Problem 1

Consider the linear, time-varying system:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

$$y(t) = C(t)u(t)$$

Recall the definition of the observability Grammian:

$$H(t_1, t_0) = \int_{t_0}^{t_1} \phi^\mathsf{T}(\tau, t_0)C^\mathsf{T}(\tau)C(\tau)\phi(\tau, t_0)d\tau$$

Consider the function from $\mathcal{R}$ to $\mathcal{R}^{n\times n}$:

$$X : t_0 \mapsto H(t_1, t_0)$$

(Two other ways to write this are: $X(t_0) = H(t_1, t_0)$, or $X(\cdot) = H(t_1, \cdot)$.)

Show that the function X satisfies the linear matrix differential equation:

$$\dot{X}(t) = -A^\mathsf{T}(t)X(t) - X(t)A(t) - C^\mathsf{T}(t)C(t) \qquad\qquad X(t_1) = 0_{n\times n}$$

Here, the initial condition $0_{n\times n}$ is the zero matrix in $\mathcal{R}^{n\times n}$.

Solution

Denoting

$$X(t) = \int_{t}^{t_1} \phi^T(\tau, t)C^T(\tau)C(\tau)\phi(\tau, t)d\tau = \int_{t}^{t_1} f(t, \tau)d\tau$$

Using the Leibniz rule:

$$\begin{aligned}
\dot{X}(t) &= -f(t,t) + \int_t^{t_1} \frac{\partial f(t,\tau)}{\partial t} d\tau \\
&= -IC^\mathsf{T}(t)C(t)I + \int_t^{t_1} \left[-A(t)^\mathsf{T}\phi^\mathsf{T}(\tau,t)C^\mathsf{T}(\tau)C(\tau)\phi(\tau,t) + \phi^\mathsf{T}(\tau,t)C^\mathsf{T}(\tau)C(\tau)\phi(\tau,t)(-A(t))\right] d\tau \\
&= -A^\mathsf{T}(t)X(t) - X(t)A(t) - C^\mathsf{T}(t)C(t) \qquad (1)
\end{aligned}$$

and we have

$$X(t_1) = \int_{t_1}^{t_1} f(t,\tau)d\tau = 0_{n\times n}$$

Problem 2

Consider a linear time-varying system with dynamics:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

$$y(t) = C(t)x(t)$$

Let's call this system R.

As we've covered before, the dual system is given by the dynamics:

$$\dot{\tilde{x}}(t) = -A^\intercal(t)\tilde{x}(t) - C^\intercal(t)\tilde{u}(t)$$

$$\tilde{y}(t) = B^\intercal(t)\tilde{x}(t)$$

Let's call this dual system $\tilde{R}$.

Consider any state x_0 that is controllable to zero on $[t_0, t_1]$ for R, and any state $\tilde{x}_0$ that is unobservable on $[t_0, t_1]$ for $\tilde{R}$. Show that x_0 and $\tilde{x}_0$ are orthogonal, i.e. $\langle x_0, \tilde{x}_0 \rangle = 0$.

Solution

Because x_0 is controllable to zero, there exists a $u(t)$ such that $x_0 = \int_{t_0}^{t_1} \phi(t_0, \tau)B(\tau)u(\tau)d\tau$. Because $\tilde{x}_0$ is unobservable, $B^\intercal(t)\phi^\intercal(t_0, t)\tilde{x}_0 = 0,\ \forall t \in [t_0, t_1]$. Therefore, $\langle x_0, \tilde{x}_0 \rangle = x_0^\intercal \tilde{x}_0 = \int_{t_0}^{t_1} u^\intercal(\tau)B^\intercal(\tau)\phi^\intercal(t_0, \tau)\tilde{x}_0 d\tau = 0$.

Problem 3

Consider:

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Is the system controllable? If not, put it in Kalman controllability canonical form.
- Is the system observable? If not, put it in Kalman observability canonical form.

Solution

- Controallability matrix $\mathcal{C} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$. Since $\text{rank}(\mathcal{C}) = 1$, it is uncontrollable. Let $P^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$. Then, $\bar{A} = PAP^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ $\quad \bar{B} = PB = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad \bar{C} = CP^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$.
- Observablity matrix $\mathcal{O} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$. Since $\text{rank}(\mathcal{O}) = 2$, it is observable.

Problem 4

Take the transfer function:

$$H(s) = \frac{s+3}{(s+1)(s+2)}$$

- Put this system into controllable canonical form.
- Using static linear state feedback ($u = -Kx$), find a K that places the poles at $-5 \pm 2j$.

Solution

- $H(s) = \frac{s+3}{s^2+3s+2}$. CCF: $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ $\quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad C = \begin{bmatrix} 3 & 1 \end{bmatrix}$.
- Let $K = [k_1 \ k_2]$, then $A - BK = \begin{bmatrix} 0 & 1 \\ -(2+k_1) & -(3+k_2) \end{bmatrix}$ and the closed-loop characteristic polynomial is $\Delta(s) = s^2 + (3+k_2)s + (2+k_1)$. Then substitute the roots, and we have $K = \begin{bmatrix} 27 & 7 \end{bmatrix}$.

Problem 5

Consider the following dynamical system, inspired by a linearization of the pendubot from Chapter 1 in the reader.

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 5 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ -5 & 6 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 2 \\ 0 \\ -3 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x$$

Design a reduced-order Luenberger observer for this system. You may freely use MATLAB or any other computer assistance to do so (and are encouraged to do so!), but still show your work.

Solution

Using the notations in the reader, we have $A_{11} = 0$, $A_{21} = \begin{bmatrix} 5 \\ 0 \\ -5 \end{bmatrix}$, $A_{12} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$, $A_{22} = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 6 & 0 & 0 \end{bmatrix}$, $B_1 = 0$, $B_2 = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$ and $y = x_1$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Then, the matrix for the observer is $A_{22} - LA_{12} = \begin{bmatrix} -\ell_1 & -2 & 0 \\ -\ell_2 & 0 & 1 \\ 6 - \ell_3 & 0 & 0 \end{bmatrix}$. The characteristic polynomial is $\Delta(s) = s^3 + \ell_1 s^2 - 2\ell_2 s - 2\ell_3 + 12$. Putting all the poles at -1, we have $L = \begin{bmatrix} 3 & -\frac{3}{2} & \frac{11}{2} \end{bmatrix}^\intercal$.

The observer is $\dot{\hat{x}}_2 = A_{22}\hat{x}_2 + A_{21}y + B_2 u + L(\dot{y} - A_{12}\hat{x}_2)$. Then, x can be recovered from y and x_2, i.e. $\hat{x} = \begin{bmatrix} y \\ \hat{x}_2 \end{bmatrix}$.