Problem 1

Consider the following LTI model:

$$\dot{x} = \begin{bmatrix} -7 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix} x$$

Check controllability using:

a) the controllability matrix.

b) the rows of $\overline{B} = M^{-1}B$, where M is chosen such that $M^{-1}AM$ is diagonal.

c) the Hautus-Rosenbrock condition.

You may use a calculator, MATLAB, or another computational device, but be sure you would know how to do this manually if needed.

Solution

a) We have $C = [B|AB|A^2B]$, and $C = \begin{bmatrix} 1 & 1 & -3 & -5 & 9 & 25 \\ 1 & -1 & -3 & 5 & 9 & -25 \\ 1 & 0 & -3 & 0 & 9 & 0 \end{bmatrix}$, and thus $\text{rank}(C) = 2$. So, the model is uncontrollable.

b) We have $M = \begin{bmatrix} 0.707 & 0.707 & -0.577 \\ 0 & -0.707 & -0.577 \\ 0.707 & 0 & -0.577 \end{bmatrix}$, and $\overline{B} = M^{-1}B = \begin{bmatrix} 0 & 0 \\ 0 & 1.414 \\ -1.732 & 0 \end{bmatrix}$. There is a 0-row in $\overline{B}$, so the model is uncontrollable.

c) The eigen values of A are $s = [-1, -5, -3]$. For $s_1 = -1$, we have $\text{rank}(-I - A|B) = 2$, and thus the model is uncontrollable.

Problem 2

Let A be an $n \times n$ matrix and B be an $n \times r$ matrix, both with real entries. Suppose (A, B) is controllable. Prove or disprove the following statements. (If the statement is false, then producing a counterexample will suffice.)

a) The pair (A^2, B) is controllable.

b) Let $k(\cdot)$ be a known n-dimensional function, piecewise continuous in t. Consider the model:

$$\dot{x} = Ax + Bu + k(t)$$

This model is controllable, in the sense that for any initial state x_0 and any target final state x_f, there exists a control $u(\cdot)$ that directs the system from x_0 to x_f in finite time.

c) Given that the model $\dot{x} = Ax + Bu$ has the initial condition $x(0) = x_0 \neq 0$, it is possible to find a piecewise continuous control, defined on $[0, \infty)$, such that the model is brought to rest at $t = 1$, i.e. $x(t) = 0$ for all $t \geq 1$.

d) Suppose the model starts at $x(0) = 0$; there exists a piecewise continuous control which will bring the state to $x_f \in \mathbb{R}^n$ by time $t \geq 1$ and maintain that value $x(t) = x_f$ for all $t \geq 1$.

Solution

a) False. Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then $A^2 = I$. Let $B = [1, 3]^T$, then $\text{rank}([B, AB]) = 2$, $\text{rank}([B, A^2B]) = 1$.

b) True. Decompose x into two components, $x = x_1 + x_2$, and $\dot{x_1} = Ax_1 + Bu$, $\dot{x_2} = Ax_2 + k(t)$. Because (A, B) is controllable, there exists t_f, such that for any x_0 and x_f we can find a control $u(t)$. Fix this t_f and set the initial state of x_2 to 0, i.e. $x_2(0) = 0$, then $x_2(t_f)$ is fixed. Then, let $x_1(0) = x_0$, we can find a $u(t)$ such that $x_1(t_f) = x_f - x_2(t_f)$. This $u(t)$ will dirve the original model from x_0 to x_f.

c) True. According to the part (ii) of the proof of Theorem 5.3.1 in the reader, we know if there exists $t_f > 0$ such that $W(0, t_f)$ is singular, then we have $\text{rank}(\mathbb{C}) < n$. This implies that if $\text{rank}(\mathbb{C}) = n$, then $\forall t_f > 0$, $W(0, t_f)$ is nonsingular. So, we know that if a LTI model is controllable, then t_f can be any positive real number. For this specific problem, we choose $t_f = 1$. Then, we can find a control $u(t)$ which will direct the model from x_0 to 0 when $t = 0$. For $t \geq 1$, set $u(t) = 0$ such that $\dot{x} = 0$, and thus the model rest on 0.

d) False. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then, we have $\text{rank}(\mathbb{C}) = 2$, and model is controllable. However, for $x_f = [1, 1]^T$, there doesn't exist a u such that $\dot{x} = Ax_f + Bu = 0$. Therefore, the model can not rest on x_f.

Problem 3

Define the operator $\mathcal{A}$ as follows:

$$\mathcal{A}(u) = \int_{t_0}^{t_f} \phi(t_f, \tau)B(\tau)u(\tau)d\tau$$

The domain of $\mathcal{A}$ is the set of all piecewise continuous time functions on $[t_0, t_f]$. The co-domain is $\mathbb{R}^n$.

a) Compute the adjoint $\mathcal{A}^*$.

b) Compute the composition $V = \mathcal{A} \circ \mathcal{A}^*$. Note $V : \mathbb{R}^n \to \mathbb{R}^n$, so V should be a $n \times n$ matrix. What is the relationship with V and the controllability Grammian W? Adapt Theorem 5.2.2 from the reader to a statement about V.

Solution

a) According to the definition of adjoint, for all $x \in \mathbb{R}^n$ and u, a piecewise continuous function on $[t_0, t_f]$, $\langle \mathcal{A}(u), x\rangle = \langle u, \mathcal{A}^*(x)\rangle$.

$$\langle \mathcal{A}(u), x\rangle = (\mathcal{A}(u))^* x = \int_{t_0}^{t_f} u^*(\tau)B^*(\tau)\phi^*(t_f, \tau)x d\tau$$
$$= \langle u, \mathcal{A}^*(x)\rangle = \int_{t_0}^{t_f} u^*(\tau)\mathcal{A}^*(x)(\tau)d\tau$$

Because x and u are chosen arbitrarily, we have $\mathcal{A}^*(x)(t) = B^*(t)\phi^*(t_f, t)x,\ t \in [t_0, t_f]$.

b)

$$V(x) = \mathcal{A}(\mathcal{A}^*(x)) = \mathcal{A}(B^*(t)\phi^*(t_f, t)x)$$
$$= (\int_{t_0}^{t_f} \phi(t_f, \tau)B(\tau)B^*(t)\phi^*(t_f, t)d\tau)x = \phi(t_f, t_0)W(t_0, t_f)\phi^*(t_f, t_0)x$$

Because $\phi(t_f, t_0)$ is invertible, we can say "The LTV model is controllable at time t_0 if and only if there exists a finite time $t_f > t_0$ such that $\forall x \neq \vartheta$, $V(x) \neq \vartheta$."