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On the convergence of the Warnsdorff's algorithm on the rectangular boards

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Abstract

The Knight's Tour problem, dating back centuries, challenges mathematicians to find a sequence of moves for a knight to visit every square on a chessboard exactly once. Warnsdorff's rule, proposed in 1823, offers a heuristic approach to solving this problem by prioritizing moves to squares with the fewest onward moves. In our study, we explore the convergence rates of Warnsdorff's algorithm on rectangular and square boards. Our findings reveal intriguing patterns in convergence rates, suggesting potential correlations between board geometry and algorithm accuracy. Through empirical data and theoretical discussions, we contribute to understanding the limits and effectiveness of Warnsdorff's rule in solving the Knight's Tour problem.

Keywords: Warnsdorff's algorithm; Knight tour; Simulation; Optimisation

1. Introduction

It is said, that we have toured the chessboard with a certain piece if we have done the following: placed our piece on a certain square and then made moves only to squares that have not been occupied yet, so that at some point, all squares on the board have been visited. Often, this problem can come with restrictions, for example, starting from a specific square or ending at a certain square, and so on.

A Knight's Tour is a sequence of moves of a Knight on a chessboard such that the Knight visits every square exactly once. There are two types of Knight Tours: "closed" — where the Knight ends on a square that is one Knight's move

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from the beginning square (so that it could tour the board again immediately, following the same path), and “open” — when the Knight cannot move back to the beginning square after the tour ends.

What makes these chess problems so beautiful is that, clearly shown with The Knight’s Tour, the solutions are just chess visualizations of Graph Theory problems.

This problem has fascinated mathematicians for centuries. The earliest known reference to the Knight’s Tour problem dates back to the 9th century AD! Rudrata’s presented the pattern of a Knight’s Tour on a half-board as an elaborate poetic figure called the turagapadabandha or “arrangement in the steps of a horse”. The same verse in four lines of eight syllables each can be read from left to right or by following the path of the Knight on tour. The “Knight’s Tour” of the chessboard was first proposed (solved) in a ninth century Arabic manuscript by Abu Zakariya Yahya ben Ibrahim al-Hakim. The author gives two tours, one by Ali C. Mani, an otherwise unknown chess player, and the other by al-Adli ar-Rumi, who flourished around 840 and is known to have written a book on Shatranj (the form of chess which was popular at the time).

In its classical form without restrictions, this problem has been known since at least the 18th century. Leonhard Euler worked on this problem, he titled it, “Solution to a curious problem that seems to defy any analysis”. Euler addressed this problem in a letter to Goldbach on April 26, 1757. In the letter, he stated: “The recollection of a problem once proposed to me has recently led me to some subtle investigations in which ordinary analysis seems to be of no avail. I have found, at last, a clear method of finding as many solutions as one wishes (though their number is not infinite), without making any trials”. Besides considering the problem for the knight, Euler also analyzed similar problems for other chess pieces.

While the problem was known before Euler, he was the first to recognize its mathematical essence, and therefore, the problem is often associated with his name. Euler’s method involves the knight moving along an arbitrary route until it exhausts all possible moves. Then, after a special rearrangement of its elements, the remaining unvisited squares are added to the made route.

Allen Schwenk in 1991 proved the theorem that for any $m \times n$ board with $m \leqslant n$, a closed Knight’s tour is always possible unless one or more of these three conditions holds:

- m and n are both odd;
- $m = 1, 2,$ or 4;
- $m = 3$ and $n = 4, 6,$ or 8.

In his proof, the author converts the chessboard to graph.

Many methods are known for finding knight’s tours, they are named after their discoverers — Euler’s method, Vandermonde’s method, Munk and Collin’s frame method, Polignac and Roget’s method of quartering, and others.

Warnsdorff’s rule is a heuristic designed to find a single Knight’s Tour, initially proposed by H. C. von Warnsdorff in 1823. The rule dictates that the Knight should always move to the square with the fewest onward moves, considering only squares that have not been visited before. In cases where multiple squares have the same number of onward moves, various methods can be employed to break ties. This rule can also be more broadly applied to any graph, where each move is made to the adjacent vertex with the least degree.

Let us illustrate this rule. In Figure 1, each square contains the number of moves that the Knight could make from that square.

In this case, the rule tells us to move to the square with the smallest number in it, namely 2.

Warnsdorff’s rule, proposed over 200 years ago, was initially believed to be flawless. However, later findings revealed that its second part is not entirely accurate. When the Knight has multiple options available, as mentioned in the first part of the rule, not all of them are equivalent. Experimental evidence indicates that arbitrary application of the second part of Warnsdorff’s rule can lead the Knight into a dead end.

Nevertheless, in practical applications, Warnsdorff’s rule proves to be quite effective, and, even with the loose use of its second part, the likelihood of getting lost is minimal.

In this article, we study the limits of the Warnsdorff’s rule.

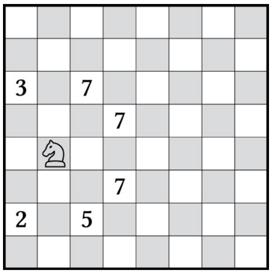

Fig. 1. Illustration of Warnsdorff's rule.

1	12	43	40	3	14	17
42	31	2	13	16	0	4
11	20	41	44	39	18	15
32	45	30	19	0	5	0
21	10	35	38	29	24	27
36	33	8	23	26	0	6
9	22	37	34	7	28	25

Fig. 2. Illustration of non convergence of Warnsdorff's rule.

2. Previous research

In 1823, Warnsdorff proposed the following rule for solving the knight's tour problem in his brochure, "The simplest and most general solution to the knight's tour problem": "On each move, place the knight on a square from which it can reach the fewest number of unvisited squares. If there are several such squares, any of them can be chosen." [3]

Finding a knight's tour is a special case of the NP-hard problem of finding Hamiltonian path of a generic graph.

The advent of Computer Science allowed the study of Warnsdorff's rule from the numerical perspective. The most studied case was the case of square boards of size m by m.

This rule is not deterministic; however, Warnsdorff claimed that no matter which random choices are made to break ties, the path produced is always a tour.

In fact, on some boards, if several incorrect choices are made, a path can be produced which is not a tour. For example, with the appropriate choices, the path from Figure 2 can be produced on the 7 by 7 board. However, cases like this one are rare on small boards.

Unfortunately, on larger boards the rule becomes, basing on the literature, rapidly less useful. For example, Parberry [2] divided the board into subsets and, using a variant of Warnsdorff's rule, produced non-tour paths in each of the subsets; then his algorithm attempted to piece the smaller paths all together and produce a tour of the full board. The whole process was tried repeatedly until a tour was found. However, Parberry reports that the largest board for which this method succeeded was the 78 by 78 board, and his experiments show an exponential decrease in success rate as the size of the board increases.

Douglas Squirrel and Paul Çull in 1996 [1] proposed another tiebreak method of resolving the undeterministic nature of the rule. They provide the partial proof of the algorithm's correctness for a single case, namely $m = 7(mod8)$. Authors claimed as well that for any m by m chessboard with $m \geqslant 5$, there exists a tour consistent with Warnsdorff's rule. The algorithm presented and partially proven by the authors, establishes the theorem for $m \geqslant 112$. To show the theorem is true for $m < 112$, they used a computer search for single move-orderings which work on these boards. Their proof was not complete; thus, we cannot verify its correctness.

However, despite the numerous discussions of the algorithm and different approaches to solve the Knight's Tour problem, there is no exact data on the convergence of this algorithm. Also, not all Knight's Tour solutions are reachable using the Warnsdorff algorithm.

3. Problem setting and implementation

We consider the rectangular boards.

For small boards (with numbers of cells less than 100) we are considering all the possible options of the Warnsdorff algorithm execution.

Moreover, for 6×6 board we provide the exhaustive search. This search gives 524486 converged options and 7083423656 not converged ones, which gives solve rate of 0.000074.

For example, if at any moment we have a *fork* (two possible moves with the same degree of freedom) in the algorithm execution, then we divide the execution into 2 branches. The value we are searching for is the expectancy of the convergence of the algorithm, thus, we need to find the total amount of options and the amount of converged ones. The expectancy of convergence is equal to the amount of converged options divided by the total amount of options.

We are looking as well at the nature of the function which shows the number of possible outcomes under the Warnsdorff algorithm execution.

The program is realised on the C++ programming language.

Main algorithm is a recursive function $SOLVE$, which takes the current board and the current move as arguments. The function $MakeMove$ tries to make the proposed move, the function $UnmakeMove$ undoes result of the previous call of $MakeMove$. Function $EnumerateMoves$ gets the list of available moves in the given board.

Variables $GoodSolves$ and $BadSolves$ are global for function $SOLVE$ and represent respectively the numbers of good and bad solutions. $GoodSolves$ contains converged tours and $BadSolves$ contains non-converged tours.

```
GoodSolves = 0, BadSolves = 0
procedure SOLVE(B : Board, m : Move)
    if B.FreeFields == 0 then                          ▷ Found a solution
        Inc(GoodSolves)
        return True
    end if
    if MakeMove(B, m) then                             ▷ Actualise the move
        if B.FreeFields == 0 then                      ▷ Register another good solution after making the move
            Inc(GoodSolves)
            UnmakeMove(B, m)
        end if
    end if
    AMoves = EnumerateMoves(B)
    if Size(AMoves) == 0 then
        Inc(BadSolves)
        UnmakeMove(B, m)
    end if
    MinCnt = ∞
    for all u ∈ Amoves do
        MinCnt = min(MinCnt, Available(B, u))          ▷ Mininum free fields if we make move u
    end for
    OK = True
    for all u ∈ Amoves  do
        if Available(B, u) == MinCnt then
            SOLVE(B, u)                                ▷ Try to find solutions with given move u
            UnmakeMove(B, m)
        end if
    end for
end procedure
```

4. Results

For the first approach, we consider the rectangular boards of sides $6 \times n$, where n is more or equal of 6. **These boards have the closed paths according the Schwenk's theorem.**

Table 1 sums up the obtained statistics.

Table 1. Rectangular boards of the size $6 \times n$

n	amount of options	amount of non converged options	percentage of converged options
6	80	0	100.00
7	1 276	160	87.46
8	2 586	178	93.12
9	12 335	1 198	90.29
10	69 084	10 800	84.37
11	253 317	32 169	87.30
12	606 056	83 454	86.23
13	1 659 989	212 086	87.22
14	7 560 366	1 180 132	84.39
15	29 978 798	5 596 764	81.33
16	72 643 219	60 347 174	83.07
17	256 657 563	45 957 973	82.09
18	1 428 197 521	345 006 305	75.84

In the rectangular boards $6 \times n$ we see that, as n increases, the total amount of options also increases. This is logical, as with more squares on the board, more moves must be done for a tour. The general trend is that as n increases, the percentage converged options decreases, we can see that, on boards greater than 6×6, the Warnsdorff method doesn't always lead to a converging solution. These findings support that, on larger boards, the Warnsdorff method is less accurate.

As can be seen in the table, the percentage converged options fluctuates. What caught our attention was that the 6×7 board had a lower percent converged options than 6×6 and 6×8 board. We hypothesised, that the fluctuation had to do with the parity of the numbers. The hypothesis was, that on boards $a \times b$ where either a or b is odd, the percentage converged options is lower than if a and b are both even. As we continued with larger numbers, our data didn't always support the hypothesis.

As we can see, 6×10 has a lower percent converged options than 6×11, 6×12, 6×13, and 6×14. Neither 10 nor 6 is odd, this doesn't support the hypothesis. However 6×16 board had a higher percent than 6×15, but 6×13 had a higher percentage than 6×12 and 6×14. As we can see, the fluctuations are not simply parity based. We cannot explain completely why there are fluctuations in the percentage of converged options.

As we had an hypothesis about the influence of parity of the number to the algorithm convergence, we considered as well the boards of size $5 \times n$. Table 2 sums up the obtained statistics.

In Figure 3 we compare the convergency rates for different board sizes. We see, that the closer is the board to the square shape, the slower is the decrease of this rate.

When we consider the square tables, the statistics is as shown in Table 3.

Again the total amount of options increases as n increases just as it was seen in the table above. The general trend as the value of n increases is again, that the percentage converged options decreases, however, as we can see there is fluctuation, as the percentage converged options for the 8×8 board is lower than for 7×7 and 9×9 board.

As clearly visible, the percentage for square boards is much higher than for rectangular boards. We hypothesise that the Warnsdorff method is more accurate on square boards than on rectangular ones, the exact reason why we do not know.

5. Conclusion and future work

In conclusion, our study sheds light on the convergence patterns of Warnsdorff's algorithm for the Knight's Tour problem. Through empirical analysis, we observed fluctuations in convergence rates across different board geometries,

Table 2. Rectangular boards of the size $5 \times n$

n	amount of options	amount of non converged options	percentage of converged options
5	32	0	100.00
6	312	0	100.00
7	1 180	433	63.31
8	3 325	686	79.37
9	7 799	1 963	74.83
10	32 079	10 528	67.18
11	88 099	28 880	67.22
12	219 685	70 437	67.94
13	1 061 750	400 708	62.30
14	2 550 231	900 143	64.70
15	6 678 567	2 668 156	60.05
16	21 100 000	8 917 175	57.82
17	52 981 765	41 831 020	55.88
18	212 400 000	87 771 257	58.69

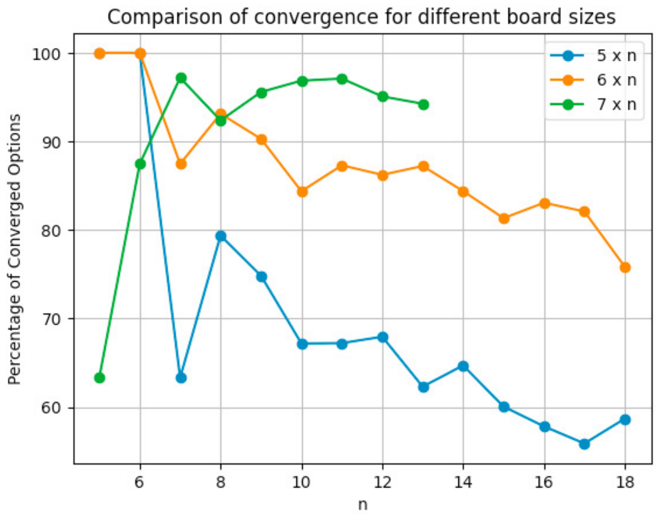

Fig. 3. Comparison of convergence for different board sizes

Table 3. Square boards of the size $n \times n$

n	amount of options	amount of converged options	percentage of converged options
5	32	32	100.00
6	80	80	100.00
7	39 024	37 926	97.19
8	144 104	137 300	95.28
9	19 578 552	18 808 408	96.07
10	2 055 858 718	1 980 656 188	96.34

suggesting the algorithm's sensitivity to board shape. Furthermore, our exploration opens avenues for future research to enhance algorithm efficiency and extend its applicability to non-rectangular boards and boards with holes, i.e. missing squares.

Moving forward, one avenue for research is to investigate methods for accelerating the algorithm to generate more data for analysis; the time taken to calculate each statistic in this paper increased exponentially as the side lengths of the boards increased, we simply didn't have the ability to generate more data, this problem could be overcome in future research. Additionally, exploring the application of Warnsdorff's rule to non-rectangular boards and boards with holes presents intriguing challenges and opportunities. Moreover, extending the algorithm to accommodate movements beyond the traditional knight's move, such as "N moves followed by 1 move," or "N moves followed by M moves," offer a promising direction for advancing the understanding of traversal problems on various grid structures. Another avenue for research could be n-dimensional boards, such as three dimensional, "X moves followed by Y moves followed by Z moves".

By addressing these areas, researchers can deepen their understanding of traversal algorithms and pave the way for innovative solutions to combinatorial problems in diverse domains.

References

[1] Douglas Squirrel and Paul Çull. *A Warnsdorff-Rule Algorithm for Knight's Tours*. 1996.

[2] Ian Parberry. *Scalability of a neural network for the knight's tour problem*. *Neurocomputing*, 12(1):19–33, 1996. Elsevier.

[3] H. C. Warnsdorff. *Des Rosselsprunges einfachste und allgemeinste Lösung*, Schmalkalden, 1823.