Hw4

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1. The problem is Nash bargaining equilibrium

suppose one son asked for s_1, the other son asked for s_2, then $s_1 + s_2 = 1{,}000{,}000$ is Nash equilibrium.

if $s_1 + s_2 \leq 1{,}000{,}000$, and the first son claims s_1, then the pure Nash equilibria is $(s_1, 1{,}000{,}000 - s_1)$.

if $s_1 + s_2 \leq 1{,}000{,}000$, and the second son claims s_2, then the pure Nash equilibria is $(1{,}000{,}000 - s_2, s_2)$.

For $(s_1, 1{,}000{,}000 - s_1)$, the problem is to find s_1 which can maximize the Nash product

$$\max_{s_1} \; s_1(1{,}000{,}000 - s_1)$$

Take the derivative with respect to s_1

$$1{,}000{,}000 - 2s_1 = 0$$

$$s_1 = 500{,}000$$

$$s_2 = 1 - s_1 = 500{,}000$$

The pure Nash equilibrium is $s_1 = 500{,}000$, $s_2 = 500{,}000$

2. (a) The payoff matrix for GRIM and ALLD in a Prisoner's Dilemma that is repeated for m rounds, on average, is given by

$$\begin{array}{cc} & \begin{array}{cc} GRIM & ALLD \end{array} \\ \begin{array}{c} GRIM \\ ALLD \end{array} & \begin{bmatrix} mR & S+(m-1)P \\ T+(m-1)P & mP \end{bmatrix} \end{array}$$

(b) if $m > \frac{T-P}{R-P}$

Since $R > P$

So, $m(R-P) > T - P$

Hence, we have $mR > T + (m-1)P$

mR is the payoff of $GRIM$, and $T + (m-1)P$ is the payoff of $ALLD$,

So that $GRIM$ is stable against invasion by $ALLD$.

(c) The payoff matrix for $GRIM^*$ vs $GRIM$ is

$$\begin{array}{cc} & \begin{array}{cc} GRIM^* & GRIM \end{array} \\ \begin{array}{c} GRIM^* \\ GRIM \end{array} & \begin{bmatrix} (m-1)R+P & T+(m-1)R \\ S+(m-1)R & mR \end{bmatrix} \end{array}$$

Since T>S, so $T + (m-1)R > S + (m-1)R$, so that $GRIM^*$ dominate $GRIM$.

(d) Define a strategy CD, CD cooperates on the first move and then defect for the rest of $m-1$ moves. The payoff matrix for CD vs $GRIM^*$ is

$$\begin{array}{cc} & \begin{array}{cc} CD & GRIM^* \end{array} \\ \begin{array}{c} CD \\ GRIM^* \end{array} & \begin{bmatrix} R+(m-1)P & R+T+(m-2)P \\ R+S+(m-2)P & (m-1)R+P \end{bmatrix} \end{array}$$

Since T>S, so $R+T+(m-2)P > R+S+(m-2)P$, so that CD dominate $GRIM^*$.

(e) If we continue the argument, then we will arrive at ALLD, the payoff matrix for $ALLD$ vs CD is

$$\begin{array}{cc} & \begin{array}{cc} ALLD & CD \end{array} \\ \begin{array}{c} ALLD \\ CD \end{array} & \begin{bmatrix} mP & T+(m-1)P \\ S+(m-1)P & R+(m-1)P \end{bmatrix} \end{array}$$

$ALLD$ dominant CD.

3. (a) The number of rounds is uncertain, and the probabilities of different rounds are implied by the continuation probability δ, which makes the probability of k repetitions follow a geometric distribution with success probability δ.

So that $X=k$, there must be $k-1$ success followed by a failure. This occurs with probability

$$P(X=k)=\delta^{k-1}(1-\delta)$$

The expected value of X which is also the expected number of rounds is

$$\langle m\rangle = E(X)=\frac{1}{1-\delta}=1+\frac{\delta}{1-\delta}$$

(b) The expected payoff is

$$\begin{array}{cc} & \begin{array}{cc} GRIM & ALLD \end{array} \\ \begin{array}{c} GRIM \\ ALLD \end{array} & \begin{bmatrix} \frac{1}{1-\delta}R & S+\left(\frac{1}{1-\delta}-1\right)P \\ T+\left(\frac{1}{1-\delta}-1\right)P & \frac{1}{1-\delta}P \end{bmatrix} \end{array}$$

Simplify to

$$\begin{array}{cc} & \begin{array}{cc} GRIM & ALLD \end{array} \\ \begin{array}{c} GRIM \\ ALLD \end{array} & \begin{bmatrix} \frac{1}{1-\delta}R & S+\frac{\delta}{1-\delta}P \\ T+\frac{\delta}{1-\delta}P & \frac{1}{1-\delta}P \end{bmatrix} \end{array}$$

(c) if $GRIM$ is stable against $ALLD$

So that

$$\frac{1}{1-\delta}R > T + \frac{\delta}{1-\delta}P$$

$$R > (1-\delta)T + \delta P$$

$$-\delta(T-P) < R - T$$

$$\delta(T-P) > T - R$$

Since $T > P$

So that

$$\delta > \frac{T-R}{T-P}$$

4. (a) transition matrix for TFT vs TFT

$$\begin{pmatrix} (1-\varepsilon)^2 & \varepsilon(1-\varepsilon) & \varepsilon(1-\varepsilon) & \varepsilon^2 \\ \varepsilon(1-\varepsilon) & \varepsilon^2 & (1-\varepsilon)^2 & \varepsilon(1-\varepsilon) \\ \varepsilon(1-\varepsilon) & (1-\varepsilon)^2 & \varepsilon^2 & \varepsilon(1-\varepsilon) \\ \varepsilon^2 & \varepsilon(1-\varepsilon) & \varepsilon(1-\varepsilon) & (1-\varepsilon)^2 \end{pmatrix}$$

Transition matrix for TFT vs GRIM

$$\begin{pmatrix} (1-\varepsilon)^2 & \varepsilon(1-\varepsilon) & \varepsilon(1-\varepsilon) & \varepsilon^2 \\ \varepsilon^2 & \varepsilon(1-\varepsilon) & \varepsilon(1-\varepsilon) & (1-\varepsilon)^2 \\ \varepsilon(1-\varepsilon) & (1-\varepsilon)^2 & \varepsilon^2 & \varepsilon(1-\varepsilon) \\ \varepsilon^2 & \varepsilon(1-\varepsilon) & \varepsilon(1-\varepsilon) & (1-\varepsilon)^2 \end{pmatrix}$$

Transition matrix for TFT vs ALLC

$$\begin{pmatrix} (1-\varepsilon)^2 & \varepsilon(1-\varepsilon) & \varepsilon(1-\varepsilon) & \varepsilon^2 \\ \varepsilon(1-\varepsilon) & \varepsilon^2 & (1-\varepsilon)^2 & \varepsilon(1-\varepsilon) \\ (1-\varepsilon)^2 & \varepsilon(1-\varepsilon) & \varepsilon(1-\varepsilon) & \varepsilon^2 \\ \varepsilon(1-\varepsilon) & \varepsilon^2 & (1-\varepsilon)^2 & \varepsilon(1-\varepsilon) \end{pmatrix}$$

Transition matrix for GRIM vs GRIM

$$\begin{pmatrix} (1-\varepsilon)^2 & \varepsilon(1-\varepsilon) & \varepsilon(1-\varepsilon) & \varepsilon^2 \\ \varepsilon^2 & \varepsilon(1-\varepsilon) & \varepsilon(1-\varepsilon) & (1-\varepsilon)^2 \\ \varepsilon^2 & \varepsilon(1-\varepsilon) & \varepsilon(1-\varepsilon) & (1-\varepsilon)^2 \\ \varepsilon^2 & \varepsilon(1-\varepsilon) & \varepsilon(1-\varepsilon) & (1-\varepsilon)^2 \end{pmatrix}$$

Transition matrix for GRIM vs ALLC

$$\begin{pmatrix} (1-\varepsilon)^2 & \varepsilon(1-\varepsilon) & \varepsilon(1-\varepsilon) & \varepsilon^2 \\ \varepsilon(1-\varepsilon) & \varepsilon^2 & (1-\varepsilon)^2 & \varepsilon(1-\varepsilon) \\ \varepsilon(1-\varepsilon) & \varepsilon^2 & (1-\varepsilon)^2 & \varepsilon(1-\varepsilon) \\ \varepsilon(1-\varepsilon) & \varepsilon^2 & (1-\varepsilon)^2 & \varepsilon(1-\varepsilon) \end{pmatrix}$$

Transition matrix for ALLC vs ALLC

$$\begin{pmatrix} (1-\varepsilon)^2 & \varepsilon(1-\varepsilon) & \varepsilon(1-\varepsilon) & \varepsilon^2 \\ (1-\varepsilon)^2 & \varepsilon(1-\varepsilon) & \varepsilon(1-\varepsilon) & \varepsilon^2 \\ (1-\varepsilon)^2 & \varepsilon(1-\varepsilon) & \varepsilon(1-\varepsilon) & \varepsilon^2 \\ (1-\varepsilon)^2 & \varepsilon(1-\varepsilon) & \varepsilon(1-\varepsilon) & \varepsilon^2 \end{pmatrix}$$

(b) Since for stationary distribution $\boldsymbol{v} = \{v_1, v_2\,, v_3, v_4\}$, such that

$$\boldsymbol{v}M = \boldsymbol{v}$$

$$v_1 + v_2 + v_3 + v_4 = 1$$

M is the transition matrix which is calculated in part (a)

For **TFT vs TFT**, in the limit $\varepsilon \to 0$, the transition matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

CC and DD are absorbing state, CD and DC are recurrent state

stationary distribution for TFT vs TFT is

$$\boldsymbol{v} = \left\{\frac{1}{4}, \frac{1}{4}\,, \frac{1}{4}, \frac{1}{4}\right\}$$

For **TFT vs GRIM**, in the limit $\varepsilon \to 0$, the transition matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

CC and DD are absorbing state, CD and DC are transient state

the stationary distribution is

$$\boldsymbol{v} = \left\{\frac{1}{2}, 0\,, 0, \frac{1}{2}\right\}$$

For **TFT vs ALLC**, in the limit $\varepsilon \to 0$, the transition matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

CC is absorbing state, others are transient state

the stationary distribution is

$$\boldsymbol{v} = \{1,0\ ,0,0\}$$

For **GRIM vs GRIM**, in the limit $\varepsilon \to 0$, the transition matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

CD and DD are absorbing state.

the stationary distribution is

$$\boldsymbol{v} = \left\{\frac{1}{2}, 0\ ,0, \frac{1}{2}\right\}$$

For **GRIM vs ALLC**, in the limit $\varepsilon \to 0$, the transition matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

CC and DC are absorbing state, others are transient state

the stationary distribution is

$$\boldsymbol{v} = \left\{\frac{1}{2}, 0\ , \frac{1}{2}, 0\right\}$$

For **ALLC vs ALLC**, in the limit $\varepsilon \to 0$, the transition matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

CC is absorbing state, cooperate all the time.

the stationary distribution is

$$\boldsymbol{v} = \{1,0\ ,0,0\}$$

(c) The payoff for the players are given by

$$S_x = v_{cc}^* R + v_{cd}^* S + v_{dc}^* T + v_{dd}^* P$$

$$S_y = v_{cc}^* R + v_{dc}^* S + v_{cd}^* T + v_{dd}^* P$$

The stationary distribution in the limit $\varepsilon \to 0$ is

TFT vs TFT, $\boldsymbol{v} = \left\{\frac{1}{4}, \frac{1}{4}\ , \frac{1}{4}, \frac{1}{4}\right\}$, the payoff for both players are

$$S_x = S_y = \frac{1}{4}(R + S + T + P)$$

TFT vs GRIM, $\boldsymbol{v} = \left\{\frac{1}{2}, 0, 0, \frac{1}{2}\right\}$, the payoff for both players are

$$S_x = \frac{1}{2}(R + P)$$

$$S_y = \frac{1}{2}(R + P)$$

TFT vs ALLC, $\boldsymbol{v} = \{1, 0, 0, 0\}$, the payoff for both players are

$$S_x = R$$

$$S_y = R$$

GRIM vs GRIM, $\boldsymbol{v} = \left\{\frac{1}{2}, 0, 0, \frac{1}{2}\right\}$, the payoff for both players are

$$S_x = \frac{1}{2}(R + P)$$

$$S_y = \frac{1}{2}(R + P)$$

GRIM vs ALLC, $\boldsymbol{v} = \left\{\frac{1}{2}, 0, \frac{1}{2}, 0\right\}$, the payoff for both players are

$$S_x = \frac{1}{2}(R + T)$$

$$S_y = \frac{1}{2}(R + S)$$

ALLC vs ALLC, $\boldsymbol{v} = \{1, 0, 0, 0\}$, the payoff for both players are

$$S_x = R$$

$$S_y = R$$

The payoff matrix in the limit $\varepsilon \to 0$ is

	TFT	GRIM	ALLC
TFT	$\frac{1}{4}(R + S + T + P)$	$\frac{1}{2}(R + P)$	R
GRIM	$\frac{1}{2}(R + P)$	$\frac{1}{2}(R + P)$	$\frac{1}{2}(R + T)$
ALLC	R	$\frac{1}{2}(R + S)$	R

(d) we limit players to only these three strategies and errors are vanishingly small

$$T > R > P > S$$

If player y choose TFT,

$$R > \frac{1}{2}(R+P)$$

Then player x choose TFT or ALLC

If player y choose GRIM, then player x choose TFT or GRIM

If player y choose ALLC, then player x choose GRIM

If player x choose TFT, player y choose TFT or ALLC

If player x choose GRIM, player y choose ALLC

If player x choose ALLC, player y choose TFT or ALLC

Compare the above, hence, there is no Nash equilibrium between the three strategies.