

Probabilistic Reasoning

[RN2] Sections 14.1, 14.2

[RN3] Sections 14.1, 14.2

University of Waterloo
CS 486/686
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Outline

- Review probabilistic inference, independence and conditional independence
- Bayesian networks
 - What are they
 - What do they mean
 - How do we create them

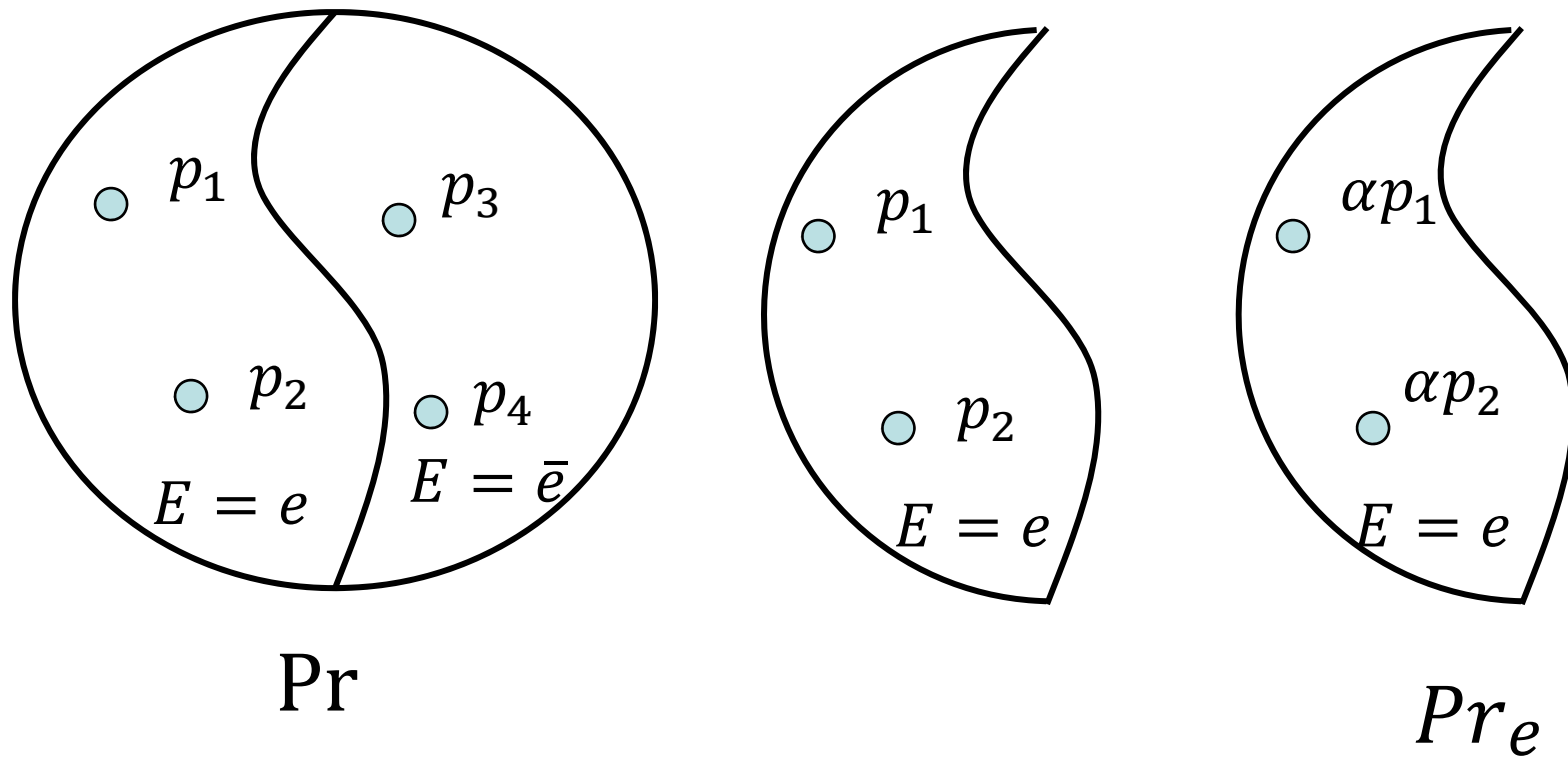
Probabilistic Inference

- By probabilistic inference, we mean
 - given a *prior* distribution Pr over variables of interest, representing degrees of belief
 - and given new evidence $E = e$ for some variable E
 - Revise your degrees of belief: *posterior* Pr_e
- How do your degrees of belief change as a result of learning $E = e$ (or more generally $E = e$, for set E)

Conditioning

- We define $Pr_e(\alpha) = \Pr(\alpha|e)$
- That is, we produce Pr_e by *conditioning* the prior distribution on the observed evidence e

Semantics of Conditioning



$$\alpha = 1/(p_1 + p_2)$$

normalizing constant

Inference: Computational Bottleneck

- Semantically/conceptually, picture is clear; but several issues must be addressed

Issue 1

- How do we specify the full joint distribution over a set of random variables X_1, X_2, \dots, X_n ?
 - **Exponential** number of possible worlds
 - e.g., if the X_i are Boolean, then 2^n numbers (or $2^n - 1$ parameters, since they sum to 1)
 - These numbers are **not robust/stable**
 - These numbers are **not natural** to assess (what is probability that "Pascal wants a cup of tea; it's not raining or snowing in Montreal; robot charge level is low; ..."?)

Issue 2

- Inference in this representation is frightfully slow
 - Must sum over exponential number of worlds to answer query $\Pr(\alpha)$ or to condition on evidence e to determine $\Pr_e(\alpha)$

Small Example: 3 Variables

sunny			~sunny		
	cold	~cold		cold	~cold
headache	0.108	0.012	headache	0.072	0.008
~headache	0.016	0.064	~headache	0.144	0.576

$$\Pr(headache) = 0.108 + 0.012 + 0.072 + 0.008 = 0.2$$

$$\begin{aligned}\Pr(headache \wedge cold | sunny) &= \Pr(headache \wedge cold \wedge sunny) / \Pr(sunny) \\ &= 0.108 / (0.108 + 0.012 + 0.016 + 0.064) = 0.54\end{aligned}$$

$$\begin{aligned}\Pr(headache \wedge cold | \sim sunny) &= \Pr(headache \wedge cold \wedge \sim sunny) / \Pr(\sim sunny) \\ &= 0.072 / (0.072 + 0.008 + 0.144 + 0.576) = 0.09\end{aligned}$$

Is there anything we can do?

- How do we avoid these two problems?
 - no solution in general
 - but in practice there is structure we can exploit
- We'll use conditional independence

Independence

- Recall that x and y are *independent* iff:

$$\Pr(x) = \Pr(x|y) \Leftrightarrow \Pr(y) = \Pr(y|x) \Leftrightarrow \Pr(xy) = \Pr(x)\Pr(y)$$

- Intuitively, learning y doesn't influence beliefs about x

- x and y are *conditionally independent given z* iff:

$$\Pr(x|z) = \Pr(x|yz) \Leftrightarrow \Pr(y|z) = \Pr(y|xz)$$

$$\Leftrightarrow \Pr(xy|z) = \Pr(x|z)\Pr(y|z) \Leftrightarrow \dots$$

- Intuitively, learning y doesn't influence your beliefs about x *if you already know z*
- e.g., learning someone's mark on 486 exam can influence the probability you assign to a specific GPA; but if you already knew the **final** 486 grade, learning the exam mark would *not* influence your GPA assessment

Variable Independence

- Two *variables* X and Y are conditionally independent given variable Z iff x, y are conditionally independent given z for all $x \in \text{Dom}(X), y \in \text{Dom}(Y), z \in \text{Dom}(Z)$
 - Also applies to sets of variables X, Y, Z
 - Also to unconditional case (X, Y independent)
- If you know the value of Z (*whatever* it is), nothing you learn about Y will influence your beliefs about X
 - these definitions differ from earlier ones (which talk about events, not variables)

What good is independence?

- Suppose (say, Boolean) variables X_1, X_2, \dots, X_n are mutually independent
 - We can specify full joint distribution using only n parameters (linear) instead of $2^n - 1$ (exponential)
- How? Simply specify $\Pr(x_1), \dots, \Pr(x_n)$
 - From this we can recover the probability of any world or any (conjunctive) query easily
 - Recall $\Pr(x, y) = \Pr(x) \Pr(y)$
and $\Pr(x|y) = \Pr(x)$ and $\Pr(y|x) = \Pr(y)$

Example

- 4 independent Boolean random vars X_1, X_2, X_3, X_4

$$\Pr(x_1) = 0.4, \Pr(x_2) = 0.2, \Pr(x_3) = 0.5, \Pr(x_4) = 0.8$$

$$\begin{aligned}\Pr(x_1, \sim x_2, x_3, x_4) &= \Pr(x_1) (1 - \Pr(x_2)) \Pr(x_3) \Pr(x_4) \\ &= (0.4)(0.8)(0.5)(0.8) \\ &= 0.128\end{aligned}$$

$$\begin{aligned}\Pr(x_1, x_2, x_3 | x_4) &= \Pr(x_1) \Pr(x_2) \Pr(x_3) \mathbf{1} \\ &= (0.4)(0.2)(0.5)(1) \\ &= 0.04\end{aligned}$$

The Value of Independence

- Complete independence reduces both *representation of joint distribution* and *inference* from $O(2^n)$ to $O(n)$!!
- **Unfortunately**, such complete mutual independence is very rare. Most realistic domains do not exhibit this property.
- **Fortunately**, most domains do exhibit a fair amount of conditional independence. We can exploit conditional independence for representation and inference as well.
- Bayesian networks do just this

An Aside on Notation

- $\Pr(X)$ for variable X (or set of variables) refers to the *(marginal) distribution* over X . $\Pr(X|Y)$ refers to family of conditional distributions over X , one for each $y \in \text{Dom}(Y)$.
- Distinguish between $\Pr(X)$ -- which is a distribution -- and $\Pr(x)$ or $\Pr(\sim x)$ (or $\Pr(x_i)$ for non-Boolean vars) -- which are numbers. Think of $\Pr(X)$ as a function that accepts any $x_i \in \text{Dom}(X)$ as an argument and returns $\Pr(x_i)$.
- Think of $\Pr(X|Y)$ as a function that accepts any x_i and y_k and returns $\Pr(x_i|y_k)$. Note that $\Pr(X|Y)$ is not a single distribution; rather it denotes the family of distributions (over X) induced by the different $y_k \in \text{Dom}(Y)$

Exploiting Conditional Independence

- Consider a story:
 - If Pascal woke up too early E , Pascal probably needs coffee C ; if Pascal needs coffee, he's likely grumpy G . If he is grumpy then it's possible that the lecture won't go smoothly L . If the lecture does not go smoothly then the students will likely be sad S .



E - Pascal woke up too early G - Pascal is grumpy S - Students are sad
 C - Pascal needs coffee L - The lecture did not go smoothly

Conditional Independence



- If you learned any of E , C , G , or L , would your assessment of $\Pr(S)$ change?
 - If any of these are seen to be true, you would increase $\Pr(s)$ and decrease $\Pr(\sim s)$.
 - So S is *not independent* of E , or C , or G , or L .
- If you knew the value of L (true or false), would learning the value of E , C , or G influence $\Pr(S)$?
 - Influence that these factors have on S is mediated by their influence on L .
 - Students aren't sad because Pascal was grumpy, they are sad because of the lecture.
 - So S is *independent* of E , C , and G , *given* L

Conditional Independence



- So S is *independent* of E , and C , and G , *given* L
- Similarly:
 - S is *independent* of E , and C , *given* G
 - G is *independent* of E , *given* C
- This means that:

$$\Pr(S|L, \{G, C, E\}) = \Pr(S|L)$$

$$\Pr(L|G, \{C, E\}) = \Pr(L|G)$$

$$\Pr(G|C, \{E\}) = \Pr(G|C)$$

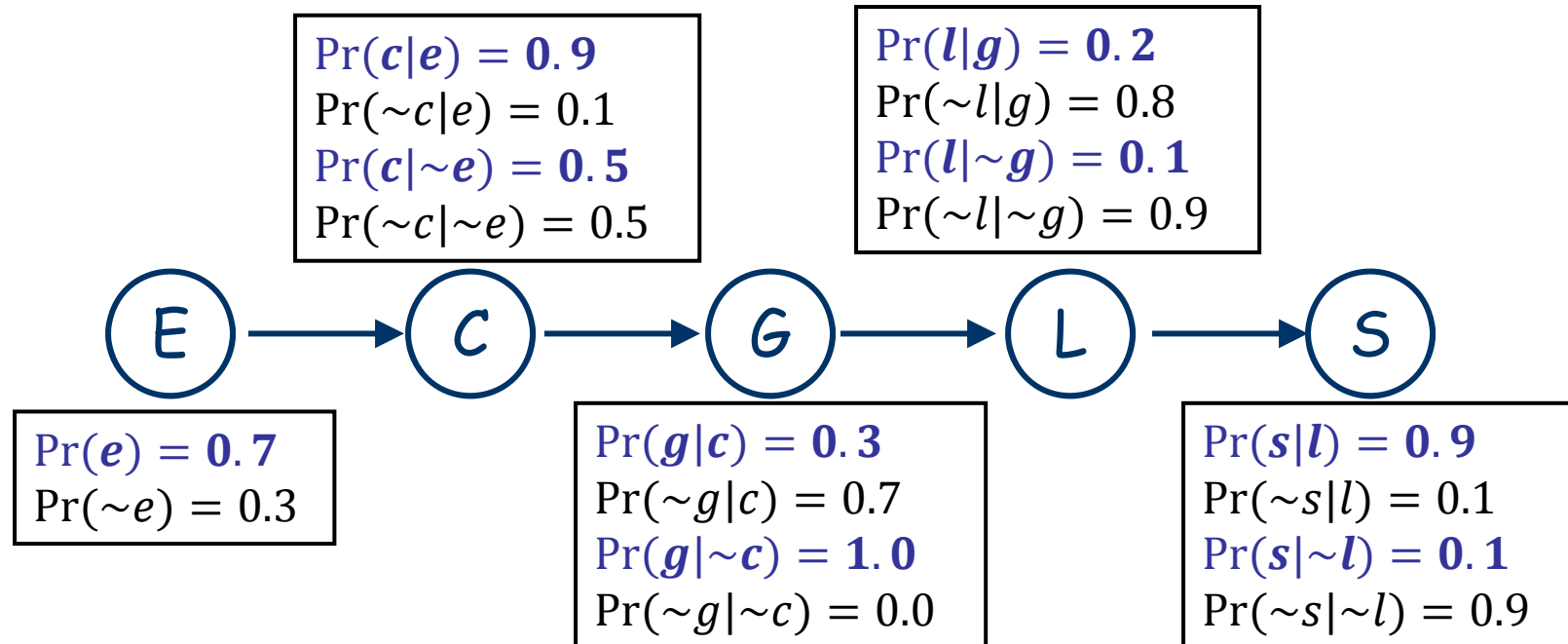
$$\Pr(C|E) \quad \text{and} \quad \Pr(E) \quad \text{don't "simplify"}$$

Conditional Independence



- By the chain rule (for any instantiation of $S \dots E$):
$$\Pr(S, L, G, C, E) = \Pr(S|L, G, C, E) \Pr(L|G, C, E) \Pr(G|C, E) \Pr(C|E) \Pr(E)$$
- By our independence assumptions:
$$\Pr(S, L, G, C, E) = \Pr(S|L) \Pr(L|G) \Pr(G|C) \Pr(C|E) \Pr(E)$$
- We can specify the full joint by specifying five *local conditional distributions*:
$$\Pr(S|L); \Pr(L|G); \Pr(G|C); \Pr(C|E); \text{ and } \Pr(E)$$

Example Quantification



- Specifying the joint requires only 9 parameters (if we note that half of these are "1 minus" the others), instead of 31 for explicit representation
 - linear in number of vars instead of exponential!
 - linear generally if dependence has a chain structure

Inference is Easy



- Want to know $\Pr(g)$? Use sum out rule:

$$\begin{aligned} P(g) &= \sum_{c_i \in \text{Dom}(C)} \Pr(g \mid c_i) \Pr(c_i) \\ &= \sum_{c_i \in \text{Dom}(C)} \Pr(g \mid c_i) \sum_{e_i \in \text{Dom}(E)} \Pr(c_i \mid e_i) \Pr(e_i) \end{aligned}$$

These are all terms specified in our local distributions!

Inference is Easy



- Computing $\Pr(g)$ in more concrete terms:

$$\Pr(c) = \Pr(c|e) \Pr(e) + \Pr(c|\sim e) \Pr(\sim e)$$

$$= 0.8 * 0.7 + 0.5 * 0.3 = 0.78$$

$$\Pr(\sim c) = \Pr(\sim c|e) \Pr(e) + \Pr(\sim c|\sim e) \Pr(\sim e) = 0.22$$

$$\Pr(\sim c) = 1 - \Pr(c), \text{ as well}$$

$$\Pr(g) = \Pr(g|c) \Pr(c) + \Pr(g|\sim c) \Pr(\sim c)$$

$$= 0.3 * 0.78 + 1.0 * 0.22 = 0.454$$

$$\Pr(\sim g) = 1 - \Pr(g) = 0.546$$

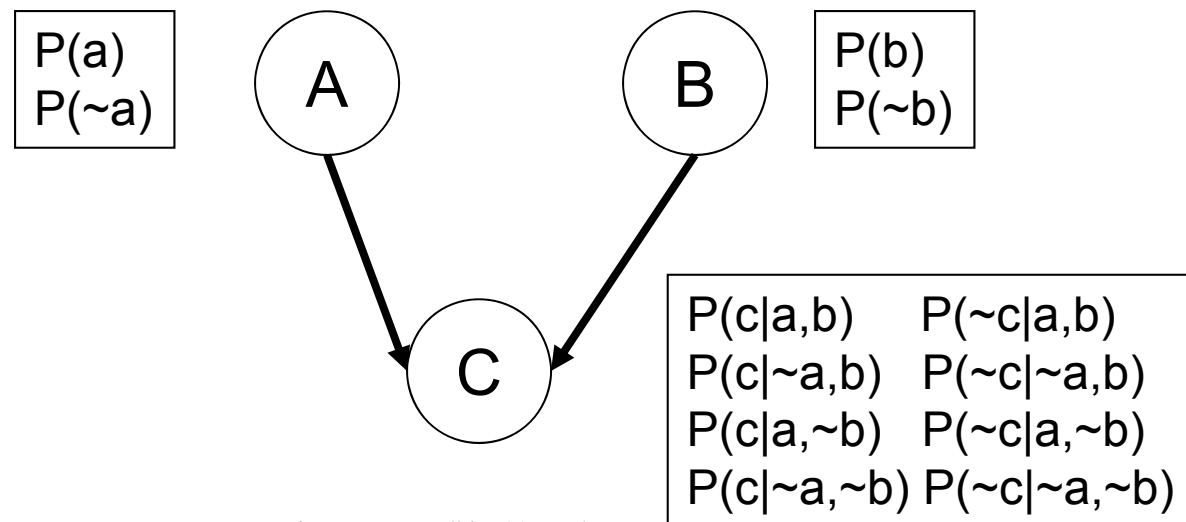
Bayesian Networks

- The structure above is a *Bayesian network*.
 - *Graphical representation* of the direct dependencies over a set of variables + a set of *conditional probability tables (CPTs)* quantifying the strength of those influences.
- Bayes nets generalize the above ideas in very interesting ways, leading to effective means of representation and inference under uncertainty.

Bayesian Networks

aka belief networks, probabilistic networks

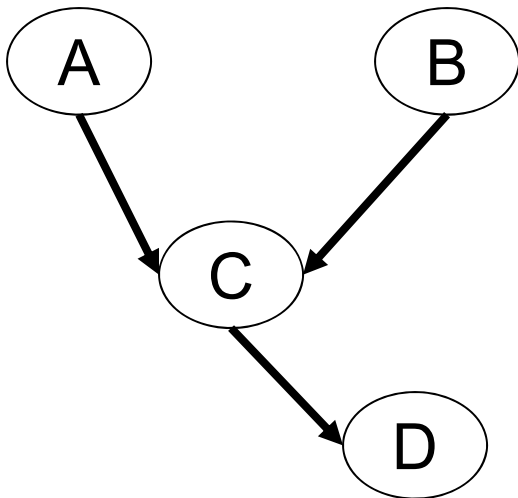
- A BN over variables $\{X_1, X_2, \dots, X_n\}$ consists of:
 - a DAG whose nodes are the variables
 - a set of CPTs $(\Pr(X_i \mid \text{Parents}(X_i)))$ for each X_i



Bayesian Networks

aka belief networks, probabilistic networks

- Key notions
 - **parents** of a node: $Par(X_i)$
 - **children** of node
 - **descendants** of a node
 - **ancestors** of a node
 - **family**: set of nodes consisting of X_i and its parents
 - CPTs are defined over families in the BN



$$Parents(C) = \{A, B\}$$

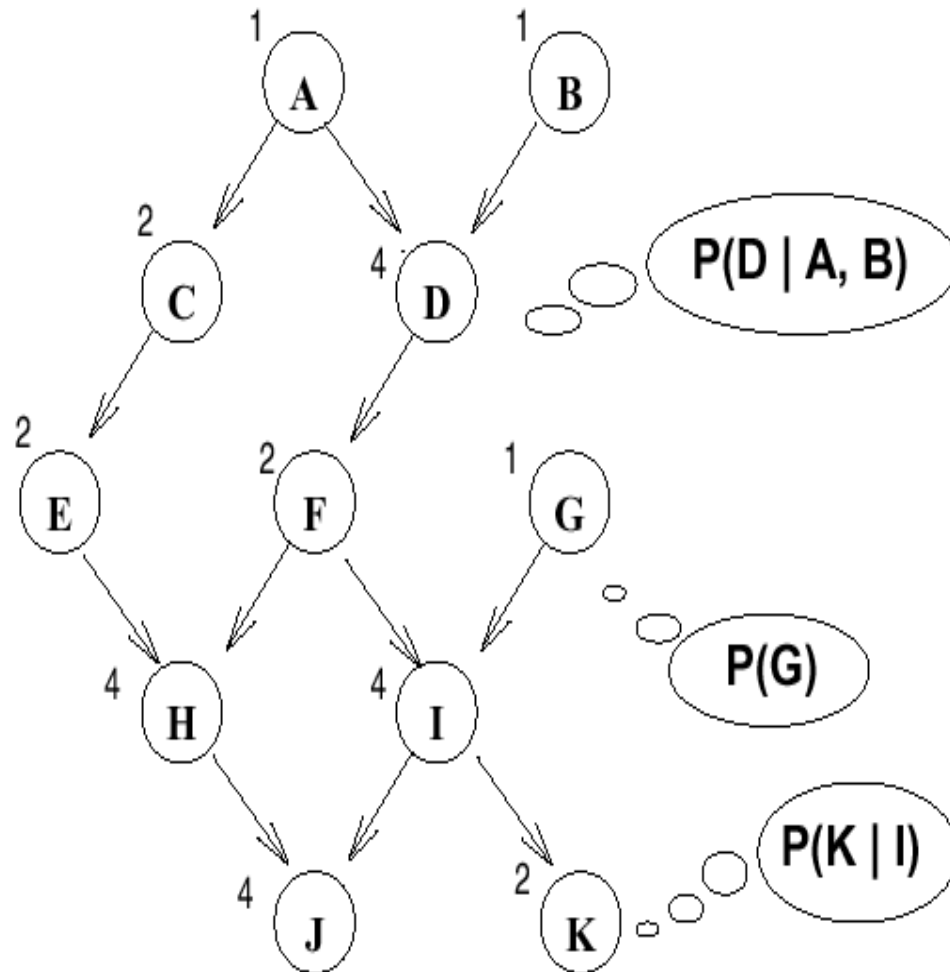
$$Children(A) = \{C\}$$

$$Descendants(B) = \{C, D\}$$

$$Ancestors\{D\} = \{A, B, C\}$$

$$Family\{C\} = \{C, A, B\}$$

An Example Bayes Net



- A few CPTs are "shown"
- Explicit joint requires $2^{11} - 1 = 2047$ params
- BN requires only 27 params (the number of entries for each CPT is listed)

Semantics of a Bayes Net

- The structure of the BN means: every X_i is *conditionally independent of all of its non-descendants given its parents*:

$$\Pr(X_i \mid S \cup \text{Par}(X_i)) = \Pr(X_i \mid \text{Par}(X_i))$$

for any subset $S \subseteq \text{NonDescendants}(X_i)$

Semantics of Bayes Nets

- If we ask for $\Pr(x_1, x_2, \dots, x_n)$
 - assuming an ordering consistent with the network

- By the chain rule, we have:

$$\begin{aligned}\Pr(x_1, x_2, \dots, x_n) \\ &= \Pr(x_n | x_{n-1}, \dots, x_1) \Pr(x_{n-1} | x_{n-2}, \dots, x_1) \dots \Pr(x_1) \\ &= \Pr(x_n | \text{Par}(x_n)) \Pr(x_{n-1} | \text{Par}(x_{n-1})) \dots \Pr(x_1)\end{aligned}$$

- Thus, the joint is recoverable using the parameters (CPTs) specified in an arbitrary BN

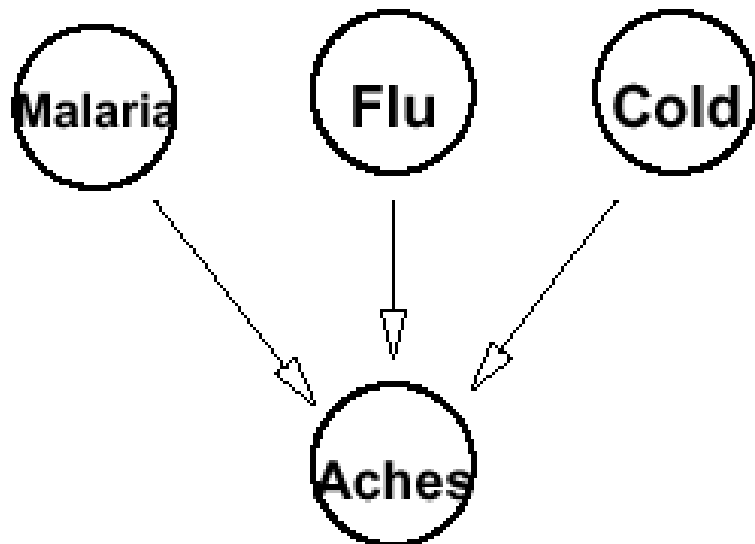
Constructing a Bayes Net

- Given any distribution over variables X_1, X_2, \dots, X_n , we can construct a Bayes net that faithfully represents that distribution.

Take any ordering of the variables (say, the order given), and go through the following procedure for X_n down to X_1 . Let $Par(X_n)$ be any subset $S \subseteq \{X_1, \dots, X_{n-1}\}$ such that X_n is independent of $\{X_1, \dots, X_{n-1}\} - S$ given S . Such a subset must exist (convince yourself). Then determine the parents of X_{n-1} in the same way, finding a similar $S \subseteq \{X_1, \dots, X_{n-2}\}$, and so on. In the end, a DAG is produced and the BN semantics must hold by construction.

Causal Intuitions

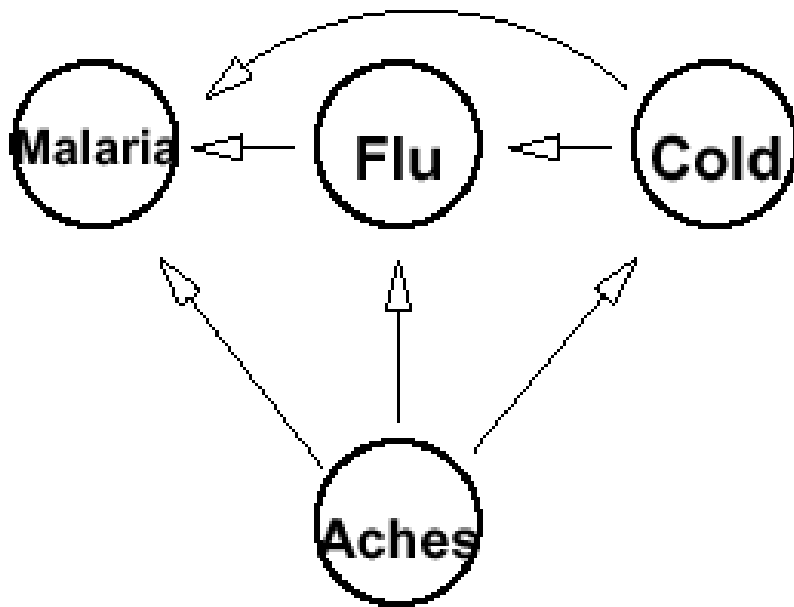
- The construction of a BN is simple
 - works with arbitrary orderings of variable set
 - but some orderings are much better than others!
 - generally, if ordering/dependence structure reflects causal intuitions, a more natural, compact BN results



- In this BN, we've used the ordering Mal, Cold, Flu, Aches to build BN for distribution P for Aches
 - Variable can only have parents that come earlier in the ordering

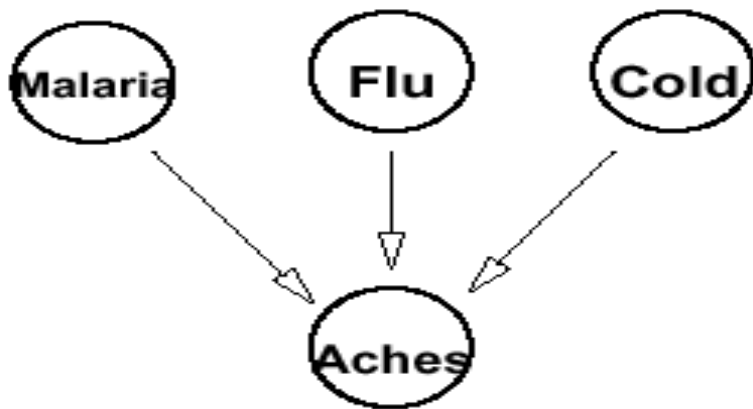
Causal Intuitions

- Suppose we build the BN for distribution P using the opposite ordering
 - i.e., we use ordering Aches, Cold, Flu, Malaria
 - resulting network is more complicated!

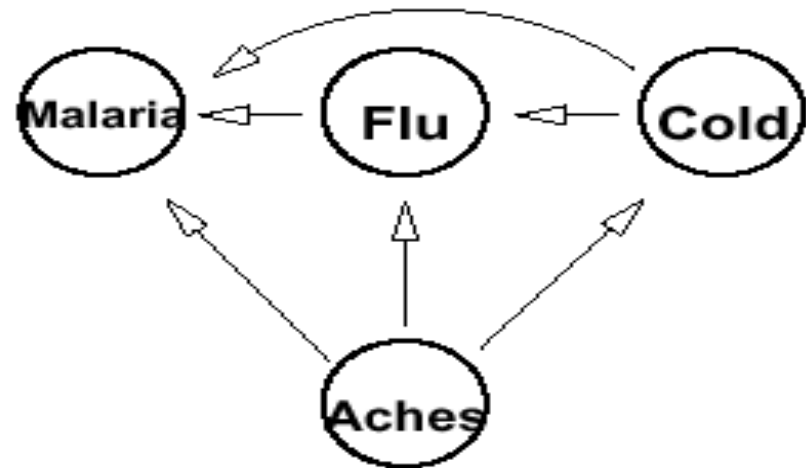


- Mal depends on Aches; but it also depends on Cold, Flu *given* Aches
 - Cold, Flu *explain away* Mal given Aches
- Flu depends on Aches; but also on Cold *given* Aches
- Cold depends on Aches

Compactness



$1+1+1+8=11$ numbers



$1+2+4+8=15$ numbers

In general, if each random variable is directly influenced by at most k others, then each CPT will be at most 2^k . Thus the entire network of n variables is specified by $n2^k$.

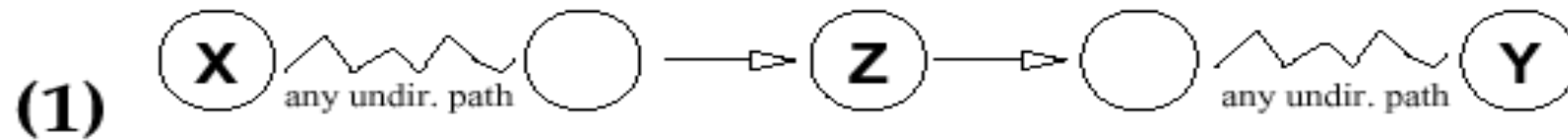
Testing Independence

- Given BN, how do we determine if two variables X , Y are independent (given evidence E)?
 - we use a (simple) graphical property
- **D-separation**: A set of variables E *d-separates* X and Y if it *blocks every undirected path* in the BN between X and Y .
- X and Y are conditionally independent given evidence E if E d-separates X and Y
 - thus BN gives us an easy way to tell if two variables are independent (set $E = \emptyset$) or cond. independent

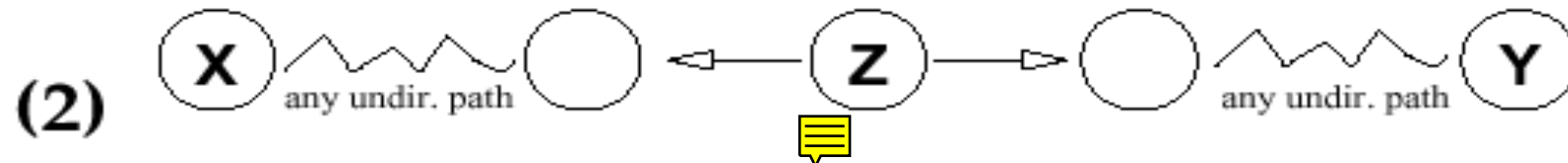
Blocking in D-Separation

- Let P be an undirected path from X to Y in a BN. Let E be an evidence set. We say E *blocks path* P iff there is some node Z on the path such that:
 - **Case 1**: one arc on P *goes into* Z and one *goes out* of Z , and $Z \in E$; or
 - **Case 2**: both arcs on P leave Z , and $Z \in E$; or
 - **Case 3**: both arcs on P enter Z and *neither Z , nor any of its descendants*, are in E .

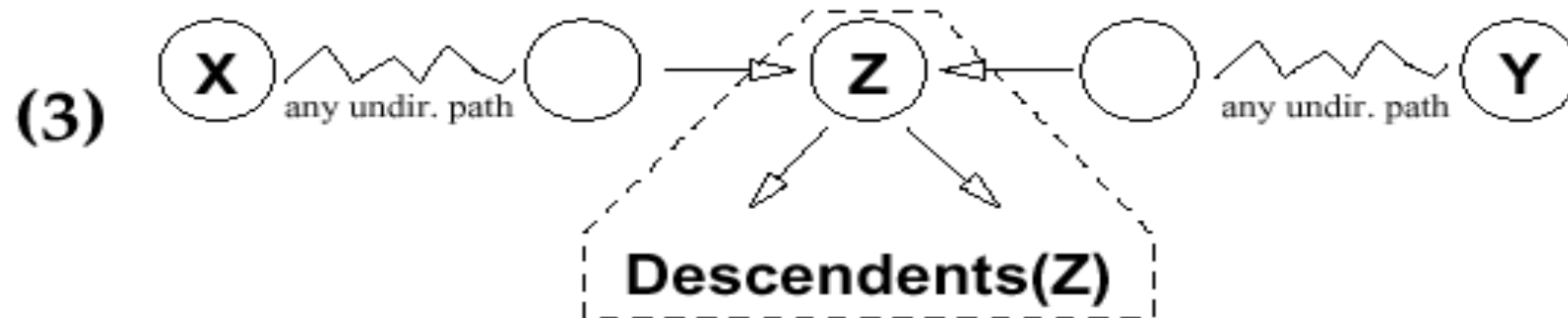
Blocking: Graphical View



If Z in evidence, the path between X and Y blocked

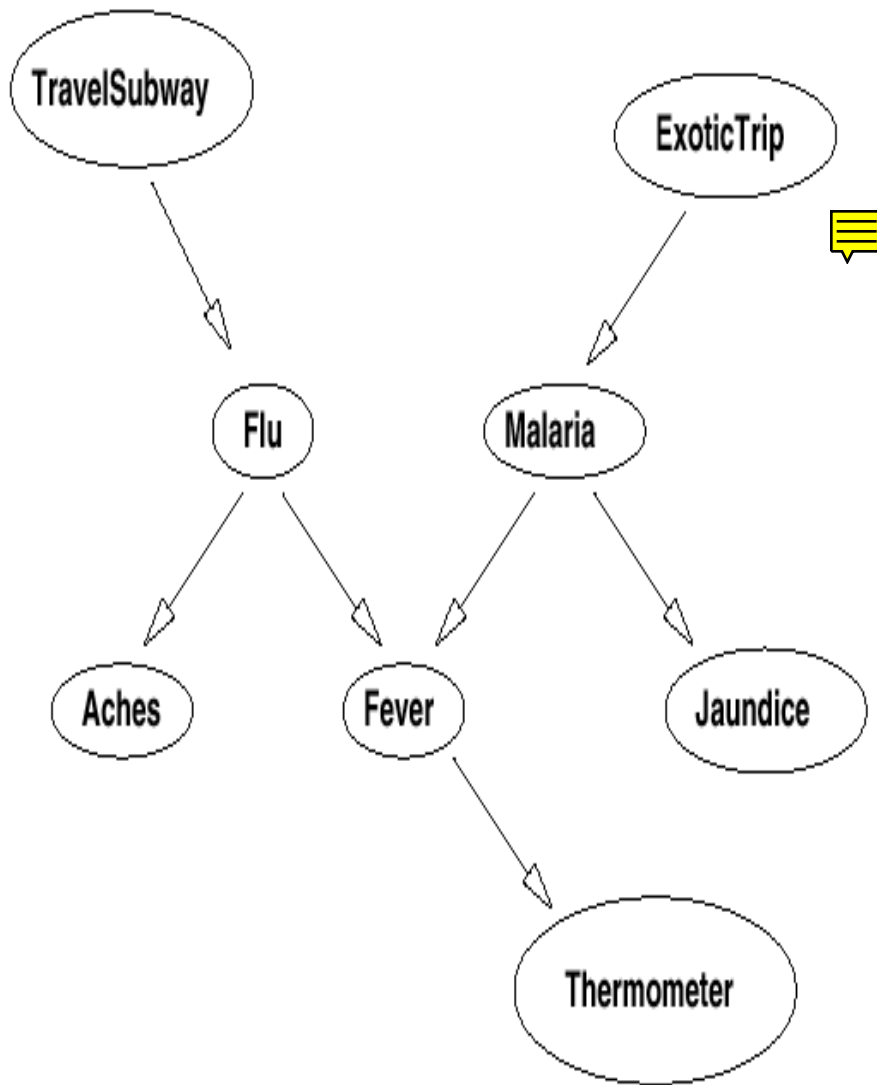


If Z in evidence, the path between X and Y blocked



If Z is **not** in evidence and **no** descendent of Z is in evidence, then the path between X and Y is blocked

D-Separation: Intuitions



1. Subway and Thermometer?
2. Aches and Fever?
3. Aches and Thermometer?
4. Flu and Malaria?
5. Subway and Exotic Trip?

D-Separation: Intuitions

- Subway and Therm are dependent; but are independent given Flu (since Flu blocks the only path)
- Aches and Fever are dependent; but are independent given Flu (since Flu blocks the only path). Similarly for Aches and Therm (dependent, but indep. given Flu).
- Flu and Mal are indep. (given no evidence): Fever blocks the path, since it is *not in evidence*, nor is its descendant Therm. Flu, Mal are dependent given Fever (or given Therm): nothing blocks path now.
- Subway, ExoticTrip are indep.; they are dependent given Therm; they are indep. given Therm and Malaria. This for exactly the same reasons for Flu/Mal above.