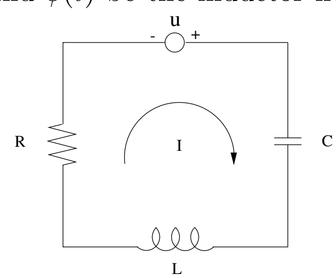
Problem 1

Consider this electrical circuit with *time-varying* characteristics $R(t)$, $L(t)$ and $C(t)$. Let $q(t)$ denote the charge in capacitor at time t, and $\phi(t)$ be the inductor flux at time t. From physical laws we know:



- Capacitor equation: $q(t) = C(t)V_C(t)$.
- Inductor equation: $\phi(t) = L(t)I(t)$.

Use these laws to derive a dynamical model of this circuit which takes the form $\dot{x} = A(t)x + B(t)u$.

Solution

Let x_1 denote the voltage of the capacitor, and x_2 be the current. Then we have $\dot{q} = \dot{C}x_1 + C\dot{x_1} = x_2$ and $\dot{\phi} = \dot{L}x_2 + L\dot{x_2} = u - x_1 - x_2R$. Therefore, we have

$$\dot{x_1} = \frac{-\dot{C}}{C}x_1 + \frac{1}{C}x_2, \quad \dot{x_2} = -\frac{1}{L}x_1 - \frac{R+\dot{L}}{L}x_2 + \frac{1}{L}u$$

$$A(t) = \begin{bmatrix} \frac{-\dot{C}}{C} & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R+\dot{L}}{L} \end{bmatrix}, \quad B(t) = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}$$

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Problem 2

Which of the following are vector spaces over $\mathbb{R}$ (with respect to the standard addition and scalar multiplication)? Justify your answers.

(a) The set of real-valued $n \times n$ matrices with nonnegative entries, where n is a given positive integer.

(b) The set of rational functions of the form $\frac{p(s)}{q(s)}$, where p and q are polynomials in the complex variable s and the degree of q does not exceed a given fixed positive integer k.

(c) The space $L^2(\mathbb{R}, \mathbb{R})$ of square-integrable functions, i.e., functions $f : \mathbb{R} \to \mathbb{R}$ with the property that $\int_{-\infty}^{\infty} f^2(t)dt < \infty$. (**Hint:** You may use the Cauchy-Schwarz inequality, and note that this inequality applies to any inner product space.)

Solution

(a) It is obvious that the ϑ in the space must be θ, an $n \times n$ matrix whose entries are all zero. However, for every matrix M in the space and $M \neq \vartheta$ there doesn't exist a matrix $\hat{M}$ so that $M + \hat{M} = \vartheta$. Therefore this is not a vector space.

(b) This is not a vector space. For example, let $k = 2$. $\frac{1}{s^2+1}$ and $\frac{1}{s+1}$ are elements in the space. However, $\frac{1}{s^2+1} + \frac{1}{s+1} = \frac{s^2+s+2}{s^3+s^2+s+2}$ is not an element in the space.
(c) The item (b)-(g) in the definition are easy to verify. We next prove that this space satisfy the constraint (a). For any $f_1 \in \mathcal{X},\ f_2 \in \mathcal{X}$, we have $\int_{-\infty}^{\infty}(f_1+f_2)^2dt = \int_{-\infty}^{\infty} f_1^2dt + \int_{-\infty}^{\infty} f_1^2dt + \int_{-\infty}^{\infty} f_1f_2dt$. It's easy to verify that $\int_{-\infty}^{\infty} f_1f_2dt$ is an inner product between f_1 and f_2 defined on $\mathcal{X}$. Therefore, according to the Cauchy-Schwarz inequality we have $\int_{-\infty}^{\infty} f_1f_2dt \leq \sqrt{\int_{-\infty}^{\infty} f_1^2dt}\sqrt{\int_{-\infty}^{\infty} f_1^2dt} < \infty$. Therefore, $\int_{-\infty}^{\infty}(f_1+f_2)^2dt < \infty$ and $f_1 + f_2 \in \mathcal{X}$. Therfore, this is a vector space.

Problem 3

A single-input, single-output linear time-invariant system is described by the transfer function:

$$G(s) = \frac{s+4}{(s+1)(s+2)(s+3)}$$

(a) Obtain a state-space representation in controllable canonical form.
(b) Now obtain one in observable canonical form.
(c) Use the partial fraction expansion of $G(s)$ to obtain a representation of this model with a diagonal state matrix A.

Solution

$$G(s) = \frac{s+4}{s^3+6s^2+11s+6}$$

(a) CCF: $\dot{x} = Ax + Bu,\ y = Cx$

where $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$; $B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$; $C = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}^T$.

(b) OCF: $\dot{x} = Ax + Bu,\ y = Cx$

where $A = \begin{bmatrix} -6 & 1 & 0 \\ -11 & 0 & 1 \\ -6 & 0 & 0 \end{bmatrix}$; $B = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$; $C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T$.

(c) $G(s) = \frac{3/2}{s+1} + \frac{-2}{s+2} + \frac{1/2}{s+3}$. Then we have:

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}; B = \begin{bmatrix} \frac{3}{2} \\ -2 \\ \frac{1}{2} \end{bmatrix}; C = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}^T; D = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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Problem 4

Let A be the linear operator in the plane corresponding to the counter-clockwise rotation around the origin by some given angle θ. Compute the matrix of A relative to the standard basis in $\mathbb{R}^2$.

Solution

Let the bases are $e_1 = [1, 0]^T, e_2 = [0, 1]^T$, then the rotated vectors are $\hat{e_1} = [\cos\theta, \sin\theta]^T, \hat{e_2} = [-\sin\theta, \cos\theta]^T$. Because $[e_1, e_2]A = [\hat{e_1}, \hat{e_2}]$, we have $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.