Problem 1

Suppose we have an LTI system with an input disturbance:

$$\dot{x} = Ax + B(u + w)$$

$$y = Cx$$

If we knew $w(t)$, we could control the system with $u(t) = -w(t) - K\widehat{x}(t)$, where $\widehat{x}(t)$ is our state estimate from some observer.

However, we don't know $w(t)$. Suppose that we know the dynamics of $w(t)$:

$$\dot{z} = A_m z$$

$$w = C_m z$$

Thus, we know A_m, C_m, but not the value of z or w. Let's augment our state-space model as follows:

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A & BC_m \\ 0 & A_m \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} C & 0 \end{bmatrix} x$$

- Is the new system controllable?
- Derive a full-state observer for this augmented system. Leave the observer gain as $L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$.
- Use the state estimate from this observer for a control law $u = -C_m\widehat{z} - K\widehat{x}$. (Note that our estimate of the unknown noise term w is $C_m\widehat{z}$.) Write out the dynamics of the controller and observer, viewing y as the input to this system and u as the output. What are the eigenvalues of these 'controller and observer' dynamics? (Again, leave the controller gain as K.)
- Write out the closed-loop dynamics for the controller, observer, and original system, using the states $(x, z, \widehat{x}, \widehat{z})$.

Solution

- The input u doesn't effect z directly, nor indirectly via x, thus it is not controllable. Formally, choose s as an eigenvalue of A_m, then last m rows of $\begin{bmatrix} sI - A & -BC_m & B \\ 0 & sI - A_m & 0 \end{bmatrix}$ is not linearly independent, and thus the rank is less than $n + m$.
- $$\begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{z}} \end{bmatrix} = \begin{bmatrix} A & BC_m \\ 0 & A_m \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{z} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} (y - C\hat{x})$$
- $$\begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{z}} \end{bmatrix} = \begin{bmatrix} A - BK - L_1C & 0 \\ -L_2C & A_m \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{z} \end{bmatrix} + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} y$$

  $$u = \begin{bmatrix} -K & -C_m \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{z} \end{bmatrix}$$

  According to a conclusion from HW2, we know the eigevalues of $\begin{bmatrix} A - BK - L_1C & 0 \\ -L_2C & A_m \end{bmatrix}$ is the union of eigenvalues of $A - BK - L_1C$ and A_m.

- $$\begin{bmatrix} \dot{x} \\ \dot{z} \\ \dot{\hat{x}} \\ \dot{\hat{z}} \end{bmatrix} = \begin{bmatrix} A & BC_m & -BK & -BC_m \\ 0 & A_m & 0 & 0 \\ L_1C & 0 & A - BK - L_1C & 0 \\ L_2C & 0 & -L_2C & A_m \end{bmatrix} \begin{bmatrix} x \\ z \\ \hat{x} \\ \hat{z} \end{bmatrix}$$

Problem 2

Suppose we have the same setup as the previous problem, with system dynamics given by

$$\dot{x} = Ax + B(u + w)$$

$$y = Cx$$

and noise dynamics given by

$$\dot{z} = A_m z$$

$$w = C_m z$$

Rather than explicitly estimate the unknown state, let's filter the output. Let's add the following states:

$$\dot{x}_a = A_m^\intercal x_a + C_m^\intercal y$$

- Derive an observer for $\widehat{x}$, ignoring the w term. Again, leave L in the equations without explicitly calculating these values.
- Stabilize the system with $u = -K_1 \widehat{x} - K_2 x_a$. Derive the dynamics for your controller and filter $(\widehat{x}, x_a)$, viewing y as an input and u as the output.
- What are the eigenvalues of your 'controller and filter' dynamics?

Solution

- $\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$
- $$\begin{bmatrix} \dot{\hat{x}} \\ \dot{x}_a \end{bmatrix} = \begin{bmatrix} A - BK_1 - LC & -BK_2 \\ 0 & A_m^\intercal \end{bmatrix} \begin{bmatrix} \hat{x} \\ x_a \end{bmatrix} + \begin{bmatrix} L \\ C_m^\intercal \end{bmatrix} y$$
  $$u = \begin{bmatrix} -K_1 & -K_2 \end{bmatrix} \begin{bmatrix} \hat{x} \\ x_a \end{bmatrix}$$
- According to a conclusion from HW2, we know the eigevalues of $\begin{bmatrix} A - BK_1 - LC & -BK_2 \\ 0 & A_m^\intercal \end{bmatrix}$ is the union of eigenvalues of $A - BK_1 - LC$ and $A_m^\intercal$.