QUATERNIONIC ANALYSIS

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Abstract. In this paper, we discuss the topic of Quaternionic Analysis, in which we attempt to extend complex analysis to the quaternions.

1. Introduction

Quaternions are a four-dimensional algebra that extend the idea of a complex numbers to create an entirely new number system. They were invented by the Irish mathematician William Rowan Hamilton in [Ham66]. Much of the modern theory of Quaternionic Analysis was developed by Fueter in [Fue35].

Definition 1.1. The set of quaternions, denoted $\mathbb{H}$, is a 4-dimensional vector space over $\mathbb{R}$ with basis $1, i, j, k$, where i, j, k are the elementary quaternions.

Definition 1.2. The elementary quaternions satisfy

$$i^2 = j^2 = k^2 = ijk = -1.$$

Furthermore, they are anticommutative, meaning that

$$ij = -ji.$$

From this, we trivially obtain the result of Theorem 1.3.

Theorem 1.3. *The elementary quaternions satisfy*

$$ij = k, jk = i, ki = j.$$

We are now ready to extend the elementary quaternions to the larger set of quaternions.

Definition 1.4. A quaternion can be represented as a linear combination in the form

$$q = w + xi + yj + zk,$$

where $x, y, z \in \mathbb{R}$.

Definition 1.5. The conjugate of a quaternion $q = w + xi + yj + zk$ is

$$\overline{q} = w - xi - yj - zk.$$

Definition 1.6. The magnitude of a quaternion $q = w + xi + yj + zk$ is

$$|q| = \sqrt{w^2 + x^2 + y^2 + z^2}.$$

Definition 1.7. If $q = t + xi + yj + zk \in H$, then the real part $\mathrm{Re}(q) = t$ and the pure quaternion part $\mathrm{Pu}(q) = xi + yj + zk$.

The quaternions $\mathbb{H}$ form a 4-dimensional algebra over $\mathbb{R}$. If q is any quaternion linearly independent from 1, the vector subspace spanned by q and 1 is a subfield of $\mathbb{H}$ that is isomorphic to $\mathbb{C}$ [Sud98]. We can also embed $\mathbb{C}$ in $\mathbb{H}$; for instance, any quaternion $q \in \mathbb{H}$ can be expressed in the form

$$q = v + jw,$$

where $v, w \in \mathbb{C}$ and j is an elementary quaternion.

2. Differential Forms

Definition 2.1. We define the differential of a function $f : \mathbb{H} \to \mathbb{H}$ as

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz.$$

If this exists, we say that f is *real-differentiable*.

Definition 2.2. The differential of $f : \mathbb{H} \to \mathbb{H}$ at a point q is then defined analogously to Definition 2.1 as a linear mapping $df_q : \mathbb{H} \to \mathbb{H}$.

Definition 2.3. We also define the quaternion gradient operator as

$$\Box = \frac{\partial}{\partial w} + i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}.$$

Definition 2.4. Finally, we define the differential of the identity function

$$dq = dt + i\ dx + j\ dy + k\ dz.$$

3. Functions

Definition 3.1. A quaternion function $f : \mathbb{H} \to \mathbb{H}$ is quaternion-differentiable on the left at a point q if

$$\frac{df}{dq} = \lim_{h \to 0} \frac{f(q+h) - f(q)}{h}$$

exists as $h \to 0$ from any direction in $\mathbb{H}$.

Theorem 3.2. *Suppose that a function $f : \mathbb{H} \to \mathbb{H}$ is defined and quaternion-differentiable on the left throughout a connected open set U. Then on U, f has the form*

$$f(q) = aq + b,$$

where $a, b \in \mathbb{H}$.

Proof. [Sud98] If f is quaternion-differentiable on the left, we can write

$$df_q(h) = h\frac{df}{dq} = dq\frac{df}{dq}.$$

We then equate coefficients of t, x, y, z to obtain

$$\begin{aligned}\frac{df}{dq} &= \frac{\partial f}{\partial t} \\ &= -i\frac{\partial f}{\partial x} \\ &= -j\frac{\partial f}{\partial y} \\ &= -k\frac{\partial f}{\partial z}.\end{aligned}$$

Now suppose that $q = v + jw$, where $v = t + ix$ and $w = y - iz$, and let $f(q) = g(v, w) + jh(v, w)$, where g and h are complex functions of v and w. We can then separate our the previous equation into two sets of equations:

$$\frac{\partial g}{\partial t} = -i\frac{\partial g}{\partial x} = \frac{\partial h}{\partial y} = i\frac{\partial h}{\partial z},$$

and

$$\frac{\partial h}{\partial t} = i\frac{\partial h}{\partial x} = -\frac{\partial g}{\partial y} = i\frac{\partial g}{\partial z}.$$

We can write these in terms of complex derivatives as

$$\frac{\partial g}{\partial \overline{v}} = \frac{\partial h}{\partial \overline{w}} = \frac{\partial h}{\partial v} = \frac{\partial g}{\partial w} = 0,$$

$$\frac{\partial g}{\partial v} = \frac{\partial h}{\partial w},$$

and

$$\frac{\partial h}{\partial \overline{v}} = -\frac{\partial g}{\partial \overline{w}}.$$

Then, by Cauchy-Riemann, g is analytic in v and $\overline{w}$, and h is analytic in $\overline{v}$ and w. Then

$$\frac{\partial^2 g}{\partial v^2} = \frac{\partial g}{\partial v}\left(\frac{\partial h}{\partial w}\right) = \frac{\partial g}{\partial w}\left(\frac{\partial h}{\partial v}\right) = 0,$$

from which we find that g is linear and the desired result follows. □

Definition 3.3. Conformal mappings in quaternionic analysis, as in complex analysis, are those which preserve angles.

4. Regular Functions

Definition 4.1. We define the Left Cauchy-Riemann-Fueter equation of a function $f : \mathbb{H} \to \mathbb{H}$ as

$$\frac{\partial_l f}{\partial \overline{q}} = \frac{\partial f}{\partial t} + i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}.$$

Definition 4.2. Similarly, we define the Right Cauchy-Riemann-Fueter equation of a function $f : \mathbb{H} \to \mathbb{H}$ as

$$\frac{\partial_r f}{\partial \overline{q}} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j + \frac{\partial f}{\partial z}k.$$

Definition 4.3. A function $f : \mathbb{H} \to \mathbb{H}$ is left-regular or right regular if

$$\frac{\partial_l f}{\partial \overline{q}} = 0$$

or

$$\frac{\partial_r f}{\partial \overline{q}} = 0,$$

respectively.

If we write $q = v + jw$, the equations simplify to

$$\frac{\partial g}{\partial \overline{v}} = \frac{\partial h}{\partial \overline{w}}, \quad \frac{\partial g}{\partial w} = -\frac{\partial h}{\partial v}.$$

Definition 4.4. A harmonic function $f : \mathbb{H} \to \mathbb{H}$ is one that satisfies

$$\frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0.$$

Definition 4.5. We define

$$\Gamma_r(df) = \frac{\partial f}{\partial t} + i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}.$$

Theorem 4.6. *The left differential ∂_l can be written as*

$$\partial_l(f) = \frac{1}{2}\overline{\Gamma}_r(df).$$

Theorem 4.7. *[Sud98] If f is a harmonic function, then*

$$\partial_l f$$

is regular.

Theorem 4.8. *[Sud98] Let u be a real-valued function defined on a star-shaped open set $U \subseteq \mathbb{H}$. If U is harmonic and has a continuous second derivative, there exists a regular function f defined on U such that $Re(f) = u$.*

5. Integrals

Definition 5.1. We define the 3-form

$$Dq(x, y, z)$$

to be the alternating trilinear form

$$\langle w, Dq(x, y, z)\rangle = v(w, x, y, z),$$

where the volume form

$$v = dt \wedge dx \wedge dy \wedge dz.$$

Geometrically, $Dq(x, y, z)$ is the quaternion perpendicular to x, y, z with magnitude equal to the volume of the parallelepiped given by x, y, z.

Theorem 5.2. *[Sud98] $Dq(a, b, c) = \frac{1}{2}(c\overline{a}b - b\overline{a}c)$.*

Theorem 5.3. *[Sud98] A differentiable function f is regular at q if and only if*

$$Dq \wedge df_q = 0.$$

From this result, we find that

$$\int_{\partial D} Dq\ f = 0$$

if f is regular and continuously differentiable on a set D.

We also obtain the Cauchy-Fueter Integral Formula:

Theorem 5.4. *If f is regular on every point of a 4-parallelopiped C and $q_0 \in C$,*

$$f(q_0) = \frac{1}{2\pi^2} \int_{\partial C} \frac{(q-q_0)^{-1}}{|q-q_0|^2} Dqf(q).$$

Finally, we get to Cauchy's theorem:

Theorem 5.5. *Suppose f is regular in an open set U, and let C be a rectifiable 3-chain which is homologous to 0 in the singular homology of U. Then*

$$\int_C Dq\ f = 0.$$

We end with one application of quaternions to number theory.

6. Lagrange's Four-Square Theorem

In this section, we examine Lagrange's Four-Squares Theorem and its proof using quaternionic analysis. The theorem statement is as follows:

Conjecture 6.1 (Lagrange's Four-Square Theorem)**.** *Every nonnegative integer p can be written as*

$$p = a^2 + b^2 + c^2 + d^2,$$

where $a, b, c,$ and d are nonnegative integers.

To prove this, we'll first need several definitions to develop the necessary theory.

Definition 6.2. A Lipschitz quaternion is a quaternion of the form

$$a + bi + cj + dk,$$

where $a, b, c, d \in \mathbb{Z}$.

Definition 6.3. The Hurwitz Integers are the set of all quaternions with all-integer or all-half-integer components; that is, any Hurwitz Integer h can be expressed in the form

$$h = \frac{1}{2}a(1+i+j+k) + bi + cj + dk,$$

where a, b, c are all integers.

The following theorem then follows trivially.

Theorem 6.4. *The sum of any two Hurwitz Integers is itself a Hurwitz integer.*

Definition 6.5. A Hurwitz integer a is a multiple of a Hurwitz integer b if and only if there is a Hurwitz integer c such that $a = bc$.

Definition 6.6. [CW11] A Hurwitz Prime is a Hurwitz Integer which has no non-trivial factors (that is, factors other than elementary quaternions and

$$\pm\frac{1}{2}\pm\frac{1}{2}i\pm\frac{1}{2}j\pm\frac{1}{2}k$$

).

From now on, we'll refer to typical primes $p \in Z$ as "regular primes", and Hurwitz Primes as "Hurwitz Primes" to clearly differentiate the two.

Lemma 6.7. *[CW11] If positive integers p and q can be written as the sum of four squares, so can their product.*

Proof. Suppose that $p = a^2 + b^2 + c^2 + d^2$ and $q = w^2 + x^2 + y^2 + z^2$. Then, if

$$\alpha = a + bi + cj + dk \text{ and } \beta = w + xi + yj + zk,$$

we know that $p = ||\alpha||^2$ and $q = ||\beta||^2$, for some Lipschitz quaternions α and β. Then $uv = ||\alpha^2||| \cdot |\beta||^2 = ||\alpha\beta||^2 = ||A||^2$ for some Lipschitz quaternion A, from which the desired result follows. $\square$

This means that it suffices to show the Lagrange Four-Square Theorem for all regular primes. The following two lemmas from [CW11] are given without proof, as said proofs are not particularly relevant to the Lagrange Four-Square Theorem.

Lemma 6.8. *Let p be a Hurwitz prime. If*

$$p|\alpha\beta,$$

either $p|\alpha$ or $p|\beta$.

Lemma 6.9. *If p is an odd regular prime, there exist integers l and m such that*

$$p|1 + l^2 + m^2.$$

Theorem 6.10. *If p is an odd regular prime, it is not a Hurwitz Prime.*

Proof. We'll use contradiction. Suppose that p is a Hurwitz prime. Now, by Lemma 6.9, there exist integers l and m such that

$$p|1 + l^2 + m^2.$$

But

$$\begin{aligned}
(1 + li + mj)(1 - li - mj) &= (1 + (li + mj))(1 - (li + mj)) \\
&= (1)^2 - (li + mj)(li + mj) \\
&= 1 - (-l^2 - m^2 + li \cdot mj + mj \cdot li) \\
&= 1 + l^2 + m^2 - lm(i \cdot j - j \cdot i) \\
&= 1 + l^2 + m^2.
\end{aligned}$$

Then, by Lemma 6.8, if $p|1 + l^2 + m^2$, we must have $p|1 + li + mj$ or $p|1 - li - mj$. But then

$$\frac{1}{p} \pm \frac{li + mj}{p}$$

is a Hurwitz Integer, which is impossible, as $p > 2$. Therefore, p is not a Hurwitz Integer. $\square$

Theorem 6.11. *Let p be a regular odd prime. Then p can be written as the sum of four squares.*

Proof. Since p is not a Hurwitz prime, it can be factored as

$$p = (a + bi + cj + dk)\alpha. \tag{6.1}$$

Then

$$\overline{p} = p = \overline{\alpha}(a - bi - cj - dk).$$

Multiplying, we obtain

$$p^2 = (a + bi + cj + dk)\alpha\overline{\alpha}(a - bi - cj - dk) = (a^2 + b^2 + c^2 + d^2)||a||^2.$$

Since p is not a Hurwitz prime, the original factorization in (1) is nontrivial; therefore, we know that both parts of the factorization

$$p^2 = (a^2 + b^2 + c^2 + d^2)||a||^2$$

are equal to p. Then $a^2 + b^2 + c^2 + d^2 = p$, as desired. □

Finally, we are ready to prove Lagrange's Four-Square Theorem.

Theorem 6.12 (Lagrange's Four-Square Theorem)**.** *Every nonnegative integer n can be written as*

$$n = a^2 + b^2 + c^2 + d^2,$$

where $a, b, c,$ and d are nonnegative integers.

Proof. We start with base cases. We have

$$1 = 0^2 + 0^2 + 0^2 + 1^2$$

and

$$2 = 1^2 + 1^2 + 0^2 + 0^2.$$

By Theorem 6.11, any odd regular prime can be expressed in this form as well. Let

$$n = \prod_{i=1}^{k} p_i^{e_i}.$$

Since each of the p_i can be written in the desired form, Lemma 6.7 produces the desired result. □

References

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