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Generalized knight's tour on 3D chessboards

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ABSTRACT

In [G.L. Chia, Siew-Hui Ong, Generalized knight's tours on rectangular chessboards, Discrete Applied Mathematics 150 (2005) 80–98], Chia and Ong proposed the notion of the generalized knight's tour problem (GKTP). In this paper, we address the 3D GKTP, that is, the GKTP on 3D chessboards of size $L \times M \times N$, where $L \leq M \leq N$. We begin by presenting several sufficient conditions for a 3D chessboard not to admit a closed or open generalized knight's tour (GKT) with given move patterns. Then, we turn our attention to the 3D GKTP with $(1, 2, 2)$ move. First, we show that a chessboard of size $L \times M \times N$ does not have a closed GKT if either (a) $L \leq 2$ or $L = 4$, or (b) $L = 3$ and $M \leq 7$. Then, we constructively prove that a chessboard of size $3 \times 4s \times 4t$ with $s \geq 2$and $t \geq 2$ must contain a closed GKT.

1. Introduction

The knight's tour problem (KTP) is an interesting mathematical problem, and its history can be dated back to Euler and De Moivre [2]. In the past decade, the KTP has received considerable interest [11,8,5,6,9,7]. Recently, Chia and Ong [3] initiated the study of the so-called generalized knight's tour problem (GKTP) by considering the more general (a, b) move rather than the $(1, 2)$ move. In [10,4], the KTP was generalized to the 3D situation, still with $(1, 2)$ move.

In this paper, we address the 3D GKTP with (a, b, c) move, that is, the GKTP on 3D chessboards of size $L \times M \times N$, where $L \leq M \leq N$. We begin by presenting several sufficient conditions for a 3D chessboard not to admit a closed or open generalized knight's tour (GKT) with given move patterns. Then, we turn our attention to the 3D GKTP with $(1, 2, 2)$ move. First, we show that a chessboard of size $L \times M \times N$ does not have a closed GKT if either (a) $L \leq 2$ or $L = 4$, or (b) $L = 3$ and $M \leq 7$. Then, we constructively prove that a chessboard of size $3 \times 4s \times 4t$ with $s \geq 2$ and $t \geq 2$ must contain a closed GKT.

2. Preliminaries

An $L \times M \times N$ chessboard is a 3-dimension array of cube cells arranged in L rows, M columns and N levels, which is plotted in a x–y–z coordinate system. Fig. 1 presents a $3 \times 4 \times 8$ 3D chessboard. Without loss of generality, we assume $L \leq M \leq N$.

We consider the following move type on 3D chessboards. Suppose the cells of the $L \leq M \leq N$ chessboard are (i, j, k) where $0 \leq i \leq L - 1$, $0 \leq j \leq M - 1$ and $0 \leq k \leq N - 1$. A move from cell (i, j, k) to cell (r, s, t) is termed an (a, b, c)-knight's move if $\{|r - i|, |s - j|, |t - k|\} = \{a, b, c\}$. For a given (a, b, c)-knight's move on an $L \times M \times N$ chessboard, there is associated with it a graph whose vertex set and edge set are $\{(i, j, k) | 0 \leq i \leq L - 1, 0 \leq j \leq M - 1, 0 \leq k \leq N - 1\}$ and $\{(i, j, k)(r, s, t) | 0 \leq i, r \leq L - 1, 0 \leq j, s \leq M - 1, 0 \leq k, t \leq N - 1, \{|r - i|, |s - j|, |t - k|\} = \{a, b, c\}\}$ respectively. Let $G((a, b, c), L, M, N)$ denote this graph, or just $G(L, M, N)$ for simplicity if the move (a, b, c) is understood or not to be emphasized.

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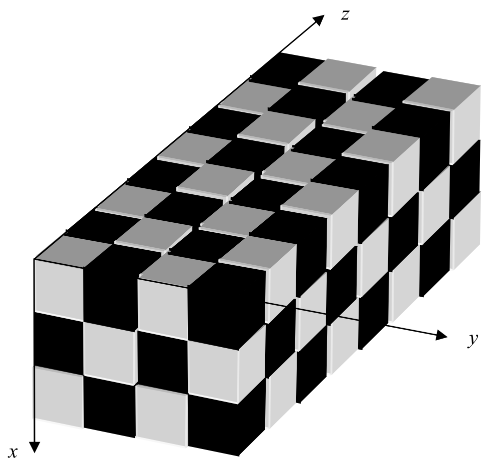

Fig. 1. The $3 \times 4 \times 8$ 3D chessboard.

A closed (a, b, c)-*knight's tour* is a series of (a, b, c)-knight's moves that visits every small cube of the $L \times M \times N$ chessboard exactly once and then returns to the starting cube. The *generalized knight's tour problem* (GKTP) asks: which $L \times M \times N$ chessboards admit a closed (a, b, c)-knight's tour? This amount to asking: which graph $G((a, b, c), L, M, N)$ is Hamiltonian?

There are two common tours to consider on the rectangular parallelepiped. One is to tour the six exterior $L \times M$, $L \times N$, or $M \times N$ boards that form the rectangular parallelepiped. Qing and Watkins [10] recently showed that a (1, 2)-knight's tour exists on the exterior of the cube $n \times n \times n$ for all n. The focus of this paper is the (a, b, c)-knight's tour within the N stacked copies of the $L \times M$ board that form the rectangular parallelepiped, that is the 3D chessboards.

3. Some results of (a, b, c)-knight's move

Theorem 1. *Suppose the $L \times M \times N$ chessboard admit a closed (a, b, c)-knight's tour, where $a < b \leq c$ and $L \leq M \leq N$. Then*

(i) *$a + b + c$ is odd and not all a, b, c are equal;*
(ii) *L or M or N is even;*
(iii) *$L \geq a + b$;*
(iv) *$N \geq 2c$.*

Proof. The theorem is an extension of the results by Chia and Ong (Ref. [3]) concerning generalized knight's tours on 2D chessboards. Obviously, the observations in Theorem 2 of Ref. [3] are easily extended to (a, b, c)-knight's tour on the $L \times M \times N$ chessboard. □

Theorem 2. *Suppose $1 < a \leq b \leq c$. Then no open (a, b, c)-knight's tour on the $(a + 1) \times (a + b + 1) \times (c + 1)$ chessboard is possible, where a, b and c are positive integers.*

Proof. Note that the vertex (a, b, c) is of degree 0 on the $(a + 1) \times (a + b + 1) \times (c + 1)$ chessboards, so that the open (a, b, c)-knight's tour is impossible. □

To draw more conclusions we introduce certain graphical concepts. When we remove a vertex v from a graph G we also remove all edges incident with v.

Theorem 3. *Suppose $c > 1$. Then no closed $(1, c, c)$-knight's tour on the $(c + 1) \times M \times N$ chessboard is possible, where $M = 2c + 1, 2c + 2, \ldots, 3c, 3c + 1, \ldots, 4c - 1$.*

Proof. (1) For the case $M = 2c+1, 2c+2, \ldots, 3c$, the graph $G((1, c, c), c+1, M, N)$ contains the subgraph in Fig. 2. Observe that the two dark vertices $(c, 1, 1)$ and $(c, 2c + 1, 1)$ are of degree 2 and they share the same neighborhood which means that these 4 vertices induce a cycle of length 4 which is contained in Hamiltonian cycle, a contradiction. So, we conclude that no Hamiltonian cycle exists for the $(c + 1) \times M \times N$ chessboard.
(2) For the case $M = 3c + 1, 3c + 2, \ldots, 4c - 1$, the graph $G((1, c, c), c + 1, M, N)$ contains the subgraph in Fig. 3. We just need to observe that the two dark vertices $(c, c, 1)$ and $(c, 3c, 1)$ are of degree 2 and they share the same neighborhood which means that these 4 vertices induce a cycle of length 4 which is contained in a Hamiltonian cycle, a contradiction. Thus, we conclude that no Hamiltonian cycle exists for the $(c + 1) \times M \times N$ chessboard. □

Theorem 4. *Suppose $c > 1$. Then no open $(1, c, c)$-knight's tour on the $(c+1) \times M \times N$ chessboard is possible, where $N < 3c+1$.*

Proof. The graph $G((1, c, c), c+1, M, N)$ contains the subgraph in Fig. 4. The vertices $(c, 1, 1)$, $(c, 1, 2c+1)$ and $(c, 2c+1, 1)$ are of degree 2. For any graph G having a Hamiltonian path, removing any set of k vertices can leave at most $k+1$ connected

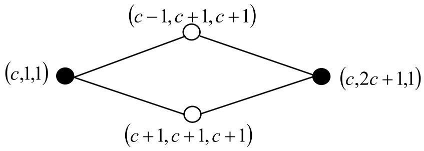

Fig. 2. A subgraph in $G((1, c, c), c + 1, M, N)$, where $M \in \{2c + 1, 2c + 2, \ldots, 3c\}$.

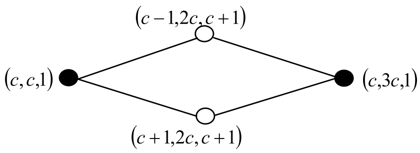

Fig. 3. A subgraph in $G((1, c, c), c + 1, M, N)$, where $M \in \{3c + 1, 3c + 2, \ldots, 4c - 1\}$.

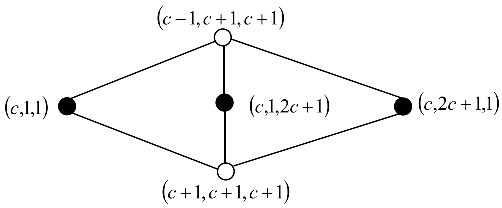

Fig. 4. A subgraph in $G((1, c, c), c + 1, M, N)$, where $M < 3c + 1$, $N < 3c + 1$.

components. Since removing vertices $(c - 1, c + 1, c + 1)$ and $(c + 1, c + 1, c + 1)$ from $G((1, c, c), c + 1, M, N)$ can leave at least four components. We must conclude that no Hamiltonian path exists for the $(c + 1) \times M \times N$ chessboard. □

Theorem 5. *Suppose $L = 2(ak + l)$, where $1 \le k \le l \le a$. Then the $L \times M \times N$ chessboard admits no closed $(a, a + 1, a + 1)$-knight's tour, where a is odd.*

Lemma 1 ([3]). *Suppose the vertices of an $m \times n$ chessboard B are colored in equal amount with two colors, red and blue. Suppose further that every red vertex is adjacent only to the blue vertices and that at least one blue vertex is adjacent to a blue vertex. Then B admits no closed (a, b)-knight's tour.*

It is clear that the Lemma 1 is true for 3D chessboards. We prove Theorem 5 in the following:

Proof. The assertion is a generalization of Theorem 8 in Ref. [3]. The argument is analogous to that for Theorem 8 in Ref. [3], with the modification that the whole $M \times N$ plane chessboard is colored instead of coloring the rows only. □

Similarly, we can prove the following result. It is a 3D version of Theorem 8 of Ref. [3].

Theorem 6. *Suppose $L = 2(ak+l)$, where $1 \le k \le l \le a$. Then the $L \times M \times N$ chessboard admits no closed $(a, a, a+1)$-knight's tour, where a is even.*

Theorem 7. *Suppose $L = a(k + 2l)$ where $1 \le l \le k/2$. Then the $L \times M \times N$ chessboard admits no closed (a, ak, ak)-knight's tour, where a is odd and k is even.*

Proof. This is a 3D version of Theorem 7 of Ref. [3]. Note that in this case, the condition that "k is even" is an imposed condition (unlike Theorem 7 of [3] where "k is even" is a forced condition). The proof is similar to the one given in Ref. [3] with rows replaced by $M \times N$ plane chessboards and with the slight modification mentioned above. □

4. (1, 2, 2)-knight's move

In this section, we shall focus our attention on the $(1, 2, 2)$-knight's move. We can prove the following:

Theorem 8. *The $L \times M \times N$ chessboard with $L \le M \le N$ does not admit a closed $(1, 2, 2)$-knight's tour if one of the following conditions holds:*

(i) *L, M and N are odd;*
(ii) *$L \le 2$;*
(iii) *$L = 3$ and $M \le 7$;*
(iv) *$L = 4$.*

Proof. (i) and (ii) follow directly from Theorem 1. (iii) follows directly from Theorem 3 with $c = 2$. (iv) follows directly from Theorem 5 with $a = k = l = 1$. □

1	46	3	92	121	44	21	24	87	42	19	26	67	38	17	28
4	93	128	45	22	91	110	43	20	25	86	39	18	27	64	37
47	2	95	120	111	122	23	90	101	88	41	66	71	68	29	16
94	5	112	127	124	109	104	107	78	83	70	85	40	65	36	63
113	48	125	96	119	106	123	102	89	100	79	76	69	72	15	30
6	51	118	115	126	103	108	105	82	77	84	73	58	33	62	35
49	114	53	8	97	116	55	10	99	80	57	12	75	60	31	14
52	7	50	117	54	9	98	81	56	11	74	59	32	13	34	61

Fig. 5. 8 × 16 chessboard closed (1, 2)-knight's tour.

1	136	7	274	361	130	61	70	259	124	55	76	199	112	49	82
10	277	382	133	64	271	328	127	58	73	256	115	52	79	190	109
139	4	283	358	331	364	67	268	301	262	121	196	211	202	85	46
280	13	334	379	370	325	310	319	232	247	208	253	118	193	106	187
337	142	373	286	355	316	367	304	265	298	235	226	205	214	43	88
16	151	352	343	376	307	322	313	244	229	250	217	172	97	184	103
145	340	157	22	289	346	163	28	295	238	169	34	223	178	91	40
154	19	148	349	160	25	292	241	166	31	220	175	94	37	100	181

(a) The first row.

284	359	140	5	68	269	332	365	122	197	302	263	86	47	212	203
335	380	281	14	311	320	371	326	209	254	233	248	107	188	119	194
8	275	2	137	62	71	362	131	56	77	260	125	50	83	200	113
383	134	11	278	329	128	65	272	257	116	59	74	191	110	53	80
158	23	146	341	164	29	290	347	170	35	296	239	92	41	224	179
149	350	155	20	293	242	161	26	221	176	167	32	101	182	95	38
374	287	338	143	368	305	356	317	236	227	266	299	44	89	206	215
353	344	17	152	323	314	377	308	251	218	245	230	185	104	173	98

(b) The second row.

3	138	9	276	363	132	63	72	261	126	57	78	201	114	51	84
12	279	384	135	66	273	330	129	60	75	258	117	54	81	192	111
141	6	285	360	333	366	69	270	303	264	123	198	213	204	87	48
282	15	336	381	372	327	312	321	234	249	210	255	120	195	108	189
339	144	375	288	357	318	369	306	267	300	237	228	207	216	45	90
18	153	354	345	378	309	324	315	246	231	252	219	174	99	186	105
147	342	159	24	291	348	165	30	297	240	171	36	225	180	93	42
156	21	150	351	162	27	294	243	168	33	222	177	96	39	102	183

(c) The third row.

Fig. 6. A closed (1, 2, 2)-knight's tour in the 3 × 8 × 16 chessboard.

Theorem 9. *Suppose $L = 3$, $M = 4s$, $N = 4t$, where $s \geq 2$, $t \geq 2$. Then the $L \times M \times N$ chessboard admits a closed* (1, 2, 2)*-knight's tour.*

Proof. First note that, by Theorem 1 of Ref. [7], the $4s \times 4t$ chessboard admits a closed (1, 2)-knight's tour.

Let C denote a closed (1, 2)-knight's tour on the $4s \times 4t$ chessboard. We wish to construct a closed (1, 2, 2)-knight's tour on the $3 \times 4s \times 4t$ chessboard based on C.

Let $C^* = v_1v_2, \ldots, v_nv_1$ denote the resulting Hamiltonian cycle where $v_1, v_4, v_7, \ldots, v_{n-2}$ (respectively v_2, v_5, $v_8, \ldots, v_{n-1}$ and v_3, v_6, $v_9, \ldots, v_n$) are vertices on the first (respectively second and third) row of the $L \times M \times N$ chessboard.

Notice that the vertex $(1, i, j)$ is adjacent to the vertex $(2, r, s)$ where $(r, s) \in \{(i+2, j+2), (i-2, j+2), (i+2, j-2), (i-2, j-2)\}$ and that vertex $(2, r, s)$ adjacent to $(3, i, j)$.

The idea of the construction is described as follows. Let $C = (i_1, j_1)(i_2, j_2) \cdots (i_{4s}, j_{4t})(i_1, j_1)$ denote the given Hamiltonian cycle. Start with the vertex (i_1, j_1) and the edge $(i_1, j_1)(i_2, j_2)$ of C, we construct a path $(1, i_1, j_1)(2, r, s)(3, i_1, j_1)(1, i_2, j_2)$ of C^* where

$$(r, s) = \begin{cases} (i_1 + 2, j_1 + 2) & \text{if } i_1 \pmod 4 \in \{1, 2\}\,,\ j_1 \pmod 4 \in \{1, 2\} \\ (i_1 + 2, j_1 - 2) & \text{if } i_1 \pmod 4 \in \{1, 2\}\,,\ j_1 \pmod 4 \in \{0, 3\} \\ (i_1 - 2, j_1 + 2) & \text{if } i_1 \pmod 4 \in \{0, 3\}\,,\ j_1 \pmod 4 \in \{1, 2\} \\ (i_1 - 2, j_1 - 2) & \text{if } i_1 \pmod 4 \in \{0, 3\}\,,\ j_1 \pmod 4 \in \{0, 3\}\,. \end{cases}$$

Repeat the above construction for the edge $(i_2, j_2)(i_3, j_3)$ by replacing i_1 and j_1 with i_2 and j_2 respectively. Continue in this manner until all the edges of C have been covered. The result is the required Hamiltonian cycle C^*. □

For example, Fig. 5 depicts a closed (1, 2)-knight's tour in the 8×16 chessboard. Fig. 6 depicts a closed (1, 2, 2)-knight's tour in the $3 \times 8 \times 16$ chessboard. Fig. 6(a)–(c) depict the first row, the second row and the third row, respectively of the chessboard.

5. Conclusions

We have considered a new GKTP with (a, b, c) move in a 3D chessboard. In addition to presenting several sufficient conditions for a 3D chessboard not to admit a closed or open GKT with the given move patterns, a constructive method that can be used to find $3 \times 4s \times 4t$ chessboard GKT is given. In [1] the knight's tour is generalized to m-dimensional chessboard and an intelligence algorithm proposed to find a solution of the m-dimensional GKTP. We are attempting to find the sufficient conditions for an m-dimensional chessboard not to admit a closed or open GKT with the given move patterns.

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