Design and Analysis of Algorithms I

Asymptotic Analysis

The Gist

Motivation

Importance: Vocabulary for the design and analysis of algorithms (e.g. "big-Oh" notation).

- "Sweet spot" for high-level reasoning about algorithms.
- Coarse enough to suppress architecture/language/compiler-dependent details.
- Sharp enough to make useful comparisons between different algorithms, especially on large inputs (e.g. sorting or integer multiplication).

Asymptotic Analysis

High-level idea: Suppress constant factors and lower-order terms

too system-dependent

irrelevant for large inputs

Example: Equate $6n \log_2 n + 6$ with just $n \log n$.

Terminology: Running time is $O(n \log n)$

["big-Oh" of $n \log n$]

where $n =$ input size (e.g. length of input array).

Example: One Loop

Problem: Does array A contain the integer t? Given A (array of length n) and t (an integer).

Algorithm 1

for $i = 1$ to n **do**
  if $A[i] == t$ **then**
    Return TRUE
Return FALSE

Question: What is the running time?

A) $O(1)$ C) $O(n)$

B) $O(\log n)$ D) $O(n^2)$

Example: Two Loops

Given A, B (arrays of length n) and t (an integer). [Does A or B contain t?]

Algorithm 2

```
for i = 1 to n do
    if A[i] == t then
        Return TRUE
for i = 1 to n do
    if B[i] == t then
        Return TRUE
Return FALSE
```

Question: What is the running time?

A) $O(1)$ C) $O(n)$

B) $O(\log n)$ D) $O(n^2)$

Example: Two Nested Loops

Problem: Do arrays A, B have a number in common? Given arrays A, B of length n.

Algorithm 3

for $i = 1$ to n **do**
  for $j = 1$ to n **do**
    if $A[i] == B[j]$ **then**
      Return TRUE
Return FALSE

Question: What is the running time?

A) $O(1)$ C) $O(n)$

B) $O(\log n)$ D) $O(n^2)$

Example: Two Nested Loops (II)

Problem: Does array A have duplicate entries? Given arrays A of length n.

Algorithm 4

```
for i = 1 to n do
    for j = i+1 to n do
        if A[i] == A[j] then
            Return TRUE
Return FALSE
```

Question: What is the running time?

A) $O(1)$ C) $O(n)$

B) $O(\log n)$ D) $O(n^2)$

Asymptotic Analysis

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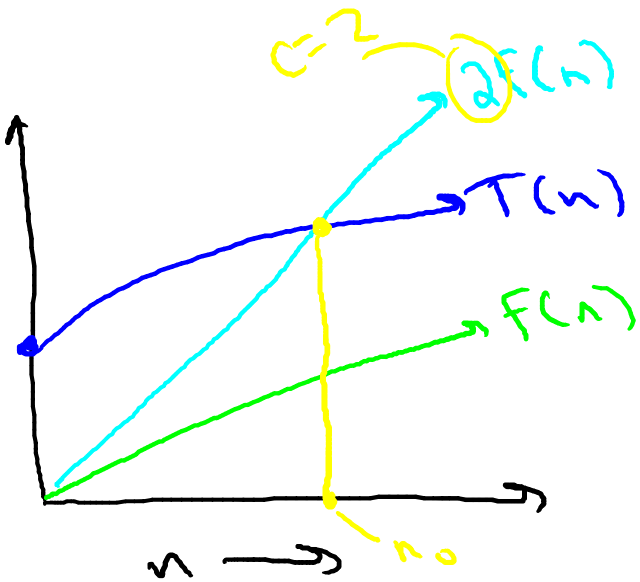
Big-Oh: Definition

Big-Oh: English Definition

Let $T(n)$ = function on $n = 1,2,3,\ldots$
[usually, the worst-case running time of an algorithm]

Q : When is $T(n) = O(f(n))$?
A : if eventually (for all sufficiently large n), $T(n)$ is bounded above by a constant multiple of $f(n)$

Big-Oh: Formal Definition



$Picture\ \ T(n) = O(f(n))$

<u>Formal Definition</u> : T(n) = O(f(n)) if and only if there exist constants $c, n_0 > 0$ such that

$$T(n) \leq c \cdot f(n)$$

For all $n \geq n_0$

<u>Warning</u> : c, n_0 cannot depend on n

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Big-Oh: Basic Examples

Example #1

Claim : if $T(n) = a_k n^k + ... + a_1 n + a_0$ then

$$T(n) = O(n^k)$$

Proof : Choose $n_0 = 1$ and $c = |a_k| + |a_{k-1}| + .. + |a_1| + |a_0|$

Need to show that $\forall n \geq 1, T(n) \leq c \cdot n^k$

We have, for every $n \geq 1,$

$$\begin{aligned} T(n) &\leq |a_k|n^k + ... + |a_1|n + |a_0| \\ &\leq |a_k|n^k + ... + |a_1|n^k + |a_0|n^k \\ &= c \cdot n^k \end{aligned}$$

Tim Roughgarden

Example #2

Claim : for every $k \geq 1,\ n^k$ is not $O(n^{k-1})$

Proof : by contradiction. Suppose $n^k = O(n^{k-1})$

Then there exist constants c, n_0 such that

$$n^k \leq c \cdot n^{k-1} \quad \forall n \geq n_0$$

But then [cancelling n^{k-1} from both sides]:

$$n \leq c \quad \forall n \geq n_0$$

Which is clearly False [contradiction].

Asymptotic Analysis

Big-Oh: Relatives (Omega & Theta)

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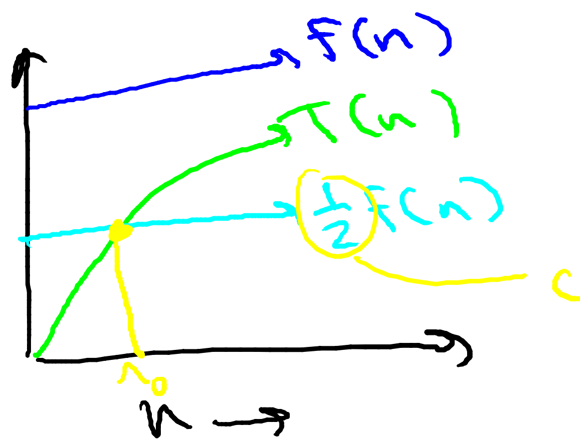
Omega Notation

<u>Definition</u> : $T(n) = \Omega(f(n))$
If and only if there exist constants c, n_0 such that

$$T(n) \geq c \cdot f(n) \quad \forall n \geq n_0 .$$

Picture



$$T(n) = \Omega(f(n))$$

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Theta Notation

<u>Definition</u> : $T(n) = \theta(f(n))$ if and only if
$T(n) = O(f(n))$ and $T(n) = \Omega(f(n))$

<u>Equivalent</u> : there exist constants c_1, c_2, n_0 such that

$$c_1 f(n) \leq T(n) \leq c_2 f(n)$$

$$\forall n \geq n_0$$

Let $T(n) = \frac{1}{2}n^2 + 3n$. Which of the following statements are true ? (Check all that apply.)

- ☐ $T(n) = O(n)$.
- ☐ $T(n) = \Omega(n)$. $[n_0 = 1, c = 1/2]$
- ☐ $T(n) = \Theta(n^2)$. $[n_0 = 1, c_1 = 1/2, c_2 = 4]$
- ☐ $T(n) = O(n^3)$. $[n_0 = 1, c = 4]$

Little-Oh Notation

<u>Definition</u> : $T(n) = o(f(n))$ if and only if for all constants c>0, there exists a constant n_0 such that

$$T(n) \leq c \cdot f(n) \quad \forall n \geq n_0$$

<u>Exercise</u> : $\forall k \geq 1, n^{k-1} = o(n^k)$

Where Does Notation Come From?

- "On the basis of the issues discussed here, I propose that members of SIGACT, and editors of compter science and mathematics journals, adopt the O, Ω, and Θ notations as defined above, unless a better alternative can be found reasonably soon".

-D. E. Knuth, "Big Omicron and Big Omega and Big Theta", SIGACT News, 1976. Reprinted in "Selected Papers on Analysis of Algorithms."

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Additional Examples

Example #1

Claim : $2^{n+10} = O(2^n)$

Proof : need to pick constants c, n_0 such that

$$(*) \quad 2^{n+10} \leq c \cdot 2^n \quad n \geq n_0$$

Note : $2^{n+10} = 2^{10} \times 2^n = (1024) \times 2^n$

So if we choose $c = 1024, n_0 = 1$ then (*) holds.

Q.E.D

Example #2

Claim : $2^{10n} \neq O(2^n)$

Proof : by contradiction. If $2^{10n} = O(2^n)$ then there exist constants $c, n_0 > 0$ such that

$$2^{10n} \leq c \cdot 2^n \quad n \geq n_0$$

But then [cancelling 2^n]

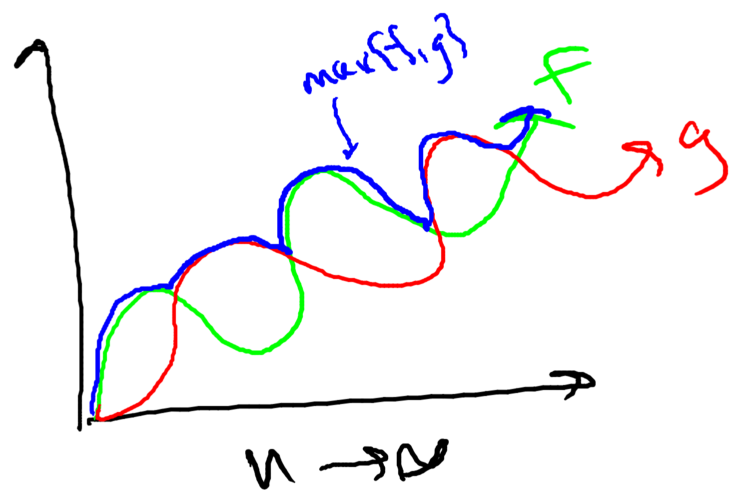
$$2^{9n} \leq c \quad \forall n \geq n_0$$

Which is certainly false. Q.E.D

Example #3

<u>Claim</u> : for every pair of (positive) functions f(n), g(n),

$$max\{f, g\} = \theta(f(n) + g(n))$$



Example #3 (continued)

Proof : $max\{f,g\} = \theta(f(n)+g(n))$

For every n, we have

$$max\{f(n),g(n)\} \leq f(n)+g(n)$$

And

$$2 * max\{f(n),g(n)\} \geq f(n)+g(n)$$

Thus $\frac{1}{2} * (f(n)+g(n)) \leq max\{f(n),g(n)\} \leq f(n)+g(n) \quad \forall n \geq 1$

$$=> max\{f,g\} = \theta(f(n)+g(n)) \quad [where \;\; n_0 = 1, c_1 = 1/2, c_2 = 1]$$

Q.E.D