Linear and Quadratic Least Squares Approach

DR. SUNEESH JACOB

FACULTY HANDLING EIE417

Linear Least Squares

- $y = a_0 + a_1 x$
  - $y_1 = a_0 + a_1 x_1$
  - $y_2 = a_0 + a_1 x_2$
  - $y_3 = a_0 + a_1 x_3$
  - $y_4 = a_0 + a_1 x_4$
  - $y_5 = a_0 + a_1 x_5$
- Error $e = a_0 + a_1 x - y$
  - $e_1 = a_0 + a_1 x_1 - y_1$
  - $e_2 = a_0 + a_1 x_2 - y_2$
  - $e_3 = a_0 + a_1 x_3 - y_3$
  - $e_4 = a_0 + a_1 x_4 - y_4$
  - $e_5 = a_0 + a_1 x_5 - y_5$

S. No.	Input	Output
1	x_1	y_1
2	x_2	y_2
3	x_3	y_3
4	x_4	y_4
5	x_5	y_5

$$E = e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_5^2$$

$$\Rightarrow E(a_0, a_1) = (a_0 + a_1 x_1 - y_1)^2 + (a_0 + a_1 x_2 - y_2)^2 + (a_0 + a_1 x_3 - y_3)^2 + (a_0 + a_1 x_4 - y_4)^2 + (a_0 + a_1 x_5 - y_5)^2$$

$$E(a_0, a_1) = (a_0 + a_1x_1 - y_1)^2 + (a_0 + a_1x_2 - y_2)^2 + (a_0 + a_1x_3 - y_3)^2 + (a_0 + a_1x_4 - y_4)^2 + (a_0 + a_1x_5 - y_5)^2$$

$$\frac{\partial E}{\partial a_0} = 2(a_0 + a_1x_1 - y_1) + 2(a_0 + a_1x_2 - y_2) + 2(a_0 + a_1x_3 - y_3) + 2(a_0 + a_1x_4 - y_4) + 2(a_0 + a_1x_5 - y_5)$$

$$\Rightarrow \frac{\partial E}{\partial a_0} = \sum_{i=1}^{5} 2(a_0 + a_1x_i - y_i) = 2{\times}5a_0 + 2a_1 \sum_{i=1}^{5} x_i - 2\sum_{i=1}^{5} y_i$$

$$\frac{\partial E}{\partial a_1} = 2(a_0 + a_1x_1 - y_1)x_1 + 2(a_0 + a_1x_2 - y_2)x_2 + 2(a_0 + a_1x_3 - y_3)x_3 + 2(a_0 + a_1x_4 - y_4)x_4 + 2(a_0 + a_1x_5 - y_5)x_5$$

$$\Rightarrow \frac{\partial E}{\partial a_1} = \sum_{i=1}^{5} 2(a_0 + a_1x_i - y_i)x_i = 2a_0 \sum_{i=1}^{5} x_i + 2a_1 \sum_{i=1}^{5} x_i^2 - 2\sum_{i=1}^{5} x_iy_i$$

$$\begin{Bmatrix} \frac{\partial E}{\partial a_0} \\ \frac{\partial E}{\partial a_1} \end{Bmatrix} = 0 \Rightarrow \begin{Bmatrix} 2{\times}5a_0 + 2a_1 \sum_{i=1}^{5} x_i - 2\sum_{i=1}^{5} y_i \\ 2a_0 \sum_{i=1}^{5} x_i + 2a_1 \sum_{i=1}^{5} x_i^2 - 2\sum_{i=1}^{5} x_iy_i \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} \frac{\partial E}{\partial a_0} \\ \frac{\partial E}{\partial a_1} \end{Bmatrix} = 0 \Rightarrow \begin{Bmatrix} 5a_0 + a_1 \sum_{i=1}^{5} x_i - \sum_{i=1}^{5} y_i \\ a_0 \sum_{i=1}^{5} x_i + a_1 \sum_{i=1}^{5} x_i^2 - \sum_{i=1}^{5} x_i y_i \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 5 & \sum_{i=1}^{5} x_i \\ \sum_{i=1}^{5} x_i & \sum_{i=1}^{5} x_i^2 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = \begin{Bmatrix} \sum_{i=1}^{5} y_i \\ \sum_{i=1}^{5} x_i y_i \end{Bmatrix} \Rightarrow \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = \begin{bmatrix} 5 & \sum_{i=1}^{5} x_i \\ \sum_{i=1}^{5} x_i & \sum_{i=1}^{5} x_i^2 \end{bmatrix}^{-1} \begin{Bmatrix} \sum_{i=1}^{5} y_i \\ \sum_{i=1}^{5} x_i y_i \end{Bmatrix}$$

For n data points:

$$\begin{Bmatrix} \frac{\partial E}{\partial a_0} \\ \frac{\partial E}{\partial a_1} \end{Bmatrix} = 0 \Rightarrow \begin{Bmatrix} na_0 + a_1 \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} y_i \\ a_0 \sum_{i=1}^{5} x_i + a_1 \sum_{i=1}^{5} x_i^2 - \sum_{i=1}^{5} x_i y_i \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} n & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = \begin{Bmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_i y_i \end{Bmatrix} \Rightarrow \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = \begin{bmatrix} n & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \end{bmatrix}^{-1} \begin{Bmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_i y_i \end{Bmatrix}$$

Quadratic Least Squares

- $y = a_0 + a_1 x + a_2 x^2$
  - $y_1 = a_0 + a_1 x_1 + a_2 x_1^2$
  - $y_2 = a_0 + a_1 x_2 + a_2 x_2^2$
  - $y_3 = a_0 + a_1 x_3 + a_2 x_3^2$
  - $y_4 = a_0 + a_1 x_4 + a_2 x_4^2$
  - $y_5 = a_0 + a_1 x_5 + a_2 x_5^2$
- Error $e = a_0 + a_1 x + a_2 x^2 - y$
  - $e_1 = a_0 + a_1 x_1 + a_2 x_1^2 - y_1$
  - $e_2 = a_0 + a_1 x_2 + a_2 x_2^2 - y_2$
  - $e_3 = a_0 + a_1 x_3 + a_2 x_3^2 - y_3$
  - $e_4 = a_0 + a_1 x_4 + a_2 x_4^2 - y_4$
  - $e_5 = a_0 + a_1 x_5 + a_2 x_5^2 - y_5$

S. No.	Input	Output
1	x_1	y_1
2	x_2	y_2
3	x_3	y_3
4	x_4	y_4
5	x_5	y_5

$$E = e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_5^2$$

$$\Rightarrow E(a_0, a_1, a_2) = (a_0 + a_1 x_1 + a_2 x_1^2 - y_1)^2 + (a_0 + a_1 x_2 + a_2 x_2^2 - y_2)^2 + (a_0 + a_1 x_3 + a_2 x_3^2 - y_3)^2 + (a_0 + a_1 x_4 + a_2 x_4^2 - y_4)^2 + (a_0 + a_1 x_5 + a_2 x_5^2 - y_5)^2$$

$$E(a_0,a_1,a_2)=(a_0+a_1x_1+a_2x_1^2-y_1)^2+(a_0+a_1x_2+a_2x_2^2-y_2)^2+(a_0+a_1x_3+a_2x_3^2-y_3)^2+(a_0+a_1x_4+a_2x_4^2-y_4)^2+\left(a_0+a_1x_5+a_2x_5^2-y_5\right)^2$$

$$\frac{\partial E}{\partial a_0}=2(a_0+a_1x_1+a_2x_1^2-y_1)+2(a_0+a_1x_2+a_2x_2^2-y_2)+2(a_0+a_1x_3+a_2x_3^2-y_3)+2(a_0+a_1x_4+a_2x_4^2-y_4)+2\left(a_0+a_1x_5+a_2x_5^2-y_5\right)$$

$$\Rightarrow\frac{\partial E}{\partial a_0}=\sum_{i=1}^{5}2\left(a_0+a_1x_i+a_2x_i^2-y_i\right)=2\times5a_0+2a_1\sum_{i=1}^{5}x_i+2a_2\sum_{i=1}^{5}x_i^2-2\sum_{i=1}^{5}y_i$$

$$\frac{\partial E}{\partial a_1}=2(a_0+a_1x_1+a_2x_1^2-y_1)x_1+2(a_0+a_1x_2+a_2x_2^2-y_2)x_2+2(a_0+a_1x_3+a_2x_3^2-y_3)x_3+2(a_0+a_1x_4+a_2x_4^2-y_4)x_4+2\left(a_0+a_1x_5+a_2x_5^2-y_5\right)x_5$$

$$\Rightarrow\frac{\partial E}{\partial a_1}=\sum_{i=1}^{5}2\left(a_0+a_1x_i+a_2x_i^2-y_i\right)x_i=2a_0\sum_{i=1}^{5}x_i+2a_1\sum_{i=1}^{5}x_i^2+2a_2\sum_{i=1}^{5}x_i^3-2\sum_{i=1}^{5}x_iy_i$$

$$\frac{\partial E}{\partial a_2}=2(a_0+a_1x_1+a_2x_1^2-y_1)x_1^2+2(a_0+a_1x_2+a_2x_2^2-y_2)x_2^2+2(a_0+a_1x_3+a_2x_3^2-y_3)x_3^2+2(a_0+a_1x_4+a_2x_4^2-y_4)x_4^2+2\left(a_0+a_1x_5+a_2x_5^2-y_5\right)x_5^2$$

$$\Rightarrow\frac{\partial E}{\partial a_2}=\sum_{i=1}^{5}2\left(a_0+a_1x_i+a_2x_i^2-y_i\right)x_i^2=2a_0\sum_{i=1}^{5}x_i^2+2a_1\sum_{i=1}^{5}x_i^3+2a_2\sum_{i=1}^{5}x_i^4-2\sum_{i=1}^{5}x_i^2y_i$$

$$\begin{Bmatrix}\frac{\partial E}{\partial a_0}\\ \frac{\partial E}{\partial a_1}\\ \frac{\partial E}{\partial a_2}\end{Bmatrix}=0\Rightarrow\begin{Bmatrix}2\times5a_0+2a_1\sum_{i=1}^{5}x_i+2a_2\sum_{i=1}^{5}x_i^2-2\sum_{i=1}^{5}y_i\\ 2a_0\sum_{i=1}^{5}x_i+2a_1\sum_{i=1}^{5}x_i^2+2a_2\sum_{i=1}^{5}x_i^3-2\sum_{i=1}^{5}x_iy_i\\ 2a_0\sum_{i=1}^{5}x_i^2+2a_1\sum_{i=1}^{5}x_i^3+2a_2\sum_{i=1}^{5}x_i^4-2\sum_{i=1}^{5}x_i^2y_i\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\end{Bmatrix}$$

$$\begin{Bmatrix} \frac{\partial E}{\partial a_0} \\ \frac{\partial E}{\partial a_1} \\ \frac{\partial E}{\partial a_2} \end{Bmatrix} = 0 \Rightarrow \begin{Bmatrix} 5a_0 + a_1 \sum_{i=1}^{5} x_i + a_2 \sum_{i=1}^{5} x_i^2 - \sum_{i=1}^{5} y_i \\ a_0 \sum_{i=1}^{5} x_i + a_1 \sum_{i=1}^{5} x_i^2 + a_2 \sum_{i=1}^{5} x_i^3 - \sum_{i=1}^{5} x_i y_i \\ a_0 \sum_{i=1}^{5} x_i^2 + a_1 \sum_{i=1}^{5} x_i^3 + a_2 \sum_{i=1}^{5} x_i^4 - \sum_{i=1}^{5} x_i^2 y_i \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 5 & \sum_{i=1}^{5} x_i & \sum_{i=1}^{5} x_i^2 \\ \sum_{i=1}^{5} x_i & \sum_{i=1}^{5} x_i^2 & \sum_{i=1}^{5} x_i^3 \\ \sum_{i=1}^{5} x_i^2 & \sum_{i=1}^{5} x_i^3 & \sum_{i=1}^{5} x_i^4 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} \sum_{i=1}^{5} y_i \\ \sum_{i=1}^{5} x_i y_i \\ \sum_{i=1}^{5} x_i^2 y_i \end{Bmatrix} \Rightarrow \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} = \begin{bmatrix} 5 & \sum_{i=1}^{5} x_i & \sum_{i=1}^{5} x_i^2 \\ \sum_{i=1}^{5} x_i & \sum_{i=1}^{5} x_i^2 & \sum_{i=1}^{5} x_i^3 \\ \sum_{i=1}^{5} x_i^2 & \sum_{i=1}^{5} x_i^3 & \sum_{i=1}^{5} x_i^4 \end{bmatrix}^{-1} \begin{Bmatrix} \sum_{i=1}^{5} y_i \\ \sum_{i=1}^{5} x_i y_i \\ \sum_{i=1}^{5} x_i^2 y_i \end{Bmatrix}$$

For n data points:

$$\begin{Bmatrix} \frac{\partial E}{\partial a_0} \\ \frac{\partial E}{\partial a_1} \\ \frac{\partial E}{\partial a_2} \end{Bmatrix} = 0 \Rightarrow \begin{Bmatrix} na_0 + a_1 \sum_{i=1}^{n} x_i + a_2 \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} y_i \\ a_0 \sum_{i=1}^{n} x_i + a_1 \sum_{i=1}^{n} x_i^2 + a_2 \sum_{i=1}^{n} x_i^3 - \sum_{i=1}^{n} x_i y_i \\ a_0 \sum_{i=1}^{n} x_i^2 + a_1 \sum_{i=1}^{n} x_i^3 + a_2 \sum_{i=1}^{n} x_i^4 - \sum_{i=1}^{n} x_i^2 y_i \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} n & \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i^3 \\ \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i^3 & \sum_{i=1}^{n} x_i^4 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_i y_i \\ \sum_{i=1}^{n} x_i^2 y_i \end{Bmatrix} \Rightarrow \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} = \begin{bmatrix} n & \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i^3 \\ \sum_{i=1}^{n} x_i^2 & \sum_{i=1}^{n} x_i^3 & \sum_{i=1}^{n} x_i^4 \end{bmatrix}^{-1} \begin{Bmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_i y_i \\ \sum_{i=1}^{n} x_i^2 y_i \end{Bmatrix}$$