Consider a rigid satellite that includes three reaction wheels. Let the inertia matrix for the satellite with the reaction wheels is $J \in \mathcal{R}^{3\times3}$, the inertia matrix for the satellite excluding the reaction wheel subsystem is $J_s \in \mathcal{R}^{3\times3}$, the angular velocity of the satellite with respect to the inertial frame is $\boldsymbol{\omega} \in \mathcal{R}^3$, the inertia matrix for the reaction wheel system is $J_w \in \mathcal{R}^{3\times3}$, the wheel axes are aligned with the satellite body axes, and the wheel rotation speed with respect to the satellite body frame is $\boldsymbol{\omega}_w \in \mathcal{R}^3$.

1. Define the total angular momentum of the satellite system, and derive the Euler's rotational equations of motion. Do not introduce new variables except for the derivatives of the given variables. [5 pts]

Once the satellite is deployed in orbit, it experiences a tumbling motion due to its initial angular velocity. In order to spin it up along with the third axis, it turns on the reaction wheel that is aligned with $\hat{\mathbf{b}}_3 \in \mathcal{R}^3$. Let the wheel axial inertia is I_w and the speed of the wheel is Ω_w.

2. Rewrite the Euler's rotational equations of motion for this stabilization using I_w, Ω_w, $\hat{\mathbf{b}}_3$, and their derivatives. Do not introduce new variables. [5 pts]

Let $J = \begin{bmatrix} 500 & 0 & 0 \\ 0 & 400 & -7 \\ 0 & -7 & 440 \end{bmatrix}$ kg $\cdot$ m^2, $I_w = 0.1$ kg$\cdot$m^2, $\boldsymbol{\omega}(t=0) = [5, 0, 0]^T$ rpm, the initial wheel momentum is zero, the nominal wheel momentum is $h_n = 55$ kg$\cdot$m^2/s, and the spin-up duration is $t_s = 5,000$ s. Assume that the constant wheel control is designed as $\dot{h}_w = h_n/t_s$.

3-1. Plot the time responses of the angular velocity of the satellite and the nutation angle $\beta(t)$, which is the angle between the satellite system angular momentum and the control axis. Also, draw a three-dimensional plot for the angular velocity of the satellite. The units for the angular velocity of the satellite and the nutation angle should be rad/s and deg, respectively. *Note that rpm must be transformed into rad/s first.* Discuss your results with findings. [20 pts]

3-2. Redo Problem 3-1 by assuming the off-diagonal elements of J are zeros. [5 pts]

The satellite is equipped with two attitude sensors, such as a sun sensor and a star tracker, and the star tracker accuracy is twice as good as the sun sensor. the At time 5,000 s, the attitude sensors measured two observation vectors ${}^{\mathcal{B}}\mathbf{v}_{\text{sun}} = [0.8, -0.5, 0.2]^T$ and ${}^{\mathcal{B}}\mathbf{v}_{\text{star}} = [-0.3, -0.1, 0.9]^T$, respectively. Let the corresponding reference unit vectors are ${}^{\mathcal{N}}\hat{\mathbf{v}}_{\text{sun}} = [1, 0, 0]^T$ and ${}^{\mathcal{N}}\hat{\mathbf{v}}_{\text{star}} = [0, 0, 1]^T$.

4. Find the satellite attitude matrices with respect to the inertial frame using TRIAD and QUEST, and compare the attitude difference with the principal rotation angle in degree. [15 pts]

Let the solution of the QUEST (i.e., Rodrigues parameters) is chosen as the satellite attitude at time 5,000 s. Using the angular velocity of the satellite and the reaction wheel at time 5,000 s, you want to control the satellite by controlling the speed of reaction wheels.

5-1. Design a MRP-based nonlinear control law to stabilize the satellite into the orientation with Lypunov method. You must prove the stability. [15 pts]

5-2. Find proper gains to stabilize the satellite into the zero attitudes and angular velocity in 50 s. Note that the norm errors for both attitude and the angular velocity at the final time must be less than 10^{-5}. Plot the time histories of MRPs, angular velocity of the satellite (rad/s), the wheel speed (rpm), and the control input torque (Nm). Discuss your results with findings. [35 pts]