Optimal strategies in Colonel Balloon Game

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Abstract

Game Theory mathematically analyzes the interaction between self-interested parties participated in competition (non-cooperative) and cooperation games. A "game" in Game Theory is any communication between people whose payoffs are affected by decisions made by others. So the field is broadly applicable in economics, social sciences, psychology, and computer science. Intuitively, Game Theory's Nash Equilibrium is a solution concept in competitive games where every player's strategy is one's best regardless of others' strategies. Nash Theorem points out that every finite game has a Nash Equilibrium [1]. In other words, there exists at least one stable situation where each player has the best response regardless of other player's strategy. Thus, Nash Equilibrium computation is crucial in Game Theory. This Junior Independent Study formally introduces mixed strategy Nash Equilibrium, and applies its computational methods to a finite two-person game called Colonel Blotto. Specifically, the computational implementation is examined in "Colonel Balloon Game", a Colonel Blotto game stimulation which aims to educate players about game theory.

Contents

1 Introduction

Historically, Game Theory situations can be found in not only religious texts such as the Bible and Talmud, but also literature, namely Caroll's *Alice in Wonderland*. The most evident appearances of this concept are in military science, where opposite forces analyze one another's behaviors and aim for victory gains with the least damage. Game theory takes shape as a branch of Mathematics and Economics with the publication of Zermelo about winning strategy in chess alongside Neumann and Morgenstern's *Theory of Games and Economics Behaviors* [7] [9]. However, it is not until the 1950s when game theory attracted the interest scholars because of its applications in warfare and politics. For example, the simulation of the Colonel Blotto game, where two parties distribute a certain number of resources to K battlefields, is used to study electoral competition in politics.

The structure of the paper consists of three main sections: (1) overview, (2) formal definition of Nash equilibrium mixed strategy, and (3) Optimal strategies in the software implementation. This project aims to introduce the game theory sub-field in Computer Science as well as its important concepts best response and Nash equilibrium. Through the Colonel Blotto game, the paper also investigates how the solution concept can be interpreted and applied to real-life situations.

2 History

2.1 Game Theory

Informally, a game in Game Theory is any communication between multiple people, and each participant's outcomes are prompted by not only their decisions but also others'. A game consists of three main elements: participants as players, their decisions as strategies, and each player's consequences for the game as payoffs. For example, the game ”Rock, Paper, Scissors” has two or more players, whose strategies are either rock, paper, or scissors, and the player’s possible payoffs are loss, win, and draw.

Game theory revolves around human’s behaviors in strategic interactions. Analyses of players’ actions take into account not only the individual’s decisions but also others’, and Nash equilibrium is widely used in this perspective. Theorists characterize real life behaviors by studying the Nash equilibrium Mixed Strategy of a finite game [3].

2.2 Nash Equilibrium and Nash Theory

Overall, Nash equilibrium is a solution concept where every player’s strategy is the best response (best replies) to every other player. Given other players’ strategies and consequences, there exists a situation where each player cannot improve the gaining amount when changing his/her current strategy, or in other words, the payoffs of each player are irrelevant to the others. This stable state of game is called Nash equilibrium.

The concept is named after John Forbes Nash Jr., who proved the Nash Theorem, which is one of the most important theorems of game theory. John Nash proves that every finite game with mixed strategies has a Nash equilibrium [1]. In other words, there exists at least one stable situation where each player has the best response regardless of other players’ strategies. The problem, however, lies in finding Nash Equilibria. Since game complexity can grow exponentially with the number of strategies and players, it is impossible to manually solve games that have more than three strategies. This is when algorithmic game theory comes into practice.

Because of its inefficiency in run-time, Nash equilibrium computation algorithm has been one of the major topics between mathematicians and computer scientists. Further discussions of computing Nash equilibrium continue in section Three.

3 Theory

Games are classified based on five types: cooperative/non-cooperative, normal form/extensive form, simultaneous/sequential, zero-sum/non-zero-sum, and symmetric/asymmetric. A comprehensive discussion of each types and its example games can be found in [7] and [9]. Cooperative (collation)/non-cooperative game types and simultaneous strategy move/sequential move game type are briefly discussed for fundamental understanding of Nash equilibrium in the next section. According to [7], unlike collation games where players work together to achieve the best outcomes, non-cooperative games are self-enforcing and between self-interested players whose goal is to maximize their payoffs. Thus, non-cooperative games focus on individual's strategies and corresponding payoffs as well as the Nash equilibrium of the game, which is further discussed in the paper. Moreover, games can also be played sequentially, e.g. chess where players play one after the other, or simultaneously e.e. Rock, Paper, Scissors where all players play at the same time regardless of earlier games.

The below sub-sections define game theory's key terms that are useful for the topic of understanding the connection between Nash equilibrium and best replies. Specifically, it consists of definitions, illustrated by examples, of the folllowig terms: (1) finite games, (2) payoff bi-matrix, (3) pure and (4) mixed strategies, (5) best reply, and (6) Nash equilibrium.

3.1 Finite games

By definition from [5] and [9], a finite non-cooperative game G with N players is a $2n+1$-tuple in the following form

$$G = (N, S_1, ...S_n, u_1, ...u_n)$$

where

- $N = \{1, 2, ...n\}, n \in \mathbb{N}, n \geq 1$, is the set of players.
- Set $\{S_1, S_2, ..., S_n\}$, is the set of each player's pure strategy set.
- $\{u_1, u_2, ..., u_n\}$, is the set of each player's payoff functions.

In other words, the game G is characterized by a non-empty set of n players, and each player has a finite strategy set S_i (an infinite game has infinite strategy sets).

"Rock, paper, Scissors" is again used to illustrate the definition. If a round of "Rock, Paper, Scissors" is G and the player set $N = \{A, B\}$, then $S_A = S_B = \{\text{Rock (R), Paper (P), Scissors (S)}\}$. The payoffs of player A and B are bound by the game's rules: rock beats scissors, scissors beat paper, and paper beats rocks. Figure 1 shows the payoff matrix of two-player Rock, Paper, Scissors game. The game is an example of a non-cooperative, zero-sum, and simultaneous game. Zero-sum games describe games where the sum of the two players' payoffs are always zero. Figure 1 shows the normal form game represented as a 3×3 matrix, whose elements are two-tuple payoffs corresponding to strategy (u_A, u_B). As 1 denotes "win", -1 "lose" -1, and "draw" 0, the sum of each tuple's elements is always zero, indicating the zero-sum characteristic of Rock, Paper, Scissors.

With the formal definition of a finite non-cooperative game in mind, a finite two-person game is defined with $N = \{A, B\}$, a set containing pure

	R	**P**	**S**
r	(0,0)	(1,-1)	(-1,1)
p	(-1,1)	(0,0)	(1,-1)
s	(1,-1)	(-1,1)	(0,0)

Figure 1: Payoff matrix (u_A, u_B) of Rock, Paper, Scissors of column player A and row player B

strategy sets is $\{S_A, S_B\}$, and payoff functions $u_A : S_A \longrightarrow \mathbb{R}$, $u_B : S_B \longrightarrow \mathbb{R}$. In terms of game representation, a finite two-person game is usually associated with a $m \times n$ payoff bi-matrix, where $|S_A| = m, |S_B| = n$.

3.2 Mixed strategies in a bi-matrix game

Formally, [5] and [6] define bi-matrix game, pure strategies, and mixed strategies in the following manners:

Definition 3.1 ***bi-matrix game*** *A bi-matrix game is a pair of $m \times n$ matrices (A, B), where integers $m, n \geq 1$.*

Bi-matrix (A,B) represents a finite two-person strategy game whose $N = \{A, B\}$, $|S_A| = m$, $|S_B| = n$. When player A (column player) plays column i and player B (row player) plays row j, then entries A_{ij}, B_{ij} are respectively payoffs for players A and B. In the case of figure 1, the bi-matrix (A,B) where $m = n = 3$ has an entry A_{Rp} of "win" (1) as payoff of player A while B_{Pr} of "lose" (-1) as payoff of B.

Another crucial terms in non-cooperative game are pure strategies and mixed strategies. Informally speaking, player i's pure strategies are all possible elements $s \in S_i$, or a complete definition of a player's move in accordance to every possible situation he/she faces. On the other hand, a mixed strategy is the probability distributions over the pure strategies. For instance, while "Rock", "Paper", and "Scissors" are pure strategies in the above game, the player's choice or probability to play Rock 30%, Paper 20%, and Scissors 50% is a mixed strategy. Furthermore, a mixed strategy Nash equilibrium occurs when each player's mixed strategy is the best reply and has the best payoff regardless others' strategies.

Player A, who has m pure strategies, has a mixed strategy which is an m-dimensional vector. The vector is a probability distribution **p** over the row of A and its components' sum are always equal to 1. With the above definition of a mixed strategy in mind, a set of mixed strategies are a set of vectors, where each vector is a probability distribution. For instance, mixed strategies of player A in Rock, Paper, Scissors is $\Delta^m = \{(0.3, 0.3, 0.4), (0, 1, 0), ...\}$.

Definition 3.2 *Player A's mixed strategies, denoted as Δ, is a set of all possible distributions:*

$$\Delta^m := \{\boldsymbol{p} = (p_1, ..., p_m) \in \mathbb{R}^m | \sum_{i=1}^{m} p_i = 1, p_i \geq 0, \forall i = 1, ..., m\}. \tag{1}$$

If player A chooses the mixed strategy $\mathbf{p} = (0, 1, 0)$. This means that he/she definitely plays paper throughout the game. In this case, the strategy $\mathbf{p}$ is also a pure strategy since $p_i = 1$, with $i = 2$, and can be denoted as $\mathbf{e}^i = \mathbf{e}^2$.

Definition 3.3 *The expected payoff of player A's strategy $\boldsymbol{p}$ over player B's strategy $\boldsymbol{q}$ is denoted as the following*

$$\boldsymbol{p}A\boldsymbol{q} = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i q_j a_{ij} \tag{2}$$

From player A's viewpoint, if A plays mixed strategy **p** and B plays **q** then A's outcome of the game is the sum of all products of each matrix A's entry a_{ij} with its corresponding probability distribution, namely $\mathbf{p}_i, \mathbf{q}_j$.

Considering the bi-matrix game Rock, Paper, Scissors' and its mixed

	1.0	**0.0**	**0.0**
0.3	(0,0)	(1,-1)	(-1,1)
0.3	(-1,1)	(0,0)	(1,-1)
0.4	(1,-1)	(-1,1)	(0,0)

Figure 2: Mixed strategy of column player A and row player B

strategy in figure 2, player A's expected payoff is

$$\mathbf{p}A\mathbf{q} = \sum_{i=1}^{3}\sum_{j=1}^{3} p_i q_j a_{ij} = \begin{bmatrix}1 & 0 & 0\end{bmatrix} \cdot A \cdot \begin{bmatrix}0.3\\0.3\\0.4\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\end{bmatrix} \cdot \begin{bmatrix}0 & 1 & -1\\-1 & 0 & 1\\1 & -1 & 0\end{bmatrix} \cdot \begin{bmatrix}0.3\\0.3\\0.4\end{bmatrix} = 0.1$$

Similarly, player B's expected payoff $\mathbf{q}$ is the matrix product

$$\mathbf{p}B\mathbf{q} = \sum_{i=1}^{3}\sum_{j=1}^{3} p_i q_j b_{ij} = \begin{bmatrix}1 & 0 & 0\end{bmatrix} \cdot B \cdot \begin{bmatrix}0.3\\0.3\\0.4\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\end{bmatrix} \cdot \begin{bmatrix}0 & -1 & 1\\1 & 0 & -1\\-1 & 1 & 0\end{bmatrix} \cdot \begin{bmatrix}0.3\\0.3\\0.4\end{bmatrix} = -0.1$$

The above bi-matrix can be interpreted as the following: if player A plays Rock all the time and player B plays Rock 30%, Paper 30%, and Scissors 40% of the time, then A is more likely to gain 0.1 while B to lose 0.1. Because Rock, Paper, Scissors is a zero-sum game, the winning amount of one player is the losing of the other.

3.3 Mixed strategy Nash equilibrium in a bi-matrix game

To further the introduction of Nash equilibrium in 1.2, the discussion of best reply that results in a formal definition of Nash equilibrium is provided. Previously stated, players in game theory are assumed to be rational, that is to say, to be selfish and to always choose strategies that maximize the expected payoffs. These strategies are formally defined as best replies. And considering definition 3.2, [6] defines best replies as the following

Definition 3.4 ***Best Reply*** *In a bi-matrix game size $m \times n$, player A's best reply to player B's $\boldsymbol{q}$ is strategy $\boldsymbol{p}$ such that*

$$\boldsymbol{p}A\boldsymbol{q} \geq \boldsymbol{p}'A\boldsymbol{q}, \forall \boldsymbol{p}' \in \Delta^m$$

and similar to player B's $\boldsymbol{q}$

In other words, a mixed strategy $\mathbf{p}$ is the best reply of A if its payoff value is no less than any other mixed strategy $\mathbf{p}\ '$ in the set Δ^m.

Definition 3.5 ***Nash equilibrium*** *in a bi-matrix game (A,B) is a pair of strategies $(\boldsymbol{p}*, \boldsymbol{q}*)$ such that $\boldsymbol{p}*$ is player A's best reply to $\boldsymbol{q}*$ and $\boldsymbol{q}*$ is player B's best reply to $\boldsymbol{p}*$.*

The Nash equilibrium in bi-matrix games centers around best replies. Specifically, it is a stable solution or game situation in which both players' strategies have expected payoff that are indifferent to each other [14]. The notations in this section help to explain Nash equilibrium.

3.4 Computing Nash equilibrium

Because of the heavy mathematical definition in finding Nash equilibrium, this section only demonstrates the concept through an example. The example hopes to provide a brief understanding of the connection between Nash equilibrium and best replies, as well as how players can use Nash equilibrium to devise their game plans. The example is tasked to find a pair of Nash equilibrium of a finite two person game (there can be multiple Nash equilibrium pairs). The computation is based on the fact that the Nash equilibria tuple $(\mathbf{p}*, \mathbf{q}*)$ happens when $\mathbf{p} * A\mathbf{q}*$ and $\mathbf{p} * B\mathbf{q}*$ are indifferent to each other, meaning the expected payoff of $\mathbf{p}*$ and $\mathbf{q}*$ are equal.

Example

We consider the example of a two-person finite game G in which player A has three pure strategies and player B has two pure strategies. The game is defined by the following information:

- Pure strategy sets $|S_A| = m = 3$, $|S_B| = n = 2$
- Mixed strategy sets of player A is $\Delta^m = \{(p_1, p_2, p_3) \mid p_3 = 1 - p_1 - p_2\}$ and of player B is $\Delta^n = \{(q_1, q_2) \mid q_2 = 1 - q_1\}$
- Pure strategy sets of player A is $\{\mathbf{e}^1 = (1,0,0),\ \mathbf{e}^2 = (0,1,0),\ \mathbf{e}^3 = (0,0,1)\}$ and of player B is $\{\mathbf{e}^1 = (1,0),\ \mathbf{e}^2 = (0,1)\}$
- We assume the bi-matrix game has the payoff bi-matrix demonstrated in figure 3.

	p1	**p2**	**p3**
q1	(1,-3)	(-3,1)	(-7,0)
q2	(0,2)	(1,1)	(2,4)

Figure 3: Bi-matrix (A,B) game

- Payoff matrix A $= \begin{bmatrix} 1 & -3 & -7 \\ 0 & 1 & 2 \end{bmatrix}$ and payoff matrix B $= \begin{bmatrix} -3 & 1 & 0 \\ 2 & 1 & 4 \end{bmatrix}$
- Expected payoffs of A over $\mathbf{q}$ is $\mathbf{p}A\mathbf{q} = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix} A \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$ and expected payoffs of B over $\mathbf{p}$ is $\mathbf{p}B\mathbf{q} = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix} B \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$
- The payoff functions are

$$u_A : f(\mathbf{p}) \longrightarrow \mathbb{R}\ where\ f(\mathbf{p}) = \mathbf{p}A\mathbf{q} \tag{3}$$

$$u_B : f(\mathbf{q}) \longrightarrow \mathbb{R}\ where\ f(\mathbf{q}) = \mathbf{p}B\mathbf{q} \tag{4}$$

Referring to definition 3.5 and 3.4, Nash equilibrium centers on players' best replies, and best replies on the expected payoff functions. We know that a pair of Nash equilibrium $(\mathbf{p}*, \mathbf{q}*)$ has the same expected payoffs. Thus, in order to know when this situation occurs, we analyze the behaviors of each player's payoff value, specifically, their value in accordance to the mixed strategy $\mathbf{p}, \mathbf{q}$. To do so, we construct graphs of player A and player B's payoff functions and analyze them, respectively.

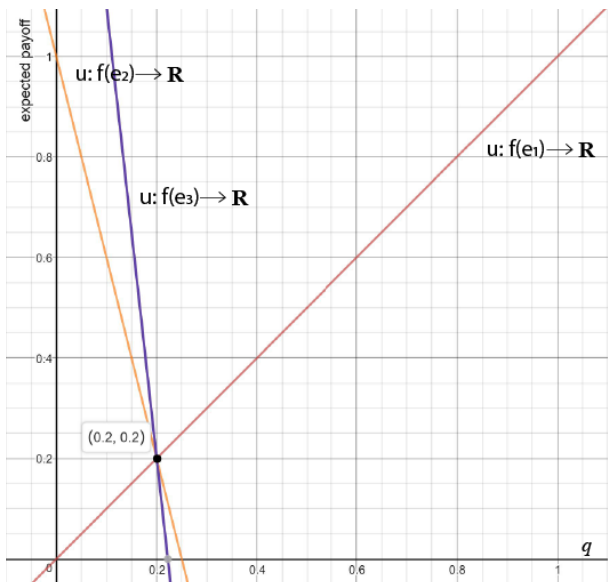

Figure 4: Expected payoff of player A over strategy $\mathbf{q}$

First to analyze the game from player A's viewpoint, a graph that displays A's expected payoff functions in accordance to player B's q_1 is constructed (figure 4). The graph is two-dimensional because player B has two pure strategies. It can be intuitively explained by answering the following question: if player B plays a mixed strategy $\mathbf{q}$ ($\mathbf{q}$ relies on q_1), how does player A's expected payoff behave when their mixed strategy are $\mathbf{p} = (1, 0, 0) = \mathbf{e}^1$, $\mathbf{p} = (0, 1, 0) = \mathbf{e}^2$, and $\mathbf{p} = (0, 0, 1) = \mathbf{e}^3$? Respectively, player A's expected payoff of $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$ are red line, yellow line, and purple line.

From the graph, we can see that the purple line has the highest value when $0 < q_1 < 0.2$, the red line has the highest value when $0.2 < q_1 < 1$, and the three lines hold the same value when $q_1 = 0.2$. Mathematically speaking, if we denote $\beta_A(\mathbf{q})$ as the set of best replies of player A against $\mathbf{q} = (q_1, 1-q_1)$, the graph shows that

$$\beta_A(\mathbf{q}) = \beta_A((q_1, 1-q_1)) = \begin{cases} e^2 & \text{if } 0 \leq q_1 < 0.2 \\ \Delta^3 & \text{if } q_1 = 0.2 \\ e^1 & \text{if } 0.2 < q_1 \leq 1 \end{cases}$$

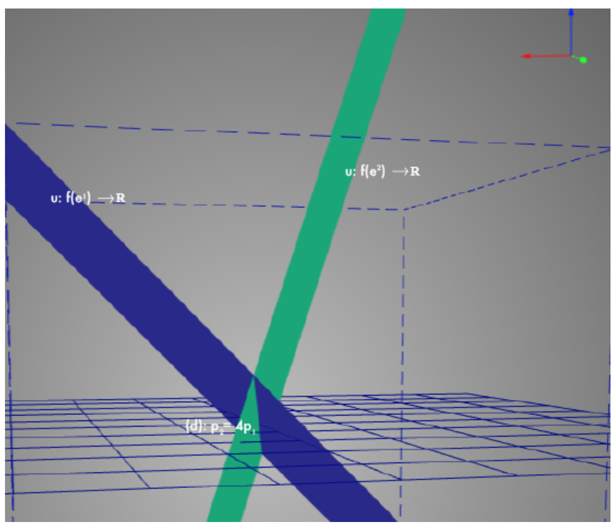

Figure 5: Expected payoff of player B over strategy $\mathbf{p}$

We now analyze the best replies from B's viewpoint. The 3-dimensional graph demonstrates player B's expected payoff function in accordance to player A's p_1 and p_2. Figure 5 shows how player B gains/loses when playing $\mathbf{e}^1, \mathbf{e}^2$. Considering player A plays a mixed strategy $\mathbf{p} = (p_1, p_2, 1 - p_1 - p_2)$, player B's $\mathbf{e}^1$ and $\mathbf{e}^2$ alternatively be the best reply except when $f(\mathbf{e}^1) = f(\mathbf{e}^2)$ or $-4p_1 + p_2 = 0$, where both have the same payoff.

Mathematically, we denote $\beta_B(\mathbf{p})$ as the set of best replies of player B against strategy $\mathbf{p}$.

$$\beta_B(\mathbf{p}) = \beta_B((p_1, p_2, 1-p_1-p_2))) = \begin{cases} e^1 & \text{if } p_2 > 4p_1 \\ \Delta^2 & \text{if } p_2 = 4p_1 \\ e^2 & \text{if } p_2 < 4p_1 \end{cases}$$

Corollary 13.6 of [5] explicitly points out the way to compute Nash equilibrium in a bi-matrix game (A,B).

Corollary 3.6 *A strategy pair* $(\boldsymbol{p}, \boldsymbol{q})$ *is a Nash equilibrium in a bi-matrix game (A,B) if and only if* $C(\boldsymbol{p}) \subseteq PB_A(\boldsymbol{q})$ *and* $C(\boldsymbol{q}) \subseteq PB_B(\boldsymbol{p})$*.*

where

- $PB_A(\mathbf{q})$ is the set of pure best replies of player A to $\mathbf{q}$, in other words, decisions that gains the most when facing mixed strategy $\mathbf{q}$:

$$PB_A(\mathbf{q}) = \{i \ \in \{1,..,m\} \mid \mathbf{e}^i A\mathbf{q} = \max_k \mathbf{e}^k A\mathbf{q}\}$$

- $C(\mathbf{p})$ of mixed strategy $\mathbf{p} \in \Delta^m$ is a set of coordinates i where p_i is positive, i.e.

$$C(\mathbf{p}) = \{i \in \{1, ..., m\} \mid p_i > 0\}$$

Assuming player B plays a mixed strategy $\mathbf{q}$, where $q_1 = 0.2$. So $\mathbf{q} = (q_1, 1-q_1) = (0.2, 0.8)$, then

$$PB_A((0.2, 0.8)) = \{1, 2\} \tag{5}$$

Considering $\mathbf{p}_2 = 4\mathbf{p}_1$, or strategy $\mathbf{p} = (p_1, 4p_1, 1-5p_1)$, we have

$$PB_B(\mathbf{p}) = \{i \ \in \{1, 2, 3\} \mid \mathbf{e}^i B\mathbf{q} = \max_k \mathbf{e}^k A\mathbf{q}\} \tag{6}$$

If player B plays $(0.2, 0.8)$ and player A plays $\mathbf{p}$ where $p_2 = 4\mathbf{p}_1$, then a Nash equilibrium pair $(\mathbf{p}*, \mathbf{q}*)$ occurs if and only if C$((p_1, 4p_1, 1 - 5p_1))$ $\subseteq PB_A(\mathbf{q}) = \{1, 2\}$ and C((0.2, 0.8)) $\subseteq PB_B(\mathbf{p})$. So p_1, p_2 must be positive and $p_3 = 0$ or $p_1 = 0.2$. Therefore, $\mathbf{q} = (0.2, 0.8)$ and $\mathbf{p} = (0.2, 0.8, 0)$ is one of Nash equilibriums of the game.

3.5 Solving complex games

As mentioned in 2.1, Nash Theorem states that there exists at least one Nash equilibrium in a bi-matrix game. However, computing Nash equilibrium in large two-person games is humanly impossible due to two problems. First, the graphical representation of bi-matrix games, which is an n-dimension support, is based on the number of strategies each player has. Thus, games with more than three strategies means 4-dimension graphs or more are excessive to visualize. Additionally, games with large strategy spaces have polynomial run-time [8] so real-time Nash equilibrium computation is only feasible for small dimensional support, roughly speaking, twenty-two dimensions.

4 The software: Colonel Balloon Game

4.1 Colonel Blotto in Game Theory

The Blotto game is first introduced as a game where "the psychology of the players matters" by Borel in 1921 and further researched by Gross and Wagner in 1950. The finite two-person game rule is simple: player A and B both have a limited size of n resources; not knowing what the others do, both distribute n troops to K battlefields; the player who wins most battlefields wins the game. One famous application of this particular game is its model in electoral politics discussed by Laslier and Picard (2002), i.e. when running for presidents, two candidates allocate limited budgets to fifty-two states for PR campaigns.

The game classifications are non-cooperative, simultaneous (strategic game), zero-sum game. This means the game can be represented as a bi-matrix payoffs and one player's payoffs (utility) are the opposite of the other player's. The Nash equilibrium of Colonel Blotto game is straightforward. Borel [2] shows that if $K = 3$, Nash equilibrium occurs when the probability distribution on each strategy is a uniform distribution over $[0, \frac{2}{3}]$. In other words, each player's best reply is to randomize the strategy so that the marginal distribution in each battlefield is uniform in $[0, \frac{2 \cdot n}{K}]$ [4].

Despite having simple rules, Blotto game can be extremely complex if K and n are large. For instance, with $n = 6$ and $K = 3$, the pure strategy set S_A, S_B have cardinality of 10, but for $n = 12$ and $K = 3$, the cardinality is up to 55. Because of this, Blotto's Nash equilibrium is hard to generalize and solve efficiently in a reasonable run-time. As demonstrated in section Three, Nash equilibrium centers on best replies (optimal strategies), so we solve the game by analyzing their strategies.

A good way to analyze Colonel Blotto game is to abstract the strategic behaviors [13]. For instance, a strategy where players can distribute the n troops evenly across K battlefields are called focal points. Additionally, players can focus on either left or right sides of K battlefields, and those with more than $\frac{n}{K}$ troops are called "reinforced battlefield". As the project focuses on Nash equilibrium and optimal strategies in a finite two-person game, "Colonel Balloon", an educational stimulation of the Colonel Blotto game is built.

4.2 Colonel Balloon in the software

4.2.1 Software Description

This section provides an overview of the game engine of *Colonel Balloon Game*. The Blotto game, written in Python, is an educational game that aims to help players understand the basics of game theory and strategic abstraction. The *Colonel Balloon* is a one-user software where two players play against each other in a water balloon fight. The game is inspired by the web-based experience *The Evolution of Trust* [10]. The software is written in Python 2.7, and using PyQT4 for User Interface. The graphics are drawn and designed manually for a pleasant and informal player experience.

Having the possibility to be extremely complex, Colonel Balloon's rules are more specific for a simpler game strategy set:

- Player P and computer C has n balloons, where n is randomized in a set N $= \{9, 12, 15, 18\}$.
- Not knowing each other's strategies, P and C simultaneously prepare their resources across $K = 3$ fields.
- The number of balloons distributed in a field can be zero, but the balloons in the current field cannot be greater than ones after it.
- User P are expected to play repeatedly against three different types of opponents and each game's result is independent from one to another.

Using Kohli's strategic abstraction approach, the game customizes three types of strategy behaviors to let users play against:

- Uniform strategy: the computer always distributes n troops evenly across K=3 battlefields. In other words, its mixed strategy is a pure strategy behavior where $s = (\frac{n}{3}, \frac{n}{3}, \frac{n}{3})$.
- Right-focused strategy: the computer tends to distribute troops at the right battlefields. This means that its mixed strategy is a fixed probability distribution whose pure strategies at the beginning such as $\{(0, 0, n), (0, 1, n-1), ...\}$ have higher probability - the more right-focused a strategy is, the more likely it is to be played.

- Random strategy: the computer randomly distributes n troops across K battlefields. So its mixed strategy is an equal probability distribution over all pure strategies. This mixed strategy is the best reply in a Colonel Blotto game.

4.2.2 Software Implementation

The game engine of *Colonel Balloon* deals with three main classes: *MyGame*, *MyPlayer*, and *Strategy.*

The *MyPlayer* class specifies player types. Its main function is to manage a player's basic information such as name and score, as well as a player's mixed strategy profile and choice of playing. Aside from the child class *ThePlayer* which only manages the user's information, other child classes namely *UniformPlayer*, *RightPlayer*, and *RandomPlayer* all have predefined mixed strategy profile in accordance to their strategic abstracted behaviors.

As the strategy in *Colonel Balloon* is a three-tuple that follows the previously mentioned rule, *Strategy* first generates all 3-tuples of a (n, 3) game using the combinatoric *Stars and bars* principle. The strategy enumeration algorithm (developed by [12]) is based on Theorem 2 of *Bars and stars.* The theorem states that the number of ways to distribute N stars between $K + 1$ bars is the same as to the that to select N out of $N + K - 1$ positions [11]. After generating all strategies using *Stars and bars*, the strategy list is filtered based on the game's rule and input into the already-constructed empty game.

MyGame class constructs and manages a finite two-person game with three levels, each level is occupied by each computer player. Because *Colonel Balloon* is a zero-sum game where payoffs and strategies stay the same in the bi-matrix even if column players switch places with row players, the game's strategy space is optimize into one strategy set and payoff matrix instead of two. In other words, after constructing a *Strategy* object in accordance to the randomized n, *MyGame* manages the strategy space and let *MyPlayer* uses the strategy space when necessary.

5 Conclusion

Nash Theorem and Nash equilibrium are crucial in Game theory as they link tightly with best replies and strategic behaviors. For example, when playing Rock, Paper, Scissors, if the strategy for player A is assumed to be 0.3 Rock, 0.3 Paper, and 0.4 Scissors, the expected outcomes of the game when played by that strategy in real life is what interests game theorists. In addition, the intersection of Game theory and Computer Science, algorithmic game theory, results in many practical solutions in both fields. Computer scientists solve problems in machine learning, computer networking, and algorithm analysis by conceptualize the issues into games. For example, spam detection in machine learning applies sequential game between two players: spam detector and spammers. Specifically, the spam detector records and analyzes the behaviors of senders and after a sequence of time, decide if the sender is a spammer. More illustrations of game theory applications in computer science can be found in [9].

Despite being a sub-field of Mathematics and Economics, game theory in practice largely involves human behaviors and psychology logic. This makes the field versatile, hence its wide application in not only math, computer science, and economics but also sociology, psychology, and political science. In strategic interactions, game theory aims to seek optimal strategies. Thus, Nash equilibrium being a situation where players maximize their payoff, is fundamental in fulfilling this goal. Theoretically speaking, mixed strategy Nash equilibrium means a player can randomize their pure strategies to obtain the best replies for the game. However, questions about the accuracy of optimal strategies and Nash equilibrium in real life are still being explored.

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7 Appendix: Source Code