ y])$	$\Rightarrow$	by [4])
$\lambda z.\ ((\lambda f.\ f\ x)\ [x \rightarrow f\ y]\ z)$	$\Rightarrow$	by [2])
$\lambda z.\ (\lambda g.\ (g\ x)\ [x \rightarrow f\ y])\ z$	$\Rightarrow$	by [7]) since $f \in FV(f\ y)$
$\lambda z.\ (\lambda g.\ g\ (f\ y))\ z$	$\Rightarrow$	by [4], [2], and [1]

Rules

[1] $v[v \rightarrow E_1] = E_1$ for any variable v

[2] $x[v \rightarrow E_1] = x$ for any variable $x \neq v$

[3] $c[v \rightarrow E_1] = c$ for any constant c

[4] $(E_{rator}\ E_{rand})[v \rightarrow E_1] = ((E_{rator}[v \rightarrow E_1])(E_{rand}[v \rightarrow E_1]))$

[5] $(\lambda v.\ E)[v \rightarrow E_1] = (\lambda v.\ E)$

[6] $(\lambda x.\ E)[v \rightarrow E_1] = \lambda x.\ (E[v \rightarrow E_1])$ when $x \neq v$ and $x \notin FV(E_1)$

[7] $(\lambda x.\ E)[v \rightarrow E_1] = \lambda z.\ (E[x \rightarrow z][v \rightarrow E_1])$ when $x \neq v$ and $x \in FV(E_1)$, where $z \neq v$ and $z \notin FV(E\ E_1)$

Reduction

Reduction

- A λ-expression has as its meaning the λ-expression that results after all its function applications (combinations) are carried out
- Evaluating a λ-expression is called **reduction**
- Four rules of reduction
  - α-**Reduction**: Renaming rule
  - β-**Reduction**: Substitution rule
  - η-**Reduction**: Function Equality rule
  - δ-**Reduction**: Pre-defined Constants' rule

α-Reduction

- **Definition**: If v and w are variables and E is a λ-expression,

$$\lambda v.\ E \Rightarrow_\alpha \lambda w.\ E[v \rightarrow w]$$

provided that w does not occur at all in E, which makes the substitution $E[v \rightarrow w]]$ safe

- The equivalence of expressions under α-reduction is what makes part [7] of the definition of substitution correct
- The α-reduction rule simply allows the changing of bound variables as long as there is no capture of a free variable occurrence
- The two sides of the rule can be thought of as variants of each other, both members of an equivalence class of *congruent* λ-expressions

α-Reduction

- The last example contains two α-reductions:

$$\lambda y.\ (\lambda f.\ f\ x)\ y \Rightarrow_\alpha \lambda y.\ ((\lambda f.\ f\ x)\ y)[y \rightarrow z] \Rightarrow_\alpha \lambda z.\ (\lambda f.\ f\ x)\ z$$

$$\lambda z.\ (\lambda f.\ f\ x)\ z \Rightarrow_\alpha \lambda z.\ ((\lambda f.\ f\ x)\ z)[f \rightarrow g] \Rightarrow_\alpha \lambda z.\ (\lambda g.\ g\ x)\ z$$

- Now that we have a justification of the substitution mechanism, the main simplification rule can be formally defined

β-Reduction

- **Definition**: If v is a variable and E and E_1 are λ-expressions,

$$(\lambda v.\ E)\ E_1 \Rightarrow_\beta E[v \rightarrow E_1]$$

  provided that the substitution $E[v \rightarrow E_1]$ is carried out according to the rules for a safe substitution
- This β-reduction rule describes the function application rule in which the actual parameter or argument E_1 is *passed to* the function $(\lambda v.\ E)$ by substituting the argument for the formal parameter v in the function
- The left side $(\lambda v.\ E)\ E_1$ of a β-reduction is called a **β-redex** – derived from *reduction expression* – and meaning an expression that can be **β-reduced**
- β-reduction is the main rule of evaluation in the λ-calculus
- α-reduction makes the substitutions for variables valid

β-Reduction

- The evaluation of a λ-expression consists of a series of β-reductions, possibly interspersed with α-reductions to change bound variables to avoid confusion
- Take $E \Rightarrow F$ to mean $E \Rightarrow_\beta F$ or $E \Rightarrow_\alpha F$ and let $\Rightarrow^*$ be the *reflexive* and *transitive* closure of $\Rightarrow$
- Hence:
  - For any expression E, $E \Rightarrow^* E$ and
  - For any three expressions, $(E_1 \Rightarrow^* E_2$ and $E_2 \Rightarrow^* E_3)$ implies $E_1 \Rightarrow^* E_3$
- The goal of evaluation in the λ-calculus is to reduce a λ-expression via $\Rightarrow$ until it contains no more β-redexes
- To define an *equality* relation on λ-expressions, we also allow a β-reduction rule to work backward

β-Abstraction

- **Definition**: Reversing β-reduction produces the **β-abstraction** rule,

$$E[v \rightarrow E_1] \Rightarrow_\beta (\lambda v.\ E)\ E_1$$

  and the two rules taken together give **β-conversion**, denoted by $\Leftrightarrow_\beta$
- Therefore $E \Leftrightarrow_\beta F$ if $E \Rightarrow_\beta F$ or $F \Rightarrow_\beta E$
- Take $E \Leftrightarrow F$ to mean $E \Leftrightarrow_\beta F$, $E \Rightarrow_\alpha F$ or $F \Rightarrow_\alpha E$ and let $\Leftrightarrow^*$ be the *reflexive* and *transitive* closure of $\Leftrightarrow$
- Two λ-expressions E and F are **equivalent** or **equal** if $E \Leftrightarrow^* F$
- Reductions (both α and β) are allowed to sub-expressions in a λ-expression by three rules:

  [1] $E_1 \Rightarrow E_2$ implies $E_1\ E \Rightarrow E_2\ E$
  [2] $E_1 \Rightarrow E_2$ implies $E\ E_1 \Rightarrow E\ E_2$
  [3] $E_1 \Rightarrow E_2$ implies $\lambda x.\ E_1 \Rightarrow \lambda x.\ E_2$

η-Reduction

- **Definition**: If v is a variable and E is a λ-expression (denoting a function), and v has no free occurrence in E,

$$\lambda v.\ (E\ v) \Rightarrow_\eta E$$

- Example:

$$\lambda x.\ (sqr\ x) \Rightarrow_\eta sqr$$

$$\lambda x.\ (add\ 5\ x) \Rightarrow_\eta (add\ 5)$$

  Note: $(add\ 5\ x)$ abbreviates as $(add\ 5)$

- Take $E \Leftrightarrow_\eta F$ to mean $E \Rightarrow_\eta F$ or $F \Rightarrow_\eta E$

η-Reduction

- The requirement that x should have no free occurrences in E is necessary to avoid an invalid reduction such as
$$\lambda x.\ (add\ x\ x) \Rightarrow (add\ x)$$
- This rule fails when E represents some constants; for example, if 5 is a predefined constant numeral, $\lambda x.\ (5\ x)$ and 5 are not equivalent or even related
- η-reduction, justifies an extensional view of functions; that is, *two functions are equal if they produce the same values when given the same arguments*
$$\forall x, f(x) = g(x) \Rightarrow f = g$$

η-Reduction

- **Extensionality Theorem**: If $F_1\ x \Rightarrow^* E$ and $F_2\ x \Rightarrow^* E$ where $x \notin FV(F_1\ F_2)$, then $F_1 \Leftrightarrow^* F_2$ where $\Leftrightarrow^*$ includes η-reductions.
Proof:
$$F_1 \Leftrightarrow_\eta \lambda x.\ (F_1\ x) \Leftrightarrow_\eta \lambda x.\ E \Leftrightarrow_\eta \lambda x.\ (F_2\ x) \Leftrightarrow_\eta F_2$$

- The rule is not strictly necessary for reducing λ-expressions and may cause problems in the presence of constants, but included for completeness

δ-Reduction

- **Definition**: If the λ-calculus has predefined constants (that is, if it is not pure), rules associated with those predefined values and functions are called *delta* rules:
- Example:

$$(add\ 3\ 5) \Rightarrow_\delta 8 \text{ and } (not\ \ true) \Rightarrow_\delta false$$

- Example:

$$twice = \lambda f.\ \lambda x.\ f\ (f\ \ x)$$

$$\begin{array}{l}
twice\ (\lambda n.\ (add\ \ n\ 1))\ 5 \Rightarrow_\beta \\
(\lambda f.\ \lambda x.\ (f\ \ (f\ \ x)))(\lambda n.\ (add\ \ n\ 1))\ 5 \Rightarrow_\beta \\
(\lambda x.\ ((\lambda n.\ (add\ \ n\ 1))((\lambda n.\ (add\ \ n\ 1))\ x)))\ 5 \Rightarrow_\beta \\
(\lambda n.\ (add\ \ n\ 1))\ ((\lambda n.\ (add\ \ n\ 1))\ 5) \Rightarrow_\beta \\
(add\ ((\lambda n.\ (add\ \ n\ 1))\ 5)\ 1) \Rightarrow_\beta \\
(add\ (add\ 5\ 1)\ 1) \Rightarrow_\delta 7
\end{array}$$

Order of Evaluation

Evaluation Strategies

- **Call-by-Value (CBV)**
  - **C, C++, ALGOL, Scheme**: the argument expression is evaluated, and the resulting value is bound to the corresponding variable in the function (by copying the value into a new memory region)
  - *Eager Evaluation*
- **Call-by-Reference (CBR)**
  - **C++, C#**: a function receives an implicit reference to a variable used as argument, rather than a copy of its value. **Call-by-Reference-to-const (CBRc)** available in **C#** as well (array parameter)
  - **C, C++**: CBR may be simulated in languages that use CBV by making use of references, such as pointers (**Call-by-Address or CBA**)
- **Call-by-Copy-Restore (CBCR) / Value-Result**
  - **Fortran IV, Ada**: a special case of call by reference where the provided reference is unique to the caller (**Copy-in-Copy-out**)
- **Call-by-Name (CBN)**
  - **C / C++ Macro, ALGOL 60, Simula**: the arguments to a function are not evaluated before the function is called – rather, they are substituted directly into the function body
  - *Lazy Evaluation*
  - **Call-by-Need** (**Haskell, R**): a memorized variant of CBN where, if the function argument is evaluated, that value is stored for subsequent uses
  - **Call-by-Push-Value (CBPV)**: inspired by monads, allows writing semantics for λ-calculus without writing two variants to deal with the difference between **CBN** and **CBV**

Source: *Evaluation strategy*, Wikipedia

Evaluation Strategies

```
#include <iostream>
using namespace std;

void f(int a, int b) { a++; b--; return; }              // CBV
void g(int& a, int& b) { a++; b--; return; }            // CBR
void h(int* pa, int* pb) { (*pa)++; (*pb)--; return; } // CBA
#define m_f(a, b) ( a * b )                             // CBN

int main() {
    int x = 3, y = 4, z = 5;
    f(x, y);
    cout << x << " " << y << endl;          // CBV = 3 4
    g(x, y);
    cout << x << " " << y << endl;          // CBR = 4 3
    h(&x, &y); // x = 4, y = 3
    cout << x << " " << y << endl;          // CBA = 5 2
    g(z, z);
    cout << z << endl;                      // CBR = 5
                                            // CBCR = 6 (z <- a) or 4 (z <- b)
    cout << m_f(x + 1, y + 1) << endl;      // CBN = x + 1 * y + 1 = x + y + 1 = 8
}
```

Reduction Strategies

Definition: A λ-expression is in **normal form** if it contains no β-redexes (and no δ-rules in an applied λ calculus), so that it cannot be further reduced using the β-rule or the δ-rule.

An expression in normal form has no more function applications to evaluate

Reduction Strategies

Questions:

[1] Can every λ-expression be reduced to a normal form?

[2] Is there more than one way to reduce a particular λ-expression?

[3] If there is more than one reduction strategy, does each one lead to the same normal form expression?

[4] Is there a reduction strategy that will guarantee that a normal form expression will be produced?

Reduction Strategies

1. Can every λ-expression be reduced to a normal form?

No. Consider:

$(\lambda x.\ x\ x)(\lambda x.\ x\ x) \Rightarrow$
$(\lambda x.\ x\ x)(\lambda x.\ x\ x) \Rightarrow$
$(\lambda x.\ x\ x)(\lambda x.\ x\ x) \Rightarrow$
$\ldots$

Reduction Strategies

2. Is there more than one way to reduce a particular λ-expression?

Yes. Consider:

$$(\lambda x.\ \lambda y.\ (add\ y\ ((\lambda z.\ (mul\ x\ z))\ 3)))\ 7\ 5$$

Path 1: OUTERMOST

$(\lambda x.\ \lambda y.\ (add\ y\ ((\lambda z.\ (mul\ x\ z))\ 3)))\ 7\ 5 \Rightarrow_\beta$
$(\lambda y.\ (add\ y\ ((\lambda z.\ (mul\ 7\ z))\ 3)))\ 5 \Rightarrow_\beta$
$(add\ 5\ ((\lambda z.\ (mul\ 7\ z))\ 3)) \Rightarrow_\beta (add\ 5\ (mul\ 7\ 3)) \Rightarrow_\delta (add\ 5\ 21) \Rightarrow_\delta 26$

Path 2: INNERMOST

$(\lambda x.\ \lambda y.\ (add\ y((\lambda z.\ (mul\ x\ z))\ 3)))\ 7\ 5 \Rightarrow_\beta$
$(\lambda x.\ \lambda y.\ (add\ y\ (mul\ x\ 3)))\ 7\ 5 \Rightarrow_\beta (\lambda x.\ (add\ 5\ (mul\ x\ 3)))\ 7 \Rightarrow_\beta$
$(add\ 5\ (mul\ 7\ 3)) \Rightarrow_\delta (add\ 5\ 21) \Rightarrow_\delta 26$

Path 3: MIXED

$(\lambda x.\ \lambda y.\ (add\ y((\lambda z.\ (mul\ x\ z))\ 3)))\ 7\ 5 \Rightarrow_\beta$
$(\lambda x.\ \lambda y.\ (add\ y\ (mul\ x\ 3)))\ 7\ 5 \Rightarrow_\beta (\lambda y.\ (add\ y\ (mul\ 7\ 3)))\ 5 \Rightarrow_\delta$
$(\lambda y.\ (add\ y\ 21))\ 5 \Rightarrow_\beta (add\ 5\ 21) \Rightarrow_\delta 26$

Reduction Strategies

3. **If there is more than one reduction strategy, does each one lead to the same normal form expression?**

No. Consider:

$$(\lambda y.\ 5)((\lambda x.\ x\ x)(\lambda x.\ x\ x))$$

Path 1:

$(\lambda y.\ 5)((\lambda x.\ x\ x)(\lambda x.\ x\ x)) \Rightarrow 5$

Path 2:

$(\lambda y.\ 5)((\lambda x.\ x\ x)(\lambda x.\ x\ x)) \Rightarrow$
$(\lambda y.\ 5)((\lambda x.\ x\ x)(\lambda x.\ x\ x)) \Rightarrow$
$(\lambda y.\ 5)((\lambda x.\ x\ x)(\lambda x.\ x\ x)) \Rightarrow$
...

Reduction Strategies

4. **Is there a reduction strategy that will guarantee that a normal form expression will be produced?**

Mathematician Curry proved that *if an expression has a normal form, then it can be found by leftmost reduction*.

A *normal order reduction* can have either of the following outcomes:

[1] It reaches a unique (up to α-conversion) normal form λ-expression

[2] It never terminates

Unfortunately, there is no algorithmic way to determine for an arbitrary λ-expression which of these two outcomes will occur

Reduction Strategies: Normal and Applicative Order

Two important orders of rewriting:

- **Normal Order** – rewrite the *outermost (leftmost)* occurrence of a function application.
  - *This is equivalent to call by name*
- **Applicative Order** – rewrite the *innermost (leftmost)* occurrence of a function application first
  - *This is equivalent to call by value*

Normal order evaluation always gives the same results as lazy evaluation, but may end up evaluating an expression more times

Order of Evaluation

- **Applicative Order** (*leftmost innermost*)

  $((\lambda n.\ (add\ 5\ n))\ 8) \Rightarrow$
  $((\lambda n.\ (add5\ n))\ 8) \Rightarrow //\ add5 : N \rightarrow N$ is curried
  $(add5\ 8) \Rightarrow 13$

  - Eager Evaluation
  - Call-by-Value (CBV)
  - Curried functions $(f\ x\ y\ z)$ use eager reduction

- **Normal Order** (*leftmost outermost*)

  $((\lambda n.\ (add\ 5\ n))\ 8) \Rightarrow (add\ 5\ 8) \Rightarrow 13$

  - Lazy Evaluation
  - Call-by-Name (CBN)
  - Function $f(x, y, z)$ use lazy reduction

Order of Evaluation

- Example:

$$\begin{aligned} \textit{double } x &= x + x \\ \textit{average } x\ y &= (x + y)/2 \end{aligned}$$

- Using prefix notation:

$$\begin{aligned} \textit{double } x &= \textit{plus } x\ x \\ \textit{average } x\ y &= \textit{divide } (\textit{plus } x\ y)\ 2 \end{aligned}$$

- Evaluate:

$$\textit{double } (\textit{average } 2\ 4)$$

Order of Evaluation

- Evaluate:

$$double\ (average\ 2\ 4)$$

- **Using normal order of evaluation**:

  $double\ (average\ 2\ 4) \Rightarrow plus\ (average\ 2\ 4)\ (average\ 2\ 4) \Rightarrow$
  $plus\ (divide\ (plus\ 2\ 4)\ 2)\ (average\ 2\ 4) \Rightarrow plus\ (divide\ 6\ 2)\ (average\ 2\ 4) \Rightarrow$
  $plus\ 3\ (average\ 2\ 4) \Rightarrow plus\ 3\ (divide\ (plus\ 2\ 4)\ 2) \Rightarrow plus\ 3\ (divide\ 6\ 2) \Rightarrow$
  $plus\ 3\ 3 \Rightarrow 6$

  - Notice that $(average\ 2\ 4)$ was evaluated twice ... lazy evaluation would cache the results of the first evaluation

- **Using applicative order of evaluation**:

  $double\ (average\ 2\ 4) \Rightarrow double\ (divide\ (plus\ 2\ 4)\ 2) \Rightarrow double\ (divide\ 6\ 2) \Rightarrow$
  $double\ 3 \Rightarrow plus\ 3\ 3 \Rightarrow 6$

Order of Evaluation

- **Consider**:
  $my\_if \;\; True \; x \; y = x$
  $my\_if \;\; False \; x \; y = y$
- **Evaluate**: $my\_if \;\; (less \; 3 \; 4) \; (plus \; 5 \; 5) \; (divide \; 1 \; 0)$
- **Using normal order of evaluation**:

  $my\_if \;\; (less \; 3 \; 4) \; (plus \; 5 \; 5) \; (divide \; 1 \; 0) \Rightarrow$
  $my\_if \;\; True \; (plus \; 5 \; 5) \; (divide \; 1 \; 0) \Rightarrow (plus \; 5 \; 5) \Rightarrow$
  10
- **Using applicative order of evaluation**:

  $my\_if \;\; (less \; 3 \; 4) \; (plus \; 5 \; 5) \; (divide \; 1 \; 0) \Rightarrow$
  $my\_if \;\; True \; (plus \; 5 \; 5) \; (divide \; 1 \; 0) \Rightarrow$
  $my\_if \;\; True \; 10 \; (divide \; 1 \; 0) \Rightarrow$
  $DIVIDE \;\; BY \;\; ZERO \;\; ERROR$

Order of Evaluation

```
#include <iostream>
using namespace std;

int function_f(int x, int y) { return x * y; }  // CBV

#define macro_f(x, y) (x * y)                    // CBN
#define macro_f_safe(x, y) ((x) * (y))           // CBN

int main() {
    int my_if = (3 < 4)? 5 + 5: 1 / 0;      // = 10 : Only one expr. to evaluate - lazy

    int f_f = function_f(1 + 1, 1 + 2);     // = 2 * 3 = 6 : CBV
    int m_f = macro_f(1 + 1, 1 + 2);        // = 1 + 1 * 1 + 2 = 4 : CBN
    int m_f_s = macro_f_safe(1 + 1, 1 + 2); // = (1 + 1) * (1 + 2) = 2 * 3 = 6 : CBN
}
```

Properties of Order of Evaluation: Strictness

Two important properties of evaluation order:

- *If there is any evaluation order that will terminate and that will not generate an error, normal order evaluation will terminate and will not generate an error*
- *ANY evaluation order that terminates without error will give the same result as any other evaluation order that terminates without error*

Definition: A function f is *strict* in an argument if that argument is *always evaluated* whenever an application of f is evaluated.

If a function is strict in an argument, we can safely evaluate the argument first if we need the value of applying the function.

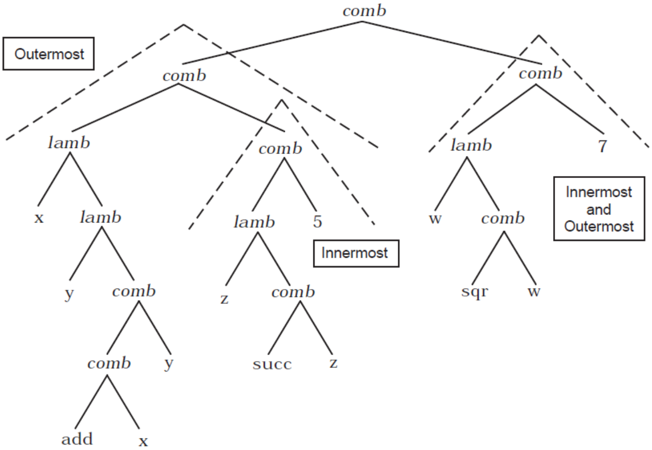
Lazy Evaluation and Strictness Analysis

- We can use lazy evaluation on an ad-hoc basis (for example, for *if*), for all arguments
- For all arguments, for some implementations of functional languages we can improve efficiency using strictness analysis
  - *plus a b* is strict in both arguments
  - *if x y z* is strict in *x*, but not in *y* and *z*
- We can do some analysis and sometimes decide if a user-defined function is strict in some of its arguments:
- Examples:
  - *double x* is strict in *x*
  - $squid\ n\ x\ =\ if\ n = 0\ then\ x + 1\ else\ x - n$ is strict in *n* and *x*
  - $crab\ n\ x\ =\ if\ n = 0\ then\ x + 1\ else\ n$ is strict in *n* but not *x*
- If a function is strict in an argument *x*, it is correct to pass *x* by value, even with normal order evaluation semantics
- It is not always decidable whether a function is strict in an argument – if we do not know, pass using lazy evaluation

Reduction Strategies: Normal and Applicative Order

- **Definition**: A **normal order reduction** always reduces the *leftmost outermost* β-redex (or δ-redex) first
- **Definition**: An **applicative order reduction** always reduces the *leftmost innermost* β-redex (or δ-redex) first
- **Definition**: For any λ-expression of the form $E = ((\lambda x.\ B)\ A)$, we say that β-redex E is outside any β-redex that occurs in B or A and that these are inside E
- A β-redex in a λ-expression is
  - *outermost* if there is no β-redex outside of it
  - *innermost* if there is no β-redex inside of it
- Use AST for detection

AST of λ-expression



β-redexes in $(((\lambda x.\ \lambda y.\ (add\ x\ y))\ ((\lambda z.\ (succ\ z))\ 5))\ ((\lambda w.\ (sqr\ w))\ 7))$

Order of Evaluation

```
((lambda (x) (+ x x)) (* 2 3))
            /       \
        lazy/        \eager
          /            \
         /              \
(+ (* 2 3) (* 2 3))   ((lambda (x) (+ x x)) 6)
        \              /
         \            /
          \          /
           \        /
            (+ 6 6)
```