0.1 F: Quadratic Extensions

If F is a field whose characteristic is $\neq 2$, any quadratic extension of F is of the form $F(\sqrt{a})$ for some $a \in F$.

Proof. Suppose $F(\alpha)$ is a quadratic extension of F. Then, by definition, there is a minimum polynomial of α over F of degree 2, $g(x)$, of the form

$$g(x) = x^2 + \beta x + \gamma \text{ where } g(\alpha) = 0 \text{ and } \beta,\, \gamma \in F.$$

By completing the square,

$$g(x) = \left(x + \frac{1}{2}\beta\right)^2 + \left(\gamma - \frac{1}{4}\beta^2\right).$$

Now, we define $p(\chi)$ so that

$$p(\chi) = \chi^2 + \gamma - \frac{1}{4}\beta^2.$$

Then, it is easy to see that

$$p(x + \frac{1}{2}\beta) = g(x).$$

. Hence, α is a root of $p(x + c)$ for some $c \in F$. Well, we know that

$$\begin{aligned} F(\alpha) &\cong F[x] \;/\; \langle g(x) \rangle \\ &\Leftrightarrow F(\alpha) \cong F[x] \;/\; \langle p(x+c) \rangle \\ &\Leftrightarrow F(\alpha) \cong F[x] \;/\; \langle p(x) \rangle \end{aligned}$$

So α also is in fact a root of $p(x)$. That is,

$$\alpha^2 + \gamma - \frac{1}{4}\beta^2 = 0$$

$$\alpha^2 = \frac{1}{4}\beta^2 - \gamma$$

$$\alpha = \sqrt{\frac{1}{4}\beta^2 - \gamma}$$

$$\alpha = \sqrt{a} \text{ where } a \in F$$

$\square$