ECE/CS 584: Embedded and cyberphysical system verification	Fall 2019
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	Due October 3^{rd}

Typeset your solutions using LaTeX zip your writeup (.pdf) and code (.py) in a single file called `nedid-584-F19.zip` *and upload this file through Compass.*

Problem 1. Convexity of reach sets (20 points) (a) For a linear autonomous system $\dot{x} = Ax$, $x \in \mathbb{R}^n$, prove that if the set of initial states Θ is a polytope with a set of vertices $v_1, \ldots, v_k \in \mathbb{R}^n$, then the set of reachable states at any time $t > 0$ is also a convex set. That is, $Reach(\Theta, t)$ is the convex hull of the points reached from the vertices of Θ at time t.

(b) Give an example of a nonlinear system for which $Reach(\Theta, t)$ is not convex, even if Θ is convex. You can give an analytical proof or a numerical example. You may consider a discrete time model $x(t+1) = f(x(t))$ as well.

Solution

(a) $\forall x^1, x^2 \in Reach(\Theta, t)$, denote x_0^1, x_0^2 as the corresponding initial state, then we have $x^1 = e^{At}x_0^1, x^2 = e^{At}x_0^2$, and thus $\forall \lambda \in [0,1]$ we have $\lambda x^1 + (1-\lambda)x^2 = e^{At}(\lambda x_0^1 + (1-\lambda)x_0^2)$. Because Θ is convex and $x_0^1, x_0^2 \in \Theta$, we have $(\lambda x_0^1 + (1-\lambda)x_0^2) \in \Theta$, and thus $e^{At}(\lambda x_0^1 + (1-\lambda)x_0^2) \in Reach(\Theta, t)$. Because x^1, x^2 are chosen arbitraily, we proved that $Reach(\Theta, t)$ is convex.

(b) Consider a two-dimensional non-linear discrete time system.

$$x(t+1) = \begin{cases} 0, & \text{if } x(t)_1 > 0, x(t)_2 > 0 \\ x(t), & \text{otherwise} \end{cases}$$

As shown in Fig. 1 starting from a convex set Θ, $Reach(\Theta, t)$ is not convex.

Problem 2. (20 points) Show that a set of piece-wise continuous functions is closed under composition and linear combinations. That is, consider any two piece-wise continuous functions of time $f, g : \mathbb{R} \to \mathbb{R}^m$ with points of discontinuity $\mathbf{DC}_f$ and $\mathbf{DC}_g$. Show that

(a) $f \circ g$ is also piece-wise continuous.

(b) For any $a_1, a_2 > 0$, $a_1 f + a_2 g$ is also piece-wise continuous.

Solution

(a) $\forall x \notin \mathbf{DC}_f \cup \mathbf{DC}_g$, $h = a_1 f + a_2 g$ is continuos at x. $\forall x \in \mathbf{DC}_f \cup \mathbf{DC}_g$ and $x \notin \mathbf{DC}_f \cap \mathbf{DC}_g$, h is not continuos at x. $\forall x \in \mathbf{DC}_f \cap \mathbf{DC}_g$, h may be continuous or discontinuos at x. Therefore, $\mathbf{DC}_h \subset \mathbf{DC}_f \cup \mathbf{DC}_g$, which means h has a finite number of points of discontinuity and is also piece-wise continuos.

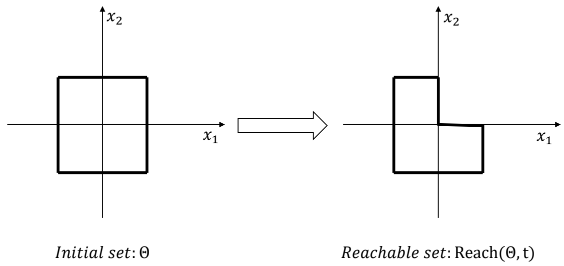

Figure 1: Starting from a convex set Θ, $Reach(\Theta, t)$ is not convex

Problem 3. Billiards model (20 points) (a) Consider an idealized billiard table of length a and width b. The table has no pockets; its surface has no friction; and its boundary bounces the balls perfectly. Write a hybrid automaton model of the position of two balls of equal mass on this table. The balls have some initial velocities. A ball bounces off a wall when its position is at the boundary. Balls collide whenever $|x_1 - x_2| \leq \varepsilon$ and $|y_1 - y_2| \leq \varepsilon$ and their velocity vectors are pointing towards each other. Here ε is some constant. Whenever a bounce occurs, the appropriate velocity changes sign. Whenever a collision occurs, the balls exchange their velocity vectors. Make all the variables internal. Wall bounces are modeled by an output action called bounce, and each collision is modeled by an output action collision.

(b) State and prove conservation of speed along each axis as an invariant property of the automaton.

Solution

(a)

automaton Billiards(a,b)
 type
 actions
  bounce; collision;
 variables
  v_{1x}:Real:=v^0_{1x}; v_{1y}:Real:=v^0_{1y}; v_{2x}:Real:=v^0_{2x}; v_{2y}:Real:v^0_{2y}; x_1:Real:=x^0_1; y_1:Real:=y^0_1; x_2:Real:x^0_2; y_2:Real:=y^0_2;
 transitions
  bounce
  pre $x_1 = 0 \vee x_1 = a$
  eff $v_{1x} := -v_{1x}$;
  bounce
  pre $x_2 = 0 \vee x_2 = a$

eff $v_{2x} := -v_{2x}$;
bounce
pre $y_1 = 0 \vee y_1 = b$
eff $v_{1y} := -v_{1y}$;
bounce
pre $y_2 = 0 \vee y_2 = b$
eff $v_{2y} := -v_{2y}$;
collision
pre $|x_1 - x_2| \leq \varepsilon \wedge |y_1 - y_2| \leq \varepsilon \wedge (v_{1x}/v_{2x} = v_{1y}/v_{2y} < 0)$
eff $v_{1x} := v_{2x}$; $v_{2x} := v_{1x}$; $v_{1y} := v_{2y}$; $v_{2y} := v_{1y}$;
trajectories
running
evolve $dx_1 = v_{1x}$; $dy_1 = v_{1y}$; $dx_2 = v_{2x}$; $dy_2 = v_{2y}$
invariant $(0 < x_1 < a) \wedge (0 < y_1 < b) \wedge (0 < x_2 < a) \wedge (0 < y_2 < b) \wedge (|x_1 - x_2| > \varepsilon \vee |y_1 - y_2| > \varepsilon \vee v_{1x}/v_{2x} \neq v_{1y}/v_{2y} \vee v_{1x}/v_{2x} \geq 0)$

(b)
$$I = [\![(|v_{1x}| + |v_{2x}| = |v_{1x}^0| + |v_{2x}^0|) \wedge (|v_{1y}| + |v_{2y}| = |v_{1y}^0| + |v_{2y}^0|)]\!]$$

(Start Condition) For the initial state, we have $|v_{1x}| + |v_{2x}| = |v_{1x}^0| + |v_{2x}^0|$ and $|v_{1y}| + |v_{2y}| = |v_{1y}^0| + |v_{2y}^0|$ according to the initialization statement in the model description.

(Transition Closure) After any "bounce" action, the direction of the speed changes, but the absolute value of the speed doesn't change. Therefore, $\forall x \xrightarrow{\text{bounce}} x'$, if $x \in I$ then $x' \in I$. After the "collision" action, v_{1x} and v_{2x} are exchanged, and v_{1y} and v_{2y} are exchanged. Thus if $|v_{1x}| + |v_{2x}| = |v_{1x}^0| + |v_{2x}^0|$ and $|v_{1y}| + |v_{2y}| = |v_{1y}^0| + |v_{2y}^0|$, then $|v'_{1y}| + |v'_{2y}| = |v_{1y}^0| + |v_{2y}^0|$ and $|v'_{1y}| + |v'_{2y}| = |v_{1y}^0| + |v_{2y}^0|$.

(Trajectory Closure) In the trjectory "running" the speed variables never change, and thus $\forall \tau \in \mathbb{T}$, if τ.fstate $\in I$, then τ.lstate $\in I$.

Problem 4. Hysteresis model (coding 40 points). (a) Model the following hysteresis-based switching system as a hybrid automaton. The automaton $\mathcal{A}$ has (at least) n continuous variables $x_1, \ldots, x_n$ and a discrete variable called m that takes the values in the set $m_1, \ldots, m_n$. For a state of the automaton, we say that the system is in *mode* m_i, if $m = m_i$. There are $n \times (n-1)$ actions $switch(i,j)$, where $i, j \in [n]$ and $j \neq i$. When the system is in mode m_i,
$$\dot{x}_i = a_i x_i,$$
where $a_i > 0$ is a positive constant and $\dot{x}_j = 0$ for all $j \neq i$. Further, when the system is in mode m_i, for any $j \neq i$, if x_i becomes greater than $(1+h)x_j$, then the automaton switches to mode m_j; otherwise, it continues in mode m_i. Here, $h > 0$ is a parameter of the model. Is your model deterministic?

(b) Write a simulator for this model and plot two executions starting from two different initial states: one where all the x_i's are same, and another where they are different. For each execution, plot the duration that $\mathcal{A}$ stays in each mode. Can you conjecture something interesting about how long $\mathcal{A}$ stays in each mode?

Solution

(a)

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automaton Switching(h)
  type Mode:enumeration $[m_1, \ldots, m_n]$
  actions
    switch(i, j), for $i, j \in \{1, 2, 3, \ldots, n\}, i \neq j$
  variables
    m:Mode; x_i:Real; for $i \in \{1, 2, 3, \ldots, n\}$
  transitions
    switch(i, j), for $i, j \in \{1, 2, 3, \ldots, n\}, i \neq j$
    pre $(m = m_i) \wedge (x_i > (1 + h)x_j)$
    eff $m := m_j$;
  trajectories
    $Mode_i$ for $i \in \{1, 2, 3, \ldots, n\}$
    evolve $dx_i = a_i x_i$;
    invariant $(m = m_i) \wedge (x_i \leq (1 + h)x_j)$ for $j \in \{1, 2, 3, \ldots, n\}, j \neq i$

This model is nondeterministic. **pre** $(m = m_i) \wedge (x_i > (1 + h)x_j)$ may be satisfied for more than one j at the same time.

(b) Let $h = 10, n = 5, a_i = 1$, for $i \in \{1, 2, 3, 4, 5\}$, execution are plot in form of (m, duration).

Execution 1: from $x_i = 1.0$, for $i \in \{1, 2, 3, 4, 5\}$
(0, 2.398) $\rightarrow$ (1, 2.398) $\rightarrow$ (2, 2.398) $\rightarrow$ (3, 2.398) $\rightarrow$ (4, 4.796) $\rightarrow$ (0, 2.398) $\rightarrow$ (1, 2.398) $\rightarrow$ (2, 2.398) $\rightarrow$ (3, 4.796) $\rightarrow \ldots$

Execution 2: from $x = [1.0, 2.0, 3.0, 4.0, 5.0]$
(0, 3.091) $\rightarrow$(1, 2.803) $\rightarrow$(2, 2.686) $\rightarrow$(3, 2.621) $\rightarrow$(4, 3.879) $\rightarrow$(0, 2.803) $\rightarrow$(1, 2.686) $\rightarrow$(2, 2.621) $\rightarrow$(3, 3.879) $\rightarrow \ldots$

When starting from states where all x_i have the same value and all a_i have the same value, the possible duration values are $\{\ln(1 + h)/a_i, 2\ln(1 + h)/a_i\}$.

Class contribution and project ideas

- Develop a simulator for the billiards example with multiple balls. Make it usable for others by paying attention to how the board, the initial positions and velocities of the balls are specified. Explicitly document how the collisions are detected in cases where multiple collisions may occur at the same time.
- Develop a model and a simulator for Newton's cradle (search online).
- Make your own hybrid modeling problems and solutions. Lots of options here as almost everything can be modeled as a hybrid automaton: cars, drones, mixed-signal circuits, oscillators. Your problem should be (a) compact—describable in a couple of short paragraphs (e.g., like the billiards problem), (b) complete—fully specify the states, trajectories and transitions, and it has to be (c) interesting—the model should do something cool or showcase some property. You also need to provide the solution, of course.