Curve Fitting using Least Squares

- Given some sampled 2D points
- Find an optimal quadratic function using least squares

SID: 2017041930
NAME: 김선동

Problem: For the 2D points, find the 2^{nd} order curve using least squares

Quadratic fit: $y = ax^2 + bx + c$

<2D points>

(-2.9, 35.4)
(-2.1, 19.7)
(-0.9, 5.7)
(1.1, 2.1)
(0.1, 1.2)
(1.9, 8.7)
(3.1, 25.7)
(4.0, 41.5)

Those points should fit into above equation.
First, we should select 6 points.

SET 1	SET 2
(-2.9, 35.4)	(-2.9, 35.4)
(-2.1, 19.7)	(-2.1, 19.7)
(-0.9, 5.7)	(1.1, 2.1)
(1.1, 2.1)	(0.1, 1.2)
(0.1, 1.2)	(3.1, 25.7)
(1.9, 8.7)	(4.0, 41.5)

Normal Equation: $A^tA\mathbf{d} = A^t\mathbf{y}$

And then, we can make matrix A and vector **d for SET 1 & SET2** Quadratic fit: $y = ax^2 + bx + c$

SET 1

(-2.9, 35.4): $c - 2.9b + 8.41a$
(-2.1, 19.7): $c - 2.1b + 4.41a$
(-0.9, 5.7): $c - 0.9b + 0.81a$
(1.1, 2.1): $c + 1.1b + 1.21a$
(0.1, 1.2): $c + 0.1b + 0.01a$
(1.9, 8.7): $c + 1.9b + 3.61a$

$$A = \begin{bmatrix} 1 & -2.9 & 8.41 \\ 1 & -2.1 & 4.41 \\ 1 & -0.9 & 0.81 \\ 1 & 1.1 & 1.21 \\ 1 & 0.1 & 0.01 \\ 1 & 1.9 & 3.61 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} 35.4 \\ 19.7 \\ 5.7 \\ 2.1 \\ 1.2 \\ 8.7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2.9 & 8.41 \\ 1 & -2.1 & 4.41 \\ 1 & -0.9 & 0.81 \\ 1 & 1.1 & 1.21 \\ 1 & 0.1 & 0.01 \\ 1 & 1.9 & 3.61 \end{bmatrix} \begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{bmatrix} 35.4 \\ 19.7 \\ 5.7 \\ 2.1 \\ 1.2 \\ 8.7 \end{bmatrix}$$

Normal Equation: $A^TA\mathbf{d} = A^T\mathbf{y}$

And then, we can make matrix A and vector **d for SET 1 & SET2** Quadratic fit: $y = ax^2 + bx + c$

SET 2

(-2.9, 35.4): c - 2.9b + 8.41a
(-2.1, 19.7): c - 2.1b + 4.41a
(1.1, 2.1): c + 1.1b + 1.21a
(0.1, 1.2): c + 0.1b + 0.01a
(3.1, 25.7): c + 3.1b + 9.61a
(4.0, 41.5): c + 4.0b + 16a

$$A = \begin{bmatrix} 1 & -2.9 & 8.41 \\ 1 & -2.1 & 4.41 \\ 1 & 1.1 & 1.21 \\ 1 & 0.1 & 0.01 \\ 1 & 3.1 & 9.61 \\ 1 & 4.0 & 16 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} 35.4 \\ 19.7 \\ 2.1 \\ 1.2 \\ 25.7 \\ 41.5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2.9 & 8.41 \\ 1 & -2.1 & 4.41 \\ 1 & 1.1 & 1.21 \\ 1 & 0.1 & 0.01 \\ 1 & 3.1 & 9.61 \\ 1 & 4.0 & 16 \end{bmatrix} \begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{bmatrix} 35.4 \\ 19.7 \\ 2.1 \\ 1.2 \\ 25.7 \\ 41.5 \end{bmatrix}$$

To get vector d, we should calculate A^TA and $A^T\mathbf{y}$ first.

SET 1.

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ -2.9 & -2.1 & -0.9 & 1.1 & 0.1 & 1.9 \\ 8.41 & 4.41 & 0.81 & 1.21 & 0.01 & 3.61 \end{bmatrix} \begin{bmatrix} 1 & -2.9 & 8.41 \\ 1 & -2.1 & 4.41 \\ 1 & -0.9 & 0.81 \\ 1 & 1.1 & 1.21 \\ 1 & 0.1 & 0.01 \\ 1 & 1.9 & 3.61 \end{bmatrix} = \begin{bmatrix} 6 & -2.8 & 18.46 \\ -2.8 & 18.46 & -26.188 \\ 18.46 & -26.188 & 51.3286 \end{bmatrix}$$

$$A^T\mathbf{y} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ -2.9 & -2.1 & -0.9 & 1.1 & 0.1 & 1.9 \\ 8.41 & 4.41 & 0.81 & 1.21 & 0.01 & 3.61 \end{bmatrix} \begin{bmatrix} 35.4 \\ 19.7 \\ 5.7 \\ 2.1 \\ 1.2 \\ 8.7 \end{bmatrix} = \begin{bmatrix} 72.8 \\ -130.2 \\ 423.168 \end{bmatrix}$$

Normal Equation: $A^TA\mathbf{d} = A^T\mathbf{y}$

$$\begin{bmatrix} 6 & -2.8 & 18.46 \\ -2.8 & 18.46 & -26.188 \\ 18.46 & -26.188 & 51.3286 \end{bmatrix} \mathbf{d} = \begin{bmatrix} 72.8 \\ -130.2 \\ 423.168 \end{bmatrix}$$

$$\mathbf{d} = (A^TA)^{-1}A^Ty$$

We get Inverse of (A^TA) by Gauss-Jordan Elimination and then we can get vector **d**

$$\mathbf{d} = \begin{bmatrix} 12469081/13092160 \\ -365571/163652 \\ 2157115/654608 \end{bmatrix} = \begin{bmatrix} 0.95240\ldots \\ -2.23383\ldots \\ 3.29527\ldots \end{bmatrix}$$

To get vector d, we should calculate A^TA and $A^T\mathbf{y}$ first.

SET 2.

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ -2.9 & -2.1 & 1.1 & 0.1 & 3.1 & 4.0 \\ 8.41 & 4.41 & 1.21 & 0.01 & 9.61 & 16 \end{bmatrix} \begin{bmatrix} 1 & -2.9 & 8.41 \\ 1 & -2.1 & 4.41 \\ 1 & 1.1 & 1.21 \\ 1 & 0.1 & 0.01 \\ 1 & 3.1 & 9.61 \\ 1 & 4.0 & 16 \end{bmatrix} = \begin{bmatrix} 6 & 3.3 & 39.65 \\ 3.3 & 39.65 & 61.473 \\ 39.65 & 61.473 & 439.9925 \end{bmatrix}$$

$$A^T\mathbf{y} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ -2.9 & -2.1 & 1.1 & 0.1 & 3.1 & 4.0 \\ 8.41 & 4.41 & 1.21 & 0.01 & 9.61 & 16 \end{bmatrix} \begin{bmatrix} 35.4 \\ 19.7 \\ 2.1 \\ 1.2 \\ 25.7 \\ 41.5 \end{bmatrix} = \begin{bmatrix} 125.6 \\ 104.07 \\ 1298.121 \end{bmatrix}$$

Normal Equation: $A^TA\mathbf{d} = A^T\mathbf{y}$

$$\begin{bmatrix} 6 & 3.3 & 39.65 \\ 3.3 & 39.65 & 61.473 \\ 39.65 & 61.473 & 439.9925 \end{bmatrix} \mathbf{d} = \begin{bmatrix} 125.6 \\ 104.07 \\ 1298.121 \end{bmatrix}$$

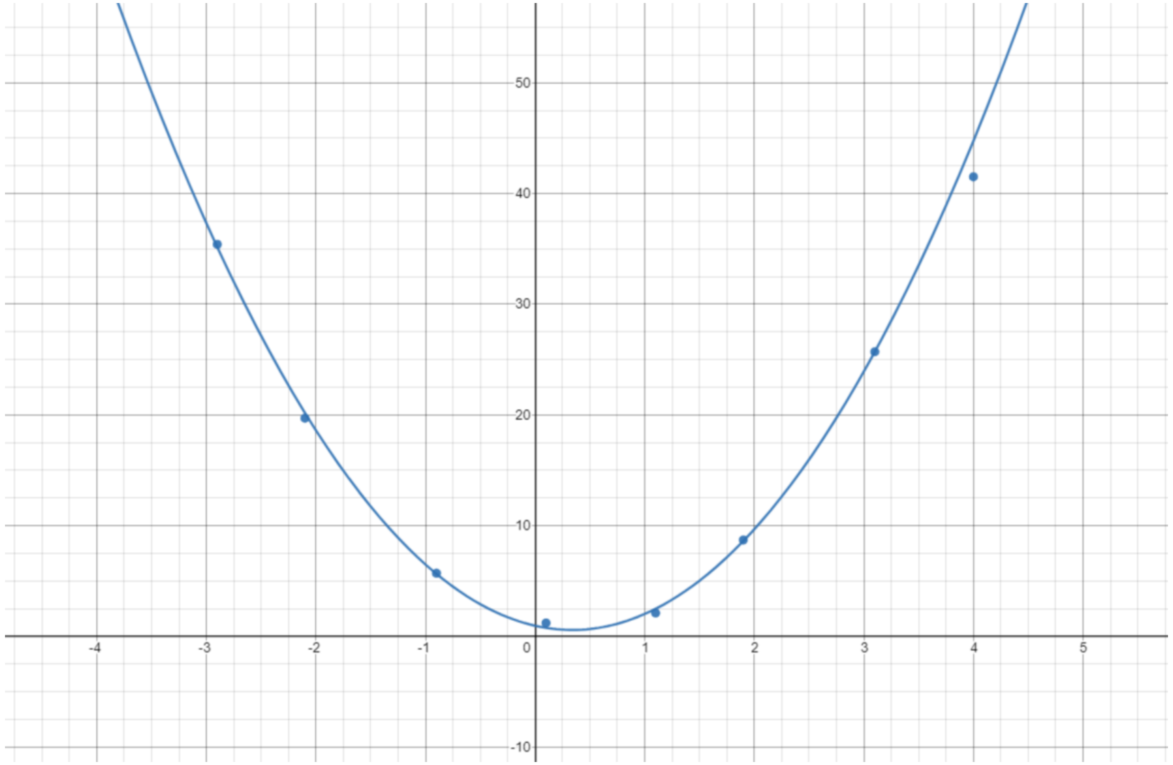
$$\mathbf{d} = (A^TA)^{-1}A^Ty$$

We get Inverse of (A^TA) by Gauss-Jordan Elimination and then we can get vector **d**

$$\mathbf{d} = \begin{bmatrix} 743447413/552880916 \\ -107936553/45133136 \\ 3497830705/1105761832 \end{bmatrix} = \begin{bmatrix} 1.34467... \\ -2.39151... \\ 3.16327... \end{bmatrix}$$

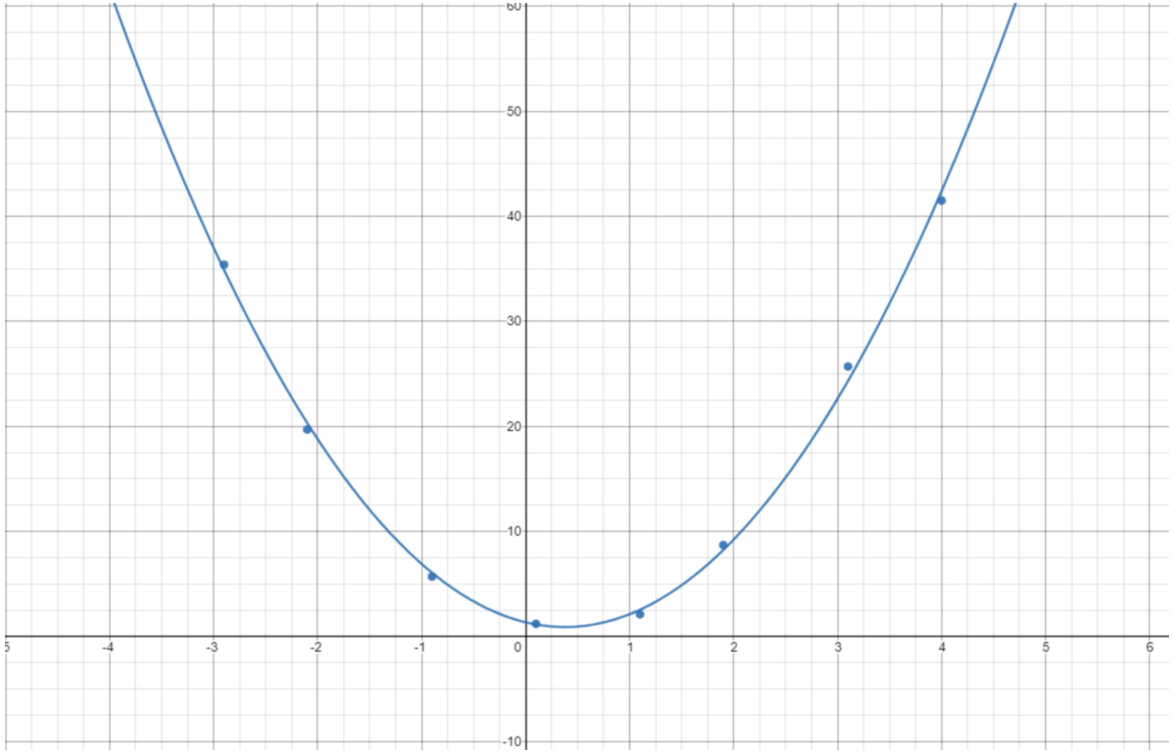
Set 1.

Get coefficients from vector D.
The Answer is…!!!
y = $\mathbf{3.29527}x^2$ $\mathbf{-2.23383}x$ + **0.95240**



Set 2.

Get coefficients from vector D.
The Answer is…!!!
y = $\mathbf{3.16327}x^2$ $\mathbf{-2.39151}x$ + **1.34467**



They almost fit to the data..!!

Analysis.

- 우리가 배웠던 Projection으로 미지수(N)보다 방정식(M)이 많은 over-constrained한 system에서 일반화된 해를 찾았다. 이는, 갖고 있는 데이터들을 기반으로 해당 system의 표현식을 구하는 것이다.

- Least Square method는 주어진 데이터를 기반으로 이들 모두를 최대한 만족시키는 line or curve에 대한 식을 찾는 방법으로, 위와 같은 과정(ppt pg #2~6)들을 통해 방정식을 풀었다.

- 행렬 A의 column vector는 independent하며 column space의 basis이다.
  그러나 vector **d**는 이 column space에 존재하지 않기 때문에 Ax=b의 해는 존재하지 않는다. (Ax=b꼴에서)

- 정확한 해는 존재하지 않지만 해를 구하기 위해서는 **d**를 column space로 projection하면 된다.

- 결론적으로, 투영시킨 vector p는 A의 column space로 projection한 것이기에 해가 존재한다. AT를 양변에 곱하고, Gauss-Jordan Elimination으로 Inverse를 구하여 **d**를 구한 것이다.

- Pg# 6을 보면 set2로 만든 그래프가 원래 데이터에 더 fit한 것으로 보인다.