# Technical Appendix

## Mathematical Derivation of Pool Skip

**The input setting:**

1. $X_{H\times W} = \{x_{i,j}\}_{H\times W}$: input matrix.
2. $W_{M\times M} = \{w_{i,j}\}_{M\times M}$: the convolutional kernel before Pool Skip. Assume $W$ is $M \times M$ kernel with $M$ being an odd number.
3. $Y_{(H-M+1)\times(W-M+1)} = \{y_{i,j}\}_{(H-M+1)\times(W-M+1)}$: the output of first convolutional computation.
4. $e$: max-pooling size which satisfies $e|H$, $e|W$, $e|H-M+1$ and $e|W-M+1$.
5. $A_{c\times d}$: the matrix obtained from max-pooling on $Y$. $c = (H-M+1)/e$ and $d = (W-M+1)/e$.
6. $\tilde{W}_{3\times 3} = \{\tilde{w}_{i,j}\}_{3\times 3}$: the convolutional kernel in the Pool Skip.
7. $O_{H-M+1,W-H+1} = \{o_{i,j}\}_{H-M+1,W-H+1}$: the output matrix.

**Detailed steps:**

**Step 1.** Apply $W_1$ to $X$ to obtain $Y$.

$$y_{i,j} = \sum_{m=0}^{M-1}\sum_{n=0}^{M-1} w_{m,n} \times x_{i+m,j+n} \tag{1}$$

Note now $Y$ is a $(H-M+1)\times(W-M+1)$ matrix.

**Step 2.** Size $e$ max-pooling.

We can write $Y$ into $c \times d$ blocks, for each block we have

$$Y^{(u,v)} = \begin{pmatrix} y^{(ue,ve)} & \cdots & y^{(ue,(v+1)e-1)} \\ \vdots & \ddots & \vdots \\ y^{((u+1)e-1,ve)} & \cdots & y^{((u+1)e-1,(v+1)e-1)} \end{pmatrix}_{e\times e} \tag{2}$$

where $u \in \{0,1,\cdots,c-1\}, v \in \{0,1,\cdots,d-1\}$. For each block $Y^{(u,v)}$, the maximum element is

$$\tilde{y}^{(u,v)} = \max_{i^{(u,v)},j^{(u,v)}\in\{0,1,\cdots,e-1\}} Y^{(u,v)}_{i^{(u,v)},j^{(u,v)}}, \tag{3}$$

with the corresponding $\tilde{i_a}^{(u,v)}, \tilde{j_a}^{(u,v)}$ as

$$(\tilde{i_a}^{(u,v)}, \tilde{j_a}^{(u,v)}) = \operatorname*{arg\,max}_{i^{(u,v)},j^{(u,v)}\in\{0,1,\cdots,e-1\}} Y^{(u,v)}_{i^{(u,v)},j^{(u,v)}}. \tag{4}$$

Therefore, we can write the matrix $A$ obtained from max-pooling on $Y$ as $A = \{\tilde{y}^{(u,v)}\}_{c\times d}$.

**Step 3.** First, we unpack $A$ to the $(H-M+1)\times(W-M+1)$ matrix $\hat{Y}$ such that the $c \times d$ blocks of $\hat{Y}$ be defined as

$$\hat{Y}^{(u,v)} = \{\hat{y}^{(u,v)}_{i_a,j_a}\}_{e\times e}, \quad \hat{y}^{(u,v)}_{i_a,j_a} = Y^{(u,v)}_{i_a,j_a}\mathbb{1}(\{(i_a,j_a) = (\tilde{i_a},\tilde{j_a})^{(u,v)}\}),\ u \in \{0,1,\cdots,c-1\}, v \in \{0,1,\cdots,d-1\}. \tag{5}$$

**Transfer local indices** $(\tilde{i_a},\tilde{j_a})^{(u,v)}$ **back to global indices**$(\tilde{i},\tilde{j})^{(u,v)}$, we have

$$\hat{y}_{i,j} = \begin{cases} \tilde{y}^{(u,v)}_{(\tilde{i_a}^{(u,v)},\tilde{j_a}^{(u,v)})}, & \text{if } e \ mod\ i = \tilde{i_a}^{(u,v)} \text{ and } e \ mod\ j = \tilde{j_a}^{(u,v)} \text{ in block } (u,v) \\ 0, & \text{o.w.} \end{cases} \tag{6}$$

where $u = \lfloor \frac{i}{e} \rfloor$, $v = \lfloor \frac{j}{e} \rfloor$, $\tilde{i_a} = i \ mod\ e$ and $\tilde{j_a} = j \ mod\ e$.

Add extra rows and columns of 0 outside $\hat{Y}$, we have a $(H-M+3)\times(W-M+3)$ matrix $Y' = \{y'_{i,j}\}$ and each $y'_{i,j}$ can be represented as

$$y'_{i,j} = \begin{cases} \hat{y}_{i+1,j+1} \\ 0, & \text{o.w.} \end{cases} \tag{7}$$

where $1 \le i \le H-M+1$ and $1 \le j \le W-M+1$.

Combine the Equation 7 and 6:

$$y'_{i,j} = \begin{cases} \tilde{y}^{(u,v)}_{(\tilde{i_a},\tilde{j_a})}, & \text{if } e \; mod \; (i-1) = \tilde{i_a}^{(u,v)} \text{ and } e \; mod \; (j-1) = \tilde{j_a}^{(u,v)} \text{ in block } Y^{(u,v)} \\ 0, & \text{o.w.} \end{cases} \tag{8}$$

where $1 \leq i \leq H - M + 1$ and $1 \leq j \leq W - M + 1$.

Combine the Equation 8 and 3:

$$y'_{i,j} = \begin{cases} \max\limits_{i_a,j_a} Y^{(u,v)}_{i_a,j_a}, & \text{if } e \; mod \; (i-1) = \tilde{i_a} \text{ and } e \; mod \; (j-1) = \tilde{j_a} \text{ in block } (u,v) \\ 0, & \text{o.w.} \end{cases} \tag{9}$$

where $1 \leq i \leq H - M + 1$ and $1 \leq j \leq W - M + 1$.

**Step 4.** $3 \times 3$ Conv.

$$y_{out,i,j} = \sum_{s=0}^{2} \sum_{t=0}^{2} \tilde{w}_{s,t} \times y'_{i+s,j+t} \tag{10}$$

Note now $Y'$ is a $(H - M + 1) \times (W - M + 1)$ matrix.

Combine the Equation 10 and 9:

$$y_{out,i,j} = \begin{cases} \sum\limits_{s=0}^{2} \sum\limits_{t=0}^{2} \tilde{w}_{s,t} \times \max\limits_{i_a,j_a} Y^{(u,v)}_{i_a,j_a}, & \\ \qquad \text{if } e \; mod \; (i-1-s) = \tilde{i_a} \text{ and } e \; mod \; (j-1-t) = \tilde{j_a} \text{ in block } (u,v) & \\ 0, & \text{o.w.} \end{cases} \tag{11}$$

where $1 \leq i \leq H - M + 1$ and $1 \leq j \leq W - M + 1$.

**Step 5.** Add Pool skip.

$$o_{i,j} = y_{i,j} + y_{out,i,j} \tag{12}$$

If $e \; mod \; (i-1-s) = \tilde{i_a}^{(u,v)}$ and $e \; mod \; (j-1-t) = \tilde{j_a}^{(u,v)}$ in block $(u,v)$, where $1 \leq i \leq H - M + 1$ and $1 \leq j \leq W - M + 1$.

$$\begin{aligned} o_{i,j} &= \sum_{m=0}^{M-1} \sum_{n=0}^{M-1} w_{m,n} \times x_{i-1+m,j-1+n} + \sum_{s=0}^{2} \sum_{t=0}^{2} \tilde{w}_{s,t} \times \max\{ \sum_{m=0}^{M-1} \sum_{n=0}^{M-1} w_{m,n} \times x_{ue+m+s,ve+n+t}, \\ &\quad \sum_{m=0}^{M-1} \sum_{n=0}^{M-1} w_{m,n} \times x_{ue+1+m+s,ve+1+n+t}, \cdots, \sum_{m=0}^{M-1} \sum_{n=0}^{M-1} w_{m,n} \times x_{(u+1)e+m+s,(v+1)e+n+t} \} \\ &= \sum_{m=0}^{M-1} \sum_{n=0}^{M-1} w_{m,n} \times x_{i-1+m,j-1+n} + \sum_{s=0}^{2} \sum_{t=0}^{2} \tilde{w}_{s,t} \times \sum_{m=0}^{M-1} \sum_{n=0}^{M-1} w_{m,n} \times x_{ue+\tilde{i_a}^{(u,v)}+m+s,ve+\tilde{j_a}^{(u,v)}+n+t} \end{aligned} \tag{13}$$

where $u \in \{0, 1, \cdots, c-1\}, v \in \{0, 1, \cdots, d-1\}$.

Denote set $K_i = \{(m,s) : ue + \tilde{i_a}^{(u,v)} + m + s \in ([i, i+M] \cap [ue + \tilde{i_a}^{(u,v)}, ue + \tilde{i_a}^{(u,v)} + M + 2])\}$ and $L_j = \{(n,t) : ve + \tilde{j_a}^{(u,v)} + n + t \in ([j, j+M] \cap [ve + \tilde{j_a}^{(u,v)}, ve + \tilde{j_a}^{(u,v)} + M + 2])\}$.

Note that when $(m,s) \in K_i$, $ue + \tilde{i_a}^{(u,v)} + m + s = i - 1 + m$, and when $(n,t) \in L_j$, $ve + \tilde{j_a}^{(u,v)} + n + t = j - 1 + n$. we have:

$$
\begin{aligned}
o_{i,j} &= \sum_{m=0}^{M-1}\sum_{n=0}^{M-1} w_{m,n} \times x_{i-1+m,j-1+n} + \sum_{s=0}^{2}\sum_{t=0}^{2} \tilde{w}_{s,t} \times \sum_{m=0}^{M-1}\sum_{n=0}^{M-1} w_{m,n} \times x_{ue+\tilde{i_a}^{(u,v)}+m+s,ve+\tilde{j_a}^{(u,v)}+n+t} \\
&= \sum_{\mathbb{1}_{K_i}((m,s))=1 \text{ and } \mathbb{1}_{L_j}((n,t))=1} (w_{m,n} \times x_{ue+\tilde{i_a}^{(u,v)}+m+s,ve+\tilde{j_a}^{(u,v)}+n+t} + \tilde{w}_{s,t} \times w_{m,n} \times x_{ue+\tilde{i_a}^{(u,v)}+m+s,ve+\tilde{j_a}^{(u,v)}+n+t}) \\
&\quad + \sum_{\mathbb{1}_{K_i}((m,s))\neq 1 \text{ or } \mathbb{1}_{L_j}((n,t))\neq 1} (w_{m,n} \times x_{i+m,j+n} + \tilde{w}_{s,t} \times w_{m,n} \times x_{ue+\tilde{i_a}^{(u,v)}+m+s,ve+\tilde{j_a}^{(u,v)}+n+t}) \\
&= \sum_{\mathbb{1}_{K_i}((m,s))=1 \text{ and } \mathbb{1}_{L_j}((n,t))=1} [(1+\tilde{w}_{s,t}) \times w_{m,n} \times x_{ue+\tilde{i_a}^{(u,v)}+m+s,ve+\tilde{j_a}^{(u,v)}+n+t}] \\
&\quad + \sum_{\mathbb{1}_{K_i}((m,s))\neq 1 \text{ or } \mathbb{1}_{L_j}((n,t))\neq 1} (w_{m,n} \times x_{i+m,j+n} + \tilde{w}_{s,t} \times w_{m,n} \times x_{ue+\tilde{i_a}^{(u,v)}+m+s,ve+\tilde{j_a}^{(u,v)}+n+t}) \\
&= \sum_{\mathbb{1}_{K_i}((m,s))=1 \text{ and } \mathbb{1}_{L_j}((n,t))=1} [(1+\tilde{w}_{s,t}) \times w_{m,n} \times x_{i-1+m,j-1+n}] \\
&\quad + \sum_{\mathbb{1}_{K_i}((m,s))\neq 1 \text{ or } \mathbb{1}_{L_j}((n,t))\neq 1} (w_{m,n} \times x_{i-1+m,j-1+n} + \tilde{w}_{s,t} \times w_{m,n} \times x_{ue+\tilde{i_a}^{(u,v)}+m+s,ve+\tilde{j_a}^{(u,v)}+n+t})
\end{aligned} \tag{14}
$$

Combine all step above, and recall set $K_i = \{(m,s) : ue + \tilde{i_a}^{(u,v)} + m + s \in ([i, i+M] \cap [ue + \tilde{i_a}^{(u,v)}, ue + \tilde{i_a}^{(u,v)} + M + 2])\}$ and $L_j = \{(n,t) : ve + \tilde{j_a}^{(u,v)} + n + t \in ([j, j+M] \cap [ve + \tilde{j_a}^{(u,v)}, ve + \tilde{j_a}^{(u,v)} + M + 2])\}$. Note that when $(m,s) \in K_i$, $ue + \tilde{i_a}^{(u,v)} + m + s = i + m$, and when $(n,t) \in L_j$, $ve + \tilde{j_a}^{(u,v)} + n + t = j + n$, we have:

$$
\begin{aligned}
o_{i,j} &= y_{i,j} + y_{out,i,j} \\
&= \begin{cases}
\sum_{\mathbb{1}_{K_i}((m,s))=1 \text{ and } \mathbb{1}_{L_j}((n,t))=1} [(1+\tilde{w}_{s,t}) \times w_{m,n} \times x_{i-1+m,j-1+n}] \\
+ \sum_{\mathbb{1}_{K_i}((m,s))\neq 1 \text{ or } \mathbb{1}_{L_j}((n,t))\neq 1} (w_{m,n} \times x_{i-1+m,j-1+n} + \tilde{w}_{s,t} \times w_{m,n} \times x_{ue+\tilde{i_a}^{(u,v)}+m+s,ve+\tilde{j_a}^{(u,v)}+n+t}) \\
\qquad \text{if } e \; mod \; (i-1) = \tilde{i_a}^{(u,v)} \text{ and } e \; mod \; (j-1) = \tilde{j_a}^{(u,v)} \text{ in block } Y^{(u,v)}, \\
\sum_{m=0}^{M-1}\sum_{n=0}^{M-1} w_{m,n} \times x_{i-1+m,j-1+n}, \qquad \text{o.w.}
\end{cases}
\end{aligned} \tag{15}
$$

for all $i \in \{1, 2, \cdots, H-M+1\}$ and $j \in \{1, 2, \cdots, W-M+1\}$.