1 Linear: $at + b$

$$
\begin{aligned}
L(a,b) &= \sum_{i=1}^{k} log(at_i + b) - \int_{t_1}^{t_k} (at + b)dt \\
&= \sum_{i=1}^{k} log(at_i + b) - \frac{a}{2}(t_k^2 - t_1^2) - b(t_k - t_1) \\
&\text{let M} = t_k - t_1 \text{ and P} = t_k + t_1 \text{ then} \\
&= \sum_{i=1}^{k} log(at_i + b) - \frac{a}{2}MP - bM \\
\frac{\partial L(a,b)}{\partial b} &= \sum_{i=1}^{k} \frac{1}{at_i + b} - M = 0 \\
\implies M &= \sum_{i=1}^{k} \frac{1}{at_i + b} \\
\frac{\partial L(a,b)}{\partial a} &= \sum_{i=1}^{k} \frac{t_i}{at_i + b} - \frac{MP}{2} = 0
\end{aligned}
$$

Use R to compute a and b.

2 Exponential: e^{at+b}

$$
\begin{aligned}
L(a,b) &= \sum_{i=1}^{k} log(e^{at_i+b}) - \int_{t_1}^{t_k} e^{at+b}dt \\
&= \sum_{i=1}^{k} (at_i+b) - \frac{1}{a}\left[e^{a(t_k-t_1)+b}\right] \\
& \text{let M} = t_k - t_1, \text{ then} \\
&= \sum_{i=1}^{k} (at_i+b) - \frac{1}{a}\left(e^{aM+b}\right) \\
\frac{\partial L(a,b)}{\partial b} &= k - \frac{1}{a}\left(e^{aM+b}\right) = 0 \\
\implies k &= \frac{1}{a}\left(e^{aM+b}\right) \quad and\ b = ln(ka) - aM \\
\frac{\partial L(a,b)}{\partial a} &= \sum_{i=1}^{k} t_i + \frac{1}{a^2}\left(e^{aM+b}\right) - \frac{M}{a}\left(e^{aM+b}\right) = 0
\end{aligned}
$$

$$substitute\ \frac{1}{a}\left(e^{aM+b}\right) = k, then\ we\ have$$

$$\sum_{i=1}^{k} t_i + \frac{k}{a} - Mk = 0$$

solve for a then substitute into b, we will have

$$a = \frac{k}{Mk - \sum t_i}$$

$$b = ln\left(\frac{k^2}{Mk - \sum t_i}\right) - \frac{Mk}{Mk - \sum t_i}$$

3 Power: at^b

$$
\begin{aligned}
L(a,b) &= \sum_{i=1}^{k} log(at_i^b) - \int_{t_1}^{t_k} at_i^b dt \\
&= kloga + \sum_{i=1}^{k} logt_i^b - \frac{a}{b+1} M^{b+1}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial L(a,b)}{\partial a} &= \frac{k}{a} - \frac{M^{b+1}}{b+1} = 0 \\
\implies \frac{k}{a} &= \frac{M^{b+1}}{b+1} \quad and\ a = \frac{k(b+1)}{M^{b+1}} \\
\frac{\partial L(a,b)}{\partial b} &= log(\prod_{i=1}^{k} t_i) + \frac{aM^{b+1}}{b+1}\left(\frac{1}{b+1} - logM\right) = 0 \\
substitute\ \frac{M^{b+1}}{b+1} &= \frac{k}{a}, then\ we\ have \\
&\frac{\partial L(a,b)}{\partial b} = log(\prod_{i=1}^{k} t_i) + k\left(\frac{1}{b+1} - logM\right) = 0 \\
&\text{solve for b then substitute into a, we will have} \\
&b = \frac{k}{klogM - log(\prod t_i)} - 1 \\
&a = \frac{k^2}{M^{klogM - log(\prod t_i)}[klogM - log(\prod t_i)]}
\end{aligned}
$$