Problem 1

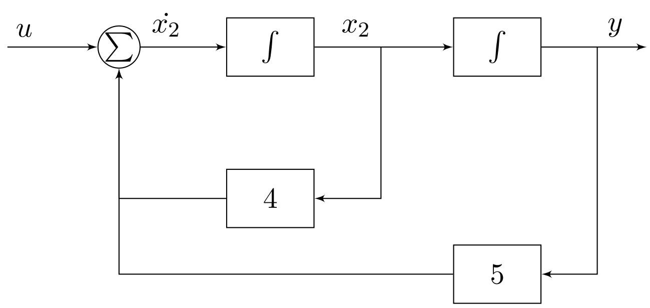
Let

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 5 & 4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x = x_1$$

- Draw a block diagram representing this system.
- Design a reduced-order Luenberger observer, and draw the block diagram for the system and the observer.

Solution

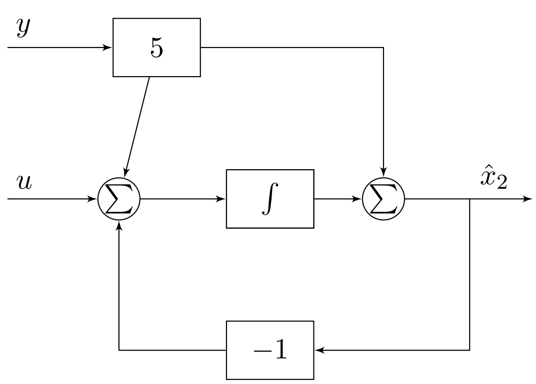
- 

- Using the notations in the reader, we have $A_{11} = 0$, $A_{12} = 1$, $A_{21} = 5$, $A_{22} = 4$, $B_1 = 0$, $B_2 = 1$ and $y = x_1$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

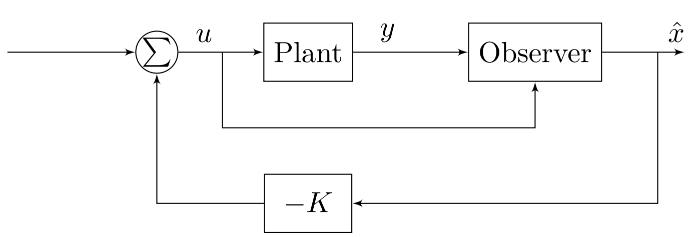
  Considering the gain of the observer feedback L, we must have $A_{22} - LA_{12} = 4 - L < 0$. Choose $L = 5$. Then, the observer equation is

$$\dot{\hat{x}}_2 = 4\hat{x}_2 + 5y + u + L(\dot{y} - \hat{x}_2) = -\hat{x}_2 + 5y + u + 5\dot{y}$$

  The diagram of the reduced-order Luenberger observer is



  The overall system diagram is



Problem 2

Take any matrix $A \in \mathbb{R}^{m \times n}$. Prove the following statements:

- $\mathcal{R}(A)^\perp = \mathcal{N}(A^\intercal)$
- $\mathcal{N}(A)^\perp = \mathcal{R}(A^\intercal)$

Solution

- ($\Rightarrow$) If α is an element of $\mathcal{R}(A)^\perp$, then for any $x \in \mathbb{R}^n$, $Ax \in \mathcal{R}(A)$, we have $\alpha^\intercal Ax = 0$. Because this equation holds for all x in $\mathbb{R}^n$, we must have $\alpha^\intercal A = 0$ and $\alpha \in \mathcal{N}(A^\intercal)$.

  ($\Leftarrow$) If α is an element of $\mathcal{N}(A^\intercal)$, then $A^\intercal \alpha = 0$. For all $x \in \mathbb{R}^n$, $Ax \in \mathcal{R}(A)$, we have $(Ax)^\intercal \alpha = x^\intercal A^\intercal \alpha = 0$ and $\alpha \in \mathcal{R}(A)^\perp$.

- First, we prove the following statement: if W is a subspace of R^n, then $(W^\perp)^\perp = W$.

  Proof: We assume that $\{v_1, \ldots, v_r\}$ is an orthogonal basis of W. Because $V = W \bigoplus W^\perp$, we can find $\{v_{r+1}, \ldots, v_n\}$ as an orthogonal basis of $W^\perp$, and $\{v_1, \ldots, v_n\}$ is an orthogonal basis of V. For all $v \in W^\perp$ and $w \in W$, $v \perp w$, and thus $W \subseteq (W^\perp)^\perp$. For all $v = \sum_{i=1}^n \alpha_i v_i \in V$, if $v \in (W^\perp)^\perp$, then $\forall i \in \{r+1, \ldots, n\}$, $v \perp v_i$, and thus $\forall i \in \{r+1, \ldots, n\}$, $\alpha_i = 0$ which implies $v \in W$. Therefore, $(W^\perp)^\perp \subseteq W$. Finally, we have $(W^\perp)^\perp = W$.

  Then, with the conclusion from the first subproblem, $\mathcal{N}(A)^\perp = (\mathcal{R}(A^\intercal)^\perp)^\perp = \mathcal{R}(A^\intercal)$.

Problem 3

Consider the following dynamics:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}$$

In this problem, you will be asked to calculate two Kalman decompositions for this system.

- Calculate $\Sigma_{c\bar{o}} = \mathcal{R}(\mathcal{C}) \cap \mathcal{N}(\mathcal{O})$
- Calculate a Σ_{co} such that $\Sigma_{c\bar{o}} \oplus \Sigma_{co} = \mathcal{R}(\mathcal{C})$
- Calculate a $\Sigma_{\bar{c}\bar{o}}$ such that $\Sigma_{c\bar{o}} \oplus \Sigma_{\bar{c}\bar{o}} = \mathcal{N}(\mathcal{O})$
- Calculate a $\Sigma_{\bar{c}o}$ such that $\Sigma_{c\bar{o}} \oplus \Sigma_{co} \oplus \Sigma_{\bar{c}\bar{o}} \oplus \Sigma_{\bar{c}o} = \mathbb{R}^n$
- As we discussed in class, this need not be unique; calculate a different Σ_{co}, $\Sigma_{\bar{c}\bar{o}}$, and $\Sigma_{\bar{c}o}$ for the same system

Solution

We have $\mathcal{C} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\mathcal{O} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 3 & 0 \\ 1 & 0 & 9 & 0 \\ 1 & 0 & 27 & 0 \end{bmatrix}$. Let $e_1,\ e_2,\ e_3,\ e_4$ be the standard basis of R^4. Then, we have $\mathcal{R}(\mathcal{C}) =< e_1, e_2 >$, $\mathcal{N}(\mathcal{O}) =< e_2, e_4 >$, where $< e_1, e_2 >$ refers to the subspace spanned by e_1 and e_2.

- $\Sigma_{c\bar{o}} = \mathcal{R}(\mathcal{C}) \cap \mathcal{N}(\mathcal{O}) =< e_2 >$
- $\Sigma_{co} =< e_1 >$
- $\Sigma_{\bar{c}\bar{o}} =< e_4 >$
- $\Sigma_{\bar{c}o} =< e_3 >$
- $\Sigma_{c\bar{o}} =< e_2 >,\ \Sigma_{co} =< e_1 + e_2 >,\ \Sigma_{\bar{c}\bar{o}} =< e_2 + e_4 >,\ \Sigma_{\bar{c}o} =< e_1 + e_2 + e_3 + e_4 >$