Dynamics systems in Physics

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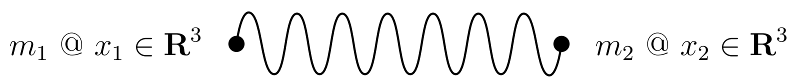

Figure 1: A spring connecting two point masses

1 Force equations

1.1 Ideal springs

Suppose (simple) rules for springs hold. That is when a spring with $k \in \mathbf{R}_{++}$ as the spring constant and $l \in \mathbf{R}_{++}$ as the natural length and two point masses $m_1 \in \mathbf{R}_{++}$ and $m_2 \in \mathbf{R}_{++}$, which are located at $x_1 \in \mathbf{R}^3$ and $x_2 \in \mathbf{R}^3$ as show in Figure 1. The force exerted on m_1 is

$$F_s(x_1; k, l, x_2) = -k(\|x_1 - x_2\| - l)v_{2,1} = -k\left(\frac{\|x_1 - x_2\| - l}{\|x_1 - x_2\|}\right)(x_1 - x_2) \tag{1}$$

where $v_{2,1} \in \mathbf{R}^3$ is a unit-length vector pointing to the direction of $x_1 - x_2$, *i.e.*,

$$v_{2,1} = \frac{1}{\|x_1 - x_2\|}(x_1 - x_2).$$

Here $\|x\|$ is the 2-norm or the size of a vector $x \in \mathbf{R}^3$ defined by

$$\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

The force exerted on m_2 $F_s(x_2; k, l, x_1)$ be can obtained using

1.2 Gravity-like forces

In a real world we live, we feel the gravity dragging us downward. But since we can assume anything here! :) let us assume that there exists gravity-like force in a sense that given an acceleration vector $a \in \mathbf{R}^3$, the force exerted on a point mass $m \in \mathbf{R}_{++}$ located at $x \in \mathbf{R}^3$ is

$$F_g(m; a) = ma \in \mathbf{R}^3 \tag{2}$$

i.e., the magnitude of the force is proportional to m (and the magnitude of the acceleration), the direction is the same as that of the acceleration. It does *not* depend on the location.

A typical example, of course, is *the* gravity where

$$a = (0, 0, -g) \in \mathbf{R}^3$$

with $g = 9.8m/s^2$.

1.3 Frictional forces

We model frictional forces exerted on bodies (not point mass) using the coefficient of friction $c \in \mathbf{R}_{++}$ where the force is modeled by

$$F_f(v) = -cv \in \mathbf{R}^3 \tag{3}$$

where $v \in \mathbf{R}^3$ is the velocity along the surface.

2 Energies

2.1 Kinetic energy of a mass

The kinetic energy of a mass m with velocity $v \in \mathbf{R}^3$ is defined by

$$E_\mathrm{k}(m) = \frac{m}{2}\|v\|^2 \tag{4}$$

2.2 Potential energy

Note that the role of the potential energies in dynamics is played in a way that the difference of them at two location, hence adding a constant to any potential energy makes no difference.

2.2.1 Potential energy by a gravity-like force

The potential energy of a mass m by a gravity-like force with acceleration $a \in \mathbf{R}^3$ is defined by

$$E_\mathrm{p,g}(m, x; a) = -mx \cdot a = -mx^T a \tag{5}$$

where $a^T b = a \cdot b$ for two vectors a and b is the inner product defined by

$$(a_1, a_2, a_3) \cdot (b_1, b_2, b_3) = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

For example, the potential energy by gravity is

$$E_\mathrm{p,g}(m, x; g) = -(0, 0, -g) \cdot (x_1, x_2, x_3) = mgx_3$$

2.2.2 Potential energy by a spring

The potential energy of a mass m by a gravity-like force with acceleration $a \in \mathbf{R}^3$ is defined by

$$E_\mathrm{p,s}(x_1, x_2; k, l) = \frac{k}{2}(\|x_1 - x_2\| - l)^2 \tag{6}$$

2.3 The law of preservation of the energy

In a system with masses, springs, and gravity-like forces with no frictional forces, the sum of the kinetic energies of all the masses and that of the potential energies of all the gravity-like forces and springs is preserved unless, for example, external forces are exerted on the system, two point masses merge into one, or some non-elastic crashes happen.

2.3.1 Proof

Suppose that the location of a point mass m is $x(t) \in \mathbf{R}^3$ and the force exerted on m is $F(t) \in \mathbf{R}^3$ where t refers to the time. These notations explicitly show that both quantities are functions of time t. The Newton's second law, $F(t) = ma(t)$ where $a(t) = \frac{d}{dt^2}x(t)$ is the acceleration of m, implies

$$\begin{aligned} \int_{t_1}^{t_2} (-F(t)) \cdot dx &= -\int_{t_1}^{t_2} ma(t) \cdot dx = -\int_{t_1}^{t_2} m \left(\frac{dv(t)}{dt} \right) \cdot v(t)dt \\ &= -\int_{t_1}^{t_2} mv(t) \cdot dv(t) = -\left(\frac{m}{2}v(t_2)^2 - \frac{m}{2}v(t_1)^2 \right) = \frac{m}{2}v(t_1)^2 - \frac{m}{2}v(t_2)^2 \end{aligned} \tag{7}$$

where we use the definition of velocity, $v(t) = dx(t)/dt$.

Now we will show that $\int (-F(t))dt$ is the same as (the difference of) the potential energies defined for the gravity-like force and the spring respectively. For the gravity-like force, we have

$$\int_{t_1}^{t_2} (-F(t)) \cdot dx = -m \int_{t_1}^{t_2} a \cdot dx = -m(a \cdot x(t_2) - a \cdot x(t_1)) = E_{\mathrm{p,g}}(t_2) - E_{\mathrm{p,g}}(t_1). \tag{8}$$

For the spring, we have

$$\int_{t_1}^{t_2} (-F(t)) \cdot dx = \cdots = E_{\mathrm{p,s}}(t_2) - E_{\mathrm{p,s}}(t_1). \tag{9}$$

Therefore for any type of force, (7) implies

$$E_{\mathrm{p}}(t_2) - E_{\mathrm{p}}(t_1) = \frac{m}{2}v(t_1)^2 - \frac{m}{2}v(t_2)^2 \quad \Leftrightarrow \quad \frac{m}{2}v(t_1)^2 + E_{\mathrm{p}}(t_1) = \frac{m}{2}v(t_2)^2 + E_{\mathrm{p}}(t_2), \tag{10}$$

hence the total energy, *i.e.*, the sum of the kinetic energy and the potential energy, is preserved for a point mass.

Because the potential energy is additive quantity and (10) holds for each mass in a given system, the law of the preservation of the energy hold for a dynamic system satisfying the conditions mentioned above.

2.4 Dissipated energy by a frictional force

The energy dissipated due to a frictional force exerted on a mass can be calculated by

$$\int F_f(v) \cdot dx \tag{11}$$

2.5 The law of preservation of the energy with frictional forces

In a system with masses, springs, and gravity-like forces *with* frictional forces, the sum of the kinetic energies and potential energies is reduced and the amount of reduction is (exactly) the same as the total dissipated energy.

2.5.1 Proof

It is quite straightforward to show this using similar derivations used in §2.3.1, thus we will now show it here.

3 Momentum

The momentum of a mass m with velocity v is defined by

$$M(m, v) = mv \in \mathbf{R}^3 \tag{12}$$

3.1 The law of preservation of the momentum

In a system with masses, the (vector) sum of the momentum of all the masses is preserved unless, for example, external forces are exerted on the system even when, for example, two point masses merge into one or some non-elastic crashes happen.

3.1.1 Proof

Suppose a point mass m. If no force is exerted, the momentum is (of course) preserved.

Now suppose two point masses m_1 and m_2. If they exert forces on each other, by the Newton's third law, the forces exerted on each point mass is the same in magnitude and the exact opposite in direction. Assume that $F(t) \in \mathbf{R}^3$ is exerted on m_1, then we have

$$\int_{t_1}^{t_2} F(t)dt = \int_{t_1}^{t_2} m_1 a_1(t)dt = \int_{t_1}^{t_2} m_1 dv_1(t) = m_1 v_1(t_2) - m_1 v_1(t_1)$$

and

$$-\int_{t_1}^{t_2} F(t)dt = \int_{t_1}^{t_2} m_2 a_2(t)dt = \int_{t_1}^{t_2} m_2 dv_2(t) = m_2 v_2(t_2) - m_2 v_2(t_1)$$

where we use $a(t) = dv(t)/dt$. Adding these two equations give us

$$m_1 v_1(t_1) + m_2 v_2(t_1) = m_1 v_1(t_2) + m_2 v_2(t_2). \tag{13}$$

Therefore the (vector) sum of the momentums of two masses is preserved.

Now (13) holds for every interacting mass pairs, the sum of the momentums of all the masses in a system is preserved while the conditions mentioned above are satisfied.

4 Equilibrium point

An equilibrium point of a system is defined by the configuration of all the masses in the system so as to minimize the total energy or equivalently, that where the force exerted on every mass, *i.e.*, the sum of all the forces exerted on every mass, is zero.

4.1 Finding the equilibrium point

For illustration, suppose that there are two point masses m_1 and m_2 whose locations are $x_1 \in \mathbf{R}^3$ and $x_2 \in \mathbf{R}^3$ and three springs with spring constants k_i and natural lengths l_i for $i = 1, 2, 3$. Assume one end of the first spring is fixed at y_1 and the other end is attached to m_1, the second spring is attached to m_1 and m_2, and one end of the third spring is fixed at y_2 and the other end is attached to m_2. This is shown in Figure 2. Suppose further that there exists a gravity-like force with an acceleration $a \in \mathbf{R}^3$. Then the total potential energy of the system is

$$\begin{aligned} f(x_1, x_2) &= \frac{k_1}{2}(\|x_1 - y_1\| - l_1)^2 + \frac{k_2}{2}(\|x_1 - x_2\| - l_2)^2 + \frac{k_3}{2}(\|x_2 - y_2\| - l_3)^2 \\ &\quad -a^T(m_1 x_1 + m_2 x_2). \end{aligned} \tag{14}$$

To minimize the function is the same as to find solutions of the following equations

$$\frac{\partial}{\partial x_1} f(x_1, x_2) = 0 \in \mathbf{R}^3, \quad \frac{\partial}{\partial x_2} f(x_1, x_2) = 0 \in \mathbf{R}^3, \tag{15}$$

which constitutes 6 equations. Since we have 6 variables and 6 equations, there exist solutions to these equations (in general).

To obtain the solution to (15), we calculate each term in (15) in the following.

$$-k_1 \left(\frac{\|x_1 - y_1\| - l_1}{\|x_1 - y_1\|} \right) (x_1 - y_1) - k_2 \left(\frac{\|x_1 - x_2\| - l_2}{\|x_1 - x_2\|} \right) (x_1 - x_2) + m_1 a = 0 \tag{16}$$

$$-k_2 \left(\frac{\|x_2 - x_1\| - l_2}{\|x_2 - x_1\|} \right) (x_2 - x_1) - k_3 \left(\frac{\|x_2 - y_2\| - l_3}{\|x_2 - y_2\|} \right) (x_2 - y_2) + m_2 a = 0 \tag{17}$$

Note that the quantity in (16) equals to the sum of the forces exerted on m_1 and that in (17) equals to the sum of the forces exerted on m_2. Therefore (at least for this case) it has been shown that the configuration where the sum of all potential energies is minimized is the same as that where the sum of the forces exerted on each mass is zero.

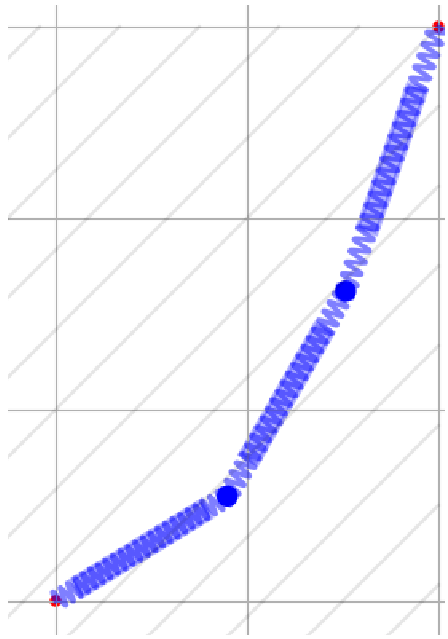

Figure 2: 2 point masses and 3 springs

4.1.1 Numerical solution

One can solve the system of equations (16) and (17) using an iterative method such as Netwon's method as follows [2]. Define a function $F : \mathbf{R}^6 \to \mathbf{R}^6$ such that

$$F(x) = F(x_1, x_2) = \begin{bmatrix} -\frac{\partial}{\partial x_1} f(x_1, x_2) \\ -\frac{\partial}{\partial x_2} f(x_1, x_2) \end{bmatrix} \in \mathbf{R}^6$$

where $x = (x_1, x_2) \in \mathbf{R}^6$ and each term is equal to (16) and (17) respectively. Given some initial point $x^0 \in \mathbf{R}^6$, one can repeat the following procedure for $k = 1, 2, \ldots$

$$x^{k+1} = x^k - \alpha_k DF(x^k)^{-1} F(x) \tag{18}$$

where $DF : \mathbf{R}^6 \to \mathbf{R}^{6\times6}$ is the Jacobian matrix of F and $0 < \alpha_k < 1$ is step lengths for each iterate.

4.1.2 Way easier, but approximate, solution method

While the function $f : \mathbf{R}^6 \to \mathbf{R}$ in (14) is not a convex function, hence it is not easy to minimize (we need some definition of *easiness* here!), we can easily, but approximately, minimize the function by letting $l_1 = l_2 = l_3 = 0$. If $l_1 = l_2 = l_3 = 0$, f in (14) becomes

$$\tilde{f}(x_1, x_2) = \frac{k_1}{2}\|x_1 - y_1\|^2 + \frac{k_2}{2}\|x_1 - x_2\|^2 + \frac{k_3}{2}\|x_2 - y_2\|^2 - a^T(m_1 x_1 + m_2 x_2), \tag{19}$$

hence a convex function [3], and the partial derivatives are

$$-\frac{\partial}{\partial x_1} f(x_1, x_2) = -k_1(x_1 - y_1) - k_2(x_1 - x_2) + m_1 a = 0 \in \mathbf{R}^3 \tag{20}$$

and

$$-\frac{\partial}{\partial x_2} f(x_1, x_2) = -k_2(x_2 - x_1) - k_3(x_2 - y_2) + m_2 a = 0 \in \mathbf{R}^3. \tag{21}$$

Note that the system of equations (20) and (21) is just a linear system because they are equivalent to

$$\begin{bmatrix} (k_1 + k_2)I_3 & -k_2 I_3 \\ -k_2 I_3 & (k_2 + k_3)I_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} m_1 a \\ m_2 a \end{bmatrix} \tag{22}$$

which is nothing but a (simple) linear system

$$Ax = b \tag{23}$$

with

$$A = \begin{bmatrix} (k_1 + k_2)I_3 & -k_2 I_3 \\ -k_2 I_3 & (k_2 + k_3)I_3 \end{bmatrix} \in \mathbf{R}^{6\times 6}, \quad b = \begin{bmatrix} m_1 a \\ m_2 a \end{bmatrix} \in \mathbf{R}^6.$$

Using a modern computer system and the numerical packages (that have been developed for decades by amazing experts based on amazing research results), any linear system can be solved extremely fast, stably, and efficiently *even for hundreds of thousands of variables*. Therefore one can obtain the solution for (22) very easily.

Hence, we can use this solution for a surrogate for the equilibrium point to find the lowest energy point fast and easily for *an arbitrarily complicated and large system*.

If you turn on the `minimize_energy` as below, `science` will calculate this *approximate* lowest energy configuration and will put the point masses at those points initially.

```
simulation_setting:
  minimize_energy: true
```

5 Simple harmonic motions

5.1 Spring

5.1.1 Without frictional force

Suppose a (point) mass m is attached to a spring with spring constant k. Suppose also hat $x : \mathbf{R}_+ \to \mathbf{R}$ is the displacement of the mass from the natural length of the spring, which is a function of time t. The Newton's second law dictates that

$$m\ddot{x}(t) = -kx(t) \tag{24}$$

where we define

$$\ddot{x}(t) = \frac{d}{dt^2} x(t).$$

This is what is called an ordinary differential equation (ODE). We could use some fancy techniques such as Laplace transform [1] (which has some advantages, *e.g.*, effortlessly taking into account the initial conditions), but we will *NOT* use such methods because we can achieve the very same thing with much easier methods as will be seen below.

Because we know the motion dictates by (24) is a harmonic motion, we can assume it is of the form

$$x(t) = A\cos(\omega t + \theta_0) = A\cos(2\pi f t + \theta_0) \tag{25}$$

for some $A > 0$, $\omega = 2\pi f > 0$, and $0 \leq \theta_0 < 2\pi$. Here ω is called an *angular frequency* and f just a *frequency*.

Now if we plug (25) in (24), we have

$$-m\omega^2 x(t) = -kx(t)$$

which gives

$$\omega = \sqrt{k/m} \quad \Leftrightarrow \quad f = \sqrt{k/m}/2\pi \tag{26}$$

We can confirm this formula by using our dynamic simulation tool, `science`. Here are some examples, which are the results of the simulation by `science`.

- `1d-spring-harmonic-motion-0.5-Hz.gif`
- `1d-spring-harmonic-motion-1-Hz.gif`
- `1d-spring-harmonic-motion-2-Hz.gif`

Note that the other two parameters, A and θ_0, are determined by the initial conditions, that is, the initial location $x(0)$ and the initial velocity $\dot{x}(0)$. Because we have 2 variables and 2 equations, those are uniquely determined by those equations.

5.1.2 With frictional force

Here we assume that the frictional force exerts on m, the magnitude of which is proportional to the velocity (and, of course, the direction is the opposite of the velocity), as described in §1.3.

Again the Newton's second law dictates that

$$m\ddot{x}(t) = -kx(t) - c\dot{x}(t) \tag{27}$$

where we define

$$\dot{x}(t) = \frac{d}{dt}x(t).$$

This is also an ODE and we also can use fancy methods like Laplace transform, but again we avoid using it here. We know that the solution to (27) is of the form

$$x(t) = Ae^{-at}\cos(\omega t + \theta_0) = Ae^{-at}\cos(2\pi f t + \theta_0) \tag{28}$$

for some $A > 0$, $a > 0$, $\omega = 2\pi f > 0$, and $0 \leq \theta_0 < 2\pi$. Note the difference between (25) and (28), that is, we have an attenuation term, which exponentially reduces the magnitude of the harmonic motion. Instead of dealing with this solution dependent of four parameters, A, a, ω, and θ_0, we can (equivalently) deal with the below solution with two complex parameters, $\alpha, \beta \in \mathbf{C}$.

$$x(t) = \exp(\alpha t + \beta) \tag{29}$$

Note that they have the same degrees of freedom (so to speak).

Once again, let us plug (29) in (27) to obtain the quadratic equation:

$$m\alpha^2 x(t) = -kx(t) - c\alpha x(t) \quad \Leftrightarrow \quad m\alpha^2 + c\alpha + k = 0 \tag{30}$$

The solution of (30) characterizes the (harmonic) attenuated motion. Using the quadratic formula, we obtain the solution.

$$\alpha = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

First, if $c > 2\sqrt{mk}$, then α is a real number, hence there is no harmonic motion, but only attenuation. This is shown at `1d-spring-attenuation-motion.gif`.

Now if $c < 2\sqrt{mk}$, we can rewrite the solution as below:

$$\alpha = -b \pm i\omega$$

where $b = c/2m$ and $\omega = \sqrt{4mk - c^2}/2m$. If we plug this solution in (29), we obtain

$$x(t) = Ae^{-bt}\cos(\omega t + \theta_0)$$

with some $A > 0$ and $0 \leq \theta_0 < 2\pi$, which are determined by β in (29), which again are determined by the initial conditions, *i.e.*, $x(0)$ and $\dot{x}(0)$.

This gives an attenuated harmonic motion, that is, it oscillates, but the amplitude of the oscillation reduces exponentially. The attenuation rate is determined by b and the frequency of the oscillation is determined by ω (of course). One such example is shown at `1d-spring-attenuated-harmonic-motion.gif`

Note that $c = 0$ (of course) gives us the same solution as in (26).

(Note that I've omitted some technical arguments here, but please trust me on that the solutions we have derived are correct.)

5.1.3 Analysis and design

From the above derivations, we note two tasks we can do.

We first note that the three numbers m, k, and c completely characterizes the motion, *e.g.*, whether or not it will oscillates, the attenuation rate, and the frequency of the harmonic motion. This means we can do the analysis of the motion if we know exact (or even approximate) values for these parameters.

On the other hand, we can design a system using the formula we have derived in §5.1.2, *i.e.*, we know how we should design the system to achieve the attenuation rate or frequency we want by deciding, *e.g.*, the stiffness of the spring, the mass, or the coefficient of friction.

References

[1] Laplace transform. https://en.wikipedia.org/wiki/Laplace_transform.

[2] Newton-raphson method. https://en.wikipedia.org/wiki/Newton%27s_method.

[3] Stephen Boyd and Lieven Vandenberghe. *Convex Optimization*. Cambridge University Press, New York, NY, USA, 2004.