BME-590
Non Linear Control Project
Tracking Control of a Robot Manipulator with Uncertainities

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1 Introduction

Robot manipulators are robots that are capable of tracking a desired trajectory quickly and accurately. It enables the joints, links or the end-effector to track a desired trajectory and it stabilizes them in the specified point. It is very important to take into consideration all the uncertainities and non-linearities along with the highly complex and coupled dynamics in the system to achieve the best tracking.

The aim of this project was to design three non-linear controllers for a robot manipulator with two links. The control objective was to achieve perfect tracking of the desired trajectory in a short time. Different desired trajectory was given to each link. Stability was analyzed while designing each controller to make sure that the designed control input is able to achieve error convergence.The Adaptive and Robust controllers were investigated in the presence of system disturbances while the Exact model knowledge controller was simulated without considering the disturbance since EMK controllers can be used only when all the model parameters are known.

2 Dynamic model

The following equations represent the dynamics of the Robot.

$$\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} = \begin{bmatrix}p_1 + 2p_3\cos(q_2) & p_2 + p_3\cos(q_2)\\ p_2 + p_3\cos(q_2) & p_2\end{bmatrix}\begin{bmatrix}\ddot{q_1}\\ \ddot{q_2}\end{bmatrix} + \begin{bmatrix}-p_3\sin(q_2)\dot{q_2} & -p_3\sin(q_2)(\dot{q_1}+\dot{q_2})\\ p_3\sin(q_2)\dot{q_1} & 0\end{bmatrix}\begin{bmatrix}\dot{q_1}\\ \dot{q_2}\end{bmatrix}$$

$$+\begin{bmatrix}f_{d_1} & 0\\ 0 & f_{d_2}\end{bmatrix} + \begin{bmatrix}\tau_{d_1}\\ \tau_{d_2}\end{bmatrix}$$

where,

Joint angle vector $- q$

Joint's Angular velocity vector $- \dot{q}$

Joint's Angular acceleration vector $- \ddot{q}$

It can be further represented in a more compact form as done below,

$$\tau = M(q)\ddot{q}\ + V_m(q,\dot{q})\dot{q}\ + F_d\dot{q}\ + \tau_d \tag{1}$$

Where,

$$\text{Inertia term, } M(q) = \begin{bmatrix} p_1 + 2p_3\cos(q_2) & p_2 + p_3\cos(q_2) \\ p_2 + p_3\cos(q_2) & p_2 \end{bmatrix}$$

$$\text{Centripetal and Coriolis term, } V_m(q,\dot{q})\dot{q} = \begin{bmatrix} -p_3\sin(q_2)\dot{q_2} & -p_3\sin(q_2)(\dot{q_1}+\dot{q_2}) \\ p_3\sin(q_2)\dot{q_1} & 0 \end{bmatrix}$$

$$\text{Damping or friction term, } F_d = \begin{bmatrix} f_{d_1} & 0 \\ 0 & f_{d_2} \end{bmatrix}$$

$$\text{Disturbance, } \tau = \begin{bmatrix} \tau_{d_1} \\ \tau_{d_2} \end{bmatrix}$$

The known parameters of the robot dynamic model are shown in Table 1

Table 1: Robot model Parameters

Parameters	Values
p_1	3.473 kg.m^2
p_2	0.196 kg.m^2
p_3	0.242 kg.m^2
f_{d_1}	5.3 N.m.sec
f_{d_2}	1.1 N.m.sec

3 Exact model knowledge controller

3.1 Nonlinear control design & Stability Analysis

3.1.1 Assumptions

In order to use Exact model knowledge controller to achieve tracking, all parameters of the robot must be known. Since this is a hard requirement of the EMK controller, we assume that there are no unknown disturbances acting on the system. The model parameters chosen are shown in Table.1

$$\tau = M(q)\ddot{q}\ + V_m(q,\dot{q})\dot{q}\ + F_d\dot{q}$$

3.1.2 Control objective

The control objective of the EMK controller is to achieve error convergence i.e. $e \rightarrow 0$ as $t \rightarrow \infty$. Since our aim is to track a trajectory which varies with time, the non-linear system under consideration is a non-autonomous system. It is also called as a time-varying system.

The error is given by,

$$e = q_d - q$$

where q_d is the desired trajectory
Differentiating with respect to time,

$$\dot{e} = \dot{q}_d - \dot{q}$$

Differentiating again with respect to time,

$$\ddot{e} = \ddot{q}_d - \ddot{q}$$

3.1.3 Auxiliary signal:

An Auxiliary signal is introduced to convert the second-order system representing the robot dynamics into a first-order system

It is given by,

$$r = \dot{e} + \alpha e$$

where, $\alpha \in R^{+} > 0$

Open loop error system:

$$\dot{r} = \ddot{e} + \alpha\dot{e}$$
$$M\dot{r} = M\ddot{e} + \alpha M\dot{e}$$
$$M\dot{r} = M(\ddot{q}_d - \ddot{q}) + \alpha M\dot{e}$$
$$M\dot{r} = M\ddot{q}_d - M\ddot{q} + \alpha M\dot{e}$$

From Eq. 1,

$$M\dot{r} = M\ddot{q}_d - (\tau - V_m\dot{q} - F_d\dot{q}) + \alpha M\dot{e}$$
$$\boldsymbol{M\dot{r} = M\ddot{q}_d - \tau + V_m\dot{q} + F_d\dot{q} + \alpha M\dot{e}} \tag{2}$$

Closed loop error system:

Design τ:

$$\tau = M\ddot{q}_d + V_m\dot{q} + \alpha M\dot{e} + F_d\dot{q} + \boxed{}_1$$

After substituting τ

$$M\dot{r} = -\boxed{}_1$$

Stability Analysis

Lyapunov function:

$$\boldsymbol{V = \frac{1}{2} r^T M(q) r + \frac{1}{2} e^T e} \tag{3}$$

$$\begin{aligned}
\dot{V} &= \frac{1}{2} r^T \dot{M} r + r^T M(q) \dot{(r)} + e^T e \\
&= \frac{1}{2} r^T \dot{M} r - r^T \boxed{}_1 + e^T (r - \alpha e) \\
&= \frac{1}{2} r^T \dot{M} r - r^T \boxed{}_1 + e^T r - e^T \alpha e \\
&= -\alpha ||e||^2 + \frac{1}{2} r^T \dot{M} r + e^T r - r^T \boxed{}_1
\end{aligned}$$

Design $\boxed{}_1 = V_m r + e + K r$

$$\begin{aligned}
\dot{V} &= -\alpha ||e||^2 + r^T (\dot{M}/2 - V_m) r + e^T r - r^T e - r^T K r \\
&= -\alpha ||e||^2 - K ||r||^2 < 0 \quad \boldsymbol{G.A.S}
\end{aligned}$$

From Theorem 4.1, If $\dot{V}(x) < 0$ in $D - \{0\}$, then the system is **Asymptotically stable**. Also, the lyapunov function $V \to \infty$ as $z \to \infty$ where, $(||z|| = [r^T \; e^T]^T)$. This shows that the lyapunov function is **Radially Unbounded.** Hence the system is **Globally Asymptotically Stable (G.A.S).**

To show G.E.S:

$$V = \frac{1}{2} r^T M(q) r + \frac{1}{2} e^T e$$

Let $z = [r^T e^T]^T$

$$V_1 = min(\frac{1}{2}\underline{m}, \frac{1}{2}) ||z||^2 \tag{4}$$

where $\lambda_1 = min(\frac{1}{2}\underline{m}, \frac{1}{2})$

$$V_2 = max(\frac{1}{2}\bar{m}, \frac{1}{2}) ||z||^2$$

where $\lambda_2 = max(\frac{1}{2}\bar{m}, \frac{1}{2})$

$$V_1 \leq V \leq V_2 \quad \text{Positive definite and decrescent}$$

$\underline{m} -$ min eigenvalue of $M(q)$

$\bar{m} -$ max eigenvalue of $M(q)$

$$
\begin{aligned}
\dot{V} &\leq -min(\alpha, K)||z||^2 \\
\text{Let } \beta &< min((\alpha, K) \\
-\beta &> -min(\alpha, K) \\
\dot{V} &\leq -\beta||z||^2 \\
\text{From Eq. 4}& \\
\lambda_1||z||^2 &\leq V \leq \lambda_2||z||^2 \\
-\lambda_1||z||^2 &\geq -V \geq -\lambda_2||z||^2 \\
-\frac{\lambda_1}{\lambda_2}||z||^2 &\geq \frac{-V}{\lambda_2} \geq -||z||^2 \\
\dot{V} &\leq -\frac{\beta V}{\lambda_2} \\
V(t) &= V(0)e^{-\frac{\beta}{\lambda_2}t} \qquad (5)
\end{aligned}
$$

From Eq. 5, it is clear that the Lyapunov function exponentially decays to zero as $t \to \infty$. It is also radially unbounded. Hence the system is **Globally Exponentially Stable (G.E.S)**

$$\text{Final Control law, } \boldsymbol{\tau = M\ddot{q}_d + V_m\dot{q} + \alpha M\dot{e} + F_d\dot{q} + V_m r + e + Kr} \qquad (6)$$

3.2 Simulation results

The designed control law from Eq.6 was tested by simulating the system in Simulink. The following plots help in understanding the performance of the EMK controller. The desired trajectory chosen is given below,

$$\text{q}_d = \begin{bmatrix} \cos(t) \\ \cos(2t) \end{bmatrix}$$

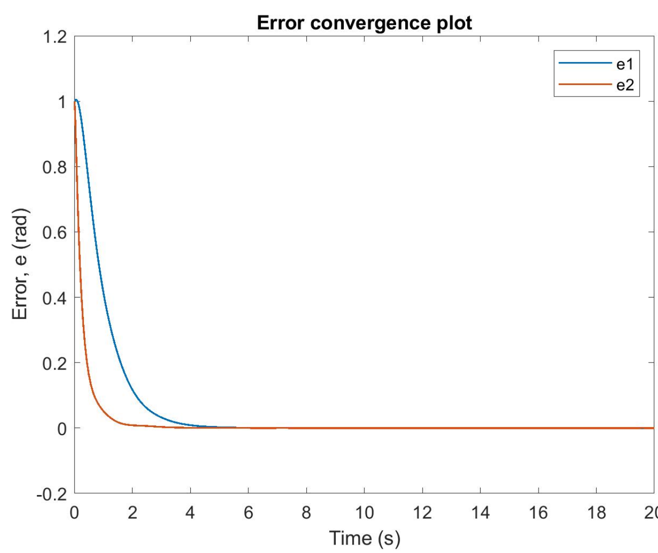

Figure 1: Error Convergence

From Fig.1 it is evident that the error between the current joint angle and the desired angle goes to zero for both the links in around 4 seconds. This is in accordance with the stability analysis performed while arriving at the control law for EMK controller.

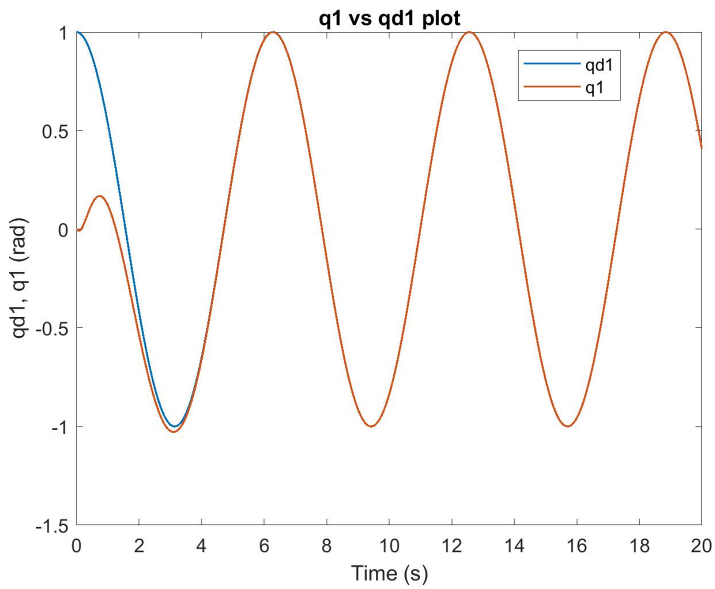

Figure 2: Joint angle 1 tracking performance

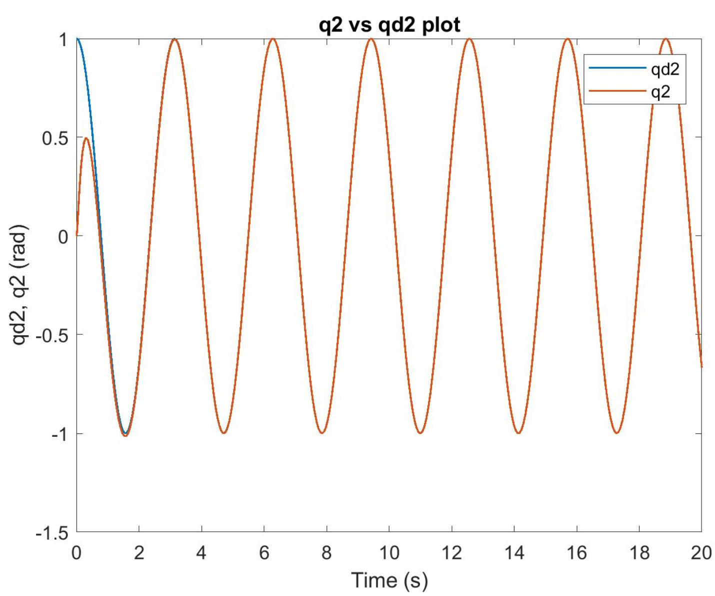

Figure 3: Joint angle 2 tracking performance

From the Fig.2 and Fig.3, it can be observed that the initial joint angle is farther away from the desired trajectory. But eventually, the control input provided by the EMK controller tries to align the links with the desired trajectory.

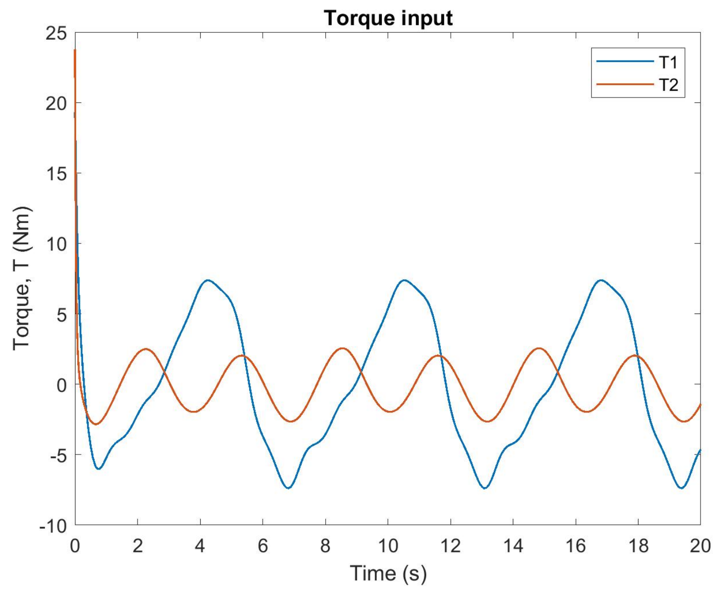

Figure 4: Control input(Torque)

From the Fig.4, we can observe that the initial torque input is high and eventually it drops to some minimal value. This is because, the tracking error is high in the beginning. To achieve quick transient response, it is necessary to have high values for control input at the start for the desired trajectory we have chosen.

4 Adaptive controller

4.1 Adaptive control design & Stability Analysis

The Adaptive Controller is more robust compared to EMK controller. Even if few robot parameters are unknown, they can be estimated. Hence we can include disturbances as a part of the system to check the efficacy of the controller. Therefore, the dynamic model is given by,

$$\tau = M(q)\ddot{q} \ + V_m(q,\dot{q})\dot{q} \ + F_d\dot{q} \ + \tau_d$$

4.1.1 Control objective

The control objective of the Adaptive controller is to achieve error convergence i.e. $e \to 0$ as $t \to \infty$. Since our aim is to track a trajectory which varies with time, the non-linear system under consideration is a non-autonomous system. It is also called as a time-varying system.

The error is given by,

$$e = q_d - q$$

where q_d is the desired trajectory
Differentiating with respect to time,

$$\dot{e} = \dot{q}_d - \dot{q}$$

Differentiating again with respect to time,

$$\ddot{e} = \ddot{q}_d - \ddot{q}$$

4.1.2 Auxiliary signal:

An Auxiliary signal is introduced to convert the second-order system representing the robot dynamics into a first-order system

It is given by,

$$r = \dot{e} + \alpha e$$

where, $\alpha \in R^+ > 0$

Open loop error system

$$\begin{aligned}
\dot{r} &= \ddot{e} + \alpha\dot{e} \\
M\dot{r} &= M\ddot{e} + \alpha M\dot{e} \\
&= M(\ddot{q}_d - \ddot{q}) + \alpha M\dot{e} \\
&= M\ddot{q}_d - M\ddot{q} + \alpha M\dot{e}
\end{aligned}$$

From Eq. 1,

$$\begin{aligned}
M\dot{r} &= M\ddot{q}_d - (\tau - V_m\dot{q} - F_d\dot{q} - \tau_d) + \alpha M\dot{e} \\
\boldsymbol{M\dot{r}} &\boldsymbol{= M\ddot{q}_d - \tau + V_m\dot{q} + F_d\dot{q} + \tau_d + \alpha M\dot{e}}
\end{aligned}$$

Adding and Subtracting $V_m r$, we get,

$$\begin{aligned}
M\dot{r} &= M\ddot{q}_d - \tau + \tau_d + V_m\dot{q} + F_d\dot{q} + \alpha M\dot{e} + V_m r - V_m r \\
&= M\ddot{q}_d - \tau + \tau_d + V_m(\dot{q} + r) + F_d\dot{q} + \alpha M\dot{e} - V_m r \\
&= M\ddot{q}_d - \tau + \tau_d + V_m(\dot{q} + \dot{e} + \alpha e) + F_d\dot{q} + \alpha M\dot{e} - V_m r \\
&= M\ddot{q}_d - \tau + \tau_d + V_m(\dot{q} + \dot{q}_d - \dot{q} + \alpha e) + F_d\dot{q} + \alpha M\dot{e} - V_m r \\
&= M\ddot{q}_d - \tau + \tau_d + V_m(\dot{q}_d + \alpha e) + F_d\dot{q} + \alpha M\dot{e} - V_m r \\
\boldsymbol{M\dot{r}} &\boldsymbol{= -V_m r + Y\theta - \tau + \tau_d}
\end{aligned}$$

where, $Y\theta = M\ddot{q}_d + V_m(\dot{q}_d + \alpha e) + F_d\dot{q} + \alpha M\dot{e}$

$$Y\theta = \begin{bmatrix} \ddot{q}_{d1} + \alpha\dot{e}_1 & \ddot{q}_{d2} + \alpha\dot{e}_2 & 2\cos q_2(\ddot{q}_{d1} + \alpha\dot{e}_1) + \cos q_2(\ddot{q}_{d2} + \alpha\dot{e}_2) - \sin q_2\dot{q}_2(\dot{q}_{d1} + \alpha e_1) - \sin q_2(\dot{q}_1 + \dot{q}_2)(\dot{q}_{d2} + \alpha e_2) & \dot{q}_1 & 0 \\ 0 & (\ddot{q}_{d1} + \alpha\dot{e}_1) + (\ddot{q}_{d2} + \alpha\dot{e}_2) & \cos q_2(\ddot{q}_{d1} + \alpha\dot{e}_1) + \sin q_2\dot{q}_1(\dot{q}_{d1} + \alpha e_1) & 0 & \dot{q}_2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ f_{d_1} \\ f_{d_2} \end{bmatrix}$$

Closed loop error system:

Design τ:

$$\tilde{\theta} = \theta - \hat{\theta}$$

$$\text{Let } \tau = Y\hat{\theta} + \boxed{}_1$$

$$\text{After substituting } \tau$$

$$M\dot{r} = -V_m r + \tau_d + Y\tilde{\theta} - \boxed{}_1$$

Stability Analysis

Lyapunov function:

$$\boldsymbol{V = \frac{1}{2} r^T M(q) r + \frac{1}{2} \tilde{\theta}^T \Gamma \tilde{\theta}} \tag{7}$$

$$\begin{aligned}
\dot{V} &= \frac{1}{2} r^T \dot{M} r + r^T M(q)\dot{(r)} + \tilde{\theta}^T \Gamma \dot{\tilde{\theta}} \\
&= \frac{1}{2} r^T \dot{M} r + r^T(-V_m r + \tau_d + Y\tilde{\theta} - \boxed{}_1) - \tilde{\theta}^T \Gamma \dot{\hat{\theta}} \\
&= r^T\left(\frac{\dot{M}}{2} - V_m\right) r + r^T \tau_d + r^T Y \tilde{\theta} - r^T \boxed{}_1 - \tilde{\theta}^T \Gamma \dot{\hat{\theta}}
\end{aligned}$$

Using Skew Symmetric property, $r^T(\frac{\dot{M}}{2} - V_m) r = 0$

$$= r^T Y \tilde{\theta} + r^T \tau_d - r^T \boxed{}_1 - \tilde{\theta}^T \Gamma \dot{\hat{\theta}}$$

Design $\boxed{}_1$ = Kr, $\dot{\hat{\theta}} = \Gamma^{-1} Y^T r$:

$$\tau = Y\hat{\theta} + Kr$$

$$\dot{V} = r^T \tau_d + r^T Y \tilde{\theta} - r^T K r - \tilde{\theta}^T Y^T r$$

$$\text{Here, } r^T Y \tilde{\theta} = \tilde{\theta}^T Y^T r$$

$$\dot{V} = -K||r||^2 + r^T \tau_d$$

$$\textbf{If } \boldsymbol{\tau_d = 0}$$

$$\dot{V} \le 0 \quad \boldsymbol{Negative\ Semi-definite}$$

Applying Barbalat's Lemma,
From the above equation, $V(t) \le V(0)$.
i.e. V(t) is upper bounded $\implies V(t) \in L_\infty$.

$$\implies r \in L_\infty \text{ and } \tilde{\theta} \in L_\infty$$
$$r \in L_\infty \implies \dot{e} \in L_\infty$$
$$\dot{r} = \ddot{e} + \alpha \dot{e}$$
$$\text{Since } \dot{e} \in L_\infty \text{ we have, } \dot{r} \in L_\infty$$
$$\text{Also, } \ddot{V} = -2r^T K \dot{r}$$
$$\implies \ddot{V} \in L_\infty$$

Therefore, $\dot{V}$ is Uniformly continuous and $\dot{V} \to 0 \implies r \to 0 \implies \dot{e} \to -\alpha e$

It is evident that the error exponentially decays to zero i.e. $e \to 0$ as $t \to \infty$ Hence the system is Asymptotically Stable. Since the lyapunov function is radially unbounded, the system is **Globally Asymptotically Stable.**

If $\boldsymbol{\tau_d > 0}$,it can still be upper bounded by some value $\beta > 0$

$$r^T \tau_d \leq ||r||\beta$$
$$\dot{V} \leq -K||r||^2 + ||r||\beta$$
$$\dot{V} \leq -K|r|(|r| - \beta/K)$$
$$\text{Since } K, |r| > 0,$$
$$\dot{V} \leq 0 \text{ iff } |r| \geq \frac{\beta}{K}$$

Using Theorem 4.18,

V(x) is positive definite, decrescent and $\dot{V} < 0$ for $|r| \geq \sqrt{\beta/K}$. Also since the lyapunov function is radially unbounded, the stability here is **Globally Uniformly Ultimately bounded (G.U.U.B)** where the $V \to 0$ and error convergence can be proved by using Barbalat's lemma as done in the case where $\tau_d = 0$

$$\text{Final Control law, } \boldsymbol{\tau} = \boldsymbol{Y\hat{\theta}} + \boldsymbol{Kr} \tag{8}$$

4.2 Simulation results

The designed control law from Eq.8 was tested by simulating the system in Simulink. The following plots help in understanding the performance of the Adaptive controller. The control gains chosen for simulation are shown in the Table.2

The desired trajectory chosen is given below,

$$\mathrm{q}_d = \begin{bmatrix} \cos(t) \\ \cos(2t) \end{bmatrix}$$

Table 2: Control gains for Adaptive control

Control gains	Values
alpha	2
K	20*eye(2)
gamma	0.5*eye(5)

4.2.1 Without disturbance

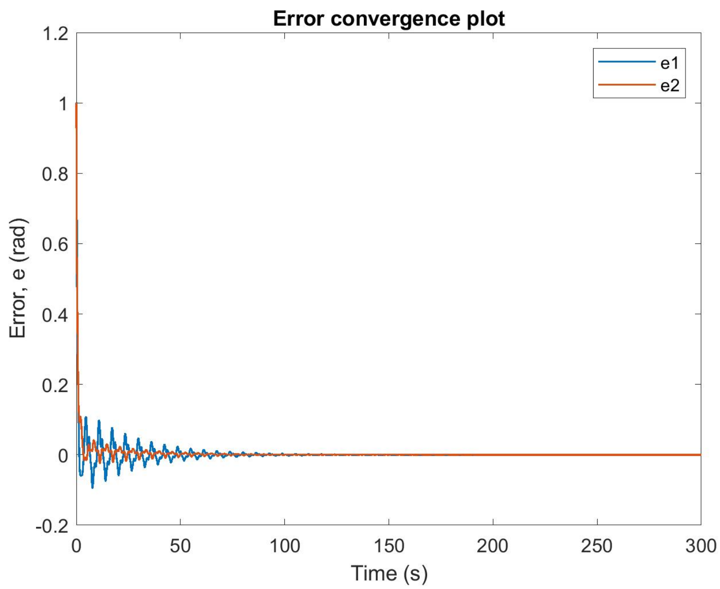

Figure 5: Error convergence

Since the model parameters $p_1, p_2, p_3, f_{d1} and\ f_{d2}$ are unknown, we have to estimate them at each time step. Therefore, it will take the controller more time than an Exact knowledge controller for error convergence. In Fig.5, it is clear that error converges to zero in around 150 seconds.

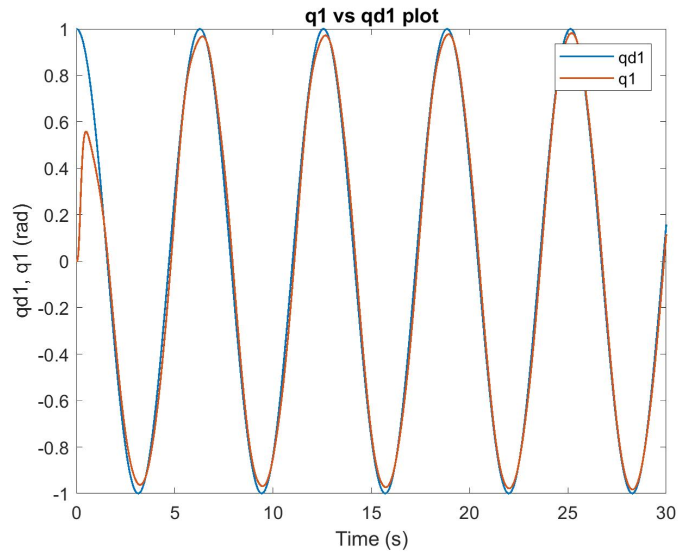

Figure 6: Joint angle 1 tracking performance

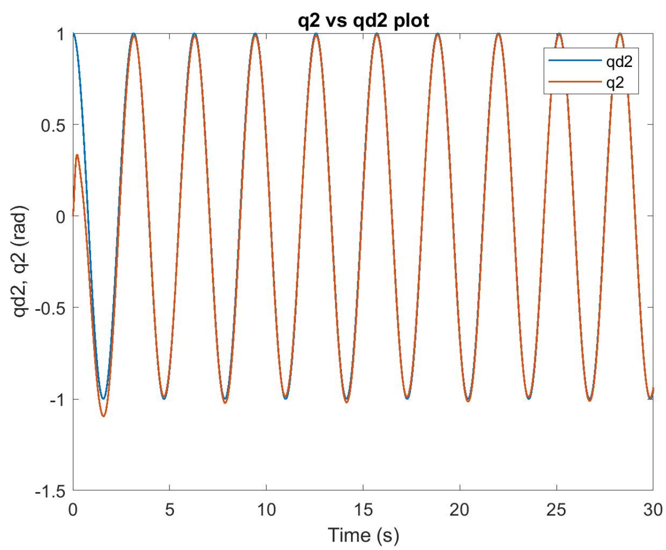

Figure 7: Joint angle 2 tracking performance

From the Fig.6 and Fig.7, it is clear that the robot almost tracks the desired trajectory with a minor error. This is due to the unknown parameters present in the robot model.

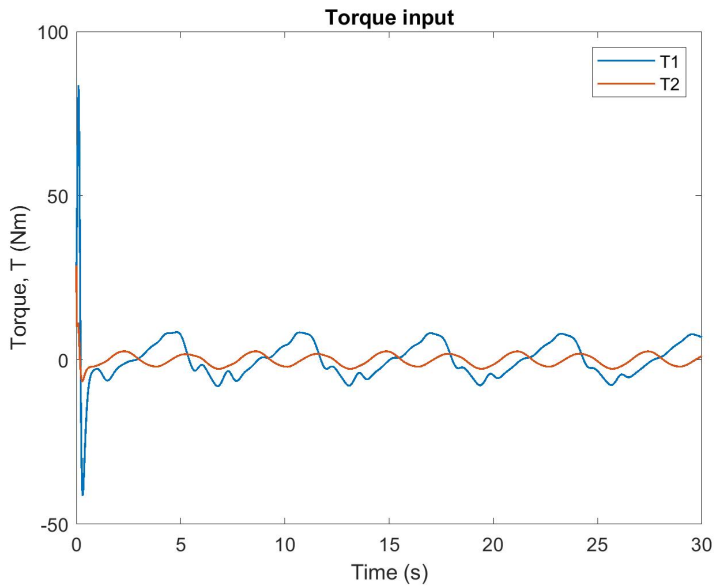

Figure 8: Control input (Torque)

As you can see from the Fig.8, the control input is high initially to make up for the high initial error.

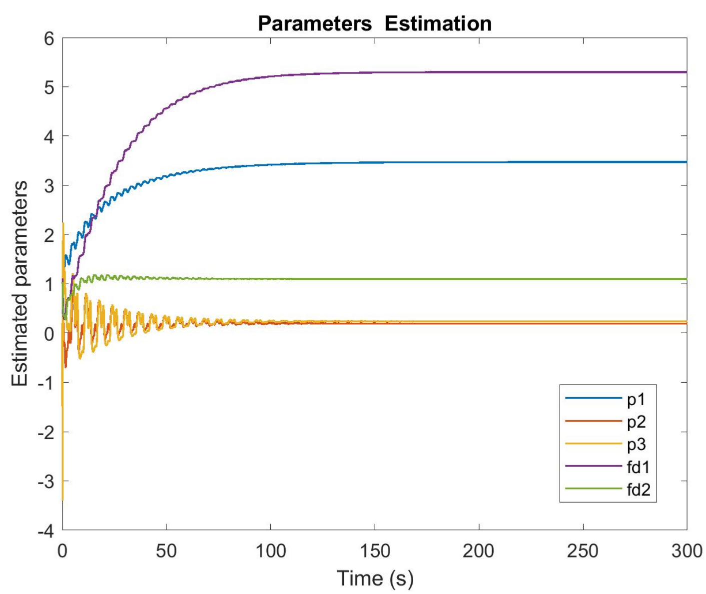

Figure 9: Parameter Convergence

From the Fig.9, we can observe that the unknown parameter values actually converge to the actual values of the robot as time progresses. And at 300 seconds, the estimates are exactly the same as the original parameter values of the robot.

4.2.2 With disturbance

The disturbance is given by,

$$\tau_d = \begin{bmatrix} 0.3 \\ 0.3 \end{bmatrix}$$

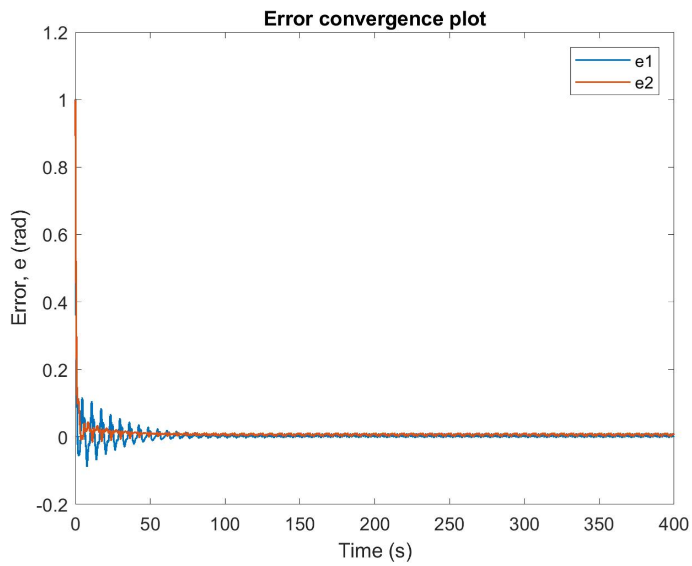

Figure 10: Error convergence

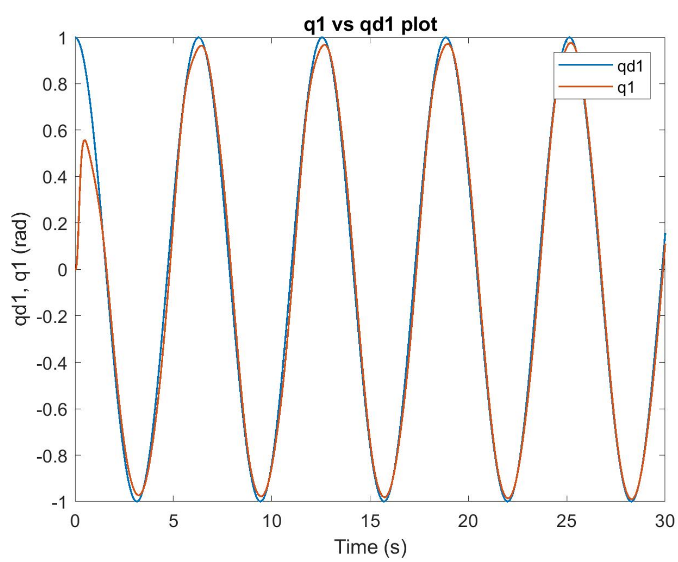

Figure 11: Joint angle 1 tracking performance

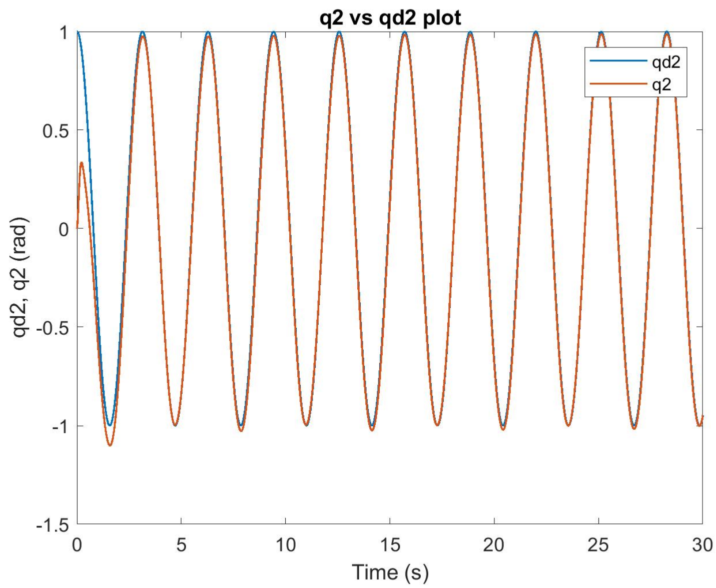

Figure 12: Joint angle 2 tracking performance

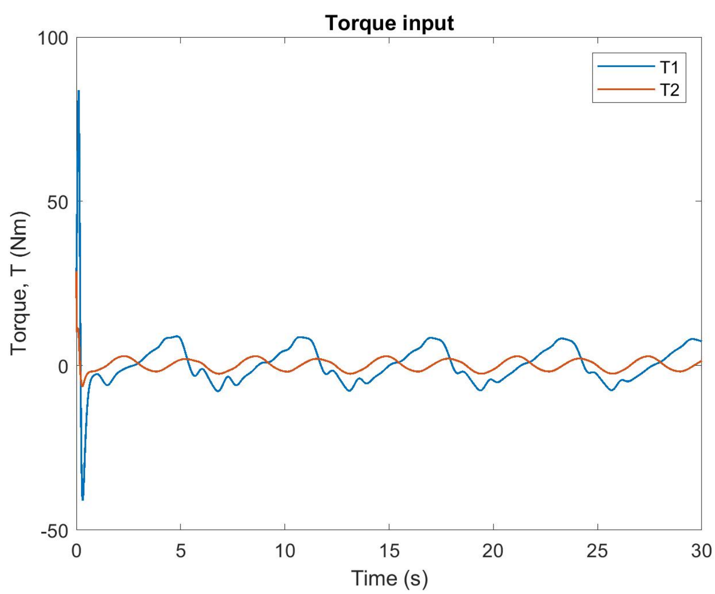

Figure 13: Control input (Torque)

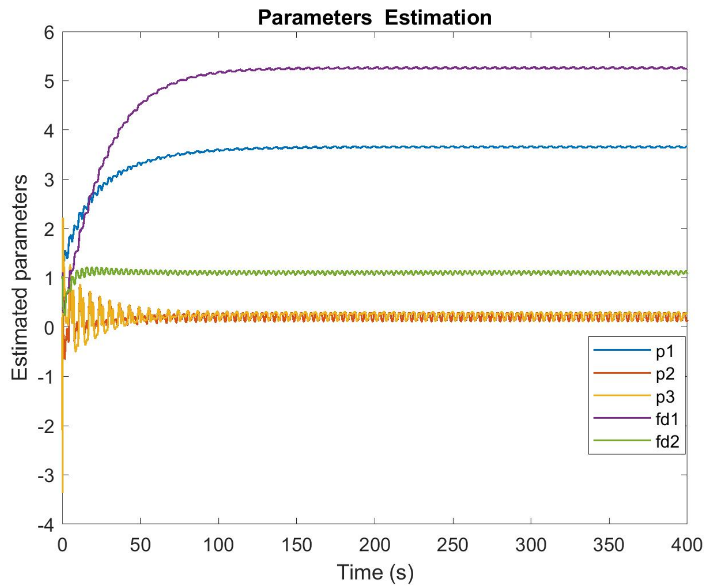

Figure 14: Parameter Convergence

Effect of the disturbance on the plots:

It can be observed from the above plots that the error doesn't completely go to zero because of the constant disturbance torque which is present in the system throughout the duration of the simulation. Due to these disturbances, the parameter estimates are not accurate even after some period of time. The robot still tracks the desired trajectory but with less accuracy compared to tracking control without disturbance. The error convergence can be improved at the cost of the input torque. Since it is necessary to keep in mind the saturation constraints on the control input, we have used reasonable control gains so as to have a feasible control input.

Also, in the presence of disturbance, the error doesn't converge to zero but stays within small bounds. This is in alignment with the stability analysis that concluded that the system is G.U.U.B in the presence of disturbances.

5 Robust controller

5.1 Robust control design & Stability Analysis

The Robust Controller is more robust compared to EMK and Adaptive controller. Even if few robot parameters are unknown, they can be upper bounded by a value and perfect tracking can be achieved. Hence we can include disturbances as a part of the system to check the efficacy of the controller. Therefore, the dynamic model is given by,

$$\tau = M(q)\ddot{q} \ + V_m(q,\dot{q})\dot{q} \ + F_d\dot{q} \ + \tau_d$$

5.1.1 Control objective

The control objective of the Robust controller is to achieve error convergence i.e. $e \to 0$ as $t \to \infty$. Since our aim is to track a trajectory which varies with time, the non-linear system under consideration is a non-autonomous system. It is also called as a time-varying system.
The error is given by,

$$e = q_d - q$$

where q_d is the desired trajectory
Differentiating with respect to time,

$$\dot{e} = \dot{q_d} - \dot{q}$$

Differentiating again with respect to time,

$$\ddot{e} = \ddot{q_d} - \ddot{q}$$

5.1.2 Auxiliary signal:

An Auxiliary signal is introduced to convert the second-order system representing the robot dynamics into a first-order system

It is given by,

$$r = \dot{e} + \alpha e$$

where, $\alpha \in R^+ > 0$

Open loop error system

$$\begin{aligned}
\dot{r} &= \ddot{e} + \alpha\dot{e} \\
M\dot{r} &= M\ddot{e} + \alpha M\dot{e} \\
&= M(\ddot{q_d} - \ddot{q}) + \alpha M\dot{e} \\
&= M\ddot{q_d} - M\ddot{q} + \alpha M\dot{e}
\end{aligned}$$

From Eq. 1,

$$\begin{aligned}
M\dot{r} &= M\ddot{q_d} - (\tau - V_m\dot{q} - F_d\dot{q} - \tau_d) + \alpha M\dot{e} \\
\boldsymbol{M\dot{r}} &= \boldsymbol{M\ddot{q_d} - \tau + V_m\dot{q} + F_d\dot{q} + \tau_d + \alpha M\dot{e}}
\end{aligned}$$

Adding and Subtracting $V_m r$, we get,

$$\begin{aligned}
M\dot{r} &= M\ddot{q_d} - \tau + \tau_d + V_m\dot{q} + F_d\dot{q} + \alpha M\dot{e} + V_m r - V_m r \\
&= M\ddot{q_d} - \tau + \tau_d + V_m(\dot{q} + r) + F_d\dot{q} + \alpha M\dot{e} - V_m r \\
&= M\ddot{q_d} - \tau + \tau_d + V_m(\dot{q} + \dot{e} + \alpha e) + F_d\dot{q} + \alpha M\dot{e} - V_m r \\
&= M\ddot{q_d} - \tau + \tau_d + V_m(\dot{q} + \dot{q_d} - \dot{q} + \alpha e) + F_d\dot{q} + \alpha M\dot{e} - V_m r \\
&= M\ddot{q_d} - \tau + \tau_d + V_m(\dot{q_d} + \alpha e) + F_d\dot{q} + \alpha M\dot{e} - V_m r \\
\boldsymbol{M\dot{r}} &= \boldsymbol{-V_m r + Y\theta - \tau + \tau_d}
\end{aligned}$$

The general idea of Robust control is to upper bound $Y\theta$ by a value $\rho(t)$. i.e.$||Y\theta|| < \rho(t)$. Most common robust control methods are High gain control, High frequency and Sliding mode control. In this project, sliding mode control has been used to achieve tracking. The control law for sliding mode control is,

$$\text{Final Control law, } \boldsymbol{\tau = \rho \ sgn(r) + Kr} \tag{9}$$

Stability analysis

Lyapunov function:

$$\begin{aligned}
\boldsymbol{V} &= \boldsymbol{\frac{1}{2} r^T M(q) r} \\
\dot{V} &= \frac{1}{2} r^T \dot{M} r + r^T M(q) \dot{(r)} \\
&= \frac{1}{2} r^T \dot{M} r + r^T(-V_m r + \tau_d + Y\theta - Kr - \rho \ sgn(r)) \\
&= r^T(\frac{\dot{M}}{2} - V_m) r + r^T Y\theta + r^T \tau_d - r^T K r - r^T \rho \ sgn(r)
\end{aligned}$$

Using Skew Symmetric property, $r^T(\frac{\dot{M}}{2} - V_m)r = 0$ and $||Y\theta|| < \rho(t)$

$$\begin{aligned}
\dot{V} &\le r^T \rho - K||r||^2 - r^T \rho + r^T \tau_d \\
\dot{V} &\le -K||r||^2 + r^T \tau_d
\end{aligned}$$

If $\tau_d = 0$,

$$\dot{V} \le -\frac{2K}{M} V \quad \boldsymbol{Negative \ definite \ and \ G.E.S} \tag{10}$$

From the Eq.10, it is evident that V(t) exponentially decays to zero as $t \to \infty$. Also the lyapunov function is radially unbounded. Hence, the system is **Globally Exponentially Stable**.

Signal Chasing to prove error convergence:

$$\begin{aligned}
&\text{Since } V(t) \to 0, r(t) \to 0 \\
&\implies \dot{e} \to -\alpha e
\end{aligned}$$

It is evident that the error exponentially decays to zero i.e. $e \to 0$ as $t \to \infty$

Hence, error convergence is achieved.

If $\boldsymbol{\tau_d > 0}$,it can still be upper bounded by some value $\beta > 0$

$$\begin{aligned}
r^T \tau_d &\le ||r||\beta \\
\dot{V} &\le -K||r||^2 + ||r||\beta \\
\dot{V} &\le -K|r|(|r| - \beta/K) \\
\text{Since } K, |r| > 0, & \\
\dot{V} &\le 0 \text{ iff } |r| \ \ge \frac{\beta}{K}
\end{aligned}$$

Using Theorem 4.18,

V(x) is positive definite, decrescent and $\dot{V} < 0$ for $|r| \geq \sqrt{\beta/K}$. Also since the lyapunov function is radially unbounded, the stability here is **Globally Uniformly Ultimately bounded (G.U.U.B)** where the $V \to 0$ and error convergence can be proved similar to the case $\tau_d = 0$

5.2 Simulation results

The designed control law from Eq.9 was tested by simulating the system in Simulink. The following plots help in understanding the performance of the Robust controller.

Table 3: Control gains for Robust control

Control gains	Values
alpha	1.8
K	42*eye(2)

The desired trajectory chosen is given below,

$$\mathrm{q}_d = \begin{bmatrix} \cos(t) \\ \cos(2t) \end{bmatrix}$$

Also, an integral part for Sliding mode control is the selection of ρ values. In this project, the strategy adopted to approximate ρ is given below,

$$\rho = \max(M(q)\ddot{q}) \ + \max(V_m(q,\dot{q})\dot{q}) \ + \max(F_d\dot{q}) + \max(\alpha M\dot{e})$$

since we have to find a value for ρ such that it upper bounds $Y\theta$

5.2.1 Without disturbance

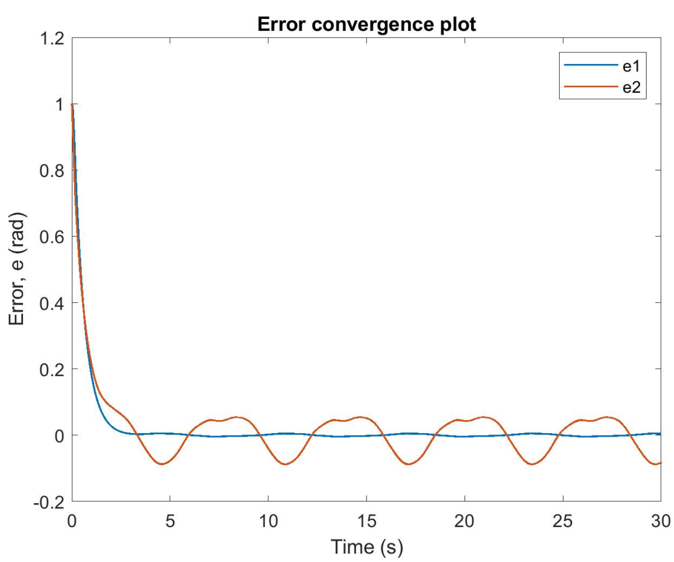

Figure 15: Error convergence

From the Fig.15, it can be seen that the error almost converges to zero.

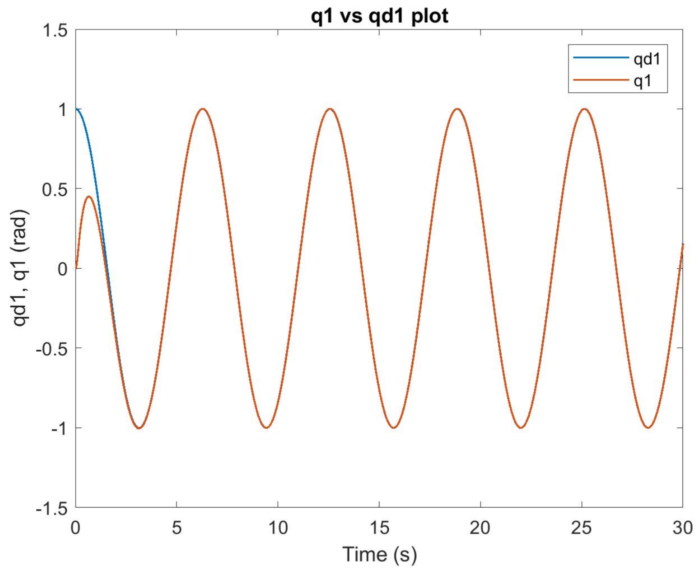

Figure 16: Joint angle 1 tracking performance

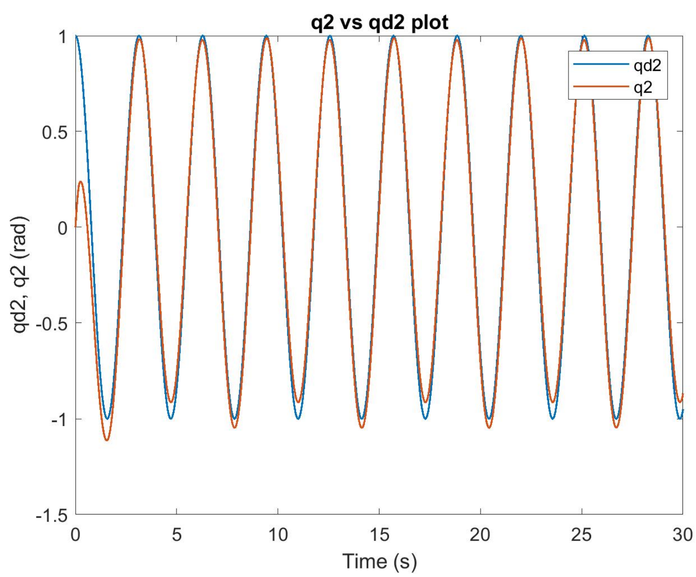

Figure 17: Joint angle 2 tracking performance

It is evident from the Fig.16 and Fig.17 that the joint angle 1 tracks the desired joint angle 1 accurately but there is some tiny deviation in the second joint angle tracking. This is because of the type of Robust control technique that has been chosen in this project. Since Sliding mode control design has infinite bandwidth, the error might get close to zero but not exactly zero. The control input keeps switching directions as error gets close to zero and hence might excite unmodeled high frequency dynamics leading to compromised accuracy.

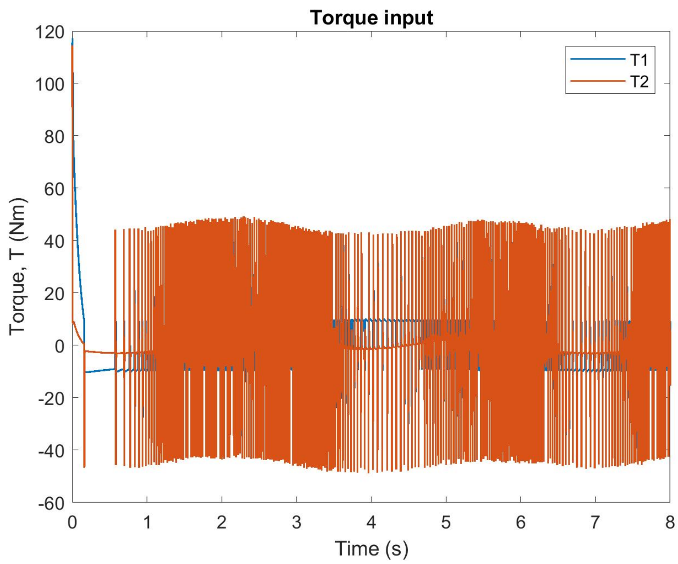

Figure 18: Control input (Torque)

From the Fig. 18, it is evident that the control input keeps switching its direction as error approaches zero and has an infinite bandwidth.

5.2.2 With disturbance

The disturbance is given by,

$$\tau_d = \begin{bmatrix} 0.3 \\ 0.3 \end{bmatrix}$$

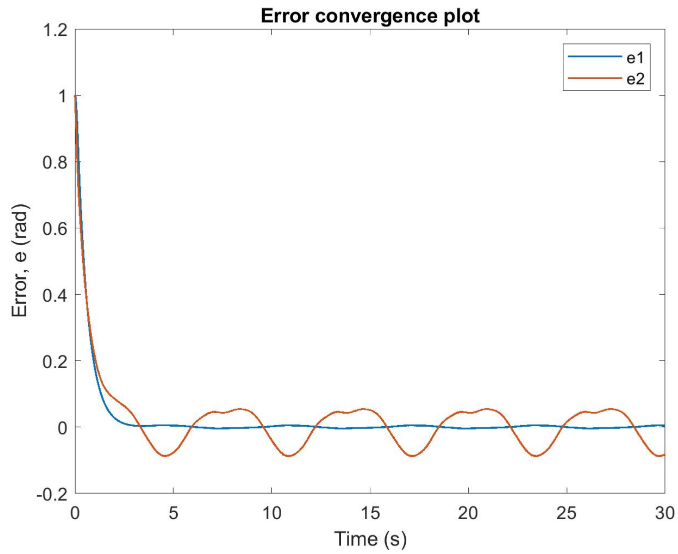

Figure 19: Error convergence

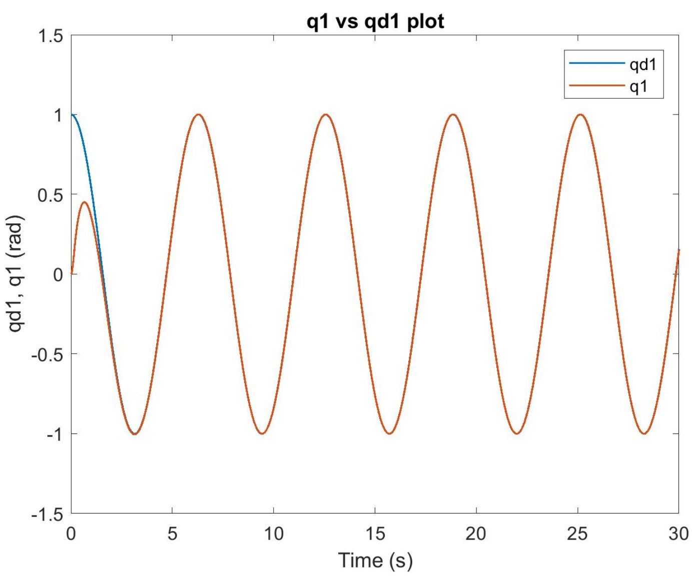

Figure 20: Joint angle 1 tracking performance

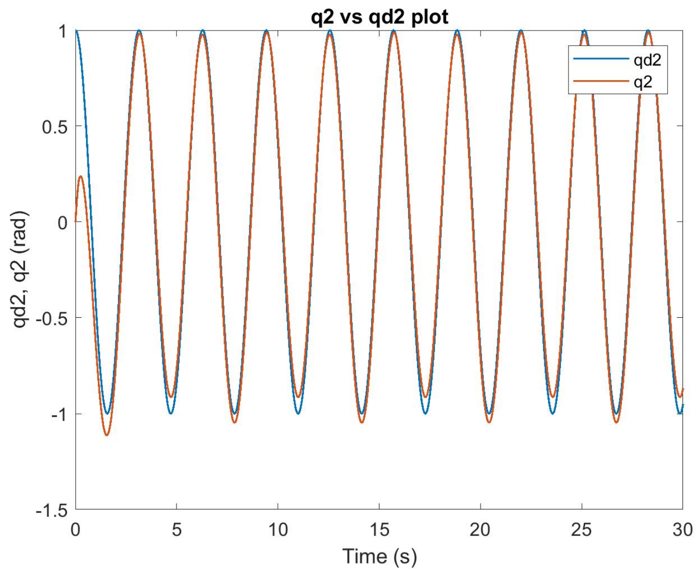

Figure 21: Joint angle 2 tracking performance

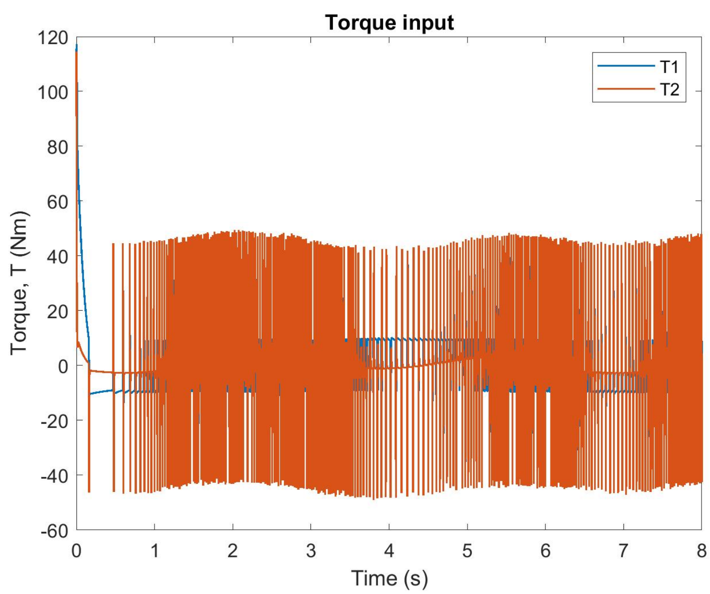

Figure 22: Control input (Torque)

Effect of disturbances on the tracking performance:

From the Fig. 19, Fig. 20, Fig. 21 and Fig. 22, it is evident that the disturbance doesn't have much effect on the error convergence of the system that uses robust controllers. Since sliding mode control doesn't depend on estimating the unknown parameters but rather upper bounds the matrices with unknown parameters, it is more robust to the presence of disturbances in the system.

6 Comparison of Controllers

In the absence of disturbances, the EMK controller and Sliding mode controller are G.E.S whereas Adaptive control is G.A.S. However, each of the controllers has its own advantages and disadvantages.

If the robot model parameters are known, then the EMK controller is the best controller since it has quick error convergence with minimal control effort.

But if there are uncertainties in the system, either robust or adaptive could be used. If our objective is to get as close as possible to desired trajectory in the shortest time, then sliding mode control design could be used since it has faster error convergence compared to Adaptive. This is because, the parameter estimation using the update law takes time.

If the objective is to achieve perfect tracking using minimal control effort and ensuring longer actuator life, then the best option would be to choose Adaptive control since it doesn't burn out the actuator quickly unlike Robust control with infinite bandwidth. Also, it estimates the unknown parameters. Therefore, after a certain period of time, perfect tracking is guaranteed.

7 Conclusion

In this project, three non-linear controllers namely, Exact model knowledge controller, Adaptive controller, and Robust controller were designed for a robot manipulator to track a desired trajectory. The control law for each of the controllers was designed using rigorous stability analysis. The performance of the designed control laws was investigated in Simulink.

Further, the Adaptive controller and Robust controller were investigated in the presence of a disturbance in the system. It was observed that the Robust controller is the most robust controller but it has infinite bandwidth and error convergence is not so good as Adaptive controller. But Adaptive controller works only when we are able to represent the unknown matrices in 'linear in parameters' form. Also, the Exact model knowledge controller gives the most perfect tracking in a very short time using minimal control input. But, it is sensitive to uncertainties and will not work if there are unknown model parameters or other uncertainities. Thus a comparative study was made on all the three non-linear controllers.