Problem 1

Let $R[x]_{\leq n}$ be the space of polynomials with real coefficients of degree at most n defined on the field of reals $\mathbb{R}$. Let $\mathcal{A} : R[x]_{\leq n} \to R[x]_{\leq n}$ be the derivative operator (e.g., $\mathcal{A}[2x^2+3x+1] = 4x+3$).

a) Show that $(R[x]_{\leq n}, \mathbb{R})$ is a vector space and $V = \{1, x, x^2, \ldots, x^n\}$ is a basis for it.

b) Show that $\mathcal{A}(\cdot)$ is a linear operator, and find the matrix representation of $\mathcal{A}$ in terms of basis V. That is, find a matrix A such that for every $f \in R[x]_{\leq n}$, $[\mathcal{A}(f)]_V = A[f]_V$, where $[g]_V \in \mathbb{R}^{n+1}$ is the representation of g with respect to the basis V.

Solution

a) There exists $\vartheta = 0(x)$, a constant function with value 0. Also $\forall \alpha, \beta \in \mathbb{R}, x_1, x_2 \in R[x]_{\leq n}$, we have $\alpha x_1 + \beta x_2$ which is also a polynomial with real coefficients of degree at most n. Therefore $(R[x]_{\leq n}, \mathbb{R})$ is a vector space.

$\sum\limits_{i=0}^{n} \alpha_i x^i = 0(x) \iff \forall i,\ \alpha_i = 0$, so $\{1, x, x^2, \ldots, x^n\}$ are linearly independent. Also for any polynomials in $R[x]_{\leq n}$, it can be represented by $\{1, x, x^2, \ldots, x^n\}$. Therefore, $\{1, x, x^2, \ldots, x^n\}$ is a basis.

b) Derivative is a linear operation, so $\forall \alpha, \beta \in \mathbb{R}, x_1, x_2 \in R[x]_{\leq n}$, $\mathcal{A}(\alpha x_1 + \beta x_2) = \alpha \mathcal{A}(x_1) + \beta \mathcal{A}(x_2)$. Therefore $\mathcal{A}$ is a linear operator. $\forall f(x) = \sum\limits_{i=0}^{n} \alpha_i x^i,\ \mathcal{A}(f(x)) = \sum\limits_{i=1}^{n} i \alpha_i x^{i-1}$, so

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 2 & \ldots & 0 \\ & & \ldots & & \\ 0 & 0 & 0 & \ldots & n \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}.$$

Problem 2

Suppose $\mathcal{A} : \mathcal{X} \to \mathcal{Y}$ is a linear operator. Show that $\dim(N(\mathcal{A})) + \dim(R(\mathcal{A})) = \dim(\mathcal{X})$. This is known as the *rank-nullity theorem*

Hint: Let $\{v_1, \ldots, v_k\}$ be a basis of $N(\mathcal{A})$. Show that this basis can be extended to a basis for $\mathcal{X}$ by adding additional independent vectors $\{v_{k+1}, \ldots, v_n\}$. Then show $A(v_{k+1}), \ldots, A(v_n)$ is a basis of $R(\mathcal{A})$.

Solution

Let the dimension of $\mathcal{X}$ be n and $\{v_1, \ldots, v_k\}$ be a basis of $N(\mathcal{A})$. Then this basis can be extended to a basis for $\mathcal{X}$ by adding additional independent vectors $\{v_{k+1}, \ldots, v_n\}$. $\forall x \in \mathcal{X}$, it can be represented by the basis, $x = \sum\limits_{i=1}^{n} \alpha_i v_i$. Then $\mathcal{A}(x) = \sum\limits_{i=1}^{n} \alpha_i \mathcal{A}(v_i) = \sum\limits_{i=k+1}^{n} \alpha_i \mathcal{A}(v_i)$. So $R(\mathcal{A})$ is the space spaned by $\{\mathcal{A}(v_{k+1}), \ldots, \mathcal{A}(v_n)\}$. Next we prove $\{\mathcal{A}(v_{k+1}), \ldots, \mathcal{A}(v_n)\}$ are linearly independent. Let $\sum\limits_{i=k+1}^{n} \beta_i \mathcal{A}(v_i) = \mathcal{A}(\sum\limits_{i=k+1}^{n} \beta_i v_i) = 0$, then $\sum\limits_{i=k+1}^{n} \beta_i v_i$ must be in $N(\mathcal{A})$ and can be represented by $\{v_1, \dots, v_k\}$. Let $\sum_{i=k+1}^{n} \beta_i v_i = \sum_{j=1}^{k} \gamma_j v_j$, because $\{v_1, \dots, v_n\}$ are linearly independent, we have $\beta_i = 0, \gamma_j = 0, \forall i, j$. So $\{\mathcal{A}(v_{k+1}), \dots, \mathcal{A}(v_n)\}$ are linearly independent. Therefore, $\dim(N(\mathcal{A})) + \dim(R(\mathcal{A})) = \dim(\mathcal{X})$.

Problem 3

As we discussed in class, every matrix can be put into Jordan form:

$$A = T \begin{bmatrix} J_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & J_k \end{bmatrix} T^{-1}$$

Here, each Jordan block J_i has the form:

$$\begin{bmatrix} \lambda_i & 1 & 0 & \dots & 0 \\ 0 & \lambda_i & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \lambda_i & 1 \\ 0 & 0 & 0 & 0 & \lambda_i \end{bmatrix}$$

a) Let J be a Jordan block of size n. Show that:

$$J^k = \begin{bmatrix} \binom{k}{0}\lambda^k & \binom{k}{1}\lambda^{k-1} & \binom{k}{2}\lambda^{k-2} & \cdots & \cdots & \binom{k}{n-1}\lambda^{k-(n-1)} \\ & \binom{k}{0}\lambda^k & \binom{k}{1}\lambda^{k-1} & \cdots & \cdots & \binom{k}{n-2}\lambda^{k-(n-2)} \\ & & \ddots & \ddots & \vdots & \vdots \\ & & & \ddots & \ddots & \vdots \\ & & & & \binom{k}{0}\lambda^k & \binom{k}{1}\lambda^{k-1} \\ & & & & & \binom{k}{0}\lambda^k \end{bmatrix}$$

Here, $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, and we use the convention that $(n-k)! = 0$ if $n - k < 0$.

Hint: You may use a proof by induction, and Pascal's rule may prove useful:

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}$$

b) If the Jordan block J_i is of size n, then for any analytic function f:

$$f(J_i) = \begin{bmatrix} f(\lambda_i) & f'(\lambda_i) & \frac{f''(\lambda_i)}{2} & \dots & \frac{f^{(n-1)}(\lambda_i)}{(n-1)!} \\ 0 & f(\lambda_i) & f'(\lambda_i) & \dots & \frac{f^{(n-2)}(\lambda_i)}{(n-2)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & f(\lambda_i) & f'(\lambda_i) \\ 0 & 0 & 0 & 0 & f(\lambda_i) \end{bmatrix}$$

Use the result from the previous problem to prove this.

Hint: Since f is analytic, we may write $f(s) = \sum_{k=0}^{\infty} \alpha_k s^k$. Using this, write out $f(J)$, and consider what the entries in $f(J)$ will be.

Note: One consequence of what you just proved is the following:

$$e^{J_i t} = \begin{bmatrix} e^{\lambda_i t} & te^{\lambda_i t} & \frac{t^2}{2}e^{\lambda_i t} & \cdots & \frac{t^{n-1}}{(n-1)!}e^{\lambda_i t} \\ 0 & e^{\lambda_i t} & te^{\lambda_i t} & \cdots & \frac{t^{n-2}}{(n-2)!}e^{\lambda_i t} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & e^{\lambda_i t} & te^{\lambda_i t} \\ 0 & 0 & 0 & 0 & e^{\lambda_i t} \end{bmatrix}$$

Thus, by answering parts (a) and (b), you have shown that:

$$e^{At} = T \begin{bmatrix} e^{J_1 t} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & e^{J_k t} \end{bmatrix} T^{-1}$$

Solution

a) Obviously, J^1 satisfies this form. Assume that J^k satisfies this form, we prove that J^{k+1} will also satisfy this form. We denote $A(i,j)$ as the element in the i-th row and j-th column of A.

$\forall 1 \le i < j \le n$, $J^{k+1}(i,j) = \sum_{h=1}^{n} J^{k+1}(i,h)J(h,j) = \binom{k}{j-i-1}\lambda^{k-(j-i-1)} + \binom{k}{j-i}\lambda^{k-(j-i)}\lambda = \binom{k+1}{j-i}\lambda^{k+1-(j-i)}$

$\forall 1 \le i = j \le n$, $J^{k+1}(i,j) = \sum_{h=1}^{n} J^{k+1}(i,h)J(h,j) = \lambda^{k+1}$

$\forall 1 \le j < i \le n$, $J^{k+1}(i,j) = \sum_{h=1}^{n} J^{k+1}(i,h)J(h,j) = 0$

So J^{k+1} also satisfies this form. Therefore, the proposition is true $\forall k \in \mathcal{Z}^+$

b) Since f is analytic, we may write $f(J) = \sum_{k=0}^{\infty} \alpha_k J^k$. We denote $A(i,j)$ as the element in the i-th row and j-th column of A.

$$\forall 1 \le i < j \le n, \quad f(J)(i,j) = \sum_{k=0}^{\infty} \alpha_k J^k(i,j) = \sum_{k=j-i}^{\infty} \alpha_k \binom{k}{j-i} \lambda^{k-(j-i)}$$
$$= \sum_{k=j-i}^{\infty} \frac{1}{(j-i)!}\alpha_k \frac{k!}{(n-(j-i))!}\lambda^{k-j-i} = \frac{f^{(j-i)}(\lambda)}{(j-i)!}$$

$$\forall 1 \le i = j \le n, \quad f(J)(i,j) = \sum_{k=0} \alpha_k J^k(i,j) = \sum_{k=0} \alpha_k \lambda^k = f(\lambda)$$

$$\forall 1 \leq j < i \leq n, \qquad f(J)(i,j) = \sum_{k=0}^{\infty} \alpha_k J^k(i,j) = 0$$

So $f(J)$ satisfies this form.

Problem 4

Suppose that A and Q are $n \times n$ matrices, and consider the matrix differential equation:

$$\dot{Z} = AZ + ZA^* \qquad Z(0) = Q \tag{1}$$

(a) Show using the product rule that the unique solution to (1) is given by:

$$Z(t) = e^{At} Q e^{A^* t}$$

(b) Show that if $e^{At} \to 0$ as $t \to \infty$, then

$$P = \lim_{t_f \to \infty} \int_0^{t_f} Z(t) dt$$

is a solution to the *Lyapunov equation*:

$$AP + PA^* + Q = 0$$

(**Hint:** Integrate both sides of (1) from 0 to t_f and use the fundamental theorem of calculus.)

Solution

(a) First, the solution satisfies the initial state, $Z(0) = e^{A0} Q e^{A^* 0} = Q$. Then we show that is satisfies the differential equation. $\dot{Z}(t) = (e^{At})' Q e^{A^* t} + e^{At} Q (e^{A^* t})' = A e^{At} Q e^{A^* t} + e^{At} Q e^{A^* t} A^* = AZ + ZA^*$.

(b) Because $e^{At} \to 0$ as $t \to \infty$, $e^{A^* t} \to 0$ and $Z(t) \to 0$ as $t \to \infty$. Integrating both sides of (1) from 0 to t_f and using the fundamental theorem of calculus, we have

$$Z(t_f) - Z(0) = A \int_0^{t_f} Z dt + \int_0^{t_f} Z dt A^*$$

Letting t_f go to ∞, we have $-Q = AP + PA^*$. Therefore P is a solution of $AP + PA^* + Q = 0$.