Uncertainty
[RN2 Sec. 13.1-13.6]
[RN3 Sec. 13.1-13.5]

CS 486/686

University of Waterloo

Lecture 6: May 21, 2015

A Decision Making Scenario

- You are considering to buy a used car…
  - Is it in good condition?
  - How much are you willing to pay?
  - Should you get it inspected by a mechanics?
  - Should you buy the car?

In the next few lectures

- Probability theory
  - Model uncertainty
- Utility theory
  - Model preferences
- Decision theory
  - Combine probability theory and utility theory

Introduction

- Logical reasoning breaks down when dealing with uncertainty
- Example: Diagnosis

$$\forall p\ Symtom(p, Toothache) \Rightarrow Disease(p, Cavity)$$

  - But not all people with toothaches have cavities…

$$\forall p\ Symtom(p, Toochache) \Rightarrow Disease(p, Cavity) \vee Disease(p, Gumdisease) \vee Disease(p, HitInTheJaw) \vee \cdots$$

  - Can't enumerate all possible causes and not very informative

$$\forall p\ Disease(p, Cavity) \Rightarrow Symptom(p, Toothache)$$

  - Does not work since not all cavities cause toothaches…

Introduction

- Logic fails because
  - We are lazy
    - Too much work to write down all antecedents and consequences
  - Theoretical ignorance
    - Sometimes there is just no complete theory
  - Practical ignorance
    - Even if we knew all the rules, we might be uncertain about a particular instance (not collected enough info yet)

Probabilities to the rescue

- For many years AI danced around the fact that the world is an uncertain place
- Then a few AI researchers decided to go back to the 18th century
  - Probabilities allow us to deal with uncertainty that comes from our laziness and ignorance
  - Clear semantics
  - Provide principled answers for
    - Combining evidence, predictive and diagnostic reasoning, incorporation of new evidence
  - Can be learned from data
  - Intuitive for humans (?)

Discrete Random Variables

- Random variable A describes an outcome that cannot be determined in advance (roll of a dice)
  - Discrete random variable means that its possible values come from a countable domain (sample space)
    - E.G If X is the outcome of a dice throw, then $X \in \{1,2,3,4,5,6\}$
  - **Boolean random variable** $A \in \{True, False\}$
    - A = The Canadian PM in 2040 will be female
    - A = You have Ebola
    - A = You wake up tomorrow with a headache

Events

- An event is a complete specification of the state of the world in which the agent is uncertain
  - Subset of the sample space

- Example:

$$Cavity = True \wedge Toothache = True$$

$$Dice = 2$$

- Events must be
  - Mutually exclusive
  - Exhaustive (at least one event must be true)

Probabilities

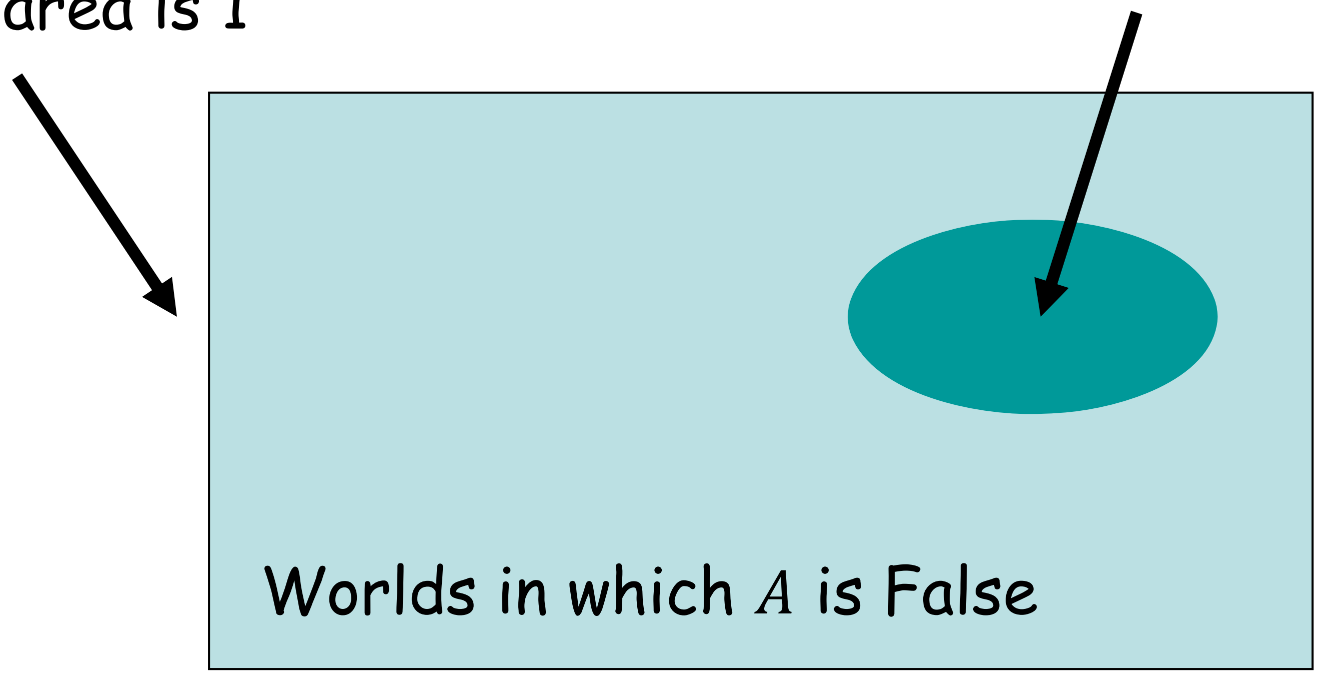
- We let $P(A)$ denote the "degree of belief" we have that statement A is true
  - Also "fraction of worlds in which A is true"
    - Philosophers like to discuss this (but we won't)

- Note:
  - $P(A)$ DOES NOT correspond to a degree of truth
  - Example: Draw a card from a shuffled deck
    - The card is of some type (e.g ace of spades)
    - Before looking at it $P(ace\ of\ spades) = 1/52$
    - After looking at it $P(ace\ of\ spades) = 1$ or 0

Visualizing A

Event space of all
possible worlds.
It's area is 1

Worlds in which A is true



$P(A)$ = Area of oval

The Axioms of Probability

- $0 \leq P(A) \leq 1$
- $P(True) = 1$
- $P(False) = 0$
- $P(A \vee B) = P(A) + P(B) - P(A \wedge B)$

- These axioms limit the class of functions that can be considered as probability functions

Interpreting the axioms

- $0 \leq P(A) \leq 1$
- $P(True) = 1$
- $P(False) = 0$
- $P(A \vee B) = P(A) + P(B) - P(A \wedge B)$

The area of A can't be smaller than 0

A zero area would mean no world could ever have A as true

Interpreting the axioms

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The area of A can't be larger than 1

An area of 1 would mean all possible worlds have A as true

Interpreting the axioms

- $0 \leq P(A) \leq 1$
- $P(True) = 1$
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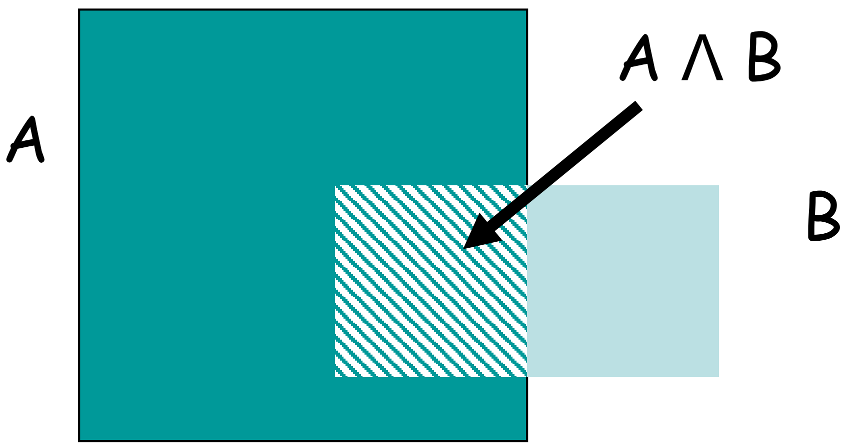
- $P(A \vee B) = P(A) + P(B) - P(A \wedge B)$



Take the axioms seriously!

- There have been attempts to use different methodologies for uncertainty
  - Fuzzy logic, three valued logic, Dempster-Shafer, non-monotonic reasoning,…
- But if you follow the axioms of probability then no one can take advantage of you ☺

A Betting Game [di Finetti 1931]

- Propositions A and B
- Agent 1 announces its "degree of belief" in A and B ($P(A)$ and $P(B)$)
- Agent 2 chooses to bet for or against A and B at stakes that are consistent with $P(A)$ and $P(B)$
- If Agent 1 does not follow the axioms, it is guaranteed to lose money

Agent 1		Agent 2		Outcome for Agent 1			
Proposition	Belief	Bet	Odds	$A \wedge B$	$A \wedge \sim B$	$\sim A \wedge B$	$\sim A \wedge \sim B$
A	0.4	A	4 to 6	-6	-6	4	4
B	0.3	B	3 to 7	-7	3	-7	3
$A \vee B$	0.8	$\sim(A \vee B)$	2 to 8	2	2	2	-8
				-11	**-1**	**-1**	**-1**

Theorems from the axioms

- Thm: $P(\sim A) = 1 - P(A)$

- Proof: $P(A \vee \sim A) = P(A) + P(\sim A) - P(A \wedge \sim A)$

$$P(True) = P(A) + P(\sim A) - P(False)$$

$$1 \;=\; P(A) + P(\sim A) - 0$$

$$P(\sim A) = 1 - P(A)$$

Multivalued Random Variables

- Assume domain of A (sample space) is $\{v_1, v_2, \dots, v_k\}$
- A can take on exactly one value out of this set

  $P(A = v_i \wedge A = v_j) = 0$ if $i \neq j$

  $P(A = v_1 \vee A = v_2 \vee \dots \vee A = v_k) = 1$

Terminology

- Probability distribution:
  - A specification of a probability for each event in our sample space
  - Probabilities must sum to 1
- Assume the world is described by two (or more) random variables
  - Joint probability distribution
    - Specification of probabilities for all combinations of events

Joint distribution

- Given two random variables A and B:
- Joint distribution:

$$\Pr(A = a \wedge B = b)\ \forall a, b$$

- Marginalisation (sumout rule):

$$\Pr(A = a) = \Sigma_b \Pr(A = a \wedge B = b)$$

$$\Pr(B = b) = \Sigma_a \Pr(A = a \wedge B = b)$$

Example: Joint Distribution

sunny

	cold	~cold
headache	0.108	0.012
~headache	0.016	0.064

~sunny

	cold	~cold
headache	0.072	0.008
~headache	0.144	0.576

$P(headache \wedge sunny \wedge cold) = 0.108$ $P(\sim headache \wedge sunny \wedge \sim cold) = 0.064$

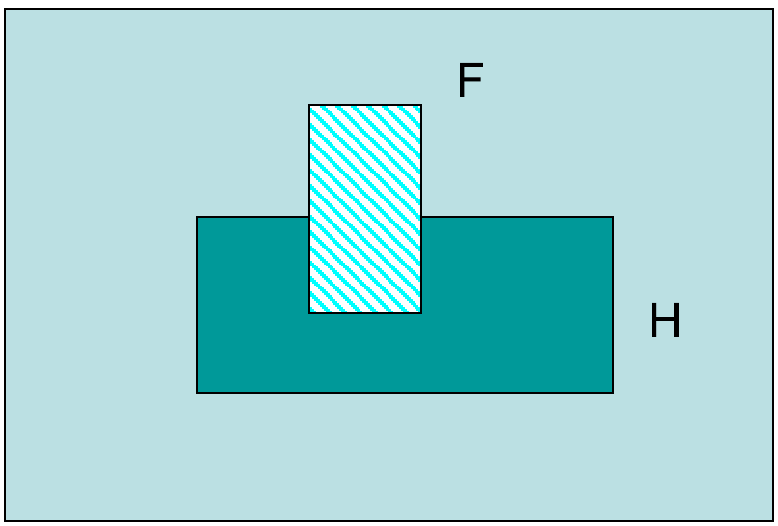
$P(headache \vee sunny) = 0.108 + 0.012 + 0.072 + 0.008 + 0.016 + 0.064 = 0.28$

$P(headache) = 0.108 + 0.012 + 0.072 + 0.008 = 0.2$

marginalization

Conditional Probability

- $P(A|B)$ fraction of worlds in which B is $true$ that also have A true

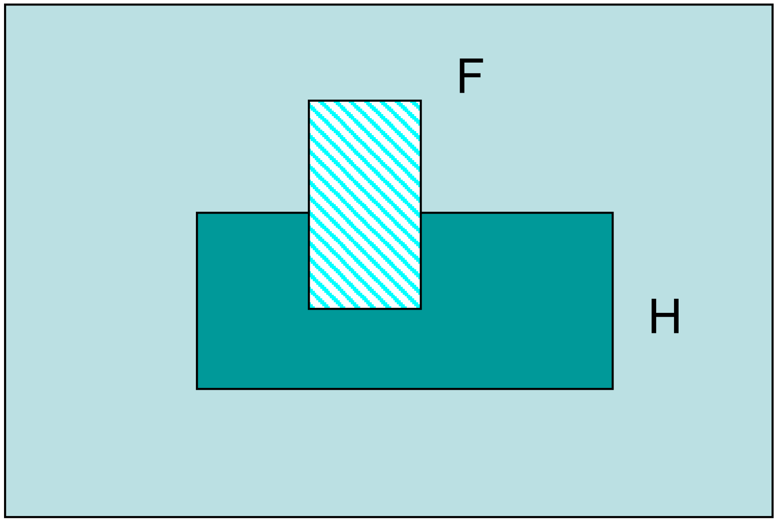


H ="Have headache"
F ="Have Flu"

$P(H) = 1/10$
$P(F) = 1/40$
$P(H|F) = 1/2$

Headaches are rare and flu is rarer, but if you have the flu, then there is a 50-50 chance you will have a headache

Conditional Probability



H ="Have headache"
F ="Have Flu"

$P(H) = 1/10$
$P(F) = 1/40$
$P(H|F) = 1/2$

$P(H|F)$ = Fraction of flu inflicted worlds in which you have a headache

=(# worlds with flu and headache)/(# worlds with flu)

= (Area of "H and F" region)/(Area of "F" region)

$$= \frac{P(H \land F)}{P(F)}$$

Conditional Probability

- Definition:

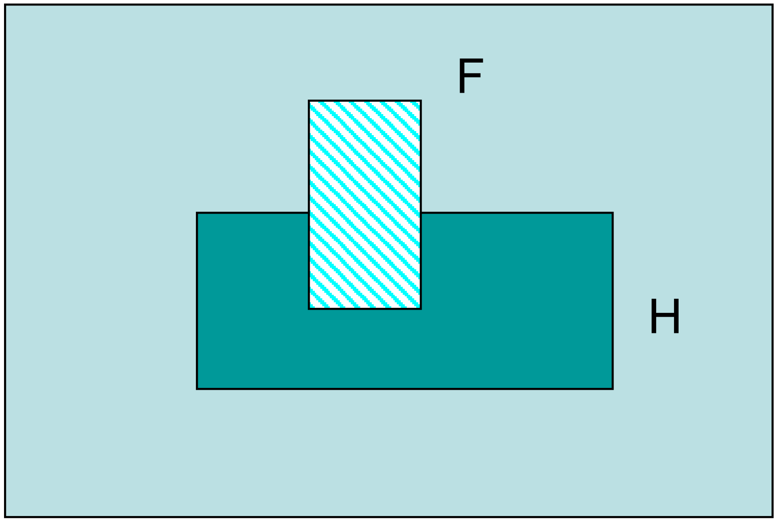
$$P(A|B) = P(A \wedge B) \, / \, P(B)$$

- Chain rule:

$$P(A \wedge B) = P(A|B) \, P(B)$$

Memorize these!

Inference



One day you wake up with a headache. You think "Drat! 50% of flues are associated with headaches so I must have a 50-50 chance of coming down with the flu"

H ="Have headache"
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$P(H) = 1/10$
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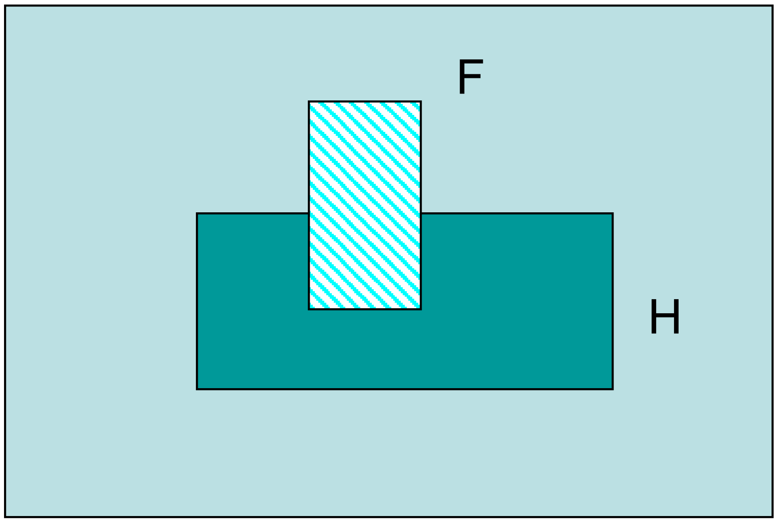
Is your reasoning correct?

Inference



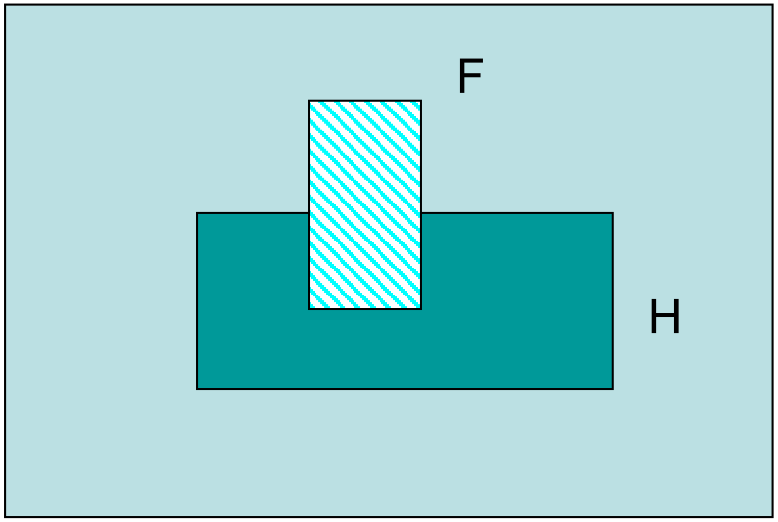
One day you wake up with a headache. You think “Drat! 50% of flues are associated with headaches so I must have a 50-50 chance of coming down with the flu”

H =“Have headache”
F =“Have Flu”

$$P(F \wedge H) = P(F)P(H|F) = 1/80$$

$P(H) = 1/10$
$P(F) = 1/40$
$P(H|F) = 1/2$

Inference



One day you wake up with a headache. You think "Drat! 50% of flues are associated with headaches so I must have a 50-50 chance of coming down with the flu"

H ="Have headache"
F ="Have Flu"

$$P(F \wedge H) = P(F)P(H|F) = 1/80$$

$$P(F|H) = P(F \wedge H)/P(H) = 1/8$$

$P(H) = 1/10$
$P(F) = 1/40$
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Example: Joint Distribution

sunny

	cold	~cold
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$$\begin{aligned} P(headache \wedge cold | sunny) &= P(headache \wedge cold \wedge sunny)/P(sunny) \\ &= 0.108/(0.108 + 0.012 + 0.016 + 0.064) \\ &= 0.54 \end{aligned}$$

$$\begin{aligned} P(headache \wedge cold | {\sim}sunny) &= P(headache \wedge cold \wedge {\sim}sunny)/P({\sim}sunny) \\ &= 0.072/(0.072 + 0.008 + 0.144 + 0.576) \\ &= 0.09 \end{aligned}$$

Bayes Rule

- Note

$$P(A|B)P(B) = P(A \wedge B) = P(B \wedge A) = P(B|A)P(A)$$

- Bayes Rule

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

Memorize this!

Using Bayes Rule for inference

- Often we want to form a hypothesis about the world based on what we have observed
- Bayes rule is vitally important when viewed in terms of stating the belief given to hypothesis H, given evidence e

Prior probability

Likelihood

$$\underbrace{P(H|e)}_{\text{Posterior probability}} = \frac{\overbrace{P(e|H)}^{\text{Likelihood}}\,\overbrace{P(H)}^{\text{Prior probability}}}{\underbrace{P(e)}_{\text{Normalizing constant}}}$$

Posterior probability

Normalizing constant

More General Forms of Bayes Rule

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|{\sim}A)P({\sim}A)}$$

$$P(A|B \wedge X) = \frac{P(B|A \wedge X)P(A|X)}{P(B|X)}$$

$$P(A = v_i|B) = \frac{P(B|A = v_i)P(A = v_i)}{\sum_{k=1}^{n} P(B|A = v_k)P(A = v_k)}$$

Example

- A doctor knows that Asian flu causes a fever 95% of the time. She knows that if a person is selected at random from the population, they have a 10^{-7} chance of having Asian flu. 1 in 100 people suffer from a fever.

- You go to the doctor complaining about the symptom of having a fever. What is the probability that Asian flu is the cause of the fever?

Example

- A doctor knows that Asian flu causes a fever 95% of the time. She knows that if a person is selected at random from the population, they have a 10^{-7} chance of having Asian flu. 1 in 100 people suffer from a fever.
- You go to the doctor complaining about the symptom of having a fever. What is the probability that Asian flu is the cause of the fever?

$A =$ Asian flu
$F =$ fever

Evidence = Symptom (F)
Hypothesis = Cause (A)

$$\boxed{P(A|F) = \frac{P(F|A)P(A)}{P(F)} = \frac{0.95 \times 10^{-7}}{0.01} = 0.95 \times 10^{-5}}$$

Computing conditional probabilities

- Often we are interested in the posterior joint distribution of some query variables Y given specific evidence e for evidence variables E
- Set of all variables: X
- Hidden variables: $H = X - Y - E$
- If we had the joint probability distribution then could marginalize
- $P(Y|E = e) = \alpha \sum_h P(Y \wedge E = e \wedge H = h)$
  where α is the normalization factor

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Problem: Joint distribution is usually too big to handle

Independence

- Two variables A and B are independent if knowledge of A does not change uncertainty of B (and vice versa)
  - $P(A|B) = P(A)$
  - $P(B|A) = P(B)$
  - $P(A \wedge B) = P(A)P(B)$
  - In general $P(X_1, X_2, \ldots, X_n) = \prod_{i=1}^{n} P(X_i)$

Need only n numbers to specify joint distribution!

Conditional Independence

- Absolute independence is often too strong a requirement

- Two variables A and B are conditionally independent given C if
  - $P(a|b,c) = P(a|c) \quad \forall a, b, c$
  - i.e. knowing the value of B does not change the prediction of A if the value of C is known

Conditional Independence

- Diagnosis problem
  $$Fl = Flu, \qquad Fv = Fever, \qquad C = Cough$$
- Full joint distribution has $2^3 - 1 = 7$ independent entries
- If someone has the flu, we can assume that the probability of a cough does not depend on having a fever
  $$P(C|Fl, Fv) = P(C|Fl)$$
- If the patient does not have the Flu, then C and Fv are again conditionally independent
  $$P(C|{\sim}Fl, Fv) = P(C|{\sim}Fl)$$

Conditional Independence

- Full distribution can be written as

$$P(C, Fl, Fv) = P(C, Fv|Fl)P(Fl)$$
$$= P(C|Fl)P(Fv|Fl)P(Fl)$$

  - That is we only need 5 numbers now!

  - Huge savings if there are lots of variables

Conditional Independence

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Such a probability distribution is sometimes called a naïve Bayes model.

In practice, they work well – even when the independence assumption is not true