# Problem 1

Consider the linear, time-varying system:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

$$y(t) = C(t)u(t)$$

Recall the definition of the observability Grammian:

$$H(t_1, t_0) = \int_{t_0}^{t_1} \phi^\intercal(\tau, t_0) C^\intercal(\tau) C(\tau) \phi(\tau, t_0) d\tau$$

Consider the function from $\mathcal{R}$ to $\mathcal{R}^{n\times n}$:

$$X : t_0 \mapsto H(t_1, t_0)$$

(Two other ways to write this are: $X(t_0) = H(t_1, t_0)$, or $X(\cdot) = H(t_1, \cdot)$.)

Show that the function $X$ satisfies the linear matrix differential equation:

$$\dot{X}(t) = -A^\intercal(t)X(t) - X(t)A(t) - C^\intercal(t)C(t) \qquad X(t_1) = 0_{n\times n}$$

Here, the initial condition $0_{n\times n}$ is the zero matrix in $\mathcal{R}^{n\times n}$.

## Solution

Denoting

$$X(t) = \int_{t}^{t_1} \phi^T(\tau, t) C^T(\tau) C(\tau) \phi(\tau, t) d\tau = \int_{t}^{t_1} f(t, \tau) d\tau$$

Using the Leibniz rule:

$$\begin{aligned}\dot{X}(t) &= -f(t,t) + \int_t^{t_1} \frac{\partial f(t,\tau)}{\partial t} d\tau \\ &= -IC^\intercal(t)C(t)I + \int_t^{t_1} \left[-A(t)^\intercal \phi^\intercal(\tau,t) C^\intercal(\tau) C(\tau)\phi(\tau,t) + \phi^\intercal(\tau,t)C^\intercal(\tau)C(\tau)\phi(\tau,t)(-A(t))\right] d\tau \\ &= -A^\intercal(t)X(t) - X(t)A(t) - C^\intercal(t)C(t) \qquad (1)\end{aligned}$$

and we have

$$X(t_1) = \int_{t_1}^{t_1} f(t,\tau) d\tau = 0_{n\times n}$$

# Problem 2

Consider a linear time-varying system with dynamics:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

$$y(t) = C(t)x(t)$$

Let's call this system $R$.

As we've covered before, the dual system is given by the dynamics:

$$\dot{\tilde{x}}(t) = -A^{\intercal}(t)\tilde{x}(t) - C^{\intercal}(t)\tilde{u}(t)$$

$$\tilde{y}(t) = B^{\intercal}(t)\tilde{x}(t)$$

Let's call this dual system $\tilde{R}$.

Consider any state $x_0$ that is controllable to zero on $[t_0, t_1]$ for $R$, and any state $\tilde{x}_0$ that is unobservable on $[t_0, t_1]$ for $\tilde{R}$. Show that $x_0$ and $\tilde{x}_0$ are orthogonal, i.e. $\langle x_0, \tilde{x}_0 \rangle = 0$.

## Solution

Because $x_0$ is controllable to zero, there exists a $u(t)$ such that $x_0 = \int_{t_0}^{t_1} \phi(t_0, \tau)B(\tau)u(\tau)d\tau$. Because $\tilde{x}_0$ is unobservable, $B^{\intercal}(t)\phi^{\intercal}(t_0, t)\tilde{x}_0 = 0$, $\forall t \in [t_0, t_1]$. Therefore, $\langle x_0, \tilde{x}_0 \rangle = x_0^{\intercal}\tilde{x}_0 = \int_{t_0}^{t_1} u^{\intercal}(\tau)B^{\intercal}(\tau)\phi^{\intercal}(t_0, \tau)\tilde{x}_0 d\tau = 0$.

# Problem 3

Consider:

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Is the system controllable? If not, put it in Kalman controllability canonical form.
- Is the system observable? If not, put it in Kalman observability canonical form.

## Solution

- Controallability matrix $\mathcal{C} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$. Since $\text{rank}(\mathcal{C}) = 1$, it is uncontrollable. Let $P^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$. Then, $\bar{A} = PAP^{-1} = \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix}$ $\bar{B} = PB = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\bar{C} = CP^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$
- Observablity matrix $\mathcal{O} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$. Since $\text{rank}(\mathcal{O}) = 2$, it is observable.

# Problem 4

Take the transfer function:

$$H(s) = \frac{s+3}{(s+1)(s+2)}$$

- Put this system into controllable canonical form.
- Using static linear state feedback ($u = -Kx$), find a $K$ that places the poles at $-5 \pm 2j$.

## Solution

- $H(s) = \frac{s+3}{s^2+3s+2}$. CCF: $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.
- Let $K = [k_1 \ k_2]$, then $A - BK = \begin{bmatrix} 0 & 1 \\ -(2+k_1) & -(3+k_2) \end{bmatrix}$ and the closed-loop characteristic polynomial is $\Delta(s) = s^2 + (2+k_1)s + (3+k_2)$. Then substitute the roots, and we have $K = [-12 \ 24]$.

# Problem 5

Consider the following dynamical system, inspired by a linearization of the pendubot from Chapter 1 in the reader.

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 5 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ -5 & 6 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 2 \\ 0 \\ -3 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x$$

Design a reduced-order Luenberger observer for this system. You may freely use MATLAB or any other computer assistance to do so (and are encouraged to do so!), but still show your work.

## Solution

Type your solution here