

Problem 1

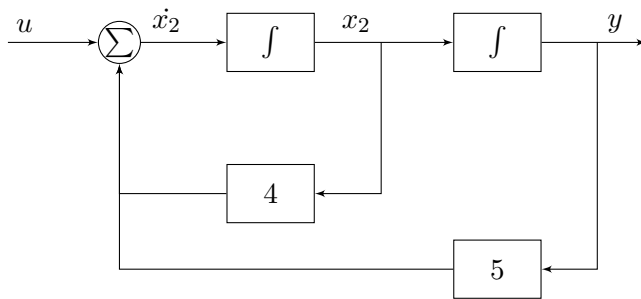
Let

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 5 & 4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x = x_1$$

- Draw a block diagram representing this system.
- Design a reduced-order Luenberger observer, and draw the block diagram for the system and the observer.

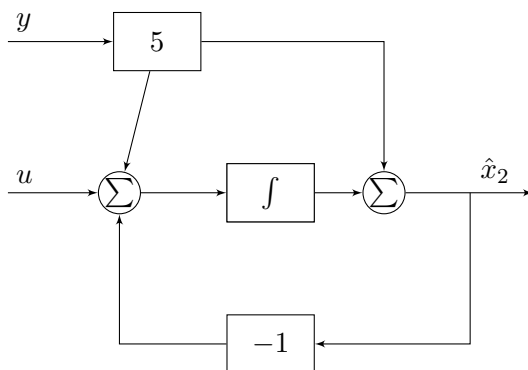
Solution



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- Using the notations in the reader, we have $A_{11} = 0$, $A_{12} = 1$, $A_{21} = 5$, $A_{22} = 4$, $B_1 = 0$, $B_2 = 1$ and $y = x_1$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Considering the gain of the observer feedback L , we must have $A_{22} - LA_{12} = 4 - L < 0$. Choose $L = 5$. Then, the observer equation is

$$\dot{\hat{x}}_2 = 4\hat{x}_2 + 5y + u + L(\dot{y} - \hat{x}_2) = -\hat{x}_2 + 5y + u + 5\dot{y}$$



Problem 2

Take any matrix $A \in \mathbb{R}^{m \times n}$. Prove the following statements:

- $\mathcal{R}(A)^\perp = \mathcal{N}(A^\top)$
- $\mathcal{N}(A)^\perp = \mathcal{R}(A^\top)$

Solution

- (\Rightarrow) If α is an element of $\mathcal{R}(A)^\perp$, then for any $x \in \mathbb{R}^n$, $Ax \in \mathcal{R}(A)$, we have $\alpha^\top Ax = 0$. Because this equation holds for all x in \mathbb{R}^n , we must have $\alpha^\top A = 0$ and $\alpha \in \mathcal{N}(A^\top)$.
- (\Leftarrow) If α is an element of $\mathcal{N}(A^\top)$, then $A^\top \alpha = 0$. For all $x \in \mathbb{R}^n$, $Ax \in \mathcal{R}(A)$, we have $(Ax)^\top \alpha = x^\top A^\top \alpha = 0$ and $\alpha \in \mathcal{R}(A)^\perp$.
- $\mathcal{N}(A)^\perp = (\mathcal{R}(A^\top)^\perp)^\perp = \mathcal{R}(A^\top)$.

Problem 3

Consider the following dynamics:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}$$

In this problem, you will be asked to calculate two Kalman decompositions for this system.

- Calculate $\Sigma_{c\bar{o}} = \mathcal{R}(\mathcal{C}) \cap \mathcal{N}(\mathcal{O})$
- Calculate a Σ_{co} such that $\Sigma_{c\bar{o}} \oplus \Sigma_{co} = \mathcal{R}(\mathcal{C})$
- Calculate a $\Sigma_{\bar{c}\bar{o}}$ such that $\Sigma_{c\bar{o}} \oplus \Sigma_{\bar{c}\bar{o}} = \mathcal{N}(\mathcal{O})$
- Calculate a $\Sigma_{\bar{c}o}$ such that $\Sigma_{c\bar{o}} \oplus \Sigma_{co} \oplus \Sigma_{\bar{c}\bar{o}} \oplus \Sigma_{\bar{c}o} = \mathbb{R}^n$
- As we discussed in class, this need not be unique; calculate a different Σ_{co} , $\Sigma_{\bar{c}\bar{o}}$, and $\Sigma_{\bar{c}o}$ for the same system

Solution

We have $\mathcal{C} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\mathcal{O} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 3 & 0 \\ 1 & 0 & 9 & 0 \\ 1 & 0 & 27 & 0 \end{bmatrix}$. Let e_1, e_2, e_3, e_4 be the standard basis of

\mathbb{R}^4 . Then, we have $\mathcal{R}(\mathcal{C}) = \langle e_1, e_2 \rangle$, $\mathcal{N}(\mathcal{O}) = \langle e_2, e_4 \rangle$, where $\langle e_1, e_2 \rangle$ refers to the subspace spanned by e_1 and e_2 .

- $\Sigma_{c\bar{o}} = \mathcal{R}(\mathcal{C}) \cap \mathcal{N}(\mathcal{O}) = \langle e_2 \rangle$
- $\Sigma_{co} = \langle e_1 \rangle$
- $\Sigma_{\bar{c}\bar{o}} = \langle e_4 \rangle$
- $\Sigma_{\bar{c}o} = \langle e_3 \rangle$
- $\Sigma_{c\bar{o}} = \langle e_2 \rangle$, $\Sigma_{co} = \langle e_1 + e_2 \rangle$, $\Sigma_{\bar{c}\bar{o}} = \langle e_2 + e_4 \rangle$, $\Sigma_{\bar{c}o} = \langle e_1 + e_2 + e_3 + e_4 \rangle$