Typeset your solutions using LATEX zip your writeup (.pdf) and code (.py) in a single file called `nedid-584-F19.zip` *and upload this file through Compass.*

Problem 1. Convexity of reach sets (20 points)   (a) For a linear autonomous system $\dot{x} = Ax$, $x \in \mathbb{R}^n$, prove that if the set of initial states Θ is a polytope with a set of vertices $v_1, \ldots, v_k \in \mathbb{R}^n$, then the set of reachable states at any time $t > 0$ is also a convex set. That is, $Reach(\Theta, t)$ is the convex hull of the points reached from the vertices of Θ at time t.

(b) Give an example of a nonlinear system for which $Reach(\Theta, t)$ is not convex, even if Θ is convex. You can give an analytical proof or a numerical example. You may consider a discrete time model $x(t + 1) = f(x(t))$ as well.

Solution

(a) $\forall x^1, x^2 \in Reach(\Theta, t)$, denote x_0^1, x_0^2 as the corresponding initial state, then we have $x^1 = e^{At}x_0^1, x^2 = e^{At}x_0^2$, and thus $\forall \lambda \in [0, 1]$ we have $\lambda x^1 + (1-\lambda)x^2 = e^{At}(\lambda x_0^1 + (1-\lambda)x_0^2)$. Because Θ is convex and $x_0^1, x_0^2 \in \Theta$, we have $(\lambda x_0^1 + (1-\lambda)x_0^2) \in \Theta$, and thus $e^{At}(\lambda x_0^1 + (1-\lambda)x_0^2) \in Reach(\Theta, t)$. Because x^1, x^2 are chosen arbitraily, we proved that $Reach(\Theta, t)$ is convex.

(b) Consider a two-dimensional non-linear discrete time system.

$$x(t+1) = \begin{cases} 0, & \text{if } x(t)_1 > 0, x(t)_2 > 0 \\ x(t), & \text{otherwise} \end{cases}$$

As shown in Fig. 1 starting from a convex set Θ, $Reach(\Theta, t)$ is not convex.

Problem 2. (20 points)   Show that a set of piece-wise continuous functions is closed under composition and linear combinations. That is, consider any two piece-wise continuous functions of time $f, g : \mathbb{R} \to \mathbb{R}^m$ with points of discontinuity $\mathbf{DC}_f$ and $\mathbf{DC}_g$. Show that

(a) $f \circ g$ is also piece-wise continuous.

(b) For any $a_1, a_2 > 0$, $a_1 f + a_2 g$ is also piece-wise continuous.

Solution

(a) $\forall x \notin \mathbf{DC}_f \cup \mathbf{DC}_g$, $h = a_1 f + a_2 g$ is continuos at x. $\forall x \in \mathbf{DC}_f \cup \mathbf{DC}_g$ and $x \notin \mathbf{DC}_f \cap \mathbf{DC}_g$, h is not continuos at x. $\forall x \in \mathbf{DC}_f \cap \mathbf{DC}_g$, h may be continuous or discontinuos at x. Therefore, $\mathbf{DC}_h \subset \mathbf{DC}_f \cup \mathbf{DC}_g$, which means h has a finite number of points of discontinuity and is also piece-wise continuos.

x_2

x_1

x_2

x_1

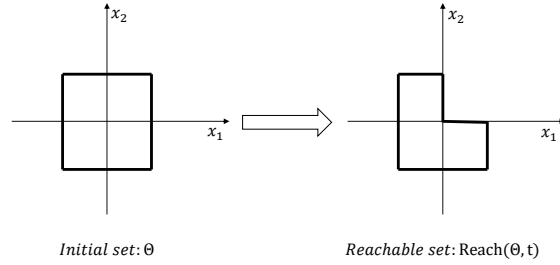*Initial set*: Θ   *Reachable set*: Reach(Θ, t)

Figure 1: Starting from a convex set Θ, $Reach(\Theta, t)$ is not convex

Problem 3. Billiards model (20 points) (a) Consider an idealized billiard table of length a and width b. The table has no pockets; its surface has no friction; and its boundary bounces the balls perfectly. Write a hybrid automaton model of the position of two balls of equal mass on this table. The balls have some initial velocities. A ball bounces off a wall when its position is at the boundary. Balls collide whenever $|x_1 - x_2| \leq \varepsilon$ and $|y_1 - y_2| \leq \varepsilon$ and their velocity vectors are pointing towards each other. Here ε is some constant. Whenever a bounce occurs, the appropriate velocity changes sign. Whenever a collision occurs, the balls exchange their velocity vectors. Make all the variables internal. Wall bounces are modeled by an output action called bounce, and each collision is modeled by an output action collision.

(b) State and prove conservation of speed along each axis as an invariant property of the automaton.

Solution

(a) ―――――――――――――――――

    automaton Billiards(a,b)

      type

      actions

        bounce; collision;

      variables

        v_{1x}:Real:=v_{1x}^0; v_{1y}:Real:=v_{1y}^0; v_{2x}:Real:=v_{2x}^0; v_{2y}:Real:v_{2y}^0; x_1:Real:=x_1^0; y_1:Real:=y_1^0; x_2:Real:x_2^0;
y_2:Real:=y_2^0;

      transitions

        bounce

        pre $x_1 = 0 \vee x_1 = a$

        eff $v_{1x} := -v_{1x}$;

        bounce

        pre $x_2 = 0 \vee x_2 = a$

2

        eff $v_{2x} := -v_{2x}$;
        bounce
        pre $y_1 = 0 \lor y_1 = b$
        eff $v_{1y} := -v_{1y}$;
        bounce
        pre $y_2 = 0 \lor y_2 = b$
        eff $v_{2y} := -v_{2y}$;
        collision
        pre $|x_1 - x_2| \le \varepsilon \land |y_1 - y_2| \le \varepsilon \land (v_{1x}/v_{2x} = v_{1y}/v_{2y} < 0)$
        eff $v_{1x} := v_{2x}$; $v_{2x} := v_{1x}$; $v_{1y} := v_{2y}$; $v_{2y} := v_{1y}$;
    trajectories
        running
        evolve $dx_1 = v_{1x}$; $dy_1 = v_{1y}$; $dx_2 = v_{2x}$; $dy_2 = v_{2y}$
        invariant $(0 < x_1 < a) \land (0 < y_1 < b) \land (0 < x_2 < a) \land (0 < y_2 < b) \land (|x_1 - x_2| >$
$\varepsilon \lor |y_1 - y_2| > \varepsilon \lor v_{1x}/v_{2x} \ne v_{1y}/v_{2y} \lor v_{1x}/v_{2x} \ge 0)$

(b)

$$I = [\![(|v_{1x}| + |v_{2x}| = |v_{1x}^0| + |v_{2x}^0|) \land (|v_{1y}| + |v_{2y}| = |v_{1y}^0| + |v_{2y}^0|)]\!]$$

(Start Condition) For the initial state, we have $|v_{1x}| + |v_{2x}| = |v_{1x}^0| + |v_{2x}^0|$ and $|v_{1y}| + |v_{2y}| = |v_{1y}^0| + |v_{2y}^0|$ according to the initialization statement in the model description.

(Transition Closure) After any "bounce" action, the direction of the speed changes, but the absolute value of the speed doesn't change. Therefore, $\forall x \xrightarrow{\text{bounce}} x'$, if $x \in I$ then $x' \in I$. After the "collision" action, v_{1x} and v_{2x} are exchanged, and v_{1y} and v_{2y} are exchanged. Thus if $|v_{1x}| + |v_{2x}| = |v_{1x}^0| + |v_{2x}^0|$ and $|v_{1y}| + |v_{2y}| = |v_{1y}^0| + |v_{2y}^0|$, then $|v_{1y}'| + |v_{2y}'| = |v_{1y}^0| + |v_{2y}^0|$ and $|v_{1y}'| + |v_{2y}'| = |v_{1y}^0| + |v_{2y}^0|$.

(Trajectory Closure) In the trjectory "running" the speed variables never change, and thus $\forall \tau \in \mathbb{T}$, if $\tau.\text{fstate} \in I$, then $\tau.\text{lstate} \in I$.

Problem 4. Hysteresis model (coding 40 points). (a) Model the following hysteresis-based switching system as a hybrid automaton. The automaton $\mathcal{A}$ has (at least) n continuous variables $x_1, \ldots, x_n$ and a discrete variable called m that takes the values in the set $m_1, \ldots, m_n$. For a state of the automaton, we say that the system is in *mode* m_i, if $m = m_i$. There are $n \times (n-1)$ actions $switch(i, j)$, where $i, j \in [n]$ and $j \ne i$. When the system is in mode m_i,

$$\dot{x}_i = a_i x_i,$$

where $a_i > 0$ is a positive constant and $\dot{x}_j = 0$ for all $j \ne i$. Further, when the system is in mode m_i, for any $j \ne i$, if x_i becomes greater than $(1 + h)x_j$, then the automaton switches to mode m_j; otherwise, it continues in mode m_i. Here, $h > 0$ is a parameter of the model. Is your model deterministic?

(b) Write a simulator for this model and plot two executions starting from two different initial states: one where all the x_i's are same, and another where they are different. For each execution, plot the duration that $\mathcal{A}$ stays in each mode. Can you conjecture something interesting about how long $\mathcal{A}$ stays in each mode?

Solution

(a)

```
automaton Switching(h)
    type Mode:enumeration [m_1, …, m_n]
    actions
        switch(i, j), for i, j ∈ {1, 2, 3, …, n}, i ≠ j
    variables
        m:Mode; x_i:Real; for i ∈ {1, 2, 3, …, n}
    transitions
        switch(i, j), for i, j ∈ {1, 2, 3, …, n}, i ≠ j
        pre (m = m_i) ∧ (x_i > (1 + h)x_j)
        eff m := m_j;
    trajectories
        Mode_i for i ∈ {1, 2, 3, …, n}
        evolve dx_i = a_i x_i;
        invariant (m = m_i) ∧ (x_i ≤ (1 + h)x_j) for j ∈ {1, 2, 3, …, n}, j ≠ i
```

This model is nondeterministic. $\textbf{pre } (m = m_i) \wedge (x_i > (1 + h)x_j)$ may be satisfied for more than one j at the same time.

(b) Let T_i be the total duration that the model stays in mode i. For every execution, we plot the variables x_i as time grows. Some ececutions are plotted in Fig. 2, Fig. 3 and Fig. 4.

After comparing the durations in Fig. 3 and Fig. 4, we can find that as the time t goes to ∞, $\frac{T_i}{T_j}$ converges to $\frac{a_j}{a_i}$, i.e. $\lim_{t\to\infty} \frac{T_i}{T_j} = \frac{a_j}{a_i}$. We will give an informal proof in the following.

After every mode has been visited at least once, the following statement will hold: $x_{min} \leq x_i \leq x_{min}(1 + h)$, $\forall i \in \{1, 2, 3, \dots, n\}$, where x_{min} is the minimal x_i at this point of time. According to the dynamics of the model, we have $T_i = \frac{1}{a_i} \ln \frac{x_i}{x_i^0}$, where x_i^0 the initial value of x_i. Therefore, $\ln x_{min} - \ln x_i^0 \leq a_i T_i \leq \ln x_{min} + \ln(1 + h) - \ln x_i^0$. As t goes to ∞, x_{min} goes to ∞, and thus the inequation (or the interval) will be dominated by $\ln x_{min}$. Therefore, $\frac{a_i T_i}{a_j T_j}$ converges to 1, which implies $\lim_{t\to\infty} \frac{T_i}{T_j} = \frac{a_j}{a_i}$.

Class contribution and project ideas

- Develop a simulator for the billiards example with multiple balls. Make it usable for others by paying attention to how the board, the initial positions and velocities of the balls are specified. Explicitly document how the collisions are detected in cases where multiple collisions may occur at the same time.

- Develop a model and a simulator for Newton's cradle (search online).

- Make your own hybrid modeling problems and solutions. Lots of options here as almost everything can be modeled as a hybrid automaton: cars, drones, mixed-signal circuits, oscillators. Your problem should be (a) compact—describable in a couple of short paragraphs
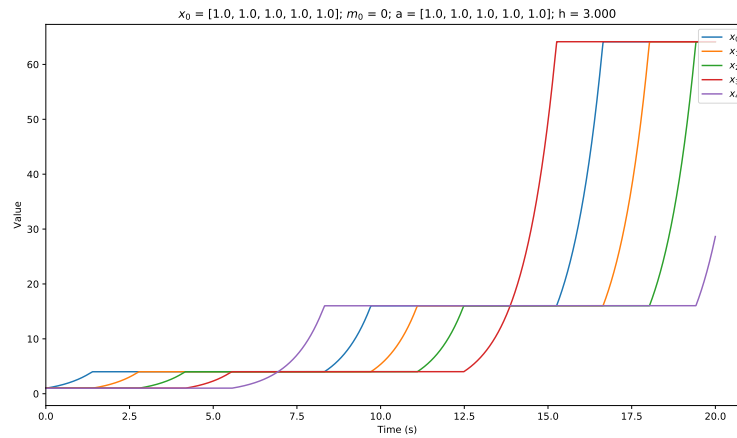
Figure 2: Starting from a state where all the x_i's are same. The durations are [4.162, 4.161, 4.161, 4.161, 3.354].

(e.g., like the billiards problem), (b) complete—fully specify the states, trajectories and transitions, and it has to be (c) interesting—the model should do something cool or showcase some property. You also need to provide the solution, of course.
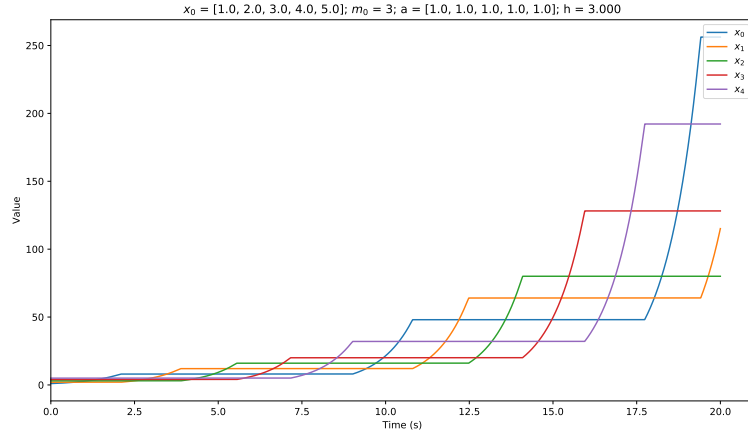
Figure 3: Starting from a state where x_i's are different. The durations are [5.546, 4.052, 3.284, 3.468, 3.649].
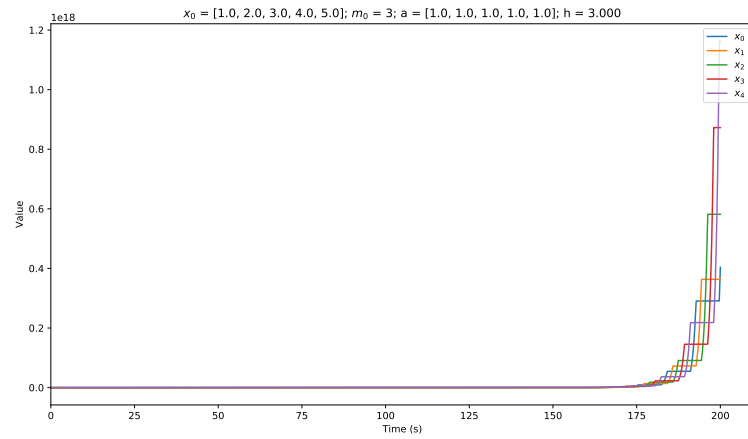

Figure 4: Starting from a state where x_i's are different (with a larger simulation budget). The durations are [40.539, 39.741, 39.806, 39.925, 39.988].