Statistical Learning
[RN2 Sec 20.1-20.2]
[RN3 Sec 20.1-20.2]

CS 486/686

University of Waterloo

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Outline

- Statistical learning
  - Bayesian learning
  - Maximum a posteriori
  - Maximum likelihood
- Learning from complete Data

Statistical Learning

- View: we have uncertain knowledge of the world

- Idea: learning simply reduces this uncertainty

Candy Example

- Favorite candy sold in two flavors:
  - Lime (hugh)
  - Cherry (yum)
- Same wrapper for both flavors
- Sold in bags with different ratios:
  - 100% cherry
  - 75% cherry + 25% lime
  - 50% cherry + 50% lime
  - 25% cherry + 75% lime
  - 100% lime

Candy Example

- You bought a bag of candy but don't know its flavor ratio
- After eating k candies:
  - What's the flavor ratio of the bag?
  - What will be the flavor of the next candy?

Statistical Learning

- Hypothesis H: probabilistic theory of the world
  - h_1: 100% cherry
  - h_2: 75% cherry + 25% lime
  - h_3: 50% cherry + 50% lime
  - h_4: 25% cherry + 75% lime
  - h_5: 100% lime
- Data D: evidence about the world
  - d_1: 1st candy is cherry
  - d_2: 2nd candy is lime
  - d_3: 3rd candy is lime
  - …

Bayesian Learning

- Prior: $\Pr(H)$
- Likelihood: $\Pr(d|H)$
- Evidence: $\mathbf{d} = \langle d_1, d_2, \ldots, d_n \rangle$

- Bayesian Learning amounts to computing the posterior using Bayes' Theorem:

$$\Pr(H|\mathbf{d}) = k \Pr(\mathbf{d}|H)\Pr(H)$$

Bayesian Prediction

- Suppose we want to make a prediction about an unknown quantity X (i.e., the flavor of the next candy)

- $Pr(X|\mathbf{d}) = \Sigma_i \, Pr(X|\mathbf{d},h_i)P(h_i|\mathbf{d})$
  $\qquad\qquad = \Sigma_i \, Pr(X|h_i)P(h_i|\mathbf{d})$

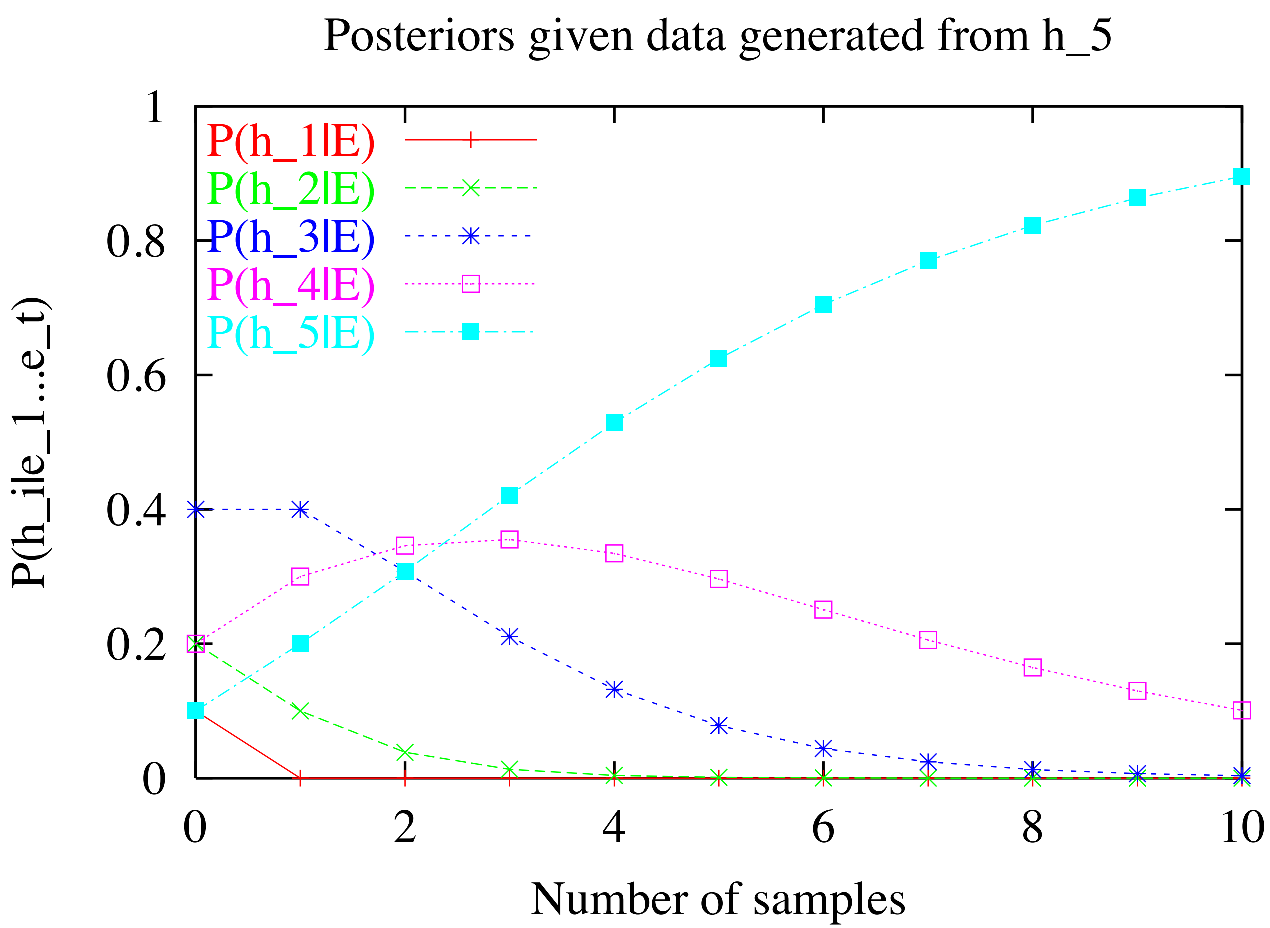
- Predictions are weighted averages of the predictions of the individual hypotheses
- Hypotheses serve as "intermediaries" between raw data and prediction

Candy Example

- Assume prior $P(H) = \langle 0.1, 0.2, 0.4, 0.2, 0.1 \rangle$
- Assume candies are i.i.d. (identically and independently distributed)
  - $P(\mathbf{d}|h) = \Pi_j P(d_j|h)$

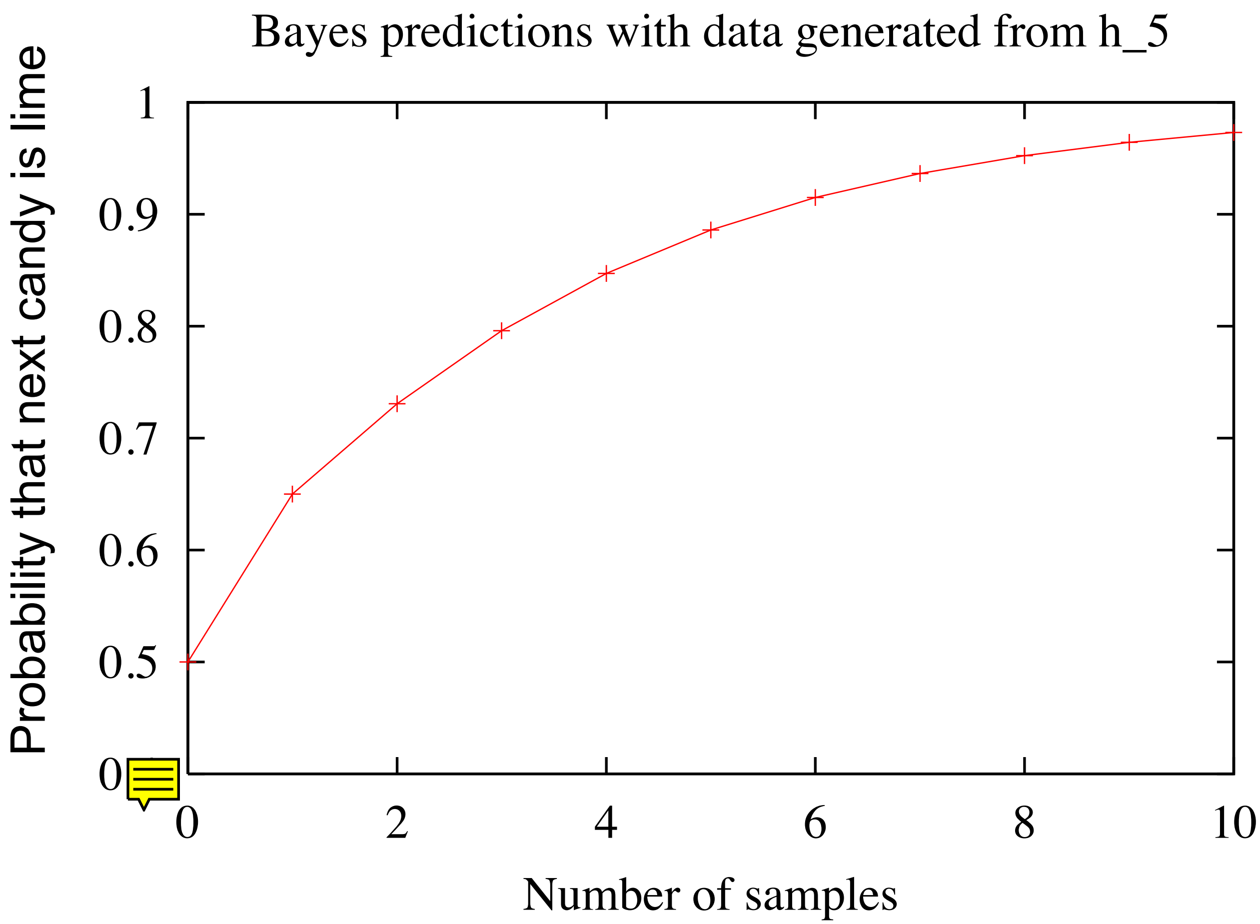
- Suppose first 10 candies all taste lime:
  - $P(\mathbf{d}|h_5) = 1^{10} = 1$
  - $P(\mathbf{d}|h_3) = 0.5^{10} = 0.00097$
  - $P(\mathbf{d}|h_1) = 0^{10} = 0$

Posterior



Posteriors given data generated from h_5
P(h_1|E)
P(h_2|E)
P(h_3|E)
P(h_4|E)
P(h_5|E)
P(h_i|e_1...e_t)
Number of samples
0
0.2
0.4
0.6
0.8
1
0
2
4
6
8
10

Prediction


Bayes predictions with data generated from h_5
Probability that next candy is lime
Number of samples
1
0.9
0.8
0.7
0.6
0.5
0
0
2
4
6
8
10

Bayesian Learning

- Bayesian learning properties:
  - **Optimal** (i.e. given prior, no other prediction is correct more often than the Bayesian one)
  - **No overfitting** (all hypotheses weighted and considered)
- There is a price to pay:
  - When hypothesis space is large Bayesian learning may be intractable
  - i.e. sum (or integral) over hypothesis often intractable
- Solution: approximate Bayesian learning

Maximum a posteriori (MAP)

- Idea: make prediction based on most probable hypothesis h_{MAP}
  - $h_{MAP} = \mathrm{argmax}_{h_i} P(h_i|\mathbf{d})$
  - $P(X|\mathbf{d}) \approx P(X|h_{MAP})$
- In contrast, Bayesian learning makes prediction based on **all** hypotheses weighted by their probability

Candy Example (MAP)

- Prediction after
  - 1 lime: $h_{MAP} = h_3$, $Pr(lime|h_{MAP}) = 0.5$
  - 2 limes: $h_{MAP} = h_4$, $Pr(lime|h_{MAP}) = 0.75$
  - 3 limes: $h_{MAP} = h_5$, $Pr(lime|h_{MAP}) = 1$
  - 4 limes: $h_{MAP} = h_5$, $Pr(lime|h_{MAP}) = 1$
  - …
- After only 3 limes, it correctly selects h_5

Candy Example (MAP)

- But what if correct hypothesis is h_4?
  - h_4: P(lime) = 0.75 and P(cherry) = 0.25

- After 3 limes
  - MAP incorrectly predicts h_5
  - MAP yields $P(lime|h_{MAP}) = 1$
  - Bayesian learning yields $P(lime|\mathbf{d}) = 0.8$

MAP properties

- MAP prediction **less accurate** than Bayesian prediction since it relies only on **one** hypothesis h_{MAP}
- But MAP and Bayesian predictions converge as data increases
- Controlled overfitting (prior can be used to penalize complex hypotheses)

- Finding h_{MAP} may be intractable:
  - $h_{MAP} = \text{argmax } P(h|\mathbf{d})$
  - Optimization may be difficult

MAP computation

- Optimization:
  - $h_{MAP} = \text{argmax}_h P(h|\mathbf{d})$
    $= \text{argmax}_h P(h) P(\mathbf{d}|h)$
    $= \text{argmax}_h P(h) \Pi_i P(d_i|h)$

- Product induces non-linear optimization
- Take the log to linearize optimization
  - $h_{MAP} = \text{argmax}_h \log P(h) + \Sigma_i \log P(d_i|h)$

Maximum Likelihood (ML)

- Idea: simplify MAP by assuming uniform prior (i.e., $P(h_i) = P(h_j)\ \forall i,j$)
  - $h_{MAP} = \text{argmax}_h\ P(h)\ P(\mathbf{d}|h)$
  - $h_{ML} = \text{argmax}_h\ P(\mathbf{d}|h)$

- Make prediction based on h_{ML} only:
  - $P(X|\mathbf{d}) \approx P(X|h_{ML})$

Candy Example (ML)

- Prediction after
  - 1 lime: $h_{ML} = h_5$, $Pr(lime|h_{ML}) = 1$
  - 2 limes: $h_{ML} = h_5$, $Pr(lime|h_{ML}) = 1$
  - …
- Frequentist: "objective" prediction since it relies only on the data (i.e., no prior)
- Bayesian: prediction based on data and uniform prior (since no prior $\equiv$ uniform prior)

ML properties

- ML prediction **less accurate** than Bayesian and MAP predictions since it ignores prior info and relies only on **one** hypothesis h_{ML}
- But ML, MAP and Bayesian predictions converge as data increases
- Subject to overfitting (no prior to penalize complex hypothesis that could exploit statistically insignificant data patterns)

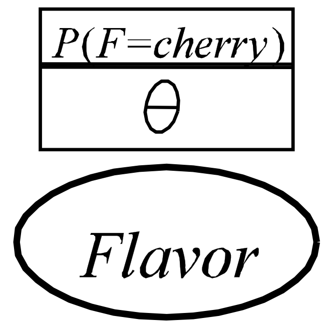
- Finding h_{ML} is often easier than h_{MAP}
  - $h_{ML} = \text{argmax}_h \Sigma_i \log P(d_i|h)$

Statistical Learning

- Use Bayesian Learning, MAP or ML
- Complete data:
  - When data has multiple attributes, **all attributes are known**
  - Easy
- Incomplete data:
  - When data has multiple attributes, **some attributes are unknown**
  - Harder

Simple ML example

- Hypothesis h_θ:
  - P(cherry)=θ & P(lime)=1-θ
- Data **d**:
  - c cherries and l limes



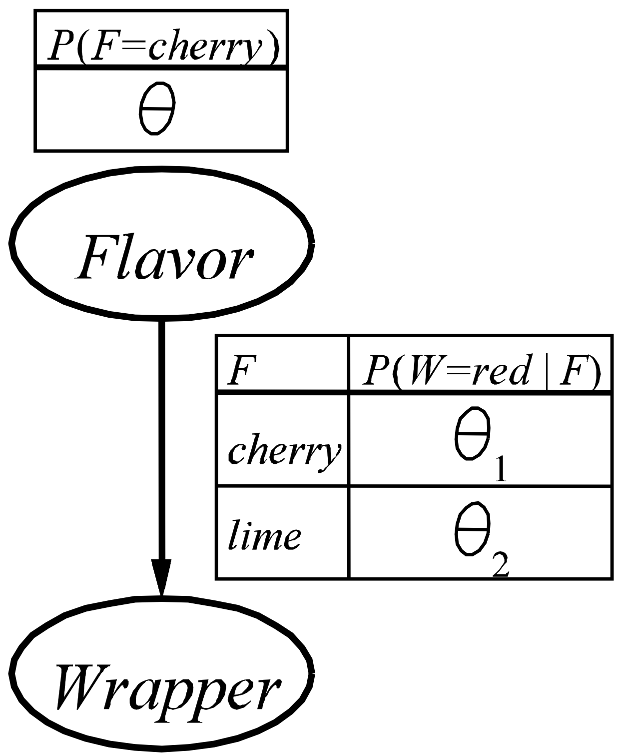
- ML hypothesis:
  - θ is relative frequency of observed data
  - θ = c/(c+l)
  - P(cherry) = c/(c+l) and P(lime)= l/(c+l)

ML computation

- 1) Likelihood expression
  - $P(\mathbf{d}|h_\theta) = \theta^c (1-\theta)^l$
- 2) log likelihood
  - $\log P(\mathbf{d}|h_\theta) = c \log \theta + l \log (1-\theta)$
- 3) log likelihood derivative
  - $d(\log P(\mathbf{d}|h_\theta))/d\theta = c/\theta - l/(1-\theta)$
- 4) ML hypothesis
  - $c/\theta - l/(1-\theta) = 0 \rightarrow \theta = c/(c+l)$

More complicated ML example

- Hypothesis: $h_{\theta,\theta_1,\theta_2}$
- Data:
  - c cherries
    - g_c green wrappers
    - r_c red wrappers
  - l limes
    - g_l green wrappers
    - r_l red wrappers



ML computation

- 1) Likelihood expression
  - $P(\mathbf{d}|h_{\theta,\theta_1,\theta_2}) = \theta^c(1-\theta)^l\,\theta_1^{r_c}(1-\theta_1)^{g_c}\,\theta_2^{r_l}(1-\theta_2)^{g_l}$
- …
- 4) ML hypothesis
  - $c/\theta - l/(1-\theta) = 0 \Rightarrow \theta = c/(c+l)$
  - $r_c/\theta_1 - g_c/(1-\theta_1) = 0 \Rightarrow \theta_1 = r_c/(r_c+g_c)$
  - $r_l/\theta_2 - g_l/(1-\theta_2) = 0 \Rightarrow \theta_2 = r_l/(r_l+g_l)$

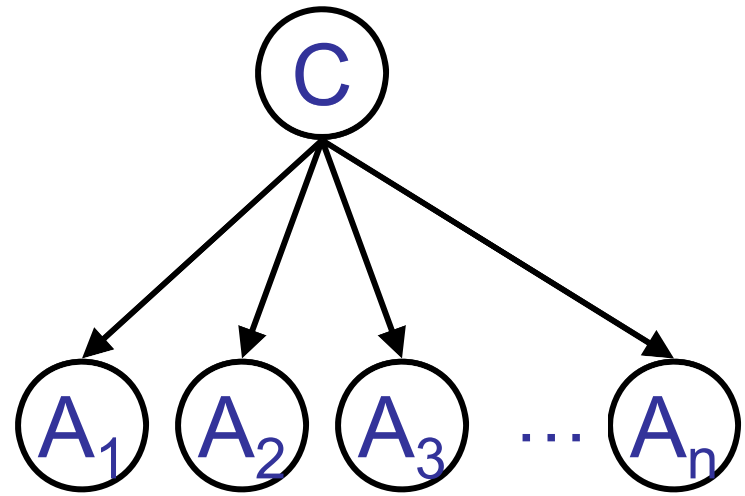
Laplace Smoothing

- An important case of overfitting happens when there is no sample for a certain outcome
  - E.g. no cherries eaten so far
  - P(cherry) = θ = c/(c+l) = 0
  - Zero prob. are dangerous: they rule out outcomes
- Solution: Laplace (add-one) smoothing
  - Add 1 to all counts
  - P(cherry) = θ = (c+1)/(c+l+2) > 0
  - Much better results in practice

Naïve Bayes model

20.2.2 Naive bayes

- Want to predict a class C based on attributes A_i
- Parameters:
  - θ = P(C=true)
  - θ_{i1} = P(A_i=true|C=true)
  - θ_{i2} = P(A_i=true|C=false)



- *Assumption: A_i's are independent given C*

Naïve Bayes model for Restaurant Problem

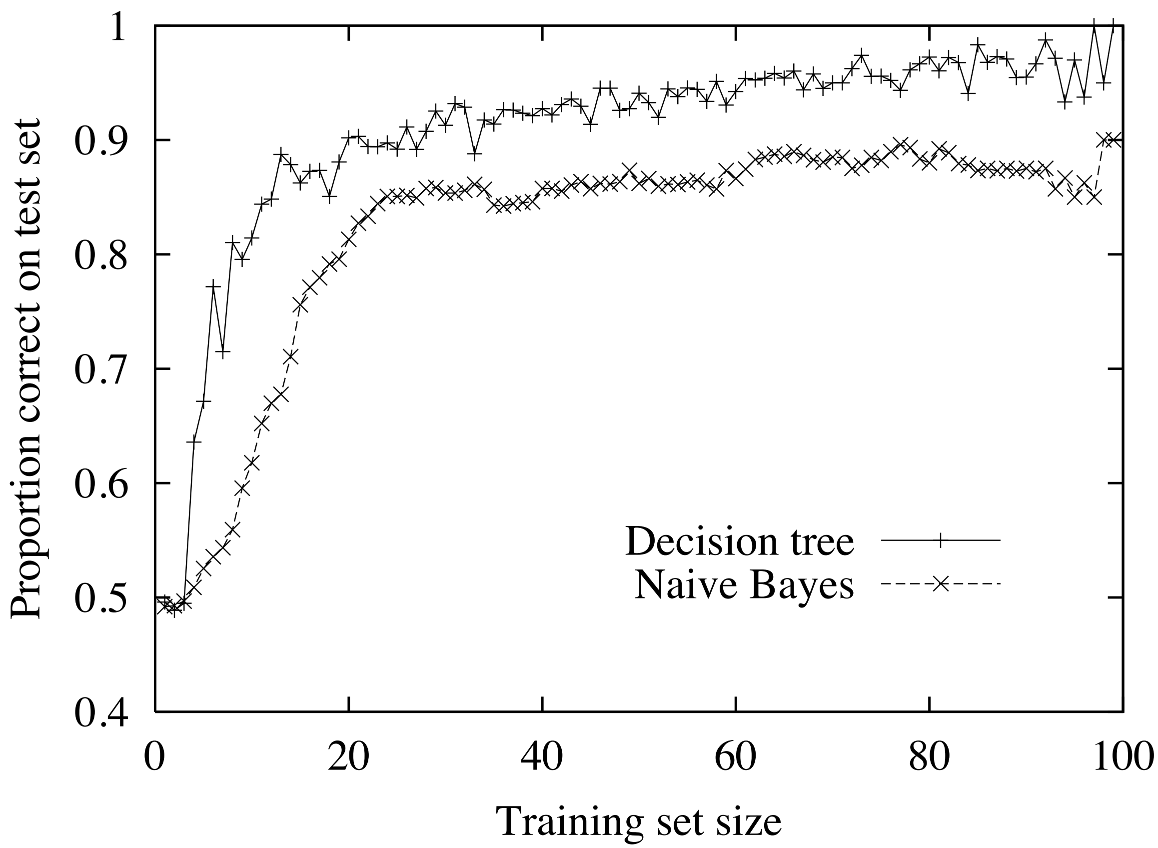
- Data:

Example	Attributes										Target
	Alt	*Bar*	*Fri*	*Hun*	*Pat*	*Price*	*Rain*	*Res*	*Type*	*Est*	*Wait*
X_1	T	F	F	T	Some	$$$	F	T	French	0–10	T
X_2	T	F	F	T	Full	$	F	F	Thai	30–60	F
X_3	F	T	F	F	Some	$	F	F	Burger	0–10	T
X_4	T	F	T	T	Full	$	F	F	Thai	10–30	T
X_5	T	F	T	F	Full	$$$	F	T	French	>60	F
X_6	F	T	F	T	Some	$$	T	T	Italian	0–10	T
X_7	F	T	F	F	None	$	T	F	Burger	0–10	F
X_8	F	F	F	T	Some	$$	T	T	Thai	0–10	T
X_9	F	T	T	F	Full	$	T	F	Burger	>60	F
X_{10}	T	T	T	T	Full	$$$	F	T	Italian	10–30	F
X_{11}	F	F	F	F	None	$	F	F	Thai	0–10	F
X_{12}	T	T	T	T	Full	$	F	F	Burger	30–60	T

- ML sets
  - θ to relative frequencies of *wait* and *~wait*
  - θ_{i1}, θ_{i2} to relative frequencies of each attribute value given *wait* and *~wait*

Naïve Bayes model vs decision trees

- Wait prediction for restaurant problem



Why is naïve Bayes less accurate than decision tree?

Assumptions independence and this assumption isn't always true, so the decision tree was better because it encoded the dependency relationship in a parent-child relation

Bayesian network parameter learning (ML)

- Parameters $\theta_{V,pa(V)=\mathbf{v}}$:
  - CPTs: $\theta_{V,pa(V)=\mathbf{v}} = P(V|pa(V)=\mathbf{v})$
- Data $\mathbf{d}$:
  - d_1 : $\langle V_1=v_{1,1}, V_2=v_{2,1}, \ldots, V_n = v_{n,1}\rangle$
  - d_2 : $\langle V_1=v_{1,2}, V_2=v_{2,2}, \ldots, V_n = v_{n,2}\rangle$
  - …
- Maximum likelihood:
  - Set $\theta_{V,pa(V)=\mathbf{v}}$ to the relative frequencies of the values of V given the values $\mathbf{v}$ of the parents of V