Support Vector Machine

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Problem

For the given dataset below, construct a linear hyperplane in Support Vector Machine

S.No	x_1	x_2	y
1	4	1	-1
2	2	4	-1
3	2	3	-1
4	3	6	-1
5	4	4	-1
6	9	10	1
7	6	8	1
8	9	5	1
9	8	7	1
10	10	8	1

Solution

We need to find a boundary hyperplane such that it separates the points into two classes with maximum margin. Let us assume hard margin here, as the points from Figure 1 can be observed to be easily separable. Since there is a free parameter, we also do the scaling. The constrained optimisation problem is

Maximise the margin

$$r$$

subject to

$$y^{(i)}\left(\boldsymbol{w}^{\boldsymbol{T}}\boldsymbol{x}+b\right) \geq r$$

$$\|\boldsymbol{w}\| = 1$$

This is equivalent to scaling the data, such that the margin is unity. The equivalent optimisation problem would be

Minimise

$$\frac{1}{2}\|\boldsymbol{w}\|^2$$

subject to

$$-y^{(i)}\left(w_1 x_1^{(i)} + w_2 x_2^{(i)} + b\right) + 1 \leq 0 \quad \text{for all } i$$

For our problem, it would be

Maximise

$$\frac{1}{2}\left(w_1^2 + w_2^2\right)$$

subject to

$$4w_1 + w_2 + b + 1 \leq 0$$

$$2w_1 + 4w_2 + b + 1 \leq 0$$

$$2w_1 + 3w_2 + b + 1 \leq 0$$

$$3w_1 + 6w_2 + b + 1 \leq 0$$

$$4w_1 + 4w_2 + b + 1 \leq 0$$

$$-(9w_1 + 10w_2 + b) + 1 \leq 0$$

$$-(6w_1 + 8w_2 + b) + 1 \leq 0$$

$$-(9w_1 + 5w_2 + b) + 1 \leq 0$$

$$-(8w_1 + 7w_2 + b) + 1 \leq 0$$

$$-(10w_1 + 8w_2 + b) + 1 \leq 0$$

These are inequality constraints, which could be either active constraints or inactive constraints in each case. Since we have 10 constraints, each having two possibilities (active or inactive), there would be 2^{10} cases. But the support vectors show us the active constraints (and the inactive constraints), and hence if we could know the support vectors, we would directly know the active constraints and the inactive constraints. From Figure 1, we can figure out – by inspection – that (3,6) and (6,8) are support vectors.

Hence, the reduced constrained optimisation problem with equality constraints alone, would be

Maximise

$$\frac{1}{2}\left(w_1^2 + w_2^2\right)$$

subject to

$$3w_1 + 6w_2 + b + 1 \leq 0$$

$$-(6w_1 + 8w_2 + b) + 1 \leq 0$$

provided that the less-than-or-equal-to directions are maintained, which can be ensured by the non-negativity of Lagrangian multipliers in the Lagrangian.

The Lagrangian of the problem would be

$$\mathcal{L}(w_1, w_2, b, \lambda_1, \lambda_2) = \frac{1}{2}\left(w_1^2 + w_2^2\right) + \lambda_1\left(3w_1 + 6w_2 + b + 1\right) + \lambda_2\left(-6w_1 - 8w_2 - b + 1\right)$$

The gradient of the Lagrangian would be

$$\nabla\mathcal{L} = \begin{Bmatrix} \frac{\partial\mathcal{L}}{\partial w_1} \\ \frac{\partial\mathcal{L}}{\partial w_2} \\ \frac{\partial\mathcal{L}}{\partial b} \\ \frac{\partial\mathcal{L}}{\partial\lambda_1} \\ \frac{\partial\mathcal{L}}{\partial\lambda_2} \end{Bmatrix} = \begin{Bmatrix} w_1 + 3\lambda_1 - 6\lambda_2 \\ w_2 + 6\lambda_1 - 8\lambda_2 \\ \lambda_1 - \lambda_2 \\ 3w_1 + 6w_2 + b + 1 \\ -6w_1 - 8w_2 - b + 1 \end{Bmatrix}$$

Setting the gradient of the Lagrangian to zero implies

$$\begin{Bmatrix} w_1 + 3\lambda_1 - 6\lambda_2 \\ w_2 + 6\lambda_1 - 8\lambda_2 \\ \lambda_1 - \lambda_2 \\ 3w_1 + 6w_2 + b + 1 \\ -6w_1 - 8w_2 - b + 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 3 & -6 \\ 0 & 1 & 0 & 6 & -8 \\ 0 & 0 & 0 & 1 & -1 \\ 3 & 6 & 1 & 0 & 0 \\ -6 & -8 & -1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} w_1 \\ w_2 \\ b \\ \lambda_1 \\ \lambda_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ -1 \end{Bmatrix}$$

By solving the above system of equations, we get

$$\begin{Bmatrix} w_1 \\ w_2 \\ b \\ \lambda_1 \\ \lambda_2 \end{Bmatrix} = \begin{Bmatrix} \frac{6}{13} \\ \frac{4}{13} \\ \frac{-55}{13} \\ \frac{2}{13} \\ \frac{2}{13} \end{Bmatrix}$$

Since we have positive values for Lagrange multipliers, the solution is valid and the weights and bias values are $w_1 = \frac{6}{13}, w_2 = \frac{4}{13}$ and $b = -\frac{55}{13}$.

Hence, the boundary line equation is

$$\frac{6}{13}x_1 + \frac{4}{13}x_2 - \frac{55}{13}x_2 = 0$$

$$\Rightarrow 6x_1 + 4x_2 - 55 = 0$$

A plot of the boundary line is shown in Figure 2.

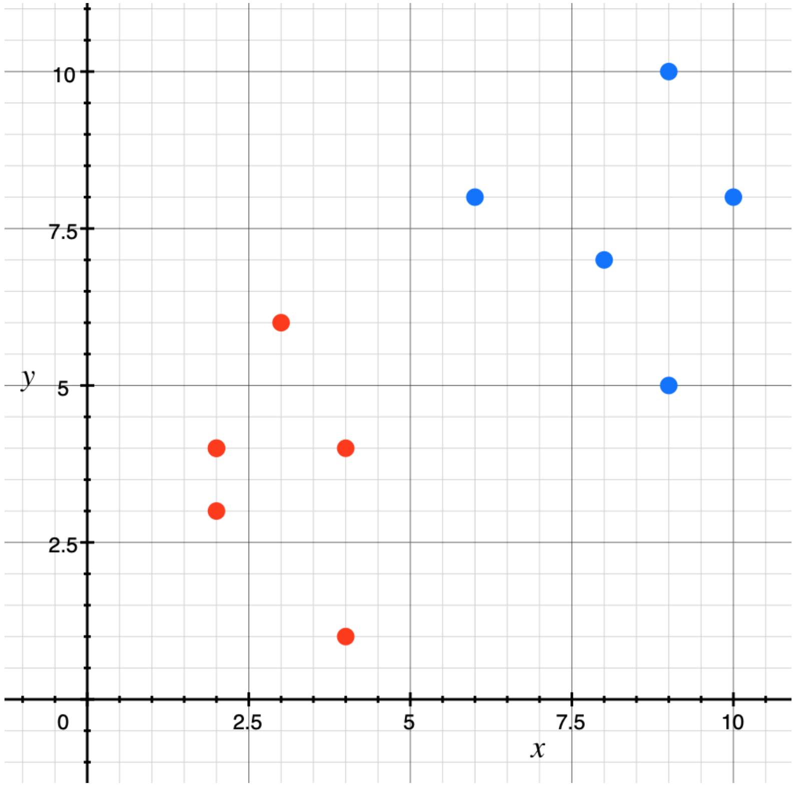

Figure 1: Points

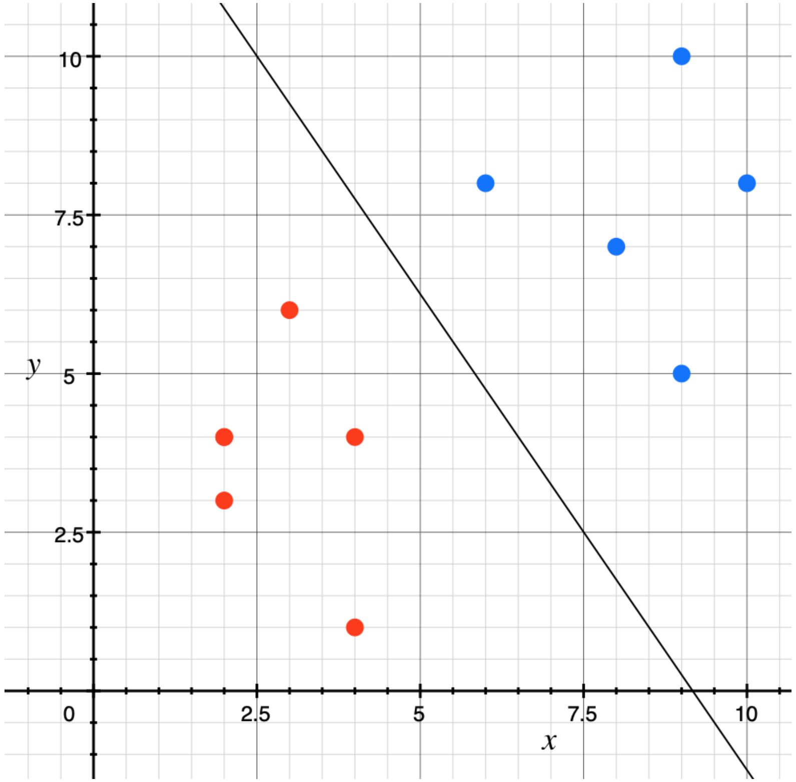

Figure 2: Points