Problem 1

Consider the nonlinear differential equation:

$$\ddot{y} = 2y - (y^2 + 1)(\dot{y} + 1) + u$$

a) Obtain a non-linear state-space representation.

b) Linearize this system of equations around its equilibrium output trajectory when $u(\cdot) \equiv 0$, and write it in state-space form.

Solution

(a) Let $x_1 = y,\ x_2 = \dot{y}$, then

$$\begin{aligned}\dot{x_1} &= x_2\\ \dot{x_2} &= 2x_1 - (x_1^2 + 1)(x_2 + 1) + u\\ y &= x_1\end{aligned}$$

(b) Letting $\dot{x_{e1}} = 0,\ \dot{x_{e2}} = 0, u = 0$, we have $x_{e1} = 1,\ x_{e2} = 0$. Linerize the system around the nominal trajectory $x^n(t) = [1, 0]^T$. Letting $x_1 = x_{e1} + \delta x_1,\ x_2 = x_{e2} + \delta x_2$, we have

$$\begin{aligned}\dot{x_1} &= \delta\dot{x_1} = x_2 = x_{e2} + \delta x_2 = \delta x_2\\ \dot{x_2} &= \delta\dot{x_2} = f(x_{e1} + \delta x_1, x_{e2} + \delta x_2, 0) = f(x_{e1}, x_{e2}, 0) + \frac{\partial f}{x_1}\delta x_1 + \frac{\partial f}{x_2}\delta x_2 = -2\delta x_2\end{aligned}.$$

The state space equation is

$$\begin{bmatrix}\delta\dot{x_1}\\ \delta\dot{x_2}\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & -2\end{bmatrix}\begin{bmatrix}\delta x_1\\ \delta x_2\end{bmatrix}$$

Problem 2

Suppose $A \in \mathbb{R}^{n\times n}$ and $D \in \mathbb{R}^{m\times m}$ are square matrices. Suppose A and D have all distinct eigenvalues. (That is, the eigenvalues of A are from both different from each other *and* the eigenvalues of D, and similarly for D.) Prove that the eigenvalues of M are the union of the eigenvalues of A and D, where:

$$M = \begin{bmatrix}A & B\\ 0_{m\times n} & D\end{bmatrix}$$

Here, $0_{m\times n} \in \mathbb{R}^{m\times n}$ is the matrix of all zeros, and $B \in \mathbb{R}^{n\times m}$ is an arbitrary matrix.

Hint: Use the eigenvectors of A and D to construct the eigenvectors of M. Note that $(sI - A)$ is invertible for any s that is *not* an eigenvalue of A.

Note: This is actually true for any A and D, but is easier to show for the distinct eigenvalue case.

Solution

Let v_i be the i-th eigenvector of A corresponding to the eigenvalue λ_i and $u_i = \begin{bmatrix} v_i \\ 0_{m\times 1} \end{bmatrix}$. Then $Mu_i = \lambda_i u_i$, which means u_i is an eigenvector of M and λ_i is an eigenvalue of M. Therefore, all the n eigenvalues of A are eigenvalues of M.
Let v_i be the i-th eigenvector of D corresponding to the eigenvalue λ_i and $C = \lambda_i I - A$. Then because λ_i is not an eigenvalue of A, C^{-1} exists. Let $u_i = \begin{bmatrix} C^{-1}Bv_i \\ v_i \end{bmatrix}$. Then $Mu_i = \lambda_i u_i$, which means u_i is an eigenvector of M and λ_i is an eigenvalue of M. Therefore, all the m eigenvalues of D are eigenvalues of M.
M has at most $n+m$ eigenvalues, so eigenvalues of M is exactly the union of the eigenvalues of A and D.

Problem 3

Consider:

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

Suppose D is invertible. Show that $\det(M) = \det(D)\det(A - BD^{-1}C)$. (This is known as the Schur complement; note how this generalizes the 2×2 equation for the determinant: $ad - bc$.)

Hint: You may use the previous problem, and you may take it for granted that $\det(AB) = \det(A)\det(B)$. (In abstract algebra terms, this means that the determinant is a group homomorphism.) Try to break down M into the product of two triangular matrices, one with determinant $\det(D)$ and one with determinant $\det\left(A - BD^{-1}C\right)$.

Solution

Since

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A - BD^{-1}C & BD^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ C & D \end{bmatrix},$$

we have $\det(M) = \det(\begin{bmatrix} A - BD^{-1}C & BD^{-1} \\ 0 & I \end{bmatrix})\det(\begin{bmatrix} I & 0 \\ C & D \end{bmatrix}) = \det(D)\det(A - BD^{-1}C)$

Problem 4

Consider the linear system:

$$\dot{x} = Ax + Bu \qquad\qquad y = Cx \tag{1}$$

For any time $T > 0$, we can view this system as a mapping $L : (x_0, (u(t))_{0\le t\le T}) \mapsto (x_f, (y(t))_{0\le t\le T})$. That is, L takes initial conditions $x(0) = x_0$ and functions $u(\cdot)$ as an input, and it outputs final states $x(T) = x_f$ and functions $y(\cdot)$, according to the differential equation (1). Let $\mathcal{U}$ denote the set of piecewise continuous, square-integrable functions from $[0, T]$ to $\mathbb{R}^{n_i}$, and similarly $\mathcal{Y}$ denote the set of piecewise continuous, square-integrable functions from $[0, T]$ to $\mathbb{R}^{n_o}$. So, $L : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^n \times \mathcal{Y}$.

The *dual system* is given by:

$$-\dot{\tilde{x}} = A^\intercal \tilde{x} + C^\intercal \tilde{u} \qquad\qquad \tilde{y} = B^\intercal \tilde{x}$$

Here, $\tilde{u} \in \tilde{\mathcal{U}} = \mathcal{Y}$ and $\tilde{y} \in \tilde{\mathcal{Y}} = \mathcal{U}$. Note the minus sign on the state dynamics; we'll actually think of the dual system moving *backward* in time. Define $L^* : (\tilde{x}_f, (\tilde{u}(t))_{0\le t\le T}) \mapsto (\tilde{x}_0, (\tilde{y}(t))_{0\le t\le T})$, mapping *final* states $\tilde{x}_f$ and dual inputs $\tilde{u}$ to initial states $\tilde{x}_0$ and dual outputs $\tilde{y}$. Note that $L^* : \mathbb{R}^n \times \tilde{\mathcal{U}} \to \mathbb{R}^n \times \tilde{\mathcal{Y}}$.

Define the inner product on $\mathbb{R}^n \times \mathcal{U}$ (which is also $\mathbb{R}^n \times \tilde{\mathcal{Y}}$) as:

$$\langle (x_0, u(\cdot)), (x_0', u'(\cdot))\rangle = x_0^\intercal x_0' + \int_0^T u(t)^\intercal u'(t)dt$$

Define the inner product on $\mathbb{R}^n \times \mathcal{Y}$ similarly. For this problem, show that L^* is the adjoint of L. (This is sometimes called the *pairing lemma*.)

Hint: Consider $\frac{d}{dt}\langle x, \tilde{x}\rangle$, and integrate on $[0, T]$.

Solution

Define $\langle x, \tilde{x}\rangle := x^T\tilde{x}$, then we have

$$\begin{aligned}\frac{d}{dt}\langle x, \tilde{x}\rangle = \langle \dot{x}, \tilde{x}\rangle + \langle x, \dot{\tilde{x}}\rangle = \langle Ax+Bu, \tilde{x}\rangle - \langle x, A^T\tilde{x}+C^T\tilde{u}\rangle = \langle Ax, \tilde{x}\rangle + \langle Bu, \tilde{x}\rangle - \langle x, A^T\tilde{x}\rangle - \langle x, C^T\tilde{u}\rangle \\ = \langle Bu, \tilde{x}\rangle - \langle x, C^T\tilde{u}\rangle = u^TB^T\tilde{x} - x^TC^T\tilde{u} = u^T\tilde{y} - y^T\tilde{u}\end{aligned}$$

From the fundamental theorem of calculus, we hace the integral equation

$$\int_0^T \frac{d}{dt}\langle x, \tilde{x}\rangle = \langle x(T), \tilde{x}(T)\rangle - \langle x(0), \tilde{x}(0)\rangle = x_f^T\tilde{x}_f - x_0^T\tilde{x}_0 = \int_0^T u^T\tilde{y}dt - \int_0^T y^T\tilde{u}dt$$

Therefore,

$$x_f^T\tilde{x}_f + \int_0^T y^T\tilde{u}dt = x_0^T\tilde{x}_0 + \int_0^T u^T\tilde{y}dt,$$

which means

$$\langle L(x_0, u_0), (\tilde{x}_f, \tilde{u})\rangle = \langle (x_0, u_0), L^*(\tilde{x}_f, \tilde{u})\rangle,$$

and L and L^* are adjoint.

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