

Searching for Universal Truths

Abstract Measure Theory

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Navigating Mathematical and Statistical Territories

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Notations

- sets of numbers
 - \mathbf{N} - set of natural numbers
 - \mathbf{Z} - set of integers
 - \mathbf{Z}_+ - set of nonnegative integers
 - \mathbf{Q} - set of rational numbers
 - \mathbf{R} - set of real numbers
 - \mathbf{R}_+ - set of nonnegative real numbers
 - \mathbf{R}_{++} - set of positive real numbers
 - \mathbf{C} - set of complex numbers
- sequences $\langle x_i \rangle$ and the like
 - finite $\langle x_i \rangle_{i=1}^n$, infinite $\langle x_i \rangle_{i=1}^\infty$ - use $\langle x_i \rangle$ whenever unambiguously understood
 - similarly for other operations, *e.g.*, $\sum x_i$, $\prod x_i$, $\cup A_i$, $\cap A_i$, $\times A_i$
 - similarly for integrals, *e.g.*, $\int f$ for $\int_{-\infty}^\infty f$
- sets
 - \tilde{A} - complement of A

- $A \sim B$ - $A \cap \tilde{B}$
- $A \Delta B$ - $(A \cap \tilde{B}) \cup (\tilde{A} \cap B)$
- $\mathcal{P}(A)$ - set of all subsets of A
- sets in metric vector spaces
 - \overline{A} - closure of set A
 - A° - interior of set A
 - **relint** A - relative interior of set A
 - **bd** A - boundary of set A
- set algebra
 - $\sigma(\mathcal{A})$ - σ -algebra generated by \mathcal{A} , *i.e.*, smallest σ -algebra containing \mathcal{A}
- norms in \mathbf{R}^n
 - $\|x\|_p$ ($p \geq 1$) - p -norm of $x \in \mathbf{R}^n$, *i.e.*, $(|x_1|^p + \cdots + |x_n|^p)^{1/p}$
 - *e.g.*, $\|x\|_2$ - Euclidean norm
- matrices and vectors
 - a_i - i -th entry of vector a
 - A_{ij} - entry of matrix A at position (i, j) , *i.e.*, entry in i -th row and j -th column
 - $\text{Tr}(A)$ - trace of $A \in \mathbf{R}^{n \times n}$, *i.e.*, $A_{1,1} + \cdots + A_{n,n}$

- symmetric, positive definite, and positive semi-definite matrices
 - $\mathbf{S}^n \subset \mathbf{R}^{n \times n}$ - set of symmetric matrices
 - $\mathbf{S}_+^n \subset \mathbf{S}^n$ - set of positive semi-definite matrices; $A \succeq 0 \Leftrightarrow A \in \mathbf{S}_+^n$
 - $\mathbf{S}_{++}^n \subset \mathbf{S}^n$ - set of positive definite matrices; $A \succ 0 \Leftrightarrow A \in \mathbf{S}_{++}^n$
- sometimes, use Python script-like notations (with serious abuse of mathematical notations)
 - use $f : \mathbf{R} \rightarrow \mathbf{R}$ as if it were $f : \mathbf{R}^n \rightarrow \mathbf{R}^n$, *e.g.*,

$$\exp(x) = (\exp(x_1), \dots, \exp(x_n)) \quad \text{for } x \in \mathbf{R}^n$$

and

$$\log(x) = (\log(x_1), \dots, \log(x_n)) \quad \text{for } x \in \mathbf{R}_{++}^n$$

which corresponds to Python code `numpy.exp(x)` or `numpy.log(x)` where `x` is instance of `numpy.ndarray`, *i.e.*, numpy array

- use $\sum x$ to mean $\mathbf{1}^T x$ for $x \in \mathbf{R}^n$, *i.e.*

$$\sum x = x_1 + \dots + x_n$$

which corresponds to Python code `x.sum()` where `x` is numpy array

- use x/y for $x, y \in \mathbf{R}^n$ to mean

$$\begin{bmatrix} x_1/y_1 & \cdots & x_n/y_n \end{bmatrix}^T$$

which corresponds to Python code `x / y` where `x` and `y` are 1-d numpy arrays

- use X/Y for $X, Y \in \mathbf{R}^{m \times n}$ to mean

$$\begin{bmatrix} X_{1,1}/Y_{1,1} & X_{1,2}/Y_{1,2} & \cdots & X_{1,n}/Y_{1,n} \\ X_{2,1}/Y_{2,1} & X_{2,2}/Y_{2,2} & \cdots & X_{2,n}/Y_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ X_{m,1}/Y_{m,1} & X_{m,2}/Y_{m,2} & \cdots & X_{m,n}/Y_{m,n} \end{bmatrix}$$

which corresponds to Python code `X / Y` where `X` and `Y` are 2-d numpy arrays

Some definitions

Definition 1. [infinitely often - i.o.] *statement P_n , said to happen infinitely often or i.o. if*

$$(\forall N \in \mathbf{N}) (\exists n > N) (P_n)$$

Definition 2. [almost everywhere - a.e.] *statement $P(x)$, said to happen almost everywhere or a.e. or almost surely or a.s. (depending on context) associated with measure space (X, \mathcal{B}, μ) if*

$$\mu\{x | P(x)\} = 1$$

or equivalently

$$\mu\{x | \sim P(x)\} = 0$$

Some conventions

- (for some subjects) use following conventions

- $0 \cdot \infty = \infty \cdot 0 = 0$

- $(\forall x \in \mathbf{R}_{++})(x \cdot \infty = \infty \cdot x = \infty)$

- $\infty \cdot \infty = \infty$

Real Analysis

Set Theory

Some principles

Principle 1. [principle of mathematical induction]

$$P(1) \& [P(n) \Rightarrow P(n+1)] \Rightarrow (\forall n \in \mathbf{N}) P(n)$$

Principle 2. [well ordering principle] *each nonempty subset of \mathbf{N} has a smallest element*

Principle 3. [principle of recursive definition] *for $f : X \rightarrow X$ and $a \in X$, exists unique infinite sequence $\langle x_n \rangle_{n=1}^{\infty} \subset X$ such that*

$$x_1 = a$$

and

$$(\forall n \in \mathbf{N}) (x_{n+1} = f(x_n))$$

- note that Principle 1 \Leftrightarrow Principle 2 \Rightarrow Principle 3

Some definitions for functions

Definition 3. [functions] for $f : X \rightarrow Y$

- terms, **map** and **function**, interchangeably used
- X and Y , called **domain of f** and **codomain of f** respectively
- $\{f(x) | x \in X\}$, called **range of f**
- for $Z \subset Y$, $f^{-1}(Z) = \{x \in X | f(x) \in Z\} \subset X$, called **preimage** or **inverse image of Z under f**
- for $y \in Y$, $f^{-1}(\{y\})$, called **fiber of f over y**
- f , called **injective** or **injection** or **one-to-one** if $(\forall x \neq v \in X) (f(x) \neq f(v))$
- f , called **surjective** or **surjection** or **onto** if $(\forall x \in X) (\exists y \in Y) (y = f(x))$
- f , called **bijective** or **bijection** if f is both injective and surjective, in which case, X and Y , said to be **one-to-one correspondence** or **bijective correspondence**
- $g : Y \rightarrow X$, called **left inverse** if $g \circ f$ is identity function
- $h : Y \rightarrow X$, called **right inverse** if $f \circ h$ is identity function

Some properties of functions

Lemma 1. [functions] for $f : X \rightarrow Y$

- f is injective if and only if f has left inverse
- f is surjective if and only if f has right inverse
- hence, f is bijective if and only if f has both left and right inverse because if g and h are left and right inverses respectively, $g = g \circ (f \circ h) = (g \circ f) \circ h = h$
- if $|X| = |Y| < \infty$, f is injective if and only if f is surjective if and only if f is bijective

Countability of sets

- set A is countable if range of some function whose domain is \mathbf{N}
- \mathbf{N} , \mathbf{Z} , \mathbf{Q} : countable
- \mathbf{R} : *not* countable

Limit sets

- for sequence, $\langle A_n \rangle$, of subsets of X
 - *limit superior or limsup of $\langle A_n \rangle$* , defined by

$$\limsup \langle A_n \rangle = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m$$

- *limit inferior or liminf of $\langle A_n \rangle$* , defined by

$$\liminf \langle A_n \rangle = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m$$

- always

$$\liminf \langle A_n \rangle \subset \limsup \langle A_n \rangle$$

- when $\liminf \langle A_n \rangle = \limsup \langle A_n \rangle$, sequence, $\langle A_n \rangle$, said to *converge to it*, denote

$$\lim \langle A_n \rangle = \liminf \langle A_n \rangle = \limsup \langle A_n \rangle = A$$

Algebras of sets

- collection \mathcal{A} of subsets of X called *algebra* or *Boolean algebra* if

$$(\forall A, B \in \mathcal{A})(A \cup B \in \mathcal{A}) \text{ and } (\forall A \in \mathcal{A})(\tilde{A} \in \mathcal{A})$$

- $(\forall A_1, \dots, A_n \in \mathcal{A})(\bigcup_{i=1}^n A_i \in \mathcal{A})$
- $(\forall A_1, \dots, A_n \in \mathcal{A})(\bigcap_{i=1}^n A_i \in \mathcal{A})$
- algebra \mathcal{A} called *σ -algebra* or *Borel field* if
 - every union of a countable collection of sets in \mathcal{A} is in \mathcal{A} , i.e.,

$$(\forall \langle A_i \rangle)(\bigcup_{i=1}^{\infty} A_i \in \mathcal{A})$$

- given sequence of sets in algebra \mathcal{A} , $\langle A_i \rangle$, exists disjoint sequence, $\langle B_i \rangle$ such that

$$B_i \subset A_i \text{ and } \bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i$$

Algebras generated by subsets

- *algebra generated by* collection of subsets of X , \mathcal{C} , can be found by

$$\mathcal{A} = \bigcap \{\mathcal{B} \mid \mathcal{B} \in \mathcal{F}\}$$

where \mathcal{F} is family of all algebras containing \mathcal{C}

– smallest algebra \mathcal{A} containing \mathcal{C} , *i.e.*,

$$(\forall \mathcal{B} \in \mathcal{F})(\mathcal{A} \subset \mathcal{B})$$

- *σ -algebra generated by* collection of subsets of X , \mathcal{C} , can be found by

$$\mathcal{A} = \bigcap \{\mathcal{B} \mid \mathcal{B} \in \mathcal{G}\}$$

where \mathcal{G} is family of all σ -algebras containing \mathcal{C}

– smallest σ -algebra \mathcal{A} containing \mathcal{C} , *i.e.*,

$$(\forall \mathcal{B} \in \mathcal{G})(\mathcal{A} \subset \mathcal{B})$$

Relation

- x said to *stand in relation* \mathbf{R} to y , denoted by $x \mathbf{R} y$
- \mathbf{R} said to *be relation on* X if $x \mathbf{R} y \Rightarrow x \in X$ and $y \in X$
- \mathbf{R} is
 - transitive if $x \mathbf{R} y$ and $y \mathbf{R} z \Rightarrow x \mathbf{R} z$
 - symmetric if $x \mathbf{R} y = y \mathbf{R} x$
 - reflexive if $x \mathbf{R} x$
 - antisymmetric if $x \mathbf{R} y$ and $y \mathbf{R} x \Rightarrow x = y$
- \mathbf{R} is
 - *equivalence relation* if transitive, symmetric, and reflexive, *e.g.*, modulo
 - *partial ordering* if transitive and antisymmetric, *e.g.*, “ \subset ”
 - *linear (or simple) ordering* if transitive, antisymmetric, and $x \mathbf{R} y$ or $y \mathbf{R} x$ for all $x, y \in X$
 - *e.g.*, “ \geq ” linearly orders \mathbf{R} while “ \subset ” does not $\mathcal{P}(X)$

Ordering

- given partial order, \prec , a is
 - a first/smallest/least element if $x \neq a \Rightarrow a \prec x$
 - a last/largest/greatest element if $x \neq a \Rightarrow x \prec a$
 - a minimal element if $x \neq a \Rightarrow x \not\prec a$
 - a maximal element if $x \neq a \Rightarrow a \not\prec x$
- partial ordering \prec is
 - strict partial ordering if $x \not\prec x$
 - reflexive partial ordering if $x \prec x$
- strict linear ordering $<$ is
 - *well ordering* for X if every nonempty set contains a first element

Axiom of choice and equivalent principles

Axiom 1. [axiom of choice] *given a collection of nonempty sets, \mathcal{C} , there exists $f : \mathcal{C} \rightarrow \cup_{A \in \mathcal{C}} A$ such that*

$$(\forall A \in \mathcal{C}) (f(A) \in A)$$

- also called *multiplicative axiom* - preferred to be called to axiom of choice by Bertrand Russell for reason write on page 20
- no problem when \mathcal{C} is finite
- need axiom of choice when \mathcal{C} is not finite

Principle 4. [Hausdorff maximal principle] *for partial ordering \prec on X , exists a maximal linearly ordered subset $S \subset X$, i.e., S is linearity ordered by \prec and if $S \subset T \subset X$ and T is linearly ordered by \prec , $S = T$*

Principle 5. [well-ordering principle] *every set X can be well ordered, i.e., there is a relation $<$ that well orders X*

- note that Axiom 1 \Leftrightarrow Principle 4 \Leftrightarrow Principle 5

Infinite direct product

Definition 4. [direct product] for collection of sets, $\langle X_\lambda \rangle$, with index set, Λ ,

$$\prod_{\lambda \in \Lambda} X_\lambda$$

called direct product

- for $z = \langle x_\lambda \rangle \in \prod X_\lambda$, x_λ called λ -th coordinate of z
- if one of X_λ is empty, $\prod X_\lambda$ is empty
- *axiom of choice* is equivalent to converse, i.e., if none of X_λ is empty, $\prod X_\lambda$ is not empty
if one of X_λ is empty, $\prod X_\lambda$ is empty
- this is why Bertrand Russell prefers *multiplicative axiom* to *axiom of choice* for name of axiom (Axiom 1)

Real Number System

Field axioms

- field axioms - for every $x, y, z \in \mathbf{F}$
 - $(x + y) + z = x + (y + z)$ - additive associativity
 - $(\exists 0 \in \mathbf{F})(\forall x \in \mathbf{F})(x + 0 = x)$ - additive identity
 - $(\forall x \in \mathbf{F})(\exists w \in \mathbf{F})(x + w = 0)$ - additive inverse
 - $x + y = y + x$ - additive commutativity
 - $(xy)z = x(yz)$ - multiplicative associativity
 - $(\exists 1 \neq 0 \in \mathbf{F})(\forall x \in \mathbf{F})(x \cdot 1 = x)$ - multiplicative identity
 - $(\forall x \neq 0 \in \mathbf{F})(\exists w \in \mathbf{F})(xw = 1)$ - multiplicative inverse
 - $x(y + z) = xy + xz$ - distributivity
 - $xy = yx$ - multiplicative commutativity
- system (set with $+$ and \cdot) satisfying axiom of field called *field*
 - *e.g.*, field of module p where p is prime, \mathbf{F}_p

Axioms of order

- axioms of order - subset, $\mathbf{F}_{++} \subset \mathbf{F}$, of positive (real) numbers satisfies
 - $x, y \in \mathbf{F}_{++} \Rightarrow x + y \in \mathbf{F}_{++}$
 - $x, y \in \mathbf{F}_{++} \Rightarrow xy \in \mathbf{F}_{++}$
 - $x \in \mathbf{F}_{++} \Rightarrow -x \notin \mathbf{F}_{++}$
 - $x \in \mathbf{F} \Rightarrow x = 0 \vee x \in \mathbf{F}_{++} \vee -x \in \mathbf{F}_{++}$
- system satisfying field axioms & axioms of order called *ordered field*
 - e.g., set of real numbers (\mathbf{R}), set of rational numbers (\mathbf{Q})

Axiom of completeness

- completeness axiom
 - every nonempty set S of real numbers which has an upper bound has a least upper bound, *i.e.*,

$$\{l | (\forall x \in S)(l \leq x)\}$$

has least element.

- use $\inf S$ and $\sup S$ for least and greatest element (when exist)

- ordered field that is complete is *complete ordered field*
 - *e.g.*, \mathbf{R} (with $+$ and \cdot)

\Rightarrow axiom of Archimedes

- given any $x \in \mathbf{R}$, there is an integer n such that $x < n$

\Rightarrow corollary

- given any $x < y \in \mathbf{R}$, exists $r \in \mathbf{Q}$ such that $x < r < y$

Sequences of \mathbf{R}

- sequence of \mathbf{R} denoted by $\langle x_i \rangle_{i=1}^{\infty}$ or $\langle x_i \rangle$
 - mapping from \mathbf{N} to \mathbf{R}
- limit of $\langle x_n \rangle$ denoted by $\lim_{n \rightarrow \infty} x_n$ or $\lim x_n$ - defined by $a \in \mathbf{R}$

$$(\forall \epsilon > 0)(\exists N \in \mathbf{N})(n \geq N \Rightarrow |x_n - a| < \epsilon)$$

– $\lim x_n$ unique if exists

- $\langle x_n \rangle$ called Cauchy sequence if

$$(\forall \epsilon > 0)(\exists N \in \mathbf{N})(n, m \geq N \Rightarrow |x_n - x_m| < \epsilon)$$

- Cauchy criterion - characterizing complete metric space (including \mathbf{R})
 - sequence converges *if and only if* Cauchy sequence

Other limits

- cluster point of $\langle x_n \rangle$ - defined by $c \in \mathbf{R}$

$$(\forall \epsilon > 0, N \in \mathbf{N})(\exists n > N)(|x_n - c| < \epsilon)$$

- limit superior or limsup of $\langle x_n \rangle$

$$\limsup x_n = \inf_n \sup_{k > n} x_k$$

- limit inferior or liminf of $\langle x_n \rangle$

$$\liminf x_n = \sup_n \inf_{k > n} x_k$$

- $\liminf x_n \leq \limsup x_n$
- $\langle x_n \rangle$ converges *if and only if* $\liminf x_n = \limsup x_n (= \lim x_n)$

Open and closed sets

- O called *open* if

$$(\forall x \in O)(\exists \delta > 0)(y \in \mathbf{R})(|y - x| < \delta \Rightarrow y \in O)$$

- intersection of finite collection of open sets is open
- union of any collection of open sets is open

- \overline{E} called *closure* of E if

$$(\forall x \in \overline{E} \ \& \ \delta > 0)(\exists y \in E)(|x - y| < \delta)$$

- F called *closed* if

$$F = \overline{F}$$

- union of finite collection of closed sets is closed
- intersection of any collection of closed sets is closed

Open and closed sets - facts

- *every open set is union of countable collection of disjoint open intervals*

- (Lindelöf) any collection \mathcal{C} of open sets has a countable subcollection $\langle O_i \rangle$ such that

$$\bigcup_{O \in \mathcal{C}} O = \bigcup_i O_i$$

- equivalently, any collection \mathcal{F} of closed sets has a countable subcollection $\langle F_i \rangle$ such that

$$\bigcap_{O \in \mathcal{F}} F = \bigcap_i F_i$$

Covering and Heine-Borel theorem

- collection \mathcal{C} of sets called *covering* of A if

$$A \subset \bigcup_{O \in \mathcal{C}} O$$

- \mathcal{C} said to *cover* A
 - \mathcal{C} called *open covering* if every $O \in \mathcal{C}$ is open
 - \mathcal{C} called *finite covering* if \mathcal{C} is finite
- *Heine-Borel theorem* - for any closed and bounded set, every open covering has finite subcovering
- corollary
 - any collection \mathcal{C} of closed sets including at least one bounded set every finite subcollection of which has nonempty intersection has nonempty intersection.

Continuous functions

- f (with domain D) called *continuous at* x if

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall y \in D)(|y - x| < \delta \Rightarrow |f(y) - f(x)| < \epsilon)$$

- f called *continuous on* $A \subset D$ if f is continuous at every point in A

- f called *uniformly continuous on* $A \subset D$ if

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x, y \in D)(|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon)$$

Continuous functions - facts

- f is continuous *if and only if* for every open set O (in co-domain), $f^{-1}(O)$ is open
- f continuous on closed and bounded set is uniformly continuous
- *extreme value theorem* - f continuous on closed and bounded set, F , is *bounded on F* and *assumes its maximum and minimum on F*

$$(\exists x_1, x_2 \in F)(\forall x \in F)(f(x_1) \leq f(x) \leq f(x_2))$$

- *intermediate value theorem* - for f continuous on $[a, b]$ with $f(a) \leq f(b)$,

$$(\forall d)(f(a) \leq d \leq f(b))(\exists c \in [a, b])(f(c) = d)$$

Borel sets and Borel σ -algebra

- *Borel set*
 - any set that can be formed from open sets (or, equivalently, from closed sets) through the operations of countable union, countable intersection, and relative complement
- *Borel algebra* or *Borel σ -algebra*
 - *smallest σ -algebra containing all open sets*
 - also
 - smallest σ -algebra containing all closed sets
 - smallest σ -algebra containing all open intervals (due to statement on page 28)

Various Borel sets

- countable union of closed sets (in \mathbf{R}), called *an F_σ* (F for closed & σ for sum)
 - thus, every countable set, every closed set, every open interval, every open sets, is an F_σ (note $(a, b) = \bigcup_{n=1}^{\infty} [a + 1/n, b - 1/n]$)
 - countable union of sets in F_σ again is an F_σ
- countable intersection of open sets called *a G_δ* (G for open & δ for durchschnitt - average in German)
 - complement of F_σ is a G_δ and vice versa
- F_σ and G_δ are simple types of Borel sets
- countable intersection of F_σ 's is $F_{\sigma\delta}$, countable union of $F_{\sigma\delta}$'s is $F_{\sigma\delta\sigma}$, countable intersection of $F_{\sigma\delta\sigma}$'s is $F_{\sigma\delta\sigma\delta}$, *etc.*, & likewise for $G_{\delta\sigma}\dots$
- below are all classes of Borel sets, but not every Borel set belongs to one of these classes

$$F_\sigma, F_{\sigma\delta}, F_{\sigma\delta\sigma}, F_{\sigma\delta\sigma\delta}, \dots, G_\delta, G_{\delta\sigma}, G_{\delta\sigma\delta}, G_{\delta\sigma\delta\sigma}, \dots,$$

Measure and Integration

Purpose of integration theory

- purpose of “measure and integration” slides
 - abstract (out) most important properties of Lebesgue measure and Lebesgue integration
- provide certain *axioms that Lebesgue measure satisfies*
- base our integration theory on these axioms
- hence, our theory valid for every system satisfying the axioms

Measurable space, measure, and measure space

- family of subsets containing \emptyset closed under countable union and complement, called *σ -algebra*
- mapping of sets to extended real numbers, called *set function*
- (X, \mathcal{B}) with set, X , and σ -algebra of X , \mathcal{B} , called *measurable space*
 - $A \in \mathcal{B}$, said to be *measurable (with respect to \mathcal{B})*
- nonnegative set function, μ , defined on \mathcal{B} satisfying $\mu(\emptyset) = 0$ and for every disjoint, $\langle E_n \rangle_{n=1}^{\infty} \subset \mathcal{B}$,

$$\mu \left(\bigcup E_n \right) = \sum \mu E_n$$

called *measure on* measurable space, (X, \mathcal{B})

- measurable space, (X, \mathcal{B}) , equipped with measure, μ , called *measure space* and denoted by (X, \mathcal{B}, μ)

Measure space examples

- $(\mathbf{R}, \mathcal{M}, \mu)$ with Lebesgue measurable sets, \mathcal{M} , and Lebesgue measure, μ
- $([0, 1], \{A \in \mathcal{M} | A \subset [0, 1]\}, \mu)$ with Lebesgue measurable sets, \mathcal{M} , and Lebesgue measure, μ
- $(\mathbf{R}, \mathcal{B}, \mu)$ with class of Borel sets, \mathcal{B} , and Lebesgue measure, μ
- $(\mathbf{R}, \mathcal{P}(\mathbf{R}), \mu_C)$ with set of all subsets of \mathbf{R} , $\mathcal{P}(\mathbf{R})$, and counting measure, μ_C
- interesting (and bizarre) example
 - (X, \mathcal{A}, μ_B) with any uncountable set, X , family of either countable or complement of countable set, \mathcal{A} , and measure, μ_B , such that $\mu_B A = 0$ for countable $A \subset X$ and $\mu_B B = 1$ for uncountable $B \subset X$

More properties of measures

- for $A, B \in \mathcal{B}$ with $A \subset B$

$$\mu A \leq \mu B$$

- for $\langle E_n \rangle \subset \mathcal{B}$ with $\mu E_1 < \infty$ and $E_{n+1} \subset E_n$

$$\mu \left(\bigcap E_n \right) = \lim \mu E_n$$

- for $\langle E_n \rangle \subset \mathcal{B}$

$$\mu \left(\bigcup E_n \right) \leq \sum \mu E_n$$

Finite and σ -finite measures

- measure, μ , with $\mu(X) < \infty$, called *finite*
- measure, μ , with $X = \bigcup X_n$ for some $\langle X_n \rangle$ and $\mu(X_n) < \infty$, called *σ -finite*
 - always can take $\langle X_n \rangle$ with disjoint X_n
- Lebesgue measure on $[0, 1]$ is finite
- Lebesgue measure on \mathbf{R} is σ -finite
- counting measure on uncountable set is *not* σ -measure

Sets of finite and σ -finite measure

- set, $E \in \mathcal{B}$, with $\mu E < \infty$, said to be *of finite measure*
- set that is countable union of measurable sets of finite measure, said to be *of σ -finite measure*
- measurable set contained in set of σ -finite measure, is of σ -finite measure
- countable union of sets of σ -finite measure, is of σ -finite measure
- when μ is σ -finite, every measurable set is of σ -finite

Semifinite measures

- roughly speaking, nearly all familiar properties of Lebesgue measure and Lebesgue integration hold for arbitrary σ -finite measure
- many treatment of abstract measure theory limit themselves to σ -finite measures
- many parts of general theory, however, do *not* required assumption of σ -finiteness
- undesirable to have development unnecessarily restrictive
- measure, μ , for which every measurable set of infinite measure contains measurable sets of arbitrarily large finite measure, said to be *semifinite*
- every σ -finite measure is semifinite measure while measure, μ_B , on page 37 is not

Complete measure spaces

- measure space, (X, \mathcal{B}, μ) , for which \mathcal{B} contains all subsets of sets of measure zero, said to be *complete*, i.e.,

$$(\forall B \in \mathcal{B} \text{ with } \mu B = 0)(A \subset B \Rightarrow A \in \mathcal{B})$$

- e.g., Lebesgue measure is complete, but Lebesgue measure restricted to σ -algebra of Borel sets is *not*
- every measure space can be *completed* by addition of subsets of sets of measure zero
- for (X, \mathcal{B}, μ) , can find *complete* measure space $(X, \mathcal{B}_0, \mu_0)$ such that
 - $\mathcal{B} \subset \mathcal{B}_0$
 - $E \in \mathcal{B} \Rightarrow \mu E = \mu_0 E$
 - $E \in \mathcal{B}_0 \Leftrightarrow E = A \cup B$ where $B, C \in \mathcal{B}, \mu C = 0, A \subset C$
- $(X, \mathcal{B}_0, \mu_0)$ called *completion* of (X, \mathcal{B}, μ)

Local measurability and saturatedness

- for (X, \mathcal{B}, μ) , $E \subset X$ for which $(\forall B \in \mathcal{B} \text{ with } \mu B < \infty)(E \cap B \in \mathcal{B})$, said to be *locally measurable*
- collection, \mathcal{C} , of all locally measurable sets is σ -algebra containing \mathcal{B}
- measure for which every locally measurable set is measurable, said to be *saturated*
- every σ -finite measure is saturated
- measure can be extended to saturated measure, but (unlike completion) extension is not unique
 - can take \mathcal{C} as extension for locally measurable sets, but measure can be extended on \mathcal{C} in more than one ways

Measurable functions

- concept and properties of measurable functions in abstract measurable space almost identical with those of Lebesgue measurable functions (page ??)
 - theorems and facts are essentially same as those of Lebesgue measurable functions
 - assume measurable space, (X, \mathcal{B})
 - for $f : X \rightarrow \mathbf{R} \cup \{-\infty, \infty\}$, following are equivalent
 - $(\forall a \in \mathbf{R})(\{x \in X | f(x) < a\} \in \mathcal{B})$
 - $(\forall a \in \mathbf{R})(\{x \in X | f(x) \leq a\} \in \mathcal{B})$
 - $(\forall a \in \mathbf{R})(\{x \in X | f(x) > a\} \in \mathcal{B})$
 - $(\forall a \in \mathbf{R})(\{x \in X | f(x) \geq a\} \in \mathcal{B})$
 - $f : X \rightarrow \mathbf{R} \cup \{-\infty, \infty\}$ for which any one of above four statements holds, called *measurable* or *measurable with respect to \mathcal{B}*
- (refer to page ?? for Lebesgue counterpart)

Properties of measurable functions

- **Theorem 1. [measurability preserving function operations]** *for measurable functions, f and g , and $c \in \mathbf{R}$*
 - $f + c, cf, f + g, fg, f \vee g$ are measurable
- **Theorem 2. [limits of measurable functions]** *for every measurable function sequence, $\langle f_n \rangle$*
 - $\sup f_n, \limsup f_n, \inf f_n, \liminf f_n$ are measurable
 - thus, $\lim f_n$ is measurable if exists

(refer to page ?? for Lebesgue counterpart)

Simple functions and other properties

- φ called *simple function* if for distinct $\langle c_i \rangle_{i=1}^n$ and measurable sets, $\langle E_i \rangle_{i=1}^n$

$$\varphi(x) = \sum_{i=1}^n c_i \chi_{E_i}(x)$$

(refer to page ?? for Lebesgue counterpart)

- for nonnegative measurable function, f , exists nondecreasing sequence of simple functions, $\langle \varphi_n \rangle$, i.e., $\varphi_{n+1} \geq \varphi_n$ such that for every point in X

$$f = \lim \varphi_n$$

- for f defined on σ -finite measure space, we may choose $\langle \varphi_n \rangle$ so that every φ_n vanishes outside set of finite measure
- for complete measure, μ , f measurable and $f = g$ a.e. imply measurability of g

Define measurable function by ordinate sets

- $\{x | f(x) < \alpha\}$ sometimes called *ordinate sets*, which is nondecreasing in α
- below says when given nondecreasing ordinate sets, we can find f satisfying

$$\{x | f(x) < \alpha\} \subset B_\alpha \subset \{x | f(x) \leq \alpha\}$$

- for nondecreasing function, $h : D \rightarrow \mathcal{B}$, for dense set of real numbers, D , i.e., $B_\alpha \subset B_\beta$ for all $\alpha < \beta$ where $B_\alpha = h(\alpha)$, exists unique measurable function, $f : X \rightarrow \mathbf{R} \cup \{-\infty, \infty\}$ such that $f \leq \alpha$ on B_α and $f \geq \alpha$ on $X \setminus B_\alpha$
- can relax some conditions and make it a.e. version as below
- for function, $h : D \rightarrow \mathcal{B}$, for dense set of real numbers, D , such that $\mu(B_\alpha \setminus B_\beta) = 0$ for all $\alpha < \beta$ where $B_\alpha = h(\alpha)$, exists measurable function, $f : X \rightarrow \mathbf{R} \cup \{-\infty, \infty\}$ such that $f \leq \alpha$ a.e. on B_α and $f \geq \alpha$ a.e. on $X \setminus B_\alpha$
 - if g has the same property, $f = g$ a.e.

Integration

- many definitions and proofs of Lebesgue integral depend only on properties of Lebesgue measure which are also true for arbitrary measure in abstract measure space (page ??)
- integral of nonnegative simple function, $\varphi(x) = \sum_{i=1}^n c_i \chi_{E_i}(x)$, on measurable set, E , defined by

$$\int_E \varphi d\mu = \sum_{i=1}^n c_i \mu(E_i \cap E)$$

– independent of representation of φ

(refer to page ?? for Lebesgue counterpart)

- for $a, b \in \mathbf{R}_{++}$ and nonnegative simple functions, φ and ψ

$$\int (a\varphi + b\psi) = a \int \varphi + b \int \psi$$

(refer to page ?? for Lebesgue counterpart)

Integral of bounded functions

- for bounded function, f , identically zero outside measurable set of finite measure

$$\sup_{\varphi: \text{simple}, \varphi \leq f} \int \varphi = \inf_{\psi: \text{simple}, f \leq \psi} \int \psi$$

if and only if $f = g$ a.e. for measurable function, g

(refer to page ?? for Lebesgue counterpart)

- but, $f = g$ a.e. for measurable function, g , if and only if f is measurable with respect to completion of μ , $\bar{\mu}$
- natural class of functions to consider for integration theory are those measurable with respect to completion of μ
- thus, shall either assume μ is complete measure or define integral with respect to μ to be integral with respect to completion of μ depending on context unless otherwise specified

Difficulty of general integral of nonnegative functions

- for Lebesgue integral of nonnegative functions (page ??)
 - first define integral for bounded measurable functions
 - define integral of nonnegative function, f as supremum of integrals of all bounded measurable functions, $h \leq f$, vanishing outside measurable set of finite measure
- unfortunately, not work in case that measure is not semifinite
 - *e.g.*, if $\mathcal{B} = \{\emptyset, X\}$ with $\mu\emptyset = 0$ and $\mu X = \infty$, we want $\int 1d\mu = \infty$, but only bounded measurable function vanishing outside measurable set of finite measure is $h \equiv 0$, hence, $\int g d\mu = 0$
- to avoid this difficulty, we define integral of nonnegative measurable function directly in terms of integrals of nonnegative simple functions

Integral of nonnegative functions

- for measurable function, $f : X \rightarrow \mathbf{R} \cup \{\infty\}$, on measure space, (X, \mathcal{B}, μ) , define *integral of nonnegative extended real-valued measurable function*

$$\int f d\mu = \sup_{\varphi: \text{simple function, } 0 \leq \varphi \leq f} \int \varphi d\mu$$

(refer to page ?? for Lebesgue counterpart)

- however, *definition of integral of nonnegative extended real-valued measurable function* can be awkward to apply because
 - taking supremum over large collection of simple functions
 - *not clear from definition that $\int (f + g) = \int f + \int g$*
- thus, first establish some convergence theorems, and determine value of $\int f$ as limit of $\int \varphi_n$ for increasing sequence, $\langle \varphi_n \rangle$, of simple functions converging to f

Fatou's lemma and monotone convergence theorem

- *Fatou's lemma* - for nonnegative measurable function sequence, $\langle f_n \rangle$, with $\lim f_n = f$ a.e. on measurable set, E

$$\int_E f \leq \liminf \int_E f_n$$

- *monotone convergence theorem* - for nonnegative measurable function sequence, $\langle f_n \rangle$, with $f_n \leq f$ for all n and with $\lim f_n = f$ a.e.

$$\int_E f = \lim \int_E f_n$$

(refer to page ?? for Lebesgue counterpart)

Integrability of nonnegative functions

- for nonnegative measurable functions, f and g , and $a, b \in \mathbf{R}_+$

$$\int (af + bg) = a \int f + b \int g \text{ \& } \int f \geq 0$$

– equality holds *if and only if* $f = 0$ a.e.

(refer to page ?? for Lebesgue counterpart)

- monotone convergence theorem together with above yields, for nonnegative measurable function sequence, $\langle f_n \rangle$

$$\int \sum f_n = \sum \int f_n$$

- measurable nonnegative function, f , with

$$\int_E f d\mu < \infty$$

said to be *integral (over measurable set, E , with respect to μ)*

(refer to page ?? for Lebesgue counterpart)

Integral

- arbitrary function, f , for which both f^+ and f^- are integrable, said to be *integrable*
- in this case, define *integral*

$$\int_E f = \int_E f^+ - \int_E f^-$$

(refer to page ?? for Lebesgue counterpart)

Properties of integral

- for f and g integrable on measure set, E , and $a, b \in \mathbf{R}$
 - $af + bg$ is integral and

$$\int_E (af + bg) = a \int_E f + b \int_E g$$

- if $|h| \leq |f|$ and h is measurable, then h is integrable
- if $f \geq g$ a.e.

$$\int f \geq \int g$$

(refer to page ?? for Lebesgue counterpart)

Lebesgue convergence theorem

- *Lebesgue convergence theorem* - for integral, g , over E and sequence of measurable functions, $\langle f_n \rangle$, with $\lim f_n(x) = f(x)$ a.e. on E , if

$$|f_n(x)| \leq g(x)$$

then

$$\int_E f = \lim \int_E f_n$$

(refer to page ?? for Lebesgue counterpart)

Setwise convergence of sequence of measures

- preceding convergence theorems assume fixed measure, μ
- can generalize by allowing measure to vary
- given measurable space, (X, \mathcal{B}) , sequence of set functions, $\langle \mu_n \rangle$, defined on \mathcal{B} , satisfying

$$(\forall E \in \mathcal{B})(\lim \mu_n E = \mu E)$$

for some set function, μ , defined on \mathcal{B} , said to *converge setwise* to μ

General convergence theorems

- *generalization of Fatou's lemma* - for measurable space, (X, \mathcal{B}) , sequence of measures, $\langle \mu_n \rangle$, defined on \mathcal{B} , converging setwise to μ , defined on \mathcal{B} , and sequence of nonnegative functions, $\langle f_n \rangle$, each measurable with respect to μ_n , converging pointwise to function, f , measurable with respect to μ (compare with Fatou's lemma on page 52)

$$\int f d\mu \leq \liminf \int f_n d\mu_n$$

- *generalization of Lebesgue convergence theorem* - for measurable space, (X, \mathcal{B}) , sequence of measures, $\langle \mu_n \rangle$, defined on \mathcal{B} , converging setwise to μ , defined on \mathcal{B} , and sequences of functions, $\langle f_n \rangle$ and $\langle g_n \rangle$, each of f_n and g_n , measurable with respect to μ_n , converging pointwise to f and g , measurable with respect to μ , respectively, such that (compare with Lebesgue convergence theorem on page 56)

$$\lim \int g_n d\mu_n = \int g d\mu < \infty$$

satisfy

$$\lim \int f_n d\mu_n = \int f \mu$$

L^p spaces

- for complete measure space, (X, \mathcal{B}, μ)
 - space of measurable functions on X with $\int |f|^p < \infty$, for which element equivalence is defined by being equal a.e., called L^p spaces denoted by $L^p(\mu)$
 - space of bounded measure functions, called L^∞ space denoted by $L^\infty(\mu)$

- norms

- for $p \in [1, \infty)$

$$\|f\|_p = \left(\int |f|^p d\mu \right)^{1/p}$$

- for $p = \infty$

$$\|f\|_\infty = \text{ess sup}|f| = \inf \{ |g(x)| \mid \text{measurable } g \text{ with } g = f \text{ a.e.} \}$$

- for $p \in [1, \infty]$, spaces, $L^p(\mu)$, are Banach spaces

Hölder's inequality and Littlewood's second principle

- *Hölder's inequality* - for $p, q \in [1, \infty]$ with $1/p + 1/q = 1$, $f \in L^p(\mu)$ and $g \in L^q(\mu)$ satisfy $fg \in L^1(\mu)$ and

$$\|fg\|_1 = \int |fg| d\mu \leq \|f\|_p \|g\|_q$$

(refer to page ?? for normed spaces counterpart)

- *complete measure space version of Littlewood's second principle* - for $p \in [1, \infty)$

$$(\forall f \in L^p(\mu), \epsilon > 0)$$

(\exists simple function φ vanishing outside set of finite measure)

$$(\|f - \varphi\|_p < \epsilon)$$

(refer to page ?? for normed spaces counterpart)

Riesz representation theorem

- *Riesz representation theorem* - for $p \in [1, \infty)$ and bounded linear functional, F , on $L^p(\mu)$ and σ -finite measure, μ , exists *unique* $g \in L^q(\mu)$ where $1/p + 1/q = 1$ such that

$$F(f) = \int f g d\mu$$

where $\|F\| = \|g\|_q$

(refer to page ?? for normed spaces counterpart)

- if $p \in (1, \infty)$, Riesz representation theorem holds without assumption of σ -finiteness of measure

Measure and Outer Measure

General measures

- consider some ways of defining measures on σ -algebra
- recall that for Lebesgue measure
 - define measure for open intervals
 - define outer measure
 - define notion of measurable sets
 - finally derive Lebesgue measure
- one can do similar things in general, *e.g.*,
 - derive measure from outer measure
 - derive outer measure from measure defined on algebra of sets

Outer measure

- set function, $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$, for space X , having following properties, called *outer measure*
 - $\mu^* \emptyset = 0$
 - $A \subset B \Rightarrow \mu^* A \leq \mu^* B$ (monotonicity)
 - $E \subset \bigcup_{n=1}^{\infty} E_n \Rightarrow \mu^* E \leq \sum_{n=1}^{\infty} \mu^* E_n$ (countable subadditivity)
- μ^* with $\mu^* X < \infty$ called *finite*
- set $E \subset X$ satisfying following property, said to be *measurable with respect to μ^**
$$(\forall A \subset X)(\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap \tilde{E}))$$
- class, \mathcal{B} , of μ^* -measurable sets is σ -algebra
- restriction of μ^* to \mathcal{B} is complete measure on \mathcal{B}

Extension to measure from measure on an algebra

- set function, $\mu : \mathcal{A} \rightarrow [0, \infty]$, defined on algebra, \mathcal{A} , having following properties, called *measure on an algebra*
 - $\mu(\emptyset) = 0$
 - $(\forall \text{ disjoint } \langle A_n \rangle \subset \mathcal{A} \text{ with } \bigcup A_n \in \mathcal{A}) (\mu(\bigcup A_n) = \sum \mu A_n)$
- *measure on an algebra, \mathcal{A} , is measure if and only if \mathcal{A} is σ -algebra*
- can extend measure on an algebra to measure defined on σ -algebra, \mathcal{B} , containing \mathcal{A} , by
 - constructing outer measure μ^* from μ
 - deriving desired extension $\bar{\mu}$ induced by μ^*
- process by which constructing μ^* from μ similar to constructing Lebesgue outer measure from lengths of intervals

Outer measure constructed from measure on an algebra

— given measure, μ , on an algebra, \mathcal{A}

- define set function, $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$, by

$$\mu^* E = \inf_{\langle A_n \rangle \subset \mathcal{A}, E \subset \bigcup A_n} \sum \mu A_n$$

- μ^* called *outer measure induced by μ*

— then

- for $A \in \mathcal{A}$ and $\langle A_n \rangle \subset \mathcal{A}$ with $A \subset \bigcup A_n$, $\mu A \leq \sum \mu A_n$
- hence, $(\forall A \in \mathcal{A})(\mu^* A = \mu A)$
- μ^* is outer measure
- every $A \in \mathcal{A}$ is measurable with respect to μ^*

Regular outer measure

- for algebra, \mathcal{A}
 - \mathcal{A}_σ denote sets that are countable unions of sets of \mathcal{A}
 - $\mathcal{A}_{\sigma\delta}$ denote sets that are countable intersections of sets of \mathcal{A}_σ
- given measure, μ , on an algebra, \mathcal{A} and outer measure, μ^* induced by μ , for every $E \subset X$ and every $\epsilon > 0$, exists $A \in \mathcal{A}_\sigma$ and $B \in \mathcal{A}_{\sigma\delta}$ with $E \subset A$ and $E \subset B$

$$\mu^* A \leq \mu^* E + \epsilon \text{ and } \mu^* E = \mu^* B$$

- outer measure, μ^* , with below property, said to be *regular*

$$(\forall E \subset X, \epsilon > 0)(\exists \mu^*\text{-measurable set } A \text{ with } E \subset A)(\mu^* A \subset \mu^* E + \epsilon)$$

- every outer measure induced by measure on an algebra is regular outer measure

Carathéodory theorem

- given measure, μ , on an algebra, \mathcal{A} and outer measure, μ^* induced by μ
- $E \subset X$ is μ^* -measurable if and only if exist $A \in \mathcal{A}_{\sigma\delta}$ and $B \subset X$ with $\mu^*B = 0$ such that

$$E = A \sim B$$

- for $B \subset X$ with $\mu^*B = 0$, exists $C \in \mathcal{A}_{\sigma\delta}$ with $\mu^*C = 0$ such that $B \subset C$
- *Carathéodory theorem* - restriction, $\bar{\mu}$, of μ^* to μ^* -measurable sets is extension of μ to σ -algebra containing \mathcal{A}
 - if μ is finite or σ -finite, so is $\bar{\mu}$ respectively
 - if μ is σ -finite, $\bar{\mu}$ is only measure on smallest σ -algebra containing \mathcal{A} which is extension of μ

Product measures

- for countable disjoint collection of measurable rectangles, $\langle (A_n \times B_n) \rangle$, whose union is measurable rectangle, $A \times B$

$$\lambda(A \times B) = \sum \lambda(A_n \times B_n)$$

- for $x \in X$ and $E \in \mathcal{R}_{\sigma\delta}$

$$E_x = \{y | \langle x, y \rangle \in E\}$$

is measurable subset of Y

- for $E \subset \mathcal{R}_{\sigma\delta}$ with $\mu \times \nu(E) < \infty$, function, g , defined by

$$g(x) = \nu E_x$$

is measurable function of x and

$$\int g d\mu = \mu \times \nu(E)$$

- XXX

Carathéodory outer measures

- set, X , of points and set, Γ , of real-valued functions on X
- two sets for which exist $a > b$ such that function, φ , greater than a on one set and less than b on the other set, said to be *separated by function, φ*
- outer measure, μ^* , with $(\forall A, B \subset X \text{ separated by } f \in \Gamma)(\mu^*(A \cup B) = \mu^*A + \mu^*B)$, called *Carathéodory outer measure with respect to Γ*
- outer measure, μ^* , on metric space, $\langle X, \rho, \cdot \rangle$ for which $\mu^*(A \cup B) = \mu^*A + \mu^*B$ for $A, B \subset X$ with $\rho(A, B) > 0$, called *Carathéodory outer measure for X* or *metric outer measure*
- for *Carathéodory outer measure, μ^* , with respect to Γ* , every function in Γ is μ^* -measurable
- for *Carathéodory outer measure, μ^* , for metric space, $\langle X, \rho, \cdot \rangle$* , every closed set (hence every Borel set) is measurable with respect to μ^*

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