ORF550 Final Project: Random Field Ising Model

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§1 Introduction

In statistical mechanics, the Ising model is a well-known model of ferromagnetism. It is known that in dimensions 2 and above, the Ising model undergoes a phase transition at the critical Curie temperature, above which the model has decay of correlations, and below which the model has long-range order [Bov06]. However, when disorder is introduced into the Ising model, its characteristics change. In this report, we study the random field Ising model (RFIM) in 2 dimensions, in which the external magnetic field consists of independent and identically distributed (iid) random variables. As it turns out, in 2 dimensions the RFIM has no phase transition. We present background on the RFIM before presenting a recent proof of a quantitative rate of decay of correlations in the case of gaussian disorder by Chatterjee, which uses tools from concentration of measure [Cha18].

§2 Ising model

In order to set the stage for the random field Ising model, we begin by providing background on the Ising model, infinite-volume Gibbs measures, and (non)uniqueness of the Gibbs measure. Throughout this section, we follow the book [Bov06].

We begin with the Ising model in finite volume. Equip $\mathbb{Z}^2$ with the nearest-neighbor graph structure. We refer to vertices x of $\mathbb{Z}^2$ as sites, and we write $x \sim y$ if the sites x and y are adjacent in the graph. Consider a finite subset $\Lambda \subseteq \mathbb{Z}^2$. A spin configuration σ on Λ is a collection of numbers σ_x for all $x \in \Lambda$ such that $\sigma_x \in \{\pm 1\}$. Physically, σ_x models the spin of a magnetic dipole at site x. We denote the set of all spin configurations on Λ by $\mathcal{S}_\Lambda = \{\pm 1\}^\Lambda$. Fix a constant $h \in \mathbb{R}$, which represents the external magnetic field. The Hamiltonian is a function $H_\Lambda : \mathcal{S}_\Lambda \to \mathbb{R}$ that measures the total energy of the configuration σ. It is given by

$$H_\Lambda(\sigma) = -\sum_{\substack{x,y\in\Lambda \\ x\sim y}} \sigma_x\sigma_y - \sum_{x\in\Lambda} h\sigma_x$$

Finally, given an inverse temperature $\beta = 1/T \in [0, \infty]$, the Ising model on Λ at inverse temperature β is the probability measure on $\mathcal{S}_\Lambda$ given by

$$\mu_{\Lambda,\beta}(\sigma) = \frac{1}{Z_{\Lambda,\beta}} e^{-\beta H_\Lambda(\sigma)}$$

Here $Z_{\Lambda,\beta} = \sum_{\sigma \in \mathcal{S}_\Lambda} e^{-\beta H_\Lambda(\sigma)}$ is a normalizing factor known as the partition function. Before continuing, let us note a few interesting properties of $\mu_{\Lambda,\beta}$. As the temperature goes to zero and β goes to infinity, the measure is supported on ground states that minimize the Hamiltonian. As temperature goes to infinity and β goes to zero, the measure becomes uniform over all $2^{|\Lambda|}$ configurations. Finally, if the external field h is zero, then the measure has spin-flip symmetry, since in this case $H_\Lambda(\sigma) = H_\Lambda(-\sigma)$.

§2.1 Infinite-volume Gibbs measures

We would now like to define the Ising model on the infinite lattice $\mathbb{Z}^2$. Such a measure is known as an infinite-volume Gibbs measure. This theory was developed by Dobrushin, Lanford, and Ruelle, and the key idea is to specify conditional distributions on finite subdomains. Such a set of conditional distributions is called a local specification. Intuitively, the local specifications describe microscopic influences on finite sets of spins. The uniqueness of the infinite-volume Gibbs measure asks whether these microscopic influences determine the macroscopic behavior of the system. We shall discuss this question in more detail in the following sections.

Let $\mathcal{S} = \{\pm 1\}^{\mathbb{Z}^2}$ denote the set of all spin configurations on $\mathbb{Z}^2$. Given $\Lambda \subseteq \mathbb{Z}^2$, let σ_Λ denote the restriction of σ to Λ. Given a pair of spin configurations $\sigma, \eta \in \mathcal{S}$, the concatenation $(\sigma_\Lambda, \eta_{\Lambda^c})$ is the spin configuration that agrees with σ on Λ and that agrees with η on Λ^c. Equip each spin with the σ-algebra $\mathcal{F}_0 = 2^{\{\pm\}}$, and let $\mathcal{F}$ denote the product σ-algebra on $\mathcal{S}$. Given $\Lambda \subseteq \mathbb{Z}^2$, let $\mathcal{F}_\Lambda$ denote the σ-algebra generated by the spins on Λ. Let ρ the uniform product measure on $\mathcal{S}$, and given $\Lambda \subseteq \mathbb{Z}^2$, let ρ_Λ denote the uniform product measure on $\mathcal{S}_\Lambda$.

Similar to the above, the Hamiltonian is a function H sending a pair (Λ, σ) of a finite domain Λ and a configuration $\sigma \in \mathcal{S}$ to a real number in the following manner:

$$H_\Lambda(\sigma) = -\sum_{\substack{x \vee y \in \Lambda \\ x \sim y}} \sigma_x \sigma_y - \sum_{x \in \Lambda} h \sigma_x$$

Note that the sum includes pairs with one site lying outside of Λ. We now define the local specification corresponding to the Ising model.

Definition 2.1. The Gibbs specification is a collection of probability kernels $\{\mu_{\Lambda,\beta}^{(\cdot)}\}_{\Lambda \subseteq \mathbb{Z}^2}$ such that for all spin configurations $\eta \in \mathcal{S}$, finite subsets $\Lambda \subseteq \mathbb{Z}^2$, and measurable functions $f : \mathcal{S} \to \mathbb{R}$, we have

$$\int \mu_{\Lambda,\beta}^{\eta}(d\sigma) f(\sigma) = \int \rho_\Lambda(d\sigma_\Lambda) \frac{e^{-\beta H_\Lambda((\sigma_\Lambda, \eta_{\Lambda^c}))}}{Z_{\Lambda,\beta}^{\eta}} f((\sigma_\Lambda, \eta_{\Lambda^c}))$$

We refer to $Z_{\Lambda,\beta}^{\eta}$ as the partition function.

Note the following two properties of the Gibbs specification. First, it is clear that for all η, $\mu_{\Lambda,\beta}^{\eta}$ is a probability measure on $\mathcal{S}$. We can think of $\mu_{\Lambda,\beta}^{\eta}$ as a the conditional probability distribution on $\mathcal{S}$ given the requirement that the configuration σ agrees with the boundary condition η on Λ^c. Second, one can show that for all Λ and for all events $\mathcal{A} \in \mathcal{F}$, the function $\mu_{\Lambda,\beta}^{(\cdot)}(\mathcal{A}) : \mathcal{S} \to \mathbb{R}$ is $\mathcal{F}_{\Lambda^c}$-measurable. We are ready to define the concept of an infinite-volume Gibbs measure.

Definition 2.2. A probability measure μ_β is compatible with the Gibbs specification if $\mu_{\Lambda,\beta}^{(\cdot)}(f) = \mu_\beta(f | \mathcal{F}_{\Lambda^c})$ for all bounded measurable functions $f : \mathcal{S} \to \mathbb{R}$. Such a measure is called an infinite-volume Gibbs measure.

In other words, μ_β is compatible with the specification if $\mu^{(\cdot)}_{\Lambda,\beta}$ equals the conditional expectation of μ_β given the σ-algebra $\mathcal{F}_{\Lambda^c}$ generated by spins outside of Λ. In fact, we have the following converse due to Dobrushin–Lanford–Ruelle.

Theorem 2.3 (DLR equations)

The probability measure μ_β is a Gibbs measure for the Gibbs specification $\{\mu^{(\cdot)}_{\Lambda,\beta}\}$ iff for all $\Lambda \subseteq \mathbb{Z}^2$ and all measurable $f : \mathcal{S} \to \mathbb{R}$, we have $\mu_\beta \mu^{(\cdot)}_{\Lambda,\beta}(f) = \mu_\beta(f)$.

§2.2 Positive association

An increasing and absorbing sequence of volumes is a sequence of subsets $\Lambda_1 \subseteq \Lambda_2 \subseteq \cdots \subseteq \mathbb{Z}^2$ such that every finite subset $\Lambda' \subseteq \mathbb{Z}^2$ is contained in some Λ_n. One approach to constructing Gibbs measures on $\mathbb{Z}^2$ is by taking limits of local specifications $\mu^{\eta}_{\Lambda_n,\beta}$ along a sequence of increasing and absorbing volumes for some fixed boundary condition η. This approach works because of the positive association property of Gibbs measures for the Ising model, whereby increasing functions are positively correlated. Formally, the result is as follows.

Theorem 2.4

Let $+$ denote the configuration $\eta_x = +1$ for all $x \in \mathbb{Z}^2$, and let $-$ denote the configuration $\eta_x = -1$ for all $x \in \mathbb{Z}^2$. Let $\{\Lambda_n\}$ be an increasing and absorbing sequence of volumes. Then:

1. The sequence of measures $\mu^{+}_{\Lambda_n,\beta}$ converges weakly to a Gibbs measure μ^{+}_{β}.
2. The sequence of measures $\mu^{-}_{\Lambda_n,\beta}$ converges weakly to a Gibbs measure μ^{-}_{β}.
3. For any Gibbs measure μ_β at inverse temperature β and for any increasing, bounded, continuous function f, we have stochastic domination: $\mu^{-}_{\beta}(f) \le \mu_\beta(f) \le \mu^{+}_{\beta}(f)$.

As an immediate corollary of the above, there is a unique Gibbs measure at inverse temperature β iff $\mu^{+}_{\beta} = \mu^{-}_{\beta}$. In fact, there is an even more compact criterion for uniqueness at any β via the order parameter. Given a Gibbs measure μ_β, the order parameter, also known as the magnetization, is simply the average spin at the origin: $\mu_\beta(\sigma_0)$. The following result will be useful in our proofs of uniqueness to come.

Theorem 2.5

There is a unique Gibbs measure at inverse temperature β iff $\mu^{+}_{\beta}(\sigma_0) = \mu^{-}_{\beta}(\sigma_0)$.

Note that expectations with respect to μ^{+}_{β} and μ^{-}_{β} are denoted $\langle \cdot \rangle_+$ and $\langle \cdot \rangle_-$ in the physics literature, and we use this notation at times below.

§2.3 Uniqueness and Nonuniqueness

As stated at the beginning of this section, we would like to understand when a local specification is compatible with a unique (infinite-volume) Gibbs measure. Dobrushin's uniqueness criterion provides a condition for when a local specification is compatible with a single Gibbs measure. Although we do not dwell on it here, Dobrushin's criterion applies when the temperature β^{-1} is large. This fits with our intuition that at high temperatures, the Ising measure behaves as a product measure.

We now search for a setting where we have nonuniqueness of the Gibbs measure. Intuitively, nonuniqueness occurs at low temperatures and zero magnetic field, and therefore we should have a phase transition when we go from high temperature to low temperature. Specifically, at $h = 0$, the Gibbs measures μ_β^+ and μ_β^- defined earlier should be distinct.

The Peierls argument establishes this fact. It utilizes a contour representation of the spin configuration. Given a configuration $\sigma \in \mathcal{S}$, the set $\Gamma(\sigma)$ denotes the set of dual edges in σ; for each xy with $\sigma_x \neq \sigma_y$, the corresponding dual bond is the unit line segment centered on the midpoint of xy and perpendicular to xy. Each connected component γ of $\Gamma(\sigma)$ is called a contour. The following theorem shows that with probability bigger than $1/2$, there is no closed contour surrounding the origin, and hence the origin is "connected to ∞." As a corollary, the boundary conditions influence the spin at the origin, and so the Gibbs measures μ_β^+ and μ_β^- corresponding to plus and minus boundary conditions are distinct.

Theorem 2.6 (Peierls argument)
Consider the Ising model at $h = 0$. Then there exists $\beta_d < \infty$ such that for $\beta > \beta_d$, we have

$$\mu_\beta^+(\exists \gamma \in \Gamma(\sigma) : 0 \in \text{int}\gamma) < \frac{1}{2}$$

Proof. We follow the exposition in [FV17]. We fix an $(2n+1) \times (2n+1)$ box Λ_n centered at the origin and work in finite volume. We begin by showing that for any given closed contour γ inside of Λ_n,

$$\mu_{\Lambda_n,\beta}^+(\gamma \in \Gamma(\sigma)) \leq e^{-2\beta|\gamma|},$$

where $|\gamma|$ denotes the length of γ.

Note that the Hamiltonian $H_{\Lambda_n}(\sigma)$ can be rewritten in terms of contours as follows:

$$\begin{aligned} H_{\Lambda_n}(\sigma) &= \sum_{\substack{x \vee y \in \Lambda_n \\ x \sim y}} (1 - \sigma_x \sigma_y) - 2(2n+2)(2n+1) \\ &= 2 \sum_{\gamma' \in \Gamma(\sigma)} |\gamma'| - 2(2n+2)(2n+1) \end{aligned}$$

It follows that the probability that $\gamma \in \Gamma(\sigma)$ is given by

$$\begin{aligned} \mu_{\Lambda_n,\beta}^+(\gamma \in \Gamma(\sigma)) &= \frac{\sum_{\sigma : \gamma \in \Gamma(\sigma)} e^{-\beta H_{\Lambda_n}(\sigma)}}{\sum_\sigma e^{-\beta H_{\Lambda_n}(\sigma)}} \\ &= e^{-2\beta|\gamma|} \frac{\sum_{\sigma:\gamma \in \Gamma(\sigma)} \exp(-2\beta \sum_{\gamma' \neq \gamma} |\gamma'|)}{\sum_\sigma \exp(-2\beta \sum_{\gamma'} |\gamma'|)} \end{aligned}$$

We now apply the key trick. Given a configuration σ containing the contour γ, consider the operation of flipping all spins in the interior of γ. Then the set of all other contours $\gamma' \neq \gamma$ remains the same, and in the resulting configuration γ is no longer a contour. In other words the numerator of the fraction above is at most $\sum_{\sigma:\gamma\notin\Gamma(\sigma)} \exp(-2\beta\sum_{\gamma'}|\gamma'|)$. It follows that the fraction is at most one and our probability is at most $e^{-2\beta|\gamma|}$.

Next, by a union bound and by splitting into cases on the length of γ,

$$\mu^+_{\Lambda_n,\beta}(\exists\gamma \in \Gamma(\sigma) : 0 \in \text{int}\gamma) \leq \sum_{\ell\geq 4} \frac{\ell}{2} \cdot 4 \cdot 3^\ell \cdot e^{-2\beta\ell},$$

where we used the fact that there are at most $\ell/2 \cdot 4 \cdot 3^\ell$ closed contours of length ℓ surrounding the origin. Since this smaller than $1/2$ for large β uniformly in n, the result follows from Theorem 2.4 by sending $n \to \infty$. □

Corollary 2.7

If $\beta > \beta_d$ (where β_d is defined in Theorem 2.6), then the two Gibbs measures μ^+_β and μ^-_β are distinct, with $\mu^+_\beta(\sigma_0) = -\mu^-_\beta(\sigma_0) > 0$.

Proof. Let $\{\Lambda_n\}$ be an increasing and absorbing sequence of domains such that $\mu^+_{\Lambda_n,\beta}$ converges to μ^+_β. Then

$$\begin{aligned}\mu^+_{\Lambda_n,\beta}(\sigma_0) &\geq 1 \cdot \mu^+_{\Lambda_n,\beta}(\nexists\gamma \in \Gamma(\sigma) : 0 \in \text{int}\gamma) + (-1) \cdot \mu^+_{\Lambda_n,\beta}(\exists\gamma \in \Gamma(\sigma) : 0 \in \text{int}\gamma) \\ &= 1 - 2\mu^+_{\Lambda_n,\beta}(\exists\gamma \in \Gamma(\sigma) : 0 \in \text{int}\gamma)\end{aligned}$$

By Theorem 2.4,

$$1 - 2\mu^+_{\Lambda_n,\beta}(\exists\gamma \in \Gamma(\sigma) : 0 \in \text{int}\gamma) \to 1 - 2\mu^+_\beta(\exists\gamma \in \Gamma(\sigma) : 0 \in \text{int}\gamma) > 0$$

as $n \to \infty$, so $\mu^+_\beta(\sigma_0) > 0$. By symmetry we find $\mu^-_\beta(\sigma_0) < 0$. Since the expected spin at the origin is different for these two measures, they are distinct, as desired. □

§3 Random field Ising model

We now define the random field Ising model with gaussian disorder, as well as random Gibbs measures for this model. Throughout this section, we follow the book [Bov06].

We introduce a probability space $(\Omega, \mathcal{B}, \mathbb{P})$, and for each $x \in \mathbb{Z}^2$, we define a random variable $\phi_x : \Omega \to \mathbb{R}$. We assume that the random variables are iid standard gaussian. Given a finite domain $\Lambda \subset \mathbb{Z}^2$ and an outcome $\omega \in \Omega$, the random Hamiltonian on Λ is given by

$$H_\Lambda[\omega](\sigma) = -\sum_{\substack{x\vee y\in\Lambda \\ x\sim y}} \sigma_x\sigma_y - \sum_{x\in\Lambda} \sqrt{v}\phi_x[\omega]\sigma_x,$$

where v is a positive constant. We see that in the RFIM, the external magnetic field is a random variable with variance v and varies from site to site. The notion of the random Gibbs specification and random Gibbs measures are natural generalizations of their nonrandom counterparts. Let $\mathcal{M}_1(\mathcal{S}, \mathcal{F})$ denote the set of all probability measures on the measurable space $(\mathcal{S}, \mathcal{F})$.

Definition 3.1. The random Gibbs specification for the RFIM is a collection of probability kernels $\{\mu^{(\cdot)}_{\Lambda,\beta}[\cdot]\}_{\Lambda\subseteq\mathbb{Z}^2}$ such that for all spin configurations $\eta \in \mathcal{S}$, outcomes $\omega \in \Omega$, finite subsets $\Lambda \subseteq \mathbb{Z}^2$, and measurable functions $f : \mathcal{S} \to \mathbb{R}$, we have

$$\int \mu^{\eta}_{\Lambda,\beta}[\omega](d\sigma)f(\sigma) = \int \rho_\Lambda(d\sigma_\Lambda)\frac{e^{-\beta H_\Lambda[\omega]((\sigma_\Lambda,\eta_{\Lambda^c}))}}{Z^{\eta}_{\Lambda,\beta}[\omega]}f((\sigma_\Lambda,\eta_{\Lambda^c}))$$

We refer to $Z^{\eta}_{\Lambda,\beta}[\omega]$ as the random partition function.

Definition 3.2. A measurable map $\mu_\beta : \Omega \to \mathcal{M}_1(\mathcal{S},\mathcal{F})$ is compatible with the random Gibbs specification if for $\mathbb{P}$-almost all ω, $\mu_\beta[\omega]$ is compatible with the local specification $\{\mu^{(\cdot)}_{\Lambda,\beta}[\omega]\}$. Such a measure is called a random infinite-volume Gibbs measure for the RFIM.

As in the case of the Ising model, we now ask the question of the uniqueness of the random Gibbs measure. First, note that by positive association, we may construct random Gibbs measures $\mu^{\pm}_\beta$ corresponding to limits of the $\pm$ boundary conditions. Also, note that we have an analogue of Theorem 2.5 for the RFIM, again via positive association. Given a random Gibbs measure μ_β, the order parameter is defined as $\mathbb{E}\mu_\beta[\omega](\sigma_0)$, where the expectation is over ω.

Theorem 3.3

There is a unique random Gibbs measure at inverse temperature β iff $\mathbb{E}\mu^{+}_\beta[\omega](\sigma_0) = \mathbb{E}\mu^{-}_\beta[\omega](\sigma_0)$.

In the case of the RFIM, the Peierls argument does not work, because the disorder breaks the symmetry of the deterministic model. In fact, in the following section we will rigorously show that in 2 dimensions, there is a unique random Gibbs measure at all β.

Before continuing, we present a nonrigorous justification of this result, known as the Imry–Ma argument. Recall that in the Peierls argument for the Ising model with zero magnetic field, contours surrounding the origin were exponentially unlikely due to the contribution of a "contour energy" of $2|\gamma|$. As a result, with probability larger than $1/2$ the origin had the same spin as ∞. In the case of RFIM, a contour still contributes $2|\gamma|$ to the Hamiltonian, but there is an additional contribution from the magnetic fields in the interior of the contour. Since the magnetic fields are iid gaussians with mean 0 and variance v, their sum yields an energy on the order of $\pm\sqrt{v|\text{int}\gamma|}$. Since in 2 dimensions, the quantity $\sqrt{|\text{int}\gamma|}$ is on the same order as the perimeter $|\gamma|$, it is plausible that the gaussian fluctuations will negate the contour energy and make the presence of contours around the origin far more likely. By this reasoning, Imry–Ma predicted that the random Gibbs measure would be unique in 2 dimensions.

The Aizenman–Wehr argument was the first proof of the uniqueness of the random Gibbs measure for RFIM. The essence of the argument is as follows. Given boundary conditions $\pm$, the free energy of the RFIM on a given finite subset $\Lambda \subseteq \mathbb{Z}^2$ for a given realization of the random field is defined as the logarithm of the partition function: $F_\pm = \log Z^{\pm}_{\Lambda,\beta}$. Fixing $x \in \Lambda$, we claim that $\partial F_\pm/\partial\phi_x = \beta\sqrt{v}\langle\sigma_x\rangle_\pm$. To see this, note that

$$\frac{\partial F_\pm}{\partial\phi_x} = \frac{\partial Z_\pm/\partial\phi_x}{Z_\pm} = \frac{1}{Z_\pm}\sum_\sigma \frac{\partial}{\partial\phi_x}e^{-\beta H(\sigma)} = \frac{1}{Z_\pm}\sum_\sigma \beta\sqrt{v}\sigma_x e^{-\beta H(\sigma)} = \beta\sqrt{v}\langle\sigma_x\rangle_\pm$$

In other words, in order to prove uniqueness of the random Gibbs measure, it suffices to control the difference $\partial F_+/\partial\phi_0 - \partial F_-/\partial\phi_0$. The Aizenman–Wehr argument considers a modified version of the free energy on Λ:

$$G_\pm = -\frac{1}{\beta}\log \mu_\beta^\pm[\omega](e^{-\beta\sum_{x\in\Lambda}\sqrt{v}\phi_x\sigma_x})$$

One can show that just as for the free energy, we have $\mathbb{E}\frac{\partial G_\pm}{\partial\phi_0} = \mathbb{E}\langle\sigma_0\rangle_\pm$. It therefore suffices to study the difference $\mathbb{E}[G_+ - G_-]$. It turns out that this difference satisfies an upper bound, as well as a central limit theorem-like lower bound that contradict each other unless $\mathbb{E}[\frac{\partial G_+}{\partial\phi_0} - \frac{\partial G_-}{\partial\phi_0}] = 0$, yielding the result.

However, the Aizenman–Wehr argument does not yield a rate of decay of the boundary influence along a sequence of finite volumes. As we shall see in the next section, we can establish uniqueness and a decay rate without the introduction of the modified free energy $G_\pm$.

§4 Quantitative decay of correlations

The first quantitative result on decay was obtained by Chatterjee in [Cha18]. In this last section, we outline his proof, which uses techniques from concentration of measure in a central way. In this section we follow the original paper [Cha18]. In a few instances we use slightly different bounds, but the end results are the same.

§4.1 Functions of gaussian variables

In the proof, we utilize two fundamental results about functions of iid gaussian variables. The first is an identity for the variance.

Theorem 4.1

Let $g = (g_1, \dots, g_n)$ be a vector of standard iid gaussian random variables, and let f be a C^∞ function of g with bounded derivatives of all orders. Then

$$\mathrm{Var} f = \sum_{k=1}^\infty \frac{1}{k!}\sum_{1\le i_1,\dots,i_k\le n}\mathbb{E}\left[\frac{\partial^k f}{\partial g_{i_1}\cdots\partial g_{i_k}}\right]^2$$

Proof. We recall the definition of the multivariate Hermite polynomials on $\mathbb{R}^n$ [AK18]. A multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ is a tuple of nonnegative integers. We define $|\alpha| = \sum_{i=1}^n \alpha_i$, $\alpha! = \prod_{i=1}^n \alpha_i!$, and $x^\alpha = \prod_{j=1}^n x_j^{\alpha_j}$. Also, given a function $f : \mathbb{R}^n \to \mathbb{R}$, we write

$$\frac{\partial^\alpha f}{\partial x^\alpha} = \frac{\partial^{|\alpha|} f}{\partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}}$$

Given a multi-index α, the Hermite polynomial associated to α is given by

$$P_\alpha(x) = \frac{1}{\sqrt{\alpha!}}(-1)^{|\alpha|}e^{\|x\|^2/2}\frac{\partial^\alpha}{\partial x^\alpha}e^{-\|x\|^2/2}$$

Note that the leading monomial in P_α is $\frac{1}{\sqrt{\alpha!}}x^\alpha$. It is easy to see that $\{P_\alpha(x)\}_\alpha$ forms an orthonormal basis for the space $L^2(\gamma^n)$, where γ^n denotes the standard gaussian measure on

$\mathbb{R}^n$. Indeed, suppose that $\alpha \neq \beta$ are two distinct multi-indices with $\alpha_1 < \beta_1$ WLOG. Then by integration by parts with respect to the coordinate x_1, if $\beta' = (0, \beta_2, \ldots, \beta_n)$, we have

$$\begin{aligned}
\langle P_\alpha, P_\beta \rangle_{L^2(\gamma^n)} &= \int_{\mathbb{R}^n} P_\alpha(x) \cdot \frac{1}{\sqrt{\beta!}} (-1)^{|\beta|} e^{\|x\|^2/2} \frac{\partial^\beta}{\partial x^\beta} e^{-\|x\|^2/2} \cdot \frac{e^{-\|x\|^2/2}}{(2\pi)^{n/2}} dx \\
&= \frac{(-1)^{|\beta|}}{(2\pi)^{n/2}\sqrt{\beta!}} \int_{\mathbb{R}^n} P_\alpha(x) \cdot \frac{\partial^\beta}{\partial x^\beta} e^{-\|x\|^2/2} dx \\
&= \frac{(-1)^{|\beta|}}{(2\pi)^{n/2}\sqrt{\beta!}} \int_{\mathbb{R}^{n-1}} (-1)^{\beta_1} \int_{\mathbb{R}} \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} P_\alpha(x) \cdot \frac{\partial^{\beta'}}{\partial x^{\beta'}} e^{-\|x\|^2/2} dx \\
&= 0,
\end{aligned}$$

where we used the fact that $\frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} P_\alpha(x) = 0$. Similarly, by integration by parts we have

$$\begin{aligned}
\langle P_\alpha, P_\alpha \rangle_{L^2(\gamma^n)} &= \int_{\mathbb{R}^n} P_\alpha(x)^2 \cdot \frac{e^{-\|x\|^2/2}}{(2\pi)^{n/2}} dx \\
&= \frac{(-1)^{|\alpha|}}{(2\pi)^{n/2}\sqrt{\alpha!}} \int_{\mathbb{R}^n} P_\alpha(x) \cdot \frac{\partial^\alpha}{\partial x^\alpha} e^{-\|x\|^2/2} dx \\
&= \frac{(-1)^{2|\alpha|}}{(2\pi)^{n/2}\sqrt{\alpha!}} \int_{\mathbb{R}^n} \frac{\partial^\alpha}{\partial x^\alpha} P_\alpha(x) \cdot e^{-\|x\|^2/2} dx \\
&= 1
\end{aligned}$$

It follows that a function $f : \mathbb{R}^n \to \mathbb{R}$ can be written as a Fourier series of the form $f(x) = \sum_\alpha \hat{f}(\alpha) P_\alpha(x)$, where $\hat{f}(\alpha) = \langle f, P_\alpha \rangle_{L^2(\alpha)}$. We claim that $\hat{f}(\alpha) = \mathbb{E}[\frac{\partial^\alpha f}{\partial x^\alpha}]$, where on the right-hand side f is a function of iid standard gaussian variables. To see this, note that by integration by parts,

$$\begin{aligned}
\hat{f}(\alpha) &= (-1)^{|\alpha|} \int_{\mathbb{R}^n} f(x) \cdot \frac{\partial^\alpha}{\partial x^\alpha} e^{-\|x\|^2/2} \cdot \frac{1}{(2\pi)^{n/2}} dx \\
&= (-1)^{2|\alpha|} \int_{\mathbb{R}^n} \frac{\partial^\alpha f}{\partial x^\alpha} \cdot d\gamma^n(x) \\
&= \mathbb{E}\left[\frac{\partial^\alpha f}{\partial x^\alpha}\right]
\end{aligned}$$

Next, note by definition that $\|f\|_{L^2(\gamma^n)}^2 = \mathbb{E}f^2$. By Parseval's formula, $\|f\|_{L^2(\gamma^n)}^2$ equals the sum of the squares of the Fourier coefficients:

$$\mathbb{E}f^2 = \sum_\alpha \frac{1}{\alpha!} \mathbb{E}\left[\frac{\partial^\alpha f}{\partial x^\alpha}\right]^2$$

Summing over $|\alpha| \geq 0$, rewriting the left-hand side, and cancelling the common term $(\mathbb{E}f)^2$, this equals

$$\mathrm{Var} f = \sum_{k \geq 1} \frac{1}{\alpha!} \sum_{|\alpha| = k} \mathbb{E}\left[\frac{\partial^\alpha f}{\partial x^\alpha}\right]^2$$

Finally, note that each derivative $\frac{\partial^\alpha f}{\partial x^\alpha}$ can be written in the form $\frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}}$ for $\frac{|\alpha|!}{\alpha!}$ tuples $1 \leq i_1, \ldots, i_k \leq n$. This implies the given identity. □

Our other tool is a result from concentration of measure: the gaussian Poincaré inequality.

Theorem 4.2 (Gaussian Poincaré inequality)
Let f and g be as in Theorem 4.1. Then

$$\mathrm{Var} f \le \mathbb{E}\left[\sum_{i=1}^{n}\left(\frac{\partial f}{\partial g_i}\right)^2\right]$$

Proof. For the sake of completeness, we present a proof using Theorem 4.1. Apply Theorem 4.1 to $\partial f/\partial g_i$ and add $E[(\partial f/\partial g_i)^2]$ to both sides to find

$$E\left[\left(\frac{\partial f}{\partial g_i}\right)^2\right] = \sum_{k=0}^{\infty}\frac{1}{k!}\sum_{1\le i_1,\dots,i_k\le n} E\left[\frac{\partial^{k+1} f}{\partial g_i \partial g_{i_1}\cdots\partial g_{i_k}}\right]$$

When we sum over $1 \le i \le n$, the right-hand side is

$$\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{1\le i_1,\dots,i_{k+1}\le n} E\left[\frac{\partial^{k+1} f}{\partial g_{i_1}\cdots\partial g_{i_{k+1}}}\right],$$

which is clearly at least

$$\sum_{k=0}^{\infty}\frac{1}{(k+1)!}\sum_{1\le i_1,\dots,i_{k+1}\le n} E\left[\frac{\partial^{k+1} f}{\partial g_{i_1}\cdots\partial g_{i_{k+1}}}\right] = \mathrm{Var} f,$$

where we again used Theorem 4.1. The claim follows. □

§4.2 Proof of decay rate

In this subsection we prove the main result, which implies the uniqueness of the random Gibbs measure at all β. For notational convenience, we drop explicit reference to the random field ω.

Theorem 4.3
Consider the RFIM on a set $\Lambda \subseteq \mathbb{Z}^2$ at inverse temperature $\beta \in [0, \infty]$. Let the random field distribution be gaussian with mean zero and variance v. Take any $x \in \Lambda$ such that $n \ge 3$, where n is the ℓ^∞ distance of x from $\partial\Lambda$. Then

$$\mathbb{E}[\langle\sigma_x\rangle_+ - \langle\sigma_x\rangle_-] \le \frac{C(1+v^{-1/2})}{\sqrt{\log\log n}},$$

where C is a universal constant. In particular, the bound has no dependence on β and holds even if $\beta = \infty$.

It turns out that it suffices to show the following result.

Lemma 4.4
Let Λ be an $n \times n$ square (i.e. with n sites per side), for some $n \ge 3$. Consider the RFIM on Λ at inverse temperature $\beta \in (0, \infty)$. Then there exists $x \in \Lambda$ such that

$$\mathbb{E}[\langle\sigma_x\rangle_+ - \langle\sigma_x\rangle_-] \le \frac{C(1+v^{-1/2})}{\sqrt{\log\log n}}$$

Before proving Lemma 4.4, let us see how it implies Theorem 4.3. Consider $\beta \in (0, \infty)$. Fix an $n \times n$ square Λ'. By Lemma 4.4, there exists $y \in \Lambda'$ such that $\mathbb{E}[\langle \sigma_y \rangle_{\Lambda',+} - \langle \sigma_y \rangle_{\Lambda',-}] \leq \frac{C(1+v^{-1/2})}{\sqrt{\log \log n}}$. Further, by translation-invariance of our model, if Λ' is translated by a given vector w, then the point $y + w$ satisfies the same bound. Since $x \in \Lambda$ is a distance n from the boundary of Λ, we may translate Λ' in such a way that Λ' is contained within Λ and y coincides with x. Therefore $\mathbb{E}[\langle \sigma_x \rangle_{\Lambda',+} - \langle \sigma_x \rangle_{\Lambda',-}] \leq \frac{C(1+v^{-1/2})}{\sqrt{\log \log n}}$. By Theorem 2.3 and comparison of boundary conditions in Theorem 2.4, $\langle \sigma_x \rangle_{\Lambda',+} - \langle \sigma_x \rangle_{\Lambda',-} \geq \langle \sigma_x \rangle_{\Lambda,+} - \langle \sigma_x \rangle_{\Lambda,-}$. The result follows.

We now prove Lemma 4.4.

Proof. Given a set $S \subseteq \Lambda$ of sites, let $M_\pm(S)$ denote the total magnetization $M_\pm(S) = \sum_{x \in S} \langle \sigma_x \rangle_\pm$ on S. By averaging over all n^2 sites in Λ, it suffices to show that

$$\mathbb{E}[M_+(\Lambda) - M_-(\Lambda)] \leq \frac{C(1+v^{-1/2})n^2}{\sqrt{\log \log n}}$$

In order to achieve this bound, we will construct a set $\mathcal{B}$ of $m \times m$ subsquares within Λ with $|\mathcal{B}| = \lfloor n/m \rfloor^2$. For our choice of m, most of the subsquares $B \in \mathcal{B}$ will satisfy

$$\mathbb{E}[M_+(B) - M_-(B)] \leq \frac{C(1+v^{-1/2})m^2}{\sqrt{\log \log n}}$$

Summing over all such $B \in \mathcal{B}$ and crudely bounding the few outlier sites will yield the desired result.

We shall choose m and $\mathcal{B}$ later in the proof. For now, fix an arbitrary $m \times m$ subsquare B. As above, let $F_\pm$ denote the free energy of the model on Λ with boundary condition $\pm$ for a given realization of the random field. Given $h \in \mathbb{R}$, consider the model obtained by replacing ϕ_x by $\phi_x + h$ for all $x \in B$. Let $F_\pm(h)$ denote the free energy of the resulting model.

We begin by showing that $M_\pm(B) = \frac{1}{\beta\sqrt{v}} F'_\pm(0)$. This is simply by the Chain Rule:

$$F'_\pm(0) = \sum_{x \in B} \frac{\partial(\phi_x + h)}{\partial h} \frac{\partial F_\pm}{\partial \phi_x} = \sum_{x \in B} \beta\sqrt{v} \langle \sigma_x \rangle_\pm = \beta\sqrt{v} M_\pm(B)$$

The above identity implies that in order to control the difference in total magnetization, it suffices to bound the difference $F'_+(0) - F'_-(0)$. If the free energy is sufficiently regular, then $F'_\pm(0)$ should be approximately the difference quotient $\frac{F_\pm(h) - F_\pm(0)}{h}$ for small h. Following this intuition, we apply the triangle inequality to find the bound $(\star)$:

$$\begin{aligned}
&\mathbb{E}[M_+(B) - M_-(B)] = \frac{1}{\beta\sqrt{v}}(\mathbb{E}[F'_+(0)] - \mathbb{E}[F'_-(0)]) \\
&\leq \frac{1}{\beta\sqrt{v}}\left[\left|\mathbb{E}[F'_+(0)] - \frac{\mathbb{E}[F_+(h)] - \mathbb{E}[F_+(0)]}{h}\right| + \left|\mathbb{E}\left[\frac{F_+(h) - F_+(0)}{h} - \frac{F_-(h) - F_-(0)}{h}\right]\right|\right. \\
&\left.+ \left|\mathbb{E}[F'_-(0)] - \frac{\mathbb{E}[F_-(h)] - \mathbb{E}[F_-(0)]}{h}\right|\right]
\end{aligned}$$

As it turns out, the second term can be easily bounded by $\frac{16\beta m}{h}$ without further assumptions on the subsquare B. Since the proof does not use any high-dimensional probability tools, we skip the proof and refer the reader to [Cha18].

We now turn to the remaining terms in ($\star$), which correspond to the error in our difference quotient approximation. We consider the first term with + boundary condition; the argument for the third term is identical and will be omitted. Note the validity of the following Taylor expansion for the free energy in h around zero:

Lemma 4.5

For any $h \geq 0$, we have

$$\mathbb{E}[F_+(h)] = \mathbb{E}[F_+(0)] + \sum_{k\geq 1} \frac{h^k}{k!}\mathbb{E}[F_+^{(k)}(0)],$$

and the same equality holds for F_- as well.

Proof. See Lemma 3.5 in [Cha18]. □

Rewriting Lemma 4.5, we find that

$$\mathbb{E}[F_+'(0)] - \frac{\mathbb{E}[F_+(h)] - \mathbb{E}[F_+(0)]}{h} = \sum_{k\geq 2} \frac{h^{k-1}}{k!}\mathbb{E}[F_+^{(k)}(0)]$$

As we have already noticed, derivatives of the free energy with respect to h can be rewritten nicely. We claim that

$$F_+^{(k)}(h) = \sum_{x_1,\ldots,x_k\in B} \frac{\partial^k}{\partial\phi_{x_1}\cdots\partial\phi_{x_k}} F_+(h)$$

This is a simple induction on k. The case $k = 1$ was essentially handled above. Given the result for $k-1$, differentiating both sides with respect to h and applying the Chain Rule yields the claim. In particular, setting $h = 0$ implies (†):

$$\mathbb{E}[F_+'(0)] - \frac{\mathbb{E}[F_+(h)] - \mathbb{E}[F_+(0)]}{h} = \sum_{k\geq 2}\sum_{x_1,\ldots,x_k\in B} \frac{h^{k-1}}{k!}\rho_+(x_1,\ldots,x_k),$$

where $\rho_+(x_1,\ldots,x_k) = \mathbb{E}[\frac{\partial^k F_+}{\partial\phi_{x_1}\cdots\partial\phi_{x_k}}]$.

At this point, we recognize a similarity between the right-hand side of (†) and the right-hand side of the identity in Theorem 4.1. This leads to the following bound.

Lemma 4.6

We have

$$\sum_{k\geq 1}\sum_{x_1,\ldots,x_k\in\Lambda} \frac{1}{k!}\rho_+(x_1,\ldots,x_k)^2 \leq \beta^2 v n^2,$$

and the same inequality holds for ρ_- as well.

Proof. Let our function of the gaussian variables $(\phi_x)_{x\in\Lambda}$ be F_+. (Note that we do not restrict to $x \in B$.) Then by Theorem 4.1,

$$\mathrm{Var} F_+ = \sum_{k\geq 1}\sum_{x_1,\ldots,x_k\in\Lambda} \frac{1}{k!}\rho_+(x_1,\ldots,x_k)^2$$

Also, by Theorem 4.2, this is bounded by the sum $\mathbb{E}[\sum_{x\in\Lambda}(\frac{\partial F_+}{\partial \phi_x})^2]$. As shown above, $\partial F_+/\partial\phi_x = \beta\sqrt{v}\langle\sigma_x\rangle$, whose square is clearly bounded by $\beta^2 v$. Summing this bound yields the claim. $\square$

Ideally, at this point one would use Lemma 4.6 to find a good bound for the same sum restricted to $x_1, \dots, x_k \in B$, and then would obtain the desired bound on (†) by an application of Cauchy–Schwarz. However, there is no reason to expect an arbitrary subsquare B to satisfy the bound in Lemma 4.6 with n replaced by m. As a result, we must choose B wisely.

We now describe our construction of the set $\mathcal{B}$. The key idea is as follows. We will split the set of k-tuples $(x_1, \dots, x_k)$ of points in Λ into tranches based on the ℓ^∞ diameter $d(x_1, \dots, x_k)$ of the set $\{x_1, \dots, x_k\}$. We will select m to be a distance such that the sum over the tranches below m is of the desired size.

Let $\varepsilon = 1/\log n$, and define the distances $m_0 = 0$ and $m_j = \varepsilon^{-j}$ for $j \geq 1$. Given $j \geq 1$, we define the sum

$$s_j = \sum_{k\geq 1} \sum_{\substack{x_1,\dots,x_k\in\Lambda \\ d(x_1,\dots,x_k)\in[m_{j-1},m_j)}} \frac{1}{k!}\rho_+(x_1,\dots,x_k)^2$$

The sum s_j represents the sum in Lemma 4.6 restricted to distances in tranche j, the interval $[m_{j-1}, m_j)$. By Lemma 4.6, we have the bound $\sum_{j\geq 1} s_j \leq \beta^2 v n^2$. We now consider the distance scale $\sqrt{n}$. Let L be the smallest integer such that $m_L \geq \sqrt{n}$. Explicitly, $L = \lceil \frac{\log n}{2\log\log n}\rceil$. Then since $\sum_{i\geq 1} s_i \leq \beta^2 v n^2$, there exists a positive integer $i \leq L$ such that $s_i \leq \frac{\beta^2 v n^2}{L}$, hence $s_i \leq 2\beta^2 v n^2 \frac{\log\log n}{\log n}$. We select m to be the largest integer less than m_i. Before continuing, note that $m_i \geq m_1 = \log n$, so $m \geq \lceil \log n\rceil - 1$. Also, $m_L \leq \sqrt{n}\log n$, so $m \leq \sqrt{n}\log n$.

Having selected m, we construct our set $\mathcal{B}$. Let Λ_0 be a $\lfloor n/m\rfloor m \times \lfloor n/m\rfloor m$ subsquare of Λ, and subdivide Λ_0 into a set $\mathcal{B}$ of $\lfloor n/m\rfloor^2$ $m\times m$ subsquares. Note that the size of $\mathcal{B}$ is at least $n^2/4m^2$. Also, note that we exclude at most $|\Lambda\setminus\Lambda_0| \leq 2nm \leq 2n^{3/2}\log n$ points, since at most m points are excluded along each axis; see Figure 1.

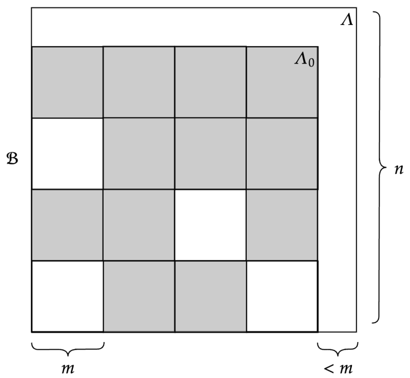

Figure 1: A possible decomposition of Λ, with "good" subsquares $B \in \mathcal{B}_0$ shaded gray.

We would now like to show that for most $B \in \mathcal{B}$, the sum

$$\sum_{k\geq 1}\sum_{x_1,\dots,x_k\in B}\frac{1}{k!}\rho_+(x_1,\dots,x_k)^2$$

can be bounded by a term on the order of $\beta^2 v m^2$. Note that since $m < m_i$, the sum equals

$s_0(B) + s_1(B)$, where

$$s_0(B) = \sum_{k\geq 1} \sum_{\substack{x_1,\dots,x_k\in B \\ d(x_1,\dots,x_k)<m_{i-1}}} \frac{1}{k!}\rho_+(x_1,\dots,x_k)^2$$

is the sum over tranches $1,\dots,i-1$ and

$$s_1(B) = \sum_{k\geq 1} \sum_{\substack{x_1,\dots,x_k\in B \\ d(x_1,\dots,x_k)\in[m_{i-1},m_i)}} \frac{1}{k!}\rho_+(x_1,\dots,x_k)^2$$

is the sum over tranche i. We will show that for most $B \in \mathcal{B}$, the values of $s_0(B)$ and $s_1(B)$ are roughly on the same order as their averages.

Let $\overline{s}_1$ denote the average of $s_1(B)$ over all $B \in \mathcal{B}$. Since s_i is a sum over tranche i for all of Λ, it is clear that $\overline{s}_1 \leq s_i/|\mathcal{B}|$. Hence $\overline{s}_1 \leq 8\beta^2 v m^2 \frac{\log\log n}{\log n}$.

Let $\overline{s}_0$ denote the average of $s_0(B)$ over all $B \in \mathcal{B}$. We crudely bound this by $\frac{1}{|\mathcal{B}|}\sum_{j\geq 1} s_j$, which by Lemma 4.6 is at most $4\beta^2 v m^2$.

Let $K = (\log n)^{1/12}$. We define the set $\mathcal{B}_0$ of "good" subsquares $B \in \mathcal{B}$ such that $s_1(B) \leq K^2\overline{s}_1$ and $s_0(B) \leq K^2\overline{s}_0$. We claim that there are few "bad" subsquares; specifically, $\frac{|\mathcal{B}\setminus\mathcal{B}_0|}{|\mathcal{B}|} \leq \frac{2}{K^2}$. To see this, equip $\mathcal{B}$ with the uniform measure, let X_0 be the random variable whose value at B equals $s_0(B)$, and let X_1 be the random variable whose value at B equals $s_1(B)$. By Markov, $\mathbb{P}[X_0 > K^2\overline{s}_0] \leq \frac{1}{K^2}$ and $\mathbb{P}[X_1 > K^2\overline{s}_1] \leq \frac{1}{K^2}$. A union bound implies the claim.

We now show that the first term in $(\star)$ can be controlled for $B \in \mathcal{B}_0$. (In the lemma below, we slightly abuse notation by suppressing the dependency of F_+ on B.)

Lemma 4.7

Let $h = \frac{\sqrt{\log\log n}}{2m}$. Then if $B \in \mathcal{B}_0$,

$$\left|\mathbb{E}[F_+'(0)] - \frac{\mathbb{E}[F_+(h)] - \mathbb{E}[F_+(0)]}{h}\right| \leq CK\beta\sqrt{v}m^2\frac{\sqrt{\log\log n}}{(\log n)^{1/4}},$$

and the same bound holds for F_-.

Proof. By (†), then left-hand side is bounded by

$$\sum_{k\geq 2}\sum_{x_1,\dots,x_k\in B}\frac{h^{k-1}}{k!}|\rho_+(x_1,\dots,x_k)|$$

We split this into a sum over tranche i and a sum over tranches $1,\dots,i-1$ and apply Cauchy–Schwarz to each sum.

For tranche i, Cauchy–Schwarz implies that

$$\sum_{k\geq 2} \sum_{\substack{x_1,\dots,x_k\in B \\ d(x_1,\dots,x_k)\in[m_{i-1},m_i)}} \frac{h^{k-1}}{k!}|\rho_+(x_1,\dots,x_k)| \leq \left[\sum_{k\geq 2} \sum_{\substack{x_1,\dots,x_k\in B \\ d(x_1,\dots,x_k)\in[m_{i-1},m_i)}} \frac{h^{2k-2}}{k!}\right]^{1/2}\sqrt{s_1(B)}$$

Since $|B| = m^2$, there are at most $(m^2)^k$ k-tuples $(x_1, \ldots, x_k)$ contained within B. Thus the expression in brackets on the right-hand side is at most $\sum_{k\geq 2} \frac{h^{2k-2}m^{2k}}{k!}$. Setting $x = h^2m^2$, this is at most $m^2 \sum_{k\geq 1} x^k/k!$, which is at most m^2e^x. By our choice of h, this is exactly $m^2(\log n)^{1/4}$. Plugging in our bound $s_1(B) \leq K^2\overline{s}_1$ for $B \in \mathcal{B}_0$, we find an upper bound of $\sqrt{8}K\beta\sqrt{v}m^2\frac{\sqrt{\log\log n}}{(\log n)^{1/4}}$ for tranche i.

We now handle the sum over tranches $1, \ldots, i-1$. If $i = 1$, then the sum is zero, so we may assume $i \geq 2$. By Cauchy–Schwarz, we have

$$\sum_{k\geq 2} \sum_{\substack{x_1,\ldots,x_k\in B \\ d(x_1,\ldots,x_k)<m_{i-1}}} \frac{h^{k-1}}{k!}|\rho_+(x_1,\ldots,x_k)| \leq \left[\sum_{k\geq 2} \sum_{\substack{x_1,\ldots,x_k\in B \\ d(x_1,\ldots,x_k)<m_{i-1}}} \frac{h^{2k-2}}{k!}\right]^{1/2} \sqrt{s_0(B)}$$

Note that if the diameter of the set $\{x_1, \ldots, x_k\}$ is less than m_{i-1}, then $\{x_2, \ldots, x_k\}$ lie inside the box centered at x_1 with sidelength $2m_{i-1}$. Such a box contains at most $(2m_{i-1}+1)^2$ sites. By our choice of i, we have $2m_{i-1}+1 = 2\varepsilon m_i + 1 \leq 2\varepsilon m + 3$. Since $m \geq \lceil \log n\rceil - 1$ and $\varepsilon = 1/\log n$ and $n \geq 3$, we have $\varepsilon m \geq c$, where $c = 1 - 1/\log 3 > 0$. Setting $p = \frac{3}{c} + 2$, we find $2\varepsilon m + 3 \leq p\varepsilon m$. Therefore given x_1, there are at most $(p\varepsilon m)^2$ points in the box, so that there are at most $m^2 \cdot (p\varepsilon m)^{2(k-1)}$ possible tuples $(x_1, \ldots, x_k)$. Thus the expression in brackets on the right-hand side is at most $m^2\sum_{k\geq 2}\frac{h^{2k-2}(p\varepsilon m)^{2k-2}}{k!}$. Setting $x = (p\varepsilon hm)^2$ and using the inequality $\sum_{k\geq 2} x^{k-1}/k! \leq xe^x$, we find a bound of $(p\varepsilon hm)^2 \cdot e^{(p\varepsilon hm)^2}$. By our choice of h, $(p\varepsilon hm)^2 = \frac{p^2}{4}\frac{\log\log n}{(\log n)^2}$, so $e^{(p\varepsilon hm)^2}$ is bounded by a constant and the expression in brackets is at most $\frac{C\log\log n}{(\log n)^2}$. Plugging in our bound $s_0(B) \leq K^2\overline{s}_0$ for $B \in \mathcal{B}_0$, we find an upper bound of $CK\beta\sqrt{v}m^2\frac{\sqrt{\log\log n}}{\log n}$ for tranches $1, \ldots, i-1$.

Clearly up to a constant, trache i contributes more, so that we have an upper bound of $CK\beta\sqrt{v}m^2\frac{\sqrt{\log\log n}}{(\log n)^{1/4}}$, as desired. $\square$

Plugging Lemma 4.7 for F_+ and F_- into $(\star)$, for "good" subsquares $B \in \mathcal{B}_0$, we find the bound

$$\begin{aligned}
\mathbb{E}[M_+(B) - M_-(B)] &\leq \frac{1}{\beta\sqrt{v}}\left[2\cdot CK\beta\sqrt{v}m^2\frac{\sqrt{\log\log n}}{(\log n)^{1/4}} + \frac{16\beta m}{h}\right] \\
&\leq CKm^2\frac{\sqrt{\log\log n}}{(\log n)^{1/4}} + \frac{Cv^{-1/2}m^2}{\sqrt{\log\log n}} \\
&\leq \frac{C(1+v^{-1/2})m^2}{\sqrt{\log\log n}}
\end{aligned}$$

It remains only to pass to Λ. It is at this point that we use the fact that most sites lie in some $B \in \mathcal{B}_0$. Note that if a site $x \in \Lambda$ is not in some $B \in \mathcal{B}_0$, then either $x \in \Lambda\setminus\Lambda_0$ or $x \in B \in \mathcal{B}\setminus\mathcal{B}_0$. We have shown that there are at most $2n^{3/2}\log n$ points outside of Λ_0. Also, we have shown that there are at most $\frac{2|\mathcal{B}|}{K^2}$ "bad" subsquares, which contain at most $|\mathcal{B}\setminus\mathcal{B}_0|m^2 \leq 2n^2/K^2$ sites. By a union bound and the fact that $|\langle\sigma_x\rangle| \leq 1$ for all x, it follows that

$$|M_+(\Lambda) - M_-(\Lambda)| \leq 2|\Lambda\setminus\Lambda_0| + 2|\mathcal{B}\setminus\mathcal{B}_0|m^2 + \sum_{B\in\mathcal{B}_0}|M_+(B) - M_-(B)|,$$

which by our choice of K is bounded by $\frac{C(1+v^{-1/2})n^2}{\sqrt{\log\log n}}$. The result follows. $\square$

§5 Acknowledgements

I would like to thank Professor Michael Aizenman for suggesting the paper [Cha18]. I would also like to thank Professor Ramon van Handel for suggesting the textbook [Bov06] and for helping explain some key concepts.

§6 Honor Pledge

I pledge my honor that I have not violated University guidelines while completing this assignment.

– Sunay Joshi

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