CONTROLLER-OBSERVER CO-DESIGN

A report submitted in partial fulfillment of the requirements of internship

in

MODERN CONTROL

by

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We dedicate this thesis to Dr. Satyanarayana Neeli, Assistant Professor, MNIT Jaipur and the Almighty without whom none of our success would be possible.

DEPARTMENT OF ELECTRICAL ENGINEERING
INDIAN INSTITUTE OF TECHNOLOGY

CERTIFICATE

This is to certify that the report entitled **Controller-Observer Co-Design** is a bonafide record of work done by **Mayank Chittora, Akshat Singhal, Sunayana Gupta** at Indian Institute of Technology Delhi, as partial fulfillment of requirements for internship in Modern Control. They have fulfill the requirements for the submission of this report, which to the best of my knowledge has reached the required standard. This report is carried out under my supervision and guidance.

Dr. S. Janardhanan

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Abstract

Name: **Mayank Chittora** **Sunayana Gupta** **Akshat Singhal**
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Report title: **Observer**
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Date: **3 July 2018**

The aim of this project is to design a functional observer.In this project Linear Matrix Inequality has been used to estimate the states of a given system. The project involved two phases, first one was to design full order and reduced order observer and the second was to design functional observer using LMI. Simulation and experimental result are also presented. Performance of the different states is also compared.

Contents

Chapter 1

Introduction to state space

A control system is an interconnection of components connected or related in such a manner as to command, direct, or regulate itself or another system. Control System is a conceptual framework for designing systems with capabilities of regulation and tracking to give a desired performance.Earlier, conventional theory was used but it had many drawbacks,such as it is only defined under zero initial conditions, can be applied only to linear time invariant systems, does not give any idea about the internal state of the system,cannot be applied to multiple input multiple output systems.

Therefore, a new control theory was devised i.e the modern control theory which is applicable to all types of systems like single input single output systems, multiple inputs and multiple outputs systems, linear and non linear systems, time varying and time invariant systems.

1.1 State Space Analysis

State-space representation is a mathematical model of a physical system as a set of input, output and state variables related by first-order differential equations or difference equations.
Basic terms related to state space analysis of modern theory of control systems :
State in State Space Analysis : It refers to smallest set of variables whose knowledge at $t = t_0$ together with the knowledge of input for $t \geq t_0$ gives the complete knowledge of the behavior of the system at any time $t \geq t_0$.
State Variables in State Space analysis : It refers to the smallest set of variables which help us to determine the state of the dynamic system. State variables are defined by $x_1(t), x_2(t)........x_n(t)$.
State Vector : Suppose there is a requirement of n state variables in order to describe the complete behavior of the given system, then these n state variables are considered to be n components of a vector x(t). Such a vector is known as state vector.
State Space : It refers to the n dimensional space which has $x_1 axis, x_2 axis.........x_n axis$.

1.2 State space representation

General state space equations are given as :

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1.1}$$

$$y(t) = Cx(t) + Du(t) \tag{1.2}$$

where
x(t) represents state vector $\in \ R^{n\times1}$
y(t) represents output vector $\in \ R^{p\times1}$
u(t) represents input vector $\in \ R^{m\times1}$
A=system matrix $\in \ R^{n\times n}$
B=input matrix $\in \ R^{m\times1}$
C=output matrix $\in \ R^{p\times n}$
D=transfer matrix$\in \ R^{p\times m}$

1.3 Controllability and Observability

Controllability and Observability :
A system is said to be controllable at time t_0 if it possible by means of unconstrained control vector to transfer the system from any initial state $x(t_0)$ to any other state in finite interval of time.

A system is said to be obsevable at time t_0, with the system in state $x(t_0)$ it is possible to determine the state from the observation of output over a finite time interval.

1.3.1 Complete State Controllability of Continuous-Time Systems

Consider the continuous-time system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1.3}$$

x(t) represents state vector $\in \ R^{n\times1}$
A=system matrix $\in \ R^{n\times n}$
B=input matrix $\in \ R^{m\times1}$
The system given above is completely state controllable if and only if the vectors $[B, AB, \ldots, A^{(n-1)}B]$ are linearly independent, or the $n \times n$ matrix
$[B \quad AB \quad ....A^{(v-1)}B]$
is of rank n
This matrix is called the controllability matrix of controllability index v

1.3.2 Complete State observbility of Continuous-Time Systems

Consider the continuous-time system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1.4}$$

$$y(t) = Cx(t) + Du(t) \tag{1.5}$$

x(t) represents state vector $\in R^{n\times 1}$
A=system matrix $\in R^{n\times n}$
B=input matrix $\in R^{m\times 1}$
The system given above is completely observable if and only if the vectors $[C'A'C'......A'^{(n-1)}C']$ are linearly independent, or the $n \times n$ matrix $[C' \quad A'C' \quad ....A'^{(v-1)}C']$ is of rank n
This matrix is called the observability matrix of observablity index v

Chapter 2

Pole Placement

Pole placement, is a method employed in feedback control system theory to place the closed-loop poles of a plant in pre-determined locations in the s-plane. Placing poles is desirable because the location of the poles corresponds directly to the eigenvalues of the system, which control the characteristics of the response of the system. The system must be considered controllable in order to implement this method. This technique is widely used in systems with multiple inputs and multiple outputs, as in active suspension systems.

The present design technique begins with a determination of the desired closed-loop poles based on the transient-response and/or frequency-response requirements, such as speed, damping ratio, or bandwidth, as well as steady-state requirements. It is assumed that the desired closed-loop poles are to be at s=μ1, s=μ2 ,p , s=μn. By choosing an appropriate gain matrix for state feedback, it is possible to force the system to have closed-loop poles at the desired locations, provided that the original system is completely state controllable.

2.1 Design by Pole Placement

A control system is considered

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{2.1}$$

We shall choose the control signal to be

$$u = -Kx \tag{2.2}$$

This means that the control signal u is determined by an instantaneous state. Such a scheme is called state feedback. The $1 \times n$ matrix K is called the state feedback gain matrix.We assume that all state variables are available for feedback. In the following analysis we assume that u is unconstrained.

This closed-loop system has no input. Its objective is to maintain the zero output. Because of the disturbances that may be present, the output will deviate from zero.The nonzero output will be returned to the zero reference input because of the state feedback scheme of the system. Such a system where the reference input is always zero is called a regulator system. Substituting Equation (2) into Equation (1) gives

$$\dot{x}(t) = (A - BK)x(t) \tag{2.3}$$

The solution of this equation is given by

$$x(t) = e^{(A-BK)t}x(0) \tag{2.4}$$

where x(0) is the initial state caused by external disturbances.The stability and transien-

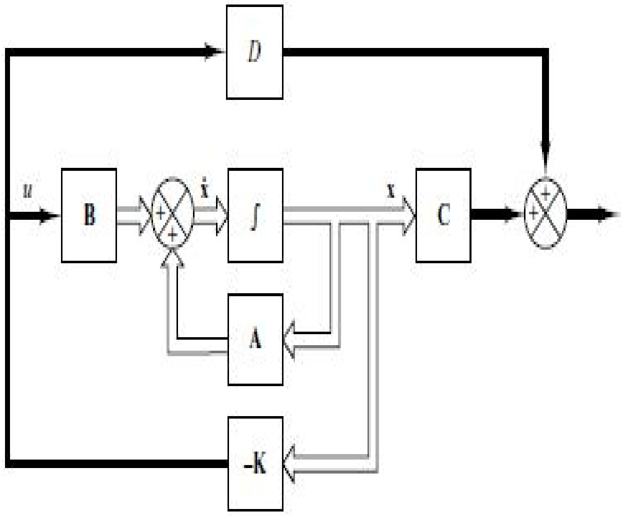

Figure 2.1: code and output

tresponse characteristics are determined by the eigenvalues of matrix A-BK.
If matrix K is chosen properly, the matrix A-BK can be made an asymptotically stable matrix, and for all $x(0) \neq 0$, it is possible to make x(t) approach 0 as t approaches infinity.The eigenvalues of matrix A-BK are called the regulator poles. If these regulator poles are placed in the left-half s plane, then x(t) approaches 0 as t approaches infinity.The problem of placing the regulator poles (closed-loop poles) at the desired location is called a pole-placement problem.

2.2 Determination of Matrix K Using Ackermanns Formula

There is a well-known formula, known as Ackermanns formula, for the determination of the state feedback gain matrix K.
Considering the same system again

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{2.5}$$

where we use the state feedback control u=Kx. We assume that the system is completely state controllable.We also assume that the desired closed-loop poles are at s=μ1, s=μ2 ,p , s=μn.
Use of the state feedback control

$$u = -Kx \tag{2.6}$$

modifies the system equation to

$$\dot{x} = (A - BK)x \tag{2.7}$$

Let us define

$$\hat{A} = A - BK \tag{2.8}$$

The desired characteristic equation is

$$\begin{aligned} |sI - A + BK| &= |sI - \hat{A}| = (s - \mu_1)(s - \mu_2)...(s - \mu_n) \\ &= s^n + \alpha_1 s^{n-1} + ... + \alpha_{n-1}s + \alpha_n \end{aligned} \tag{2.9}$$

Since the CayleyHamilton theorem states that $\hat{A}$ satisfies its own characteristic equation, we have

$$\phi(\hat{A}) = \hat{A}^n + \alpha_1 \hat{A}^{n-1} + ...\alpha_{n-1}\hat{A} + \alpha_n I = 0 \tag{2.10}$$

Using this identity we get the Ackermann's formula

$$K = \begin{bmatrix} 0 & 0 & ...0 & 1 \end{bmatrix} \begin{bmatrix} B & AB & ... & A_{n-1}B \end{bmatrix}^{-1} \phi(A) \tag{2.11}$$

In MATLAB we have placed the poles using firstly the desired poles:

$$J = \begin{bmatrix} \mu_1 & \mu_2 & ... & \mu_n \end{bmatrix} \tag{2.12}$$

Then we enter

$$K = place(A, B, J) \tag{2.13}$$

2.3 State Observers

In the pole-placement approach to the design of control systems, we assumed that all state variables are available for feedback. In practice, however, not all state variables are available for feedback. Then we need to estimate unavailable state variables.
Estimation of unmeasurable state variables is commonly called observation.A device (or a computer program) that estimates or observes the state variables is called a state observer, or simply an observer. If the state observer observes all state variables of the system, regardless of whether some state variables are available for direct measurement, it is called a full-order state observer.There are times when this will not be necessary, when we will need observation of only the unmeasurable state variables, but not of those that are directly measurable as well. For example, since the output variables are observable and they are linearly related to the state variables, we need not observe all state variables, but observe only n-mstate variables, where n is the dimension of the state vector and m is the dimension of the output vector.
An observer that estimates fewer than n state variables, where n is the dimension of the state vector, is called a reduced-order state observer or, simply, a reduced-order observer. If the order of the reduced-order state observer is the minimum possible, the observer is called a minimum-order state observer or minimum-order observer. In this section, we shall discuss both the full-order state observer and the minimum-order state observer.
A state observer estimates the state variables based on the measurements of the output and control variables. Here the concept of observability plays an important role. State observers can be designed if and only if the observability condition is satisfied.
The mathematical model of the observer is defined as:

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + K_e(y - C\hat{x}) \\ &= (A - K_e)C\hat{x} + Bu + K_e y \end{aligned} \tag{2.14}$$

where $\hat{x}$ is the estimated state and $C\hat{x}$ is the estimated output.The inputs to the observer are the output y and the control input u. Matrix K_e, which is called the observer gain matrix, is a weighting matrix to the correction term involving the difference between the measured output y and the estimated output $C\hat{x}$ .This term continuously corrects the model output and improves the performance of the observer

Chapter 3

Full-order observer

3.1 Introduction

Full order observer estimates all the states of the system. Luenberger defined this observer and introduced the conditions which must be satisfied in order for the observer to exist. It follows a system with n state variables which requires an observer of order n for full state estimation. Consider,

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{3.1}$$

$$y(t) = Cx(t) \tag{3.2}$$

For this system, it is now assumed that all state variables contain in x(t) cannot be measured directly and that an observer to reconstruct the state variables must be formed. To reconstruct all the state variables, $\hat{x}(t)$ is introduced (where $\hat{x}(t)$ denotes an approximation of the state x(t) as opposed to the actual state). Define an observer with following dynamics

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(y(t) - C\hat{x}(t)) \tag{3.3}$$

where L ∈ $R^{n \times p}$
is the observer gain matrix, to be determined as stated in Theorem 3.1 to follow.Figure 3.1 shows the input and output of a full-order state observer and also how the estimated state $\hat{x}(t)$ is used for feedback control purpose. This form of state observer uses the system input, u(t), and output, y(t), to estimate the state vector, x(t). The first two terms are a model of the system dynamics while the third term is dependent on the difference between the actual output and the expected output based on the current estimated state, and thus represents the mismatch between the system model and the actual system. Let the observer state estimation error be defined as follows

$$e(t) = x(t) - \hat{x}(t) \tag{3.4}$$

Then by taking the derivative of (4) and substituting (1), (2) and (3), the error dynamics can be defined as follows

$$\dot{e}(t) = \dot{x}(t) - \dot{\hat{x}}(t) \tag{3.5}$$

we get

$$\dot{e}(t) = (A - LC)e(t) \tag{3.6}$$

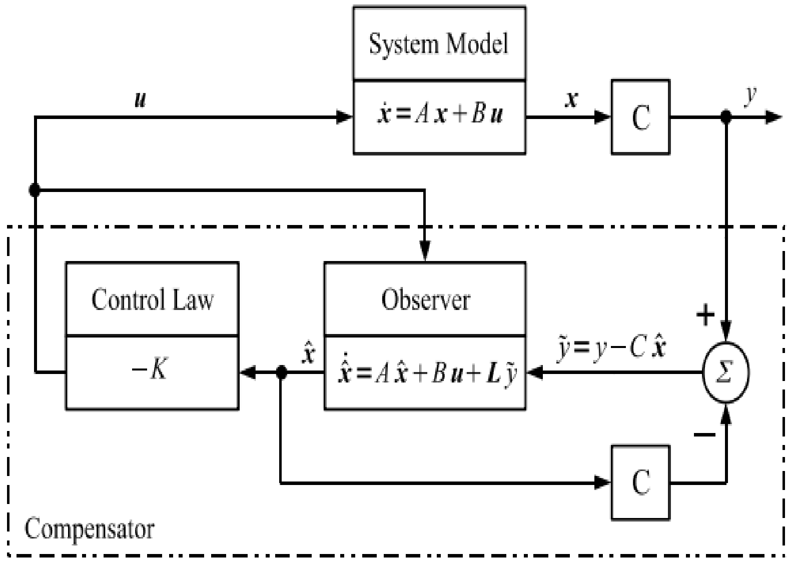

Figure 3.1: Full-order state observer

This differential equation can be readily solved, the solution being an exponential function of the form

$$e(t) = e^{(A-LC)t}e(t_0) \tag{3.7}$$

Clearly, for the estimation error to approach zero, the eigenvalues of $(A - LC)$ must have negative real parts, or equivalently $(A - LC)$ must be Hurwitz.The matrix L must therefore be chosen to satisfy those conditions via the method of pole positioning.

Theorem 3.1: Corresponding to the real matrices A and C, there is a real matrix L such that the set of eigenvalues of $(A - LC)$ can be arbitrarily assigned (subject to complex eigenvalues occurring in conjugate pairs) if and only if the pair (C,A) is completely observable.

This important result defines the requirement for a full-order state observer to exist.It highlights that if the original system is unobservable then it is not possible to construct a full-order state observer.Assuming that the system is observable, the question of how the location of the poles will affect the performance of the observer must be addressed. Since the purpose of the observer is to provide the system states to the controller, the observer must provide an estimated signal to the controller that tends toward the actual states rapidly and accurately. Otherwise the controller will yield an incorrect control signal $u(t) = K\hat{x}(t)$. The positioning of poles in $(A - LC)$ determines several performance characteristics of the observer, of which the most important is the observer transient response. As the real component of the poles becomes increasingly negative (move further left in the complex s-plane), the estimated states tend toward the actual states more rapidly. On the other hand, this also results in the elements of L becoming larger, thus amplifying sensor noise and resulting in a less accurate control signal.
Consider the state feedback using the reconstructed states with the following control law

$$u(t) = -Kx(t) \tag{3.8}$$

Assuming an appropriate choice for matrix L, the estimation error e(t) will be zero in the steady state. The initial transient response of the full-order state observer will be investigated by considering the full-order state observer presented in Figure. Of particular interest when analyzing the transient response, is whether the introduction of the observer will degrade system stability.
The closed-loop system is completed through the interconnection of the controller (8) and observer (3)to the open-loop system (2) and (1). It is described by the following composite system

$$\begin{bmatrix}\dot{x}(t)\\ \dot{\hat{x}}(t)\end{bmatrix}=\begin{bmatrix}A & -BK\\ LC & (A-LC-BK)\end{bmatrix}\begin{bmatrix}x(t)\\ \hat{x}(t)\end{bmatrix} \tag{3.9}$$

$$y(t)=\begin{bmatrix}C & 0\end{bmatrix}\begin{bmatrix}x(t)\\ \hat{x}(t)\end{bmatrix} \tag{3.10}$$

and on resolving above matrix we get

$$\dot{x}(t)=(A-BK)x(t)+BKe(t)\dot{e}(t)=(A-LC)e(t) \tag{3.11}$$

so final we have following dynamics

$$\begin{bmatrix}\dot{x}(t)\\ \dot{e}(t)\end{bmatrix}=\begin{bmatrix}A-BK & BK\\ 0 & A-LC\end{bmatrix}\begin{bmatrix}x(t)\\ e(t)\end{bmatrix} \tag{3.12}$$

and for above system to be stable eigen values must lie on left side of s-plane.

Therorem3.2 :
If the linear system is completely controllable and completely observable, there exist gain matrices K and L such that the 2n eigenvalues of the system matrix of the closed-loop system can be arbitrarily assigned, in particular to positions in the left-half of the complex s-plane.

Example

$$A=\begin{bmatrix}1 & 0 & -2 & 2 & 0 & 0 & -8 & -6 & -2\\ 0 & -1 & 3 & 18 & 41 & -10 & -19 & -28 & 4\\ 1 & 2 & 3 & 4 & 4 & 0 & -8 & -8 & 4\\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1\\ 2 & 1 & 0 & 0 & -2 & 3 & 2 & 0 & 1\\ -3 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & 0\\ -1 & 0 & -1 & 1 & 3 & 0 & -1 & 1 & 1\\ 0 & 0 & 1 & 2 & 1 & 0 & 4 & -5 & 1\\ 1 & 1 & 0 & 1 & 0 & 1 & -2 & -3 & 0\end{bmatrix}\quad B=\begin{bmatrix}1\\2\\3\\4\\5\\6\\7\\8\\9\end{bmatrix} \tag{3.13}$$

$$C=\begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix} \tag{3.14}$$

Solution
Matlab code and simulation
Since matrix $(A-LC)$ has eigenvalues in the left-half of the complex s-plane, the observer estimation error goes to zero asymptotically.plane, the observer estimation error goes to zero asymptotically.

```
CODE

A=[1 0 -2 2 0 0 -8 -6 -2;0 -1 3 18 41 -10 -19 -28 4; 1 2 3 4 4 0 -8 -8 4; 1 0
0 1 0 0 0 -1 1;2 1 0 0 -2 3 2 0 1;-3 1 0 0 1 0 -1 0 0; -1 0 -1 1 3 0 -1 1 1;
0 0 1 2 1 0 4 -5 1; 1 1 0 1 0 1 -2 -3 0];
n=size(A)
B=[1;2;3;4;5;6;7;8;9];
C=[1 0 0 0 0 0 0 0 0;0 1 0 0 0 0 0 0 0; 0 0 1 0 0 0 0 0 0];
e=[-2 -3 -4 -5 -6  -7 -8 -9 -9.2] %poles of controller%
eb=[-10 -15 -20 -25 -30 -35 -40 -45 -46];%poles of observer%
R=rank(ctrb(A,B))
R1=rank(obsv(A,C))
if R==9
k=place(A,B,e)
L=place(A',C',eb)
L=L'
else
display('state observer with arbitrary poles does not exist')
end
L
eig(A-L*C)
```

CODE-RUN

L =

1.0e+03 *

0.0741	-0.0012	0.0175
0.0045	0.0888	0.0317
0.0085	0.0039	0.0991
0.0581	-0.0885	-8.5446
-0.0685	0.1928	1.9813
-0.0772	0.3386	1.6423
-0.1785	0.0001	-0.3270
0.0517	0.0103	-2.9219
-0.1246	-0.0584	0.8384

Eig(A-L*C)=

-10.0000 -15.0000 -20.0000 -25.0000 -46.0000 -45.0000 -40.0000 -30.0000 -35.0000

Figure 3.2: code and output

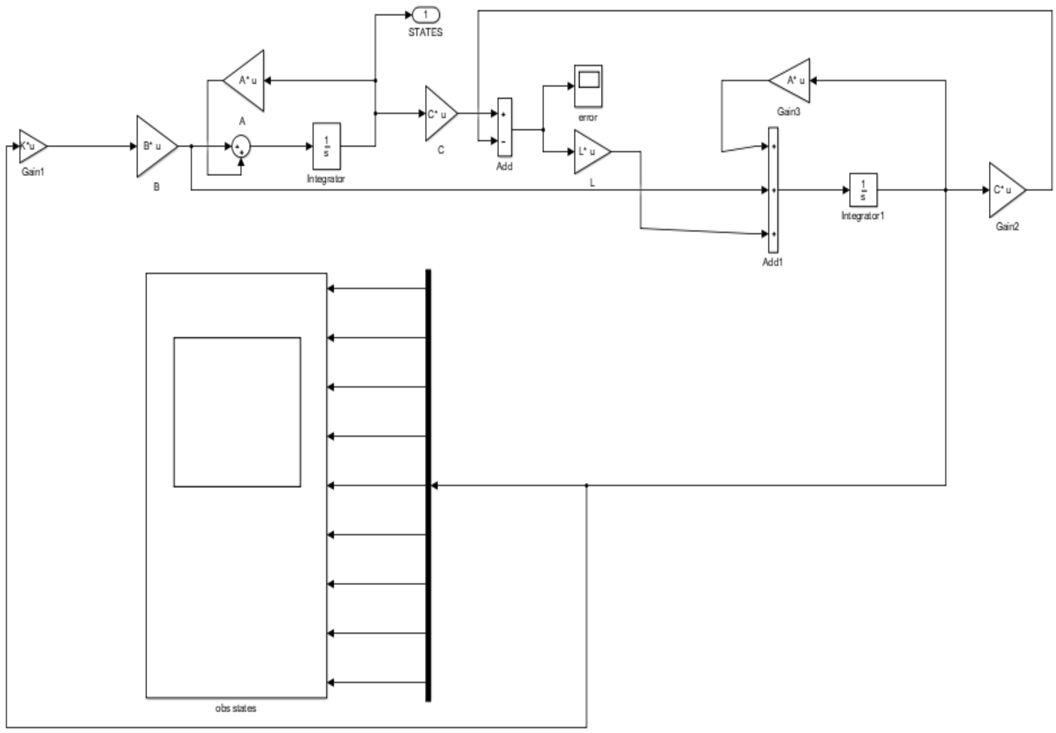

Figure 3.3: Simulation model

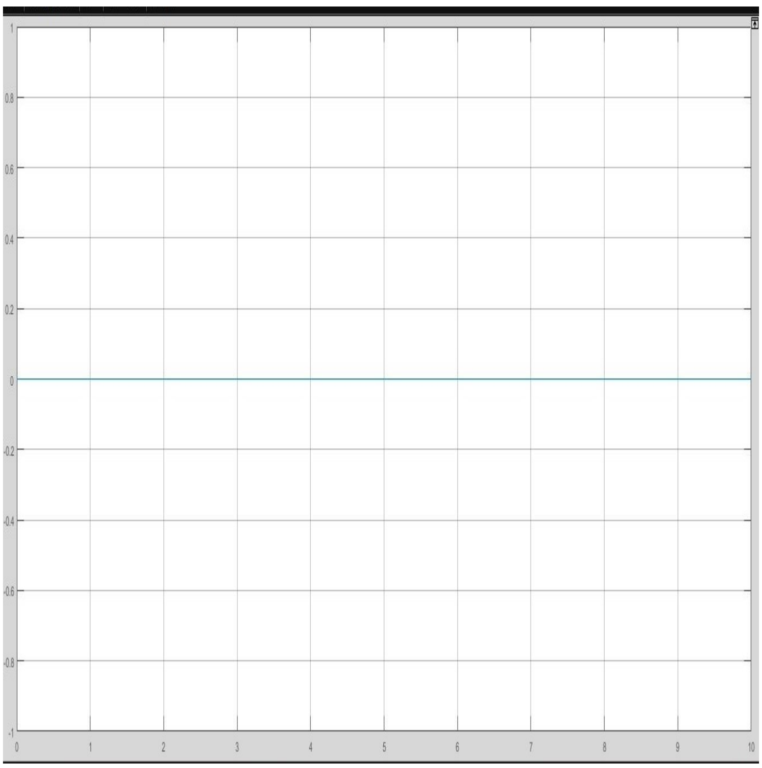

Figure 3.4: Error

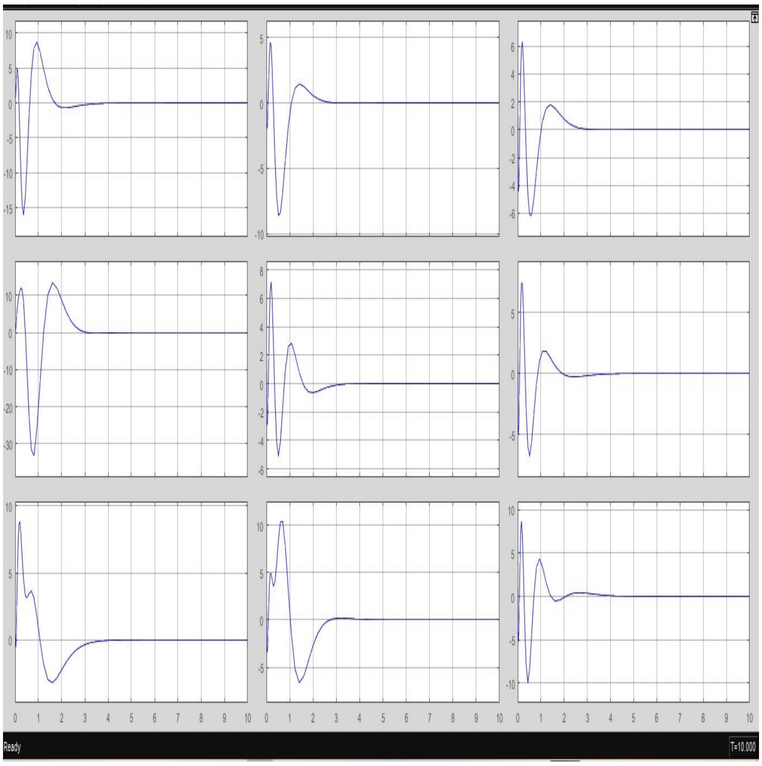

Figure 3.5: Full-order states

Chapter 4

Reduced Order Observer

The full-order state observer is of order n, being equal to the number of state variables in the original system . Although simple both conceptually and in its construction, there are some inherent redundancies in its design. Recall our objective of reconstructing the state vector x(t) and that the system output contains p linear combinations of the state variables. Intuitively, the remaining $(n-p)$ state variables should be reconstructed by an observer of order $(n-p)$. The following analysis will confirm this conjecture
The design of a reduced-order state observer is based on the following partitioned form of the system dynamics

$$\begin{bmatrix} \dot{x_p}(t) \\ \dot{x_u}(t) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_p(t) \\ x_u(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t) \tag{4.1}$$

$$y(t) = \begin{bmatrix} I_p & 0 \end{bmatrix} \begin{bmatrix} x_p(t) \\ x_u(t) \end{bmatrix} \tag{4.2}$$

where$x_p(t) \in R^p$ denotes states that are measurable and $x_u(t) \in R^{np}$ denotes states that are not measurable or unknown. Sub-matrices$A_{11} \in R^{p\times p}$,$A_{12} \in R^{p\times n-p}$ $A_{21} \in R^{n-p\times p}$,$A_{22} \in R^{n-p\times n-p}$,$B_1 \in R^{p\times m}$ and $B_2 \in R^{n-p\times m}$ are known constant. The identity matrix $I_p \quad is \quad p \times p$ matrix and the zero matrix is of dimensions $(p \times n-p)$, which gives

$$y(t) = x_p(t) \tag{4.3}$$

4.1 Reduced order

Since$x_p(t)$ is directly obtained from the output y(t), one needs to only estimate the remaining $(n-p)$ state variables of $x_u(t)$. Note that, the output equation is not always readily in the form (2). This can be achieved by using the transformation

$$x(t) = P\bar{x}(t) \tag{4.4}$$

where

$$P = \begin{bmatrix} C^+ & C^\perp \end{bmatrix} \tag{4.5}$$

$\in R^{n\times n}$ is an invertible matrix,$C^+ \in R^{n\times p}$ denotes the Moore Penrose inverse of C
i.e

$$CC^+ = I_p \tag{4.6}$$

and $C^{\perp}$ denotes an orthogonal basis for null space of C
i.e.

$$CC^{\perp} = 0 \tag{4.7}$$

Now it is clear that

$$CP = \begin{bmatrix} I_p & 0 \end{bmatrix} \tag{4.8}$$

So after using the transformation(4) , the above system is converted into

$$\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}u(t) \tag{4.9}$$

where

$$\bar{x}(t) = \begin{bmatrix} x_p(t) \\ x_u(t) \end{bmatrix} \tag{4.10}$$

$$\bar{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{4.11}$$

,

$$\bar{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \tag{4.12}$$

Now, the system yields the following two equations

$$\dot{x_p}(t) = A_{11}x_p(t) + A_{12}x_u(t) + B_1u(t) \tag{4.13}$$

$$\dot{x_u}(t) = A_{21}x_p(t) + A_{22}x_u(t) + B_2u(t) \tag{4.14}$$

The first of these equations, i.e. (13), can be re-arranged to form an intermediate output variable $\bar{y}(t) \in R^p$ where

$$\bar{y}(t) = A_{12}x_u(t) = \dot{x_p}(t) - A_{11}x_p(t) - B_1u(t) \tag{4.15}$$

The dynamics of a reduced-order state observer are defined by the following

$$\dot{\hat{x_u}}(t) = A_{22}\hat{x_u}(t) + A_{21}x_p(t) + B_2u(t) + L(\bar{y}(t) - A_{12}\hat{x_u}(t)) \tag{4.16}$$

The construction is similar to the full-order state observer case, with the last term being a corrective term. Here, matrix $L \in R^{n-p\times p}$ is of a smaller dimension than the full-order state observer case.
Consider the following reduced-order state observer for $x_u(t)$

$$\hat{x_u}(t) = z(t) + Ly(t) \tag{4.17}$$

$$\dot{z}(t) = Fz(t) + Gy(t) + Hu(t) \tag{4.18}$$

where$z(t) \in R^{n-p}$, L, F, G and H are constant matrices of appropriate dimension, to be found. They can be derived by examining the error between the estimated state variables,

$\hat{x_u}(t)$ and the actual state variables $x_u(t)$. Let an error vector$\tilde{x_u}(t) \in R^{n-p}$ be defined as follows

$$\tilde{x_u}(t) = x_u(t) - \hat{x_u}(t) \tag{4.19}$$

So after substitution, we get

$$\dot{\tilde{x_u}}(t) = \dot{x_u}(t) - \dot{\hat{x_u}}(t) = (A_{22} - LA_{12})x_u(t) - F\hat{x_u}(t) + (A_{21} + FL - G - LA_{11})y(t) + (B_2 - H - LB_1)u(t) \tag{4.20}$$

So we get following equation-

$$F = A_{22} - LA_{12} \tag{4.21}$$

$$G = A_{21} - LA_{11} + FL \tag{4.22}$$

$$H = B_2 - LB_1 \tag{4.23}$$

then error dynamics become

$$\dot{\tilde{x_u}}(t) = (A_{22} - LA_{12})\tilde{x_u}(t) \tag{4.24}$$

This result is similar to that of the full-order state observer. To make the error,$\tilde{x_u}(t)$ converges asymptotically to zero with any prescribed rate, the matrix pair (A_{12}, A_{22}) must be completely observable. It can be shown that if (C,A) is completely observable, then so is (A_{12}, A_{22}).

Thus, to drive the error, $\tilde{x_u}(t)$, asymptotically to zero, the same approach as the fullorder state observer is taken whereby the observer gain L is chosen such that the eigen values of $(A_{22}LA_{12})$ lie in the left-half of the complex s-plane. Theoretically,the eigenvalues could be moved arbitrarily towards negative infinity, which would yield rapid convergence.

A schematic of the full feedback system with the reduced-order state observer represented by(17),(18) is presented in Figure . For the case where the output of the system is already in the form (2) , P is an identity matrix. Note how y(t) acts as inputs to the dynamic part of the observer in (18) as well as contributing directly to the state estimate $\hat{x_u}(t)$ in(16) . This results in the estimate $\hat{x_u}(t)$ being more susceptible to measurement errors in y(t) than in the full-order state observer case. The feedback loop is completed with the reduced-order observer providing the unknown states to augment known states. These are then made available to the controller.

```
A=[1 0 -2 2 0 0 -8 -6 -2;0 -1 3 18 41 -10 -19 -28 4; 1 2 3 4 4 0 -8 -8 4; 1 0
0 1 0 0 0 -1 1;2 1 0 0 -2 3 2 0 1;-3 1 0 0 1 0 -1 0 0; -1 0 -1 1 3 0 -1 1 1;
0 0 1 2 1 0 4 -5 1; 1 1 0 1 0 1 -2 -3 0];
B=[1;2;3;4;5;6;7;8;9];
C=[1 0 0 0 0 0 0 0 0;0 1 0 0 0 0 0 0 0; 0 0 1 0 0 0 0 0 0];
e=[-2 -3 -4 -5 -6 -7 -8 -9 -9.2]
%e=[-0.1 -0.2 -0.3 -0.4 -0.5 -0.6 -0.7 -0.8 -0.9];
%eb=[-10 -11 -12 -13 -14 -15 -16 -17 -18]; %0.75
%eb=[-10-20j -10+20j -12 -13 -14+10j -14-10j -17-12j -17+12j -18]; %1.4
%eb=[-100+100j -100-100j -120 -140 -148 -144 -172 -180+70j -180-70j];
eb=[-2 -3 -4 -5 -6 -7 -8 -9 -9.2]

R=rank(ctrb(A,B))
if R==9
k=place(A,B,e);
L=place(A',C',eb);
L=L'
else
display('state observer with arbitrary poles does not exist')
end

R1=rank(obsv(A,C))
P=[pinv(C) null(C)]
Abar=inv(P)*A*P
Bbar= inv(P)*B
A11=Abar(1:3,1:3)
A12=Abar(1:3,4:9)
A21=Abar(4:9,1:3)
A22=Abar(4:9,4:9)
B1=Bbar(1:3)
B2=Bbar(4:9)
Cbar=C*P
```

```
ebdash=eb(4:9)
L=place(A22',A12',ebdash')
L=L'
k=place(A,B,e)

F=A22-L*A12
G=A21-L*A11+F*L
H=B2-L*B1
```

Figure 4.1: matlab code

Code Run

```
L =

   5.9079    2.2018  -21.7194

  -3.5952   -0.2496    6.6544

  -4.1217   -0.2235    7.3061

  -0.5302    0.3403   -1.9480

   1.2407    0.7392   -6.8766

  -1.2651   -0.3483    3.1614

k =

19.4923   11.1164  -43.7121    9.0823   83.1818   45.9946  -18.0662  -18.4550  -35.0546

F =

  36.4301   -3.3944   22.0175  -84.6583  -77.6584   90.8864

 -14.9337  -18.3822    0.5036   21.7307   24.6742  -31.8093

 -16.9578  -19.0602   -2.2352   20.2287   27.4604  -36.5737

   3.7266   -3.1614    3.4033  -14.3596   -8.2363    6.3704

  13.7190   -1.8017    7.3922  -27.0418  -31.8703   28.0310

  -2.8466    1.6334   -2.4827    6.5534    4.9491  -13.7828
```

```
G =

 -12.9551    3.9050  403.6779

  34.0595   -4.2487 -133.2672

  40.9823   -5.9934 -148.2085

  10.1687   -0.7241   30.4008

   2.0288    0.9710  126.4009

   6.7493   -1.1025  -46.7857


H =

  58.8467

 -10.8687

 -11.3496

  12.6936

  25.9107

   1.4774
```

Figure 4.2: code results

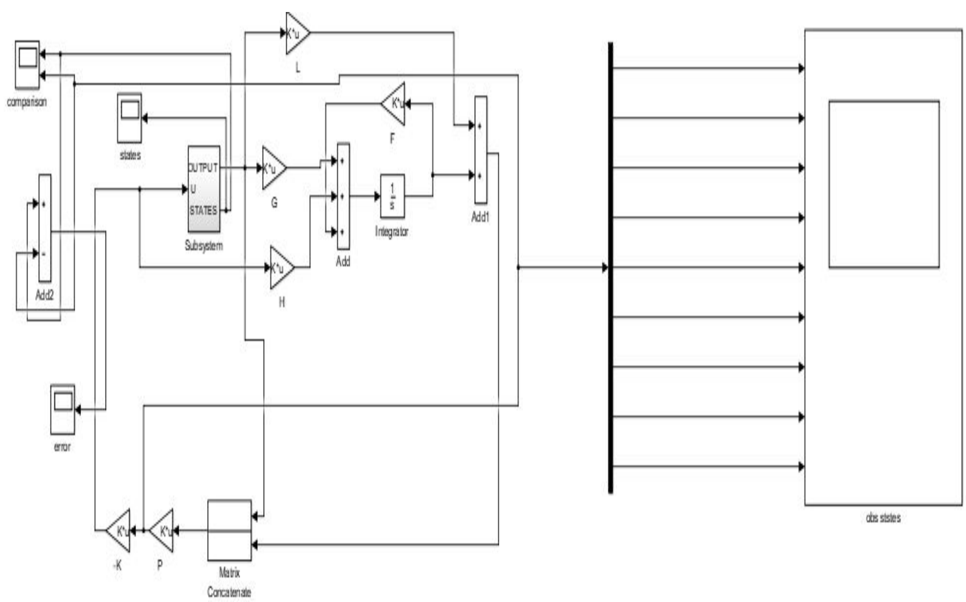

Figure 4.3: Simualtion model

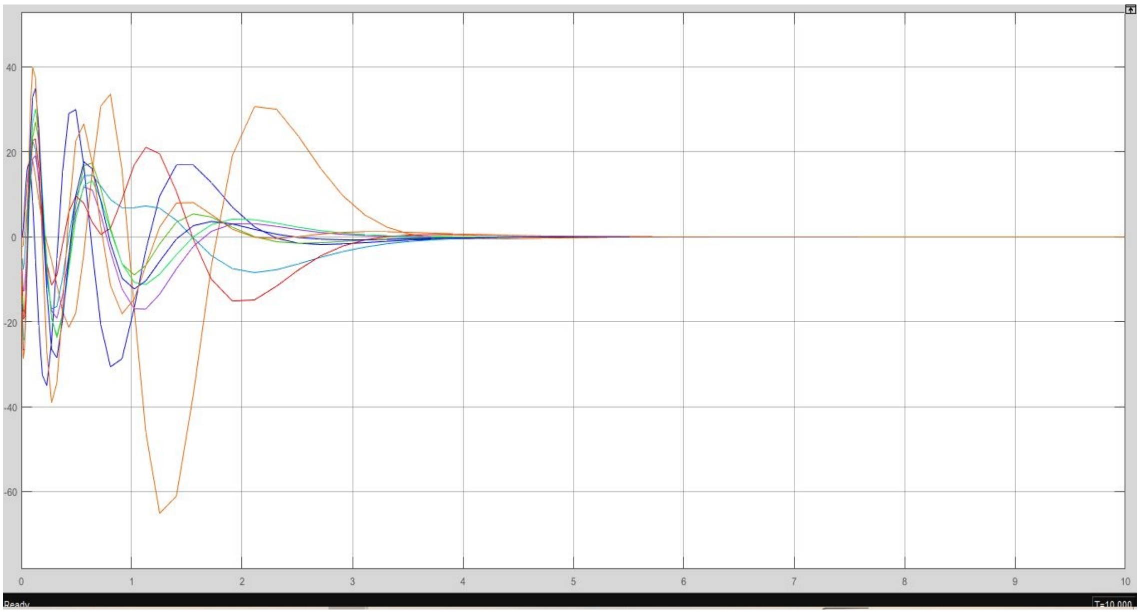

Figure 4.4: original state

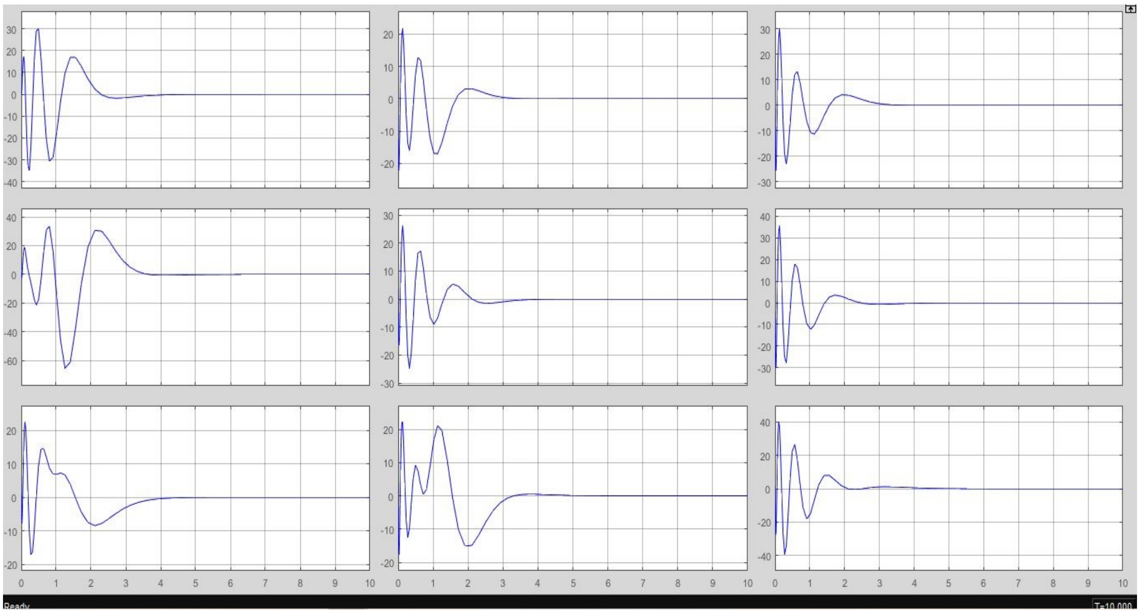

Figure 4.5: observed state

Figure 4.6: error

Chapter 5

Functional Observer

5.1 Introduction

Reduced order observer fails when only a linear combination of the state variables, i.e.,$Kx(t)$, is required, rather than a complete knowledge of the entire state vector$x(t)$. It's focus was on the reconstruction of the whole state vector or $(n-p)$ unmeasurable state variables, highlighting a redundant feature. The question therefore arises as to whether a less complex observer can be constructed to estimate a linear combination of some of the unmeasurable state variables. This is possible with the theory and construction of linear functional observers.

Full or partial state observers have been designed from the view point of exactly canceling the effect of external input signals with respect to state estimation error. In other words, the notion of approximation is not introduced in these design methods. This implies that the state-space of observers must include states having even little influence on state estimation. Therefore, it is difficult to design low-dimensional observers based on the above methods.

Furthermore, low-dimensional observer design methods have been obtained by the balanced truncation. However, in such observer design, it is not necessarily easy to obtain a reduced order observer with a specified error precision in a direct way. This is because the performance of the observation is indirectly determined throughout the approximation error by model reduction.

Therefore a need arises for functional observers.The primary aim of functional observer is to produce observers that are of further reduced-order, simpler structure and stable. This will reduce the computational time drastically.
A major result for this problem was first presented by Luenberger, where, for a linear functional observer to be able to estimate any single linear combination of the state vector, it would have an order of $(v-1)$, where v is known as the observability index. It is

defined as the least positive integer for which the following holds:

$$rank \begin{bmatrix} C \\ CA \\ CA^2 \\ . \\ . \\ CA^{v-1} \end{bmatrix} = n \tag{5.1}$$

5.2 Functional State Reconstruction Problem

Considering the system:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{5.2}$$

It is assumed that matrix C has full-row rank (i.e.,$rank(C) = p$) and the system is observable. The objective of a linear functional observer is to reconstruct a partial set of the state vector or a linear combination of the state vector with a reduced-order structure. Let $z(t) \in R^r$ be a vector that is required to be reconstructed (or estimated), where

$$z(t) = Fx(t) \tag{5.3}$$

and $F \in R^{r\times n}$ is a known matrix. It is assumed that $rank(F) = r$ and

$$rank \begin{bmatrix} F \\ C \end{bmatrix} = (r + p) \tag{5.4}$$

To reconstruct the state function, $z(t)$, the following observer structure of order $q, q \leq (n - p)$, is proposed

$$\begin{aligned} \hat{z}(t) &= Dw(t) + Ey(t) \\ \dot{w}(t) &= Nw(t) + Jy(t) + Hu(t) \end{aligned} \tag{5.5}$$

where $w(t) \in R^q$ and $\hat{z}(t) \in R^r$ is the estimate of $z(t)$. The observer matrices $D \in R^{r\times q}$, $E \in R^{r\times p}$,$N \in R^{q\times q}$, $J \in R^{q\times p}$ and$H \in R^{q\times m}$ are to be determined such that $\hat{z(t)}$ converges asymptotically to $z(t)$, i.e., $\hat{z(t)} \to z(t)$ as $t \to \infty$.

5.3 Existence Conditions

If it is assumed that any state which is unobservable can be eliminated by defining a lower dimensional observer state vector, then the order q of the observer defined in (5) should be less than or equal to the reduced-order state observer $q \leq n - p$. Let us define q as the smallest integer such that

$$rank\Sigma_q = rank \begin{bmatrix} \Sigma_q \\ FA^q \end{bmatrix}$$

With

$$\Sigma_q = \begin{bmatrix} C \\ F \\ CA \\ FA \\ . \\ . \\ . \\ CA^{q-1} \\ FA^{q-1} \\ CA^q \end{bmatrix}$$

The value of q which satisfies the above equation will be order of the observer and that will be minimum order functional observer.
Having defined the notation to be used, this section proceeds with the discussion of the linear function reconstruction problem.

In (5), the output $\hat{z}(t)$ provides an asymptotic estimate of $Fx(t)$ if

$$\lim_{x\to\infty} [\hat{z}(t) - Fx(t)] = 0$$

It stands to reason that if $\hat{z}(t)$ estimates $Fx(t)$, then$w(t)$ defined in (5) estimates other linear combination of $x(t)$, let it be called $Lx(t)$. This gives rise to the following theorem for the existence of the proposed linear functional observer (5).
Theorem 5.1. The estimate $\hat{z}(t)$ will converge asymptotically to $(Fx(t))$ for any initial condition $w(0)$ and any $u(t)$ if and only if the following conditions hold

$$N \quad is \quad Hurwitz \tag{5.6}$$

$$NL + JC - LA = 0 \tag{5.7}$$

$$H - LB = 0 \tag{5.8}$$

$$F - DL - EC = 0 \tag{5.9}$$

Here the reason underlying the above observer conditions will be provided. Define the error between $w(t)$ and $Lx(t)$ as

$$e(t) \doteq w(t) - Lx(t). \tag{5.10}$$

Taking the derivative of (10) and then substitute the observer equation (5) and system equations (2), the following is obtained

$$\begin{aligned} \dot{e}(t) &= \dot{w}(t) - L\dot{x}(t) \\ &= Nw(t) + JCx(t) + Hu(t) - LAx(t) - LBu(t) \\ &= Ne(t) + (NL + JC - LA)x(t) + (H - LB)u(t) \end{aligned} \tag{5.11}$$

Applying conditions (6) and (7) of Theorem 2.1 to the above equation yields

$$\dot{e}(t) = Ne(t). \tag{5.12}$$

Clearly, N controls the dynamics of the observer. Applying the condition (6) the following result is obtained

$$\lim_{x\to\infty}[e(t)] = \lim_{x\to\infty}[w(t) - Lx(t)] = 0 \tag{5.13}$$

Our main objective is to verify that the observer, $\hat{z}(t)$, correctly estimates the linear function Fx(t) which is the equivalent to (10). From the substitution of the observer equation, the substitution of the system output equation, and finally the application of condition (9), the following is obtained

$$\begin{aligned} e_z(t) &= \hat{z}(t) - Fx(t) \\ &= Dw(t) + ECx(t) - Fx(t) \\ &= D(w(t) - Lx(t)) \end{aligned} \tag{5.14}$$

From above, it is expected that $e_z(t)$ asymptotically approaches to zero as a result of (13). From the above analysis, it is clear that the first three conditions (6), (7) and (8) are conditions that ensure asymptotic convergence of some estimates of linear functions of the state vector, $Lx(t)$.
Its solution, L has a unique solution provided that A and N do not share common eigenvalues. The last condition (9) ensures that $\hat{z}(t)$ provides the correct estimation of the linear function $Fx(t)$.
Consider a closed-loop feedback system with a control input $u(t) = Fx(t)$, which is estimated by the observer (5), $\dot{x}(t)$ and $\dot{e}(t)$ can be derived as follows

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B\hat{u}(t) = Ax(t) + B(Dw(t) + Ey(t)) \\ &= Ax(t) + BDw(t) + B(F - DL)x(t) \\ &= (A + BF)x(t) + BDe(t) \\ \dot{e}(t) &= Ne(t). \end{aligned} \tag{5.15}$$

This yields the following composite closed-loop system

$$\begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} = \begin{bmatrix} (A + BF) & BD \\ 0 & N \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} \tag{5.16}$$

Clearly, the eigenvalues of the above composite closed-loop system are the union of the eigenvalues of $(A + BF)$ and of the observer matrix N. Since N is required to be Hurwitz and the control law stabilizes (A+BF), the overall composite closed-loop system is therefore stable. This reaffirms the separation principle of the proposed linear functional observer in (5).

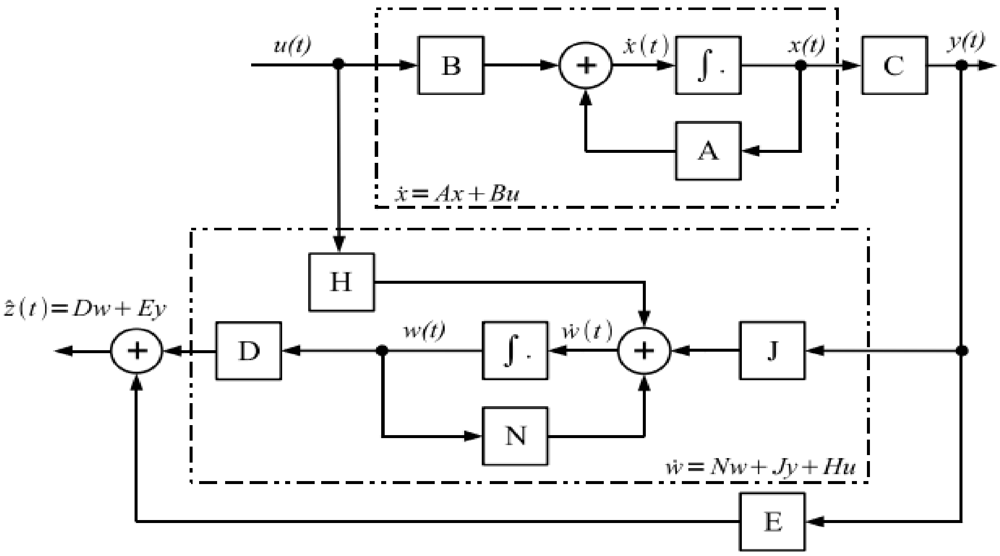

Figure 5.1: Linear Functional Observer

Chapter 6

Approach

6.1 Linear Matrix Inequality

Linear Matrix inequalities are:

- Inequalities involving matrix variables
- Matrix variable appear linearly
- Represent convex sets
- tool in modern control theory

thus
LMIs are matrix inequalities which are linear or affine in a set of matrix variables.They are essentially convex constraints and therefore many optimization problems with convex objective functions and LMI constraints can easily be solved efficiently using many existing software. This method has been very popular among control engineers in recent years. This is because a wide variety of control problems can be formulated as LMI problems. LMI has following general form

$$F(x) = F_0 + x_1F_1 + ... + x_nF_n \geq 0$$

where $x \in \Re^m$ is the vector of decision variables and are given constant symmetric real matrices, i.e. $F_i = {F_i}^T, i = 0 \ldots m$ The inequality symbol in the equation means $F(x)$ is positive definite, i.e.,$u^T F(x)u > 0$ for all nonzero $u \in \Re^n$.This matrix inequality is linear in the variables x_i. This matrix inequality is linear in the variables x_i.

$$A^T P + PA < 0$$

where $A \in \Re^{n\times n}$ is given and $X = X^T$ is the decision variable that can be expressed in the form of LMI as follows:
Let $P_1, P_2 \ldots P_m$ be a basis for the symmetric $n\times n$ matrices $(m = n(n+1)/2)$,take $F_0 = 0$ and $F_i = -A^T P_i - P_iA$
Example:Determine the stability of linear time-invariant systems.Consider the linear time-invariant system

$$\dot{x} = Ax$$

The system is stable provided the following inequalities are satisfied:

$$P > 0, \quad A^T P + PA < 0$$

These two inequalities can be combined into a single LMI as

$$\begin{bmatrix} A^T P + PA & 0 \\ 0 & -P \end{bmatrix} < 0$$

The MATLAB code for solving this stability problem for $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ is given as follows

Matlab code After running this programme we get $t_{min} = -2.615451$ and $P = \begin{bmatrix} 65.9992 & 12.8946 \\ 12.8946 & 15.1836 \end{bmatrix}$

Matlab Code

```
clc
clear

A=[0 1;-2 -3];

%declaring LMIs%

setlmis([])

P=lmivar(1,[size(A,1) 1]);

lmiterm([1 1 1 P],1,A,'s');

lmiterm([1 1 2 0],1);

lmiterm([1 2 2 P], -1,1);

LMISYS= getlmis;

[tmin,Psol]=feasp(LMISYS);

P=dec2mat(LMISYS,Psol,P);
```

Figure 6.1: code

So based on above approach functional observer is design as follows:

$$A = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & -5 & 3 & 4 \\ 1 & 1 & -8 & 3 \\ -4 & 0 & 2 & -6 \end{bmatrix} \tag{6.1}$$

$$C = \begin{bmatrix} 1 & 0 & 1 & 1 \end{bmatrix} \tag{6.2}$$

$$F = \begin{bmatrix} 1 & 4 & 8 & -4 \end{bmatrix} \tag{6.3}$$

Sol.
Here we have design the functional observer using LMIs in iteration loops and loop continues to run till we get N as Hurwitz i.e eigen values on left hand side and the other equations are satisfied.For given system observability index is 4 and we have taken observer order 2.

```
clc
clear
A=[-1 0 0 1;0 -5 3 4;1 1 -8 3;-4 0 2 6];
B=[1;2;3;4];
C=[1 0 1 1];
F=[1 4 8 -4];

N=cell(1000,1);
L=cell(1000,1);
J=cell(1000,1);
L{1}=[1 0 0 0;2 0 1 1];
%for observability index%
n=4
M=cell(n,1);
j=1;
while (j<n)
   M{j}=C*A^(j-1)

    Q=cell2mat(M)
    if rank(Q)==n
    break
else j=j+1
    end
end
v=j
```

```
%iterative loop for suitable value of N%
i=1;k=1;
while(1)

    setlmis([])
N{k}=lmivar(2,[2 2]);
J{k}=lmivar(2,[2 1]);
lmiterm([i 1 1 0],-0.0000001)
lmiterm([i 1 2 N{k}],1,L{k})
lmiterm([i 1 2 J{k}],1,C)
lmiterm([i 1 2 0],-L{k}*A)
lmiterm([i 2 2 0],-1)
lmisys = getlmis;
[tmin,xfeas] = feasp(lmisys);
 N{k} = dec2mat(lmisys,xfeas,N{k})
 J{k} = dec2mat(lmisys,xfeas,J{k})

 if eig(N{k})<0
    break
else    i=i+1;k=k+1

m=sym('m',[2,4]);
eq=N{k-1}*m+J{k-1}*C-m*A==0;
S=solve(eq,m);
V=[S.m1_1,S.m1_2,S.m1_3,S.m1_4;S.m2_1,S.m2_2,S.m2_3,S.m2_4]
L{k}=double(V)

 end
end
```

```
m=sym('m',[2,4]);
eq=N{k}*m+J{k}*C-m*A==0;
S=solve(eq,m);
V=[S.m1_1,S.m1_2,S.m1_3,S.m1_4;S.m2_1,S.m2_2,S.m2_3,S.m2_4]
L{k+1}=double(V)

setlmis([])
E=lmivar(2,[1 1]);
D=lmivar(2,[1 2]);
lmiterm([i+1 1 1 0],-0.001)
lmiterm([i+1 1 2 0],F)
lmiterm([i+1 1 2 D],-1,L{k+1})
lmiterm([i+1 1 2 E],-1,C)
lmiterm([i+1 2 2 0],-1)
lmisys = getlmis
[tmin,xfeas] = feasp(lmisys)
D = dec2mat(lmisys,xfeas,D)
E = dec2mat(lmisys,xfeas,E)
H=L{k+1}*B
```

N= -2.0000 -1.0000

-3.9999 -5.0000

L=

0.5455 0.3636 -0.3636 0.0909

2.1818 1.4545 -1.4545 0.3636

J=

2.0000

8.0000

D =

1.0072 -1.0435

E =

1.9545

H =

0.5455

2.1818

Chapter 7

Conclusion

It is observed that real part of pole of observer should be more negative than that of controller and it should not be much more negative otherwise bandwidth of system increases and which led to increase of noise in the system.
We used the condition for existence of minimum order Functional observer for the designing of functional observer. Our algorithm was based on designing of Scalar Functional Observer using iterative technique through LMI. Moreover, the parameters of the functional observer have been designed to achieve asymptotic stability . The upper bound for the order of Functional Observer was $(v-1)$ but we have designed it for order less than $(v-1)$.

Future work can be done by using BMI which can be implemented on YALMIP and TOMLAB software.

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