Hierarchical Bayesian Modeling

Benjamin Johnson (Harvard/CfA)

My name is what? I use HBM for x

we'll talk about HBM and some examples

Introduction

- **When is HBM appropriate? (with examples)**
- **Bayesian inference refresher/terminology/notation (individual/single-level)**

Hierarchical inference

- **Individuals & Populations**
- **Conditional independence & graphs**
- **Inference of the prior (pooling)**
- **Improvements in individual posteriors (shrinkage)**

Break & questions

Hierarchical Examples

- **Galaxy Mass Function**
- **Star Clusters**

Implementation Notes

- **Gibbs sampling**
- **Psuedo-importance sampling**
- **Selection functions**

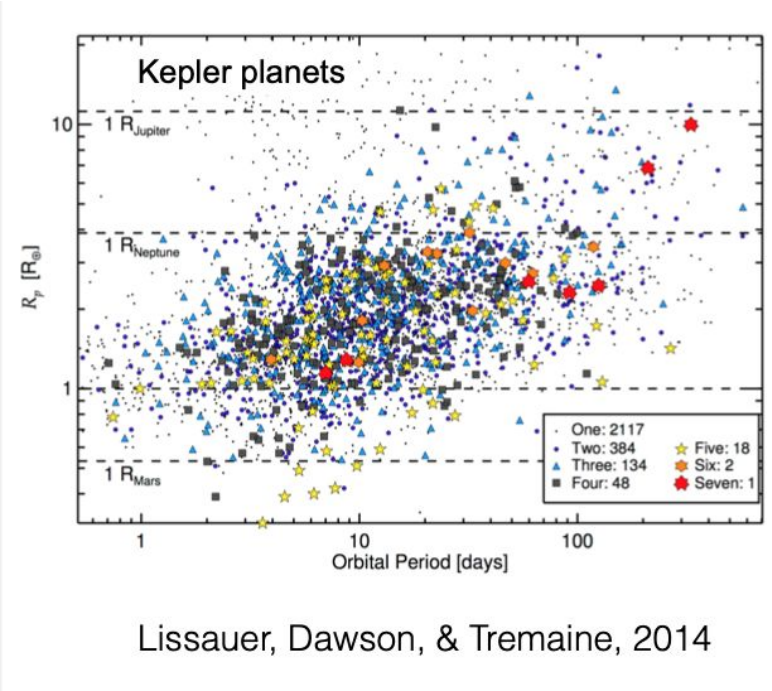
When is HBM appropriate?

- You are interested in the ***population distribution*** of properties of individual objects, but the individual properties are uncertain.

- You want to combine many individual objects or observations to ***infer parameters of the population*** especially if the individuals carry different kinds of information.

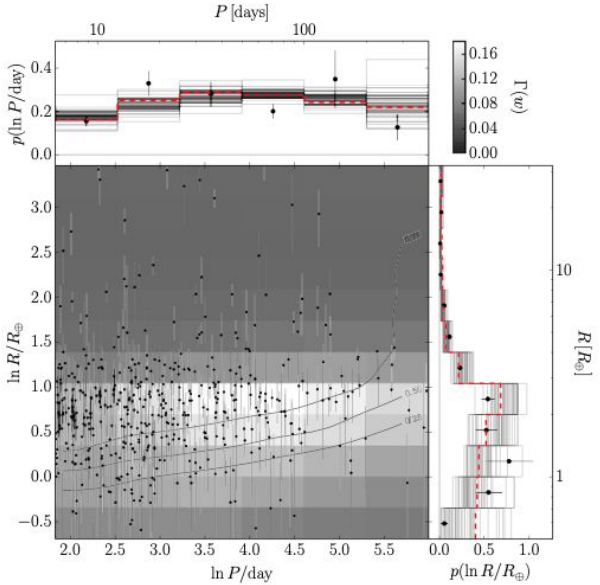
- You want to use an observed population to ***improve estimates for individuals***

When is HBM appropriate?



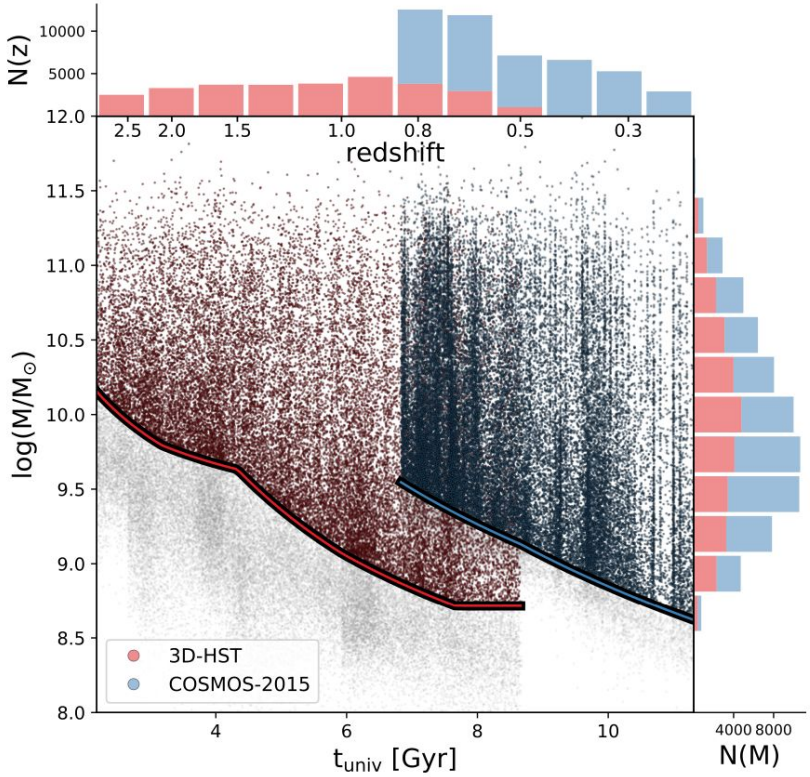
Lissauer, Dawson, & Tremaine, 2014

Given observations of many individual planets, what is the rate of earthlike planets around sunlike stars?



Foreman-Mackey et al. 2014

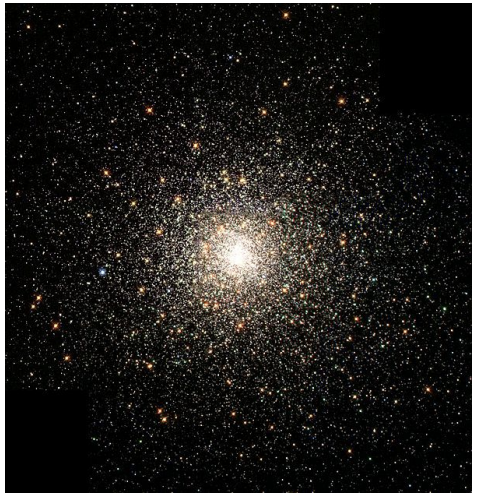
When is HBM appropriate?



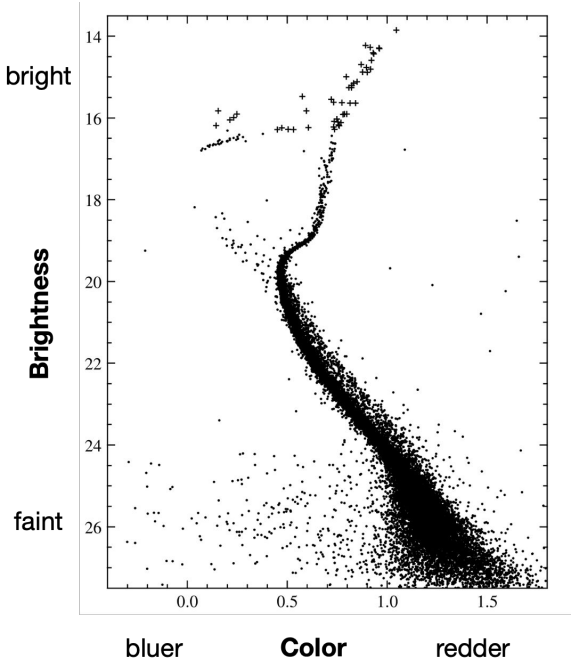
Given photometry and redshifts of many galaxies, what is the distribution of galaxy stellar masses, as a function of redshift?

Leja, Speagle, BDJ et al., 2020

When is HBM appropriate?



Given photometry (and parallaxes) of individual member stars, what are the properties -- age, metallicity, distance -- of a star cluster?



When is HBM appropriate?

Can also be used for “calibration”:

Inferring global ‘offset’ parameters from a population of many individuals while simultaneously applying the offset to individuals

Bayesian Inference Refresher

Joint Probability

conditional probability marginal probability

$$p(A, B) = p(A \mid B) \times p(B)$$

Bayes Theorem

$$p(A \mid B)\, p(B) = p(B \mid A)\, p(A)$$

Joint Probability for 3 variables

$$\begin{aligned} p(A, B, C) &= p(A, B \mid C) \times p(C) \\ &= p(A \mid B, C) \times p(B \mid C) \times p(C) \end{aligned}$$

Bayesian inference refresher

Conditional independence

A and B are independent conditional on C iff

$$p(A \mid B, C) = p(A \mid C) \quad \text{s.t.}$$
$$p(A, B, C) = p(A \mid C) \times p(B \mid C) \times p(C)$$

which also means

$$p(A, B \mid C) = p(A \mid C) \times p(B \mid C)$$

Bayesian inference refresher

The standard Bayesian inference uses Bayes theorem as follows:

$$p(\theta \mid \mathrm{D})\, p(\mathrm{D}) = p(\mathrm{D} \mid \theta)\, p(\theta)$$

posterior evidence likelihood prior

θ

parameters

D = $\{d_j\}$

dataset individual data points or observations

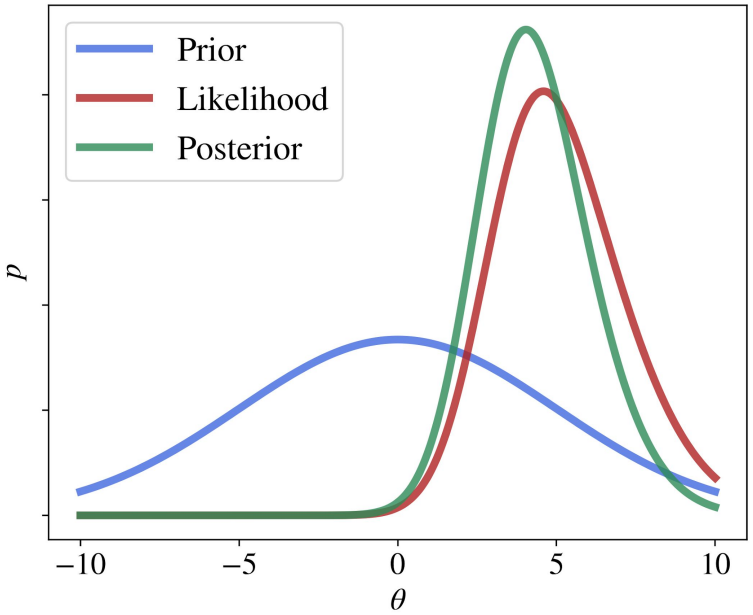
The goal is to obtain an estimate for the ***posterior*** distribution.

Bayesian inference refresher

The standard Bayesian inference uses Bayes theorem as follows:

$$p(\theta \mid \mathrm{D}) \propto p(\theta)\, p(\mathrm{D} \mid \theta)$$

posterior prior likelihood



N.B. All normalized to integrate to 1

The prior may be physically motivated, or based on previous studies, or weakly informative (flat), but it is more or less ***arbitrarily constructed*** at this point.

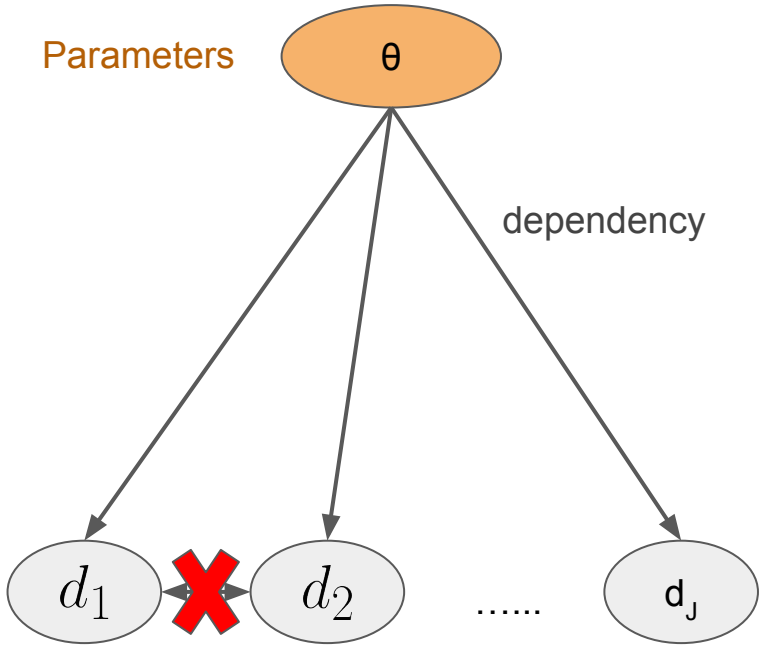
Sneak peek: In HBM we can use the actual observed population to inform the prior

Bayesian inference refresher

Iff the data points are ***conditionally independent*** of each other, we can rewrite this as something like

$$
\begin{aligned}
p(\theta \,|\, \mathrm{D})\, p(\mathrm{D}) &= p(\theta)\, p(\mathrm{D} \,|\, \theta) \\
&= p(\theta)\, p(d_1, d_2, \ldots, d_J \,|\, \theta) \\
&= p(\theta)\, p(d_1 \,|\, \theta)\, p(d_2 \,|\, \theta) \ \ldots \ p(d_J \,|\, \theta) \\
&= p(\theta) \prod_{j=1}^{J} p(d_j \,|\, \theta)
\end{aligned}
$$

conditional independence



Data or observables

N.B. If *p(θ)*=1 and the individual likelihoods are Gaussian, and you take the log of both sides, this quickly leads to the familiar chi-squared expression.

Now on to Hierarchical Inference

HBM: individuals and populations

The previous few slides are relevant for inferring the properties of an *individual*. However, if you have many individuals, you may wish to:

1) Learn properties of the population ***from*** the individuals
2) Use the population to improve priors ***for*** the individuals

How do we do this self-consistently when we don't know the properties of the individuals before-hand?

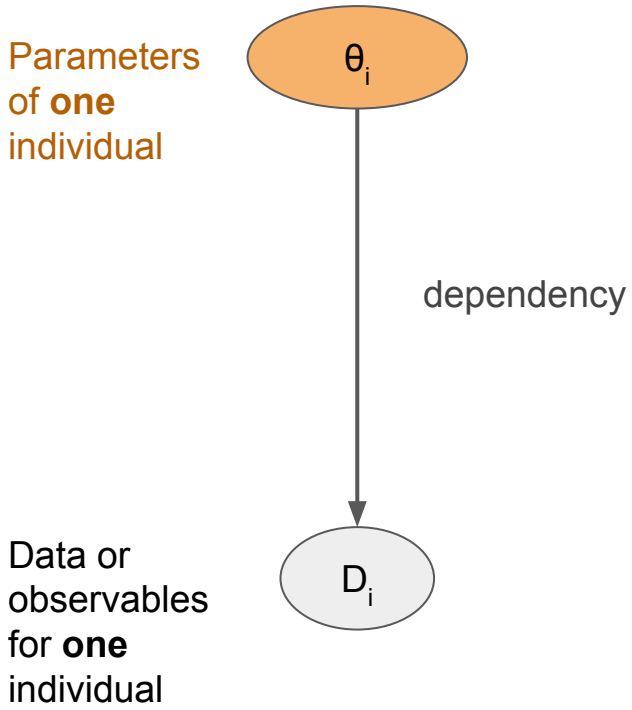
Hierarchical Bayesian Modeling

HBM: individuals and populations

The equations on the previous few slides are relevant for inferring the properties of an individual.

If you have N individuals, you might write for one individual *i*

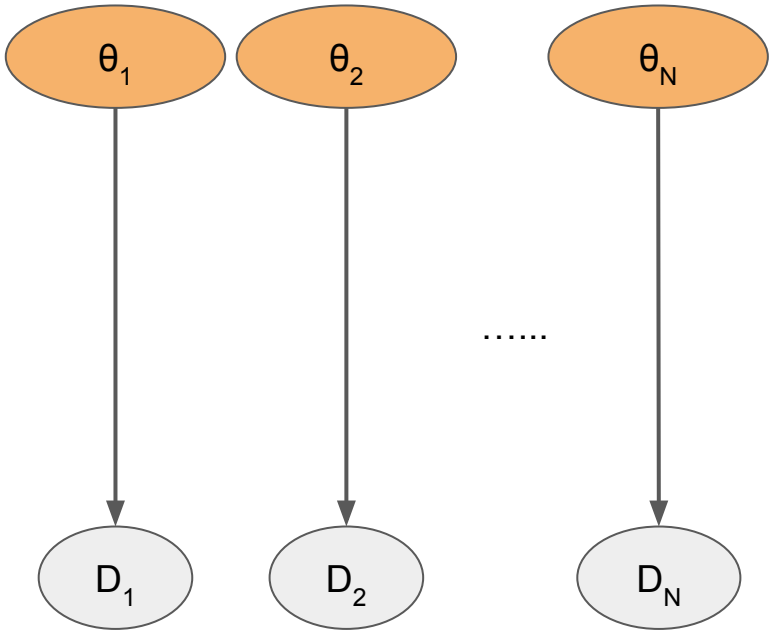
$$p(\theta_i \mid \mathrm{D}_i)\, p(\mathrm{D}_i) = p(\theta_i)\, p(\mathrm{D}_i \mid \theta_i)$$



HBM: individuals and populations

Because the individuals are independent, we can also write a simple expression for the probability of ***all*** the object parameters given ***all*** the data.

$$
\begin{aligned}
p(\{\theta_i\}\,|\,\{\mathrm{D}_i\})\,p(\{\mathrm{D}_i\}) &= p(\{\theta_i\})\,p(\{D_i\}\,|\,\{\theta_i\}) \\
&= p(\{\theta_i\})\,p(D_1\,|\,\theta_1)\,p(D_2\,|\,\theta_2)\,\ldots\,p(D_N\,|\,\theta_N) \\
&= p(\{\theta_i\}) \times \prod_{i=1}^{N} p(D_i\,|\,\theta_i) \\
&= p(\theta_1)\,p(\theta_2)\,\ldots\,p(\theta_N) \times \prod_{i=1}^{N} p(D_i\,|\,\theta_i) \\
&= \prod_{i=1}^{N} p(\theta_i)\,p(D_i|\theta_i) \quad .
\end{aligned}
$$



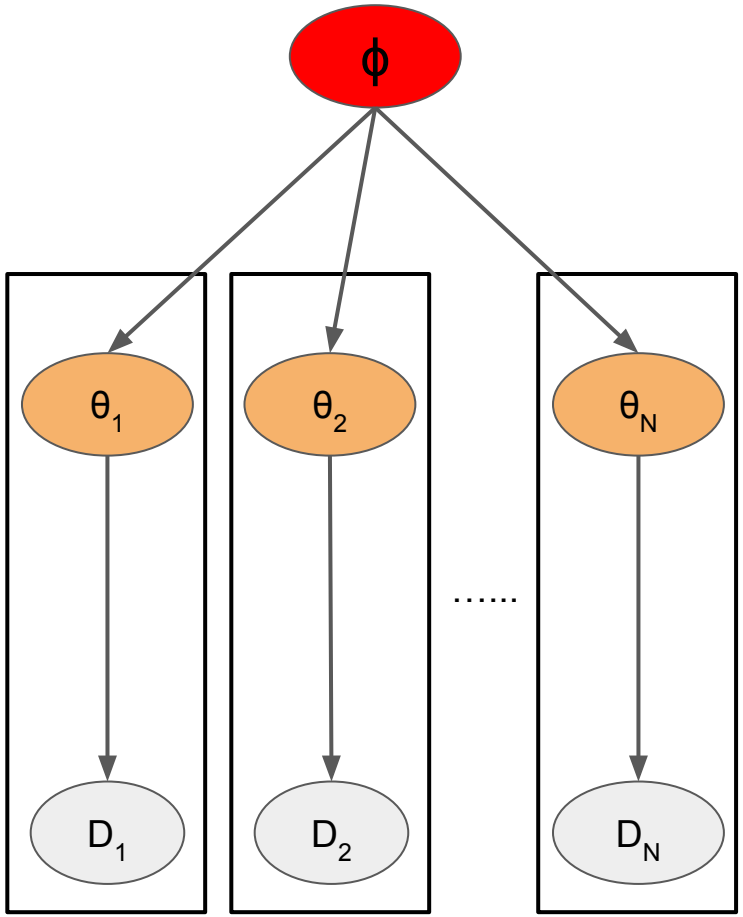
HBM: individuals and populations

But, what if the distribution or probability of any θ depends on properties of the population (e.g. the mass function)? This means that the prior probability on all the individuals $p(\{\theta_i\})$ depends on some other, **population** parameters ϕ. So now we have to write

$$p(\{\theta_i\},\{\mathrm{D}_i\},\phi) = p(\{\mathrm{D}_i\})\,p(\{\theta_i\},\phi\,|\,\{\mathrm{D}_i\})$$
$$= p(\phi)\,p(\{\theta_i\}\,|\,\phi)\,p(\{\mathrm{D}_i\}\,|\,\{\theta_i\},\phi)$$

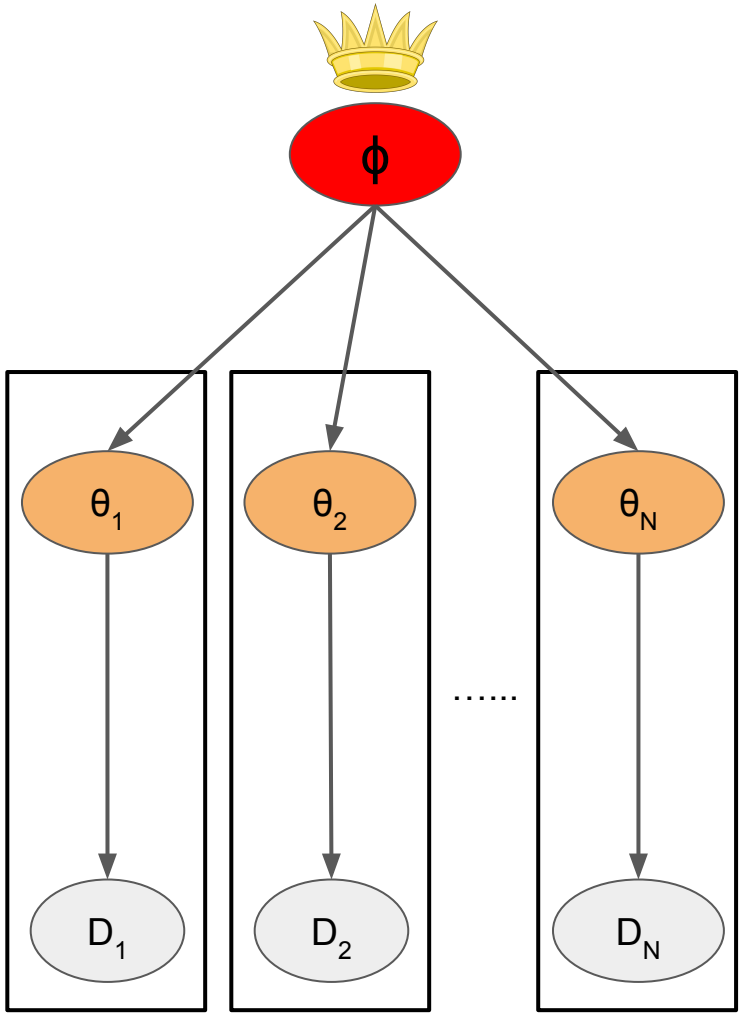
And iff the θ_i are conditionally independent, we can write this as

$$p(\{\mathrm{D}_i\})\,p(\{\theta_i\},\phi\,|\,\{\mathrm{D}_i\}) = p(\phi)\,p(\{\theta_i\}\,|\,\phi)\,p(\{\mathrm{D}_i\}\,|\,\{\theta_i\},\phi)$$
$$= p(\phi)\prod_{i=1}^{N} p(\theta_i\,|\,\phi)\,p(D_i|\theta_i,\phi)$$



HBM: individuals and populations

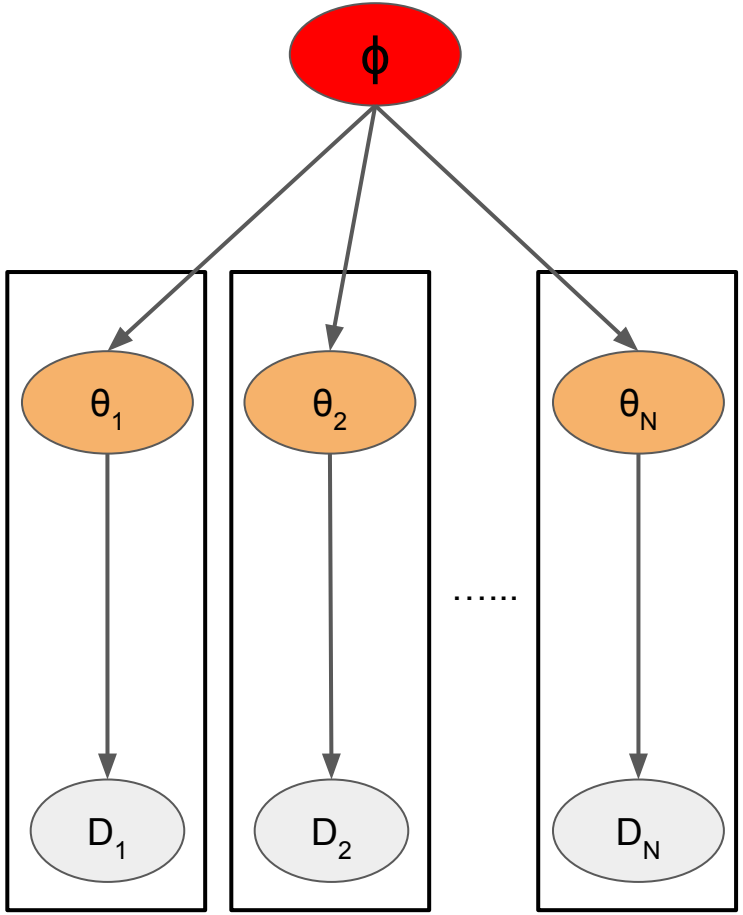
Now we have a ***hierarchy*** of parameters



HBM: individuals and populations

$$p(\{\theta_i\}, \phi \,|\, \{D_i\}) = p(\phi) \prod_{i=1}^{N} p(\theta_i \,|\, \phi)\, p(D_i \,|\, \theta_i)$$

$$p(\Theta, \phi \,|\, \mathbf{D}) \propto p(\phi)\, p(\Theta \,|\, \phi)\, p(\mathbf{D} \,|\, \Theta, \phi)$$

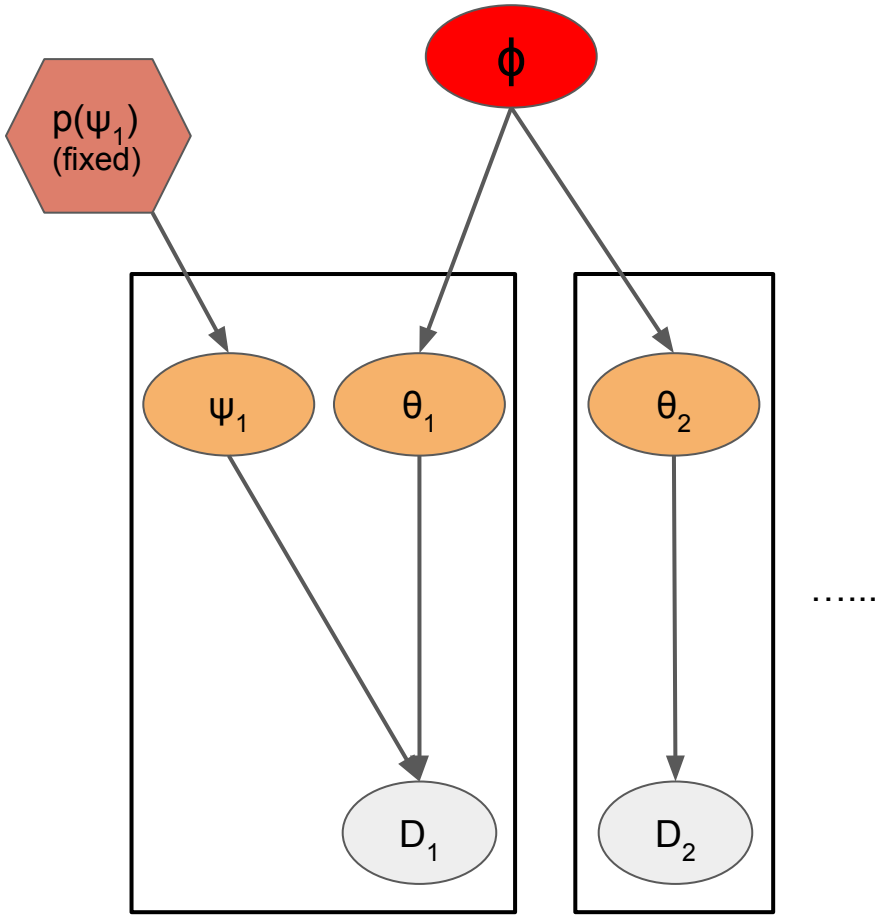


HBM: individuals and populations

Note that each individual may have additional parameters (ψ) that are ***not*** informed by the population.

These are effectively ***marginalized*** over when inferring ϕ

$$p(D_i \mid \theta_i) = \int d\psi_i \, p(D_i \mid \theta_i, \psi_i) \, p(\psi_i)$$



HBM: individuals and populations

Since we have many individuals, we should be able to infer the value of ϕ (marginalizing over all the individual parameters)

$$p(\phi \mid \mathbf{D}) \propto p(\phi) \int d\Theta \, p(\Theta \mid \phi) \, p(\mathbf{D} \mid \Theta, \phi)$$

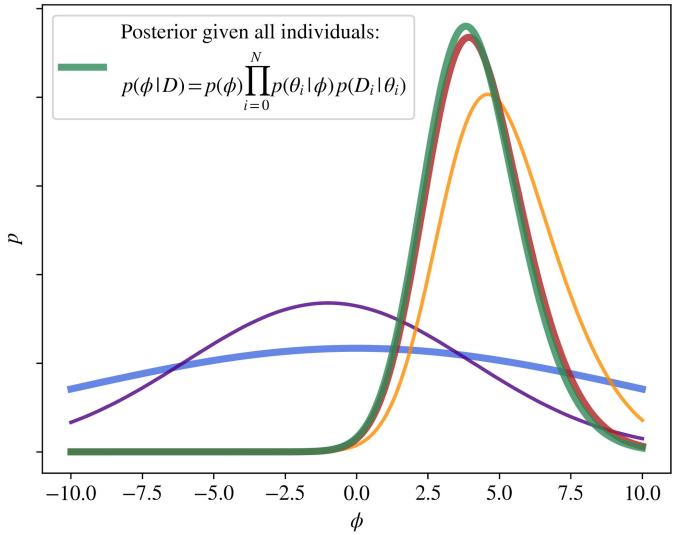
This is called ***pooling*** (since we are pooling estimates from individuals together)

HBM: individuals and populations

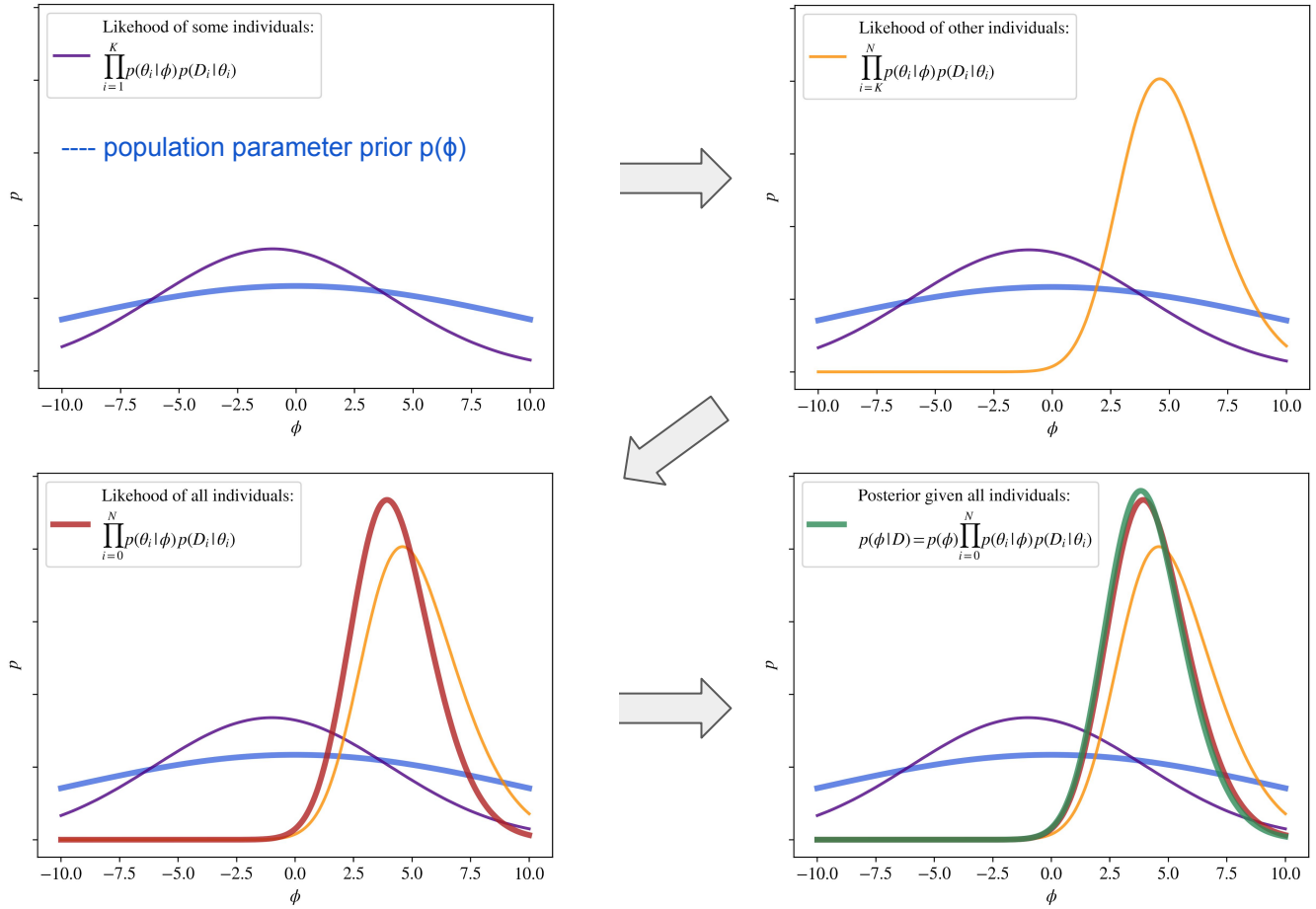
Since we have many individuals, we should be able to infer the value of ϕ (marginalizing over all the individual parameters)

This is called ***pooling*** (since we are pooling estimates from individuals together)

The constraints on ϕ will come from the most *informative* individuals, but are applied to *all* individuals



HBM: individuals and populations ***pooling***



HBM: individuals and populations

Since we can infer and constrain ϕ using the pooled estimate, the priors for each θ_i are better known (often narrower) and we get better inference of θ_i than if we treated the objects separately.

This effect of pooling is called ***shrinkage*** since the individual estimates will 'shrink' towards values predicted by the rest of the population.

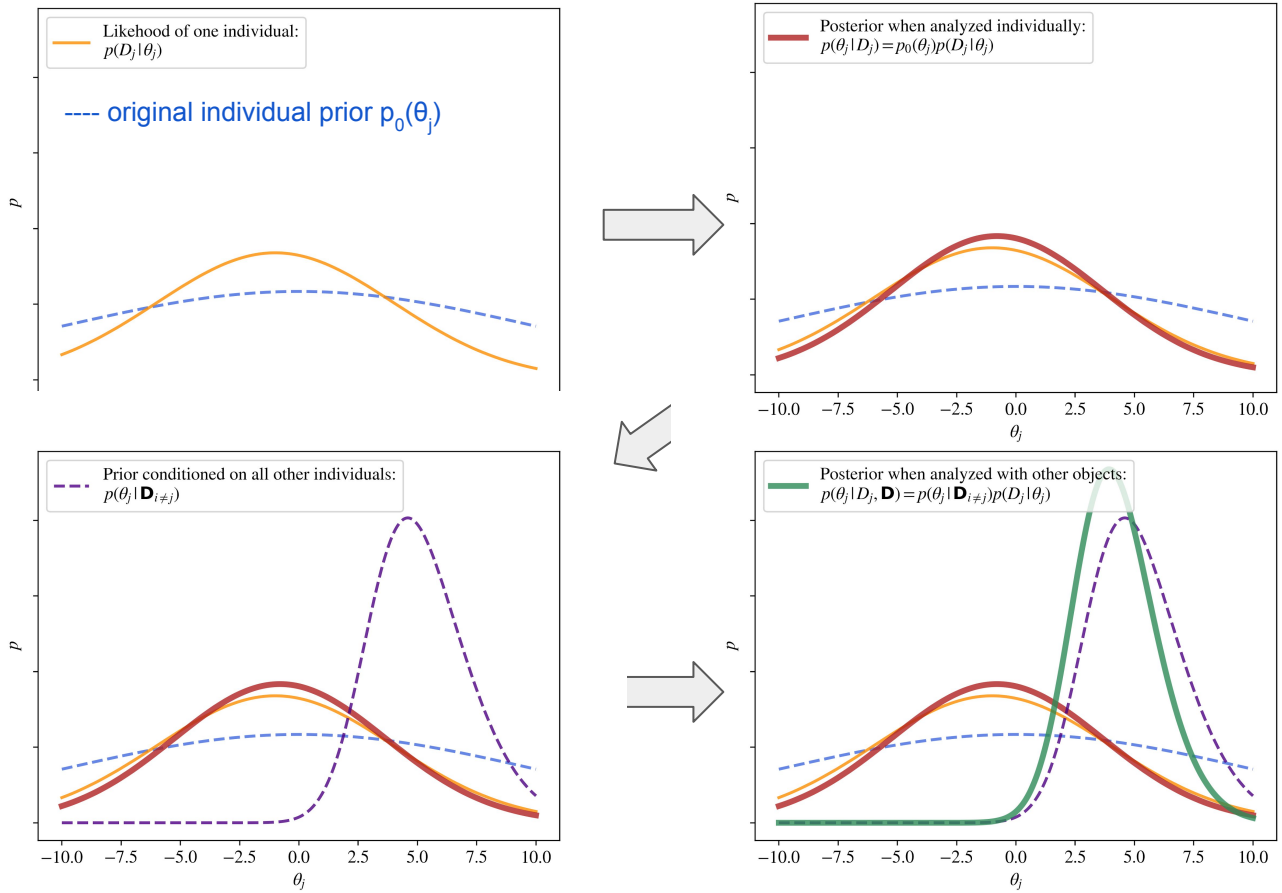
<u>This is a key advantage of HBM!</u>

Expressed mathematically as marginalization over all *other* individual level parameters; looks complex but often easy to do in practice with MCMC samples

$$\underbrace{p(\theta_j \mid D_j, \mathbf{D}_{i\neq j})}_{\text{posterior}} \propto \int d\phi \underbrace{p(D_j \mid \theta_j)}_{\text{likelihood of jth data}} p(\theta_j \mid \phi) \underbrace{p(\phi \mid \mathbf{D}_{i\neq j})}_{\text{posterior for } \phi \text{ given all } \textit{other} \text{ data}}$$

HBM: individuals and populations

shrinkage



Break & Questions

HBM: Examples

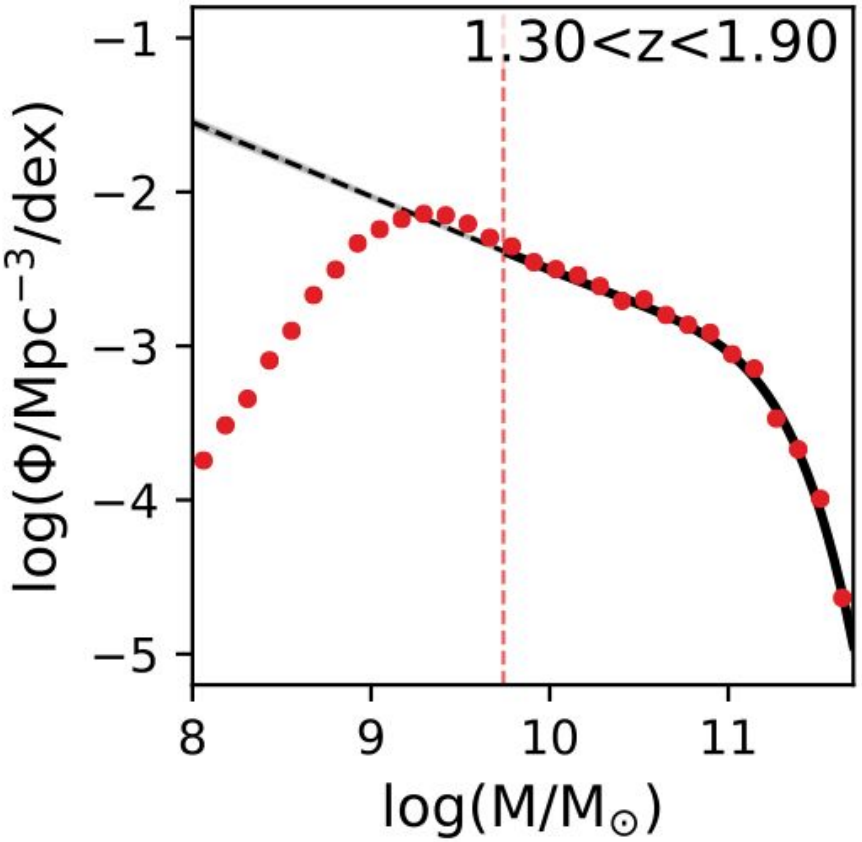
1) Galaxy Stellar Mass Function (Leja, Speagle, BDJ et al. 2020)

2) Star cluster properties

HBM: Galaxy stellar mass function

A very common use for HBM is to determine the prior probability of individual properties from the distribution of observed examples.

The galaxy stellar mass function (GSMF) is a key constraint on theories of galaxy evolution, describing the ***distribution*** of galaxy stellar masses.



Leja, Speagle, BDJ et al., 2020

HBM: Galaxy stellar mass function

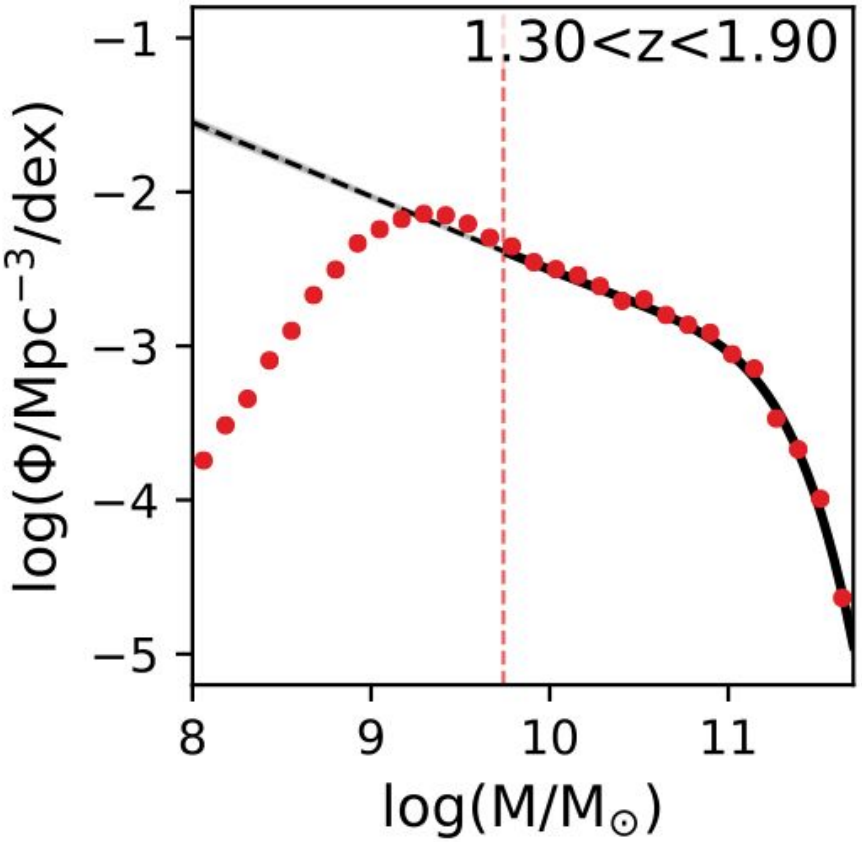
The galaxy stellar mass function (GSMF) is a key constraint on theories of galaxy evolution, describing the ***distribution*** of galaxy stellar masses.

It is often described by a 3-parameter function, the Schecter function:

$$\Phi(M) = \rho_* \left(M/M_*\right)^{\alpha} e^{-M}$$

$$M_* : \text{knee}$$
$$\alpha : \text{faint end slope}$$
$$\rho_* : \text{normalization}$$

The parameters of this function may evolve with redshift as galaxies grow and merge
(see Leja et al. 2020 for details)
but for simplicity we consider a single redshift.



Leja, Speagle, BDJ et al., 2020

HBM: Galaxy stellar mass function

<u>Observables</u>
Photometry of each galaxy, grism redshifts:
D_i = {magnitudes}, redshift

<u>Individual</u>
For each galaxy:
θ_i = (Mass, redshift, metallicity, A_V, SFH, etc..)

<u>Population</u>
For all the galaxies:

$\phi = (\alpha, M_*)$

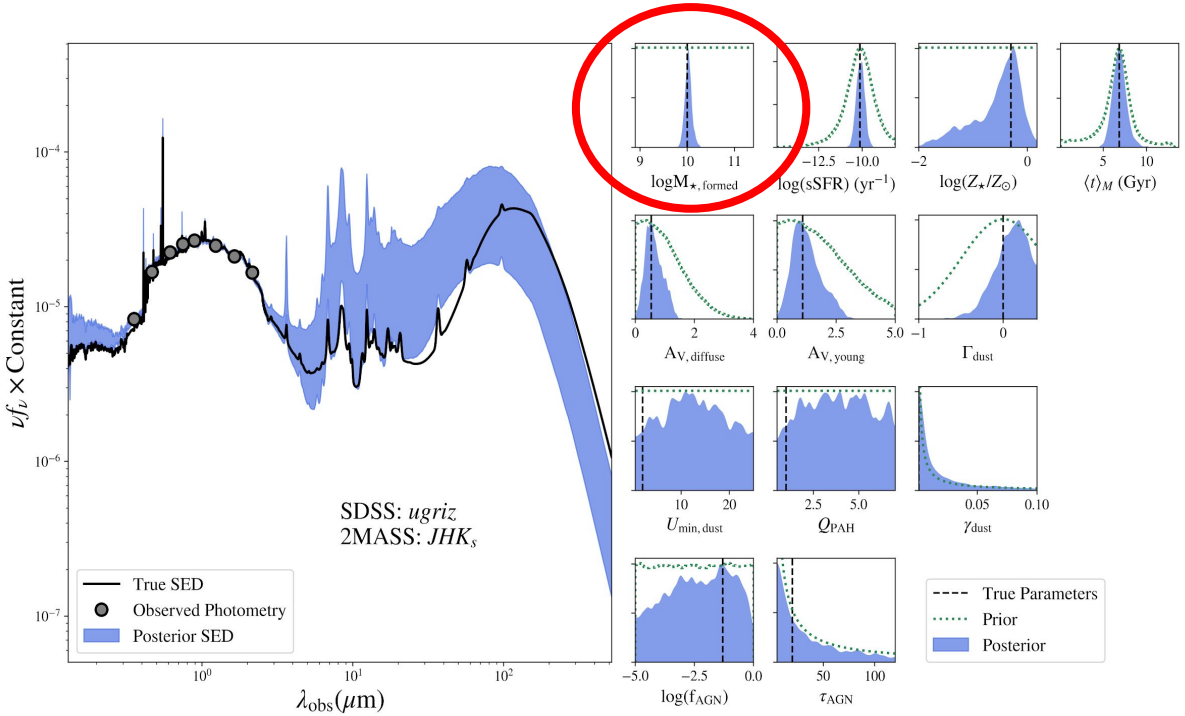
We neglect ρ_* for the moment

HBM: Galaxy stellar mass function

We can use stellar population synthesis modeling to infer the properties of individual galaxies from observed magnitudes.

In this case we have an very weakly informative prior.

$$p(\log M_i) = \text{constant}$$



Johnson et al 2020

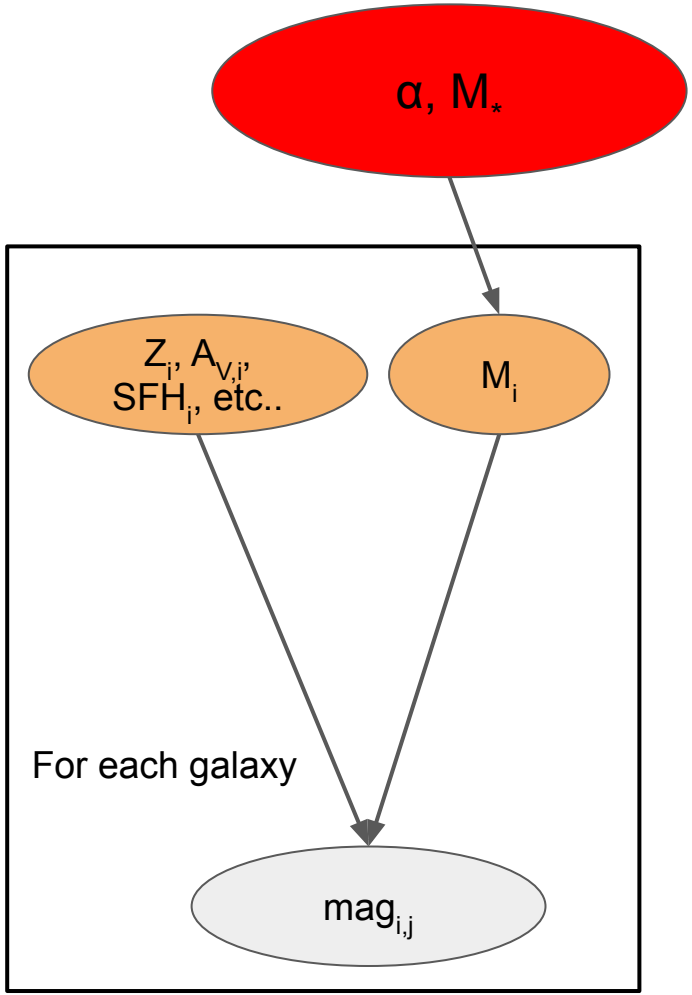
HBM: Galaxy stellar mass function

But we can combine all the galaxies to obtain a constraint on the GSMF parameters, which forms a more *realistic* prior.

$$p(M_i \mid \phi) = p(M_i \mid \alpha, M_*)$$
$$= \frac{(M/M_*)^{\alpha}\, e^{-M/M_*}}{\Gamma(\alpha+1, M_c/M_*)}$$

This is just the Schecter function normalized to 1 by the term in the denominator. M_c is the completeness limit.

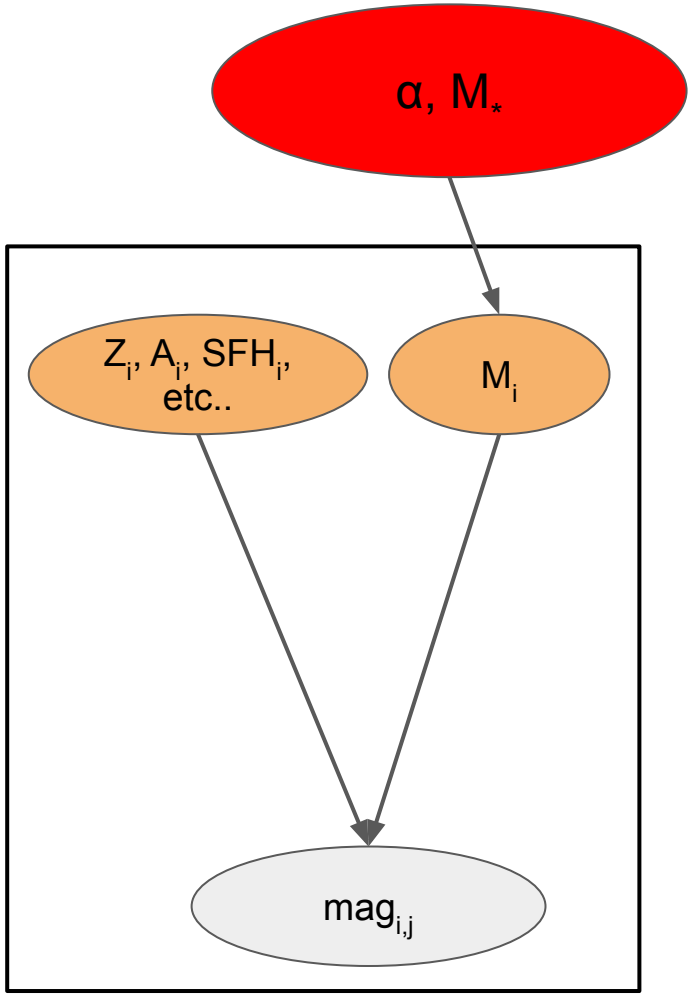
NOTE: If you are interested in the normalization parameter (ρ_*) then you need another term in the probability related to the expected total number of objects: see Leja+20 or Foreman-Mackey+14 for details



HBM: Galaxy stellar mass function

But we can combine all the galaxies to obtain a constraint on the GSMF parameters, which forms a more accurate prior.

$$p(\{M_i\}, \alpha, M_* \,|\, \mathbf{D}) \propto p(\alpha, M_*) \prod_{i=1}^{N} \frac{(M_i/M_*)^{\alpha}\, e^{-M_i/M_*}}{\Gamma(\alpha+1, M_c/M_*)}\, p(D_i \,|\, M_i)$$



HBM: Galaxy stellar mass function

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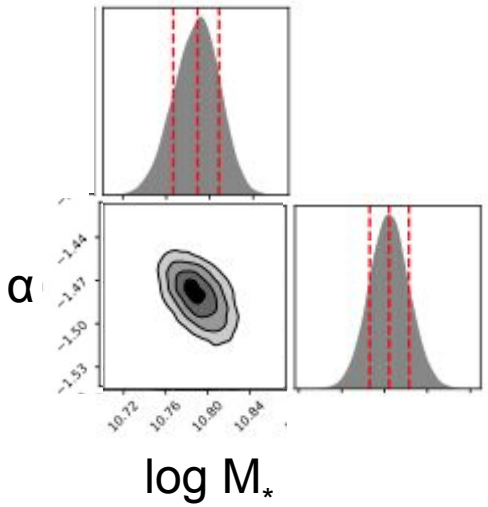
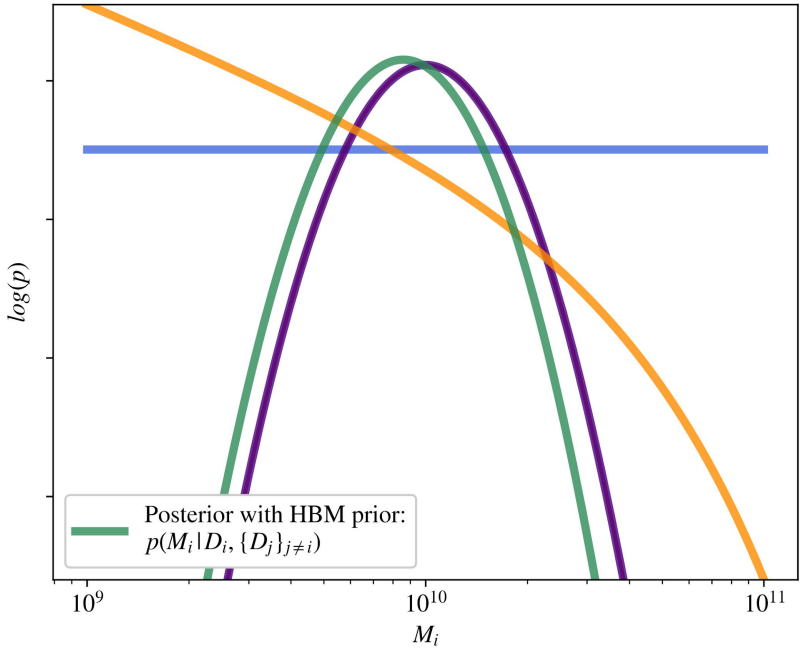


HBM: Galaxy stellar mass function

This changes (improves) the prior and posterior for the mass of individual galaxies - ***shrinkage***

Though in this case the posterior does not get much narrower, it based on a more realistic prior.



---- original uninformative prior

HBM: Galaxy stellar mass function

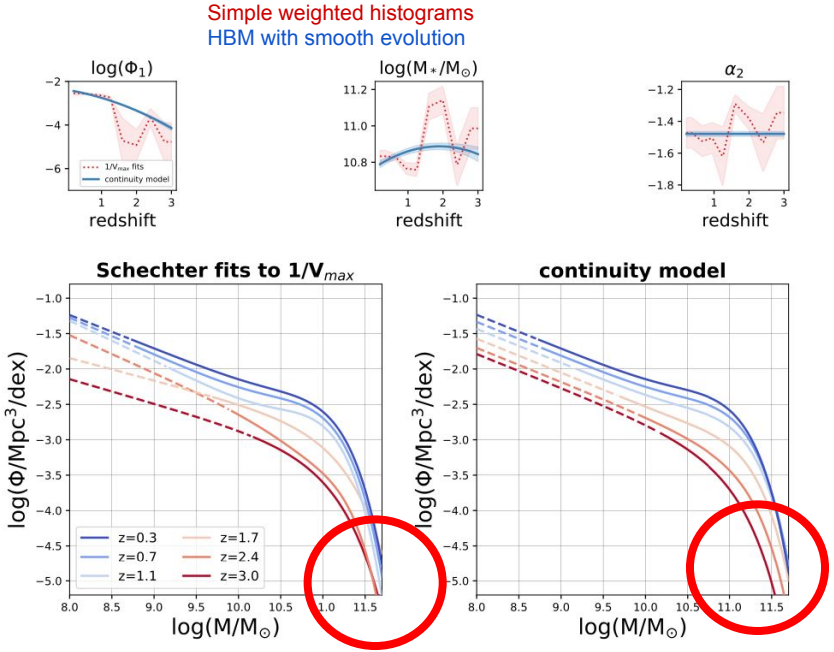
Other benefits:

1) When considering many redshifts, you can impose a smoothness constraint on the parameter evolution. E.g.

$$M_*(z) = M_{*,0} + M_{*,1} \times z$$

Then the GSMF at any redshift incorporates information from galaxies at all redshifts, and avoids binning effects.

2) Accounts for measurement uncertainties and uncertainties on individual galaxy masses



Leja, Speagle, BDJ et al., 2020

HBM: Galaxy stellar mass function

Implementation Note:

This now involves sampling for 2 + $N_{galaxies} \times (N_{\psi} + 1)$ parameters!!!

In practice we performed inference to obtain samples of individual posteriors based on the initial uninformative prior $p_0(M_i)$, and then ***re-weighted*** each sample *k* to get the HBM weighted likelihood.

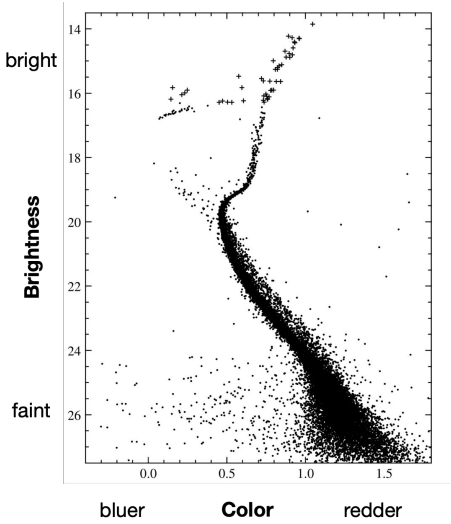
This requires having ‘enough’ samples near the bulk of the new likelihood (and knowing the original prior distribution)

$$w_{i,k} = \frac{p(M_{i,k} \mid \alpha, M_*)}{p_0(M_{i,k})}$$

HBM: Star clusters

Star clusters are gravitationally bound groups of stars, **co-located** and assumed to be **coeval** and born with the **same metallicity** and elemental abundances.

However, each star has a different initial mass -- distributed according to the **IMF** -- and is therefore in a different stage of stellar evolution at the present time, and has different physical parameters (luminosity, effective temperature, radius) and therefore different values of the observables.



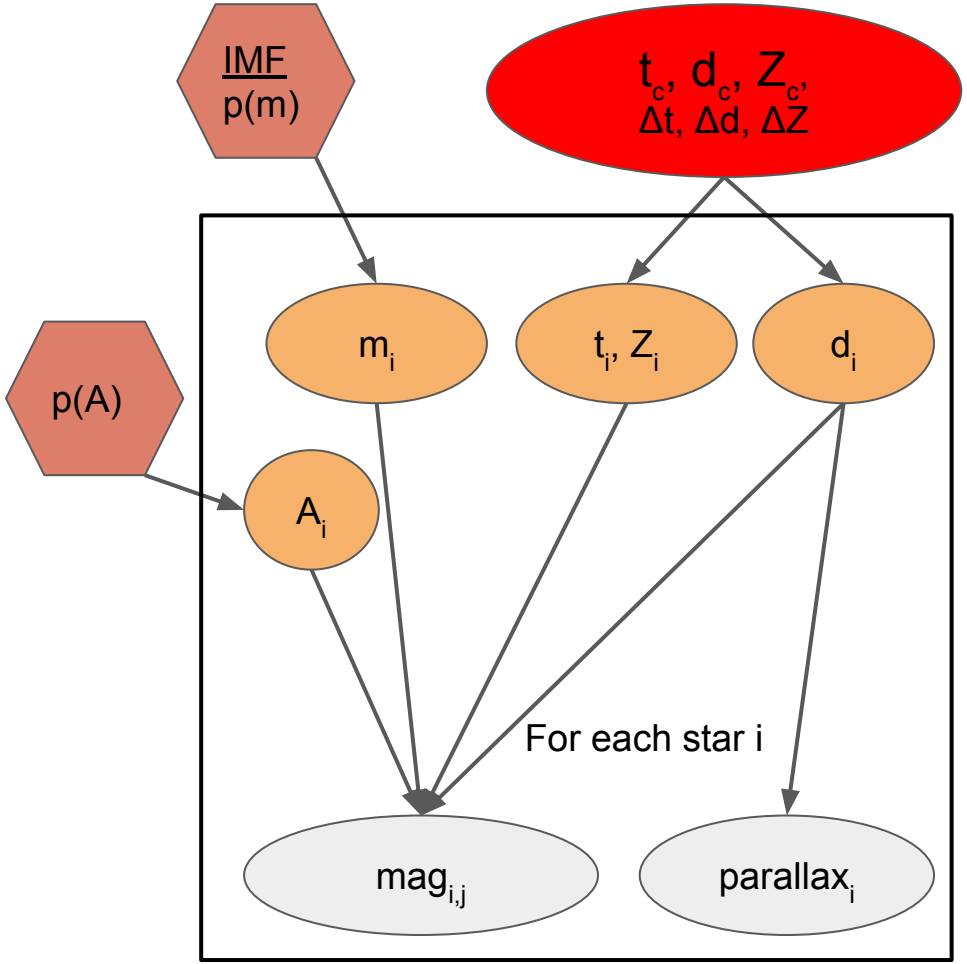
HBM: Star clusters

This naturally lends itself to a hierarchical model where the individuals are the stars

with **observables**
D_i = magnitude, colors [, parallax]

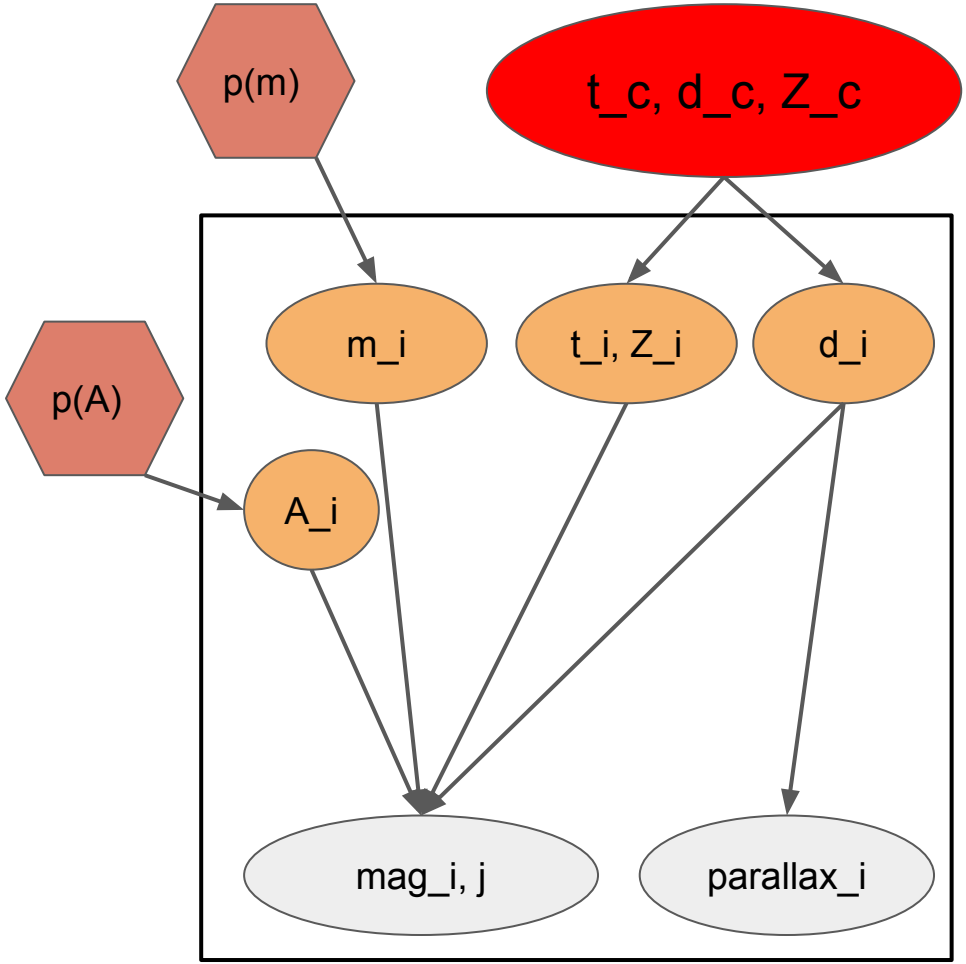
and fundamental **individual parameters**
θ_i = (distance, age, metallicity, mass, A_V)

The **population parameters** are the cluster parameters:
ϕ = (cluster age, cluster distance, cluster metallicity, widths of each distribution)



HBM: Star clusters

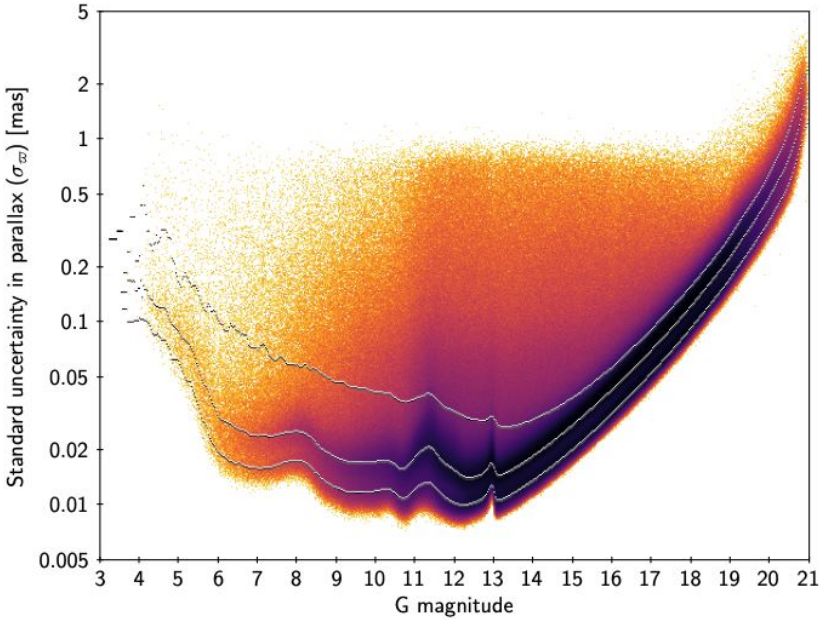
Equation here



HBM: Star clusters

Pooling information:

The brightest stars will have the best parallax and be most informative about ***distance***



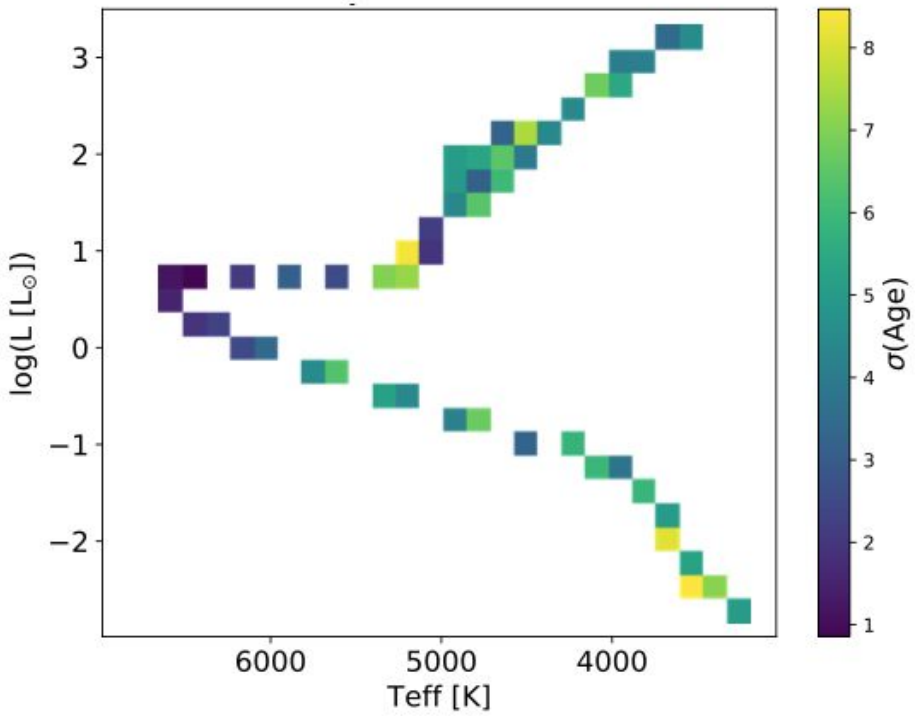
Lindegren et al. 2020

HBM: Star clusters

Pooling information:

The brightest stars will have the best parallax and be most informative about ***distance***

The bluest MS stars are very informative about ***age*** (given a distance)



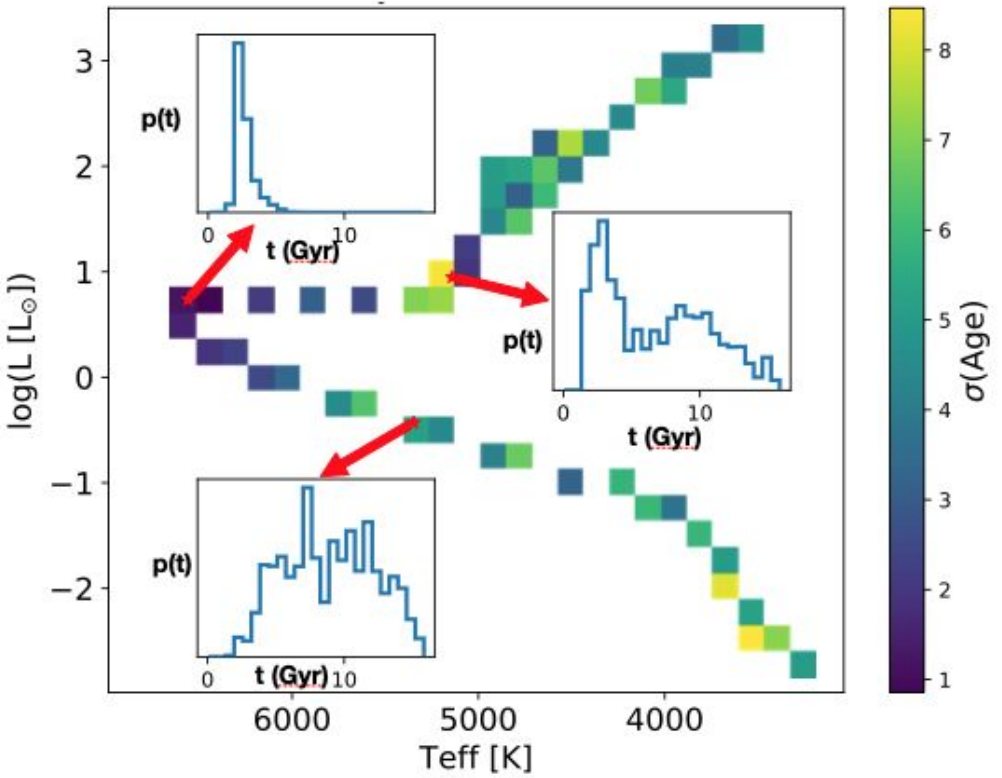
see e.g Cargile, Conroy, BDJ et al. 2020

HBM: Star clusters

Pooling information:

The brightest stars will have the best parallax and be most informative about ***distance***

The bluest MS stars are very informative about ***age*** (given a distance)



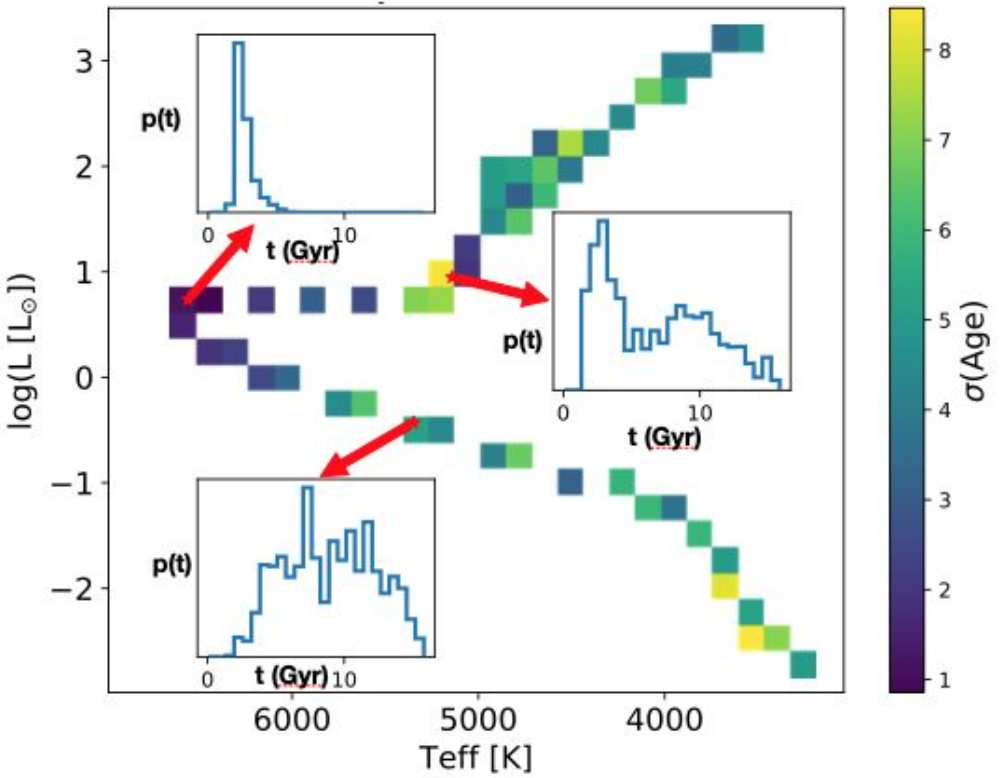
HBM: Star clusters

Pooling information:

The brightest stars will have the best parallax and be most informative about ***distance***

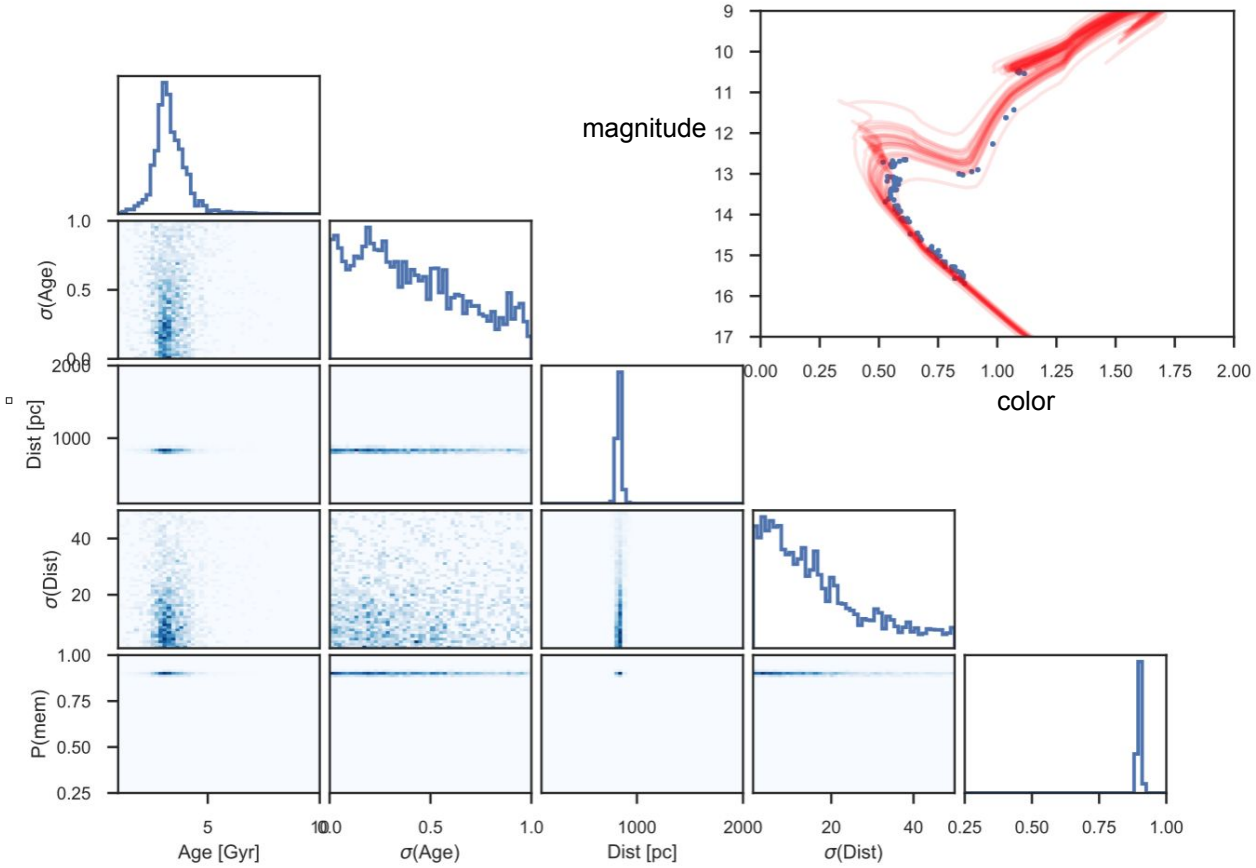
The bluest MS stars are very informative about ***age*** (given a distance)

All stars are informative about ***metallicity*** (given age and distance)



HBM: Star clusters

Results (posterior density) for M67 using Gaia DR1 data



HBM: Star clusters

PLOT HERE showing shrinkage of mass or FeH estimates

HBM: Practical advice

Sampling can be challenging, since there are many parameters (population parameters plus all individual parameters.)

Clever strategies often required.

Pseudo-importance sampling (e.g. Hogg et al. 2010, Foreman-Mackey et al. 2014)

- re-weight existing samples based on new priors, requires samples to exist where necessary and knowledge of priors

Gibbs sampling (e.g. Mandel 2011, Gelman et al.)

- MCMC sample parameters of individuals conditional on population parameters, then sample population parameters conditional on individuals, repeat ad-infinitum
- Exploits conditional independence of the graphs

Hamiltonian Monte Carlo (R. Neal 2012 https://arxiv.org/abs/1206.1901 , M. Betancourt 2017)

- use gradients to enable sampling in large numbers of parameters (individual & population) simultaneously

HBM: Practical advice

- Carefully consider dependencies in your model.
- Beware selection effects; these can be included in the model but may be complex.
- Usually good to try on a mock population first!

HBM: Practical advice

General Resources:

Bayesian Data Analysis (textbook)
Gelman, et al.
(advanced & detailed)

Krushke & Vanpaemel (2015)
https://jkkweb.sitehost.iu.edu/articles/KruschkeVanpaemel2015.pdf

Loredo 2013
https://arxiv.org/abs/1208.3036

Loredo & Hendry 2019 (excellent!)
https://arxiv.org/abs/1911.12337

Some notable examples in Astronomy Literature:

von Hippel, T. et al. 2006 (star clusters)
https://arxiv.org/abs/astro-ph/0603493

van Dyk et al 2009 (star clusters)
https://arxiv.org/abs/0905.2547

Andreon & Hurn 2010
(galaxy cluster scaling relations)
https://arxiv.org/abs/1001.4639

Foreman-Mackey et al. 2014 (exoplanets)
https://arxiv.org/abs/1406.3020

Wolfgang et al 2016 (exoplanet scaling)
https://arxiv.org/abs/1504.07557

Mandel et al 2011 (supernovae)
https://arxiv.org/abs/1011.5910