Bandits

CS 486 / 686

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Exploration/Exploitation Tradeoff

- Fundamental problem of RL due to the active nature of the learning process

- Consider one-state RL problems known as bandits

Stochastic Bandits

- Formal definition:
  - Single state: S = {s}
  - A: set of actions (also known as arms)
  - Space of rewards (typically assumed to be [0,1])

- No transition function to be learned since there is a single state

- We simply need to learn the **stochastic** reward function

Origin

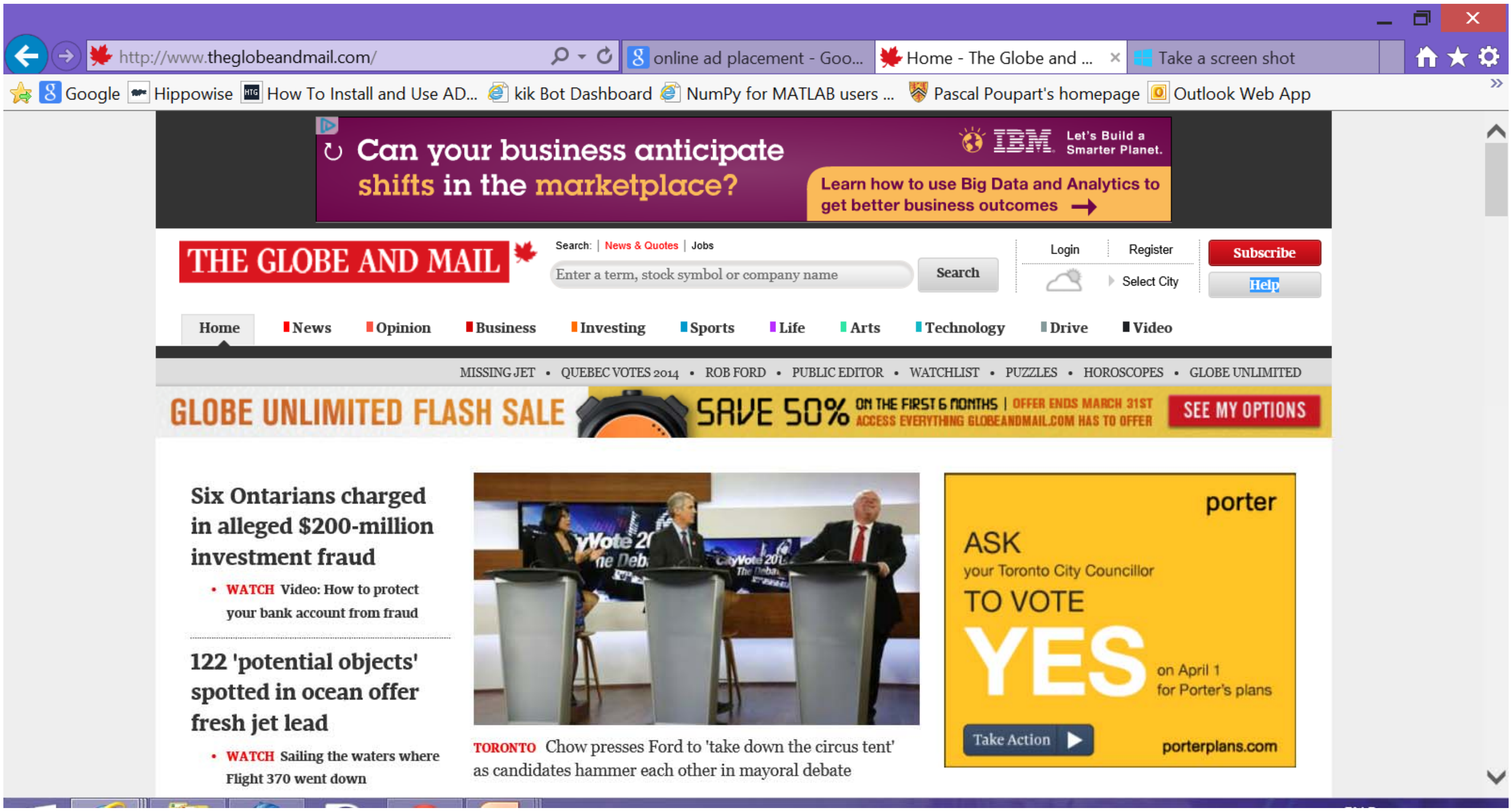
- The term bandit comes from gambling where slot machines can be thought as one-armed bandits.

- Problem: which slot machine should we play at each turn when their payoffs are not necessarily the same and initially unknown?

Examples

- Design of experiments (Clinical Trials)
- Online ad selection
- Games
- Networks (packet routing)

Online Ad Optimization



Online Ad Optimization

- Problem: which ad should be presented?
- Answer: present ad with highest payoff

$$payoff = clickThroughRate \times payment$$

  - Click through rate: probability that user clicks on ad
  - Payment: $$ paid by advertiser
    - Amount determined by an auction

Simplified Problem

- Assume payment is 1 unit for all ads
- Need to estimate click through rate

- Formulate as a bandit problem:
  – Arms: the set of possible ads
  – Rewards: 0 (no click) or 1 (click)

- In what order should ads be presented to maximize revenue?
  – How should we balance exploitation and exploration?

Simple yet difficult problem

- Simple: description of the problem is short
- Difficult: no known tractable optimal solution

Simple heuristics

- Greedy strategy: select the arm with the highest average so far
  - May get stuck in local optimum due to lack of exploration
- ϵ-greedy: select an arm at random with probability ϵ and otherwise do a greedy selection
  - Convergence rate depends on choice of ϵ

Regret

- Let $R(a)$ be the unknown average reward of a
- Let $r^* = \max_a R(a)$ and $a^* = argmax_a\, R(a)$

- Denote by $loss(a)$ the expected regret of a

$$loss(a) = r^* - R(a)$$

- Denote by $Loss_n$ the expected cumulative regret for n time steps

$$Loss_n = \sum_{t=1}^{n} loss(a_t)$$

Theoretical Guarantees

- When ϵ is constant, then
  - For large enough t: $\Pr(a_t \neq a^*) \approx \epsilon$
  - Expected cumulative regret: $Loss_n = O(n)$
    - Linear regret
- When $\epsilon_{\mathrm{t}} \propto 1/t$
  - For large enough t: $\Pr(a_t \neq a^*) \approx \epsilon_t = O\left(\frac{1}{t}\right)$
  - Expected cumulative regret: $Loss_n = O(\log\, n)$
    - Logarithmic regret

$Pr(a_k \neq a_*) \to \epsilon$
$\Delta = r^* - r_{min}$
Loss in the constant ϵ.
$Loss = \sum_{t=1}^{n} \epsilon\Delta = n\epsilon\Delta = O(n)$
Loss in the decreasing ϵ.
$Loss = \sum_{t=1}^{n} \epsilon_t\Delta = \sum_{t=1}^{n} \frac{1}{t}\Delta = \log\Delta = O(\log n)$

Empirical mean

- Problem: how far is the empirical mean $\tilde{R}(a)$ from the true mean $R(a)$?

- If we knew that $|R(a) - \tilde{R}(a)| \leq bound$
  - Then we would know that $R(a) < \tilde{R}(a) + bound$
  - And we could select the arm with best $\tilde{R}(a) + bound$

- Overtime, additional data will allow us to refine $\tilde{R}(a)$ and compute a tighter $bound$.

Positivism in the Face of Uncertainty

- Suppose that we have an oracle that returns an upper bound $UB_n(a)$ on $R(a)$ for each arm based on n trials of arm a.
- Suppose the upper bound returned by this oracle converges to $R(a)$ in the limit:
  - i.e. $\lim_{n\to\infty} UB_n(a) = R(a)$
- Optimistic algorithm
  - At each step, select $argmax_a \; UB_n(a)$

Convergence

- Theorem: An optimistic strategy that always selects $\mathrm{argmax_a} UB_n(a)$ will converge to a^*

- Proof by contradiction:
  - Suppose that we converge to suboptimal arm a after infinitely many trials.
  - Then $R(a) = UB_\infty(a) \geq UB_\infty(a') = R(a')\ \forall a'$
  - But $R(a) \geq R(a')\ \forall a'$ contradicts our assumption that a is suboptimal.

Probabilistic Upper Bound

- Problem: We can't compute an upper bound with certainty since we are sampling
- However we can obtain measures f that are upper bounds most of the time
  - i.e., $\Pr\big(R(a) \leq f(a)\big) \geq 1-\delta$
  - Example: Hoeffding's inequality

$$\boxed{\Pr\left(R(a) \leq \tilde{R}(a) + \sqrt{\frac{\log\left(\frac{1}{\delta}\right)}{2n_a}}\right) \geq 1-\delta}$$

where n_a is the number of trials for arm a

Upper Confidence Bound (UCB)

- Set $\delta_n = 1/n^4$ in Hoeffding's bound
- Choose a with highest Hoeffding bound

UCB(h)

$V \leftarrow 0,\ \ n \leftarrow 0,\ \ n_a \leftarrow 0\ \ \ \forall a$

Repeat until $n = h$

Execute $\mathrm{argmax_a}\ \tilde{R}(a) + \sqrt{\frac{2\log\ n}{n_a}}$

Receive r

$V \leftarrow V + r$

$\tilde{R}(a) \leftarrow \frac{n_a \tilde{R}(a)+r}{n_a+1}$

$n \leftarrow n+1,\ \ \ n_a \leftarrow n_a + 1$

Return V

UCB Convergence

- Theorem: Although Hoeffding's bound is probabilistic, UCB converges
- Proof: As n increases, the term $\sqrt{\frac{2\log n}{n_a}}$ increases, ensuring that all arms are tried infinitely often
- Expected cumulative regret: $Loss_n = O(\log n)$
  - Logarithmic regret

Summary

- Stochastic bandits
  - Exploration/exploitation tradeoff
- ϵ-greedy and UCB
  - Theory: logarithmic expected cumulative regret

- In practice:
  - UCB often performs better than ϵ-greedy
  - Many variants of UCB improve performance