Searching for Universal Truths
Abstract Measure Theory

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Navigating Mathematical and Statistical Territories

- Notations & definitions & conventions
  - notations - 2
  - some definitions - 6
  - some conventions - 7
- Real analysis - 8
  - set theory - 9
  - real number system - 21
  - measure and integration - 34
  - measure and outer measure - 63
- Proof & references & indices
  - references - 72
  - index - 74

Notations

- sets of numbers
  - **N** - set of natural numbers
  - **Z** - set of integers
  - $\mathbf{Z}_+$ - set of nonnegative integers
  - **Q** - set of rational numbers
  - **R** - set of real numbers
  - $\mathbf{R}_+$ - set of nonnegative real numbers
  - $\mathbf{R}_{++}$ - set of positive real numbers
  - **C** - set of complex numbers
- sequences $\langle x_i \rangle$ and the like
  - finite $\langle x_i \rangle_{i=1}^n$, infinite $\langle x_i \rangle_{i=1}^\infty$ - use $\langle x_i \rangle$ whenever unambiguously understood
  - similarly for other operations, $e.g.$, $\sum x_i$, $\prod x_i$, $\cup A_i$, $\cap A_i$, $\bigtimes A_i$
  - similarly for integrals, $e.g.$, $\int f$ for $\int_{-\infty}^{\infty} f$
- sets
  - $\tilde{A}$ - complement of A

- $A \sim B$ - $A \cap \tilde{B}$
- $A \Delta B$ - $(A \cap \tilde{B}) \cup (\tilde{A} \cap B)$
- $\mathcal{P}(A)$ - set of all subsets of A

- sets in metric vector spaces
  - $\overline{A}$ - closure of set A
  - A° - interior of set A
  - $\mathbf{relint}\, A$ - relative interior of set A
  - $\mathbf{bd}\, A$ - boundary of set A

- set algebra
  - $\sigma(\mathcal{A})$ - σ-algebra generated by $\mathcal{A}$, *i.e.*, smallest σ-algebra containing $\mathcal{A}$

- norms in $\mathbf{R}^n$
  - $\|x\|_p$ $(p \geq 1)$ - p-norm of $x \in \mathbf{R}^n$, *i.e.*, $(|x_1|^p + \cdots + |x_n|^p)^{1/p}$
  - *e.g.*, $\|x\|_2$ - Euclidean norm

- matrices and vectors
  - a_i - i-th entry of vector a
  - A_{ij} - entry of matrix A at position (i, j), *i.e.*, entry in i-th row and j-th column
  - $\mathbf{Tr}(A)$ - trace of $A \in \mathbf{R}^{n \times n}$, *i.e.*, $A_{1,1} + \cdots + A_{n,n}$

- symmetric, positive definite, and positive semi-definite matrices
  - $\mathbf{S}^n \subset \mathbf{R}^{n\times n}$ - set of symmetric matrices
  - $\mathbf{S}^n_+ \subset \mathbf{S}^n$ - set of positive semi-definite matrices; $A \succeq 0 \Leftrightarrow A \in \mathbf{S}^n_+$
  - $\mathbf{S}^n_{++} \subset \mathbf{S}^n$ - set of positive definite matrices; $A \succ 0 \Leftrightarrow A \in \mathbf{S}^n_{++}$
- sometimes, use Python script-like notations (with serious abuse of mathematical notations)
  - use $f : \mathbf{R} \to \mathbf{R}$ as if it were $f : \mathbf{R}^n \to \mathbf{R}^n$, *e.g.*,

$$\exp(x) = (\exp(x_1), \ldots, \exp(x_n)) \quad \text{for } x \in \mathbf{R}^n$$

    and

$$\log(x) = (\log(x_1), \ldots, \log(x_n)) \quad \text{for } x \in \mathbf{R}^n_{++}$$

    which corresponds to Python code `numpy.exp(x)` or `numpy.log(x)` where `x` is instance of `numpy.ndarray`, *i.e.*, `numpy` array
  - use $\sum x$ to mean $\mathbf{1}^T x$ for $x \in \mathbf{R}^n$, *i.e.*

$$\sum x = x_1 + \cdots + x_n$$

    which corresponds to Python code `x.sum()` where `x` is `numpy` array

– use x/y for $x, y \in \mathbf{R}^n$ to mean

$$\left[\begin{array}{ccc} x_1/y_1 & \cdots & x_n/y_n \end{array}\right]^T$$

which corresponds to Python code `x / y` where `x` and `y` are 1-d `numpy` arrays

– use X/Y for $X, Y \in \mathbf{R}^{m\times n}$ to mean

$$\left[\begin{array}{cccc} X_{1,1}/Y_{1,1} & X_{1,2}/Y_{1,2} & \cdots & X_{1,n}/Y_{1,n} \\ X_{2,1}/Y_{2,1} & X_{2,2}/Y_{2,2} & \cdots & X_{2,n}/Y_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ X_{m,1}/Y_{m,1} & X_{m,2}/Y_{m,2} & \cdots & X_{m,n}/Y_{m,n} \end{array}\right]$$

which corresponds to Python code `X / Y` where `X` and `Y` are 2-d `numpy` arrays

Some definitions

Definition 1. [infinitely often - i.o.] *statement P_n, said to happen* infinitely often *or* i.o. *if*

$$(\forall N \in \mathbf{N})\,(\exists n > N)\,(P_n)$$

Definition 2. [almost everywhere - a.e.] *statement $P(x)$, said to happen* almost everywhere *or* a.e. *or* almost surely *or* a.s. *(depending on context) associated with measure space $(X, \mathscr{B}, \mu)$ if*

$$\mu\{x|P(x)\} = 1$$

or equivalently

$$\mu\{x| \sim P(x)\} = 0$$

Some conventions

- (for some subjects) use following conventions
  - $0 \cdot \infty = \infty \cdot 0 = 0$
  - $(\forall x \in \mathbf{R}_{++})(x \cdot \infty = \infty \cdot x = \infty)$
  - $\infty \cdot \infty = \infty$

Real Analysis

Set Theory

Some principles

Principle 1. [principle of mathematical induction]

$$P(1)\&[P(n \Rightarrow P(n+1)] \Rightarrow (\forall n \in \mathbf{N})P(n)$$

Principle 2. [well ordering principle] *each nonempty subset of* $\mathbf{N}$ *has a smallest element*

Principle 3. [principle of recursive definition] *for* $f : X \to X$ *and* $a \in X$, *exists unique infinite sequence* $\langle x_n \rangle_{n=1}^{\infty} \subset X$ *such that*

$$x_1 = a$$

and

$$(\forall n \in \mathbf{N}) \left(x_{n+1} = f(x_n)\right)$$

- note that Principle 1 $\Leftrightarrow$ Principle 2 $\Rightarrow$ Principle 3

Some definitions for functions

Definition 3. [functions] *for $f : X \to Y$*

- *terms,* map *and* function*, exterchangeably used*
- *X and Y, called* domain of f *and* codomain of f *respectively*
- *$\{f(x)|x \in X\}$, called* range of f
- *for $Z \subset Y$, $f^{-1}(Z) = \{x \in X|f(x) \in Z\} \subset X$, called* preimage *or* inverse image of Z under f
- *for $y \in Y$, $f^{-1}(\{y\})$, called* fiber of f over y
- *f, called* injective *or* injection *or* one-to-one *if $(\forall x \neq v \in X)\,(f(x) \neq f(v))$*
- *f, called* surjective *or* surjection *or* onto *if $(\forall x \in X)\,(\exists y in Y)\,(y = f(x))$*
- *f, called* bijective *or* bijection *if f is both injective and surjective, in which case, X and Y, said to be* one-to-one correspondece *or* bijective correspondece
- *$g : Y \to X$, called* left inverse *if $g \circ f$ is identity function*
- *$h : Y \to X$, called* right inverse *if $f \circ h$ is identity function*

Some properties of functions

Lemma 1. [functions] *for $f : X \to Y$*

- f *is injective* if and only if f *has left inverse*

- f *is surjective* if and only if f *has right inverse*

- *hence,* f *is bijective* if and only if f *has both left and right inverse because if* g *and* h *are left and right inverses respectively,* $g = g \circ (f \circ h) = (g \circ f) \circ h = h$

- *if* $|X| = |Y| < \infty$, f *is injective* if and only if f *is surjective* if and only if f *is bijective*

Countability of sets

- set A is countable if range of some function whose domain is **N**

- **N**, **Z**, **Q**: countable

- **R**: *not* countable

Limit sets

- for sequence, $\langle A_n \rangle$, of subsets of X
  - *limit superior or limsup of $\langle A_n \rangle$*, defined by

$$\limsup \langle A_n \rangle = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m$$

  - *limit inferior or liminf of $\langle A_n \rangle$*, defined by

$$\liminf \langle A_n \rangle = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m$$

- always

$$\liminf \langle A_n \rangle \subset \limsup \langle A_n \rangle$$

- when $\liminf \langle A_n \rangle = \limsup \langle A_n \rangle$, sequence, $\langle A_n \rangle$, said to *converge to it*, denote

$$\lim \langle A_n \rangle = \liminf \langle A_n \rangle = \limsup \langle A_n \rangle = A$$

Algebras of sets

- collection $\mathscr{A}$ of subsets of X called *algebra* or *Boolean algebra* if

$$(\forall A, B \in \mathscr{A})(A \cup B \in \mathscr{A}) \text{ and } (\forall A \in \mathscr{A})(\tilde{A} \in \mathscr{A})$$

  - $(\forall A_1, \ldots, A_n \in \mathscr{A})(\cup_{i=1}^n A_i \in \mathscr{A})$
  - $(\forall A_1, \ldots, A_n \in \mathscr{A})(\cap_{i=1}^n A_i \in \mathscr{A})$
- algebra $\mathscr{A}$ called *σ-algebra* or *Borel field* if
  - every union of a countable collection of sets in $\mathscr{A}$ is in $\mathscr{A}$, *i.e.*,

$$(\forall \langle A_i \rangle)(\cup_{i=1}^\infty A_i \in \mathscr{A})$$

- given sequence of sets in algebra $\mathscr{A}$, $\langle A_i \rangle$, exists disjoint sequence, $\langle B_i \rangle$ such that

$$B_i \subset A_i \text{ and } \bigcup_{i=1}^\infty B_i = \bigcup_{i=1}^\infty A_i$$

Algebras generated by subsets

- *algebra generated by* collection of subsets of X, $\mathcal{C}$, can be found by

$$\mathscr{A} = \bigcap \{\mathscr{B} | \mathscr{B} \in \mathcal{F}\}$$

  where $\mathcal{F}$ is family of all algebras containing $\mathcal{C}$
  - smallest algebra $\mathscr{A}$ containing $\mathcal{C}$, *i.e.*,

$$(\forall \mathscr{B} \in \mathcal{F})(\mathscr{A} \subset \mathscr{B})$$

- *σ-algebra generated by* collection of subsets of X, $\mathcal{C}$, can be found by

$$\mathscr{A} = \bigcap \{\mathscr{B} | \mathscr{B} \in \mathcal{G}\}$$

  where $\mathcal{G}$ is family of all σ-algebras containing $\mathcal{C}$
  - smallest σ-algebra $\mathscr{A}$ containing $\mathcal{C}$, *i.e.*,

$$(\forall \mathscr{B} \in \mathcal{G})(\mathscr{A} \subset \mathscr{B})$$

Relation

- x said to *stand in relation* $\mathbf{R}$ to y, denoted by $x\ \mathbf{R}\ y$
- $\mathbf{R}$ said to *be relation on* X if $x\ \mathbf{R}\ y \Rightarrow x \in X$ and $y \in X$
- $\mathbf{R}$ is
  - transitive if $x\ \mathbf{R}\ y$ and $y\ \mathbf{R}\ z \Rightarrow x\ \mathbf{R}\ z$
  - symmetric if $x\ \mathbf{R}\ y = y\ \mathbf{R}\ x$
  - reflexive if $x\ \mathbf{R}\ x$
  - antisymmetric if $x\ \mathbf{R}\ y$ and $y\ \mathbf{R}\ x \Rightarrow x = y$
- $\mathbf{R}$ is
  - *equivalence relation* if transitive, symmetric, and reflexive, $e.g.$, modulo
  - *partial ordering* if transitive and antisymmetric, $e.g.$, "$\subset$"
  - *linear (or simple) ordering* if transitive, antisymmetric, and $x\ \mathbf{R}\ y$ or $y\ \mathbf{R}\ x$ for all $x, y \in X$
    - $e.g.$, "$\geq$" linearly orders **R** while "$\subset$" does not $\mathcal{P}(X)$

Ordering

- given partial order, $\prec$, a is
  - a first/smallest/least element if $x \neq a \Rightarrow a \prec x$
  - a last/largest/greatest element if $x \neq a \Rightarrow x \prec a$
  - a minimal element if $x \neq a \Rightarrow x \not\prec a$
  - a maximal element if $x \neq a \Rightarrow a \not\prec x$
- partial ordering $\prec$ is
  - strict partial ordering if $x \not\prec x$
  - reflexive partial ordering if $x \prec x$
- strict linear ordering $<$ is
  - *well ordering* for X if every nonempty set contains a first element

Axiom of choice and equivalent principles

Axiom 1. [axiom of choice] *given a collection of nonempty sets, $\mathcal{C}$, there exists $f : \mathcal{C} \to \cup_{A\in\mathcal{C}} A$ such that*

$$(\forall A \in \mathcal{C})\,(f(A) \in A)$$

- also called *multiplicative axiom* - preferred to be called to axiom of choice by Bertrand Russell for reason writte on page 20
- no problem when $\mathcal{C}$ is finite
- need axiom of choice when $\mathcal{C}$ is not finite

Principle 4. [Hausdorff maximal principle] *for particial ordering $\prec$ on X, exists a maximal linearly ordered subset $S \subset X$, $i.e.$, S is linearity ordered by $\prec$ and if $S \subset T \subset X$ and T is linearly ordered by $\prec$, $S = T$*

Principle 5. [well-ordering principle] *every set X can be well ordered, $i.e.$, there is a relation $<$ that well orders X*

- note that Axiom 1 $\Leftrightarrow$ Principle 4 $\Leftrightarrow$ Principle 5

Infinite direct product

Definition 4. [direct product] *for collection of sets, $\langle X_\lambda \rangle$, with index set, Λ,*

$$\bigtimes_{\lambda \in \Lambda} X_\lambda$$

called direct product

- *for* $z = \langle x_\lambda \rangle \in \bigtimes X_\lambda$, x_λ *called* λ-th coordinate of z

- if one of X_λ is empty, $\bigtimes X_\lambda$ is empty
- *axiom of choice* is equivalent to converse, *i.e.*, if none of X_λ is empty, $\bigtimes X_\lambda$ is not empty
  if one of X_λ is empty, $\bigtimes X_\lambda$ is empty
- this is why Bertrand Russell prefers *multiplicative axiom* to *axiom of choice* for name of axiom (Axiom 1)

Real Number System

Field axioms

- field axioms - for every $x, y, z \in \mathbf{F}$
  - $(x + y) + z = x + (y + z)$ - additive associativity
  - $(\exists 0 \in \mathbf{F})(\forall x \in \mathbf{F})(x + 0 = x)$ - additive identity
  - $(\forall x \in \mathbf{F})(\exists w \in \mathbf{F})(x + w = 0)$ - additive inverse
  - $x + y = y + x$ - additive commutativity
  - $(xy)z = x(yz)$ - multiplicative associativity
  - $(\exists 1 \neq 0 \in \mathbf{F})(\forall x \in \mathbf{F})(x \cdot 1 = x)$ - multiplicative identity
  - $(\forall x \neq 0 \in \mathbf{F})(\exists w \in \mathbf{F})(xw = 1)$ - multiplicative inverse
  - $x(y + z) = xy + xz$ - distributivity
  - $xy = yx$ - multiplicative commutativity
- system (set with $+$ and $\cdot$) satisfying axiom of field called *field*
  - $e.g.$, field of module p where p is prime, $\mathbf{F}_p$

Axioms of order

- axioms of order - subset, $\mathbf{F}_{++} \subset \mathbf{F}$, of positive (real) numbers satisfies
  - $x, y \in \mathbf{F}_{++} \Rightarrow x + y \in \mathbf{F}_{++}$
  - $x, y \in \mathbf{F}_{++} \Rightarrow xy \in \mathbf{F}_{++}$
  - $x \in \mathbf{F}_{++} \Rightarrow -x \not\in \mathbf{F}_{++}$
  - $x \in \mathbf{F} \Rightarrow x = 0 \vee x \in \mathbf{F}_{++} \vee -x \in \mathbf{F}_{++}$
- system satisfying field axioms & axioms of order called *ordered field*
  - $e.g.$, set of real numbers ($\mathbf{R}$), set of rational numbers ($\mathbf{Q}$)

Axiom of completeness

- completeness axiom
  - every nonempty set S of real numbers which has an upper bound has a least upper bound, *i.e.*,
  $$\{l | (\forall x \in S)(l \leq x)\}$$
  has least element.
  - use $\inf S$ and $\sup S$ for least and greatest element (when exist)
- ordered field that is complete is *complete ordered field*
  - *e.g.*, **R** (with $+$ and $\cdot$)

$\Rightarrow$ axiom of Archimedes
  - given any $x \in$ **R**, there is an integer n such that $x < n$

$\Rightarrow$ corollary
  - given any $x < y \in$ **R**, exists $r \in$ **Q** such tat $x < r < y$

Sequences of R

- sequence of **R** denoted by $\langle x_i \rangle_{i=1}^{\infty}$ or $\langle x_i \rangle$
  - mapping from **N** to **R**
- limit of $\langle x_n \rangle$ denoted by $\lim_{n\to\infty} x_n$ or $\lim x_n$ - defined by $a \in \mathbf{R}$ such that

$$(\forall \epsilon > 0)(\exists N \in \mathbf{N})(n \geq N \Rightarrow |x_n - a| < \epsilon)$$

  - $\lim x_n$ unique if exists
- $\langle x_n \rangle$ called Cauchy sequence if

$$(\forall \epsilon > 0)(\exists N \in \mathbf{N})(n, m \geq N \Rightarrow |x_n - x_m| < \epsilon)$$

- Cauchy criterion - characterizing complete metric space (including **R**)
  - sequence converges *if and only if* Cauchy sequence

Other limits

- cluster point of $\langle x_n \rangle$ - defined by $c \in \mathbf{R}$

$$(\forall \epsilon > 0, N \in \mathbf{N})(\exists n > N)(|x_n - c| < \epsilon)$$

- limit superior or limsup of $\langle x_n \rangle$

$$\limsup x_n = \inf_n \sup_{k>n} x_k$$

- limit inferior or liminf of $\langle x_n \rangle$

$$\liminf x_n = \sup_n \inf_{k>n} x_k$$

- $\liminf x_n \leq \limsup x_n$
- $\langle x_n \rangle$ converges *if and only if* $\liminf x_n = \limsup x_n$ $(=\lim x_n)$

Open and closed sets

- O called *open* if

$$(\forall x \in O)(\exists \delta > 0)(y \in \mathbf{R})(|y - x| < \delta \Rightarrow y \in O)$$

  - intersection of finite collection of open sets is open
  - union of any collection of open sets is open

- $\overline{E}$ called *closure* of E if

$$(\forall x \in \overline{E} \;\&\; \delta > 0)(\exists y \in E)(|x - y| < \delta)$$

- F called *closed* if

$$F = \overline{F}$$

  - union of finite collection of closed sets is closed
  - intersection of any collection of closed sets is closed

Open and closed sets - facts

- *every open set is union of countable collection of disjoint open intervals*

- (Lindelöf) any collection $\mathcal{C}$ of open sets has a countable subcollection $\langle O_i \rangle$ such that

$$\bigcup_{O \in \mathcal{C}} O = \bigcup_i O_i$$

  – equivalently, any collection $\mathcal{F}$ of closed sets has a countable subcollection $\langle F_i \rangle$ such that

$$\bigcap_{O \in \mathcal{F}} F = \bigcap_i F_i$$

Covering and Heine-Borel theorem

- collection $\mathcal{C}$ of sets called *covering* of A if

$$A \subset \bigcup_{O \in \mathcal{C}} O$$

  - $\mathcal{C}$ said to *cover* A
  - $\mathcal{C}$ called *open covering* if every $O \in \mathcal{C}$ is open
  - $\mathcal{C}$ called *finite covering* if $\mathcal{C}$ is finite

- *Heine-Borel theorem* - for any closed and bounded set, every open covering has finite subcovering

- corollary
  - any collection $\mathcal{C}$ of closed sets including at least one bounded set every finite subcollection of which has nonempty intersection has nonempty intersection.

Continuous functions

- f (with domain D) called *continuous at* x if

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall y \in D)(|y - x| < \delta \Rightarrow |f(y) - f(x)| < \epsilon)$$

- f called *continuous on* $A \subset D$ if f is continuous at every point in A

- f called *uniformly continuous on* $A \subset D$ if

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x, y \in D)(|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon)$$

Continuous functions - facts

- f is continuous *if and only if* for every open set O (in co-domain), $f^{-1}(O)$ is open

- f continuous on closed and bounded set is uniformly continuous

- *extreme value theorem* - f continuous on closed and bounded set, F, is *bounded on F and assumes its maximum and minimum on F*

$$(\exists x_1, x_2 \in F)(\forall x \in F)(f(x_1) \leq f(x) \leq f(x_2))$$

- *intermediate value theorem* - for f continuous on $[a, b]$ with $f(a) \leq f(b)$,

$$(\forall d)(f(a) \leq d \leq f(b))(\exists c \in [a, b])(f(c) = d)$$

Borel sets and Borel σ-algebra

- *Borel set*
  - any set that can be formed from open sets (or, equivalently, from closed sets) through the operations of countable union, countable intersection, and relative complement

- *Borel algebra* or *Borel σ-algebra*
  - *smallest σ-algebra containing all open sets*

  - also
    - smallest σ-algebra containing all closed sets
    - smallest σ-algebra containing all open intervals (due to statement on page 28)

Various Borel sets

- countable union of closed sets (in **R**), called *an* F_σ (F for closed & σ for sum)
  - thus, every countable set, every closed set, every open interval, every open sets, is an F_σ (note $(a, b) = \bigcup_{n=1}^\infty [a + 1/n, b - 1/n]$)
  - countable union of sets in F_σ again is an F_σ
- countable intersection of open sets called *a* G_δ (G for open & δ for durchschnitt - average in German)
  - complement of F_σ is a G_δ and vice versa
- F_σ and G_δ are simple types of Borel sets
- countable intersection of F_σ's is $F_{\sigma\delta}$, countable union of $F_{\sigma\delta}$'s is $F_{\sigma\delta\sigma}$, countable intersection of $F_{\sigma\delta\sigma}$'s is $F_{\sigma\delta\sigma\delta}$, *etc.*, & likewise for $G_{\delta\sigma\ldots}$
- below are all classes of Borel sets, but not every Borel set belongs to one of these classes

$$F_\sigma, F_{\sigma\delta}, F_{\sigma\delta\sigma}, F_{\sigma\delta\sigma\delta}, \ldots, G_\delta, G_{\delta\sigma}, G_{\delta\sigma\delta}, G_{\delta\sigma\delta\sigma}, \ldots,$$

Measure and Integration

Purpose of integration theory

- purpose of "measure and integration" slides
  - abstract (out) most important properties of Lebesgue measure and Lebesgue integration
- provide certain *axioms that Lebesgue measure satisfies*
- base our integration theory on these axioms
- hence, our theory valid for every system satisfying the axioms

Measurable space, measure, and measure space

- family of subsets containing $\emptyset$ closed under countable union and completement, called *σ-algebra*
- mapping of sets to extended real numbers, called *set function*
- $(X, \mathscr{B})$ with set, X, and σ-algebra of X, $\mathscr{B}$, called *measurable space*
  - $A \in \mathscr{B}$, said to be *measurable (with respect to $\mathscr{B}$)*
- nonnegative set function, μ, defined on $\mathscr{B}$ satisfying $\mu(\emptyset) = 0$ and for every disjoint, $\langle E_n \rangle_{n=1}^{\infty} \subset \mathscr{B}$,
$$\mu\left(\bigcup E_n\right) = \sum \mu E_n$$
called *measure on* measurable space, $(X, \mathscr{B})$
- measurable space, $(X, \mathscr{B})$, equipped with measure, μ, called *measure space* and denoted by $(X, \mathscr{B}, \mu)$

Measure space examples

- $(\mathbf{R}, \mathcal{M}, \mu)$ with Lebesgue measurable sets, $\mathcal{M}$, and Lebesgue measure, μ
- $([0,1], \{A \in \mathcal{M} | A \subset [0,1]\}, \mu)$ with Lebesgue measurable sets, $\mathcal{M}$, and Lebesgue measure, μ
- $(\mathbf{R}, \mathscr{B}, \mu)$ with class of Borel sets, $\mathscr{B}$, and Lebesgue measure, μ
- $(\mathbf{R}, \mathcal{P}(\mathbf{R}), \mu_C)$ with set of all subsets of $\mathbf{R}$, $\mathcal{P}(\mathbf{R})$, and counting measure, μ_C
- interesting (and bizarre) example
  - $(X, \mathcal{A}, \mu_B)$ with any uncountable set, X, family of either countable or complement of countable set, $\mathcal{A}$, and measure, μ_B, such that $\mu_B A = 0$ for countable $A \subset X$ and $\mu_B B = 1$ for uncountable $B \subset X$

More properties of measures

- for $A, B \in \mathscr{B}$ with $A \subset B$

$$\mu A \leq \mu B$$

- for $\langle E_n \rangle \subset \mathscr{B}$ with $\mu E_1 < \infty$ and $E_{n+1} \subset E_n$

$$\mu \left(\bigcap E_n \right) = \lim \mu E_n$$

- for $\langle E_n \rangle \subset \mathscr{B}$

$$\mu \left(\bigcup E_n \right) \leq \sum \mu E_n$$

Finite and σ-finite measures

- measure, μ, with $\mu(X) < \infty$, called *finite*
- measure, μ, with $X = \bigcup X_n$ for some $\langle X_n \rangle$ and $\mu(X_n) < \infty$, called *σ-finite*
  - always can take $\langle X_n \rangle$ with disjoint X_n
- Lebesgue measure on $[0, 1]$ is finite
- Lebesgue measure on **R** is σ-finite
- countering measure on uncountable set is *not* σ-measure

Sets of finite and σ-finite measure

- set, $E \in \mathscr{B}$, with $\mu E < \infty$, said to be *of finite measure*
- set that is countable union of measurable sets of finite measure, said to be *of σ-finite measure*
- measurable set contained in set of σ-finite measure, is of σ-finite measure
- countable union of sets of σ-finite measure, is of σ-finite measure
- when μ is σ-finite, every measurable set is of σ-finite

Semifinite measures

- roughly speacking, nearly all familiar properties of Lebesgue measure and Lebesgue integration hold for arbitrary σ-finite measure
- many treatment of abstract measure theory limit themselves to σ-finite measures

- many parts of general theory, however, do *not* required assumption of σ-finiteness
- undesirable to have development unnecessarily restrictive

- measure, μ, for which every measurable set of infinite measure contains measurable sets of arbitrarily large finite measure, said to be *semifinite*
- every σ-finite measure is semifinite measure while measure, μ_B, on page 37 is not

Complete measure spaces

- measure space, $(X, \mathscr{B}, \mu)$, for which $\mathscr{B}$ contains all subsets of sets of measure zero, said to be *complete*, $i.e.$,

$$(\forall B \in \mathscr{B} \text{ with } \mu B = 0)(A \subset B \Rightarrow A \in \mathscr{B})$$

  - $e.g.$, Lebesgue measure is complete, but Lebesgue measure restricted to σ-algebra of Borel sets is *not*
- every measure space can be *completed* by addition of subsets of sets of measure zero
- for $(X, \mathscr{B}, \mu)$, can find *complete* measure space $(X, \mathscr{B}_0, \mu_0)$ such that

  - $\mathscr{B} \subset \mathscr{B}_0$
  - $E \in \mathscr{B} \Rightarrow \mu E = \mu_0 E$
  - $E \in \mathscr{B}_0 \Leftrightarrow E = A \cup B$ where $B, C \in \mathscr{B}, \mu C = 0, A \subset C$

  - $(X, \mathscr{B}_0, \mu_0)$ called *completion* of $(X, \mathscr{B}, \mu)$

Local measurability and saturatedness

- for $(X, \mathscr{B}, \mu)$, $E \subset X$ for which $(\forall B \in \mathscr{B}$ with $\mu B < \infty)(E \cap B \in \mathscr{B})$, said to be *locally measurable*

- collection, $\mathscr{C}$, of all locally measurable sets is σ-algebra containing $\mathscr{B}$

- measure for which every locally measurable set is measurable, said to be *saturated*

- every σ-finite measure is saturated

- measure can be extended to saturated measure, but (unlike completion) extension is not unique
  - can take $\mathscr{C}$ as extension for locally measurable sets, but measure can be extended on $\mathscr{C}$ in more than one ways

Measurable functions

- concept and properties of measurable functions in abstract measurable space almost identical with those of Lebesgue measurable functions (page **??**)
- theorems and facts are essentially same as those of Lebesgue measurable functions
- assume measurable space, $(X, \mathscr{B})$
- for $f : X \to \mathbf{R} \cup \{-\infty, \infty\}$, following are equivalent
  - $(\forall a \in \mathbf{R})(\{x \in X | f(x) < a\} \in \mathscr{B})$
  - $(\forall a \in \mathbf{R})(\{x \in X | f(x) \leq a\} \in \mathscr{B})$
  - $(\forall a \in \mathbf{R})(\{x \in X | f(x) > a\} \in \mathscr{B})$
  - $(\forall a \in \mathbf{R})(\{x \in X | f(x) \geq a\} \in \mathscr{B})$
- $f : X \to \mathbf{R} \cup \{-\infty, \infty\}$ for which any one of above four statements holds, called *measurable* or *measurable with respect to* $\mathscr{B}$

(refer to page **??** for Lebesgue counterpart)

Properties of measurable functions

- **Theorem 1. [measurability preserving function operations]** *for measurable functions, f and g, and $c \in \mathbf{R}$*
  - $f + c$, cf, $f + g$, fg, $f \vee g$ *are measurable*

- **Theorem 2. [limits of measurable functions]** *for every measurable function sequence, $\langle f_n \rangle$*
  - $\sup f_n$, $\limsup f_n$, $\inf f_n$, $\liminf f_n$ *are measurable*
  - *thus,* $\lim f_n$ *is measurable if exists*

(refer to page **??** for Lebesgue counterpart)

Simple functions and other properties

- φ called *simple function* if for distinct $\langle c_i \rangle_{i=1}^n$ and measurable sets, $\langle E_i \rangle_{i=1}^n$

$$\varphi(x) = \sum_{i=1}^n c_i \chi_{E_i}(x)$$

(refer to page **??** for Lebesgue counterpart)

- for nonnegative measurable function, f, exists nondecreasing sequence of simple functions, $\langle \varphi_n \rangle$, $i.e.$, $\varphi_{n+1} \geq \varphi_n$ such that for every point in X

$$f = \lim \varphi_n$$

  - for f defined on σ-finite measure space, we may choose $\langle \varphi_n \rangle$ so that every φ_n vanishes outside set of finite measure

- for complete measure, μ, f measurable and $f = g$ a.e. imply measurability of g

Define measurable function by ordinate sets

- $\{x|f(x) < \alpha\}$ sometimes called *ordinate sets*, which is nondecreasing in α
- below says when given nondecreasing ordinate sets, we can find f satisfying

$$\{x|f(x) < \alpha\} \subset B_\alpha \subset \{x|f(x) \leq \alpha\}$$

- for nondecreasing function, $h : D \to \mathscr{B}$, for dense set of real numbers, D, *i.e.*, $B_\alpha \subset B_\beta$ for all $\alpha < \beta$ where $B_\alpha = h(\alpha)$, exists unique measurable function, $f : X \to \mathbf{R} \cup \{-\infty, \infty\}$ such that $f \leq \alpha$ on B_α and $f \geq \alpha$ on $X \sim B_\alpha$
- can relax some conditions and make it a.e. version as below
- for function, $h : D \to \mathscr{B}$, for dense set of real numbers, D, such that $\mu(B_\alpha \sim B_\beta) = 0$ for all $\alpha < \beta$ where $B_\alpha = h(\alpha)$, exists measurable function, $f : X \to \mathbf{R} \cup \{-\infty, \infty\}$ such that $f \leq \alpha$ a.e. on B_α and $f \geq \alpha$ a.e. on $X \sim B_\alpha$
  - if g has the same property, $f = g$ a.e.

Integration

- many definitions and proofs of Lebesgue integral depend only on properties of Lebesgue measure which are also true for arbitrary measure in abstract measure space (page **??**)
- integral of nonnegative simple function, $\varphi(x) = \sum_{i=1}^n c_i \chi_{E_i}(x)$, on measurable set, E, defined by

$$\int_E \varphi d\mu = \sum_{i=1}^n c_i \mu(E_i \cap E)$$

  - independent of representation of φ

(refer to page **??** for Lebesgue counterpart)

- for $a, b \in \mathbf{R}_{++}$ and nonnegative simple functions, φ and ψ

$$\int (a\varphi + b\psi) = a \int \varphi + b \int \psi$$

(refer to page **??** for Lebesgue counterpart)

Integral of bounded functions

- for bounded function, f, identically zero outside measurable set of finite measure

$$\sup_{\varphi:\ \text{simple},\ \varphi \leq f} \int \varphi = \inf_{\psi:\ \text{simple},\ f \leq \psi} \int \psi$$

if and only if $f = g$ a.e. for measurable function, g

(refer to page **??** for Lebesgue counterpart)

- but, *$f = g$ a.e. for measurable function, g,* if and only if *f is measurable with respect to completion of μ, $\bar{\mu}$*
- *natural class of functions to consider for integration theory are those measurable with respect to completion of μ*
- thus, shall either assume μ is complete measure or define integral with respect to μ to be integral with respect to completion of μ depending on context unless otherwise specified

Difficulty of general integral of nonnegative functions

- for Lebesgue integral of nonnegative functions (page **??**)
  - first define integral for bounded measurable functions
  - define integral of nonnegative function, f as supremum of integrals of all bounded measurable functions, $h \leq f$, vanishing outside measurable set of finite measure

- unfortunately, not work in case that measure is not semifinite
  - $e.g.$, if $\mathscr{B} = \{\emptyset, X\}$ with $\mu\emptyset = 0$ and $\mu X = \infty$, we want $\int 1 d\mu = \infty$, but only bounded measurable function vanishing outside measurable set of finite measure is $h \equiv 0$, hence, $\int g d\mu = 0$

- to avoid this difficulty, we define integral of nonnegative measurable function directly in terms of integrals of nonnegative simple functions

Integral of nonnegative functions

- for measurable function, $f : X \to \mathbf{R} \cup \{\infty\}$, on measure space, $(X, \mathscr{B}, \mu)$, define *integral of nonnegative extended real-valued measurable function*

$$\int f d\mu = \sup_{\varphi:\ \text{simple function},\ 0 \leq \varphi \leq f} \int \varphi d\mu$$

(refer to page **??** for Lebesgue counterpart)

- however, *definition of integral of nonnegative extended real-valued measurable function* can be awkward to apply because
  - taking supremum over large collection of simple functions
  - *not clear from definition that* $\int (f+g) = \int f + \int g$

- thus, first establish some convergence theorems, and determine value of $\int f$ as limit of $\int \varphi_n$ for increasing sequence, $\langle \varphi_n \rangle$, of simple functions converging to f

Fatou's lemma and monotone convergence theorem

- *Fatou's lemma* - for nonnegative measurable function sequence, $\langle f_n \rangle$, with $\lim f_n = f$ a.e. on measurable set, E
$$\int_E f \leq \liminf \int_E f_n$$

- *monotone convergence theorem* - for nonnegative measurable function sequence, $\langle f_n \rangle$, with $f_n \leq f$ for all n and with $\lim f_n = f$ a.e.
$$\int_E f = \lim \int_E f_n$$

(refer to page **??** for Lebesgue counterpart)

Integrability of nonnegative functions

- for nonnegative measurable functions, f and g, and $a, b \in \mathbf{R}_+$

$$\int (af + bg) = a \int f + b \int g \;\&\; \int f \geq 0$$

  – equality holds *if and only if* $f = 0$ a.e.

(refer to page **??** for Lebesgue counterpart)

- monotone convergence theorem together with above yields, for nonnegative measurable function sequence, $\langle f_n \rangle$

$$\int \sum f_n = \sum \int f_n$$

- measurable nonnegative function, f, with

$$\int_E f d\mu < \infty$$

  said to be *integral (over measurable set, E, with respect to μ)*

(refer to page **??** for Lebesgue counterpart)

Integral

- arbitrary function, f, for which both f^+ and f^- are integrable, said to be *integrable*

- in this case, define *integral*

$$\int_E f = \int_E f^+ - \int_E f^-$$

(refer to page **??** for Lebesgue counterpart)

Properties of integral

- for f and g integrable on measure set, E, and $a, b \in \mathbf{R}$
  - $af + bg$ is integral and

$$\int_E (af + bg) = a \int_E f + b \int_E g$$

  - if $|h| \leq |f|$ and h is measurable, then h is integrable
  - if $f \geq g$ a.e.

$$\int f \geq \int g$$

(refer to page **??** for Lebesgue counterpart)

Lebesgue convergence theorem

- *Lebesgue convergence theorem* - for integral, g, over E and sequence of measurable functions, $\langle f_n \rangle$, with $\lim f_n(x) = f(x)$ a.e. on E, if

$$|f_n(x)| \leq g(x)$$

then

$$\int_E f = \lim \int_E f_n$$

(refer to page **??** for Lebesgue counterpart)

Setwise convergence of sequence of measures

- preceding convergence theorems assume fixed measure, μ

- can generalize by allowing measure to vary

- given measurable space, $(X, \mathscr{B})$, sequence of set functions, $\langle \mu_n \rangle$, defined on $\mathscr{B}$, satisfying
$$(\forall E \in \mathscr{B})(\lim \mu_n E = \mu E)$$
for some set function, μ, defined on $\mathscr{B}$, said to *converge setwise* to μ

General convergence theorems

- *generalization of Fatou's leamma* - for measurable space, $(X, \mathscr{B})$, sequence of measures, $\langle \mu_n \rangle$, defined on $\mathscr{B}$, converging setwise to μ, defined on $\mathscr{B}$, and sequence of nonnegative functions, $\langle f_n \rangle$, each measurable with respect to μ_n, converging pointwise to function, f, measurable with respect to μ (compare with Fatou's lemma on page 52)

$$\int f d\mu \leq \liminf \int f_n d\mu_n$$

- *generalization of Lebesgue convergence theorem* - for measurable space, $(X, \mathscr{B})$, sequence of measures, $\langle \mu_n \rangle$, defined on $\mathscr{B}$, converging setwise to μ, defined on $\mathscr{B}$, and sequences of functions, $\langle f_n \rangle$ and $\langle g_n \rangle$, each of f_n and g_n, measurable with respect to μ_n, converging pointwise to f and g, measurable with respect to μ, respectively, such that (compare with Lebesgue convergence theorem on page 56)

$$\lim \int g_n d\mu_n = \int g d\mu < \infty$$

satisfy

$$\lim \int f_n d\mu_n = \int f \mu$$

L^p spaces

- for complete measure space, $(X, \mathscr{B}, \mu)$
  - space of measurable functions on X with with $\int |f|^p < \infty$, for which element equivalence is defined by being equal a.e., called *L^p spaces* denoted by $L^p(\mu)$
  - space of bounded measure functions, called *L^∞ space* denoted by $L^\infty(\mu)$

- norms
  - for $p \in [1, \infty)$

$$\|f\|_p = \left(\int |f|^p d\mu \right)^{1/p}$$

  - for $p = \infty$

$$\|f\|_\infty = \mathrm{ess\ sup} |f| = \inf \{ |g(x)| | \text{ measurable } g \text{ with } g = f \text{ a.e.} \}$$

- for $p \in [1, \infty]$, spaces, $L^p(\mu)$, are Banach spaces

Hölder's inequality and Littlewood's second principle

- *Hölder's inequality* - for $p, q \in [1, \infty]$ with $1/p + 1/q = 1$, $f \in L^p(\mu)$ and $g \in L^q(\mu)$ satisfy $fg \in L^1(\mu)$ and

$$\|fg\|_1 = \int |fg| d\mu \leq \|f\|_p \|g\|_q$$

(refer to page **??** for normed spaces counterpart)

- *complete measure space version of Littlewood's second principle* - for $p \in [1, \infty)$

$$(\forall f \in L^p(\mu), \epsilon > 0)$$

$$(\exists \text{ simple function } \varphi \text{ vanishing outside set of finite measure})$$

$$(\|f - \varphi\|_p < \epsilon)$$

(refer to page **??** for normed spaces counterpart)

Riesz representation theorem

- *Riesz representation theorem* - for $p \in [1, \infty)$ and bounded linear functional, F, on $L^p(\mu)$ and σ-finite measure, μ, exists *unique* $g \in L^q(\mu)$ where $1/p + 1/q = 1$ such that
$$F(f) = \int fg d\mu$$
where $\|F\| = \|g\|_q$

(refer to page **??** for normed spaces counterpart)

- if $p \in (1, \infty)$, Riesz representation theorem holds without assumption of σ-finiteness of measure

Measure and Outer Measure

General measures

- consider some ways of defining measures on σ-algebra

- recall that for Lebesgue measure
  - define measure for open intervals
  - define outer measure
  - define notion of measurable sets
  - finally derive Lebesgue measure

- one can do similar things in general, $e.g.$,
  - derive measure from outer measure
  - derive outer measure from measure defined on algebra of sets

Outer measure

- set function, $\mu^* : \mathcal{P}(X) \to [0, \infty]$, for space X, having following properties, called *outer measure*
  - $\mu^* \emptyset = 0$
  - $A \subset B \Rightarrow \mu^* A \leq \mu^* B$ (monotonicity)
  - $E \subset \bigcup_{n=1}^{\infty} E_n \Rightarrow \mu^* E \leq \sum_{n=1}^{\infty} \mu^* E_n$ (countable subadditivity)

- μ^* with $\mu^* X < \infty$ called *finite*

- set $E \subset X$ satisfying following property, said to be *measurable with respect to μ^**

$$(\forall A \subset X)(\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap \tilde{E}))$$

- class, $\mathscr{B}$, of μ^*-measurable sets is σ-algebra

- restriction of μ^* to $\mathscr{B}$ is complete measure on $\mathscr{B}$

Extension to measure from measure on an algebra

- set function, $\mu : \mathscr{A} \to [0, \infty]$, defined on algebra, $\mathscr{A}$, having following properties, called *measure on an algebra*
  - $\mu(\emptyset) = 0$
  - $(\forall$ disjoint $\langle A_n \rangle \subset \mathscr{A}$ with $\bigcup A_n \in \mathscr{A})$ $(\mu(\bigcup A_n) = \sum \mu A_n)$

- *measure on an algebra, $\mathscr{A}$*, is measure *if and only if* $\mathscr{A}$ is σ-algebra

- can extend measure on an algebra to measure defined on σ-algebra, $\mathscr{B}$, containing $\mathscr{A}$, by
  - constructing outer measure μ^* from μ
  - deriving desired extension $\bar{\mu}$ induced by μ^*

– process by which constructing μ^* from μ similar to constructing Lebesgue outer measure from lengths of intervals

Outer measure constructed from measure on an algebra

— given measure, μ, on an algebra, $\mathscr{A}$

- define set function, $\mu^* : \mathcal{P}(X) \to [0, \infty]$, by

$$\mu^* E = \inf_{\langle A_n \rangle \subset \mathscr{A},\ E \subset \bigcup A_n} \sum \mu A_n$$

- μ^* called *outer measure induced by μ*

— then

- for $A \in \mathscr{A}$ and $\langle A_n \rangle \subset \mathscr{A}$ with $A \subset \bigcup A_n$, $\mu A \leq \sum \mu A_n$
- hence, $(\forall A \in \mathscr{A})(\mu^* A = \mu A)$
- μ^* is outer measure
- every $A \in \mathscr{A}$ is measurable with respect to μ^*

Regular outer measure

- for algebra, $\mathscr{A}$
  - $\mathscr{A}_\sigma$ denote sets that are countable unions of sets of $\mathscr{A}$
  - $\mathscr{A}_{\sigma\delta}$ denote sets that are countable intersections of sets of $\mathscr{A}_\sigma$

- given measure, μ, on an algebra, $\mathscr{A}$ and outer measure, μ^* induced by μ, for every $E \subset X$ and every $\epsilon > 0$, exists $A \in \mathscr{A}_\sigma$ and $B \in \mathscr{A}_{\sigma\delta}$ with $E \subset A$ and $E \subset B$

$$\mu^* A \leq \mu^* E + \epsilon \text{ and } \mu^* E = \mu^* B$$

- outer measure, μ^*, with below property, said to be *regular*

$$(\forall E \subset X, \epsilon > 0)(\exists\ \mu^*\text{-measurable set } A \text{ with } E \subset A)(\mu^* A \subset \mu^* E + \epsilon)$$

- every outer measure induced by measure on an algebra is regular outer measure

Carathéodory theorem

— given measure, μ, on an algebra, $\mathscr{A}$ and outer measure, μ^* induced by μ

- $E \subset X$ is μ^*-measurable *if and only if* exist $A \in \mathscr{A}_{\sigma\delta}$ and $B \subset X$ with $\mu^* B = 0$ such that
$$E = A \sim B$$
  - for $B \subset X$ with $\mu^* B = 0$, exists $C \in \mathscr{A}_{\sigma\delta}$ with $\mu^* C = 0$ such that $B \subset C$

- *Carathéodory theorem -* restriction, $\bar{\mu}$, of μ^* to μ^*-measurable sets if extension of μ to σ-algebra containing $\mathscr{A}$
  - if μ is finite or σ-finite, so is $\bar{\mu}$ respectively
  - if μ is σ-finite, $\bar{\mu}$ is only measure on smallest σ-algebra containing $\mathscr{A}$ which is extension of μ

Product measures

- for countable disjoint collection of measurable rectangles, $\langle (A_n \times B_n) \rangle$, whose union is measurable rectangle, $A \times B$

$$\lambda(A \times B) = \sum \lambda(A_n \times B_n)$$

- for $x \in X$ and $E \in \mathscr{R}_{\sigma\delta}$

$$E_x = \{y | \langle x, y \rangle \in E\}$$

  is measurable subset of Y
- for $E \subset \mathscr{R}_{\sigma\delta}$ with $\mu \times \nu(E) < \infty$, function, g, defined by

$$g(x) = \nu E_x$$

  is measurable function of x and

$$\int g d\mu = \mu \times \nu(E)$$

- XXX

Carathéodory outer measures

- set, X, of points and set, Γ, of real-valued functions on X
- two sets for which exist $a > b$ such that function, φ, greater than a on one set and less than b on the other set, said to be *separated by function, φ*
- outer measure, μ^*, with $(\forall A, B \subset X$ separated by $f \in \Gamma)(\mu^*(A \cup B) = \mu^* A + \mu^* B)$, called *Carathéodory outer measure with respect to Γ*
- outer measure, μ^*, on metric space, $\langle X, \rho \rangle$, for which $\mu^*(A \cup B) = \mu^* A + \mu^* B$ for $A, B \subset X$ with $\rho(A, B) > 0$, called *Carathéodory outer measure for X* or *metric outer measure*
- for *Carathéodory outer measure, μ^*, with respect to* Γ, every function in Γ is μ^*-measurable
- for *Carathéodory outer measure, μ^*, for metric space,* $\langle X, \rho, \rangle$, every closed set (hence every Borel set) is measurable with respect to μ^*

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Index

L^∞ space

complete measure spaces, 60

L^p spaces

complete measure spaces, 60

σ-algebra, 15, 36

generated by subsets, 16

smallest containing subsets, 16

a.e.

almost everywhere, 6

a.s.

almost surely, 6

algebra, 15

generated by, 16

smallest containing subsets, 16

algebra generated by, 16

almost everywhere, 6

almost everywhere - a.e., 6

almost surely, 6

axiom

multiplicative axiom, 19, 20

axiom of choice, 19

axioms

axiom of choice, 19

bijection

functions, 11

bijective

functions, 11

bijective correspondece
functions, 11

Boolean algebra, 15

Borel σ-algebra, 32

Borel field, 15

Borel sets, 32
F_σ, 33
$F_{\sigma\delta}$, 33
G_δ, 33
$G_{\delta\sigma}$, 33

Borel, Félix Édouard Justin Émile
σ-algebra, 32
field, 15
Heine-Borel theorem, 29
sets, 32

boundary
set, 3

Carathéodory outer measure
for metric space, 71
with respect to function set, 71

Carathéodory theorem, 69

Carathéodory, Constantin
Carathéodory theorem, 69
outer measure, 71

closure
set, 3

codomain
functions, 11

complement

set, 2

complete

measure, 42

measure spaces, 42

metric spaces, 25

ordered field, 24

complex number, 2

convergence

of set, 14

definitions

almost everywhere - a.e., 6

direct product, 20

functions, 11

infinitely often - i.o., 6

difference

set, 3

direct product, 20

domain

functions, 11

equivalence relation, 17

Fatou's lemma

generalization, 58

integral, 52

Fatou, Pierre Joseph Louis

Fatou's lemma, 52, 58

fiber

functions, 11

finite sequence, 2

functions, 11, 12
 bijection, 11
 bijective, 11
 bijective correspondece, 11
 codomain, 11
 domain, 11
 fiber, 11
 injection, 11
 injective, 11
 inverse image, 11
 left inverse, 11
 one-to-one, 11
 one-to-one correspondece, 11
 onto, 11
 preimage, 11
 range, 11
 right inverse, 11
 surjection, 11
 surjective, 11

general measure
 product measure, 70

generated by
 σ-algebra
 by subsets, 16
 algebra, 16

Hölder's inequality
 complete measure spaces, 61

Hölder, Ludwig Otto
 Hölder's inequality
 complete measure spaces, 61

Hausdorff maximal principle, 19

Hausdorff, Felix
maximal principle, 19

Heine, Heinrich Eduard
Heine-Borel theorem, 29

Heine-Borel theorem, 29

i.o.
infinitely often, 6

infinite sequence, 2

infinitely often, 6

infinitely often - i.o., 6

injection
functions, 11

injective
functions, 11

integer, 2

integrable, 54
nonnegative functions, 53

integral, 54
bounded functions, 49
Fatou's lemma, 52
generalization, 58
integrable, 54
nonnegative functions, 53
Lebesgue convergence theorem, 56
generalization, 58
monotone convergence theorem, 52
nonnegative functions, 51
integrable, 53
properties, 55

simple functions, 48

interior
set, 3

inverse image
functions, 11

Lebesgue convergence theorem
generalization, 58
integral, 56

Lebesgue, Henri Léon
convergence theorem, 56, 58

left inverse
functions, 11

lemmas
functions, 12

limit inferior (liminf)
set, 14

limit superior (limsup)
set, 14

limits of measurable functions, 45

linear ordering, 17

Littlewood's three principles
second principle
complete measure spaces, 61

maps, 11

matrix
positive definite, 4
positive semi-definite, 4
symmetric, 4

trace, 3

measurability preserving function operations, 45

measurable functions

abstract measurable spaces, 44

defined by ordinate sets, 47

ordinate sets, 47

properties, 45

simple, 46

measurable sets, 36, 65

measurable spaces, 36

measure, 36

σ-finite, 39

complete, 42

countable additivity, 36

finite, 39

measurable spaces, 36

measure, 36

on an algebra, 66

semifinite, 41

sets of σ-finite measure, 40

sets of finite measure, 40

measure spaces, 36

complete, 42

completed, 42

completion, 42

examples, 37

local measurability, 43

saturatedness, 43

setwise convergence, 57

metric outer measure, 71

monotone convergence theorem

integral, 52

multiplicative axiom, 19, 20

natural number, 2

norm
vector, 3

number
complex number, 2
integer, 2
natural number, 2
rational number, 2
real number, 2

one-to-one
functions, 11

one-to-one correspondece
functions, 11

onto
functions, 11

ordering
linear, 17
partial, 17
simple, 17

ordinate sets
measurable functions, 47

outer measure, 65
Carathéodory, 71
finite, 65
induced by measure on an algebra, 67
regular, 68

partial ordering, 17

positive definite matrix, 4

positive semi-definite matrix, 4

preimage
- functions, 11

principle of mathematical induction, 10

principle of recursive definition, 10

principles
- Hausdorff maximal principle, 19
- principle of mathematical induction, 10
- principle of recursive definition, 10
- well ordering principle, 10
- well-ordering principle, 19

product measure
- general measure, 70

range
- functions, 11

rational number, 2

real number, 2

relation
- be relation on, 17
- stand in relation, 17

relative interior
- set, 3

Riesz representation theorem, 62
- complete measure spaces, 62

Riesz, Frigyes
- Riesz representation theorem, 62
  - complete measure spaces, 62

right inverse
functions, 11

separated by function, 71

sequence, 2
finite sequence, 2
infinite sequence, 2

set
boundary, 3
closure, 3
complement, 2
convergence of sequence, 14
difference, 3
interior, 3
limit inferior (liminf), 14
limit superior (limsup), 14
relative interior, 3

simple ordering, 17

smallest σ-algebra containing subsets, 3, 16

smallest algebra containing subsets, 16

surjection
functions, 11

surjective
functions, 11

symmetric matrix, 4

theorems
limits of measurable functions, 45
measurability preserving function operations, 45

trace
matrix, 3

vector

norm, 3

well ordering principle, 10

well-ordering principle, 19

ZZ-important

open set in **R** is union of countable collection of disjoint open intervals, 28

ZZ-revisit

every outer measure induced by measure on an algebra is regular outer measure, 68

ZZ-todo

0 - apply new comma conventions, 0

1 - convert bullet points to proper theorem, definition, lemma, corollary, proposition, etc., 0

CANCELED - $<$ 2024 0421 - python script extracting important list, 0

DONE - 2024 0324 - change tocpageref and funpageref to hyperlink, 0

DONE - 2024 0324 - python script extracting figure list $\rightarrow$ using "list of figures" functionality on doc, 0

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DONE - 2025 0414 - 1 - change mathematicians' names, 0