

The background is a solid blue color. It features several faint, white, abstract circular patterns. On the left side, there is a large, semi-circular scale with tick marks and numbers ranging from 150 to 260. The numbers are oriented vertically. There are also several concentric circles and arcs with arrows indicating a clockwise direction, scattered across the left and bottom-left areas.

OPTIMISATION

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OPTIMISATION - EXAMPLES

Minimise

$$f(x) = x^3 - 5x^2 + 7x + 1$$

OPTIMISATION - EXAMPLES

Minimise

$$f(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$$

subject to

$$x_1 + 2x_2 \leq 1$$

$$x_1^2 + x_2 \leq 1$$

$$x_1^2 - x_2 \leq 1$$

$$2x_1^2 + x_2 = 1$$

$$0 \leq x_1 \leq 1$$

$$-0.5 \leq x_2 \leq 2$$

WHY OPTIMIZATION?

- Multiple possible solutions
 - Infinitely many
- Satisfaction of constraints

TYPICAL OPTIMISATION PROBLEM

Minimise

$$f(x)$$

subject to

$$g_1(x) \leq 0$$

$$g_2(x) \leq 0$$

$$\vdots$$

$$h_1(x) = 0$$

$$h_2(x) = 0$$

$$\vdots$$

TYPICAL MULTI-OBJECTIVE OPTIMISATION PROBLEM

Minimise

$$f_1(x)$$

$$f_2(x)$$

$$\vdots$$

subject to

$$g_1(x) \leq 0$$

$$g_2(x) \leq 0$$

$$\vdots$$

$$h_1(x) = 0$$

$$h_2(x) = 0$$

$$\vdots$$

METHODS (UNCONSTRAINED)

- Gradient descent method
- L-BFGS method
- Steihaug-Toint method
- Trust-region policy optimisation method
- Nelder-Mead

METHODS (CONSTRAINED)

- Penalty Methods
- Lagrange Method
- Sequential Quadratic Programming
- Augmented Lagrangian

CONDITION FOR OPTIMUM

$$f(x)$$

$$f'(x) = 0$$

$$f''(x) > 0$$

CONDITION FOR OPTIMUM

$$f(x) = x^3 - 5x^2 + 7x + 1$$

$$f'(x) = 0$$

$$f''(x) > 0$$

$$f'(x) = 3x^2 - 10x + 7 = 0 \Rightarrow x = \frac{7}{3}, 1$$

$$f''\left(\frac{7}{3}\right) = 4, \quad f''(1) = -4$$

CONDITION FOR OPTIMUM

$$f(x) = x^3 - 5x^2 + 7x + 1$$

$$f'(x) = 0$$

$$f''(x) > 0$$

$$f'(x) = 3x^2 - 10x + 7 = 0 \Rightarrow x = \frac{7}{3}, 1$$

$$f''\left(\frac{7}{3}\right) = 4 > 0, \quad f''(1) = -4 < 0$$

CONDITION FOR OPTIMUM

$$f(x + \delta x) = f(x) + \frac{\delta x}{1!} f'(x) + \frac{\delta x^2}{2!} f''(x) + \dots$$

$$f(x + \delta x) = f(x) + \frac{\delta x}{1!} f'(x) + \mathcal{O}(\delta x^2)$$

$$f(x + \delta x) - f(x) = \frac{\delta x}{1!} f'(x) + \mathcal{O}(\delta x^2)$$

$$f'(x) = 0$$

CONDITION FOR OPTIMUM

$$f(x + \delta x) = f(x) + \frac{\delta x}{1!} f'(x) + \frac{\delta x^2}{2!} f''(x) + \dots$$

$$f(x + \delta x) = f(x) + \frac{\delta x}{1!} f'(x) + \frac{\delta x^2}{2!} f''(x) + \mathcal{O}(\delta x^3)$$

$$f(x + \delta x) = f(x) + 0 + \frac{\delta x^2}{2!} f''(x) + \mathcal{O}(\delta x^3)$$

$$f(x + \delta x) - f(x) = \frac{\delta x^2}{2!} f''(x) + \mathcal{O}(\delta x^3)$$

$$f''(x) > 0$$

CONDITION FOR OPTIMUM

Condition for minimum

$$f'(x) = 0$$

$$f''(x) > 0$$

CONDITION FOR OPTIMUM

If $f''(x)$ is 0 then

$$f'''(x) = 0$$

$$f^{iv}(x) > 0$$

CONDITION FOR OPTIMUM

For a multivariate function:

$$f(\mathbf{x} + \delta\mathbf{x}) = f(\mathbf{x}) + \frac{\delta\mathbf{x}^T}{1!} \nabla_{\mathbf{x}} f + \frac{1}{2!} \delta\mathbf{x}^T \nabla_{\mathbf{xx}} f \delta\mathbf{x} + \dots$$

$$\nabla_{\mathbf{x}} f = \mathbf{0}$$

$$\nabla_{\mathbf{xx}} f \rightarrow \text{positive definite}$$

(All eigenvalues should be greater than zero)

CONDITION FOR OPTIMUM

$$f(\mathbf{x} + \delta \mathbf{x}) = f(\mathbf{x}) + \frac{\delta \mathbf{x}^T}{1!} \nabla_{\mathbf{x}} f + \frac{1}{2!} \delta \mathbf{x}^T \nabla_{\mathbf{x}\mathbf{x}} f \delta \mathbf{x} + \dots$$

If $\mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$ then

$$f(\mathbf{x} + \delta \mathbf{x}) = f(\mathbf{x}) + \begin{Bmatrix} \delta x_1 & \delta x_2 \end{Bmatrix} \begin{Bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{Bmatrix} + \frac{1}{2} \begin{Bmatrix} \delta x_1 & \delta x_2 \end{Bmatrix} \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} \begin{Bmatrix} \delta x_1 \\ \delta x_2 \end{Bmatrix} + \dots$$

CONDITION FOR OPTIMUM

$$f(\mathbf{x} + \delta \mathbf{x}) = f(\mathbf{x}) + \{\delta x_1 \quad \delta x_2\} \begin{Bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{Bmatrix} + \frac{1}{2} \{\delta x_1 \quad \delta x_2\} \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} \begin{Bmatrix} \delta x_1 \\ \delta x_2 \end{Bmatrix} + \dots$$

$$\begin{Bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{Bmatrix} = 0$$

$$\begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} \rightarrow \text{positive definite} \\ \text{(all eigenvalues should be positive)}$$

LOCAL OPTIMUM VS GLOBAL OPTIMUM

GLOBAL OPTIMUM IS THE MOST OPTIMAL
POINT AMONG ALL THE LOCAL OPTIMA

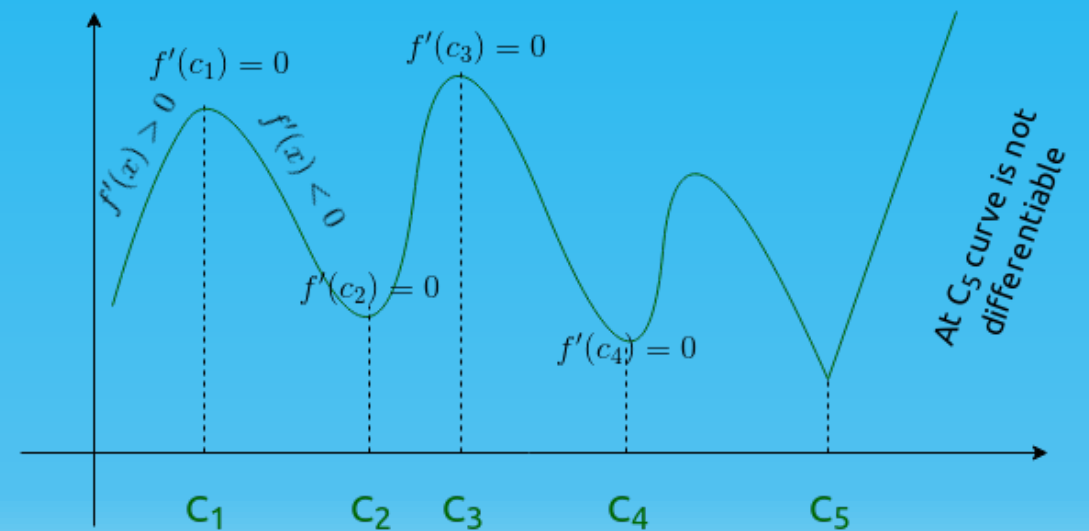


Image taken from: <https://media.geeksforgeeks.org/wp-content/uploads/20190520123301/diff1.png>

LOCAL OPTIMUM VS GLOBAL OPTIMUM

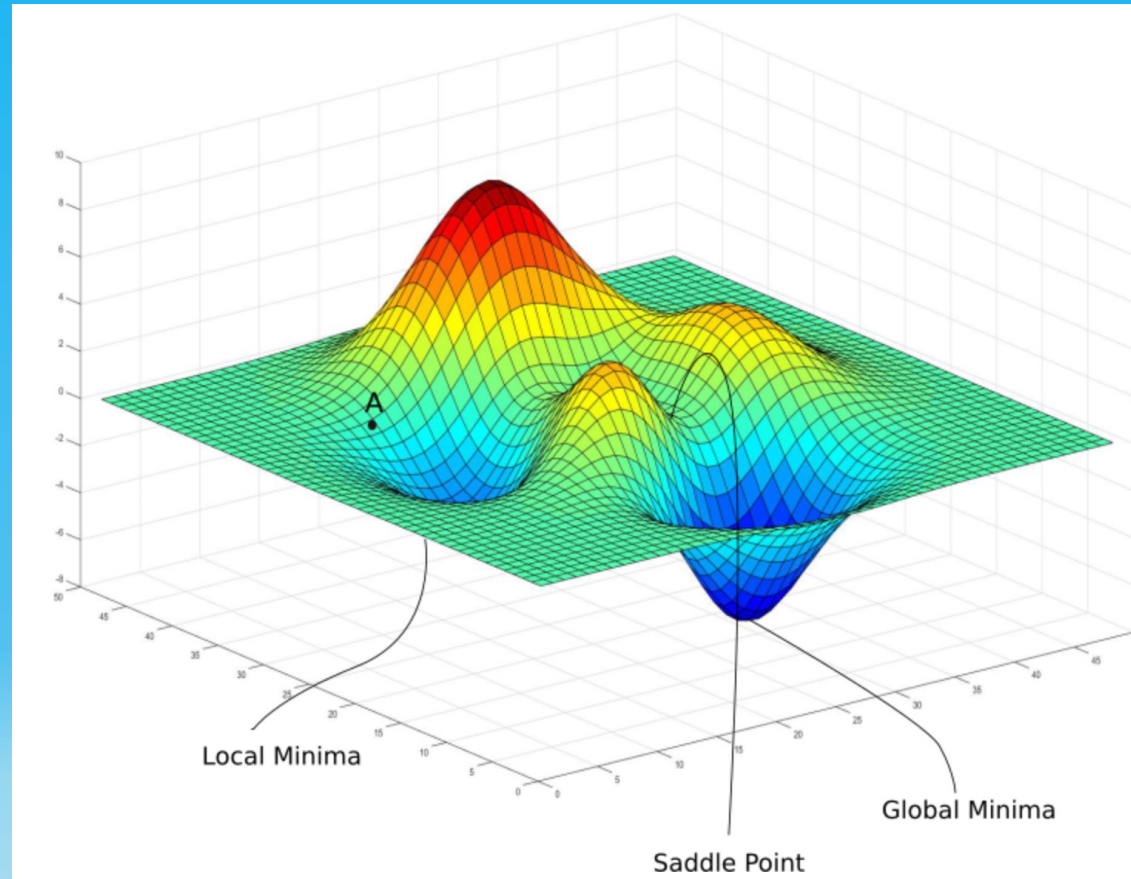


Image taken from: https://wngaw.github.io/images/local_vs_global_minima.png

CONSTRAINED OPTIMISATION

with equality constraints alone

Minimise

$$f(\mathbf{x})$$

subject to

$$h_1(\mathbf{x}) = 0$$

$$h_2(\mathbf{x}) = 0$$

CONSTRAINED OPTIMISATION

with equality constraints alone

Lagrangian

Minimise

$$f(\mathbf{x})$$

subject to

$$h_1(\mathbf{x}) = 0$$

$$h_2(\mathbf{x}) = 0$$

$$L(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) - \lambda_1 h_1(\mathbf{x}) - \lambda_2 h_2(\mathbf{x})$$

$$L(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) - \boldsymbol{\lambda}^T \mathbf{h}(\mathbf{x}) = f(\mathbf{x}) - H(\mathbf{x})$$

$$\text{where } H(\mathbf{x}) = \boldsymbol{\lambda}^T \mathbf{h}(\mathbf{x})$$

$$\boldsymbol{\lambda} = \begin{Bmatrix} \lambda_1 \\ \lambda_2 \end{Bmatrix}$$

$$\mathbf{h}(\mathbf{x}) = \begin{Bmatrix} h_1(\mathbf{x}) \\ h_2(\mathbf{x}) \end{Bmatrix}$$

CONSTRAINED OPTIMISATION

with equality constraints alone

Lagrangian

$$L(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) - H(\mathbf{x})$$

$$f(\mathbf{x} + \boldsymbol{\delta x}) = f(\mathbf{x}) + \frac{\boldsymbol{\delta x}^T}{1!} \nabla_{\mathbf{x}} f + \frac{1}{2!} \boldsymbol{\delta x}^T \nabla_{\mathbf{x}\mathbf{x}} f \boldsymbol{\delta x} + \dots$$

$$f(\mathbf{x} + \boldsymbol{\delta x}) = f(\mathbf{x}) + \frac{\boldsymbol{\delta x}^T}{1!} \nabla_{\mathbf{x}} f + \mathcal{O}(\delta x_i^2)$$

$$f(\mathbf{x} + \boldsymbol{\delta x}) = f(\mathbf{x}) + \frac{\boldsymbol{\delta x}^T}{1!} (\nabla_{\mathbf{x}} L + \nabla_{\mathbf{x}} H) + \mathcal{O}(\delta x_i^2)$$

$$f(\mathbf{x} + \boldsymbol{\delta x}) - f(\mathbf{x}) = \boldsymbol{\delta x}^T \nabla_{\mathbf{x}} L + \boldsymbol{\delta x}^T \nabla_{\mathbf{x}} H + \mathcal{O}(\delta x_i^2)$$

$$\nabla_{\mathbf{x}} L = \mathbf{0}$$

$$\text{Also, } h(\mathbf{x}) = 0 \Rightarrow \nabla_{\boldsymbol{\lambda}} L = \mathbf{0}$$

CONSTRAINED OPTIMISATION

with equality constraints alone

Decision variable as a function of parameter t :

$$\mathbf{x}_{(t)}$$

Function

$$f(\mathbf{x}_{(t)}) = f(\mathbf{x}_{(0)}) + \frac{\nabla_{\mathbf{x}} \mathbf{f}_{(0)}^T}{1!} \dot{\mathbf{x}}_{(0)} t + \frac{1}{2!} (\dot{\mathbf{x}}_{(0)}^T \nabla_{\mathbf{x}\mathbf{x}} \mathbf{f}_{(0)} \dot{\mathbf{x}}_{(0)} + \nabla_{\mathbf{x}} \mathbf{f}_{(0)}^T \ddot{\mathbf{x}}_{(0)}) t^2 + \mathcal{O}(t^3)$$

$$f(\mathbf{x}_{(t)}) = f(\mathbf{x}_{(0)}) + \frac{\nabla_{\mathbf{x}} \mathbf{f}_{(0)}^T}{1!} \dot{\mathbf{x}}_{(0)} t + \frac{1}{2} \dot{\mathbf{x}}_{(0)}^T \nabla_{\mathbf{x}\mathbf{x}} \mathbf{f}_{(0)} \dot{\mathbf{x}}_{(0)} t^2 + \frac{1}{2} \nabla_{\mathbf{x}} \mathbf{f}_{(0)}^T \ddot{\mathbf{x}}_{(0)} t^2 + \mathcal{O}(t^3)$$

$$f(\mathbf{x}_{(t)}) = f(\mathbf{x}_{(0)}) + \frac{\nabla_{\mathbf{x}} \mathbf{f}_{(0)}^T}{1!} \dot{\mathbf{x}}_{(0)} t + \frac{1}{2} \dot{\mathbf{x}}_{(0)}^T \nabla_{\mathbf{x}\mathbf{x}} \mathbf{f}_{(0)} \dot{\mathbf{x}}_{(0)} t^2 + \frac{1}{2} (\nabla_{\mathbf{x}} \mathbf{L}_{(0)}^T - \nabla_{\mathbf{x}} \mathbf{H}_{(0)}^T) \ddot{\mathbf{x}}_{(0)} t^2 + \mathcal{O}(t^3)$$

$$f(\mathbf{x}_{(t)}) = f(\mathbf{x}_{(0)}) + \frac{\nabla_{\mathbf{x}} \mathbf{f}_{(0)}^T}{1!} \dot{\mathbf{x}}_{(0)} t + \frac{1}{2} \dot{\mathbf{x}}_{(0)}^T \nabla_{\mathbf{x}\mathbf{x}} \mathbf{f}_{(0)} \dot{\mathbf{x}}_{(0)} t^2 + \frac{1}{2} \nabla_{\mathbf{x}} \mathbf{L}_{(0)}^T \ddot{\mathbf{x}}_{(0)} t^2 - \frac{1}{2} \nabla_{\mathbf{x}} \mathbf{H}_{(0)}^T \ddot{\mathbf{x}}_{(0)} t^2 + \mathcal{O}(t^3)$$

CONSTRAINED OPTIMISATION

with equality constraints alone

Decision variable as a function of parameter t :

$$\mathbf{x}_{(t)}$$

Weighted constraint function

$$H(\mathbf{x}_{(t)}) = H(\mathbf{x}_{(0)}) + \frac{\nabla_{\mathbf{x}} H_{(0)}^T}{1!} \dot{\mathbf{x}}_{(0)} t + \frac{1}{2!} (\dot{\mathbf{x}}_{(0)}^T \nabla_{\mathbf{x}\mathbf{x}} H_{(0)} \dot{\mathbf{x}}_{(0)} + \nabla_{\mathbf{x}} H_{(0)}^T \ddot{\mathbf{x}}_{(0)}) t^2 + \mathcal{O}(t^3)$$
$$-\frac{1}{2} \nabla_{\mathbf{x}} H_{(0)}^T \ddot{\mathbf{x}}_{(0)} t^2 = H(\mathbf{x}_{(0)}) - H(\mathbf{x}_{(t)}) + \frac{\nabla_{\mathbf{x}} H_{(0)}^T}{1!} \dot{\mathbf{x}}_{(0)} t + \frac{1}{2} \dot{\mathbf{x}}_{(0)}^T \nabla_{\mathbf{x}\mathbf{x}} H_{(0)} \dot{\mathbf{x}}_{(0)} t^2 + \mathcal{O}(t^3)$$

CONSTRAINED OPTIMISATION

with equality constraints alone

Decision variable as a function of parameter t :

$$\mathbf{x}_{(t)}$$

Function

$$f(\mathbf{x}_{(t)}) = f(\mathbf{x}_{(0)}) + \frac{\nabla_{\mathbf{x}} f_{(0)}^T}{1!} \dot{\mathbf{x}}_{(0)} t + \frac{1}{2} \dot{\mathbf{x}}_{(0)}^T \nabla_{\mathbf{x}\mathbf{x}} f_{(0)} \dot{\mathbf{x}}_{(0)} t^2 + \frac{1}{2} \nabla_{\mathbf{x}} L_{(0)}^T \ddot{\mathbf{x}}_{(0)} t^2 - \frac{1}{2} \nabla_{\mathbf{x}} H_{(0)}^T \ddot{\mathbf{x}}_{(0)} t^2 + \mathcal{O}(t^3)$$

$$\Rightarrow f(\mathbf{x}_{(t)}) = f(\mathbf{x}_{(0)}) + \frac{\nabla_{\mathbf{x}} f_{(0)}^T}{1!} \dot{\mathbf{x}}_{(0)} t + \frac{1}{2} \dot{\mathbf{x}}_{(0)}^T \nabla_{\mathbf{x}\mathbf{x}} f_{(0)} \dot{\mathbf{x}}_{(0)} t^2 + \frac{1}{2} \nabla_{\mathbf{x}} L_{(0)}^T \ddot{\mathbf{x}}_{(0)} t^2 + \left(H(\mathbf{x}_{(0)}) - H(\mathbf{x}_{(t)}) + \frac{\nabla_{\mathbf{x}} H_{(0)}^T}{1!} \dot{\mathbf{x}}_{(0)} t + \frac{1}{2} \dot{\mathbf{x}}_{(0)}^T \nabla_{\mathbf{x}\mathbf{x}} H_{(0)} \dot{\mathbf{x}}_{(0)} t^2 + \mathcal{O}(t^3) \right) + \mathcal{O}(t^3)$$

CONSTRAINED OPTIMISATION

with equality constraints alone

Decision variable as a function of parameter t :

$$\mathbf{x}_{(t)}$$

Function

$$\begin{aligned}\Rightarrow f(\mathbf{x}_{(t)}) &= f(\mathbf{x}_{(0)}) + \frac{\nabla_{\mathbf{x}} f_{(0)}^T}{1!} \dot{\mathbf{x}}_{(0)} t + \frac{1}{2} \dot{\mathbf{x}}_{(0)}^T \nabla_{\mathbf{x}\mathbf{x}} f_{(0)} \dot{\mathbf{x}}_{(0)} t^2 + \frac{1}{2} \nabla_{\mathbf{x}} L_{(0)}^T \ddot{\mathbf{x}}_{(0)} t^2 + \left(H(\mathbf{x}_{(0)}) - H(\mathbf{x}_{(t)}) + \frac{\nabla_{\mathbf{x}} H_{(0)}^T}{1!} \dot{\mathbf{x}}_{(0)} t + \frac{1}{2} \dot{\mathbf{x}}_{(0)}^T \nabla_{\mathbf{x}\mathbf{x}} H_{(0)} \dot{\mathbf{x}}_{(0)} t^2 + \mathcal{O}(t^3) \right) + \mathcal{O}(t^3) \\ \Rightarrow f(\mathbf{x}_{(t)}) &= f(\mathbf{x}_{(0)}) + (\nabla_{\mathbf{x}} f_{(0)}^T + \nabla_{\mathbf{x}} H_{(0)}^T) \dot{\mathbf{x}}_{(0)} t + \frac{1}{2} \dot{\mathbf{x}}_{(0)}^T (\nabla_{\mathbf{x}\mathbf{x}} f_{(0)} + \nabla_{\mathbf{x}\mathbf{x}} H_{(0)}) \dot{\mathbf{x}}_{(0)} t^2 + \frac{1}{2} \nabla_{\mathbf{x}} L_{(0)}^T \ddot{\mathbf{x}}_{(0)} t^2 + H(\mathbf{x}_{(0)}) - H(\mathbf{x}_{(t)}) + \mathcal{O}(t^3) \\ \Rightarrow f(\mathbf{x}_{(t)}) &= f(\mathbf{x}_{(0)}) + \nabla_{\mathbf{x}} L_{(0)}^T \dot{\mathbf{x}}_{(0)} t + \frac{1}{2} \dot{\mathbf{x}}_{(0)}^T \nabla_{\mathbf{x}\mathbf{x}} L_{(0)} \dot{\mathbf{x}}_{(0)} t^2 + \frac{1}{2} \nabla_{\mathbf{x}} L_{(0)}^T \ddot{\mathbf{x}}_{(0)} t^2 + H(\mathbf{x}_{(0)}) - H(\mathbf{x}_{(t)}) + \mathcal{O}(t^3)\end{aligned}$$

Note: Here, despite $\mathcal{O}(t^3)$ at the end of RHS, the term $H(\mathbf{x}_{(t)})$ is not limited to second order terms of t but includes all the orders of t !!

CONSTRAINED OPTIMISATION

with equality constraints alone

Let $\mathbf{x}_{(t)}$ be always a point in the feasible space. $\therefore \mathbf{h}(\mathbf{x}_{(t)}) = 0$. This implies that $H(\mathbf{x}_{(t)})$ is always zero for any t .

$$\begin{aligned} f(\mathbf{x}_{(t)}) &= f(\mathbf{x}_{(0)}) + \nabla_{\mathbf{x}} L_{(0)}^T \dot{\mathbf{x}}_{(0)} t + \frac{1}{2} \dot{\mathbf{x}}_{(0)}^T \nabla_{\mathbf{xx}} L_{(0)} \dot{\mathbf{x}}_{(0)} t^2 + \frac{1}{2} \nabla_{\mathbf{x}} L_{(0)}^T \ddot{\mathbf{x}}_{(0)} t^2 + H(\mathbf{x}_{(0)}) - H(\mathbf{x}_{(t)}) + \mathcal{O}(t^3) \\ &\Rightarrow f(\mathbf{x}_{(t)}) = f(\mathbf{x}_{(0)}) + \nabla_{\mathbf{x}} L_{(0)}^T \dot{\mathbf{x}}_{(0)} t + \frac{1}{2} \dot{\mathbf{x}}_{(0)}^T \nabla_{\mathbf{xx}} L_{(0)} \dot{\mathbf{x}}_{(0)} t^2 + \frac{1}{2} \nabla_{\mathbf{x}} L_{(0)}^T \ddot{\mathbf{x}}_{(0)} t^2 + \mathcal{O}(t^3) \end{aligned}$$

CONSTRAINED OPTIMISATION

with equality constraints alone

$$f(\mathbf{x}_{(t)}) = f(\mathbf{x}_{(0)}) + \nabla_{\mathbf{x}} \mathbf{L}_{(0)}^T \dot{\mathbf{x}}_{(0)} t + \frac{1}{2} \dot{\mathbf{x}}_{(0)}^T \nabla_{\mathbf{x}\mathbf{x}} \mathbf{L}_{(0)} \dot{\mathbf{x}}_{(0)} t^2 + \frac{1}{2} \nabla_{\mathbf{x}} \mathbf{L}_{(0)}^T \ddot{\mathbf{x}}_{(0)} t^2 + \mathcal{O}(t^3)$$

Condition for $\mathbf{x}_{(0)}$ to be a local minimum:

We already know that at the local minimum we have $\nabla_{\mathbf{x}} \mathbf{L} = 0$, which implies $\nabla_{\mathbf{x}} \mathbf{L}_{(0)}^T = 0$.

$$\Rightarrow f(\mathbf{x}_{(t)}) - f(\mathbf{x}_{(0)}) = \frac{1}{2} \dot{\mathbf{x}}_{(0)}^T \nabla_{\mathbf{x}\mathbf{x}} \mathbf{L}_{(0)} \dot{\mathbf{x}}_{(0)} t^2 + \mathcal{O}(t^3)$$

$\nabla_{\mathbf{x}\mathbf{x}} \mathbf{L}_{(0)}$ should be positive definite in the feasible space, i.e., in the tangent space, i.e., in the space normal to $\nabla_{\mathbf{x}} \mathbf{h}$.

CONSTRAINED OPTIMISATION

with equality constraints alone

Minimise

$$f(\mathbf{x})$$

subject to

$$h_1(\mathbf{x}) = 0$$

$$h_2(\mathbf{x}) = 0$$

First order conditions

$$\nabla_{\mathbf{x}} L(\mathbf{x}, \boldsymbol{\lambda}) = 0$$

$$\nabla_{\boldsymbol{\lambda}} L(\mathbf{x}, \boldsymbol{\lambda}) = 0$$

Second order conditions

$\nabla_{\mathbf{xx}} L(\mathbf{x}, \boldsymbol{\lambda}) \rightarrow$ positive definite in tangent plane

$\mathbf{d}^T \nabla_{\mathbf{xx}} L(\mathbf{x}, \boldsymbol{\lambda}) \mathbf{d} > 0$ for all \mathbf{d} satisfying $\mathbf{d}^T \nabla \mathbf{h} = 0$

CONSTRAINED OPTIMISATION

with equality and inequality constraints

Minimise

$$f(x)$$

subject to

$$h_1(x) = 0$$

$$h_2(x) = 0$$

$$g_1(x) \leq 0$$

$$g_2(x) \leq 0$$

Lagrangian

$$L(x, \lambda) = f(x) + \lambda_1 h_1(x) + \lambda_2 h_2(x) + \mu_1 g_1(x) + \mu_2 g_2(x)$$

$$L(x, \lambda, \mu) = f(x) + \lambda^T h(x) + \mu^T g(x)$$

CONSTRAINED OPTIMISATION

Minimise

$$f(x)$$

subject to

$$h_1(x) = 0$$

$$h_2(x) = 0$$

$$g_1(x) \leq 0$$

$$g_2(x) \leq 0$$

Rule: Each inequality constraint can be either active or inactive at a given point.

If it is active then it is considered as an equality constraint for that step, and if it is inactive then it is considered as if the constraint does not exist for that step.

Thus, the optimization problem reduces to a problem with either equality constraints alone or an unconstrained problem.

KKT conditions:

$$\nabla_x L(x) = 0$$

$$h_i = 0$$

$$g_i \leq 0$$

$$\mu_i g_i = 0$$

$$\mu_i \geq 0$$

$$\nabla_x L(x) = 0$$

$$\nabla_\lambda L(x) = 0$$

Either $\mu_i = 0$ (inactive) or $g_i = 0$ (active)

$\nabla_{\mu_i} L(x) = 0$ for all i values where $g_i = 0$

$\mu_i \geq 0$ (signifies the direction of $g_i(x) \leq 0$)

Second-order conditions:

$\nabla_{xx} L(x, \lambda) \rightarrow$ positive definite in tangent space with respect to active set of constraints

CONSTRAINED OPTIMISATION

Penalty approach

$$F(x) = f(x) + c \cdot P(x)$$

Minimise

$$F(x)$$

CONSTRAINED OPTIMISATION

Minimise

$$f(x)$$

subject to

$$h_1(x) = 0$$

$$h_2(x) = 0$$

$$P(x) = h_1^2 + h_2^2$$

Minimise

$$F(x) = f(x) + c \cdot P(x)$$

$$F(x) = f + c(h_1^2 + h_2^2)$$

CONSTRAINED OPTIMISATION

Minimise

$$f(x)$$

subject to

$$g_1(x) \leq 0$$

$$g_2(x) \leq 0$$

$$h_1(x) = 0$$

$$P(x) = (\max(0, g_1))^2 + (\max(0, g_2))^2 + h_1^2$$

CONSTRAINED OPTIMISATION

Minimise

$$f(x)$$

subject to

$$g_1(x) \leq 0$$

$$g_2(x) \leq 0$$

$$g_3(x) \leq 5$$

$$g_4(x) \geq 7$$

$$g_5(x) \geq 0$$

$$h_1(x) = 0$$

$$h_2(x) = -8$$

$$P(x) = (\max(0, g_1))^2 + (\max(0, g_2))^2 + (\max(0, g_3 - 5))^2 + (\max(0, 7 - g_4))^2 + (\max(0, -g_5))^2 + h_1^2 + (h_2 + 8)^2$$

GRADIENT DESCENT

Minimise

$$f(x)$$

- Start with an initial guess x_0
- Set $x_n = x_0$.
- If $\nabla_x f(x_n) \neq 0$, find the minimum of $f(x_n - \alpha \nabla_x f(x_n))$ and find the corresponding minimiser α . Else terminate with the solution $x^* = x_n$
- Put $x_n \leftarrow x_n - \alpha \nabla_x f(x_n)$ and repeat the second step until termination.

CONSTRAINED OPTIMISATION

Example

Minimise

$$f(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$$

subject to

$$x_1 + 2x_2 \leq 1$$

$$x_1^2 + x_2 \leq 1$$

$$x_1^2 - x_2 \leq 1$$

$$2x_1^2 + x_2 = 1$$

$$0 \leq x_1 \leq 1$$

$$-0.5 \leq x_2 \leq 2$$

Minimise

$$\begin{aligned} F(x) &= 100(x_2 - x_1^2)^2 + (1 - x_1)^2 \\ &+ c \left((\max(0, x_1 + 2x_2 - 1))^2 + (\max(0, x_1^2 + x_2 - 1))^2 + (\max(0, x_1^2 - x_2 - 1))^2 \right. \\ &\quad \left. + (2x_1^2 + x_2 - 1)^2 + (\max(0, -x_1))^2 \right. \\ &\quad \left. + (\max(0, x_1 - 1))^2 + (\max(0, -0.5 - x_2))^2 + (\max(0, x_2 - 2))^2 \right) \end{aligned}$$

PYTHON IMPLEMENTATION

```
def F(x,c=0):

    f = 100*(x[1]-x[0]**2)**2+(1-x[0])**2

    g1 = x[0]+2*x[1]-1
    g2 = x[0]**2+x[1]-1
    g3 = x[0]**2-x[1]-1
    h1 = 2*x[0]+x[1]-1
    g4 = -x[0]
    g5 = x[0]-1
    g6 = -0.5-x[1]
    g7 = x[1]-2

    P = max(0,g1)**2+max(0,g2)**2+max(0,g3)**2+max(0,g4)**2+max(0,g5)**2+max(0,g6)**2+max(0,g7)**2+h1**2

    return f+c*P

res = minimize(lambda x:F(x,c=1), [1,1], method='nelder-mead', options={'xatol': 1e-8, 'disp': True})
```

Minimise

$$\begin{aligned} F(x) &= 100(x_2 - x_1^2)^2 + (1 - x_1)^2 \\ &+ c \left((\max(0, x_1 + 2x_2 - 1))^2 + (\max(0, x_1^2 + x_2 - 1))^2 + (\max(0, x_1^2 - x_2 - 1))^2 \right. \\ &\quad \left. + (2x_1^2 + x_2 - 1)^2 + (\max(0, -x_1))^2 \right. \\ &\quad \left. + (\max(0, x_1 - 1))^2 + (\max(0, -0.5 - x_2))^2 + (\max(0, x_2 - 2))^2 \right) \end{aligned}$$

DISCRETE VARIABLE OPTIMISATION

$x_1 \rightarrow$ number of chips

The optimized result is $x_1 = 3.219$

Branch and Bound method
Heuristic methods

Example:

Maximise

$$f(x) = 100x_1 + 150x_2$$

subject to

$$2x_1 + x_2 \leq 10$$

$$3x_1 + 6x_2 \leq 40$$

$$x_1, x_2 \geq 0$$

x_1 and x_2 are integers.

GENETIC ALGORITHMS

Minimise

$$f(x) = 3x^2 - 5x + 2$$

Bounds of x

$$[-10,10]$$

$$-10 \rightarrow 00000000000000$$

$$+10 \rightarrow 11111111111111$$

GENETIC ALGORITHMS

- Initial population (Generation 0)
- New generation
 - Fitness function and selection
 - Can be the objective function itself
 - Pairs of individuals
 - Crossover
 - 1001010100101101 ► 1001010110101010
 - 0101001010101010 ► 0101001000101101
 - Mutation
 - Flipping of a random bit

PYTHON IMPLEMENTATION

```
import numpy as np

from pymoo.algorithms.soo.nonconvex.ga import GA

from pymoo.core.problem import Problem

from pymoo.operators.crossover.sbx import SBX

from pymoo.operators.mutation.pm import PM

from pymoo.operators.repair.rounding import RoundingRepair

from pymoo.operators.sampling.rnd import IntegerRandomSampling

from pymoo.optimize import minimize
```

```
class MyProblem(Problem):
```

```
    def __init__(self):
```

```
        super().__init__(n_var=2, n_obj=1, n_ieq_constr=4, xl=[-10,-10], xu=[10,10], vtype=int)
```

```
    def _evaluate(self, x, out, *args, **kwargs):
```

```
        out["F"] = -100*x[:,0]-150*x[:,1]
```

```
        out["G"] = [2*x[:,0]+x[:,1]-10, 3*x[:,0]+6*x[:,1]-40,-x[:,0],-x[:,1]]
```

```
problem = MyProblem()
```

```
method = GA(pop_size=20, sampling=IntegerRandomSampling(), crossover=SBX(prob=1.0, eta=3.0, vtype=float, repair=RoundingRepair),
mutation=PM(prob=1.0, eta=3.0, vtype=float, repair=RoundingRepair()), eliminate_duplicates=True)
```

```
res = minimize(problem, method, termination=('n_gen', 40), seed=1, save_history=True)
```

The background is a solid blue color. In the top right corner, there is a large, faint, circular graphic resembling a clock face or a radar chart, with concentric circles and radial lines. In the bottom left corner, there is a smaller, faint, circular graphic with concentric circles. A horizontal gradient bar, transitioning from blue to a lighter blue, is positioned across the middle of the image. The word "Queries?" is centered in the middle of the image, below the gradient bar.

Queries?

The background is a solid blue gradient, lighter at the bottom. It features several faint, white, circular patterns. In the top right, there is a large circular scale with degree markings from 80 to 210 and concentric circles with arrows. In the bottom right, there are smaller concentric circles with arrows. In the bottom left, there are partial circular patterns. The text "Thank you!" is centered in the middle of the image.

Thank you!