0.1 B: Further Examples of Finite Extensions

Let F be a field of characteristic $\neq 2$. Let $a, b \in F$ and $a \neq b$.

0.1.1

Prove that any field F containing $\sqrt{a}+\sqrt{b}$ also contains $\sqrt{a}$ and $\sqrt{b}$. Conclude that $F(\sqrt{a}+\sqrt{b}) = F(\sqrt{a}, \sqrt{b})$.

Proof. By definition, $\tau = \sqrt{a}+\sqrt{b} \in F(\sqrt{a}+\sqrt{b})$. Since $F(\sqrt{a}+\sqrt{b})$ is a field, $\tau^2 = a+2\sqrt{ab}+b \in F(\sqrt{a}+\sqrt{b})$. Then, it must be that $\sqrt{ab} \in F(\sqrt{a}+\sqrt{b})$. Well, since the product of any two elements in a field is an element of the field, it must be that $\tau\sqrt{ab} = a\sqrt{b} + b\sqrt{a} \in F(\sqrt{a} + \sqrt{b})$. Hence, we have shown that any field F containing $\sqrt{a} + \sqrt{b}$ also contains $\sqrt{a}$ and $\sqrt{b}$.
Since by definition, $F(\sqrt{a}, \sqrt{b})$ is the minimum field that contains $\sqrt{a}$ and $\sqrt{b}$, it follows that $F(\sqrt{a}, \sqrt{b}) \subseteq F(\sqrt{a} + \sqrt{b})$. The reverse inclusion is also true because $\sqrt{a} + \sqrt{b} \in F(\sqrt{a}, \sqrt{b})$ and $F(\sqrt{a} + \sqrt{b})$ is the minimum field containing $\sqrt{a} + \sqrt{b}$. $\square$