Statistical Methods in Finance Project

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1 Summary

The goal of this project was to apply financial statistics methods to monthly closing prices for twenty assets of technology companies between January 2015 and December 2019. The assets that were chosen are shown in Table 1. We used S&P 500 as the market portfolio and 3 month US treasury bills as the risk-free rate.

Our results can be summarized as follows:

- Mastercard (MA) stock produced the optimal risk-adjusted returns during the period considered.
- With portfolio theory, Mastercard(MA) has the largest portfolio weight whether short-sales are or are not allowed.
- The efficientportfolio considering T-Bills has a smaller VaR than the portfolio with only risky assets.
- With factor analysis, we find that there are three factors that account for variance in return performance - one for telecommunications, one for software and hardware technologies, and one broad index of technology companies.
- Minimum variance portfolio with long only has the lowest value at risk and expected shortfall.
- t-copula fits the joint distribution of returns best accounting for outlier values.

2 Descriptive Statistics

From Table 1, we observe that the best performing assets, in terms of expected value, are NVDA, NFLX, and AMZN. We find that SPY, MA, and IBM are all moderately negatively skewed while remaining assets have no skew. IBM also has a relatively large kurtosis value of 3.2, indicatig it has heavy tails. Finally, the assets with largest beta values are: NVDA, BABA, and AMZN suggesting these stocks are the most volatile i.e. have largest risk-reward.

key	mean	sd	min	max	skewness	kurtosis	beta	IsStationary	ShapiroPval
SPY	0.0091	0.0345	-0.0922	0.0816	-0.6006	0.7254	1.0000	TRUE	0.024
RF	0.0108	0.0088	0.0001	0.0245	0.2299	-1.5296	0.0000	FALSE	0.000
AAPL	0.0178	0.0761	-0.1999	0.1827	-0.5228	0.0692	1.1939	TRUE	0.225
MSFT	0.0220	0.0610	-0.1462	0.1792	-0.1234	0.5823	1.1562	TRUE	0.269
AMZN	0.0298	0.0804	-0.2259	0.2156	-0.2009	0.9572	1.5501	TRUE	0.296
GOOGL	0.0155	0.0570	-0.1015	0.1968	0.5549	0.6971	1.0439	TRUE	0.109
FB	0.0160	0.0683	-0.1186	0.2403	0.4346	0.5821	1.0551	TRUE	0.408
V	0.0180	0.0443	-0.0851	0.1151	-0.2228	-0.3492	0.8993	TRUE	0.554
MA	0.0213	0.0486	-0.1177	0.1143	-0.5607	0.2113	0.9579	TRUE	0.180
INTC	0.0106	0.0634	-0.1414	0.1778	-0.1859	-0.0364	0.8979	TRUE	0.682
T	0.0069	0.0490	-0.1394	0.0987	-0.3945	0.1081	0.6583	TRUE	0.546
VZ	0.0082	0.0485	-0.1245	0.0935	-0.2196	-0.5035	0.5194	TRUE	0.439
CSCO	0.0116	0.0624	-0.1685	0.1218	-0.4617	-0.1144	1.2055	TRUE	0.307
CMCSA	0.0091	0.0592	-0.1609	0.1172	-0.5668	0.0203	1.0466	TRUE	0.115
ADBE	0.0252	0.0569	-0.1034	0.1605	-0.0400	-0.4242	1.0682	TRUE	0.904
ORCL	0.0042	0.0529	-0.1021	0.1186	0.1413	-0.5767	1.0953	TRUE	0.564
CRM	0.0168	0.0655	-0.1474	0.2062	-0.0097	0.4999	1.1945	TRUE	0.588
NVDA	0.0414	0.1225	-0.2887	0.3260	-0.4847	0.8377	2.2040	TRUE	0.049
NFLX	0.0312	0.1165	-0.2195	0.3422	0.2332	0.0581	1.3786	TRUE	0.373
BABA	0.0120	0.1051	-0.2177	0.3518	0.2984	0.5626	2.1438	TRUE	0.545
IBM	0.0003	0.0665	-0.2700	0.1677	-0.7549	3.2042	1.3767	TRUE	0.004
AVGO	0.0212	0.0787	-0.2353	0.2365	-0.0329	1.2924	0.9338	TRUE	0.187

Table 1: Summary statistics for S&P 500, risk free, and our 20 stocks. We report the mean, standard deviation, minimum, maximum, skewness, kurtosis, and beta values for each asset. We also report whether each return time series is stationary and p-value for Shaprio-Wilk test for normality.

We also plot the individual assets below in Figures 8 and 9. First notice that all log returns ranges are in [-0.2, 0.2]. As expected the risk free returns have least variance and therefore exhibit a smooth curve. To check for outliers we report boxplots of all of the assets in Figure 10. Next to see which assets were individually best performing over the full five year stretch from 2015 to 2019 we observe the equity curves shown in Figures 11 and 12.

Applying KPSS test for stationarity, we find that the risk-free asset has non-stationary distribution (p-val: 0.0355). And several assets have non-normal distribution as determined by Shapiro-Wilk test for normality: SPY, RF, NVDA, and IBM as reported in Table 1.

3 Portfolio Theory

Firstly, by considering the equitable-weighted portfolio, we can find that the variance of this equitable-weighted portfolio is 0.001938. Furthermore, its return is 1.657%, and its standard deviation rate indicates the risk is 4.402%. Figure 1 below shows the weight of this equitable-weighted portfolio.

	AAPL	MSFT	AMZN	GOOGL	FB	V	MA	INTC	T	VZ	CSCO	CMCSA	ADBE	ORCL	CRM	NVDA	NFLX	BABA	IBM	AVGO
Weight	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
sum	1																			

Figure 1: Summary statistics for equitable-weighted portfolio

Secondly, we determine the minimum variance portfolio (MVP) when short sales are not permitted. The weights of the MVP are shown in Figure 2. The variance of the MVP is 0.001, its return as the monthly mean is 1.20%, and its standard deviation rate as the monthly risk is 3.21%. Then, its value at risk (VaR) is -4.08%.

Now if we are required to invest $100,000, we can obtain -$4,080 as the expected shortfall by considering the monthly value-at-risk of the $100,000 investment. In particular, if we compare this VaR of MVP with each value of 20 assets, we can observe that the MVPs VaR is the lowest. If we annualize the return and standard deviastion of this MVP, the results are as follows. The return as the annual mean is 14.40%, and the standard deviation as the annual risk is 11.12%. Then, its value at risk (VaR) is -3.89%. Provided that we compare the monthly mean and standard deviation of MVP with equitable-weighted portfolio, the mean is increased and the standard deviation is decreased. This shows that the return of MVP is increased, and the risk of MVP is lowered.

	AAPL	MSFT	AMZN	GOOGL	FB	V	MA	INTC	T	VZ	CSCO	CMCSA	ADBE	ORCL	CRM	NVDA	NFLX	BABA	IBM	AVGO
Weight	0	0	0	0.0665	0.0186	0.00758	0.2103	0.0606	0.1999	0.32908	0	0	0	0.10001	0	0	0	0	0	0.0073
Sum	1																			

Figure 2: Summary statistics for weights of minimum varaince portfolio(MVP) when long only is considered

From Figure 2, we can know that the percentage of our investment should be as follows: (1) 32.91% to VZ (2) 21.03% to MA (3) 19.99% to T (4) 10.00% to ORCL (5) 6.65% to GOOGL (6) 6.06% to INTC (7) 1.86% to FB (8) 0.76% to V (9) 0.73% to AVGO.

When short sales are not permitted for the tangency portfolio, the Sharpe ratio is 0.3431. Furthermore, the variance of this tangent portfolio is 0.0022, its return is 2.67%, and its standard deviation is 4.67%. Figure 3 below shows the weight of this tangent portfolio.

	AAPL	MSFT	AMZN	GOOGL	FB	V	MA	INTC	T	VZ	CSCO	CMCSA	ADBE	ORCL	CRM	NVDA	NFLX	BABA	IBM	AVGO
Weight	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
Sum	1																			

Figure 3: Summary statistics for weights of tangent portfolio when long only strategies are considered

From Figure 3, it seems that the best investment strategy is to invest only in Mastercard (MA). To explain this, we should explore the Sharpe ratio of each stocks.

Sharpe ratio	0.1336	0.137	0.2073	0.0631	0.0457	0.25467	0.3384	-0.033	-0.131	-0.0157	0.03848	-0.082	0.2322	-0.1456	0.1385	0.0065	0.1895	-0.0268	-0.176	0.0864

Figure 4: Statistics for the Sharpe Ratio of each stocks

As you can see from Figure 4 above, the Sharpe ratio of MA is larger than the Sharpe ratio of the other stocks. This suggests that MA stock has the best risk-adjusted performance of all the twenty risky assets.

When short sales are allowed, the variance of the minimum variance portfolio (MVP) is 0.0003. Furthermore, its return is 2.04%, and its standard deviation is 1.65%. Its VaR is -0.67% andits Sharpe ratio is 0.5904. To explain further, compared with the Sharpe ratio when short sales are not allowed, the value of the Sharpe ratio with short sales is increased. Figure 5 below shows the weight of the minimum variance portfolio (MVP).

	AAPL	MSFT	AMZN	GOOGL	FB	V	MA	INTC	T	VZ	CSCO	CMCSA	ADBE	ORCL	CRM	NVDA	NFLX	BABA	IBM	AVGO
Weight	0.0862	-0.413	-0.004	0.2531	0.0107	0.05843	0.7051	0.1354	0.1796	0.40228	-0.1195	-0.23	0.2756	0.33652	-0.116	-0.222	-0.093	-0.0758	-0.165	-0.006
Sum	1																			

Figure 5: Summary statistics for weights of minimum variance portfolio (MVP) when short sales are allowed

From Figure 5, we find that the percentage of our investment should be as follows: (1) 71% to MA (2) 40% to VZ (3) 34% to ORCL (4) 28% to ADBE (5) 25% to GOOGL (6) 18% to T (7) 14% to INTC (8) 9% to AAPL (9) 6% to V (10) 1% to FB. On the other hand, the percentage of short-sales should be as follows: (1) -41% to MSFT (2) -23% to CMCSA (3) -22% to NVDA (4) -16% to IBM (5) -12% to CRM (6) -12% to CSCO (7) -9% to NFLX (8) -8% to BABA (9) -1% to AVGO.

If we analyze the weights of this MVP, we notice that our investment on MA, VZ, ORCL and others can exceed the portfolio size. However, this is fine because, by selling stocks, we can make enough money for investment. When short sales are allowed in the tangency portfolio, the Sharpe ratios is 1.1018 and the variance is 0.00009. Furthermore, its return is 4.46%, its standard deviation is 3.08%, and its VaR is -0.60%. Figure 6 below shows the weight of the tangency portfolio.

	AAPL	MSFT	AMZN	GOOGL	FB	V	MA	INTC	T	VZ	CSCO	CMCSA	ADBE	ORCL	CRM	NVDA	NFLX	BABA	IBM	AVGO
Weight	0.1622	-0.375	0.2352	0.4166	-0.228	-0.3723	1.9289	0.004	0.0492	0.45914	0.0332	-0.37	0.2621	0.19424	-0.481	-0.316	-0.2	0.07948	-0.641	0.1597
Sum	1																			

Figure 6: Summary statistics for weights of tangency portfolio when short sales are allowed

From Figure 6, we find that the percentage of our investment should be as follows: (1) 193% to MA (2) 46% to VZ (3) 42% to GOOGL (4) 26% to ADBE (5) 24% to AMZN (6) 19% to ORCL (7) 16% to AAPL (8) 16% to AVGO (9) 8% to BABA (10) 5% to T (10) 3% to CSCO. On the other hand, the percentage of short-sales should be as follows: (1) -64% to IBM (2) -48% to CRM (3) -37%

to MSFT (4) -37% to V (5) -37% to CMCSA (6) -32% to NVDA (7) -23% to FB (8) -20% to NFLX

If we analyze those results, MA is the largest part of investment for this portfolio. We find that short-sales of IBM, CRM, and others help raise money for the investment in MA.

4 Asset Allocation

First of all, suppose we wanted to achieve a target expected return of 12.2% per year, which corresponds to an expected return of 1.1% per month, using only the twenty tech stocks. Furthermore, suppose that short-sales are not allowed. The efficient portfolio that achieves 1.1% as the target return is the intersection point between 'y=2' and the portfolio where twenty tech stocks have the same weight.

	AAPL	MSFT	AMZN	GOOGL	FB	V	MA	INTC	T	VZ	CSCO	CMCSA	ADBE	ORCL	CRM	NVDA	NFLX	BABA	IBM	AVGO
weight	3.59%	3.77%	3.13%	4.05%	4.06%	3.54%	3.24%	4.45%	4.73%	4.34%	4.13%	4.64%	3.33%	4.85%	3.71%	4.24%	2.81%	4.47%	5.13%	3.79%

Figure 7: Statistics for weights of efficient portfolio

From Figure 7, each of the assets in this efficient portfolio should be invested by following the above percentages. While calculating the weights, we can find that the monthly risk on this efficient portfolio is 1.58%, which means that the standard deviation of this portfolio is 1.58%. Furthermore, the monthly 5% value-at-risk is -1.50%. By using this VaR, we can know that expected shortfall based on an initial $100,000 investment is -$1499.

On top of that, suppose we wanted to achieve a target expected return of 1.1% per month by using a combination of T-Bills and the tangency portfolio that does not allow short sales. Here, T-bills are regarded as risk free assets. We already know that the mean of T-Bills is 1.068% from calculation. In this allocation, T-bills should be invested at a rate of 75.76%, which means that the tangency portfolio should be invested at a rate of 24.242%. To put it another way, the investment of T-Bills takes bigger place than the investment of the tangency portfolio. In particular, this efficient porfolio of this allocation has the standard deviation as 0.77%. In other words, the monthly risk on this efficient portfolio is 0.77%. Furthermore, 5% value-at-risk of this portfolio is -0.17%, and the expected shortfall based on an initial $100,000 investment is -$166.5.

Let's compare the VaR of this combination considering T-Bills with the VaR computed from risky assets allocation without short sales. Then, we found that the former is -0.17%, and the latter is -1.50%. Therefore, we can conclude that the VaR of the combination considering T-Bills is lesser than the VaR computed from the allocation of risky assets without short sales.

5 Principal Component Analysis

Assets with largest correlations included Visa (V) - Mastercard (MA) with correlation of 0.83, Salesforce (CRM) - Mastercard (MA) with correlation of 0.74, and Adobe (ADBE) - Mastercard (MA) with correlation of 0.74. Based on all of the estimated correlation values, diversification is guaranteed to reduce expected risk. Running PCA analysis, we find that first three components acount for 62.3% of the variance. More importantly we find that the second and third components separate the twenty stocks into three distrinct groups: one with negative loadings on the PC2 (e.g. VZ, T, CMC), another one with positive loading on the PC3 (e.g. ORCL, IBM, NVDA, AAPL), and the last one with positive loading on the PC2 and negative loading on the PC3 (e.g. GOOGL, AMZN, NFLX, CRM) as seen in Figure 13.

Next we fit a three-factor factor model wuth maximum likelihood. Looking at Figure 14, we see that three factors is sufficient and combined they account for 55.4% of the variance of all stock returns. For factor 1, the largest loadings are for AMZN, V, and CRM suggesting these stocks have largest influence on this factor. For factor 2, the maximum loadings are with Intel (INTC), Adobe (ADBE), and Broadcom (AVGO), and for factor 3 the maximum loadings are with Verizon (VZ) and Comcast (CMCST). These loadings suggest that factor 3 is modeling overall performance of telecommunications industry, factor 2 performance of software and hardware technologies, and factor 1 is a broad tracker of technology industries.

6 Risk Management

Assume that we have $100,000 to invest, and estimate the 5% value-at-risk (VaR) and expected shortfall (ES) for each asset on this investment over a one month investment horizon. We first use the parametric method to estimate the 5% VaR and ES for the monthly returns of each asset based on the normal distribution. The VaR and ES under this parametric method are denoted as VaR_par and ES_par. Then, we use the nonparametric method to estimate the VaR and ES agian by denoting them as VaR_nonp and ES_nonp. In addition, we also use the bootstrap method to compute the estimated standard errors and 95% confidence intervals for our 5% VaR and ES.

From Table 2, it is apparent that BABA has the highest VaR and V has the lowest VaR under the parametric method. Although V still has the lowest VaR under the nonparametric method, NVDA has the highest value of VaR with nonparametric method. By looking at the Table 3 of expected shortfalls, under both parametric and nonparametric methods, NVDA has the highest value of ES and V has the lowest one. These results indicates that NVDA can bear a worse loss than other assets while V has the lowest ability to bear a loss. In other words, NVDA could bear more risk than other assets while V could not.

In Tables 4 and 5 below, we report the VaR and ES results for the previously considered portfolios. We find that MVP has the lowest VaR and ES because

	name	VaR_par	VaR_nonp	sd.error	2.5%CI	97.5%CI
1	BABA	15948.14	16051.98	2660.04	11178.34	21605.49
2	NFLX	15889.03	13166.31	3010.37	6116.30	17916.75
3	NVDA	15842.20	20485.25	7885.94	7570.51	38482.85
4	IBM	10825.21	8772.75	2451.78	3781.61	13392.42
5	AVGO	10711.55	9038.75	2318.85	4349.47	13439.18
6	AAPL	10636.92	12451.58	3068.76	7363.96	19393.28
7	AMZN	10131.41	8368.43	3529.74	882.94	14719.28
8	FB	9543.24	8670.52	1511.15	5482.83	11406.45
9	INTC	9291.61	10476.29	1723.28	7680.99	14436.11
10	CSCO	9017.45	9392.37	2168.35	5334.18	13833.94
11	CRM	9008.79	8904.11	3112.07	3202.46	15401.57
12	CMCSA	8743.26	8034.04	2295.98	2645.75	11645.81
13	ORCL	8209.79	7698.89	777.52	6195.33	9243.16
14	GOOGL	7757.15	6884.28	874.65	5170.17	8598.74
15	MSFT	7743.90	7614.27	1631.76	4018.28	10414.68
16	T	7304.57	7610.93	1745.41	4303.30	11145.17
17	VZ	7087.73	6642.82	1076.02	4661.13	8879.04
18	ADBE	6767.66	6563.75	1550.26	3716.84	9793.74
19	MA	5792.83	6483.61	1849.83	2749.45	10000.64
20	V	5419.76	6235.54	992.79	4696.47	8588.15

Table 2: Summary of Value at Risk (VaR) for each of the twenty different stocks using both parametric and nonparametric methods. Confidence intervals are computed based on parametric method i.e. assuming normality of asset returns

MVP is contructed with the lowest variance compared with other portfolios. This is reasonable due to MVP's lowest risk. The tangency portfolio with short sales allowed has the largest estimated VaR and ES because more risky methods of investing are allowed this time. Based on the VaR and ES results of different portfolios, we observe that the more risky portfolio we have, the higher its standard error is, and therefore higher the value of the loss it could bear.

	name	ES_par	ES_nonp	sd.error	2.5%CI	97.5%CI
1	NVDA	20917.85	27657.36	4719.02	20408.29	38906.51
2	NFLX	20717.08	19425.76	2798.27	14093.31	25062.33
3	BABA	20304.60	19332.20	1908.32	15695.53	23175.99
4	AVGO	13971.16	14642.60	5025.14	4416.76	24114.93
5	AAPL	13790.21	16109.19	2804.86	10852.98	21847.83
6	IBM	13582.39	15234.22	6534.28	1527.48	27141.37
7	AMZN	13462.85	16164.80	4648.56	7441.88	25663.91
8	FB	12373.09	10974.94	1030.17	9179.83	13218.01
9	INTC	11920.50	12366.05	1368.60	9685.73	15050.54
10	CRM	11724.47	13207.64	2172.63	9457.81	17974.36
11	CSCO	11602.45	13094.81	2676.18	7915.18	18405.61
12	CMCSA	11195.68	13127.13	2594.47	8363.68	18533.81
13	ORCL	10401.68	8993.44	852.16	7301.81	10642.22
14	MSFT	10271.25	11202.51	2323.69	6584.78	15693.48
15	GOOGL	10120.77	8553.32	1089.02	6455.77	10724.63
16	T	9335.83	10549.88	2158.29	6367.61	14827.95
17	ADBE	9127.39	8825.28	1523.49	5983.78	11955.75
18	VZ	9096.62	8978.22	2037.22	4896.83	12882.60
19	MA	7806.02	9679.76	1685.84	6639.15	13247.54
20	V	7253.34	7312.30	875.62	5602.43	9034.79

Table 3: Summary of expected shortfall for each of the twenty different stocks using both parametric and nonparametric methods. Confidence intervals are computed based on parametric method i.e. assuming normality of asset returns

	names	estimate	sd.error	2.5%CI	97.5%CI
1	Tangency(short sales)	9649.93	3754.64	1180.33	15898.26
2	Equal Weight	6594.12	2347.62	2248.32	11450.81
3	Tangency(long only)	6483.61	1849.83	2749.45	10000.64
4	MVP(short sales)	5572.92	1911.15	1405.31	8896.87
5	MVP(Long only)	3622.17	754.06	1966.11	4921.96

Table 4: VaR of different portfolios

	names	estimate	sd.error	2.5%CI	97.5%CI
1	Tangency(short sales)	16826.80	3591.38	9971.26	24049.20
2	Tangency(long only)	9679.76	1685.84	6639.15	13247.54
3	Equal Weight	9403.71	1408.35	7161.00	12681.63
4	MVP(short sales)	9096.94	1780.60	5817.76	12797.57
5	MVP(Long only)	5425.78	1140.61	3260.89	7732.00

Table 5: ES of different portfolios

7 Copulas

In this section, we use different types of copulas to model the joint distribution of the returns. The following table shows the results of maximum log-likelihood and AIC by fitting these different copulas into the data.

	Copula Family	Maximized log-likelihood	AIC
1	Students t	232.96	-461.91
2	Gaussian	216.70	-431.40
3	Clayton	193.56	-385.12
4	Frank	168.24	-334.48

Table 6: Summary of different copula and their performance in terms of log-likelihood and AIC

From the results above, t copula fits the data better than other types of copulas because it has the highest value of the maximum log-likelihood and lowest value of AIC. Hence, the codependency of the returns of 20 assets tends to be a Students t-distribution.

8 Conclusion

We find that with univariate analyses Mastercard produces best risk-adjusted returns. With asset allocation, we found that using treasury bills as risk free assets allow for a more efficient portfolio i.e. one with a smaller value at risk. Next by applying factor analysis, we found that there are several factors which can account for the different return performances. Further exploration of other factors may yield a portfolio with larger Sharpe ratio. We also observe that the more risky a portfolio is, the higher its standard error is, and therefore the higher the value of the loss it could bear. Finally, a t-copula fits the dependence structure of returns best in terms of log likelihood, as it accounts for outlier values.

Contributions

Mark Aksen worked on sections 1, 2, 5, and 8. Sungbin Park worked on sections 1, 3, 4, and 8. And Qizhen Yang worked on sections 1, 6, 7 and 8.

Appendix

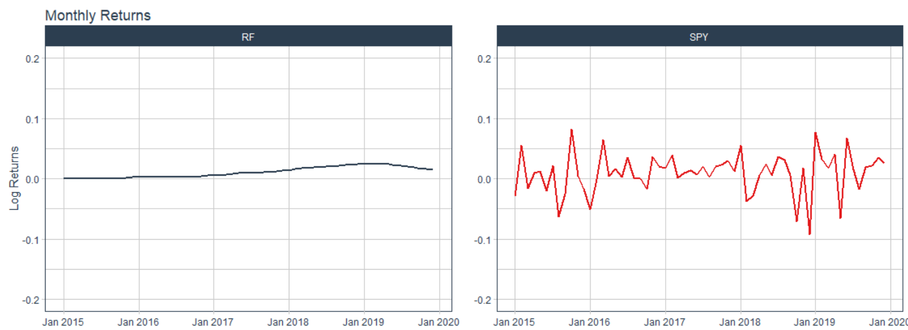

Figure 8: Log returns for S&P 500 and risk free assets

Figure 9: Log returns for our twenty risky assets

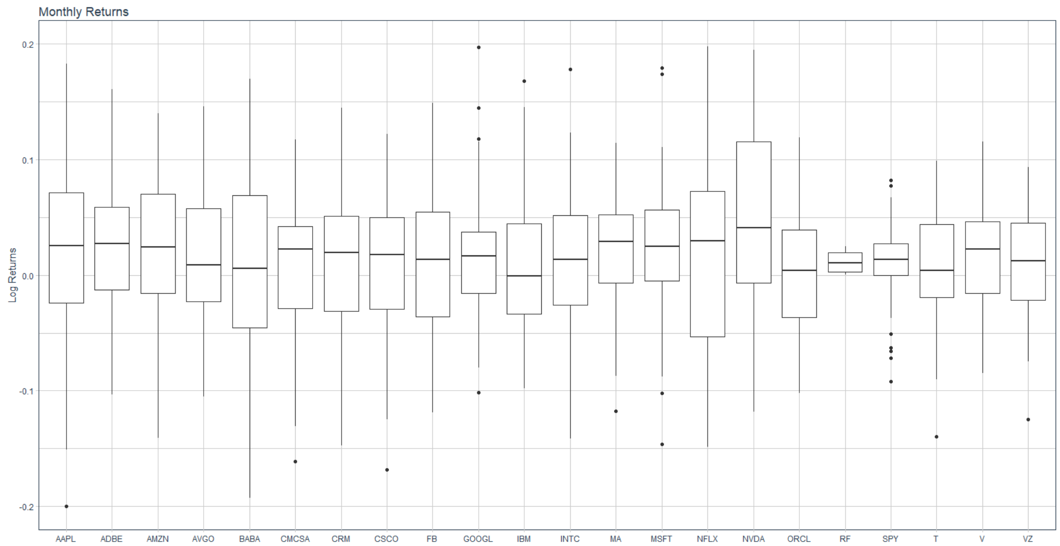

Figure 10: Boxplot of all assets (twenty stocks, SPY, RF)

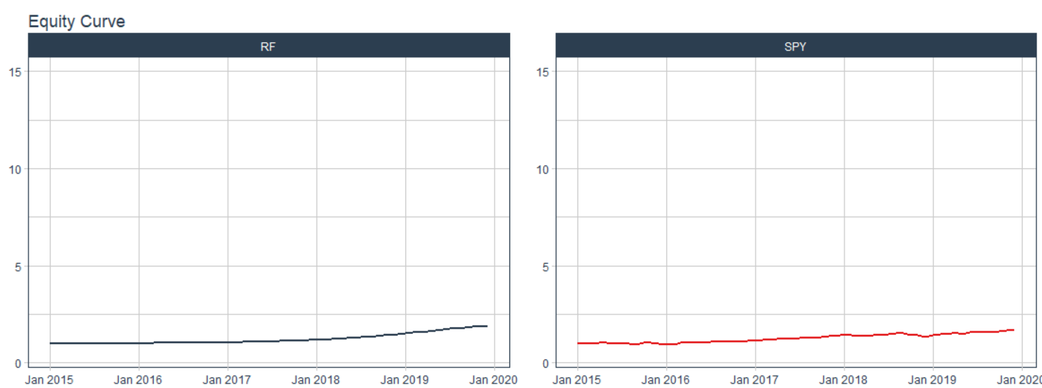

Figure 11: Equity curves for S&P 500 and risk free assets

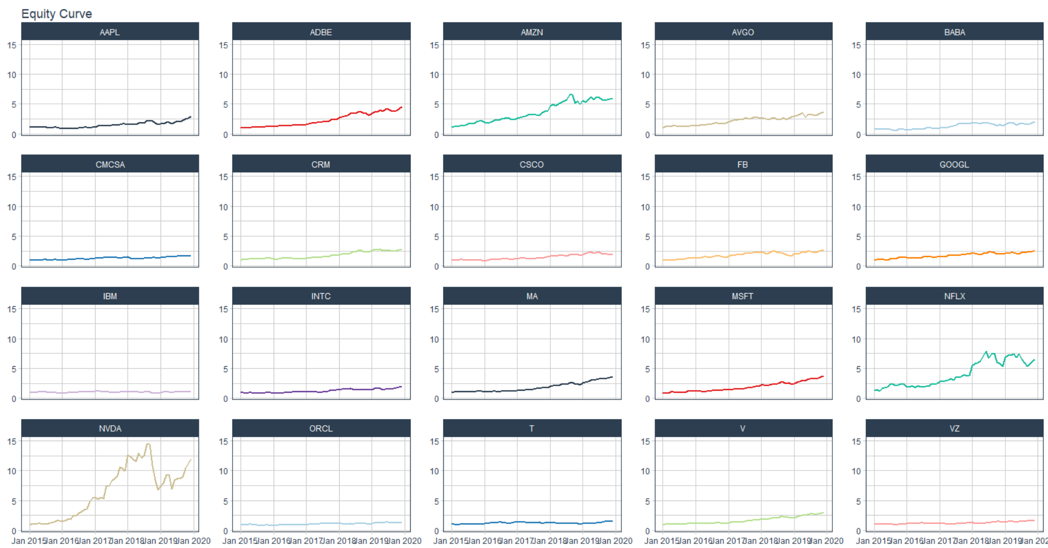

Figure 12: Equity curves for our twenty risky assets

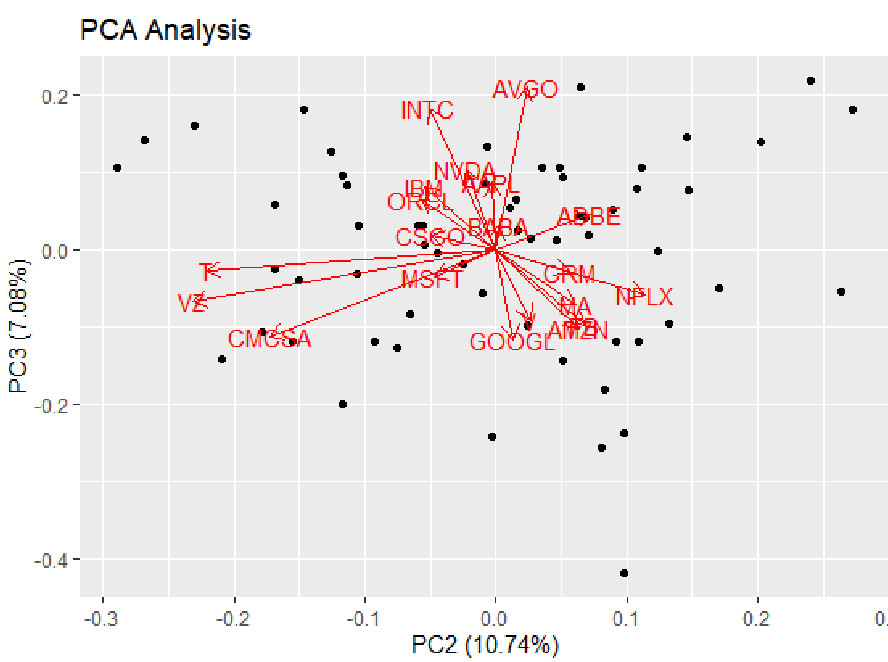

Figure 13: PCA components 2 and 3 separate the stocks into three distinct groups.

```
Loadings:
      Factor1 Factor2 Factor3
AAPL   0.377   0.527   0.112
MSFT   0.598   0.420   0.359
AMZN   0.741   0.256
GOOGL  0.655   0.195   0.222
FB     0.591   0.208
V      0.839   0.259   0.205
MA     0.853   0.305
INTC   0.169   0.664   0.189
T              0.135   0.638
VZ                     0.803
CSCO   0.465   0.423   0.266
CMCSA  0.267   0.162   0.640
ADBE   0.656   0.600
ORCL   0.331   0.566   0.252
CRM    0.714   0.392
NVDA   0.316   0.577   0.153
NFLX   0.613   0.213  -0.136
BABA   0.518   0.507   0.141
IBM    0.339   0.547   0.196
AVGO   0.155   0.623

               Factor1 Factor2 Factor3
SS loadings      5.512   3.582   1.981
Proportion Var   0.276   0.179   0.099
Cumulative Var   0.276   0.455   0.554

Test of the hypothesis that 3 factors are sufficient.
The chi square statistic is 155.4 on 133 degrees of freedom.
The p-value is 0.0899
```

Figure 14: 3-Factor model results