National Physical Laboratory

NPL Time and Frequency User Club

A Beginner's Guide to Kalman Filters

J.A.Davis

9th February 2006

WHAT IS A KALMAN FILTER

- An recursive analytical technique to estimate time dependent physical parameters in the presence of noise processes

- Example of a time and frequency application:

  Offset between two clocks

PREDICTORS, FILTERS AND SMOOTHING ALGORITHMS

- A predictor provides estimates of the physical parameter’s current values using previous measurements

- A filter provides estimates of the physical parameter’s using previous and current measurements

- A smoothing algorithm provides estimates of the physical parameter’s current values using previous, current and future measurements

NPL

WHY USE A KALMAN FILTER?

- Provides optimal estimates when noise model is exact
- Computationally efficient, update filter when adding new measurements to the existing data set
- Very simple to implement
- Recursive algorithm

WHY USE A KALMAN FILTER?

- May be used as a predictor
- Easy to verify performance using simulated data
- Uncertainty estimates are provided as part of the filter
- May be easily modified to provide a steering algorithm or other application

ELEMENTS OF THE KALMAN FILTER

- Physical parameters being estimated represented by the components of the state vector.

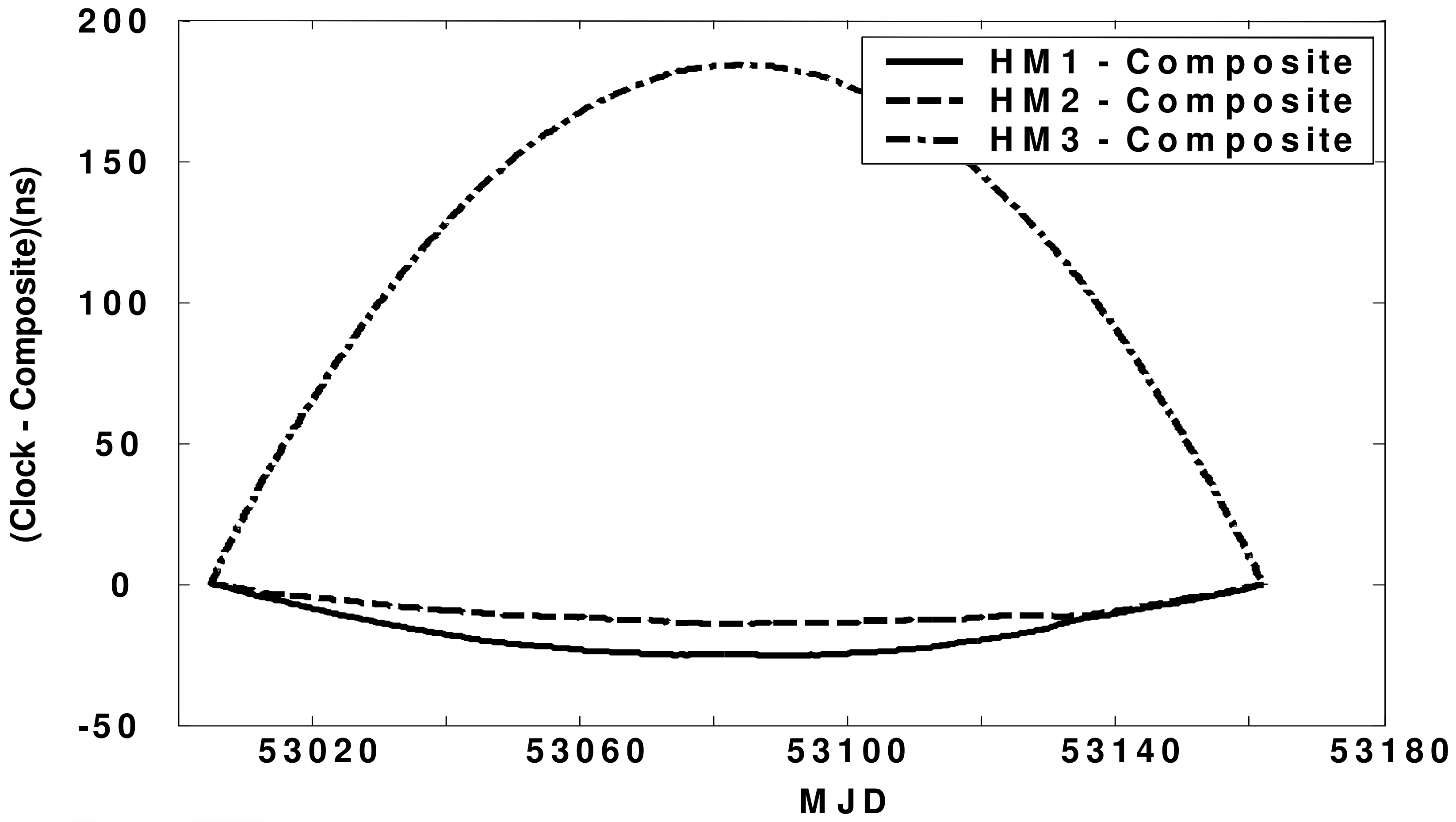
- $\hat{x}^+$ is the state vector with the current measurements include

- $\hat{x}^-$ is the state vector extrapolated to next data point where the next set of measurements have not yet been included

- Example of the elements of a state vector (Phase offset, Normalised frequency offset, Linear frequency drift)

NPL

DETERMINISTIC CLOCK CHARACTERISTICS (ACTIVE HYDROGEN MASERS)



ELEMENTS OF THE KALMAN FILTER

- *y* is the measurement vector.
- *H* is the design matrix that relates the measurements to the physical parameters being estimated $y = Hx$.
- In our simple example we have phase measurements and $H = (1,0,0)$.
- *K* is the Kalman gain, the higher the “weighting” of current measurements used to estimates the physical parameters. *K* is determined by the Kalman filter.

NPL

STATE PROPAGATION MATRIX

- State propagation matrix is used to extrapolate current estimate:

$$\Phi = \begin{pmatrix} 1 & \tau & \dfrac{\tau^2}{2} \\ 0 & 1 & \tau \\ 0 & 0 & 1 \end{pmatrix}$$

NPL

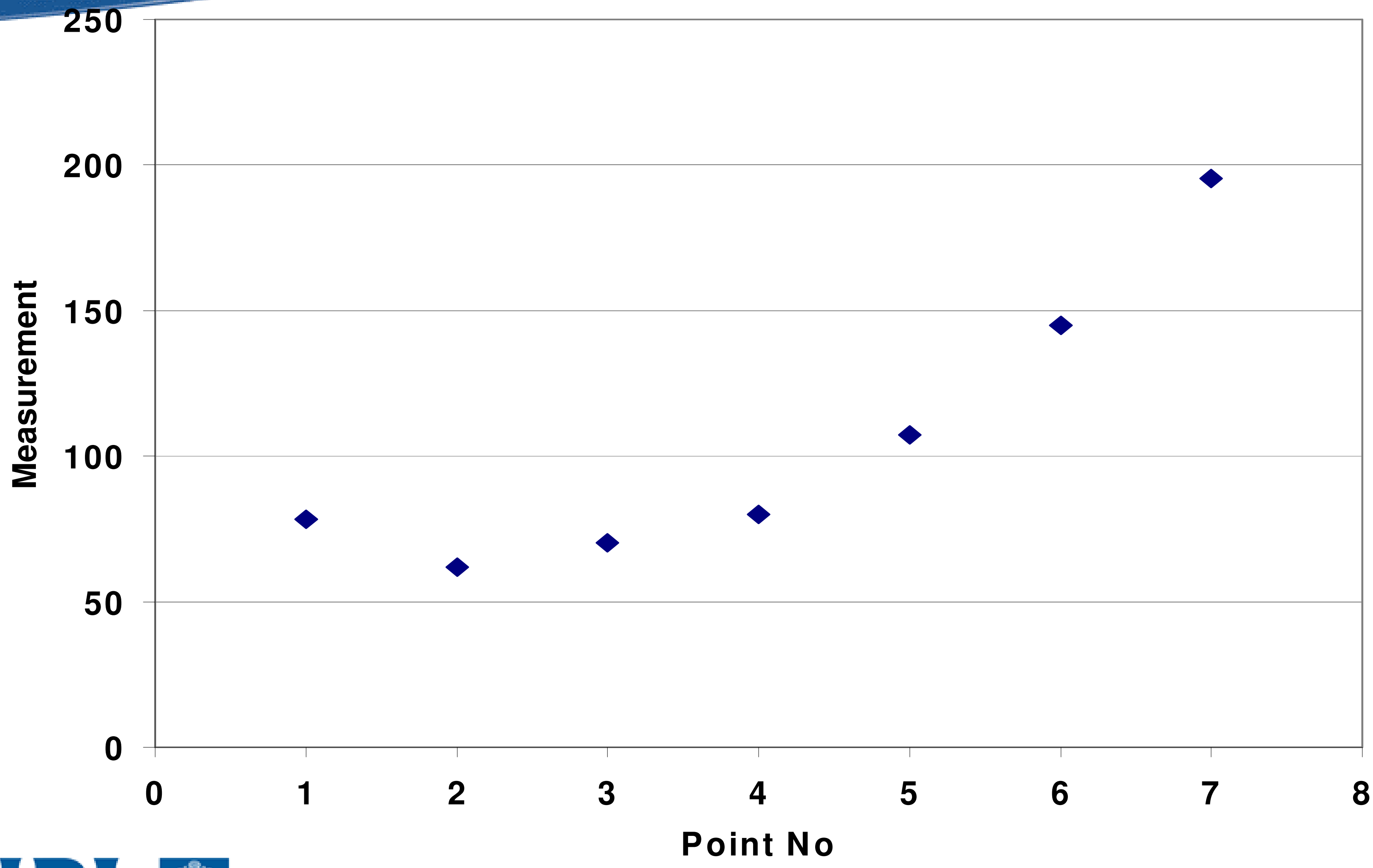
COVARIANCE MATRICES

- Process covariance matrix *Q*:
- Describes the process noise added at each measurement. This is the clock noise

- *R* is the measurement noise covariance matrix. Noise is assumed to be white and may be correlated.

- *P* is the parameter covariance matrix, describes the uncertainties in the state vector estimates. P^+ and P^- is the matrix after and before the measurement update

SIMPLE MEASUREMENT SET
CURRENT POINT = 3



HOW IT ALL WORKS

- State vector extrapolation (1)

$$\underline{\hat{x}}(t_n^-) = \underline{\underline{\Phi}}(\tau)\underline{\hat{x}}(t_{n-1}^+)$$

- Parameter covariance matrix extrapolation (2)

$$\underline{\underline{P}}(t_n^-) = \underline{\underline{\Phi}}(\tau)P(t_{n-1}^+)\underline{\underline{\Phi}}^T(\tau) + \underline{\underline{Q}}(t_n)$$

HOW IT ALL WORKS

- Kalman gain determination (3)

$$\underline{\underline{K}}(t_n) = \underline{\underline{P}}(t_n^-)\underline{\underline{H}}^T\left[\underline{\underline{H}}\,\underline{\underline{P}}(t_n^-)\underline{\underline{H}}^T + \underline{\underline{R}}(t_n)\right]^{-1}$$

- Incorporating current measurements (4)

$$\underline{\hat{x}}(t_n^+) = \underline{\hat{x}}(t_n^-) + \underline{\underline{K}}(t_n)\left[\underline{y}(t_n) - \underline{\underline{H}}\,\underline{\hat{x}}(t_n^-)\right]$$

- Parameter covariance matrix update (5)

$$\underline{\underline{P}}(t_n^+) = \left[\underline{\underline{I}} - \underline{\underline{K}}(t_n)\underline{\underline{H}}\right]\underline{\underline{P}}(t_n^-)$$

NPL

WHAT ARE THE PITFALLS?

- Literature is full of descriptions of very badly constructed Kalman filters (including time and frequency literature)

- Require a reasonably accurate model of underlying physical processes

- Require a reasonably accurate model of the noise processes

- Physical parameters must be observable

  Clock ensemble algorithm physical parameters are only partly observable. Problem reported 1987, problem solved 2002

NPL

PROPERTIES OF THE KALMAN FILTER

- Equations 2, 3 and 5 may be run independently of the rest, and do not include actual measurements

- Uncertainty of the state vector determination depends only on the covariance matrices Q and R, the measurement matrix H and the number of iterations

- Kalman gain has similar properties

ASSUMPTIONS MADE WHEN USING THE KALMAN FILTER

- Process noise added on successive epochs does not correlate

- Measurement noise is assumed to be white, zero mean and distributed normally

WHERE DO WE USE KALMAN FILTER CLOCK ALGORITHMS?

- Clock ensemble algorithm
  - Input from three or more clock, outputs a “composite” that should be more stable than any individual clock

- Clock predictor
  - Predicts the future offset between two clocks based on current and previous measurements

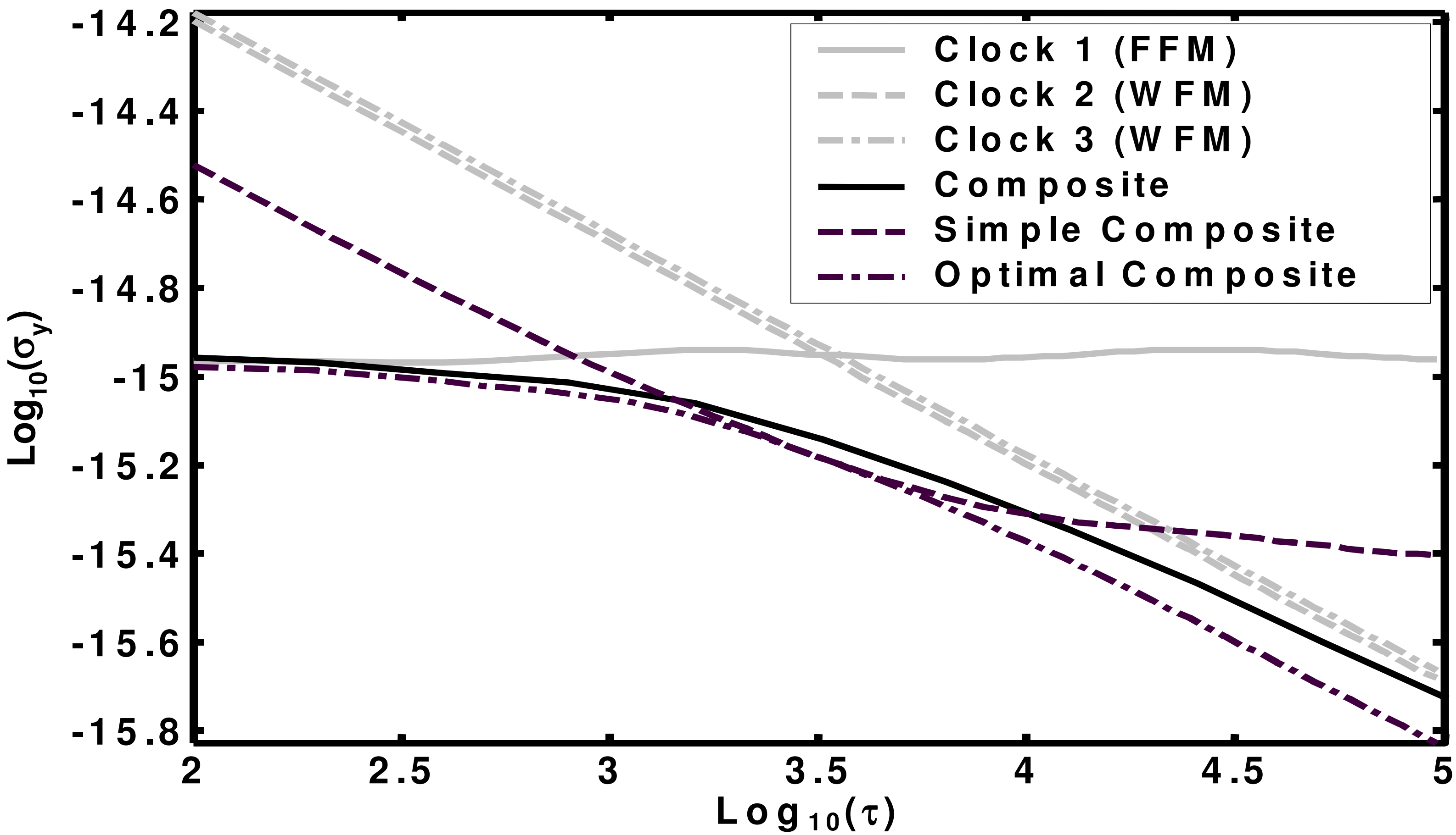
NPL

WHERE DO WE USE KALMAN FILTER CLOCK ALGORITHMS?

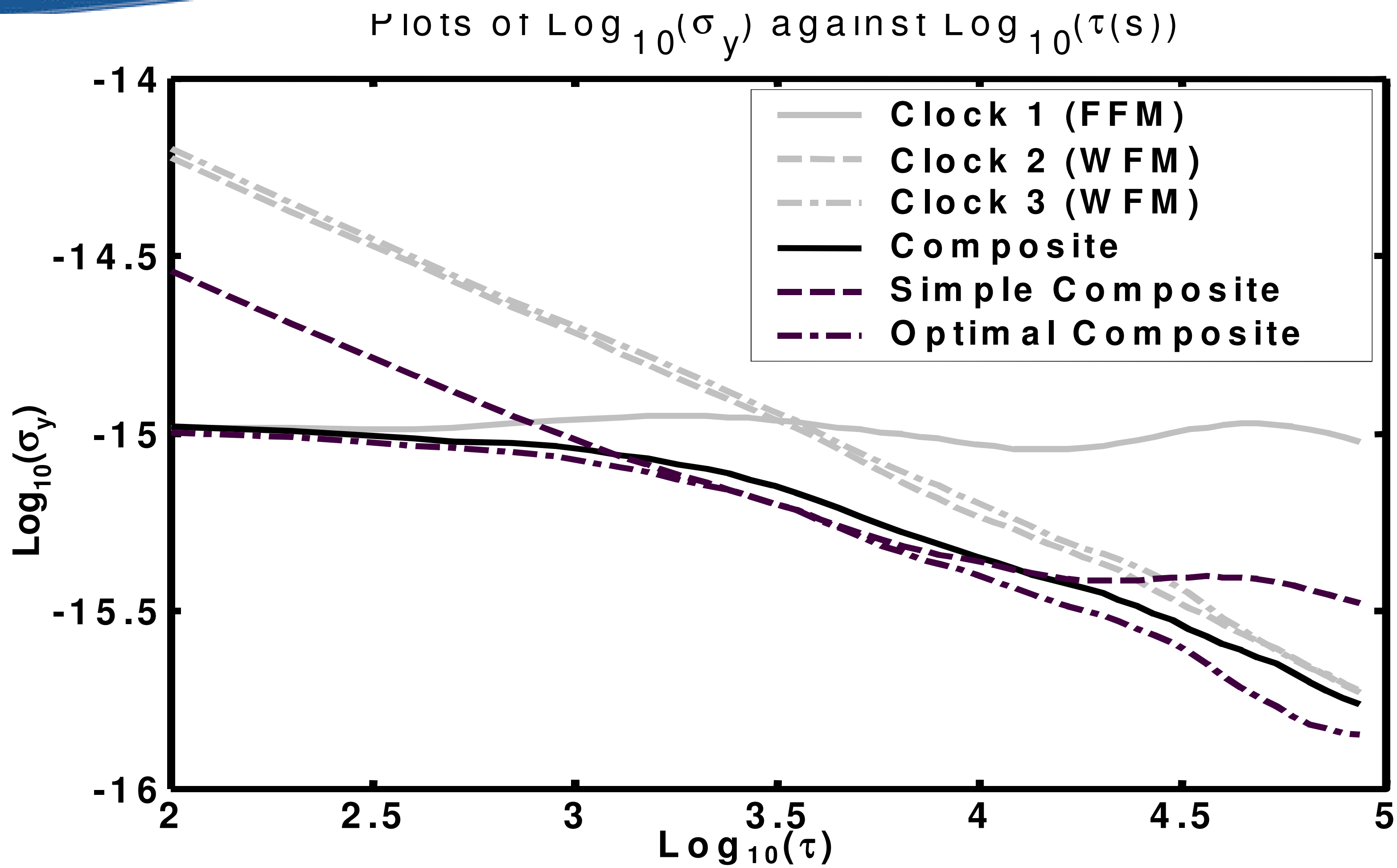
- Clock steering algorithm
  - Steers a hardware or software clock to stay close to a reference timescale. Trade-off of frequency stability and time offset
- Time transfer combining algorithm
  - Combines measurements from several time transfer links between the same two clocks to form an optimal composite

NPL

APPLICATION TO NPL'S CLOCK ENSEMBLE ALGORITHM

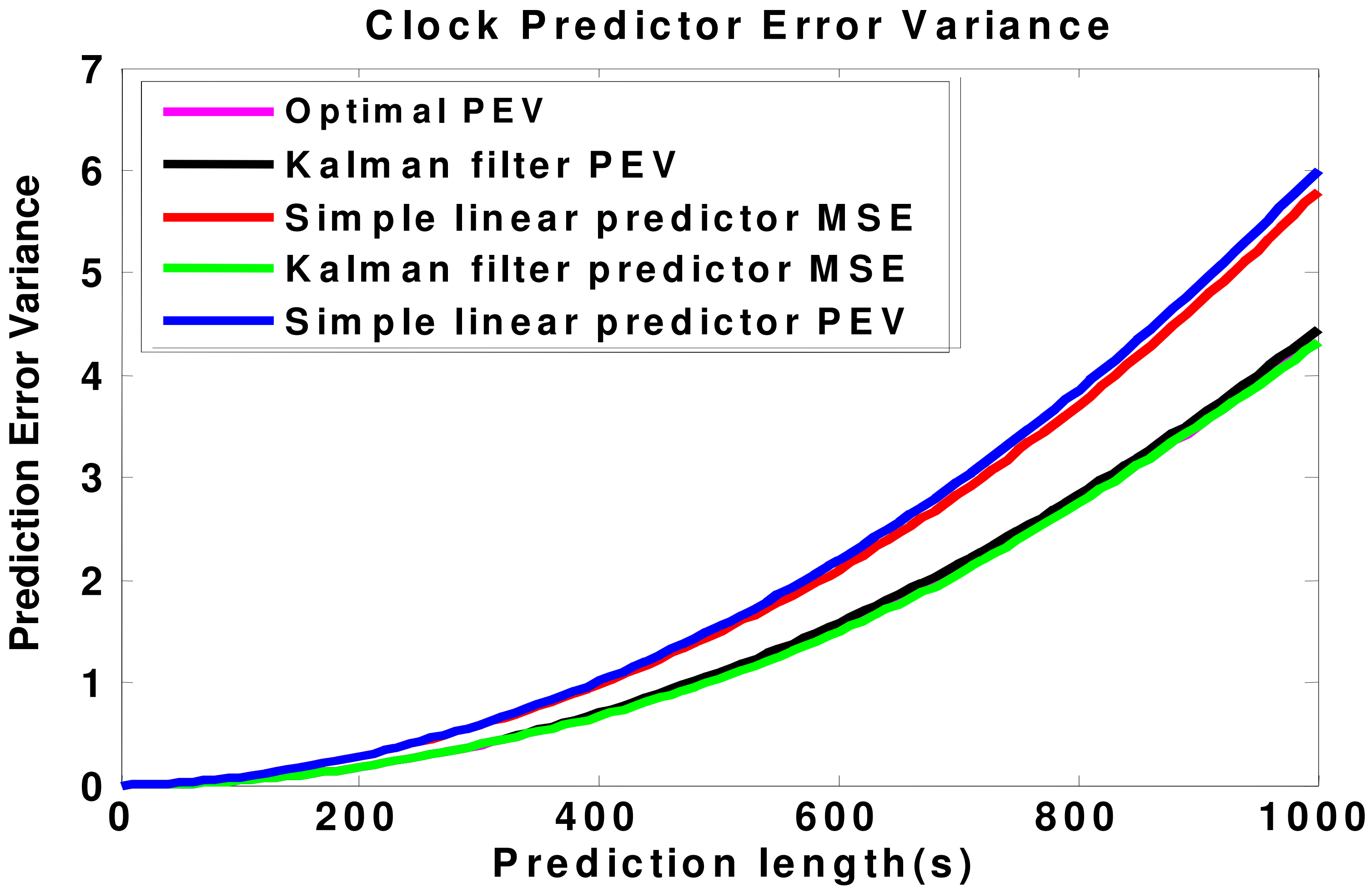


APPLICATION TO NPL'S CLOCK ENSEMBLE ALGORITHM



NPL

CLOCK PREDICTOR APPLICATION



NPL

CONCLUSIONS

- Kalman filters may be used as a very effective in time and frequency analysis

- Must understand their limitations

- Choice of noise models and the state vector components is critical