CS452 : SPRING 2015

THE KINEMATICS OF TRAIN CALIBRATION

BILL COWAN
UNIVERSITY OF WATERLOO

1. INTRODUCTION

Implementing a train project produces an application program that writes bytes to interfaces and reads bytes from interfaces. Whatever is connected to the interface determines how the bytes are interpreted. From your previous courses you are familiar with the terminal's interpretation of bytes you write to it, and the meaning of bytes you read from it.

At a high level you understand the interpretation of bytes sent to and from the train controller: they set speeds and switches; they tell your application which sensors have been recently triggered. You learned which bytes mean what in assignment zero and in the fourth part of the kernel, and used this information to form mental plans that you carried out by monitoring the track with your eyes and typing commands at the terminal. Turning the commands into appropriate bytes is easy.

In the train project we replace you by an application that monitors the track without eyes, forms a plan without a brain and carries out the plan without fingers. Carrying out the plan you already know how to do; making it is not much harder; but you can't do much of either without the ability to determine the state of the trains and track, which is performed effortlessly by human vision and cognition. The remainder of this document describes a few techniques for doing so.

2. KINEMATICS

You probably remember whichever elementary physics course you took dividing basic mechanics into two subjects, kinematics, the mathematics of motion, and dynamics, the mathematics of forces. The trains we use have simulated dynamics: a small light locomotive driven by an electric stepper motor has dynamics that very different from those of a real freight train: several two hundred ton locomotives pulling two hundred ore cars, each of a hundred tons. The latter takes fifteen minutes and a dozen kilometres to reach travelling speed, and even more to stop, which we can't tolerate within a thirty minute demo.

Typical buyers of model trains, of which the trains course is not one, find the ordinary dynamics of electric trains unsatisfactory, and would like dynamics similar to real trains, but scaled down to fit into a basement. The trains we use accomplish this by using a microcontroller inside the locomotive. When a command arrives asking for a change in speed the microcontroller runs a program that gradually changes the velocity of the locomotive with an acceleration that more or less models the scaled down performance of a real train. The desire of the model railroader for realism plays a role in the pseudo-dynamics of train behaviour we will use when considering what assumptions are reasonable, but that will be the extent of our interest in dynamics.

The basis of kinematics is the observation that the location of the train on the track is a time-varying function, $x(t)$. In practice, denoting the exact position of a train on a track with turn-outs[*] that divide the track into many connected segments. This problem is discussed, probably too extensively, in class. Discussing kinematics we usually are interested in local properties, so describing the position of a train by a single real number is quite satisfactory.

The velocity of a train[†], $v(t)$, is the rate at which the train's location changes, its derivative:

$$v(t) \;=\; \frac{d}{dt}x(t)\,.$$

Newton used the dot notation for the time derivative, and it continues to this day because it keeps equations more compact. I'll use it in this document because what was good enough for Newton is good enough for me. Integrating this equation gives us the distance travelled by a train between one time and another:

$$x(t) - x(t_0) \;=\; \int_{t_0}^{t} v(t')dt'\,.$$

In our discussion we also use higher order derivatives of location:

- acceleration, the derivative of velocity, $a(t) \;=\; \frac{d}{dt}v(t)$, and

- jerk, the derivative of acceleration, $j(t) \;=\; \frac{d}{dt}a(t)$.

---

[*] You probably know a turn-out as a switch, which allows one track to diverge into two or two tracks to merge into one. 'Switch', of course, is both a noun and a verb, an ambiguity that can make writing confusing. In class, and in these notes I try consistently to use 'switch' as a verb and the less common term 'turn-out' as a noun. That is, your software switches a turn-out.

[†] There are two closely related, but different concepts associated with a moving train. One is the number in the `tr` command, an integer between zero and fourteen, which is used by the microcontroller to decide how quickly to turn the wheels: we call this the 'speed' of the train. The other is the rate at which the train moves along the track, usually measured in cm/sec. We call this quantity the 'velocity'.

3. Measurement

The project application receives a sensor notification $[0, t_p]$ after the sensor was triggered, where t_p is the time since the previous sensor query. During this time the train has travelled a distance $v[0, t_p] = D$ beyond the sensor, where D is a random variable uniformly distributed on $[0, t_p]$.

Thus, $E(D) = \frac{1}{2} v t_p$. This is the constant, or systematic, part of the error. The random part of the error is then uniformly distributed on $\left[-\frac{1}{2} v r_p, \frac{1}{2} v r_p\right]$. Normally we would write $D = \frac{1}{2} r_p \pm \frac{1}{2} r_p$ to indicate the two ends of the interval within which values of D can appear, which is what we need to ensure that deadlines are met.

4. Infinite Acceleration

The first model we examine is the simplest: in it we change the velocity from v_0 to v_1 some time between t_0 and t_1; call that time t_a. The basic equation of this model is

$$v(t) = \begin{cases} v_0 & t < t_a \\ v_1 & t > t_a \end{cases}.$$

$$x(t) = \begin{cases} v_0(t - t_0) & t < t_a \\ v_0(t_a - t_0) + v_1(t - t_a) & t > t_a \end{cases}.$$

That's all there is to it. The definition of the model ensures that the velocities are correct at both boundaries, and the position is continuous throughout the interval of acceleration.

5. Finite (Constant) Acceleration

5.a. The model
Basic equations:

$$x(t) - x(t_0) = \int_{t_0}^{t} v(t')dt'$$

$$v(t) - v(t_0) = a\int_{t_0}^{t} dt'$$

$$= a(t - t_0)$$

Conditions to be satisfied:

$$v(t_0) = v_0$$
$$v(t_1) = v_1$$
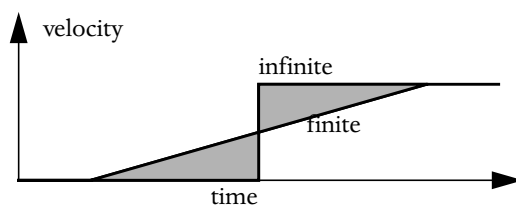
Solution:

$$v_1 \;=\; v(t_1) \;=\; a(t_1 - t_0) + v_0$$

$$a \;=\; \frac{v_1 - v_0}{t_1 - t_0}$$

$$v(t) \;=\; (v_1 - v_0)\frac{t - t_0}{t_1 - t_0} + v_0$$

$$x(t) - x_0 \;=\; \frac{1}{2}\frac{v_1 - v_0}{t_1 - t_0}(t - t_0)^2 + v_0(t - t_0)$$

$$=\; \frac{1}{2}(t - t_0)\frac{(v_1 - v_0)(t - t_0) + 2v_0(t_1 - t_0)}{t_1 - t_0}$$

$$=\; \frac{1}{2}(t - t_0)\frac{v_1(t - t_0) - v_0(t + t_0) + 2v_0 t_1}{t_1 - t_0}$$

5.B. COMPARISON TO INFINITE ACCELERATION MODEL

Comparing the two models is easily visualized by comparing graphs of the velocity over time during acceleration. (See figure directly below.)

In a diagram that plots velocity distances are areas. It is easy to see that the two shaded triangles have the same area when the step is mid-way between the starting point and the ending point. Thus, choosing $t_a = (t_1 - t_0)/2$ makes the distance travelled the same in each model, so that differences between the two models are restricted to the time during which acceleration occurs in the finite acceleration model.

Note also that the distance travelled according to the finite acceleration model moves ahead of the infinite acceleration because the train moves faster. At t_a this discrepancy is a maximum: in this section we calculate the maximum discrepancy between these too models.

The infinite acceleration model has the train located at

$$x(t) - x_0 = \begin{cases} v_0(t - t_0) & t < t_a \\ v_0(t_a - t_0) + v_1(t - t_a) & t > t_a \end{cases}$$

$$= \begin{cases} v_0(t - t_0) \\ \frac{1}{2}v_0(t_1 - t_0) + \frac{1}{2}v_1(2t - (t_1 + t_0)) \end{cases}$$

while the constant acceleration model has the train at

$$x(t) - x_0 = \frac{1}{2}\frac{v_1 - v_0}{t_1 - t_0}(t - t_0)^2 + v_0(t - t_0).$$

The difference between these two expressions is the difference between the train's location in the different models. When $t < t_a$ the difference at $t = t_a$ is

$$\Delta = \frac{1}{2}\frac{v_1 - v_0}{t_1 - t_0}(t - t_0)^2$$

$$= \frac{1}{8}(v_1 - v_0)(t_1 - t_0)$$

and when $t > t_a$ the difference at $t = t_a$ is

$$\Delta = \frac{1}{2}\frac{v_1 - v_0}{t_1 - t_0}(t - t_0)^2 + v_0(t - t_0) - \left(\frac{1}{2}v_0(t_1 - t_0) + \frac{1}{2}v_1(2t - (t_1 + t_0))\right)$$

$$= \frac{1}{8}(v_1 - v_0)(t_1 - t_0) + \frac{1}{2}v_0(t - t_0) - \frac{1}{2}v_0(t_1 - t_0)$$

$$= \frac{1}{8}(v_1 - v_0)(t_1 - t_0)$$

Thus,

6. Zero Acceleration

Another type of model, more realistic, is constrained to have zero acceleration at the beginning and end of the velocity change.

6.A. The Problem and its solution
Basic equation:

$$x(t) - x(t_0) = \frac{1}{24}A(t - t_0)^4 + \frac{1}{6}B(t - t_0)^3 + \frac{1}{2}C(t - t_0)^2 + D(t - t_0)$$

$$v(t) - v(t_0) = \frac{1}{6}A(t - t_0)^3 + \frac{1}{2}B(t - t_0)^2 + C(t - t_0)$$

$$a(t) - a(t_0) = \frac{1}{2}A(t - t_0)^2 + B(t - t_0)$$

$$j(t) - j(t_0) = A(t - t_0)$$

Conditions to be satisfied:

$$v(t_0) = v_0$$
$$v(t_1) = v_1$$
$$a(t_0) = 0$$
$$a(t_1) = 0$$

Solution:

$$D = v_0$$
$$C = 0$$
$$\frac{1}{6}A(t_1 - t_0)^3 + \frac{1}{2}B(t_1 - t_0)^2 = v_1 - v_0,$$
$$\frac{1}{2}A(t_1 - t_0)^2 + B(t_1 - t_0) = 0$$

and

$$B = -\frac{1}{2}A(t_1 - t_0)$$
$$\left(\frac{1}{6} - \frac{1}{4}\right)A(t_1 - t_0)^3 = v_1 - v_0$$

and

$$A = -12\frac{v_1 - v_0}{(t_1 - t_0)^3}$$
$$B = 6\frac{v_1 - v_0}{(t_1 - t_0)^2}$$

Finally

$$x(t) = -\frac{t_1 - t_0}{2}(v_1 - v_0)\left(\frac{t - t_0}{t_1 - t_0}\right)^4 + (t_1 - t_0)(v_1 - v_0)\left(\frac{t - t_0}{t_1 - t_0}\right)^3 + v_0(t - t_0)$$
$$v(t) = -2(v_1 - v_0)\left(\frac{t - t_0}{t_1 - t_0}\right)^3 + 3(v_1 - v_0)\left(\frac{t - t_0}{t_1 - t_0}\right)^2 + v_0$$

The amount of time taken by the change of velocity, t_1, must somehow be measured.

6.B. COMPARISON TO CONSTANT ACCELERATION
The constant acceleration solution is

$$x(t) - x(t_0) = (t - t_0)\left(\frac{1}{2}(v_1 - v_0)\left(\frac{t - t_0}{t_1 - t_0}\right) + v_0\right),$$

and the zero acceleration solution is

$$x(t) - x(t_0) = -\frac{t_1 - t_0}{2}(v_1 - v_0)\left(\frac{t - t_0}{t_1 - t_0}\right)^4 + (t_1 - t_0)(v_1 - v_0)\left(\frac{t - t_0}{t_1 - t_0}\right)^3$$
$$+ v_0(t - t_0)$$
$$= (t - t_0)\left((v_1 - v_0)\left(-\frac{1}{2}\left(\frac{t - t_0}{t_1 - t_0}\right)^3 + \left(\frac{t - t_0}{t_1 - t_0}\right)^2\right) + v_0\right)$$

The difference between the solutions is

$$\Delta(t) = (t - t_0)(v_1 - v_0)\left(-\frac{1}{2}\left(\frac{t - t_0}{t_1 - t_0}\right)^3 + \left(\frac{t - t_0}{t_1 - t_0}\right)^2 - \frac{1}{2}\left(\frac{t - t_0}{t_1 - t_0}\right)\right)$$
$$= \left(-\frac{1}{2}\frac{(t - t_0)^2(v_1 - v_0)}{t_1 - t_0}\right)\left(\frac{t - t_1}{t_1 - t_0}\right)^2$$
$$= \left(-\frac{1}{2}(v_1 - v_0)(t_1 - t_0)\right)\left(\frac{t - t_0}{t_1 - t_0}\right)^2\left(\frac{t - t_1}{t_1 - t_0}\right)^2$$

We want to find where the difference is greatest and how big it is there. The condition for an extremum is

$$(t - t_0)(t - t_1)^2 + (t - t_0)^2(t - t_1) = 0$$
$$(t - t_0)(t - t_1)\left(t - \frac{1}{2}(t_1 + t_0)\right) = 0$$

By inspection the solutions are $t = t_0, t_1, \frac{1}{2}(t_1 + t_0)$. The third is the

maximum; the other two are minima.

The maximum difference is then

$$\Delta_{max} = \left(-\frac{1}{32}\left(\frac{v_1 - v_0}{(t_1 - t_0)^3}\right)\right)(t_1 - t_0)^4$$
$$= -\frac{1}{32}(v_1 - v_0)(t_1 - t_0)$$

7. Zero Acceleration and Zero Jerk

7.a. Solving this model
Notational simplification:

$$t_0 = 0$$
$$x(0) = 0$$

Undo notational simplification:
- Replace t by $t - t_0$.
- Replace $x(t)$ by $x(t - t_0) - x(t_0)$.

Using the notational simplification above, the basic equation is

$$x(t) = \frac{1}{720}At^6 + \frac{1}{120}Bt^5 + \frac{1}{24}Ct^4 + \frac{1}{6}Dt^3 + \frac{1}{2}Et^2 + Ft.$$

Conditions to be satisfied:

$$
\begin{aligned}
x(0) &= 0 \\
x(t_1) &= x_1 \\
v(0) &= v_0 \\
v(t_1) &= v_1 \\
a(0) &= 0 \\
a(t_1) &= 0 \\
j(0) &= 0 \\
j(t_1) &= 0
\end{aligned}
$$

Solution:

$$
\begin{aligned}
F &= v_0 \\
E &= 0 \\
D &= 0 \\
\frac{1}{720}At_1^6 + \frac{1}{120}Bt_1^5 + \frac{1}{24}Ct_1^4 + v_0t_1 &= x_1 \\
\frac{1}{120}At_1^5 + \frac{1}{24}Bt_1^4 + \frac{1}{6}Ct_1^3 &= v_1 - v_0 \\
\frac{1}{24}At_1^4 + \frac{1}{6}Bt_1^3 + \frac{1}{2}Ct_1^2 &= 0 \\
\frac{1}{6}At_1^3 + \frac{1}{2}Bt_1^2 + Ct_1 &= 0
\end{aligned}
$$

and

$$
\begin{aligned}
C &= -\frac{1}{6}At_1^2 - \frac{1}{2}Bt_1 \\
0 &= \frac{1}{24}At_1^2 + \frac{1}{6}Bt_1 + \frac{1}{2}\left(-\frac{1}{6}At_1^2 - \frac{1}{2}Bt_1\right) \\
&= -\frac{1}{24}At_1^2 - \frac{1}{12}Bt_1 \\
B &= -\frac{1}{2}At_1
\end{aligned}
$$

Continuing,

$$C = -\frac{1}{6}At_1^2 + \frac{1}{4}At_1^2$$

$$= \frac{1}{12}At_1^2$$

$$v_1 - v_0 = \frac{1}{120}At_1^5 - \frac{1}{48}At_1^5 + \frac{1}{72}At_1^5$$

$$= \frac{6 - 15 + 10}{720}At_1^5$$

$$= \frac{1}{720}At_1^5$$

$$x_1 = \frac{1}{720}At_1^6 + \frac{1}{120}Bt_1^5 + \frac{1}{24}Ct_1^4 - v_0t_1 \, .$$

Then

$$A = \frac{720(v_1 - v_0)}{t_1^5}$$

$$B = -\frac{360(v_1 - v_0)}{t_1^4} \, ,$$

$$C = \frac{60(v_1 - v_0)}{t_1^3}$$

$$F = v_0$$

and the solution is:

$$x(t) = \frac{720(v_1 - v_0)t^6}{720t_1^5} - \frac{360(v_1 - v_0)t^5}{120t_1^4} + \frac{60(v_1 - v_0)t^4}{24t_1^3} + v_0t$$

$$= (v_1 - v_0)t_1\left(\left(\frac{t}{t_1}\right)^6 - 3\left(\frac{t}{t_1}\right)^5 + \frac{5}{2}\left(\frac{t}{t_1}\right)^4\right) + v_0t$$

$$v(t) - v_0 = (v_1 - v_0)\left(6\left(\frac{t}{t_1}\right)^5 + (-15)\left(\frac{t}{t_1}\right)^4 + 10\left(\frac{t}{t_1}\right)^3\right)$$

$$v(0) = v_0$$

$$v(t_1) = v_1$$

As above, t_1 remains undetermined.

7.B. FINDING THE MAXIMUM DIFFERENCE

How does this solution compare with the solution above? Particularly how far apart in distance can they get.

The solution zero acceleration above is

$$x(t) = -\frac{t_1}{2}(v_1 - v_0)\left(\frac{t}{t_1}\right)^4 + t_1(v_1 - v_0)\left(\frac{t}{t_1}\right)^3 + v_0t \, ,$$

and the zero jerk solution is

$$x(t) = (v_1 - v_0)t_1\left(\left(\frac{t}{t_1}\right)^6 - 3\left(\frac{t}{t_1}\right)^5 + \frac{5}{2}\left(\frac{t}{t_1}\right)^4\right) + v_0 t \, .$$

The difference between them is

$$\Delta = (v_1 - v_0)t_1\left(\left(\frac{t}{t_1}\right)^6 - 3\left(\frac{t}{t_1}\right)^5 + 3\left(\frac{t}{t_1}\right)^4 - \left(\frac{t}{t_1}\right)^3\right)$$

$$= (v_1 - v_0)t_1\left(\frac{t}{t_1}\right)^3\left(\frac{t}{t_1} - 1\right)^3 \, .$$

It has extrema at solutions of

$$\left(\frac{t}{t_1}\right)^2\left(\frac{t}{t_1} - 1\right)^3 + \left(\frac{t}{t_1}\right)^3\left(\frac{t}{t_1} - 1\right)^2 = 0$$

$$\left(\frac{t}{t_1}\right)^2\left(\frac{t}{t_1} - 1\right)^2\left(2\frac{t}{t_1} - 1\right) = 0 \, .$$

Four solutions are at the ends of the interval where the difference is zero; the fifth is in the middle of the interval, $\frac{t}{t_1} = \frac{1}{2}$. The fifth is the one we want. At that point the difference is

$$\Delta = (v_1 - v_0)t_1\left(\frac{1}{2}\right)^3\left(-\frac{1}{2}\right)^3$$

$$= -\frac{1}{64}(v_1 - v_0)(t_1 - t_0) \, .$$

8. Closing Comments

This document has developed three models. The first solution works well for many projects. The second is believed to be the one that is actually in the trains. The third is not useful; the mathematics above only shows that more elaborate models, which one might expect to follow actual locomotive dynamics more closely[*], are not used in the Marklin trains. Assuming they aren't used, of course.

---

[*] Most passenger trains on which I travel leave the station with zero jerk at the onset of motion. That's why it's so easy to mistake the motion of my train from the backwards motion of a neighbouring train.