# Problem 1

## Solution

**Symmetric** $K(x, y) = \langle K(x, \cdot), K(y, \cdot)\rangle = \overline{\langle K(y, \cdot), K(x, \cdot)\rangle} = \overline{K(y, x)} = K(y, x)$.

**Positive semi-definite** For any $n \in \mathbb{N}^+$ and $x_1,\ x_2,\ \cdots,\ x_n \in X$, construct a matrix $M$ such that $M_{i,j} = K(x_i, x_j)$. For all $v \in \mathbb{R}^n$, we have $v^\intercal M v = \langle \sum_{i=1}^n v_i K(x_i, \cdot), \sum_{i=1}^n v_i K(x_i, \cdot)\rangle \geq 0$. Therefore, $M$ is positive semi-definite.

# Problem 2

## Solution

(a) For all $x,\ y \in X$, construct a matrix $M = \begin{bmatrix} K(x,x) & K(x,y) \\ K(x,y) & K(y,y) \end{bmatrix}$. Then, $M$ is positive semi-definite. Thus, $\det(M) = K(x, x)K(y, y) - K(x, y)^2 \geq 0$.

(b) (1) $\langle f, g\rangle = \sum_{i=1}^n \sum_{j=1}^m \alpha_i \beta_j K(x_i, x_j) = \sum_{i=1}^n \sum_{j=1}^m \alpha_i \beta_j K(x_j, x_i) = \langle g, f\rangle = \overline{\langle g, f\rangle}$; (2) For all $a, b \in \mathbb{R}$ and $f = \sum_{i=1}^n \alpha_i K(x_i, \cdot)$, $g = \sum_{i=1}^m \beta_i K(x_i, \cdot)$, $h = \sum_{i=1}^k \gamma_i K(x_i, \cdot)$, we have $\langle af + bg, h\rangle = a \sum_{i=1}^n \sum_{j=1}^k \alpha_i \gamma_j K(x_i, x_j) + b \sum_{i=1}^m \sum_{j=1}^k \beta_i \gamma_j K(x_i, x_j) = a\langle f, h\rangle + b\langle g, h\rangle$; (3) For all $f = \sum_{i=1}^n \alpha_i K(x_i, \cdot)$, construct matrix $M$ such that $M_{i,j} = K(x_i, x_j)$. Then, $\langle f, f\rangle = \alpha^\intercal M \alpha \geq 0$, and $\langle f, f\rangle = 0$ only if ???

# Problem 3

## Solution

Let $S$ be the subspace spanned by $K(x_1, \cdot), \cdots, K(x_n, \cdot)$. Then, for all $f$ we can decomose it into $f = f_s + v$ such that $f_s \in S$ and $v \perp f_s$. We have $||f||^2 = ||f_s||^2 + ||v||^2 \geq ||f_s||^2$ and $g(||f_s||) \leq ||f||^2$. We also have for all $x_i$, $f(x_i) = \langle f, K(x_i, \cdot)\rangle = \langle f_s, K(x_i, \cdot)\rangle + \langle v, K(x_i, \cdot)\rangle = \langle f_s, K(x_i, \cdot)\rangle = f_s(x_i)$. Thus, for a minimizer $f^*$, we must have $||v|| = 0$, and thus $v = 0$???