Conformal Invariance in Percolation

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We review properties of critical percolation in 2 dimensions. We state the conformal invariance hypothesis and derive a simplification of Cardy's formula. Finally, we apply Cardy's formula to compute the 1-arm exponent of the half-annulus.

§1 Introduction

In this note we investigate a number of intriguing features of critical percolation in 2 dimensions. We begin by considering Bernoulli bond percolation on the integer lattice, where sites are located at the points of $\mathbb{Z}^2$ and where bonds are edges between sites separated by a distance of 1. Each bond is independently open with probability p and closed with probability $1-p$. The resulting random graph has markedly different properties for different values of p in $[0,1]$. In particular, the model undergoes a sharp phase transition at the critical point $p_c \in (0,1)$, such that for $p < p_c$ there is almost surely no infinite connected component, whereas for $p > p_c$ there is almost surely an infinite cluster [AB87]. We consider the behavior of the horizontal crossing probability of a given $L \times H$ rectangle in each of the three regimes $p < p_c$, $p > p_c$, and $p = p_c$. After demonstrating scale-invariant properties of the crossing probability at the critical point, we introduce the concept of the scaling limit of a critical percolation model and the conformal invariance hypothesis. We state Cardy's formula and a useful simplification, and we conclude with an application to the 1-arm crossing probability of a half-annulus.

§2 Comparison of regimes

The sharpness of the percolation threshold implies that there are three regimes of a percolation model: subcritical ($p < p_c$), supercritical ($p > p_c$), and critical ($p = p_c$). In this section, our object of study will be the horizontal crossing probability of a rectangle with length L and height H. We will observe different behavior in each of the three regimes.

We first establish some useful notation. Let $\mathbb{P}_p$ denote the natural probability measure on the set Ω of all percolation configurations where each edge is open with probability p. (We drop the subscript when p is clear by context.) Given sites $x, y \in \mathbb{Z}^2$, let $|x-y|$ denote the ℓ_1-distance between x, y. Let $\{x \leftrightarrow y\}$ denote the event that x and y are connected by a path of open edges. Let the "two-point function" $\tau(x,y) = \mathbb{P}(x \leftrightarrow y)$ denote the probability of sites x and y being connected. Let $\theta(p) = \mathbb{P}(0 \leftrightarrow \infty)$ denote the probability that 0 is contained in an infinite connected component. The sharpness of the percolation threshold implies that $\theta(p) = 0$ for $p < p_c$ and $\theta(p) > 0$ for $p > p_c$.

All rectangles considered in this section are axis-aligned and have as their vertices sites of $\mathbb{Z}^2$. An $L \times H$ rectangle R has length L and height H. The event of a horizontal crossing of R from the left vertical edge to the right vertical edge is denoted $\mathcal{H}(R)$. Similarly, the event of a vertical crossing is denoted $\mathcal{V}(R)$.

Our main tool below will be the FKG inequality [DC18]. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a partial order $\succ$ on Ω, an event A is called increasing if $\omega \in A$ and $\omega' \succ \omega$ implies $\omega' \in A$.

Theorem 2.1 (FKG Inequality)
Let A and B be increasing events. Then $\mathbb{P}(A \cap B) \geq \mathbb{P}(A)\mathbb{P}(B)$.

Thus increasing events are positively correlated. As an example, note that the event $\mathcal{H}(R)$ is increasing; adding edges to a configuration cannot destroy the existence of a horizontal crossing.

§2.1 Subcritical regime

In the subcritical regime, the probability $\mathbb{P}(\mathcal{H}(R))$ exhibits exponential decay in L. We obtain this result through the exponential decay of the two-point function τ.

Theorem 2.2 (Exponential Decay)
If $p < p_c$, then $\tau(x, y) \leq Ae^{-m|x-y|}$ for constants $A, m > 0$.

Proof. See Section 5.3 of [AN84]. □

Corollary 2.3
Let R be an $L \times H$ rectangle with $L > H$. If $p < p_c$, then $\mathbb{P}(\mathcal{H}(R)) \leq Ae^{-mL}$ for $A, m > 0$.

Proof. Note that $\mathcal{H}(R)$ is a subset of the event that there exists a pair x, y of sites that are connected, where x, y lie on the left and right edges E_ℓ, E_r of R, respectively. By the union bound, we have

$$\mathbb{P}(\mathcal{H}(R)) \leq \sum_{x \in E_\ell, y \in E_r} \mathbb{P}(x \leftrightarrow y).$$

By Theorem 2.2, we have $\mathbb{P}(\mathcal{H}(R)) \leq H^2 \cdot Ae^{-mL}$ for $A, m > 0$. Since $H < L$, there exists $A' > 0$ such that $AH^2e^{-mL} \leq A'e^{-mL/2}$. Setting $m' = m/2$ yields $\mathbb{P}(\mathcal{H}(R)) \leq A'e^{-m'L}$. □

§2.2 Supercritical regime

In the supercritical regime, the crossing probability of a rectangle does not decay. We refer to this phenomenon as "long range order". Yet again we begin by studying the two-point function.

Theorem 2.4 (Long Range Order)
If $\theta(p) > 0$, then $\tau(x, y) \geq \theta(p)^2$ for all $x, y \in \mathbb{Z}^2$.

Proof. We assume the uniqueness of an infinite cluster; for a proof, see [BK89]. Let C_z denote the cluster containing z. Since $\{|C_x| = \infty\}, \{|C_y| = \infty\}$ are increasing, FKG implies that

$$\mathbb{P}(x \leftrightarrow y) \geq \mathbb{P}(\{|C_x| = \infty\} \cap \{|C_y| = \infty\}) \geq \mathbb{P}(|C_x| = \infty) \cdot \mathbb{P}(|C_y| = \infty) = \theta(p) \cdot \theta(p),$$

as desired. □

Corollary 2.5

Let R be an $L \times H$ rectangle. If $p > p_c$, then $\mathbb{P}(\mathcal{H}(R)) \geq \theta(p)^2$.

Proof. Fix x on the left edge of R, and fix y on the right edge of R. If $\{x \leftrightarrow y\}$ occurs, then clearly $\mathcal{H}(R)$ occurs. Hence $\mathbb{P}(\mathcal{H}(R)) \geq \mathbb{P}(x \leftrightarrow y) \geq \theta(p)^2$ by Theorem 2.4. □

§2.3 Critical regime

At the critical point, the crossing probability of a rectangle exhibits exponential decay in the aspect ratio L/H. Unlike in the previous two subsections, we do not use the two-point function to study crossing probabilities. Instead we make use of the following deep result.

Theorem 2.6 (Russo–Seymour–Welsh Theorem)

Let R be a $\rho H \times H$ rectangle with $\rho > 0$. Then there exists $c > 0$ such that $c \leq \mathbb{P}(\mathcal{H}(R)) \leq 1 - c$.

Proof. See Section 4.2 of [DC18]. □

We may bootstrap Theorem 2.6 to obtain the scale-invariant decay of probabilities at criticality.

Theorem 2.7

Let R be an $L \times H$ rectangle with $L > H$. If $p = p_c$, then there exist $c_1, c_2, C_1, C_2 > 0$ such that

$$C_1 e^{-c_1 L/H} \leq \mathbb{P}(\mathcal{H}(R)) \leq C_2 e^{-c_2 L/H}.$$

Proof. Assume WLOG that the vertices of R are $(0,0)$, $(L,0)$, (L,H), $(0,H)$. We begin with the upper bound. Construct $k = \lfloor L/H \rfloor$ disjoint $H \times H$ squares $S_1, \ldots, S_k$ in the interior of R, where square S_j has vertices $((j-1)H, 0)$, $(jH, 0)$, (jH, H), $((j-1)H, H)$; see Figure 1(b). Any horizontal crossing of R yields a horizontal crossing of each S_j, hence $\mathcal{H}(R) \subseteq \bigcap_{j=1}^{k} \mathcal{H}(S_j)$. Since $\{\mathcal{H}(S_j)\}$ are independent, $\mathbb{P}(\mathcal{H}(R)) \leq \prod_{j=1}^{k} \mathbb{P}(\mathcal{H}(S_j))$. By Theorem 2.6, $\mathbb{P}(\mathcal{H}(S_j)) \leq 1 - c$ for some $c > 0$, implying $\mathbb{P}(\mathcal{H}(R)) \leq (1-c)^{\lfloor L/H \rfloor}$, as desired.

We now consider the lower bound. Given an $H \times H$ square S, let $\mathcal{A}(S)$ denote the event that there is an annular loop around the center $\frac{H}{3} \times \frac{H}{3}$ square S_0, depicted in Figure 1(a). To make this precise, subdivide S into a grid of nine $\frac{H}{3} \times \frac{H}{3}$ squares. Let R_1, R_2, R_3, R_4 denote the rectangles formed by the first column, the first row, the third column, and the third row of the 3-by-3 grid R. We define $\mathcal{A}(S)$ as the event that there exists a connected path of open edges contained in $S \setminus S_0$ whose winding number around the center of S_0 is one. It is immediate that $\mathbb{P}(\mathcal{A}(S)) \geq \mathbb{P}(\mathcal{V}(R_1) \cap \mathcal{H}(R_2) \cap \mathcal{V}(R_3) \cap \mathcal{H}(R_4))$. By Theorem 2.6, there exists $c > 0$ such that $\mathbb{P}(\mathcal{V}(R_1)), \mathbb{P}(\mathcal{V}(R_3)) \geq c$ and $\mathbb{P}(\mathcal{H}(R_2)), \mathbb{P}(\mathcal{H}(R_4)) \geq c$. Since $\mathcal{V}(R_1), \mathcal{H}(R_2), \mathcal{V}(R_3), \mathcal{H}(R_4)$ are increasing, FKG implies that the right-hand side of the above inequality is bounded below by $c^4 > 0$.

We now use events of the form $\mathcal{A}(S)$ to generate a crossing of R. Construct $\ell = \lceil L/(H/3) \rceil$ overlapping $H \times H$ squares $S_1, \ldots, S_\ell$, where S_j has vertices $((j-1)\frac{H}{3}, 0)$, $(j\frac{H}{3}, 0)$, $(j\frac{H}{3}, H)$, $((j-1)\frac{H}{3}, H)$. We claim that

$$\mathbb{P}(\mathcal{H}(R)) \geq \mathbb{P}\left(\mathcal{H}(S_1) \cap \mathcal{H}(S_\ell) \cap \bigcap_{j=1}^{\ell} \mathcal{A}(S_j)\right).$$

This is most easily seen by inspecting Figure 1(c) below. If the event on the right-hand side occurs, then adjacent annular loops formed by the events $\mathcal{A}(S_j)$ must intersect. Hence all loops lie in a single connected cluster $\mathcal{C}$. If $\mathcal{H}(S_1)$ occurs, then the horizontal crossing of S_1 must also belong to $\mathcal{C}$, and likewise for the horizontal crossing of S_ℓ. Since the horizontal crossing of S_1 touches the left edge of R and the horizontal crossing of S_ℓ touches the right edge of R, it follows that the left and right edges of R are connected, as claimed.

To finish, we apply FKG to the right hand side of the above to find

$$\mathbb{P}(\mathcal{H}(R)) \geq \mathbb{P}(\mathcal{H}(S_1)) \cdot \mathbb{P}(\mathcal{H}(S_\ell)) \cdot \mathbb{P}(\mathcal{A}(S_1))^\ell.$$

By our work above and Theorem 2.6, there exists $c > 0$ such that each term on the right-hand side is bounded below by c. Thus $\mathbb{P}(\mathcal{H}(R)) \geq c^\ell = c^{\lceil 3L/H \rceil}$ and the claim follows. □

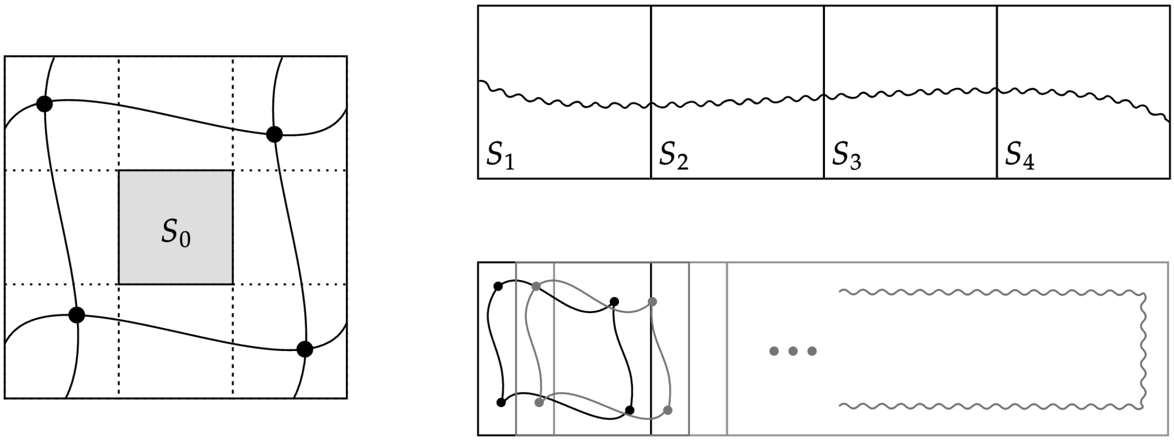

Figure 1: (a) Left, annular loop. (b) Top right, crossings of interior squares. (c) Bottom right, intersection of annular loops.

§3 Conformal invariance and Cardy's formula

The scale-invariance observed in the previous section at criticality is indicative of a fascinating phenomenon in critical percolation. The scaling limit of critical percolation is the model obtained by sending the lattice spacing to zero. In the scaling limit, Aizenman conjectured that crossing probabilities of regions should be invariant under conformal mappings. Formally, the conjecture is as follows [LPSA94].

Conjecture 3.1 (Conformal invariance hypothesis)**.** Let C be a simple closed curve and let α, β be arcs on C. Fix $\varepsilon > 0$. Given $A > 0$, let $\mathcal{D}_A C = \{\lambda x | x \in C, \lambda \in [A, A+\varepsilon]\}$ be the region between the dilations of C with respect to the origin by factors $A, A+\varepsilon$. Let $\pi_A(C, \alpha, \beta)$ denote the probability that there is a connection between the sets $\mathbb{Z}^2 \cap \mathcal{D}_A \alpha$ and $\mathbb{Z}^2 \cap \mathcal{D}_A \beta$ in the interior of $\{(A+\varepsilon)x | x \in C\}$. Then $\pi(C, \alpha, \beta) := \lim_{A \to \infty} \pi_A(C, \alpha, \beta)$ exists, and for any conformal map ψ we have $\pi(C, \alpha, \beta) = \pi(C', \alpha', \beta')$, where $C' = \psi(C)$, $\alpha' = \psi(\alpha)$, $\beta' = \psi(\beta)$.

The conformal invariance hypothesis was numerically verified by Langlands, Pouliot, and Saint-Aubin. Using methods from field theory, Cardy proposed a differential equation for the crossing probability between the segments $x_1 x_2$ and $x_3 x_4$ in the upper half-plane $\mathbb{H}$, as depicted in Figure 2(a). Writing

$u(x_1,x_2,x_3,x_4)=\frac{x_1-x_2}{x_1-x_3}\cdot\frac{x_4-x_3}{x_4-x_2}$ for the conformally-invariant cross-ratio of the points $\{x_1,x_2,x_3,x_4\}$, Cardy proposed that the crossing probability $f(x_1,x_2,x_3,x_4)$ depends only on u as follows [Aiz96].

Conjecture 3.2 (Cardy's formula)**.** Consider the scaling limit of critical percolation. Given x_1,x_2,x_3,x_4 on the extended real line, the probability of a connection between x_1x_2 and x_3x_4 in the upper half-plane $\mathbb{H}$ is given by $f(x_1,x_2,x_3,x_4)=\phi(u(x_1,x_2,x_3,x_4))$, where ϕ satisfies $u(1-u)\phi''(u)+\frac{2}{3}(1-2u)\phi'(u)=0$ with boundary conditions $\phi(0)=0$, $\phi(1)=1$.

We can write down the solution to the above ODE explicitly: if $K=\int_0^1\frac{dx}{(x(1-x))^{2/3}}$, then

$$\phi(u)=\frac{1}{K}\int_0^u\frac{dx}{(x(1-x))^{2/3}}.$$

It is easy to see that ϕ solves the equation. The Fundamental Theorem of Calculus implies $\phi'(u)=\frac{1}{K}\frac{1}{(u(1-u))^{2/3}}$ and $\phi''(u)=-\frac{1}{K}\frac{2}{3}(1-2u)\frac{1}{(u(1-u))^{5/3}}$, hence

$$u(1-u)\phi''(u)+\frac{2}{3}(1-2u)\phi'(u)=u(1-u)\cdot\left(-\frac{1}{K}\frac{2}{3}(1-2u)\frac{1}{(u(1-u))^{5/3}}\right)+\frac{2}{3}(1-2u)\cdot\frac{1}{K}\frac{1}{(u(1-u))^{2/3}}=0,$$

as desired. By our choice of K, the boundary conditions are attained.

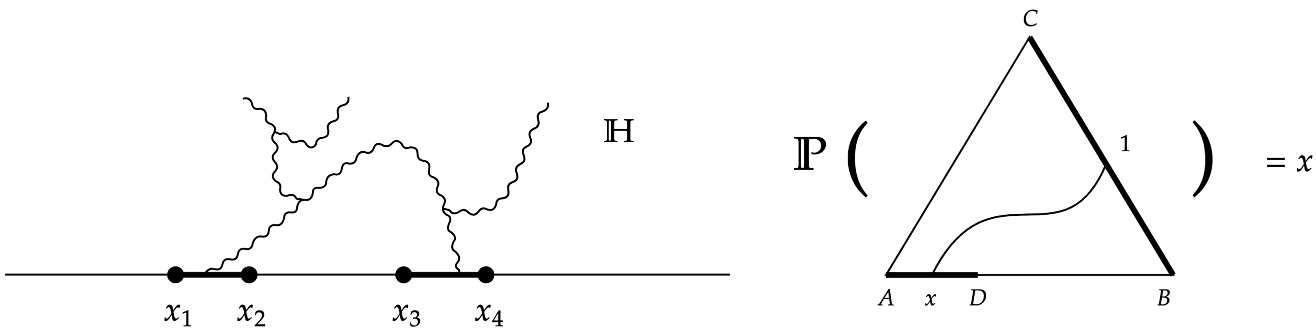

Figure 2: (a) Left, the event that x_1x_2 is connected to x_3x_4 in $\mathbb{H}$, reproduced from [Aiz96]. (b) Right, Cardy's formula for the equilateral triangle.

Cardy's formula matched the numerical results of Langlands, Pouliot, and Saint-Aubin and provided support for the hypothesis of conformal invariance [Aiz96]. In addition, the formula simplifies considerably if one applies a particular Schwarz–Christoffel mapping. The following, depicted in Figure 2(b), will be the version we use in the next section.

Proposition 3.3 (Cardy's formula for the triangle)

Consider the scaling limit of critical percolation. Let ABC be an equilateral triangle with unit sidelength, and let point D lie on AB with $AD=x$. Assuming conformal invariance and Cardy's formula, the probability that segments AD and BC are connected is x.

Proof. Let $x_1=0$, $x_2=x$, $x_3=1$, and $x_4=\infty$. Note that the general Schwarz–Christoffel mapping $g:\mathbb{H}\to\mathfrak{p}$ sending the upper half-plane to a polygon $\mathfrak{p}$ is given by

$$g(z)=\int_0^z\frac{C\,dw}{(w-a_1)^{1-\alpha_1/\pi}\cdots(w-a_k)^{1-\alpha_k/\pi}},$$

where integration is performed along any path from the origin to z and C is a constant [SS10]. The real numbers $a_1<\cdots<a_k$ are sent to the vertices of $\mathfrak{p}$ and $\alpha_1,\ldots,\alpha_k$ are the interior angles of $\mathfrak{p}$.

Setting $a_1 = x_4$, $a_2 = x_1$, $a_3 = x_3$, setting $\alpha_j = 1/3\pi$ for all j, and choosing $C = 1/K$, we find the Schwarz–Christoffel mapping

$$g(z) = \int_0^z \frac{1/K dw}{(w(1-w))^{2/3}}$$

whose image is a unit equilateral triangle $\mathfrak{p}$ with two vertices at $g(a_2) = g(0) = 0$ and $g(a_3) = \int_0^1 \frac{1/K dw}{(w(1-w))^{2/3}} = 1 \in \mathbb{R}$. The point $x_2 = x$ is sent to the real number $g(x) = \int_0^x \frac{1/K dw}{(w(1-w))^{2/3}}$ on the segment between $g(a_2)$ and $g(a_3)$. Let s_1 denote the segment between $g(x)$ and $g(a_2)$, and let s_2 denote the segment between $g(a_3)$ and $g(a_1)$. By conformal invariance, the probability of a connection between s_1 and s_2 equals the probability of a connection between a_1a_2 and a_3a_4 in $\mathbb{H}$. By Cardy's formula, the latter equals $\frac{1}{K}\int_0^u \frac{dw}{(w(1-w))^{2/3}}$, and it is easy to check that $u(0, x, 1, \infty) = x$. But we also know that the length $\ell(s_1)$ is precisely $g(x) - g(a_2) = \int_0^x \frac{1/K dw}{(w(1-w))^{2/3}}$. Thus the probability of the crossing equals the length of s_1 and the result follows. □

§4 Characteristic exponent of half-annulus

We now apply the above to calculate, up to constants, the crossing probability of a half-annulus with inner diameter ϵ and outer radius 1 in the scaling limit of critical percolation. Specifically, we consider the event of a connection between the "inner arc" of the semicircle of diameter ϵ and the "outer arc" of the semicircle of radius 1, depicted in Figure 3(a). Such an event is also referred to as the 1-arm crossing probability of the half-annulus. We show that this probability is on the order of $\epsilon^{1/3}$, and we refer to 1/3 as the characteristic exponent [Aiz96].

Proposition 4.1

Assuming conformal invariance and Cardy's formula, the 1-arm crossing probability q of a half-annulus with inner diameter ε and outer radius 1 satisfies $c_1\varepsilon^{1/3} \le q \le c_2\varepsilon^{1/3}$ for $c_1, c_2 > 0$.

Proof. Our proof is summarized in Figure 3. Orient the complex plane to have origin at the center of the semicircle with real axis along the diameter. Starting with the original crossing event, we apply three transformations to reduce it to an equilateral triangle. The three transformations are labeled **D1**, **M1**, and **D2**. Transformation **M1** is the conformal map $z \mapsto z^{1/3}$, and under the conformal invariance hypothesis, the crossing probability remains unchanged. Transformations **D1** and **D2** are what we call *deformations*. A deformation is simply a transformation that changes the probability by a nonzero multiplicative constant independent of ϵ. We prove the result assuming that **D1** and **D2** are deformations, and justify this at the end of this section.

The first deformation **D1** replaces the inner arc with a segment ℓ between 0 and ϵ on the real axis. The resulting crossing probability q' satisfies $c_1q \le q' \le c_2q$ for $c_1, c_2 > 0$. Next, we apply the conformal map **M1**. Since $re^{i\theta} \mapsto r^{1/3}e^{i\theta/3}$, the semicircle is sent to the sector of angle $\pi/3$, the segment $[0, \epsilon]$ is sent to $[0, \epsilon^{1/3}]$, and the outer arc is sent to the outer arc of the sector. By conformal invariance, the crossing probability remains q'. Finally, **D2** replaces the outer arc of the sector with a unit line segment, forming an equilateral triangle. The crossing probability q'' satisfies $c_3q' \le q'' \le c_4q'$ for $c_3, c_3 > 0$, hence $c_5q \le q'' \le c_6q$ for $c_5, c_6 > 0$. But by Proposition 3.3, q'' is precisely $\varepsilon^{1/3}$. The claim follows.

We now show that **D1** and **D2** are indeed deformations. Our main tool will be Theorem 2.6. For **D1**, let $\mathcal{E}_1$ denote the event that the inner arc is connected to the outer arc and let $\mathcal{E}_2$ denote the event that the segment ℓ is connected to the outer arc. Construct rectangles R_j, $1 \le j \le 6$, as follows:

- R_1: vertices at $-\epsilon, -0.5\epsilon, -0.5\epsilon + \epsilon i, -\epsilon + \epsilon i$;
- R_2: vertices at $-\epsilon + 0.5\epsilon i, \epsilon + 0.5\epsilon i, \epsilon + \epsilon i, -\epsilon + \epsilon i$;

- R_3: vertices at $0.5\epsilon, \epsilon, \epsilon + \epsilon i, 0.5\epsilon + \epsilon i$;
- R_4: vertices at $-0.5\epsilon, 0, \epsilon i, -0.5\epsilon + \epsilon i$;
- R_5: vertices at $-0.5\epsilon + 0.5\epsilon i, 1.5\epsilon + 0.5\epsilon i, 1.5\epsilon + \epsilon i, -0.5\epsilon + \epsilon i$;
- R_6: vertices at $\epsilon, 1.5\epsilon, 1.5\epsilon + \epsilon i, \epsilon + \epsilon i$.

Visually, rectangles R_1, R_2, R_3 form a "moat" around the inner arc, while rectangles R_4, R_5, R_6 form a moat around ℓ. As can be seen from Figure 4(a), if $\mathcal{V}(R_1)$, $\mathcal{H}(R_2)$, $\mathcal{V}(R_3)$, $\mathcal{E}_1$ occur, then $\mathcal{E}_2$ must occur. Thus $\mathbb{P}(\mathcal{E}_2) \geq \mathbb{P}(\mathcal{V}(R_1) \cap \mathcal{H}(R_2) \cap \mathcal{V}(R_3) \cap \mathcal{E}_1)$. Since all events are increasing, FKG implies $\mathbb{P}(\mathcal{E}_2) \geq \mathbb{P}(\mathcal{V}(R_1))\mathbb{P}(\mathcal{H}(R_2))\mathbb{P}(\mathcal{V}(R_3))\mathbb{P}(\mathcal{E}_1)$. Finally, by Theorem 2.6, there exists $c > 0$ such that $\mathbb{P}(\mathcal{V}(R_1)), \mathbb{P}(\mathcal{H}(R_2)), \mathbb{P}(\mathcal{V}(R_3)) \geq c$. It follows that $\mathbb{P}(\mathcal{E}_2) \geq c^3\mathbb{P}(\mathcal{E}_1)$. Similarly, from Figure 4(a), we see that if $\mathcal{V}(R_1)$, $\mathcal{H}(R_2)$, $\mathcal{V}(R_3)$, $\mathcal{E}_1$ occur, then $\mathcal{E}_2$ must occur. As above there exists $c' > 0$ such that $\mathbb{P}(\mathcal{E}_2) \geq (c')^3\mathbb{P}(\mathcal{E}_1)$, and the claim follows.

For **D2**, let $\mathcal{F}_1$ denote the event that the segment $[0, \epsilon^{1/3}]$ is connected to the outer arc of the sector, and let $\mathcal{F}_2$ denote the event that this segment is connected to the opposite side s of the equilateral triangle. It is clear that $\mathcal{F}_2$ occurs whenever $\mathcal{F}_1$ occurs, hence $\mathbb{P}(\mathcal{F}_2) \geq \mathbb{P}(\mathcal{F}_1)$. For the other direction, construct rectangles R_7, R_8 in the following way. Let R_7 have one edge along s and an orthogonal edge of sidelength $1/3$, as in the right diagram of Figure 4(b). Let $s' \neq s$ denote the edge of R_7 parallel to s. Let R_8 be a square of sidelength 1 with one edge along s', as in Figure 4(b). Let $\mathcal{G}_1$ be the event of a crossing of R_7 parallel to s, and let $\mathcal{G}_2$ be the event of a crossing of R_8 perpendicular to s. Because R_7, R_8 are rotations of axis-aligned rectangles, we may apply Theorem 2.6 to conclude that $\mathbb{P}(\mathcal{G}_1), \mathbb{P}(\mathcal{G}_2) \geq c > 0$. It is easy to see that if $\mathcal{G}_1, \mathcal{G}_2, \mathcal{F}_2$ occur, then so does $\mathcal{F}_1$. FKG implies $\mathbb{P}(\mathcal{F}_1) \geq c^2\mathbb{P}(\mathcal{F}_2)$, as desired. □

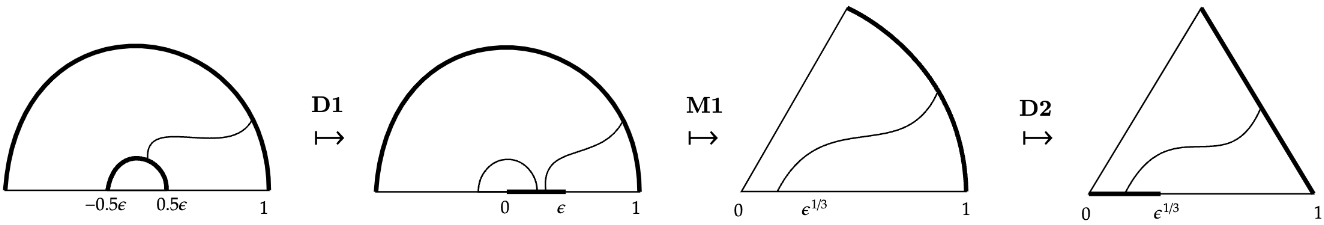

Figure 3: (a) First, the 1-arm crossing event. (b) Second, the event of a crossing between a segment and the outer arc. (c) Third, the event of crossing a sector. (d) Fourth, the event of crossing a triangle.

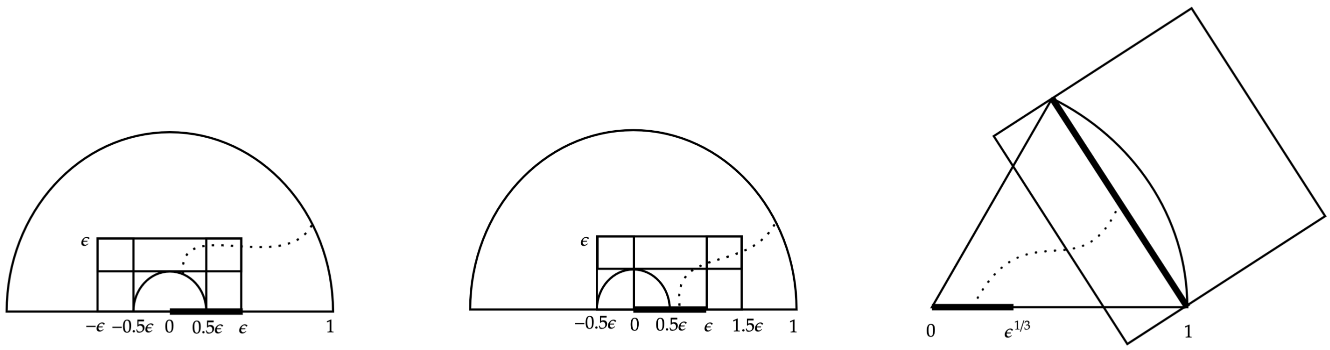

Figure 4: (a) Left/middle, proof of **D1** deformation. (b) Right, proof of **D2** deformation.

§5 Conclusion

In summary, this note has provided intuition for the validity of the conformal invariance hypothesis as well as applications. The hypothesis has been proven in scaling limits of certain percolation models, such as triangular site percolation [Smi09]. A general version of the k-arm problem has also been investigated and explicit characteristic exponents have been derived [BN11]. Overall, conformal invariance remains an important feature of critical percolation with many remaining avenues of exploration.

§6 Acknowledgements and Honor Pledge

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"I pledge my honour that this paper represents my own work in accordance with University regulations."

– Sunay Joshi

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