From the figure,

$$T = \begin{array}{c|cc} & \text{Hot} & \text{Cold} \\ \text{Hot} & 0.7 & 0.3 \\ \text{Cold} & 0.4 & 0.6 \end{array} = \begin{bmatrix} P(H|H) & P(C|H) \\ P(H|C) & P(C|C) \end{bmatrix}$$

$$E = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \text{Hot} & 0.2 & 0.4 & 0.4 \\ \text{Cold} & 0.5 & 0.4 & 0.1 \end{array} = \begin{bmatrix} P(1|H) & P(2|H) & P(3|H) \\ P(1|C) & P(2|C) & P(3|C) \end{bmatrix}$$

$$\pi = \begin{array}{cc} \text{Hot} & \text{Cold} \\ 0.8 & 0.2 \end{array} = \begin{bmatrix} P(H) & P(C) \end{bmatrix}$$

Observation = [3, 1, 3]

Hidden states - possibilities:

For three days, the possible sequences are

(i) HHH

(ii) HHC

(iii) HCH

(iv) HCC

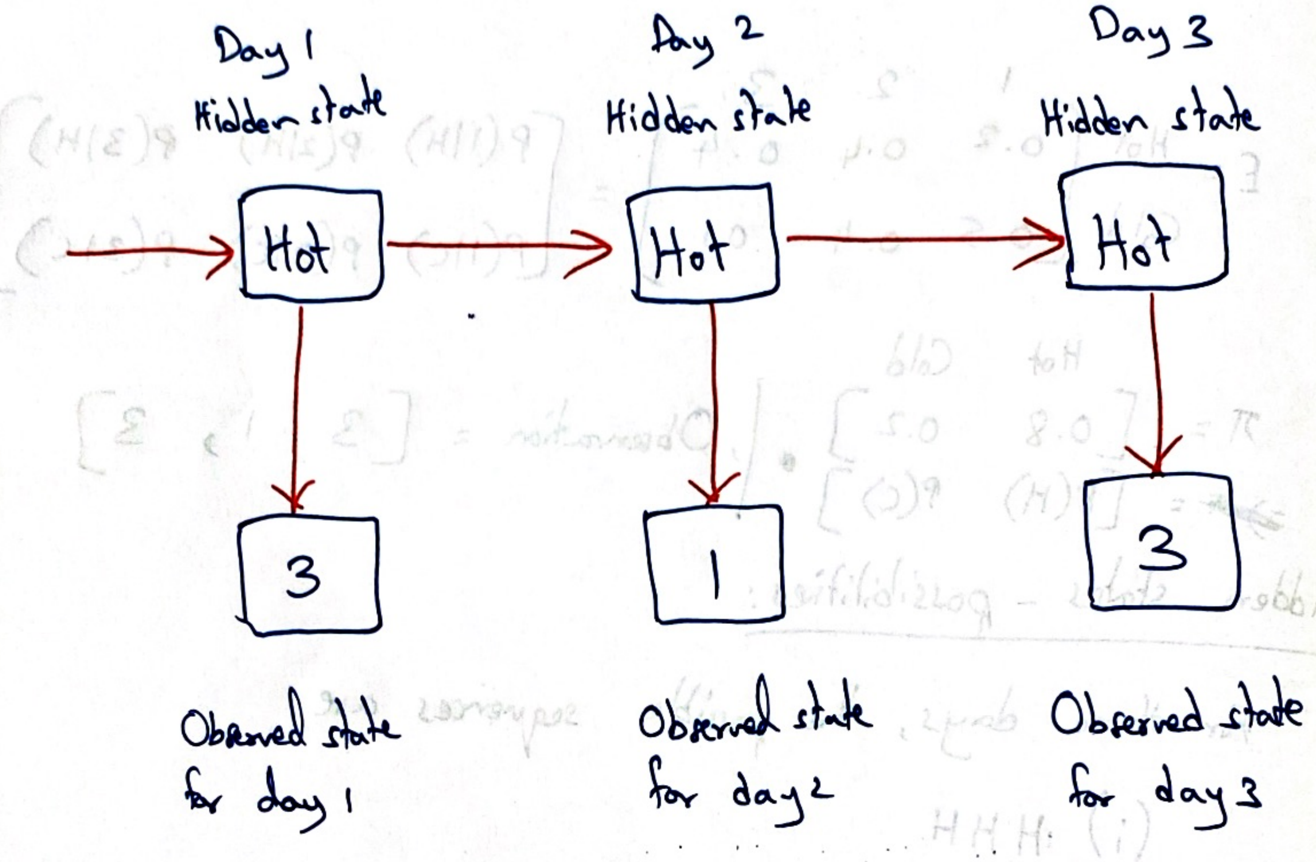
(v) CHH

(vi) CHC

(vii) CCH

(viii) CCC

Let us consider the first possible case, which is HHH. The dependencies in the sequence are shown in the figure below:



$$P\left(\underset{\text{given the observations}}{HHH}\right) = P\left(\begin{matrix}\text{Day1} = \text{Hot} \cap \\ \text{Day2} = \text{Hot} \cap \\ \text{Day3} = \text{Hot}\end{matrix}\right)_{\text{given the observations}} = P\left(\begin{matrix}H_1 = H \cap \\ H_2 = H \cap \\ H_3 = H\end{matrix} \;\middle|\; \begin{matrix}O_1 = 3 \cap \\ O_2 = 1 \cap \\ O_3 = 3\end{matrix}\right)$$

$$= P(H_1 = H) \cdot P(O_1 = 3 \mid H_1 = H) \cdot P\left(H_2 = H \;\middle|\; \begin{matrix}H_1 = H \cap \\ O_1 = 3\end{matrix}\right) \cdot P\left(O_2 = 1 \;\middle|\; \begin{matrix}H_1 = H \cap \\ O_1 = 3 \cap \\ H_2 = H\end{matrix}\right) \cdot$$

$$P\left(H_3 = H \;\middle|\; \begin{matrix}H_1 = H \cap \\ O_1 = 3 \cap \\ H_2 = H \cap \\ O_2 = 1\end{matrix}\right) \cdot P\left(O_3 = 3 \;\middle|\; \begin{matrix}H_1 = H \cap \\ O_1 = 3 \cap \\ H_2 = H \cap \\ O_2 = 1 \cap \\ H_3 = H\end{matrix}\right) \cdot$$

This is difficult to compute. However, if we make the Markov assumption that the current state ~~only~~ depends only on its next previous state variable, then the probability formulation can be reduced to

$$P(HHH) = P(H_1=H) \cdot P(O_1=3 \mid H_1=H) \cdot P(H_2=H \mid H_1=H \cap \cancel{O_1=3}) \cdot$$

$$P(O_2=1 \mid \cancel{H_1=H \cap} \; \cancel{O_1=3 \cap} \; H_2=H) \cdot P(H_3=H \mid \cancel{H_1=H \cap} \; \cancel{O_1=3 \cap} \; H_2=H \cap \cancel{O_2=1}) \cdot P(O_3=3 \mid \cancel{H_1=H \cap} \; \cancel{O_1=3 \cap} \; \cancel{H_2=H \cap} \; \cancel{O_2=1 \cap} \; H_3=H)$$

(Based on the dependencies shown in the figure).

$$\Rightarrow P(HHH) = P(H_1=H) \cdot P(O_1=3 \mid H_1=H) \cdot P(H_2=H \mid H_1=H) \cdot P(O_2=1 \mid H_2=H) \cdot$$
$$P(H_3=H \mid H_2=H) \cdot P(O_3=3 \mid H_3=H)$$

$$= P(H) \cdot P(3 \mid H) \cdot P(H \mid H) \cdot P(1 \mid H) \cdot P(H \mid H) \cdot P(3 \mid H)$$

$$= 0.8 \times 0.4 \times 0.7 \times 0.2 \times 0.7 \times 0.4$$

$$= 0.012544.$$

Similarly, ~~oth~~ the probabilities of other possibilities need to be found. The sequence with the highest probability will be the predicted hidden sequence for the given ~~observation~~ observed sequence.

$$P(HHH) = P(H).P(3|H).P(H|H).P(1|H).P(H|H).P(3|H) = 0.8\times0.4\times0.7\times0.2\times0.7\times0.4 = 0.012544$$

$$P(HHC) = P(H).P(3|H).P(H|H).P(1|H).P(C|H).P(3|C) = 0.8\times0.4\times0.7\times0.2\times0.3\times0.1 = 0.001344$$

$$P(HCH) = P(H).P(3|H).P(C|H).P(1|C).P(H|C).P(3|H) = 0.8\times0.5\times0.3\times0.5\times0.4\times0.4 = 0.0096$$

$$P(HCC) = P(H).P(3|H).P(C|H).P(1|C).P(C|C).P(3|C) = 0.8\times0.5\times0.3\times0.5\times0.6\times0.1 = 0.0036$$

$$P(CHH) = P(C).P(3|C).P(H|C).P(1|H).P(H|H).P(3|H) = 0.2\times0.4\times0.4\times0.2\times0.7\times0.4 = 0.001792$$

$$P(CHC) = P(C).P(3|C).P(H|C).P(1|H).P(C|H).P(3|C) = 0.2\times0.4\times0.4\times0.2\times0.3\times0.1 = 0.000192$$

$$P(CCH) = P(C).P(3|C).P(C|C).P(1|C).P(H|C).P(3|H) = 0.2\times0.5\times0.6\times0.5\times0.4\times0.4 = 0.0048$$

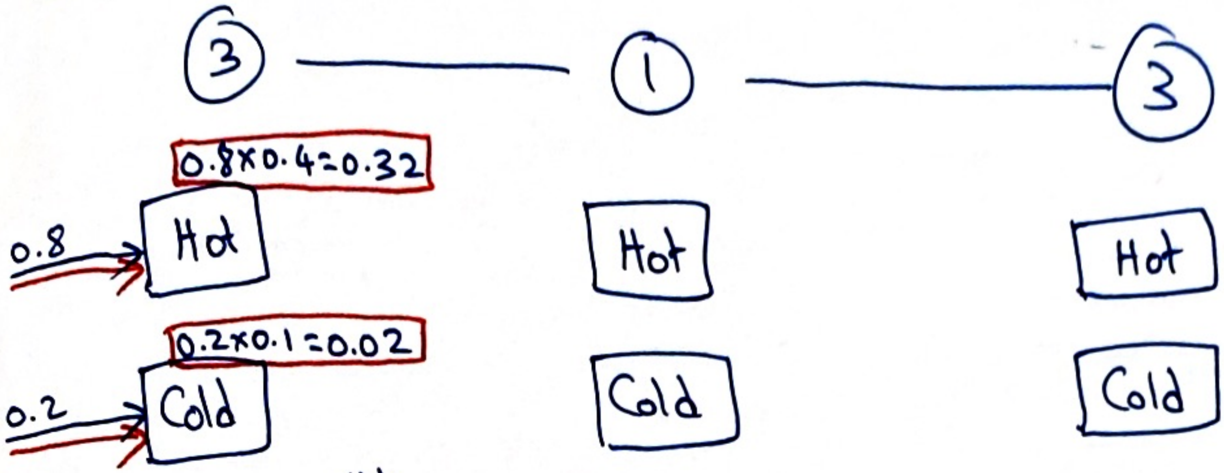
$$P(CCC) = P(C).P(3|C).P(C|C).P(1|C).P(C|C).P(3|C) = 0.2\times0.5\times0.6\times0.5\times0.6\times0.1 = 0.0018$$

From the probabilities, it can be seen that $P(HHH)$ is the highest.

Hence, the predicted hidden sequence for the given observed sequence is H - H - H.

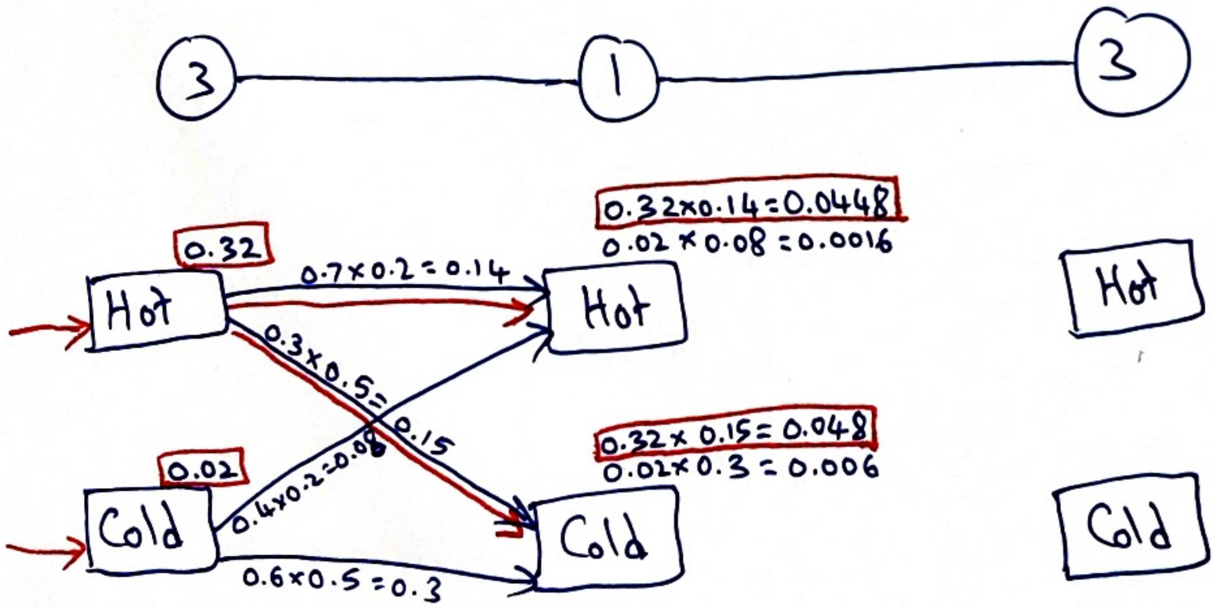
Viterbi algorithm for the same problem:

Step 1:



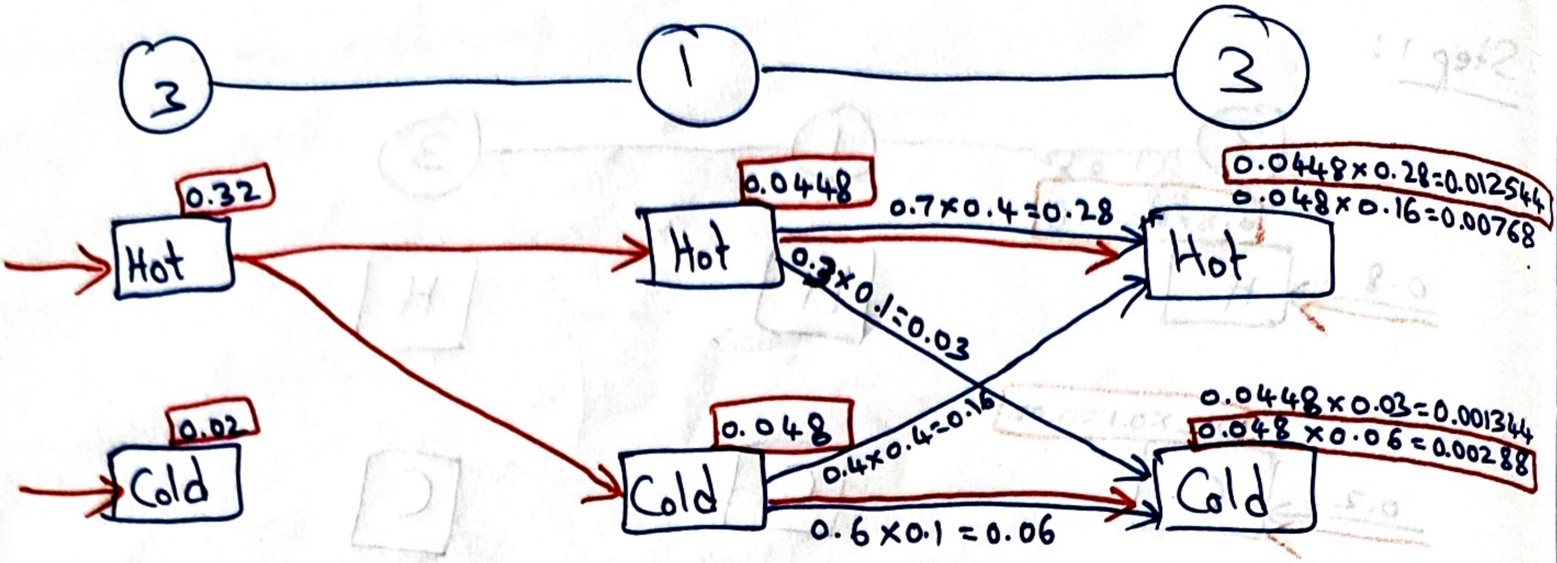
Here, each possible hidden value for the first day, has only one path to reach through. Hence, the single path is highlighted with red colour for both the possible hidden values of the first day.

Step 2:



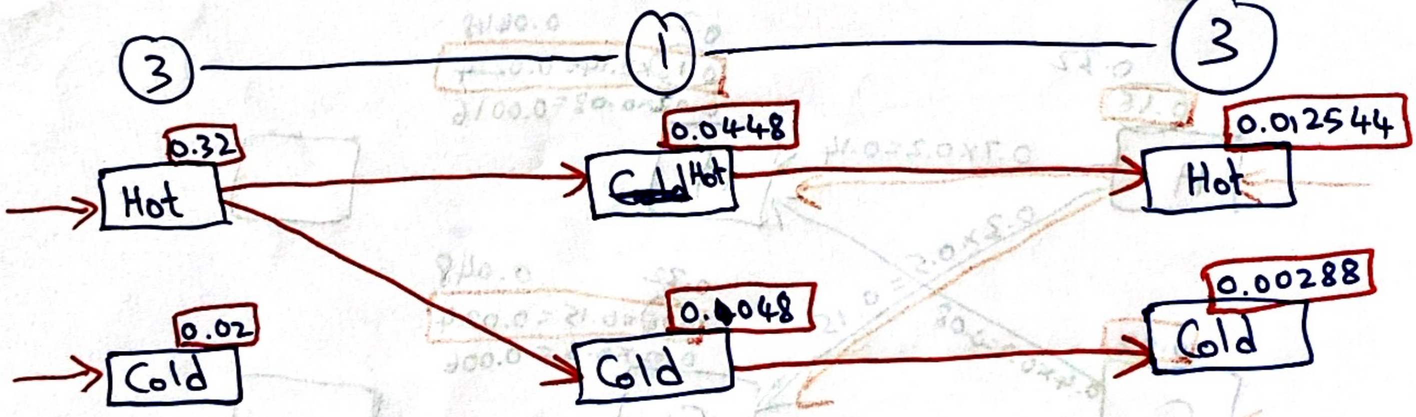
Here, among the two paths that lead to 'Hot' hidden state of the second day, the Hot→Hot path is found to be having the highest probability (0.0448). Hence, this path is marked with red arrow. Similarly, to reach the 'Cold' hidden state of the second day, the Hot-Cold path is found to be with the highest probability (0.048). Hence, this path is also marked with red arrow.

Step 3:



Similarly, even here, for the path leading to the 'Hot' hidden state of the third day, the Hot → Hot path has the highest probability (0.012544), and for the path leading to the 'Cold' hidden state of the third day, the Cold → Cold path has the highest probability (0.00288). Hence, these two paths are marked with red arrows.

Step 4:



Now that we reached the last day's computations, we need to do the backtracking to get the hidden sequence. For the last day (third day), 'Hot' has a probability of 0.012544 and 'Cold' has a probability of 0.00288. Since 0.012544 is the maximum of these two, the last day's hidden state would be 'Hot'. Now, by backtracking through the red arrows, we can predict the hidden sequence to be Hot → Hot → Hot.