1. Proof: Define $P_n(t)=Pr\{X(t)=n\}$, assuming $X(0)=0$.

The differential equations satisfied by $P_n(t)$ for $t\geq 0$

$P_0'(t)=-\lambda_0 P_0(t)$

$P_n'(t)=-\lambda_n P_n(t)+\lambda_{n-1}P_{n-1}(t)$ for $n\geq 1$.

$P_0(0)=1$, $P_n(0)=0$ $n\geq 0$

$P_0(t)=e^{-\lambda_0 t}$

Let $Q_n(t)=e^{\lambda_n t}P_n(t)$ for $n=0,1,\dots$

$$Q_n'(t)=\lambda_n e^{\lambda_n t}P_n(t)+e^{\lambda_n t}P_n'(t)$$
$$=e^{\lambda_n t}[\lambda_n P_n(t)+P_n'(t)]$$
$$=e^{\lambda_n t}\lambda_{n-1}P_{n-1}(t)$$

Integrating both side and using boundary condition

$Q_n(0)=0$ for $n\geq 1$

So $Q_n(t)=\int_0^t e^{\lambda_n t}\lambda_{n-1}P_{n-1}(x)\,dx$

$\Rightarrow P_n(t)=\lambda_{n-1}e^{-\lambda_n t}\int_0^t e^{\lambda_n x}P_{n-1}(x)\,dx$ $n=1,2,\dots$

So $P_k(t)\geq 0$.

but there is still a possibility that

$\sum_{n=0}^{\infty}P_n(t)\leq 1$

To secure the validity of the process, i.e., to assure that $\sum_{n=0}^{\infty}P_n(t)=1$ for all t. we must restrict $\sum_{n=0}^{\infty}\frac{1}{\lambda_n}=\infty$.

$\sum_{n=0}^{\infty}\frac{1}{\lambda_n}=\infty$ means the expected time for the population to become infinite is infinite. i.e., the population will never goes to infinite.

i.e., the probability that $X(t)=\infty$ is 0.

i.e. $1-\sum_{n=0}^{\infty} P_n(t)=0$

$$\sum_{n=0}^{\infty} P_n(t)=1.$$

The program is attached in another file.

2. Proof: The differential equation for the process

$P_0'(t) = -\nu P_0(t) + \mu P_1(t)$

$P_n'(t) = -(n\lambda + n\mu + \nu)P_n(t) + ((n-1)\lambda + \nu)P_{n-1}(t)$
$\qquad + (n+1)\mu P_{n+1}(t)$ for $n \geq 1$.

where $P_k(t) = P\{X(t) = k \mid X(0) = n_0\}$

The mean of the stochastic model.

$m(t) = E\{X(t) \mid X(0) = n_0\} = \sum_{n=0}^{\infty} n P_n(t)$

differentiating with respect to t

$m'(t) = \sum_{n=1}^{\infty} n P_n'(t)$

substituting $P_n'(t)$ in $m'(t)$

$m'(t) = \sum_{n=1}^{\infty} n[-(n\lambda + n\mu + \nu)P_n(t) + ((n-1)\lambda + \nu)P_{n-1}(t)$
$\qquad + (n+1)\mu P_{n+1}(t)]$

$= (\lambda - \mu)m(t) + \nu$

$\Rightarrow \frac{dm(t)}{dt} = (\lambda - \mu)m(t) + \nu$

Since the determintstic model is

$\frac{dn}{dt} = (\lambda - \mu)n + \nu$

So they have the same form.

Hence the solution of the mean equals the solution of the differential equation with appropriete initial condition.

~~If $t\to\infty$~~

Since $\frac{dm(t)}{dt} = (\lambda-\mu)\, m(t) + \nu$

If $X(0) = n_0$, then $m(0) = n_0$

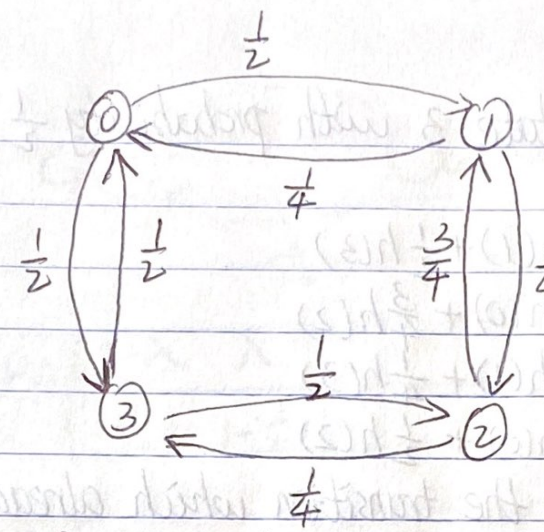
The solution of this equation is

$m(t) = \nu t + n_0$ if $\lambda = \mu$

$m(t) = \frac{\nu}{\lambda-\mu}\{e^{(\lambda-\mu)t} - 1\} + n_0 e^{(\lambda-\mu)t}$ if $\lambda \neq \mu$

The program is attached in another file.

3. (a)



(b) Proof: A Markov chain is called irreducible if and only if all states belong to one communication class.

In the graph-representation of the chain, there are directed paths from i to j and from j to i.

So $i \leftrightarrow j$

And the graph-representation of the chain is a strongly connected graph.

So the Markov chain is irreducible. $\boxtimes$

The period

$$d(k) = \gcd\{m \geq 1 : P^m_{k,k} > 0\}$$

The period for each state:

$d(0)=2 \quad d(1)=2 \quad d(2)=2 \quad d(3)=2$ $\boxtimes$

Since the period $d(k) > 1$

So the chain is periodic. $\boxtimes$

$h(k)$ — the number of transitions.

$h(0)$ assume start in state 0 and make a transition

It will either land on state 1 with probability $\frac{1}{2}$

or land in state 3 with probability $\frac{1}{2}$

then

$$\begin{cases} h(0)=1+\frac{1}{2}h(1)+\frac{1}{2}h(3) \\ h(1)=1+\frac{1}{4}h(0)+\frac{3}{4}h(2) \\ h(2)=1+\frac{3}{4}h(1)+\frac{1}{4}h(3) \\ h(3)=1+\frac{1}{2}h(0)+\frac{1}{2}h(2) \end{cases}$$

1 stands for the transition which already made.

So solve the equations

$h(0)<\infty \quad h(1)<\infty \quad h(2)<\infty \quad h(3)<\infty$

All the states are positive recurrent.

So the chain positive recurrent.

(c) $\pi P=\pi$

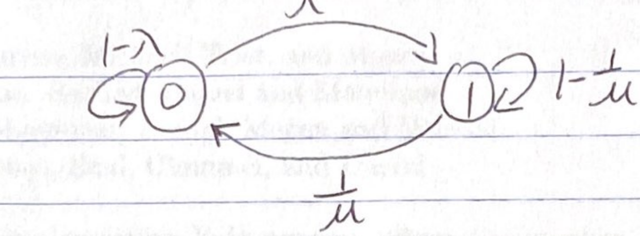
$$(\pi_1\ \pi_2\ \pi_3\ \pi_4)\begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 0 & \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix}=(\pi_1\ \pi_2\ \pi_3\ \pi_4)$$

$$\begin{cases} \frac{1}{2}\pi_2+\frac{1}{2}\pi_4=\pi_1 \\ \frac{1}{4}\pi_1+\frac{3}{4}\pi_3=\pi_2 \\ \frac{3}{4}\pi_2+\frac{1}{4}\pi_4=\pi_3 \\ \frac{1}{2}\pi_1+\frac{1}{2}\pi_3=\pi_4 \\ \pi_1+\pi_2+\pi_3+\pi_4=1 \end{cases} \Longrightarrow (\pi_1,\pi_2\ \pi_3\ \pi_4)=(\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{4})$$

The unique stationary probability distribution is

$(\pi_1\ \pi_2\ \pi_3\ \pi_4)=(\frac{1}{4}\ \frac{1}{4}\ \frac{1}{4}\ \frac{1}{4})$

4.



$$P=\begin{pmatrix} 1-\lambda & \lambda \\ \frac{1}{\mu} & 1-\frac{1}{\mu} \end{pmatrix}$$

$$\pi P = \pi$$

$$(\pi_0 \ \ \pi_1)\begin{pmatrix} 1-\lambda & \lambda \\ \frac{1}{\mu} & 1-\frac{1}{\mu} \end{pmatrix} = (\pi_0 \ \ \pi_1)$$

$$\begin{cases} (1-\lambda)\pi_0 + \frac{1}{\mu}\pi_1 = \pi_0 \\ \lambda\pi_0 + (1-\frac{1}{\mu})\pi_1 = \pi_1 \\ \pi_0 + \pi_1 = 1 \end{cases} \Rightarrow \begin{aligned} \pi_0 &= \frac{1}{1+\lambda\mu} \\ \pi_1 &= \frac{\lambda\mu}{1+\lambda\mu} \end{aligned}$$

The long time fraction of time that the promoter is unoccupied is $\pi_0 = \frac{1}{1+\lambda\mu}$

5. Assume the sequence of AP fired by a neuron can be described as a $Poiss(\lambda)$

Then the probability density function of $T(t)$ is

$T(t) \sim Exp(\lambda)$

The mean of $T(t)$ is $\frac{1}{\lambda}$