Searching for Universal Truths

Topological Spaces

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Navigating Mathematical and Statistical Territories

Notations

- sets of numbers
  - **N** - set of natural numbers
  - **Z** - set of integers
  - $\mathbf{Z}_+$ - set of nonnegative integers
  - **Q** - set of rational numbers
  - **R** - set of real numbers
  - $\mathbf{R}_+$ - set of nonnegative real numbers
  - $\mathbf{R}_{++}$ - set of positive real numbers
  - **C** - set of complex numbers
- sequences $\langle x_i \rangle$ and the like
  - finite $\langle x_i \rangle_{i=1}^n$, infinite $\langle x_i \rangle_{i=1}^\infty$ - use $\langle x_i \rangle$ whenever unambiguously understood
  - similarly for other operations, *e.g.*, $\sum x_i$, $\prod x_i$, $\cup A_i$, $\cap A_i$, $\bigtimes A_i$
  - similarly for integrals, *e.g.*, $\int f$ for $\int_{-\infty}^{\infty} f$
- sets
  - $\tilde{A}$ - complement of A

- $A \sim B$ - $A \cap \tilde{B}$
- $A \Delta B$ - $(A \cap \tilde{B}) \cup (\tilde{A} \cap B)$
- $\mathcal{P}(A)$ - set of all subsets of A

- sets in metric vector spaces
  - $\overline{A}$ - closure of set A
  - A° - interior of set A
  - $\mathbf{relint}\, A$ - relative interior of set A
  - $\mathbf{bd}\, A$ - boundary of set A

- set algebra
  - $\sigma(\mathcal{A})$ - σ-algebra generated by $\mathcal{A}$, *i.e.*, smallest σ-algebra containing $\mathcal{A}$

- norms in $\mathbf{R}^n$
  - $\|x\|_p$ $(p \geq 1)$ - p-norm of $x \in \mathbf{R}^n$, *i.e.*, $(|x_1|^p + \cdots + |x_n|^p)^{1/p}$
  - *e.g.*, $\|x\|_2$ - Euclidean norm

- matrices and vectors
  - a_i - i-th entry of vector a
  - A_{ij} - entry of matrix A at position (i, j), *i.e.*, entry in i-th row and j-th column
  - $\mathbf{Tr}(A)$ - trace of $A \in \mathbf{R}^{n \times n}$, *i.e.*, $A_{1,1} + \cdots + A_{n,n}$

- symmetric, positive definite, and positive semi-definite matrices
  - $\mathbf{S}^n \subset \mathbf{R}^{n\times n}$ - set of symmetric matrices
  - $\mathbf{S}^n_+ \subset \mathbf{S}^n$ - set of positive semi-definite matrices; $A \succeq 0 \Leftrightarrow A \in \mathbf{S}^n_+$
  - $\mathbf{S}^n_{++} \subset \mathbf{S}^n$ - set of positive definite matrices; $A \succ 0 \Leftrightarrow A \in \mathbf{S}^n_{++}$
- sometimes, use Python script-like notations (with serious abuse of mathematical notations)
  - use $f : \mathbf{R} \to \mathbf{R}$ as if it were $f : \mathbf{R}^n \to \mathbf{R}^n$, *e.g.*,

$$\exp(x) = (\exp(x_1), \ldots, \exp(x_n)) \quad \text{for } x \in \mathbf{R}^n$$

    and

$$\log(x) = (\log(x_1), \ldots, \log(x_n)) \quad \text{for } x \in \mathbf{R}^n_{++}$$

    which corresponds to Python code `numpy.exp(x)` or `numpy.log(x)` where `x` is instance of `numpy.ndarray`, *i.e.*, `numpy` array
  - use $\sum x$ to mean $\mathbf{1}^T x$ for $x \in \mathbf{R}^n$, *i.e.*

$$\sum x = x_1 + \cdots + x_n$$

    which corresponds to Python code `x.sum()` where `x` is `numpy` array

- use x/y for $x, y \in \mathbf{R}^n$ to mean

$$\left[\begin{array}{ccc} x_1/y_1 & \cdots & x_n/y_n \end{array}\right]^T$$

  which corresponds to Python code `x / y` where `x` and `y` are 1-d `numpy` arrays
- use X/Y for $X, Y \in \mathbf{R}^{m\times n}$ to mean

$$\left[\begin{array}{cccc} X_{1,1}/Y_{1,1} & X_{1,2}/Y_{1,2} & \cdots & X_{1,n}/Y_{1,n} \\ X_{2,1}/Y_{2,1} & X_{2,2}/Y_{2,2} & \cdots & X_{2,n}/Y_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ X_{m,1}/Y_{m,1} & X_{m,2}/Y_{m,2} & \cdots & X_{m,n}/Y_{m,n} \end{array}\right]$$

  which corresponds to Python code `X / Y` where `X` and `Y` are 2-d `numpy` arrays

Some definitions

Definition 1. [infinitely often - i.o.] *statement P_n, said to happen* infinitely often *or* i.o. *if*

$$(\forall N \in \mathbf{N})\,(\exists n > N)\,(P_n)$$

Definition 2. [almost everywhere - a.e.] *statement $P(x)$, said to happen* almost everywhere *or* a.e. *or* almost surely *or* a.s. *(depending on context) associated with measure space $(X, \mathscr{B}, \mu)$ if*

$$\mu\{x | P(x)\} = 1$$

or equivalently

$$\mu\{x | \sim P(x)\} = 0$$

Some conventions

- (for some subjects) use following conventions
  - $0 \cdot \infty = \infty \cdot 0 = 0$
  - $(\forall x \in \mathbf{R}_{++})(x \cdot \infty = \infty \cdot x = \infty)$
  - $\infty \cdot \infty = \infty$

Real Analysis

Set Theory

Some principles

Principle 1. [principle of mathematical induction]

$$P(1)\&[P(n \Rightarrow P(n+1)] \Rightarrow (\forall n \in \mathbf{N})P(n)$$

Principle 2. [well ordering principle] *each nonempty subset of* $\mathbf{N}$ *has a smallest element*

Principle 3. [principle of recursive definition] *for* $f : X \to X$ *and* $a \in X$*, exists unique infinite sequence* $\langle x_n \rangle_{n=1}^{\infty} \subset X$ *such that*

$$x_1 = a$$

and

$$(\forall n \in \mathbf{N})\,(x_{n+1} = f(x_n))$$

- note that Principle 1 $\Leftrightarrow$ Principle 2 $\Rightarrow$ Principle 3

Some definitions for functions

Definition 3. [functions] *for* $f : X \to Y$

- *terms,* map *and* function*, exterchangeably used*
- X *and* Y*, called* domain of f *and* codomain of f *respectively*
- $\{f(x)|x \in X\}$*, called* range of f
- *for* $Z \subset Y$, $f^{-1}(Z) = \{x \in X|f(x) \in Z\} \subset X$*, called* preimage *or* inverse image of Z under f
- *for* $y \in Y$, $f^{-1}(\{y\})$*, called* fiber of f over y
- f*, called* injective *or* injection *or* one-to-one *if* $(\forall x \neq v \in X)\,(f(x) \neq f(v))$
- f*, called* surjective *or* surjection *or* onto *if* $(\forall x \in X)\,(\exists y in Y)\,(y = f(x))$
- f*, called* bijective *or* bijection *if* f *is both injective and surjective, in which case,* X *and* Y*, said to be* one-to-one correspondece *or* bijective correspondece
- $g : Y \to X$*, called* left inverse *if* $g \circ f$ *is identity function*
- $h : Y \to X$*, called* right inverse *if* $f \circ h$ *is identity function*

Some properties of functions

Lemma 1. [functions] *for* $f : X \to Y$

- f *is injective* if and only if f *has left inverse*
- f *is surjective* if and only if f *has right inverse*
- *hence,* f *is bijective* if and only if f *has both left and right inverse because if* g *and* h *are left and right inverses respectively,* $g = g \circ (f \circ h) = (g \circ f) \circ h = h$
- *if* $|X| = |Y| < \infty$, f *is injective* if and only if f *is surjective* if and only if f *is bijective*

Countability of sets

- set A is countable if range of some function whose domain is $\mathbf{N}$

- $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{Q}$: countable

- $\mathbf{R}$: *not* countable

Limit sets

- for sequence, $\langle A_n \rangle$, of subsets of X
  - *limit superior or limsup of $\langle A_n \rangle$*, defined by

$$\limsup \langle A_n \rangle = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m$$

  - *limit inferior or liminf of $\langle A_n \rangle$*, defined by

$$\liminf \langle A_n \rangle = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m$$

- always

$$\liminf \langle A_n \rangle \subset \limsup \langle A_n \rangle$$

- when $\liminf \langle A_n \rangle = \limsup \langle A_n \rangle$, sequence, $\langle A_n \rangle$, said to *converge to it*, denote

$$\lim \langle A_n \rangle = \liminf \langle A_n \rangle = \limsup \langle A_n \rangle = A$$

Algebras of sets

- collection $\mathscr{A}$ of subsets of X called *algebra* or *Boolean algebra* if

$$(\forall A, B \in \mathscr{A})(A \cup B \in \mathscr{A}) \text{ and } (\forall A \in \mathscr{A})(\tilde{A} \in \mathscr{A})$$

  - $(\forall A_1, \ldots, A_n \in \mathscr{A})(\cup_{i=1}^n A_i \in \mathscr{A})$
  - $(\forall A_1, \ldots, A_n \in \mathscr{A})(\cap_{i=1}^n A_i \in \mathscr{A})$
- algebra $\mathscr{A}$ called *σ-algebra* or *Borel field* if
  - every union of a countable collection of sets in $\mathscr{A}$ is in $\mathscr{A}$, *i.e.*,

$$(\forall \langle A_i \rangle)(\cup_{i=1}^\infty A_i \in \mathscr{A})$$

- given sequence of sets in algebra $\mathscr{A}$, $\langle A_i \rangle$, exists disjoint sequence, $\langle B_i \rangle$ such that

$$B_i \subset A_i \text{ and } \bigcup_{i=1}^\infty B_i = \bigcup_{i=1}^\infty A_i$$

Algebras generated by subsets

- *algebra generated by* collection of subsets of X, $\mathcal{C}$, can be found by

$$\mathscr{A} = \bigcap \{\mathscr{B} | \mathscr{B} \in \mathcal{F}\}$$

  where $\mathcal{F}$ is family of all algebras containing $\mathcal{C}$
  - smallest algebra $\mathscr{A}$ containing $\mathcal{C}$, $i.e.$,

$$(\forall \mathscr{B} \in \mathcal{F})(\mathscr{A} \subset \mathscr{B})$$

- *σ-algebra generated by* collection of subsets of X, $\mathcal{C}$, can be found by

$$\mathscr{A} = \bigcap \{\mathscr{B} | \mathscr{B} \in \mathcal{G}\}$$

  where $\mathcal{G}$ is family of all σ-algebras containing $\mathcal{C}$
  - smallest σ-algebra $\mathscr{A}$ containing $\mathcal{C}$, $i.e.$,

$$(\forall \mathscr{B} \in \mathcal{G})(\mathscr{A} \subset \mathscr{B})$$

Relation

- x said to *stand in relation* $\mathbf{R}$ to y, denoted by $x \ \mathbf{R} \ y$
- $\mathbf{R}$ said to *be relation on* X if $x \ \mathbf{R} \ y \Rightarrow x \in X$ and $y \in X$
- $\mathbf{R}$ is
  - transitive if $x \ \mathbf{R} \ y$ and $y \ \mathbf{R} \ z \Rightarrow x \ \mathbf{R} \ z$
  - symmetric if $x \ \mathbf{R} \ y = y \ \mathbf{R} \ x$
  - reflexive if $x \ \mathbf{R} \ x$
  - antisymmetric if $x \ \mathbf{R} \ y$ and $y \ \mathbf{R} \ x \Rightarrow x = y$
- $\mathbf{R}$ is
  - *equivalence relation* if transitive, symmetric, and reflexive, $e.g.$, modulo
  - *partial ordering* if transitive and antisymmetric, $e.g.$, "$\subset$"
  - *linear (or simple) ordering* if transitive, antisymmetric, and $x \ \mathbf{R} \ y$ or $y \ \mathbf{R} \ x$ for all $x, y \in X$
    - $e.g.$, "$\geq$" linearly orders **R** while "$\subset$" does not $\mathcal{P}(X)$

Ordering

- given partial order, $\prec$, a is
  - a first/smallest/least element if $x \neq a \Rightarrow a \prec x$
  - a last/largest/greatest element if $x \neq a \Rightarrow x \prec a$
  - a minimal element if $x \neq a \Rightarrow x \not\prec a$
  - a maximal element if $x \neq a \Rightarrow a \not\prec x$
- partial ordering $\prec$ is
  - strict partial ordering if $x \not\prec x$
  - reflexive partial ordering if $x \prec x$
- strict linear ordering $<$ is
  - *well ordering* for X if every nonempty set contains a first element

Axiom of choice and equivalent principles

Axiom 1. [axiom of choice] *given a collection of nonempty sets, $\mathcal{C}$, there exists $f : \mathcal{C} \to \cup_{A \in \mathcal{C}} A$ such that*

$$(\forall A \in \mathcal{C})\,(f(A) \in A)$$

- also called *multiplicative axiom* - preferred to be called to axiom of choice by Bertrand Russell for reason writte on page 21
- no problem when $\mathcal{C}$ is finite
- need axiom of choice when $\mathcal{C}$ is not finite

Principle 4. [Hausdorff maximal principle] *for particial ordering $\prec$ on X, exists a maximal linearly ordered subset $S \subset X$, $i.e.$, S is linearity ordered by $\prec$ and if $S \subset T \subset X$ and T is linearly ordered by $\prec$, $S = T$*

Principle 5. [well-ordering principle] *every set X can be well ordered, $i.e.$, there is a relation $<$ that well orders X*

- note that Axiom 1 $\Leftrightarrow$ Principle 4 $\Leftrightarrow$ Principle 5

Infinite direct product

Definition 4. [direct product] *for collection of sets, $\langle X_\lambda \rangle$, with index set, Λ,*

$$\bigtimes_{\lambda \in \Lambda} X_\lambda$$

called direct product

- *for $z = \langle x_\lambda \rangle \in \bigtimes X_\lambda$, x_λ called* λ-th coordinate of z

- if one of X_λ is empty, $\bigtimes X_\lambda$ is empty
- *axiom of choice* is equivalent to converse, *i.e.*, if none of X_λ is empty, $\bigtimes X_\lambda$ is not empty
  if one of X_λ is empty, $\bigtimes X_\lambda$ is empty
- this is why Bertrand Russell prefers *multiplicative axiom* to *axiom of choice* for name of axiom (Axiom 1)

Real Number System

Field axioms

- field axioms - for every $x, y, z \in \textbf{F}$
  - $(x + y) + z = x + (y + z)$ - additive associativity
  - $(\exists 0 \in \textbf{F})(\forall x \in \textbf{F})(x + 0 = x)$ - additive identity
  - $(\forall x \in \textbf{F})(\exists w \in \textbf{F})(x + w = 0)$ - additive inverse
  - $x + y = y + x$ - additive commutativity
  - $(xy)z = x(yz)$ - multiplicative associativity
  - $(\exists 1 \neq 0 \in \textbf{F})(\forall x \in \textbf{F})(x \cdot 1 = x)$ - multiplicative identity
  - $(\forall x \neq 0 \in \textbf{F})(\exists w \in \textbf{F})(xw = 1)$ - multiplicative inverse
  - $x(y + z) = xy + xz$ - distributivity
  - $xy = yx$ - multiplicative commutativity
- system (set with $+$ and $\cdot$) satisfying axiom of field called *field*
  - $e.g.$, field of module p where p is prime, $\textbf{F}_p$

Axioms of order

- axioms of order - subset, $\mathbf{F}_{++} \subset \mathbf{F}$, of positive (real) numbers satisfies
  - $x, y \in \mathbf{F}_{++} \Rightarrow x + y \in \mathbf{F}_{++}$
  - $x, y \in \mathbf{F}_{++} \Rightarrow xy \in \mathbf{F}_{++}$
  - $x \in \mathbf{F}_{++} \Rightarrow -x \not\in \mathbf{F}_{++}$
  - $x \in \mathbf{F} \Rightarrow x = 0 \vee x \in \mathbf{F}_{++} \vee -x \in \mathbf{F}_{++}$

- system satisfying field axioms & axioms of order called *ordered field*
  - $e.g.$, set of real numbers ($\mathbf{R}$), set of rational numbers ($\mathbf{Q}$)

Axiom of completeness

- completeness axiom
  - every nonempty set S of real numbers which has an upper bound has a least upper bound, *i.e.*,
  $$\{l|(\forall x \in S)(l \leq x)\}$$
  has least element.
  - use $\inf S$ and $\sup S$ for least and greatest element (when exist)
- ordered field that is complete is *complete ordered field*
  - *e.g.*, **R** (with $+$ and $\cdot$)

$\Rightarrow$ axiom of Archimedes
  - given any $x \in$ **R**, there is an integer n such that $x < n$

$\Rightarrow$ corollary
  - given any $x < y \in$ **R**, exists $r \in$ **Q** such tat $x < r < y$

Sequences of R

- sequence of **R** denoted by $\langle x_i \rangle_{i=1}^{\infty}$ or $\langle x_i \rangle$
  - mapping from **N** to **R**
- limit of $\langle x_n \rangle$ denoted by $\lim_{n\to\infty} x_n$ or $\lim x_n$ - defined by $a \in$ **R** such that

$$(\forall \epsilon > 0)(\exists N \in \mathbf{N})(n \geq N \Rightarrow |x_n - a| < \epsilon)$$

  - $\lim x_n$ unique if exists
- $\langle x_n \rangle$ called Cauchy sequence if

$$(\forall \epsilon > 0)(\exists N \in \mathbf{N})(n, m \geq N \Rightarrow |x_n - x_m| < \epsilon)$$

- Cauchy criterion - characterizing complete metric space (including **R**)
  - sequence converges *if and only if* Cauchy sequence

Other limits

- cluster point of $\langle x_n \rangle$ - defined by $c \in \mathbf{R}$

$$(\forall \epsilon > 0, N \in \mathbf{N})(\exists n > N)(|x_n - c| < \epsilon)$$

- limit superior or limsup of $\langle x_n \rangle$

$$\limsup x_n = \inf_n \sup_{k>n} x_k$$

- limit inferior or liminf of $\langle x_n \rangle$

$$\liminf x_n = \sup_n \inf_{k>n} x_k$$

- $\liminf x_n \leq \limsup x_n$
- $\langle x_n \rangle$ converges *if and only if* $\liminf x_n = \limsup x_n$ $(=\lim x_n)$

Open and closed sets

- O called *open* if

$$(\forall x \in O)(\exists \delta > 0)(y \in \mathbf{R})(|y - x| < \delta \Rightarrow y \in O)$$

  - intersection of finite collection of open sets is open
  - union of any collection of open sets is open

- $\overline{E}$ called *closure* of E if

$$(\forall x \in \overline{E} \;\&\; \delta > 0)(\exists y \in E)(|x - y| < \delta)$$

- F called *closed* if

$$F = \overline{F}$$

  - union of finite collection of closed sets is closed
  - intersection of any collection of closed sets is closed

Open and closed sets - facts

- *every open set is union of countable collection of disjoint open intervals*

- (Lindelöf) any collection $\mathcal{C}$ of open sets has a countable subcollection $\langle O_i \rangle$ such that

$$\bigcup_{O \in \mathcal{C}} O = \bigcup_i O_i$$

  – equivalently, any collection $\mathcal{F}$ of closed sets has a countable subcollection $\langle F_i \rangle$ such that

$$\bigcap_{O \in \mathcal{F}} F = \bigcap_i F_i$$

Covering and Heine-Borel theorem

- collection $\mathcal{C}$ of sets called *covering* of A if

$$A \subset \bigcup_{O \in \mathcal{C}} O$$

  - $\mathcal{C}$ said to *cover* A
  - $\mathcal{C}$ called *open covering* if every $O \in \mathcal{C}$ is open
  - $\mathcal{C}$ called *finite covering* if $\mathcal{C}$ is finite

- *Heine-Borel theorem* - for any closed and bounded set, every open covering has finite subcovering

- corollary
  - any collection $\mathcal{C}$ of closed sets including at least one bounded set every finite subcollection of which has nonempty intersection has nonempty intersection.

Continuous functions

- f (with domain D) called *continuous at* x if

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall y \in D)(|y - x| < \delta \Rightarrow |f(y) - f(x)| < \epsilon)$$

- f called *continuous on* $A \subset D$ if f is continuous at every point in A

- f called *uniformly continuous on* $A \subset D$ if

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x, y \in D)(|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon)$$

Continuous functions - facts

- f is continuous *if and only if* for every open set O (in co-domain), $f^{-1}(O)$ is open

- f continuous on closed and bounded set is uniformly continuous

- *extreme value theorem* - f continuous on closed and bounded set, F, is *bounded on F and assumes its maximum and minimum on F*

$$(\exists x_1, x_2 \in F)(\forall x \in F)(f(x_1) \leq f(x) \leq f(x_2))$$

- *intermediate value theorem* - for f continuous on $[a, b]$ with $f(a) \leq f(b)$,

$$(\forall d)(f(a) \leq d \leq f(b))(\exists c \in [a, b])(f(c) = d)$$

Borel sets and Borel σ-algebra

- *Borel set*
  - any set that can be formed from open sets (or, equivalently, from closed sets) through the operations of countable union, countable intersection, and relative complement

- *Borel algebra* or *Borel σ-algebra*
  - *smallest σ-algebra containing all open sets*
  - also
    - smallest σ-algebra containing all closed sets
    - smallest σ-algebra containing all open intervals (due to statement on page 29)

Various Borel sets

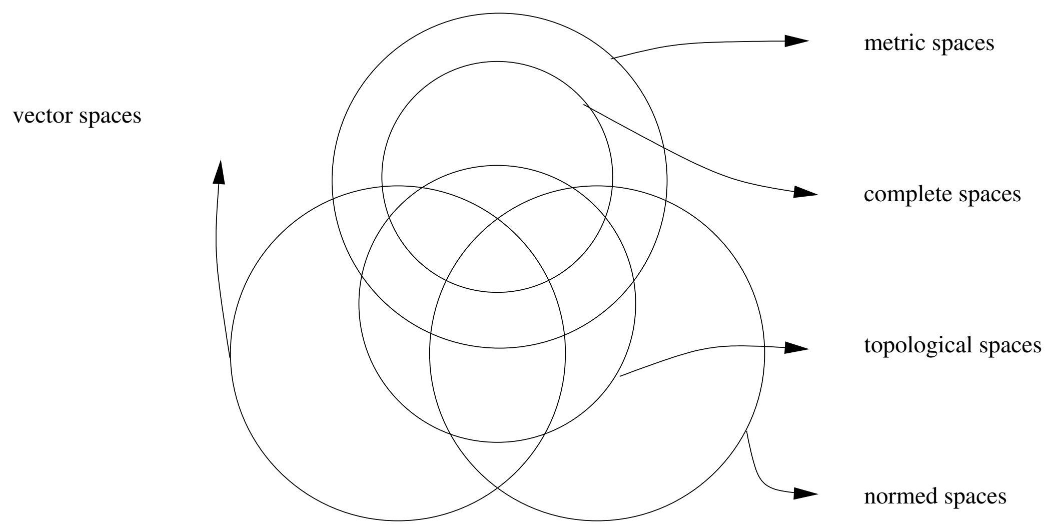
- countable union of closed sets (in **R**), called *an F_σ* (F for closed & σ for sum)
  - thus, every countable set, every closed set, every open interval, every open sets, is an F_σ (note $(a,b) = \bigcup_{n=1}^\infty [a+1/n, b-1/n]$)
  - countable union of sets in F_σ again is an F_σ
- countable intersection of open sets called *a G_δ* (G for open & δ for durchschnitt - average in German)
  - complement of F_σ is a G_δ and vice versa
- F_σ and G_δ are simple types of Borel sets
- countable intersection of F_σ's is $F_{\sigma\delta}$, countable union of $F_{\sigma\delta}$'s is $F_{\sigma\delta\sigma}$, countable intersection of $F_{\sigma\delta\sigma}$'s is $F_{\sigma\delta\sigma\delta}$, $etc.$, & likewise for $G_{\delta\sigma\cdots}$
- below are all classes of Borel sets, but not every Borel set belongs to one of these classes

$$F_\sigma, F_{\sigma\delta}, F_{\sigma\delta\sigma}, F_{\sigma\delta\sigma\delta}, \ldots, G_\delta, G_{\delta\sigma}, G_{\delta\sigma\delta}, G_{\delta\sigma\delta\sigma}, \ldots,$$

Space Overview

Diagrams for relations among various spaces

- note from the figure
  - metric should be defined to utter completeness
  - metric spaces can be induced from normed spaces



Classical Banach Spaces

Normed linear space

- X called *linear space* if

$$(\forall x, y \in X, a, b \in \mathbf{R})(ax + by \in X)$$

- linear space, X, called *normed space* with associated norm $\|\cdot\| : X \to \mathbf{R}_+$ if
  -
$$(\forall x \in X)(\|x\| = 0 \Rightarrow x \equiv 0)$$
  -
$$(\forall x \in X, a \in \mathbf{R})(\|ax\| = |a|\|x\|)$$
  - subadditivity
$$(\forall x, y \in X)(\|x + y\| \leq \|x\| + \|y\|)$$

L^p spaces

- $L^p = L^p[0,1]$ denotes space of (Lebesgue) measurable functions such that

$$\int_{[0,1]} |f|^p < \infty$$

- define $\|\cdot\| : L^p \to \mathbf{R}_+$

$$\|f\| = \|f\|_p = \left(\int_{[0,1]} |f|^p\right)^{1/p}$$

- L^p are *linear normed spaces* with norm $\|\cdot\|_p$ when $p \geq 1$ because
  - $|f(x)|^p + |g(x)|^p \leq 2^p(|f(x)|^p + |g(x)|^p)$ implies $(\forall f, g \in L^p)(f + g \in L^p)$
  - $|\alpha f(x)|^p = |a|^p|f(x)|^p$ implies $(\forall f \in L^p, a \in \mathbf{R})(af \in L^p)$
  - $\|f\| = 0 \Rightarrow f = 0$ a.e.
  - $\|af\| = |a|\|f\|$
  - $\|f + g\| \geq \|f\| + \|g\|$ (Minkowski inequality)

L^∞ space

- $L^\infty = L^\infty[0,1]$ denotes space of measurable functions bounded a.e.

- L^∞ is linear normed space with norm

$$\|f\| = \|f\|_\infty = \operatorname{ess\,sup}|f| = \inf_{g:g=f \text{ a.e}} \sup_{x\in[0,1]} |g(x)|$$

  - thus

$$\|f\|_\infty = \inf\{M|\mu\{x|f(x) > M\} = 0\}$$

Inequalities in L^∞

- *Minkowski inequality* - for $p \in [1, \infty]$

$$(\forall f, g \in L^p)(\|f + g\|_p \leq \|f\|_p + \|g\|_p)$$

  - if $p \in (1, \infty)$, equality holds *if and only if* $(\exists a, b \geq 0 \text{ with } ab \neq 0)(af = bg \text{ a.e.})$

- Minkowski inequality for $0 < p < 1$:

$$(\forall f, g \in L^p)(f, g \geq 0 \text{ a.e.} \Rightarrow \|f + g\|_p \geq \|f\|_p + \|g\|_p)$$

- *Hölder's inequality* - for $p, q \in [1, \infty]$ with $1/p + 1/q = 1$

$$(\forall f \in L^p, g \in L^q)\left(fg \in L^1 \text{ and } \int_{[0,1]} |fg| \leq \int_{[0,1]} |f|^p \int_{[0,1]} |g|^q\right)$$

  - equality holds *if and only if* $(\exists a, b \geq 0 \text{ with } ab \neq 0)(a|f|^p = b|g|^q \text{ a.e.})$

(refer to page **??** for complete measure spaces counterpart)

Convergence and completeness in normed linear spaces

- $\langle f_n \rangle$ in normed linear space
  - said to *converge* to f, *i.e.*, $\lim f_n = f$ or $f_n \to f$, if

$$(\forall \epsilon > 0)(\exists N \in \mathbf{N})(\forall n > N)(\|f_n - f\| < \epsilon)$$

  - called *Cauchy sequence* if

$$(\forall \epsilon > 0)(\exists N \in \mathbf{N})(\forall n, m > N)(\|f_n - f_m\| < \epsilon)$$

  - called *summable* if $\sum_{i=1}^{n} f_i$ converges
  - called *absolutely summable* if $\sum_{i=1}^{n} |f_i|$ converges
- normed linear space called *complete* if every Cauchy sequence converges
- normed linear space is *complete if and only if* every absolutely summable series is summable

Banach space

- *complete normed linear space* called *Banach space*

- (Riesz-Fischer) L^p spaces are compact, hence Banach spaces

- convergence in L^p called *convergence in mean of order p*

- convergence in L^∞ implies nearly uniformly converges

Approximation in L^p

- $\Delta = \langle d_i \rangle_{i=0}^{n}$ with $0 = d_1 < d_2 < \cdots < d_n = 1$ called *subdivision* of $[0,1]$ (with $\Delta_i = [d_{i-1}, d_i]$)
- $\varphi_{f,\Delta}$ for $f \in L^p$ called *step function* if

$$\varphi_{f,\Delta}(x) = \frac{1}{d_i - d_{i+1}} \int_{d_{i-1}}^{d_i} f(t)dt \text{ for } x \in [d_{i-1}, d_i)$$

- for $f \in L^p$ $(1 < p \leq \infty)$, exist $\varphi_{f,\Delta}$ and continuous function, ψ such that

$$\|\varphi_{f,\Delta_i} - f\| < \epsilon \text{ and } \|\psi - f\| < \epsilon$$

  - L^p version of Littlewood's second principle (page **??**)
  (refer to page **??** for complete measure spaces counterpart)
- for $f \in L^p$, $\varphi_{f,\Delta} \to f$ as $\max \Delta_i \to 0$, *i.e.*,

$$(\forall \epsilon > 0)(\exists \delta > 0)(\max \Delta_i < \delta \Rightarrow \|\varphi_{f,\Delta} - f\|_p < \epsilon)$$

Bounded linear functionals on L^p

- $F : X \in \mathbf{R}$ for normed linear space X called *linear functional* if

$$(\forall f, g \in F, a, b \in \mathbf{R})(F(af + bg) = aF(f) + bF(g))$$

- linear functional, F, said to be *bounded* if

$$(\exists M)(\forall f \in X)(|F(f)| \leq M\|f\|)$$

- smallest such constant called *norm of F*, $i.e.$,

$$\|F\| = \sup_{f \in X, f \neq 0} |F(f)|/\|f\|$$

Riesz representation theorem

- for every $g \in L^q$ $(1 \leq p \leq \infty)$, following defines a bounded linear functional in L^p
$$F(f) = \int fg$$
where $\|F\| = \|g\|_q$

- *Riesz representation theorem* - for every bounded linear functional in L^p, F, $(1 \leq p < \infty)$, there exists $g \in L^q$ such that
$$F(f) = \int fg$$
where $\|F\| = \|g\|_q$

(refer to page **??** for complete measure spaces counterpart)

- for each case, L^q is dual of L^p (refer to page 131 for definition of dual)

Metric Spaces

Metric spaces

- $\langle X, \rho \rangle$ with nonempty set, X, and *metric* $\rho : X \times X \to \mathbf{R}_+$ called *metric space* if for every $x, y, z \in X$
  - $\rho(x, y) = 0 \Leftrightarrow x = y$
  - $\rho(x, y) = \rho(y, x)$
  - $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ (triangle inequality)
- examples of metric spaces
  - $\langle \mathbf{R}, |\cdot| \rangle$, $\langle \mathbf{R}^n, \|\cdot\|_p \rangle$ with $1 \leq p \leq \infty$
- for $f \subset X$, $S_{x,r} = \{y | \rho(y, x) < r\}$ called *ball*
- for $E \subset X$, $\sup\{\rho(x, y) | x, y \in E\}$ called diameter of E defined by
- ρ called *pseudometric* if 1st requirement removed
- ρ called *extended metric* if $\rho : X \times X \to \mathbf{R}_+ \cup \{\infty\}$

Cartesian product

- for two metric spaces $\langle X, \rho \rangle$ and $\langle Y, \sigma \rangle$, metric space $\langle X \times Y, \tau \rangle$ with $\tau : X \times Y \to$ $\mathbf{R}_+$ such that

$$\tau((x_1, y_1), (x_2, y_2)) = (\rho(x_1, x_2)^2 + \sigma(y_1, y_2)^2)^{1/2}$$

called *Cartesian product metric space*

- τ satisfies all properties required by metric
  - $e.g.$, $\mathbf{R}^n \times \mathbf{R}^m = \mathbf{R}^{n+m}$

Open sets - metric spaces

- $O \subset X$ said to be open *open* if

$$(\forall x \in O)(\exists \delta > 0)(\forall y \in X)(\rho(y, x) < \delta \Rightarrow y \in O)$$

  - X and $\emptyset$ are open
  - intersection of *finite* collection of open sets is open
  - union of *any* collection of open sets is open

Closed sets - metric spaces

- $x \in X$ called *point of closure of $E \subset X$* if

$$(\forall \epsilon > 0)(\exists y \in E)(\rho(y, x) < \epsilon)$$

  - $\overline{E}$ denotes set of points of closure of E; called *closure* of E
  - $E \subset \overline{E}$

- $F \subset X$ said to be *closed* if

$$F = \overline{F}$$

  - X and $\emptyset$ are closed
  - union of *finite* collection of closed sets is closed
  - intersection of *any* collection of closed sets is closed

- complement of closed set is open

- complement of open set is closed

Dense sets and separability - metric spaces

- $D \subset X$ said to be dense if

$$\overline{D} = X$$

- X is said to be separable if exists finite dense subset, $i.e.$,

$$(\exists D \subset X)(|D| < \infty \;\&\; \overline{D} = X)$$

- X is separable *if and only if* exists countable collection of open sets $\langle O_i \rangle$ such that for all open $O \subset X$

$$O = \bigcup_{O_i \subset O} O_i$$

Continuous functions - metric spaces

- $f : X \to Y$ for metric spaces $\langle X, \rho \rangle$ and $\langle Y, \sigma \rangle$ called *mapping* or *function* from X into Y
- f said to be *onto* if
$$f(X) = Y$$
- f said to be *continuous* at $x \in X$ if
$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall y \in X)(\rho(y, x) < \delta \Rightarrow \sigma(f(y), f(x)) < \epsilon)$$
- f said to be *continuous* if f is continuous at every $x \in X$
- f is continuous *if and only if* for every open $O \subset Y$, $f^{-1}(O)$ is open
- if $f : X \to Y$ and $g : Y \to Z$ are continuous, $g \circ f : X \to Z$ is continuous

Homeomorphism

- one-to-one mapping of X onto Y (or equivalently, one-to-one correspondece between X and Y), f, said to be *homeomorphism* if
  - both f and f^{-1} are continuous
- X and Y said to be *homeomorphic* if exists homeomorphism
- *topology* is study of properties unaltered by homeomorphisms and such properties called *topological*
- one-to-one correspondece X and Y is homeomorphism *if and only if* it maps open sets in X to open sets in Y and vice versa
- every property defined by means of *open sets* (or equivalently, *closed sets*) or/and being *continuous functions* is *topological one*
  - $e.g.$, f is continuous on X is homeomorphism, then $f \circ h^{-1}$ is continuous function on Y

Isometry

- homeomorphism preserving distance called *isometry*, $i.e.$,

$$(\forall x, y \in X)(\sigma(h(x), h(y)) = \rho(x, y))$$

- X and Y said to be *isometric* if exists isometry

- (from abstract point of view) two isometric spaces are exactly *same*; it's nothing but relabeling of points

- two metrics, ρ and σ on X, said to be *equivalent* if identity mapping of $\langle X, \rho \rangle$ onto $\langle X, \sigma \rangle$ is homeomorphism
  - hence, two metrics are equivalent *if and only if* set in one metric is open whenever open in the other metric

Convergence - metric spaces

- $\langle x_n \rangle$ defined for metric space, X
  - said to *converge* to x, *i.e.*, $\lim x_n = x$ or $x_n \to x$, if

$$(\forall \epsilon > 0)(\exists N \in \mathbf{N})(\forall n > N)(\rho(x_n, x) < \epsilon)$$

    - equivalently, every ball about x contains all but finitely many points of $\langle x_n \rangle$
  - said to have cluster point, x, if

$$(\forall \epsilon > 0, N \in \mathbf{N})(\exists n > N)(\rho(x_n, x) < \epsilon)$$

    - equivalently, every ball about x contains infinitely many points of $\langle x_n \rangle$
    - equivalently, every ball about x contains at least one point of $\langle x_n \rangle$
- every convergent point is cluster point
  - converse not true

Completeness - metric spaces

- $\langle x_n \rangle$ of metric space, X, called *Cauchy sequence* if

$$(\forall \epsilon > 0)(\exists N \in \mathbf{N})(\forall n, m > N)(\rho(x_n, x_m) < \epsilon)$$

- convergence sequence is Cauchy sequence

- X said to be *complete* if every Cauchy sequence converges
  - $e.g.$, $\langle \mathbf{R}, \rho \rangle$ with $\rho(x, y) = |x - y|$

- for incomplete $\langle X, \rho \rangle$, exists complete X^* where X is isometrically embedded in X^* as dense set

- if X contained in complete Y, X^* is isometric with $\overline{X}$ in Y

Uniform continuity - metric spaces

- $f : X \to Y$ for metric spaces $\langle X, \rho \rangle$ and $\langle Y, \sigma \rangle$ said to be *uniformly continuous* if

$$(\forall \epsilon > 0)(\exists \delta)(\forall x, y \in X)(\rho(x, y) < \delta \Rightarrow \sigma(f(x), f(y)) < \epsilon)$$

  - example of continuous, but not uniformly continuous function
  - $h : [0, 1) \to \mathbf{R}_+$ with $h(x) = x/(1 - x)$
  - h maps Cauchy sequence $\langle 1 - 1/n \rangle_{n=1}^{\infty}$ in $[0, 1)$ to $\langle n - 1 \rangle_{n=1}^{\infty}$ in $\mathbf{R}_+$, which is *not* Cauchy sequence

- homeomorphism f between $\langle X, \rho \rangle$ and $\langle Y, \sigma \rangle$ with both f and f^{-1} uniformly continuous called *uniform homeomorphism*

Uniform homeomorphism

- uniform homeomorphism f between $\langle X, \rho \rangle$ and $\langle Y, \sigma \rangle$ maps every Cauchy sequence $\langle x_n \rangle$ in X mapped to $\langle f(x_n) \rangle$ in Y which is Cauchy
  - being Cauchy sequence, hence, being complete preserved by uniform homeomorphism
  - being uniformly continuous also preserved by uniform homeomorphism

- each of three properties (being Cauchy sequence, being complete, being uniformly continuous) called *uniform property*

- uniform properties are *not* topological properties, $e.g.$, h on page 58
  - is *homeomorphism* between incomplete space $[0, 1)$ and complete space $\mathbf{R}_+$
  - maps Cauchy sequence $\langle 1 - 1/n \rangle_{n=1}^{\infty}$ in $[0, 1)$ to $\langle n - 1 \rangle_{n=1}^{\infty}$ in $\mathbf{R}_+$, which is not Cauchy sequence
  - its inverse maps uniformly continuous function $\sin$ back to non-uniformly continuity function on $[0, 1)$

Uniform equivalence

- two metrics, ρ and σ on X, said to be *uniformly equivalent* if identity mapping of $\langle X, \rho \rangle$ onto $\langle X, \sigma \rangle$ is uniform homeomorphism, $i.e.$,

$$(\forall \epsilon, \delta > 0, x, y \in X)(\rho(x, y) < \delta \Rightarrow \sigma(x, y) < \epsilon \,\&\, \sigma(x, y) < \delta \Rightarrow \rho(x, y) < \epsilon)$$

- example of uniform equivalence on $X \times Y$
  - any two of below metrics are uniformly equivalent on $X \times Y$

$$\tau((x_1, y_1), (x_2, y_2)) = (\rho(x_1, x_2)^2 + \sigma(y_1, y_2)^2)^{1/2}$$
$$\rho_1((x_1, y_1), (x_2, y_2)) = \rho(x_1, x_2) + \sigma(y_1, y_2)$$
$$\rho_\infty((x_1, y_1), (x_2, y_2)) = \max\{\rho(x_1, x_2), \sigma(y_1, y_2)\}$$

- for $\langle X, \rho \rangle$ and complete $\langle Y, \sigma \rangle$ and $f : X \to Y$ uniformly continuous on $E \subset X$ into Y, exists unique continuous extension g of f on $\overline{E}$, which is uniformly continuous

Subspaces

- for metric space, $\langle X, \rho \rangle$, metric space $\langle S, \rho_S \rangle$ with $S \subset X$ and ρ_S being restriction of ρ to S, called *subspace* of $\langle X, \rho \rangle$
  - $e.g.$ (with standard Euclidean distance)
    - **Q** is subspace of **R**
    - $\left\{ (x, y) \in \mathbf{R}^2 \middle| \, y = 0 \right\}$ is subspace of $\mathbf{R}^2$, which is isometric to **R**
- for metric space, X, and its subspace, S,
  - $\overline{E} \subset S$ is closure of E relative to S.
  - $A \subset S$ is closure relative to S *if and only if* $(\exists \overline{F} \subset A)(A = \overline{F} \cap S)$
  - $A \subset O$ is open relative to S *if and only if* $(\exists$ open $O \subset A)(A = O \cap S)$
- also
  - every subspace of separable metric space is separable
  - every complete subset of metric space is closed
  - every closed subset of complete metric space is complete

Compact metric spaces

- motivation - want metric spaces where
  - conclusion of Heine-Borel theorem (page 30) are valid
  - many properties of $[0, 1]$ are true, $e.g.$, Bolzano-Weierstrass property (page 64)

- $e.g.$,
  - bounded closed set in **R** has *finite open covering property*

- metric space X called *compact metric space* if every open covering of X, $\mathcal{U}$, contains finite open covering of X, $e.g.$,

$$(\forall \text{ open covering of } X, \mathcal{U})(\exists \{O_1, \ldots, O_n\} \subset \mathcal{U})(X \in \cup O_i)$$

- $A \subset X$ called *compact* if compact as subspace of X
  - $i.e.$, every open covering of A contains finite open covering of A

Compact metric spaces - alternative definition

- collection, $\mathcal{F}$, of sets in X said to have *finite intersection property* if every finite subcollection of $\mathcal{F}$ has nonempty intersection

- if rephrase definition of compact metric spaces in terms of *closed* instead of *open*
  - X is called *compact metric space* if every collection of closed sets with empty intersection contains finite subcollection with empty intersection

- thus, X is compact *if and only if* every collection of closed sets with *finite intersection property* has nonempty intersection

Bolzano-Weierstrass property and sequential compactness

- metric space said to
  - have *Bolzano-Weierstrass property* if every sequence has cluster point
  - X said to be *sequentially compact* if every sequence has convergent subsequence

- *X has Bolzano-Weierstrass property* if and only if *sequentially compact* (proof can be found in Proof 1)

Compact metric spaces - properties

- following three statements about metric space are equivalent *(not true for general topological sets)*
  - being compact
  - having Bolzano-Weierstrass property
  - being sequentially compact

- compact metric spaces have corresponding to some of those of complete metric spaces (compare with statements on page 61)
  - every compact subset of metric space is closed *and bounded*
  - every closed subset of compact metric space is compact

- (will show above in following slides)

Necessary condition for compactness

- compact metric space is sequentially compact (proof can be found in Proof 2)

- equivalently, compact metric space has Bolzano-Weierstrass property (page 64)

Necessary conditions for sequentially compactness

- every continuity real-valued function on sequentially compact space is *bounded and assumes its maximum and minimum*

- sequentially compact space is *totally bounded*

- every open covering of sequentially compact space has *Lebesgue number*

Sufficient conditions for compactness

- metric space that is totally bounded and has Lebesgue number for every covering is compact

Borel-Lebesgue theorem

- conditions on pages 66, 67, and 68 imply the following equivalent statements
  - X is *compact*
  - X has *Bolzano-Weierstrass property*
  - X is *sequentially compact*

- above called *Borel-Lebesgue theorem*

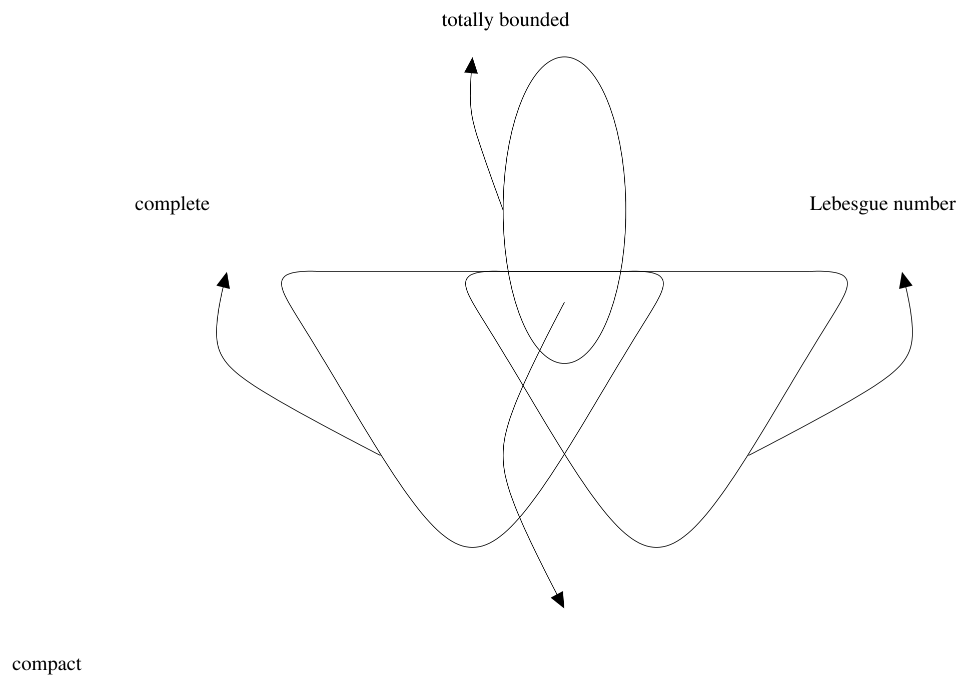
- hence, can drop *sequentially* in every statement on page 67, *i.e.*,
  - every continuity real-valued function on ~~sequentially~~ compact space is *bounded and assumes its maximum and minimum*
  - ~~sequentially~~ compact space is *totally bounded*
  - every open covering of ~~sequentially~~ compact space has *Lebesgue number*

Compact metric spaces - other facts

- closed subset of compact space is compact
- compact subset of metric space is *closed and bounded*
  - hence, Heine-Borel theorem (page 30) implies
    set of $\mathbf{R}$ *is compact* if and only if *closed and bounded*
- metric space is compact *if and only if* it is complete and totally bounded
- thus, *compactness can be viewed as absolute type of closedness*
  - refer to page 105 for exactly same comments for general topological spaces
- continuous image of compact set is compact
- continuous mapping of compact metric space into metric space is uniformly continuous

Diagrams for relations among metric spaces

- the figure shows relations among metric spaces stated on pages 67, 68, 69, and 70



Baire category

- do (more) deeply into certain aspects of complete metric spaces, namely, *Baire theory of category*

- subset E in metric space where $\sim(\overline{E})$ is dense, said to be *nowhere dense*
  - equivalently, $\overline{E}$ contains no nonempty open set

- union of countable collection of *nowhere open sets*, said to be *of first category or meager*

- set not of first category, said to be *of second category or nonmeager*

- complement of set of first category, called *residual or co-meager*

Baire category theorem

- *Baire theorem* - for complete metric space, X, and countable collection of dense open subsets, $\langle O_k \rangle \subset X$, the intersection of the collection

$$\bigcap O_k$$

  is dense
  - refer to page 116 for locally compact space version of Baire theorem

- *Baire category theorem* - no nonempty open subset of complete metric space is of first category, *i.e.*, union of countable collection of nowhere dense subsets

- Baire category theorem is *unusual* in that *uniform property, i.e., completeness of metric spaces, implies purely topological nature*[1]

[1] "no nonempty open subset of complete metric space is of first category" is purely topological nature because if two spaces are (topologically) homeomorphic, and no nonempty open subsets of one space is of first category, then neither is any nonempty open subset of the other space

Second category everywhere

- metric or topological spaces with property that "no nonempty open subset of complete metric space is of first category", said to be *of second category everywhere* (with respect to themselves)

- Baire category theorem says *complete metric space* is of second category everywhere

- locally compact Hausdorff spaces are of second category everywhere, too (refer to page 113 for definition of locally compact Hausdorff spaces)
  - for these spaces, though, many of results of category theory follow directly from *local compactness*

Sets of first category

- collection of sets with following properties, called *a σ-ideal of sets*
  - countable union of sets in the collection is, again, in the collection
  - subset of any in the collection is, again, in the collection

- both of below collections are σ-ideal of sets
  - sets of first category in topological space
  - measure zero sets in complete measure space

- sets of first category regards as "small" sets
  - such sets in complete metric spaces no interior points

- interestingly! set of first category in $[0,1]$ can have Lebesgue measure 1, hence complement of which is residual set of measure zero

Some facts of category theory

- for open set, O, and closed set, F, $\overline{O} \sim O$ and $F \sim F^\circ$ are nowhere dense
- closed set of first category in complete metric space is nowhere dense
- subset of complete metric space is residual *if and only if* contains dense G_δ, hence subset of complete metric space is of first category *if and only if* contained in F_σ whose complement is dense
- for countable collection of closed sets, $\langle F_n \rangle$, $\bigcup {F_n}^\circ$ is residual open set; if $\bigcup F_n$ is complete metric space, O is dense
- some applications of category theory to analysis seem almost too good to be belived; here's one:
- *uniform boundedness principle* - for family, $\mathcal{F}$, of real-valued continuous functions on complete metric space, X, with property that $(\forall x \in X)(\exists M_x \in \mathbf{R})(\forall f \in \mathcal{F})(|f(x)| \leq M_x)$

$$(\exists \text{ open } O, M \in \mathbf{R})(\forall x \in O, f \in \mathcal{F})(|f(x)| \leq M)$$

Topological Spaces

Motivation for topological spaces

- want to have something like
  - notion of open set is fundamental
  - other notions defined in terms of open sets
  - more general than metric spaces

- why not stick to metric spaces?
  - certain notions have natural meaning *not* consistent with topological concepts derived from metric spaces
    - $e.g.$. weak topologies in Banach spaces

Topological spaces

- $\langle X, \mathfrak{J} \rangle$ with nonempty set X of points and family $\mathfrak{J}$ of subsets, which we call open, having the following properties called *topological spaces*
  - $\emptyset, X \in \mathfrak{J}$
  - $O_1, O_2 \in \mathfrak{J} \Rightarrow O_1 \cap O_2 \in \mathfrak{J}$
  - $O_\alpha \Rightarrow \cup_\alpha O_\alpha \in \mathfrak{J}$

- family, $\mathfrak{J}$, is called *topology*

- for X, *always exist two topologies* defined on X
  - *trivial topology* having only $\emptyset$ and X
  - *discrete topology* for which every subset of X is an open set

Topological spaces associated with metric spaces

- can associate topological space, $\langle X, \mathfrak{J}\rangle$, to any metric space $\langle X, \rho\rangle$ where $\mathfrak{J}$ is family of open sets in $\langle X, \rho\rangle$
  - $\because$ because properties in definition of topological space satisfied by open sets in metric space

- $\langle X, \mathfrak{J}\rangle$ assiaciated with metric space, $\langle X, \rho\rangle$ said to be *metrizable*
  - ρ called *metric for* $\langle X, \mathfrak{J}\rangle$

- distinction between metric space and associated topological space is *essential*
  - $\because$ because different metric spaces associate same topological space
  - in this case, these metric spaces are equivalent

- metric and topological spaces are couples

Some definitions for topological spaces

- subset $F \subset X$ with $\tilde{F}$ is open called *closed*

- intersection of all closed sets containing $E \subset X$ called *closure* of E denoted by $\overline{E}$
  - $\overline{E}$ is smallest closed set containing E

- $x \in X$ called *point of closure* of $E \subset X$ if every open set containing x meets E, $i.e.$, has nonempty intersection with E

- union of all open sets contained in $E \subset X$ is called *interior* of E denoted by E°

- $x \in X$ called *interior point* of E if exists open set, E, with $x \in O \subset E$

Some properties of topological spaces

- $\emptyset$, X are closed
- union of closed sets is closed
- intersection of any collection of closed sets is closed
- $E \subset \overline{E}$, $\overline{\overline{E}} = \overline{E}$, $\overline{A \cup B} = \overline{A} \cup \overline{B}$
- F closed *if and only if* $\overline{F} = F$
- $\overline{E}$ is set of *points of closure* of E
- $E^\circ \subset E$, $(E^\circ)^\circ = E^\circ$, $(A \cup B)^\circ = A^\circ \cup B^\circ$
- E° is set of *interior points* of E
- $(\tilde{E})^\circ =\sim \overline{E}$

Subspace and convergence of topological spaces

- for subset of $\langle X, \mathfrak{J}\rangle$, A, define *topology $\mathfrak{S}$ for A* with $\mathfrak{S} = \{A \cap O | O \in \mathfrak{J}\}$
  - $\mathfrak{S}$ called *topology inherited from $\mathfrak{J}$*
  - $\langle A, \mathfrak{S}\rangle$ called *subspace* of $\langle X, \mathfrak{J}\rangle$
- $\langle x_n\rangle$ said to *converge* to $x \in X$ if

$$(\forall O \in \mathfrak{J} \text{ containing } x)(\exists N \in \mathbf{N})(\forall n > N)(x_n \in O)$$

  - denoted by

$$\lim x_n = x$$

- $\langle x_n\rangle$ said to have $x \in X$ as *cluster point* if

$$(\forall O \in \mathfrak{J} \text{ containing } x, N \in \mathbf{N})(\exists n > N)(x_n \in O)$$

- $\langle x_n\rangle$ has converging subsequence to $x \in X$, then x is cluster point of $\langle x_n\rangle$
  - converse is *not* true for arbitrary topological space

Continuity in topological spaces

- mapping $f : X \to Y$ with $\langle X, \mathfrak{J} \rangle$, $\langle Y, \mathfrak{S} \rangle$ said to be *continuous* if

$$(\forall O \in \mathfrak{S})(f^{-1}(O) \in \mathfrak{J})$$

- $f : X \to Y$ said to be *continuous at* $x \in X$ if

$$(\forall O \in \mathfrak{S} \text{ containing } f(x))(\exists U \in \mathfrak{J} \text{ containing } x)(f(U) \subset O)$$

- f is continuous *if and only if* f is continuous at every $x \in X$
- for continuous f on $\langle X, \mathfrak{J} \rangle$, restriction g on $A \subset X$ is continuous (proof can be found in Proof 3)
- for A with $A = A_1 \cup A_2$ where both A_1 and A_2 are either open or closed, $f : A \to Y$ with each of both restrictions, $f|A_1$ and $f|A_2$, continuous, is continuous

Homeomorphism for topological spaces

- one-to-one continuous function of X onto Y, f, with continuous inverse function, f^{-1}, called *homeomorphism* between $\langle X, \mathfrak{J} \rangle$ and $\langle Y, \mathfrak{S} \rangle$

- $\langle X, \mathfrak{J} \rangle$ and $\langle Y, \mathfrak{S} \rangle$ said to be *homeomorphic* if exists homeomorphism between them

- homeomorphic spaces are indistinguishable where homeomorphism amounting to relabeling of points (from abstract pointp of view)

- thus, below roles are same
  - role that *homeomorphism plays for topological spaces*
  - role that *isometry plays for metric spaces*
  - role that *isomorphism plays for algebraic systems*

Stronger and weaker topologies

- for two topologies, $\mathfrak{J}$ and $\mathfrak{S}$ for same X with $\mathfrak{S} \supset \mathfrak{J}$
  - $\mathfrak{S}$ said to be *stronger or finer* than $\mathfrak{J}$
  - $\mathfrak{J}$ said to be *weaker or coarser* than $\mathfrak{S}$

- $\mathfrak{S}$ is stronger than $\mathfrak{J}$ *if and only if* identity mapping of $\langle X, \mathfrak{S} \rangle$ to $\langle Y, \mathfrak{J} \rangle$ is continuous

- for two topologies, $\mathfrak{J}$ and $\mathfrak{S}$ for same X, $\mathfrak{J} \cap \mathfrak{S}$ also topology

- for any collection of topologies, $\{\mathfrak{J}_\alpha\}$ for same X, $\cap_\alpha \mathfrak{J}_\alpha$ is topology

- for nonempty set, X, and any collection of subsets of X, $\mathcal{C}$
  - *exists weakest topology containing $\mathcal{C}$*, $i.e.$, weakest topology where all subsets in $\mathcal{C}$ are open
  - it is intersection of all topologies containing $\mathcal{C}$

Bases for topological spaces

- collection $\mathcal{B}$ of open sets of $\langle X, \mathfrak{J} \rangle$ called *a base for topology,* $\mathfrak{J}$, of X if

$$(\forall O \in \mathfrak{J}, x \in O)(\exists B \in \mathcal{B})(x \in B \subset O)$$

- collection $\mathcal{B}_x$ of open sets of $\langle X, \mathfrak{J} \rangle$ containing x called *a base at* x if

$$(\forall O \in \mathfrak{J} \text{ containing } x)(\exists B \in \mathcal{B}_x)(x \in B \subset O)$$

  - elements of $\mathcal{B}_x$ often called *neighborhoods of* x
  - when no base given, *neighborhood of* x is an open set containing x

- thus, $\mathcal{B}$ of open sets is a base *if and only if* contains a base for every $x \in X$

- for topological space that is also metric space
  - all balls from a base
  - balls centered at x form a base at x

Characterization of topological spaces in terms of bases

- *definition of open sets in terms of base* - when $\mathcal{B}$ is base of $\langle X, \mathfrak{J} \rangle$

$$(O \in \mathfrak{J}) \Leftrightarrow (\forall x \in O)(\exists B \in \mathcal{B})(x \in B \subset O)$$

- often, convenient to specify topology for X by
  – specifying a base of open sets, $\mathcal{B}$, and
  – using above criterion to define open sets

- collection of subsets of X, $\mathcal{B}$, is base for some topology *if and only if*

$$(\forall x \in X)(\exists B \in \mathcal{B})(x \in B)$$

and

$$(\forall x \in X, B_1, B_2 \in \mathcal{B} \text{ with } x \in B_1 \cap B_2)(\exists B_3 \in \mathcal{B})(x \in B_3 \subset B_1 \cap B_2)$$

  – *condition of collection to be basis for some topology*

Subbases for topological spaces

- for $\langle X, \mathfrak{J} \rangle$, collection of open sets, $\mathcal{C}$ called *a subbase* for topology $\mathfrak{J}$ if

$$(\forall O \in \mathfrak{J}, x \in O)(\exists \langle C_i \rangle_{i=1}^n \subset \mathcal{C})(x \in \cap C_i \subset O)$$

  - sometimes convenient to define topology in terms of subbase

- for subbase for $\mathfrak{J}$, $\mathcal{C}$, collection of finite intersections of sets from $\mathcal{C}$ forms base for $\mathfrak{J}$

- any collection of subsets of X is subbase for weakest topology where sets of the collection are open

Axioms of countability

- topological space said to satisfy *first axiom of countability* if exists countable base at every point
  - every metric space satisfies first axiom of countability because for every $x \in X$, set of balls centered at x with rational radii forms base for x

- topological space said to satisfy *second axiom of countability* if exists countable base for the space
  - every metric space satisfies second axiom of countability *if and only if* separable (refer to page 52 for definition of separability)

Topological spaces - facts

- given base, $\mathcal{B}$, for $\langle X, \mathfrak{J} \rangle$
  - $x \in \overline{E}$ *if and only if* $(\exists B \in \mathcal{B})(x \in B \;\&\; B \cap E \neq \emptyset)$
- given base at x for $\langle X, \mathfrak{J} \rangle$, $\mathcal{B}_x$, and base at y for $\langle Y, \mathfrak{S} \rangle$, $\mathfrak{C}_y$
  - $f : X \to Y$ continuous at x *if and only if* $(\forall C \in \mathfrak{C}_y)(\exists B \in \mathcal{B}_x)(B \subset f^{-1}(C))$
- if $\langle X, \mathfrak{J} \rangle$ satisfies *first axiom of countability*
  - $x \in \overline{E}$ *if and only if* $(\exists \langle x_n \rangle$ from $E)(\lim x_n = x)$
  - x cluster point of $\langle x_n \rangle$ *if and only if* exists its subsequence converging to x
- $\langle X, \mathfrak{J} \rangle$ said to be *Lindelöf space* or have *Lindelöf property* if every open covering of X has countable subcover
- second axiom of countability implies *Lindelöf property*

Separation axioms

- why separation axioms
  - properties of topological spaces are (in general) quite different from those of metric spaces
  - often convenient assume additional conditions true in metric spaces
- separation axioms
  - *T_1 - Tychonoff spaces*
    - $(\forall x \neq y \in X)(\exists \text{ open } O \subset X)(y \in O, x \not\in O)$
  - *T_2 - Hausdorff spaces*
    - $(\forall x \neq y \in X)(\exists \text{ open } O_1, O_2 \subset X \text{ with } O_1 \cap O_2 = \emptyset)(x \in O_1, y \in O_2)$
  - *T_3 - regular spaces*
    - T_1 & $(\forall \text{ closed } F \subset X, x \not\in F)(\exists \text{ open } O_1, O_2 \subset X \text{ with } O_1 \cap O_2 = \emptyset)(x \in O_1, F \subset O_2)$
  - *T_4 - normal spaces*
    - T_1 & $(\forall \text{ closed } F_1, F_2 \subset X)(\exists \text{ open } O_1, O_2 \subset X \text{ with } O_1 \cap O_2 = \emptyset)(F_1 \subset O_1, F_2 \subset O_2)$

Separation axioms - facts

- necessary and sufficient condition for T_1
  - topological space satisfies T_1 *if and only if* every singletone, $\{x\}$, is closed
- important consequences of normality, T_4
  - *Urysohn's lemma -* for normal topological space, X

$$(\forall \text{ disjoint closed } A, B \subset X)(\exists f \in C(X,[0,1]))(f(A) = \{0\}, f(B) = \{1\})$$

  - *Tietze's extension theorem -* for normal topological space, X

$$(\forall \text{ closed } A \subset X, f \in C(A, \mathbf{R}))(\exists g \in C(X, \mathbf{R}))(\forall x \in A)(g(x) = f(x))$$

  - *Urysohn metrization theorem - normal* topological space satisfying *second axiom of countability* is *metrizable*

Weak topology generated by functions

- given any set of points, X & any collection of functions of X into **R**, $\mathcal{F}$, exists weakest totally on X such that all functions in $\mathcal{F}$ is continuous
  - it is weakest topology containing - refer to page 86

$$\mathcal{C} = \bigcup_{f\in\mathcal{F}} \bigcup_{O\subset\mathbf{R}} f^{-1}(O)$$

  - called *weak topology generated by* $\mathcal{F}$

Complete regularity

- for $\langle X, \mathfrak{J} \rangle$ and continuous function collection $\mathcal{F}$, *weak topology* generated by $\mathcal{F}$ is weaker than $\mathfrak{J}$
  - however, if

$$(\forall \text{ closed } F \subset X, x \not\in F)(\exists f \in \mathcal{F})(f(A) = \{0\}, f(x) = 1)$$

    then, *weak topology generated by* $\mathcal{F}$ coincides with $\mathfrak{J}$
  - if condition satisfied by $\mathcal{F} = C(X, \mathbf{R})$, X said to be *completely regular* provided X satisfied T_1 (Tychonoff space)

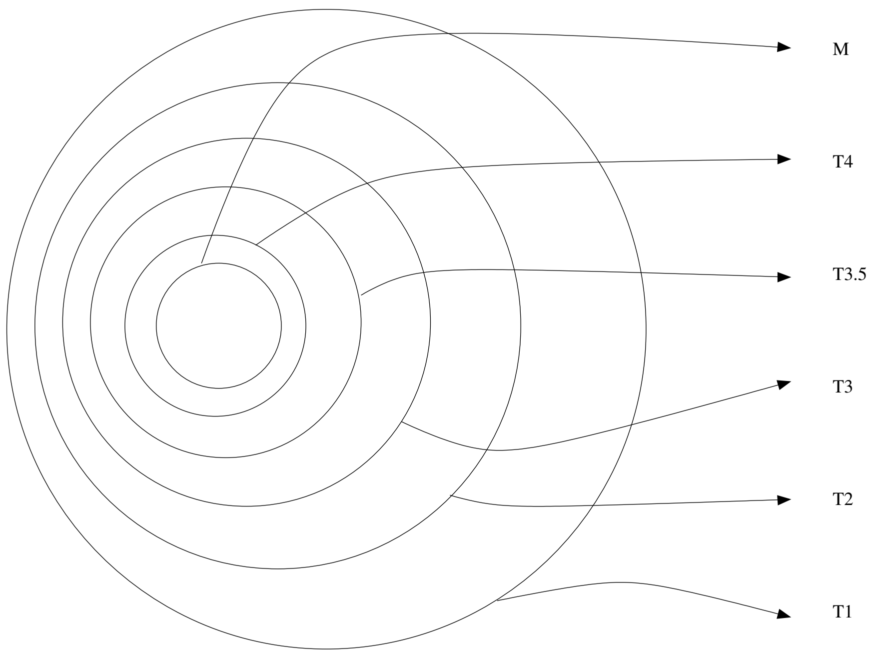
- every normal topological (T_4) space is completely regular (Urysohn's lemma)

- every completely regular space is regular space (T_3)

- complete regularity sometimes called $T_{3\frac{1}{2}}$

Diagrams for separation axioms for topological spaces

- the figure shows $T_4 \Rightarrow T_{3\frac{1}{2}} \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1$
- every metric spaces is normal space



Topological spaces of interest

- very general topological spaces quite bizarre
  - do *not* seem to be much needed in analysis
- only topological spaces (Royden) found useful for analysis are
  - metrizable topological spaces
  - locally compact Hausdorff spaces
  - topological vector spaces
- all above are *completely regular*
- algebraic geometry, however, uses Zariski topology on affine or projective space, topology giving us compact T_1 space which is not Hausdorff

Connectedness

- topological space, X ,said to be *connected* if *not* exist two nonempty disjoint open sets, O_1 and O_2, such that $O_1 \cup O_2 = X$
  - such pair, (O_1, O_2), if exist, called *separation of* X
  - pair of disjoint nonempty closed sets, (F_1, F_2), with $F_1 \cup F_2 = X$ is also *separation of* X - because they are also open
- X is connected *if and only if* only subsets that are both closed and open are $\emptyset$ and X
- subset $E \subset X$ said to be *connected* if connected in topology inherited from $\langle X, \mathfrak{J} \rangle$
  - thus, E is connected if not exist two nonempty open sets, O_1 and O_2, such that $E \subset O_1 \cup O_2$ and $E \cap O_1 \cap O_2 = \emptyset$

Properties of connected space, component, and local connectedness

- if exists continuous mapping of connected space to topological space, Y, Y is connected
- *(generalized version of) intermediate value theorem* - for $f : X \to \mathbf{R}$ where X is connected

$$(\forall x, y \in X, c \in \mathbf{R} \text{ with } f(x) < c < f(y))(\exists z \in X)(z = f(z))$$

- subset of **R** is connected *if and only if* is either interval or singletone
- for $x \in X$, union of all connected sets containing x is called *component*
  - component is *connected and closed*
  - two components containing same point coincide
  - thus, *X is disjoint union of components*
- X said to be *locally connected* if exists base for X consisting of connected sets
  - components of locally connected space are *open*
  - space *can be connected, but not locally connected*

Product topological spaces

- for $\langle X, \mathfrak{J} \rangle$ and $\langle Y, \mathfrak{S} \rangle$, topology on $X \times Y$ taking as *a base* the following

$$\{O_1 \times O_2 | O_1 \in \mathfrak{J}, O_2 \in \mathfrak{S}\}$$

  called *product topology* for $X \times Y$
  - for metric spaces, X and Y, *product topology is product metric*
- for indexed family with index set, $\mathcal{A}$, $\langle X_\alpha, \mathfrak{J}_\alpha \rangle$, product topology on $\bigtimes_{\alpha \in \mathcal{A}} X_\alpha$ defined as taking as *a base* the following

$$\left\{ \bigtimes X_\alpha \middle| O_\alpha \in \mathfrak{J}_\alpha, O_\alpha = X_\alpha \text{ except finite number of } \alpha \right\}$$

- $\pi_\alpha : \bigtimes X_\alpha \to X_\alpha$ with $\pi_\alpha(y) = x_\alpha$, *i.e.*, α-th coordinate, called *projection*
  - every π_α continuous
  - $\bigtimes X_\alpha$ *weakest topology* with continuous π_α's
- if $(\forall \alpha \in \mathcal{A})(X_\alpha = X)$, $\bigtimes X_\alpha$ denoted by $X^{\mathcal{A}}$

Product topology with countable index set

- for countable $\mathcal{A}$
  - $\bigtimes X_\alpha$ *denoted by* X^ω or $X^{\mathbf{N}}$ $\because$ only $\#$ elements of $\mathcal{A}$ important
    - $e.g.$, $\mathbf{2}^\omega$ is *Cantor set* if denoting discrete topology with two elements by $\mathbf{2}$

- if X is metrizable, X^ω is metrizable

- $\mathbf{N}^\omega = \mathbf{N}^{\mathbf{N}}$ *is topology space homeomorphic to* $\mathbf{R} \sim \mathbf{Q}$ when denoting discrete topology with countable set also by $\mathbf{N}$

Product topologies induced by set and continuous functions

- for $I = [0,1]$, $I^{\mathcal{A}}$ called *cube*
- I^ω is metrizable, and called *Hilbert cube*
- for any set X and any collection of $f : X \to [0,1]$, $\mathcal{F}$ with $(\forall x \neq y \in X)(\exists f \in \mathcal{F})(f(x) \neq f(y))$
  - can define *one-to-one mapping of $\mathcal{F}$ into I^X* with $f(x)$ as x-th coordinate of f
    - $\pi_x : \mathcal{F} \to I$ (mapping of $\mathcal{F}$ into I) with $\pi_x(f) = f(x)$
    - topology that $\mathcal{F}$ inherits as subspace of I^X called *topology of pointwise convergence* (because π_x is project, hence continuous)
  - can define *one-to-one mapping of X into $I^{\mathcal{F}}$* with $f(x)$ as f-th coordinate of x
    - topology of X as subspace of $I^{\mathcal{F}}$ is *weak topology generated by $\mathcal{F}$*
    - if every $f \in \mathcal{F}$ is continuous,
      - topology of X into $I^{\mathcal{F}}$ is continuous
      - if for every closed $F \subset X$ and for each $x \not\in F$, exists $f \in \mathcal{F}$ such that $f(x) = 1$ and $f(F) = \{0\}$, then X *is homeomorphic to image of $I^{\mathcal{F}}$*

Compact and Locally Compact Spaces

Compact spaces

- compactness for metric spaces (page 62) can be generalized to topological spaces
  - things are very much similar to those of metrics spaces
- for subset $K \subset X$, collection of open sets, $\mathcal{U}$, the union of which K is contained in called *open covering* of K
- topological space, X, said to be *compact* if every open convering of contains finite subcovering
- $K \subset X$ said to be *compact* if compact as subspace of X
  - or equivalently, K is compact if every covering of K *by open sets of* X has finite subcovering
  - thus, Heine-Borel (page 30) says every closed and bounded subset of $\mathbf{R}$ is compact
- for $\mathcal{F} \subset \mathcal{P}(X)$ any finite subcollection of which has nonempty intersection called *finite intersection property*
- thus, topological space compact *if and only if* every collection with *finite intersection property* has nonempty intersection

Compact spaces - facts

- *compactness can be viewed as absolute type of closedness* because
  - closed subset of compact space is compact
  - compact subset of Hausdorff space is closed

- refer to page 70 for exactly the same comments for metric spaces

- thus, every compact set of $\mathbf{R}$ is closed and bounded

- continuous image of compact set is compact

- one-to-one continuous mapping of compact space into Hausdorff space is homeomorphism

Refinement of open covering

- for open covering of X, $\mathcal{U}$, open covering of X every element of which is subset of element of $\mathcal{U}$, called *refinement* of $\mathcal{U}$ or said to *refine* $\mathcal{U}$

- X is cmopact *if and only if* every open covering has finite refinement

- any two open covers, $\mathcal{U}$ and $\mathcal{V}$, have common refinement, *i.e.*,

$$\{U \cap V | U \in \mathcal{U}, V \in \mathcal{V}\}$$

Countable compactness and Lindelöf

- topological space for which every open covering has countable subcovering said to be *Lindelöf*
- topological space for which every countable open covering has finite subcovering said to be *countably compact* space
- thus, topological space is compact *if and only if* both Lindelöf and countably compact
- every second countable space is Lindelöf
- thus, countable compactness coincides with compactness if second countable ($i.e.$, satisfying second axiom of countability)
- continuous image of compact countably compact space is countably compact

Bolzano-Weierstrass property and sequential compactness

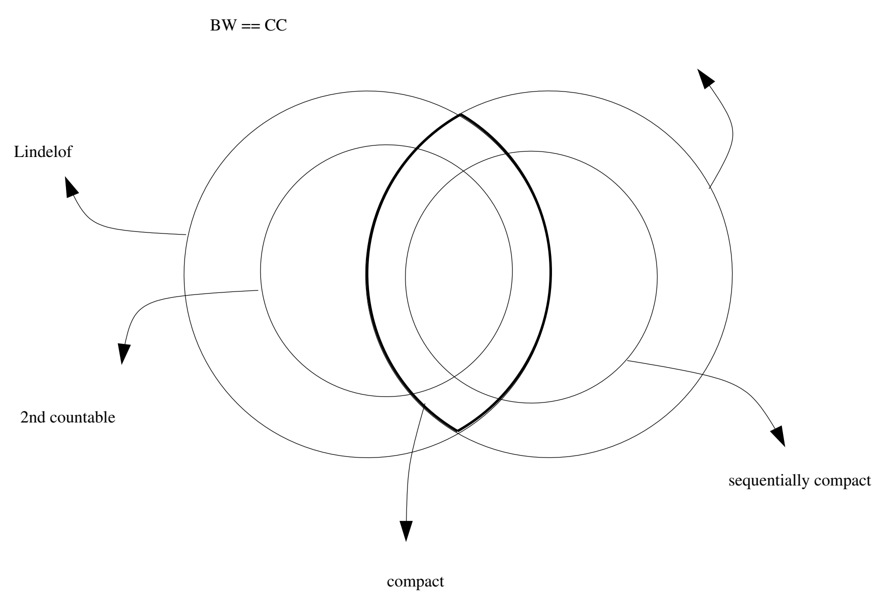
- topological space, X, said to have *Bolzano-Weierstrass property* if every sequence, $\langle x_n \rangle$, in X has at least one cluster point, $i.e.$,

  $$(\forall \langle x_n \rangle)(\exists x \in X)(\forall \epsilon > 0, N \in \mathbf{N})(\exists n > N, O \subset X)(x \in O, O \text{ is open}, x_n \in O)$$

- topological space has *Bolzano-Weierstrass properties if and only if* countably compact
- topological space said to be *sequentially compact* if every sequence has converging subsequence
- sequentially compact space is countably compact
- thus, Lindelöf coincides with compactness if sequentially compact
- countably compact and first countable ($i.e.$, satisfying first axiom of countability) space is sequentially compact

Diagrams for relations among topological spaces

- the figure shows relations among topological spaces stated on pages 107 and 108



Real-valued functions on topological spaces

- continuous real-valued function on countably compact space is bounded and assumes maximum and minimum

- $f : X \to \mathbf{R}$ with topological space, X, called *upper semicontinuous* if $\{x \in X | f(x) < \alpha\}$ is open for every $\alpha \in \mathbf{R}$

- stronger statement - upper semicontinuous real-valued function on countably compact space is bounded (from above) and assumes maximum

- *Dini* - for sequence of upper semicontinuous real-valued functions on countably compact space, $\langle f_n \rangle$, with property that $\langle f_n(x) \rangle$ decreases monotonically to zero for every $x \in X$, $\langle f_n \rangle$ converges to zero uniformly

Products of compact spaces

- *Tychonoff theorem - (probably) most important theorem in general topology*
- most applications in analysis need only special case of product of (closed) intervals, but this special case does not seem to be easire to prove than general case, $i.e.$, Tychonoff theorem
- lemmas needed to prove Tychonoff theorem
  - for collection of subsets of X with finite intersection property, $\mathcal{A}$, exists collection $\mathcal{B} \supset \mathcal{A}$ with finite intersection property that is maximal with respect to this property, $i.e.$, no collection with finite intersection property properly contains $\mathcal{B}$
  - for collection, $\mathcal{B}$, of subsets of X that is maximal with respect to finite intersection property, each intersection of finite number of sets in $\mathcal{B}$ is again in $\mathcal{B}$ and each set that meets each set in $\mathcal{B}$ is itself in $\mathcal{B}$
- *Tychonoff theorem -* product space $\bigtimes X_\alpha$ is compact for indexed family of compact topological spaces, $\langle X_\alpha \rangle$

Locally compact spaces

- topological space, X, with

$$(\forall x \in X)(\exists \text{ open } O \subset X)(x \in O, \overline{O} \text{ is compact})$$

  called *locally compact*

- topological space is locally compact *if and only if* set of all open sets with compact closures forms base for the topological space

- every compact space is locally compact
  - but converse it *not* true
    - $e.g.$, Euclidean spaces $\mathbf{R}^n$ are locally compact, but not compact

Locally compact Hausdorff spaces

- *locally compact Hausdorff spaces* constitute one of most important classes of topological spaces

- so useful is combination of Hausdorff separation axioms in connection with compactness that French usage (following Bourbaki) reserves term 'compact space' for those compact and Hausdorff, using term 'pseudocompact' for those not Hausdorff!

- following slides devote to establishing some of their basic properties

Support and subordinateness

- for function, f, on topological spaces, closure of $\{x|f(x) \neq 0\}$, called *support* of f, $i.e.$,

$$\mathbf{support}\, f = \overline{\{x|f(x) \neq 0\}}$$

- given covering $\{O_\lambda\}$ of X, collection $\{\varphi_\alpha\}$ with $\varphi_\alpha : X \to \mathbf{R}$ satisfying

$$(\forall \varphi_\alpha)(\exists O_\lambda)(\mathbf{support}\, \varphi_\alpha \subset O_\lambda)$$

said to be *subordinate to* $\{O_\lambda\}$

Some properties of locally compact Hausdorff spaces

- for compact subset, K, of locally compact Hausdorff space, X
  - exists open subset with compact closure, $O \subset X$, containing K
  - exists continuous nonnegative function, f, on X, with

$$(\forall x \in K)(f(x) = 1) \text{ and } (\forall x \not\in O)(f(x) = 0)$$

  if K is G_δ, may take $f < 1$ in $\tilde{K}$

- for open covering, $\{O_\lambda\}$, for compact subset, K, of locally compact Hausdorff space, exists $\langle \varphi_i \rangle_{i=1}^n \subset C(X, \mathbf{R}_+)$ subordinate to $\{O_\lambda\}$ such that

$$(\forall x \in K)(\varphi_1(x) + \cdots + \varphi_n(x) = 1)$$

Local compactness and second Baire category

- for locally compact space, X, and countable collection of dense open subsets, $\langle O_k \rangle \subset X$, the intersection of the collection

$$\bigcap O_k$$

  is dense
  - analogue of Baire theorem for complete metric spaces (refer to page 73 for Baire theorem)

- thus, *every locally compact space is locally of second Baire category with respect to itself*

Local compactness, Hausdorffness, and denseness

- for countable union, $\bigcup F_n$, of closed sets containing open subset, O, in locally compact space, union of interiors, $\bigcup {F_n}^\circ$, is open set dense in O

- dense subset of Hausdorff space, X, which is locally compact in its subspace topology, is open subset of X

- subset, Y, of locally compact Hausdorff space is locally compact in its subspace topology *if and only if* Y is relatively open subset of $\overline{Y}$

Alexandroff one-point compactification

- for locally compact Hausdorff space, X, can form X^* by adding single point $\omega \not\in X$ to X and take set in X^* to be open if it is either open in X or complement of compact subset in X, then
  - X^* is compact Hausdorff spaces
  - identity mapping of X into X^* is homeomorphism of X and $X^* \sim \{\omega\}$
  - X^* called *Alexandroff one-point compactification of X*
  - ω often referred to as *infinity in X^**

- continuous mapping, f, from topological space to topological space inversely mapping compact set to compact set, said to be *proper*

- proper maps from locally compact Hausdorff space into locally compact Hausdorff space are precisely those continuous maps of X into Y tha can be extended to continuous maps f^* of X^* into Y^* by taking point at infinity in X^* to point at infinity in Y^*

Manifolds

- connected Hausdorff space with each point having neighborhood homeomorphic to ball in $\mathbf{R}^n$ called n-dimensional *manifold*

- sometimes say manifold is connected Hausdorff space that is *locally Euclidean*

- thus, manifold has all local properties of Euclidean space; particularly *locally compact and locally connected*

- neighborhood homeomorphic to ball called *coordinate neighborhood* or *coordinate ball*

- pair $\langle U, \varphi \rangle$ with coordinate ball, U, with homeomorphism from U onto ball in $\mathbf{R}^n$, φ, called *coodinate chart*; φ called *coordinate map*

- coordinate (in $\mathbf{R}^n$) of point, $x \in U$, under φ said to be *coordinate of x* in the chart

Equivalent properties for manifolds

- for manifold, M, the following are equivalent
  - M is paracompact
  - M is σ-compact
  - M is Lindelöf
  - every open cover of M has star-finite open refinement
  - exist sequence of open subsets of M, $\langle O_n \rangle$, with $\overline{O_n}$ compact, $\overline{O_n} \subset O_{n+1}$, and $M = \bigcup O_n$
  - exists proper continuous map, $\varphi : M \to [0, \infty)$
  - M is second countable

Banach Spaces

Vector spaces

- set X with $+ : X \times X \to X$, $\cdot : \mathbf{R} \times X \to X$ satisfying the following properties called *vector space* or *linear space* or *linear vector space* over $\mathbf{R}$

  - for all $x, y, z \in X$ and $\lambda, \mu \in \mathbf{R}$

$$x + y = y + x$$ - additive commutativity

$$(x + y) + z = x + (y + z)$$ - additive associativity

$$(\exists 0 \in X)\ x + 0 = x$$ - additive identity

$$\lambda(x + y) = \lambda x + \lambda y$$ - distributivity of multiplication over addition

$$(\lambda + \mu)x = \lambda x + \mu x$$ - distributivity of multiplication over addition

$$\lambda(\mu x) = (\lambda\mu)x$$ - multiplicative associativity

$$0 \cdot x = 0 \in X$$

$$1 \cdot x = x$$

Norm and Banach spaces

- $\|\cdot\| : X \to \mathbf{R}_+$ with vector space, X, called *norm* if
  for all $x, y \in X$ and $\alpha \in \mathbf{R}$

  $$\|x\| = 0 \Leftrightarrow x = 0 \quad \text{- positive definiteness / positiveness /point-separating}$$

  $$\|x + y\| \geq \|x\| + \|y\| \quad \text{- triangle inequality / subadditivity}$$

  $$\|\alpha x\| = |\alpha| \|x\| \quad \text{- Absolute homogeneity}$$

- *normed vector space* that is *complete metric space* with metric induced by norm, *i.e.*, $\rho : X \times X \to \mathbf{R}_+$ with $\rho(x, y) = \|x - y\|$, called *Banach space*
  – can be said to be class of spaces endowed with both topological and algebraic structure
- examples include
  – L^p with $1 \leq p \leq \infty$ (page 43),
  – $C(X) = C(X, \mathbf{R})$, *i.e.*, space of all continuous real-valued functions on *compact* space, X

Properties of vector spaces

- normed vector space is complete *if and only if* every absolutely summable sequence is summable

Subspaces of vector spaces

- nonempty subset, S, of vector space, X, with $x, y \in S \Rightarrow \lambda x + \mu y \in S$, called *subspace* or *linear manifold*

- intersection of any family of linear manifolds is linear manifold

- hence, for $A \subset X$, exists smallest linear manifold containing A, often denoted by $\{A\}$

- if S is closed as subset of X, called *closed linear manifold*

- some definitions
  - $A + x$ defined by $\{y + x | y \in A\}$, called *translate* of A by x
  - λA defined by $\{\lambda x | x \in A\}$
  - $A + B$ defined by $\{x + y | x \in A, y \in B\}$

Linear operators on vector spaces

- mapping of vector space, X, to another (possibly same) vector space called *linear mapping*, or *linear operator*, or *linear transformation* if

$$(\forall x, y \in X, \alpha, \beta \in \mathbf{R})(A(\alpha x + \beta y y) = \alpha(Ax) + \beta(Ay))$$

- linear operator called *bounded* if

$$(\exists M)(\forall x \in X)(\|Ax\| \leq M\|x\|)$$

- least such bound called *norm* of linear operator, $i.e.$,

$$M = \sup_{x \in X, x \neq 0} \|Ax\|/\|x\|$$

  – linearity implies
$$M = \sup_{x \in X, \|x\|=1} \|Ax\| = \sup_{x \in X, \|x\| \leq 1} \|Ax\|$$

Isomorphism and isometrical isomorphism

- bounded linear operator from X to Y called *isomorphism* if exists bounded inverse linear operator, $i.e.$,

$$(\exists A : X \to Y, B : Y \to X)(AB \text{ and } BA \text{ are identity})$$

- isomorphism between two normed vector spaces that preserve norms called *isometrical isomorphism*

- from abstract point of view, isometrically isomorphic spaces are *identical*, $i.e.$, isometrical isomorphism merely amounts to *element renaming*

Properties of linear operators on vector spaces

- for linear operators, point continuity $\Rightarrow$ boundedness $\Rightarrow$ uniform continuity, *i.e.*,
  - bounded linear operator is uniformly continuous
  - linear operator continuous at one point is bounded

- *space of all bounded linear operators from normed vector space to Banach space is Banach space*

Linear functionals on vector spaces

- linear operator from vector space, X, to **R** called *linear functional*, $i.e.$, $f : X \to \mathbf{R}$ such that for all $x, y \in X$ and $\alpha, \beta \in \mathbf{R}$

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

- want to extend linear functional from subspace to whole vector space while preserving properties of functional

Hahn-Banach theorem

- *Hahn-Banach theorem* - for *vector space*, X, and linear functional, $p : X \to \mathbf{R}$ with

$$(\forall x, y \in X, \alpha \geq 0)(p(x+y) \leq p(x) + p(y) \text{ and } p(\alpha x) = \alpha p(x))$$

  and for subspace of X, S, and linear functional, $f : S \to \mathbf{R}$, with

$$(\forall s \in S)(f(s) \leq p(s))$$

  exists linear functional, $F : X \to \mathbf{R}$, such that

$$(\forall s \in S)(F(s) = f(s)) \text{ and } (\forall x \in X)(F(x) \leq p(x))$$

- corollary - for normed vector space, X, exists bounded linear functional, $f : X \to \mathbf{R}$

$$f(x) = \|f\| \|x\|$$

Dual spaces of normed spaces

- space of *bounded linear functionals* on *normed space*, X, called *dual* or *conjugate* of X, denoted by X^*

- every dual is Banach space (refer to page 128)

- dual of L^p is (isometrically isomorphic to) L^q for $1 \leq p < \infty$
  - exists natural representation of bounded linear functional on L^p by L^q (by Riesz representation theorem on page 46)

- *not* every bounded linear functionals on L^∞ has natural representation (proof can be found in Proof 4)

Natural isomorphism

- define linear mapping of normed space, X, to X^{**} (*i.e.*, dual of dual of X), $\varphi : X \to X^{**}$ such that for $x \in X$, $(\forall f \in X^*)((\varphi(x))(f) = f(x))$
  - then, $\|\varphi(x)\| = \sup_{\|g\|=1, g \in X^*} g(x) \leq \sup_{\|g\|=1, g \in X^*} \|g\| \|x\| = \|x\|$
  - by corollary on page 130, there exists $f \in X^*$ such that $f(x) = \|x\|$, then $\|f\| = 1$, and $f(x) = \|x\|$, thus $\|\varphi(x)\| = \sup_{\|g\|=1, g \in X^*} g(x) \geq f(x) = \|x\|$
  - thus, $\|\varphi(x)\| = \|x\|$, hence φ is isometrically isomorphic linear mapping of X onto $\varphi(X) \subset X^{**}$, which is subspace of X^{**}
  - φ called *natural isomorphism* of X into X^{**}
  - X said to be *reflexive* if $\varphi(X) = X^{**}$
- thus, L^p with $1 < p < \infty$ is reflexive, but L^1 and L^∞ are not
- note X may be isometric with X^{**} without reflexive

Completeness of natural isomorphism

- for natural isomorphism, φ

- X^{**} is complete, hence Banach space
  - because bounded linear functional to **R** (refer to page 128)

- thus, closure of $\varphi(X)$ in X^{**}, $\overline{\varphi(X)}$, complete (refer to page 61)

- therefore, *every normed vector space (X) is isometrically isomorphic to dense subset of Banach spaces (X^{**})*

Hahn-Banach theorem - complex version

- *Bohnenblust and Sobczyk* - for *complex* vector space, X, and linear functional, $p : X \to \mathbf{R}$ with

$$(\forall x, y \in X, \alpha \in \mathbf{C})(p(x+y) \leq p(x) + p(y) \text{ and } p(\alpha x) = |\alpha| p(x))$$

and for subspace of X, S, and (complex) linear functional, $f : S \to \mathbf{C}$, with

$$(\forall s \in S)(|f(s)| \leq p(s))$$

exists linear functional, $F : X \to \mathbf{R}$, such that

$$(\forall s \in S)(F(s) = f(s))$$

and

$$(\forall x \in X)(|F(x)| \leq p(x))$$

Open mapping on topological spaces

- mapping from topological space to another topological space the image of each open set by which is open called *open mapping*

- hence, one-to-one continuous open mapping is *homeomorphism*

- (will show) continuous linear transformation of Banach space onto another Banach space is always open mapping

- (will) use above to provide criteria for continuity of linear transformation

Closed graph theorem (on Banach spaces)

- every continuous linear transformation of Banach space onto Banach space is open mapping
  - in particular, if the mapping is one-to-one, it is isomorphism
- for linear vector space, X, complete in two norms, $\|\cdot\|_A$ and $\|\cdot\|_B$, with $C \in \mathbf{R}$ such that $(\forall x \in X)(\|x\|_A \leq C\|x\|_B)$, two norms are equivalent, *i.e.*, $(\exists C' \in \mathbf{R})(\forall x \in X)(\|x\|_B \leq C'\|x\|_A)$
- *closed graph theorem* - linear transformation, A, from Banach space, A, to Banach space, B, with property that "if $\langle x_n \rangle$ converges in X to $x \in X$ and $\langle Ax_n \rangle$ converges in Y to $y \in Y$, then $y = Ax$" is continuous
  - equivalent to say, if graph $\{(x, Ax) | x \in X\} \subset X \times Y$ is closed, A is continuous

Principle of uniform boundedness (on Banach spaces)

- *principle of uniform boundedness -* for family of bounded linear operators, $\mathcal{F}$ from Banach space, X, to normed space, Y, with

$$(\forall x \in X)(\exists M_x)(\forall T \in \mathcal{F})(\|Tx\| \leq M_x)$$

then operators in $\mathcal{F}$ is uniformly bounded, $i.e.$,

$$(\exists M)(\forall T \in \mathcal{F})(\|T\| \leq M)$$

Topological vector spaces

- just as notion of metric spaces generalized to notion of topological spaces

- *notion of normed linear space generalized to notion of topological vector spaces*

- linear vector space, X, with topology, $\mathfrak{J}$, equipped with continuous addition, $+ : X \times X \to X$ and continuous multiplication by scalars, $+ : \mathbf{R} \times X \to X$, called *topological vector space*

Translation invariance of topological vector spaces

- for topological vector space, translation by $x \in X$ is homeomorphism (due to continuity of addition)
  - hence, $x + O$ of open set O is open
  - every topology with this property said to be *translation invariant*
- for translation invariant topology, $\mathfrak{J}$, on X, and base, $\mathcal{B}$, for $\mathfrak{J}$ at 0, set

$$\{x + U | U \in \mathcal{B}\}$$

  forms *a base* for $\mathfrak{J}$ at x

- hence, sufficient to give a base at 0 to determine *translation invariance of topology*
- base at 0 often called *local base*

Sufficient and necessarily condition for topological vector spaces

- for topological vector space, X, can find base, $\mathcal{B}$, satisfying following properties

$$(\forall U, V \in \mathcal{B})(\exists W \in \mathcal{B})(W \subset U \cap V)$$

$$(\forall U \in \mathcal{B}, x \in U)(\exists V \in \mathcal{B})(x + V \subset U)$$

$$(\forall U \in \mathcal{B})(\exists V \in \mathcal{B})(V + V \subset U)$$

$$(\forall U \in \mathcal{B}, x \in X)(\exists \alpha \in \mathbf{R})(x \in \alpha U)$$

$$(\forall U \in \mathcal{B}, 0 < |\alpha| \leq 1 \in \mathbf{R})(\alpha U \subset U, \alpha U \subset \mathcal{B})$$

- conversely, for collection, $\mathcal{B}$, of subsets containing 0 satisfying above properties, exists topology for X making X *topological vector space* with $\mathcal{B}$ as base at 0
  - this topology is Hausdorff *if and only if*

$$\bigcap \{U \in \mathcal{B}\} = \{0\}$$

- for normed linear space, can take $\mathcal{B}$ to be set of spheres centered at 0, then $\mathcal{B}$ satisfies above properties, hence can form *topological vector space*

Topological isomorphism

- in topological vector space, can compare neighborhoods at one point with neighborhoods of another point by translation
- for mapping, f, from topological vector space, X, to topological vector space, Y, such that

$$
\begin{aligned}
&(\forall \text{ open } O \subset Y \text{ with } 0 \in O)(\exists \text{ open } U \subset X \text{ with } 0 \in U)\\
&(\forall x \in X)(f(x+U) \subset f(x) + O)
\end{aligned}
$$

  said to be *uniformly continuous*
- linear transformation, f, is uniformly continuous if continuous at one point
- continuous one-to-one mapping, φ, from X onto Y with continuous φ^{-1} called *(topological) isomorphism*
  - in abstract point of view, isomorphic spaces are *same*
- *Tychonoff -* finite-dimensional Hausdorff topological vector space is topologically isomorphic to $\mathbf{R}^n$ for some n

Weak topologies

- for vector space, X, and collection of linear functionals, $\mathcal{F}$, weakest topology generated by $\mathcal{F}$, $i.e.$, in way that each functional in $\mathcal{F}$ is continuous in that topology, called *weak topology generated by* $\mathcal{F}$
  - translation invariant
  - base at 0 given by sets

$$\{x \in X | \forall f \in \mathcal{G}, |f(x)| < \epsilon\}$$

    for all finite $\mathcal{G} \subset \mathcal{F}$ and $\epsilon > 0$
  - basis satisfies properties on page 140, hence, (above) weak topology makes *topological vector space*
- for *normed* vector space, X, and collection of continuous functionals, $\mathcal{F}$, $i.e.$, $\mathcal{F} \subset X^*$, weak topology generated by $\mathcal{F}$ *weaker than* (fewer open sets) norm topology of X
- metric topology generated by norm called *strong topology of* X
- weak topology generated by X^* called *weak topology of* X

Strongly and weakly open and closed sets

- open and closed sets of strong topology called *strongly open* and *strongly closed*
- open and closed sets of weak topology called *weakly open* and *weakly closed*

- wealy closed set is strongly closed, but converse not true
- however, these coincides for linear manifold, $i.e.$, linear manifold is weakly closed *if and only if* strongly closed

- every strongly convergent sequence (or net) is weakly convergent

Weak* topologies

- for normed space, *weak topology of X^** is weakest topology for which all functionals in X^{**} are continuous
- turns out that weak topology of X^* is less useful than weak topology generated by X, $i.e.$, that generated by $\varphi(X)$ where φ is the natural embedding of X into X^{**} (refer to page 132)
- (above) weak topology generated by $\varphi(X)$ called *weak* topology for X^**
  - even *weaker than* weak topology of X^*
  - thus, weak* closed subset of is weakly closed, and weak convergence implies weak* convergence
- base at 0 for weak* topology given by sets

$$\{f|\forall x \in A, |f(x)| < \epsilon\}$$

  for all finite $A \subset X$ and $\epsilon > 0$
- *when X is reflexive, weak and weak* topologies coincide*
- *Alaoglu* - unit ball $S^* = \{f \in X^*|\|f\| \geq 1\}$ is compact in weak* topology

Convex sets

- for vector space, X and $x, y \in X$

$$\{\lambda x + (1-\lambda)y | \lambda \in [0,1]\} \subset X$$

  called *segmenet joining x and y*

- set $K \subset X$ said to be *convex* or *convex set* if every segment joining any two points in K is in K, $i.e.$, $(\forall x, y \in K)$(segment joining $x, y \subset X$)

- every $\lambda x + (1-\lambda)y$ for $0 < \lambda < 1$ called *interior point of segment*

- point in $K \subset X$ where intersection with K of every line going through x contains open interval about x, said to be *internal point*, $i.e.$,

$$(\exists \epsilon > 0)(\forall y \in K, |\lambda| < \epsilon)(x + yx \in K)$$

- convex set examples - linear manifold & ball, ellipsoid in normed space

Properties of convex sets

- for convex sets, K_1 and K_2, following are also convex sets

$$K_1 \cap K_2,\ \lambda K_1,\ K_1 + K_2$$

- for linear operators from vector space, X, and vector space, Y,
  - image of convex set (or linear manifold) in X is convex set (or linear manifold) in Y,
  - inverse image of convex set (or linear manifold) in Y is convex set (or linear manifold) in X

- closure of convex set in topological vector space is convex set

Support functions of and separated convex sets

- for subset K of vector space X, $p : K \to \mathbf{R}_+$ with $p(x) = \inf \lambda | \lambda^{-1} x \in K, \lambda > 0$ called *support functions*

- for convex set $K \subset X$ containing 0 as internal point
  - $(\forall x \in X, \lambda \geq 0)(p(\lambda x) = \lambda p(x))$
  - $(\forall x, y \in X)(p(x + y) \leq p(x) + p(y))$
  - $\{x \in X | p(x) < 1\} \subset K \subset \{x \in X | p(x) \leq 1\}$

- two convex sets, K_1 and K_2 such that exists linear functional, f, and $\alpha \in \mathbf{R}$ with $(\forall x \in K_1)(f(x) \leq \alpha)$ and $(\forall x \in K_2)(f(x) \geq \alpha)$, said to be *separated*

- for two disjoint convex sets in vector space with at least one of them having internal point, exists *nonzero linear functional* that separates two sets

Local convexity

- topological vector space with base for topology consisting of convest sets, said to be *locally convex*
- for family of convex sets, $\mathcal{N}$, in vector space, following conditions are sufficient for being able to translate sets in $\mathcal{N}$ to form base for topology to make topological space into locally convex topological vector space

$$(\forall N \in \mathcal{N})(x \in N \Rightarrow x \text{ is internal})$$

$$(\forall N_1, N_2 \in \mathcal{N})(\exists N_3 \in \mathcal{N})(N_3 \subset N_1 \cap N_2)$$

$$(\forall N \in \mathcal{N}, \alpha \in \mathbf{R} \text{ with } 0 < |\alpha| < 1)(\alpha N \in \mathcal{N})$$

- conversely, for every locally convex topological vector space, exists base at 0 satisfying above conditions
- follows that
  - weak topology on vector space generated by linear functionals is locally convex
  - normed vector space is locally convex topological vector space

Facts regarding local convexity

- for locally convex topological vector space closed convex subset, F, with point, x, not in F, exists continuous linear functional, f, such that

$$f(x) < \inf_{y \in F} f(y)$$

- corollaries
  - convex set in locally convex topological vector space is strongly closed *if and only if* weakly closed
  - for distinct points, x and y, in locally convex Hausdorff vector space, exists continuous linear functional, f, such that $f(x) \neq f(y)$

Extreme points and supporting sets of convex sets

- point in convex set in vector space that is not interior point of any line segment lying in the set, called *extreme point*

- thus, x is extreme point of convex set, K, *if and only if* $x = \lambda y + (1-\lambda)z$ with $0 < \lambda < 1$ implies $y \notin K$ or $z \notin K$

- closed and convex subset, S, of convex set, K, with property that for every interior point of line segment in K belonging to S, entire line segment belongs to S, called *supporting set of K*

- for closed and convex set, K, set of points *a* continuous linear functional assumes maximum on K, is *supporting set of K*

Convex hull and convex convex hull

- for set E in vector space, intersection of all convex sets containing set, E, called *convex hull of E*, which is convex set

- for set E in vector space, intersection of all closed convex sets containing set, E, called *closed convex hull of E*, which is closed convex set

- *Krein-Milman theorem -* compact convex set in locally convex topologically vector space is *closed convex hull of its extreme points*

Hilbert spaces

- Banach space, H, with function $\langle\cdot,\cdot\rangle : H \times H \to \mathbf{R}$ satisfying following properties, called *Hilbert space*

$$(\forall x, y, z \in H, \alpha, \beta \in \mathbf{R})(\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle)$$
$$(\forall x, y \in H)(\langle x, y\rangle = \langle y, z\rangle)$$
$$(\forall x \in H)(\langle x, x\rangle = \|x\|^2)$$

- $\langle x, y\rangle$ called *inner product* for $x, y \in H$
  – examples - $\langle x, y\rangle = x^T y = \sum x_i y_i$ for $\mathbf{R}^n$, $\langle x, y\rangle = \int x(t)y(t)dt$ for L^2
- *Schwarz or Cauchy-Schwarz or Cauchy-Buniakowsky-Schwarz inequality -*

$$\|x\|\|y\| \geq \langle x, y\rangle$$

  – hence,
    - linear functional defined by $f(x) = \langle x, y\rangle$ bounded by $\|y\|$
    - $\langle x, y\rangle$ is continuous function from $H \times H$ to $\mathbf{R}$

Inner product in Hilbert spaces

- x and y in H with $\langle x, y \rangle = 0$ said to be *orthogonal* denoted by $x \perp y$

- set S of which any two elements orthogonal called *orthogonal system*

- orthogonal system called *orthonormal* if every element has unit norm

- any two elements are $\sqrt{2}$ apart, hence *if H separable, every orthonormal system in H must be countable*

- shall deal only with *separable Hilbert spaces*

Fourier coefficients

- assume orthonormal system expressed as sequence, $\langle \varphi_n \rangle$ - may be finite or infinite
- for $x \in H$

$$a_n = \langle x, \varphi_n \rangle$$

  called *Fourier coefficients*
- for $n \in \mathbf{N}$, we have

$$\|x\|^2 \geq \sum_{i=1}^{n} a_i^2$$

Proof:

$$\left\| x - \sum_{i=1}^{n} a_i \varphi_i \right\|^2 = \left\langle x - \sum a_i \varphi_i, x - \sum a_i \varphi_i \right\rangle$$

$$= \quad \langle x, x \rangle - 2 \left\langle x, \sum a_i \varphi_i \right\rangle + \left\langle \sum a_i \varphi_i, \sum a_i \varphi_i \right\rangle$$

$$= \quad \|x\|^2 - 2 \sum a_i \langle x, \varphi_i \rangle + \sum a_i^2 \|\varphi_i\|^2 = \|x\|^2 - \sum a_i^2 \geq 0$$

Fourier coefficients of limit of x

- *Bessel's inequality* - for $x \in H$, its Fourier coefficients, $\langle a_n \rangle$

$$\sum_{n=1}^{\infty} a_n^2 \leq \|x\|^2$$

- then, $\langle z_n \rangle$ defined by following is *Cauchy sequence* $z_n = \sum_{i=1}^{n} a_i \varphi_i$
- completeness (of Hilbert space) implies $\langle z_n \rangle$ converges - let $y = \lim z_n$

$$y = \lim z_n = \sum_{i=1}^{\infty} a_i \varphi_i$$

- continuity of inner product implies $\langle y, \varphi_n \rangle = \lim(z_n, \varphi_n) = a_n$, *i.e.*, Fourier coefficients of $y \in H$ are a_n, *i.e.*,
- *y has same Fourier coefficients as x*

Complete orthonormal system

- orthonormal system, $\langle \varphi_n \rangle_{n=1}^{\infty}$, of Hilbert spaces, H, is said to be *complete* if

$$(\forall x \in H, n \in \mathbf{N})(\langle x, \varphi_n \rangle = 0) \Rightarrow x = 0$$

- orthonormal system is complete *if and only if* maximal, $i.e.$,

$$\langle \varphi_n \rangle \text{ is complete} \Leftrightarrow ((\exists \text{ orthonormal } R \subset H)(\forall n \in \mathbf{N})(\varphi_n \in R) \Rightarrow R = \langle \varphi_n \rangle)$$

(proof can be found in Proof 5)
- Hausdorff maximal principle (Principle 4) implies existence of maximal orthonormal system, hence following statement
- for separable Hilbert space, H, every orthonormal system is separable and exists *a* complete orthonormal system. any such system, $\langle \varphi_n \rangle$, and $x \in H$

$$x = \sum a_n \varphi_n$$

with $a_n = \langle x, \varphi_n \rangle$, and $\|x\| = \sum a_n^2$

Dimensions of Hilbert spaces

- every complete orthonormal system of separable Hilbert space has same number of elements, $i.e.$, has same cardinality

- hence, every complete orthonormal system has either finite or countably infinite complete orthonormal system

- this number called *dimension of separable Hilbert space*
  - for Hilbert space with countably infinite complete orthonormal system, we say, $\dim H = \aleph_0$

Isomorphism and isometry between Hilbert spaces

- *isomorphism, Φ, of Hilbert space onto another Hilbert space* is linear mapping with property, $\langle \Phi x, \Phi y \rangle = \langle x, y \rangle$
- hence, every *isomorphism between Hilbert spaces is isometry*
- every n-dimensional Hilbert space is isomorphic to $\mathbf{R}^n$
- every $\aleph_0$-dimensional Hilbert space is isomorphic to l^2, which again is isomorphic to L^2
- $L^2[0, 1]$ is separable and $\langle \cos(n\pi t) \rangle$ is infinite orthogonal system
- every bounded linear functional, f, on Hilbert space, H, has unique y such that

$$(\forall x \in H)(f(x) = \langle x, y \rangle)$$

  and $\|f\| = \|y\|$

Selected Proofs

Selected proofs

- **Proof 1.** *(Proof for "Bolzano-Weierstrass-implies-seq-compact" on page 64)*
  if sequence, $\langle x_n \rangle$, has cluster point, x, every ball centered at x contains at one least point in sequence, hence, can choose subsequence converging to x. conversely, if $\langle x_n \rangle$ has subsequence converging to x, x is cluster point. ∎

- **Proof 2.** *(Proof for "compact-in-metric-implies-seq-compact" on page 66)*
  for $\langle x_n \rangle$, $\left\langle \overline{A_n} \right\rangle$ with $A_m = \langle b_n \rangle_{n=m}^{\infty}$ has *finite intersection property* because any finite subcollection $\{A_{n_1}, \ldots, A_{n_k}\}$ contains x_{n_k}, hence

$$\bigcap \overline{A_n} \neq \emptyset,$$

  thus, there exists $x \in X$ contained in every A_n. x is cluster point because for every $\epsilon > 0$ and $N \in \mathbf{N}$, then $x \in \overline{A_{N+1}}$, hence there exists $n > N$ such that x_n contained in ball about x with radius, ϵ. hence it's sequentially compact. ∎

- **Proof 3.** *(Proof for "restriction-of-continuous-topology-continuous" on page 84)* because for every open set O, $g^{-1}(O) \in \mathfrak{J}$, $A \cap g^{-1}(O)$ is open by definition of *inherited topology*. ∎

- **Proof 4.** *(Proof for "l-infinity-not-have-natural-representation" on page 131)*
  $C[0,1]$ is *closed* subspace of $L^\infty[0,1]$. define $f(x)$ for $x \in C[0,1]$ such that $f(x) = x(0) \in \mathbf{R}$. f is linear functional because $f(\alpha x + \beta y) = \alpha x(0) + \beta y(0) = \alpha f(x) + \beta(y)$. because $|f(x)| = |x(0)| \leq \|x\|_\infty$, $\|f\| \leq 1$. for $x \in C[0,1]$ such that $x(t) = 1$ for $0 \leq t \leq 1$, $|f(x)| = 1 = \|x\|_\infty$, hence achieves supremum, thus $\|f\| = 1$.
  if we define linear functional p on $L^\infty[0,1]$ such that $p(x) = f(x)$, $p(x+y) = x(0) + y(0) = p(x) + p(y) \leq p(x) + p(y)$, $p(\alpha x) = \alpha x(0) = \alpha p(x)$, and $f(x) \leq p(x)$ for all $x, y \in L^\infty[0,1]$ and $\alpha \geq 0$, and $f(s) = p(s) \leq p(s)$ for all $s \in C[0,1]$. Hence, Hahn-Banach theorem implies, exists $F : L^\infty[0,1] \to \mathbf{R}$ such that $F(x) = f(x)$ for every $x \in C[0,1]$ and $F(x) \leq f(x)$ for every $x \in L^\infty[0,1]$. Now assume $y \in L^1[0,1]$ such that $F(x) = \int_{[0,1]} xy$ for $x \in C[0,1]$. If we define $\langle x_n \rangle$ in $C[0,1]$ with $x_n(0) = 1$ vanishing outside $t = 0$ as $n \to \infty$, then $\int_{[0,1]} x_n y \to 0$ as $n \to \infty$, but $F(x_n) = 1$ for all n, hence, contradiction. Therefore there is not natural representation for F. ∎

- **Proof 5.** *(Proof for "orthonormal-system" on page 156)*

  Assume $\langle \varphi_n \rangle$ is complete, but not maximal. Then there exists orthonormal system, R, such that $\langle \varphi_n \rangle \subset R$, but $\langle \varphi_n \rangle \neq R$. Then there exists another $z \in R$ such that $z \not\in \langle \varphi_n \rangle$. But definition $\langle z, \varphi_n \rangle = 0$, hence $z = 0$. But $\|z\| = 0$, hence, cannot be member of orthonormal system. contraction, hence proved right arrow, $i.e.$, sufficient condition (of the former for the latter).

  Now assume that it is maximal. Assume there exists $z \neq 0 \in H$ such that $\langle z, \varphi_n \rangle = 0$. Then $\langle \varphi_n \rangle_{n=0}^{\infty}$ with $\varphi_0 = z/\|z\|$ is anoter orthogonal system containing $\langle \varphi_n \rangle$, hence contradiction, thus proved left arrow, $i.e.$, necessarily condition. ∎

References

References

[Roy88] H.L. Royden. *Real Analysis*. Prentice-Hall, Inc., Englewood Cliffs, New Jersey 07632, USA, 3rd edition, 1988.

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