



Design and Analysis  
of Algorithms I

# QuickSort

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## Analysis I: A Decomposition Principle

# Necessary Background

**Assumption:** you know and remember (finite) sample spaces, random variables, expectation, linearity of expectation. For review:

- Probability Review I (video)
- Lehman-Leighton notes (free PDF)
- Wikibook on Discrete Probability

# Average Running Time of QuickSort

QuickSort Theorem : for every input array of length  $n$ , the average running time of QuickSort (with random pivots) is  $O(n \log(n))$ .

Note : holds for every input. [no assumptions on the data ]

- recall our guiding principles !
- “average” is over random choices made by the algorithm (i.e., the pivot choices )

# Preliminaries

Fix input array  $A$  of length  $n$

Sample Space  $\Omega$  = all possible outcomes of random choices in QuickSort (i.e., pivot sequences)

Key Random Variable : for  $\sigma \in \Omega$

$C(\sigma)$  = # of comparisons between two input elements made by QuickSort (given random choices  $\sigma$ )

Lemma: running time of QuickSort dominated by comparisons.

Remaining goal :  $E[C] = O(n \log(n))$

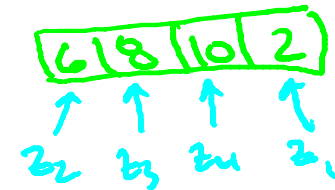
There exist constant  $c$  s.t. for all  $\sigma \in \Omega$ ,  $RT(\sigma) \leq c \cdot C(\sigma)$   
(see notes)

Tim Roughgarden

# Building Blocks

Note can't apply Master Method [random, unbalanced subproblems]

[A = final input array ]



Notation :  $z_i = i^{\text{th}}$  smallest element of A

For  $\sigma \in \Omega$ , indices  $i < j$

$X_{ij}(\sigma)$  = # of times  $z_i, z_j$  get compared in  
QuickSort with pivot sequence  $\sigma$

Fix two elements of the input array. How many times can these two elements get compared with each other during the execution of QuickSort?

☐ 1

☒ 0 or 1

☐ 0, 1, or 2

☐ Any integer between 0 and  $n - 1$

Reason : two elements compared only when one is the pivot, which is excluded from future recursive calls.

Thus : each  $X_{ij}$  is an “indicator” (i.e., 0-1) random variable

# A Decomposition Approach

So :  $C(\sigma)$  = # of comparisons between input elements

$X_{ij}(\sigma)$  = # of comparisons between  $z_i$  and  $z_j$

Thus :  $\forall \sigma, C(\sigma) = \sum_{i=1}^{n-1} \sum_{j=i+1}^n X_{ij}(\sigma)$

By Linearity of Expectation :  $E[C] = \sum_{i=1}^{n-1} \sum_{j=i+1}^n E[X_{ij}]$

*complicated* (pointing to  $E[C]$ )      *simple* (pointing to  $E[X_{ij}]$ )

Since  $E[X_{ij}] = 0 \cdot \Pr[X_{ij} = 0] + 1 \cdot \Pr[X_{ij} = 1] = \Pr[X_{ij} = 1]$

Thus :  $E[C] = \sum_{i=1}^{n-1} \sum_{j=i+1}^n \Pr[z_i, z_j \text{ get compared}] \quad (*)$

*Next video* (pointing to the phrase "get compared" in the previous block)

# A General Decomposition Principle


1. Identify random variable  $Y$  that you really care about
2. Express  $Y$  as sum of indicator random variables :

$$Y = \sum_{l=1}^m X_e$$

3. Apply Linearity of expectation :

$$E[Y] = \sum_{l=1}^m \Pr[X_e = 1]$$

“just” need to  
understand these!







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## Analysis II: The Key Insight

# Average Running Time of QuickSort

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Note : holds for every input. [no assumptions on the data ]

- recall our guiding principles !
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# The Story So Far

$C(\sigma)$  = # of comparisons between input elements  
 $X_{ij}(\sigma)$  = # of comparisons between  $\underbrace{z_i \text{ and } z_j}_{\substack{\text{i}^{\text{th}}, \text{j}^{\text{th}} \text{ smallest entries in array}}}$

Recall :  $E[C] = \sum_{i=1}^{n-1} \sum_{k=i+1}^n \underbrace{Pr[X_{ij} = 1]}_{= Pr[z_i \text{ } z_j \text{ get compared}]}$

Key Claim : for all  $i < j$ ,  $Pr[z_i, z_j \text{ get compared}] = 2/(j-i+1)$

# Proof of Key Claim

$$\Pr[z_i, z_j \text{ get compared}] = \frac{2}{(j-i+1)}$$


Fix  $z_i, z_j$  with  $i < j$

Consider the set  $z_i, z_{i+1}, \dots, z_{j-1}, z_j$

Inductively : as long as none of these are chosen as a pivot, all are passed to the same recursive call.

Consider the first among  $z_i, z_{i+1}, \dots, z_{j-1}, z_j$  that gets chosen as a pivot.

1. If  $z_i$  or  $z_j$  gets chosen first, then  $z_i$  and  $z_j$  get compared
2. If one of  $z_{i+1}, \dots, z_{j-1}$  gets chosen first then  $z_i$  and  $z_j$  are never compared [split into different recursive calls]

KEY  
INSIGHT

# Proof of Key Claim (con'd)

1.  $z_i$  or  $z_j$  gets chosen first  $\Rightarrow$  they get compared
2. one of  $z_{i+1}, \dots, z_{j-1}$  gets chosen first  $\Rightarrow z_i, z_j$  never compared

Note : Since pivots always chosen uniformly at random, each of  $z_i, z_{i+1}, \dots, z_{j-1}, z_j$  is equally likely to be the first

$$\Rightarrow \Pr[z_i, z_j \text{ get compared}] = \frac{2}{(j-i+1)}$$

Choices that lead to case (1)  $\swarrow$   
Total # of choices  $\nwarrow$

So :  $E[C] = \sum_{i=1}^{n-1} \sum_{j=1}^n \frac{2}{j-i+1}$   $\leftarrow$  [Still need to show this is  $O(n \log(n))$ ]



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## Analysis III: Final Calculations

# Average Running Time of QuickSort

QuickSort Theorem : for every input array of length  $n$ , the average running time of QuickSort (with random pivots) is  $O(n \log(n))$

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- recall our guiding principles !
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# The Story So Far

$$E[C] = 2 \sum_{i=1}^{n-1} \sum_{j=1}^n \frac{1}{j-i+1}$$

$\leq n$  choices for  $i$

$\theta(n^2)$  terms

How big can this be ?

(\*)

Note : for each fixed  $i$ , the inner sum is

$$\sum_{j=i+1}^n \frac{1}{j-i+1} = 1/2 + 1/3 + \dots$$

So  $E[C] \leq 2 \cdot n \cdot \sum_{k=2}^n \frac{1}{k}$

Claim : this is  $\leq \ln(n)$



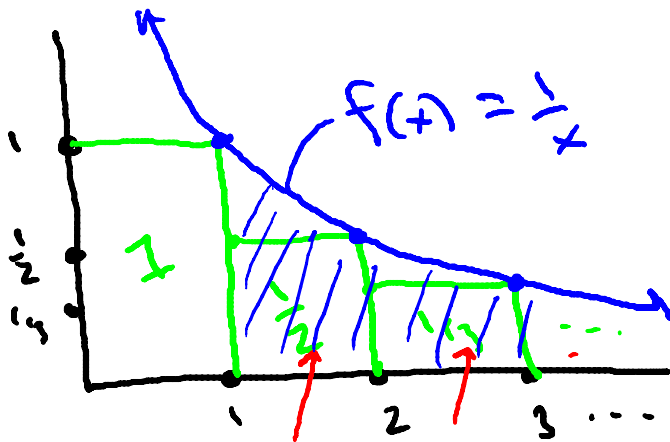
# Completing the Proof

$$E[C] \leq 2 \cdot n \cdot \sum_{k=2}^n \frac{1}{k}$$

$$\text{Claim } \sum_{k=2}^n \frac{1}{k} \leq \ln n$$

Proof of Claim

$$\text{So } \sum_{k=2}^n \frac{1}{n} \leq \int_1^n \frac{1}{x} dx$$



So :  
 $E[C] \leq 2n \ln n$

**Q.E.D.**

$$= \ln x \Big|_1^n$$

$$= \ln n - \ln 1$$

$$= \ln n$$

**Q.E.D. (CLAIM)**