THE END OF NUMERICAL ERROR

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We do not need 10^{18} sloppy operations per second that produce rounding errors of unknown size; we need a new foundation for computer arithmetic.

Analogy: Printing in 1970 vs. 2015

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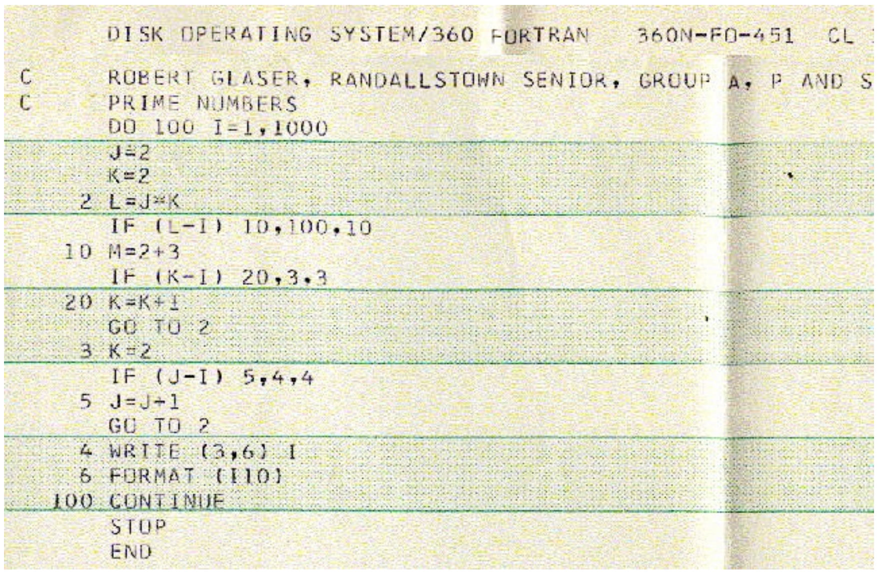
1970: 30 sec per page



```
      DISK OPERATING SYSTEM/360 FORTRAN    360N-FO-451   CL
C     ROBERT GLASER, RANDALLSTOWN SENIOR, GROUP A, P AND S
C     PRIME NUMBERS
      DO 100 I=1,1000
      J=2
      K=2
    2 L=J*K
      IF (L-I) 10,100,10
   10 M=2+3
      IF (K-I) 20,3,3
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      GO TO 2
    3 K=2
      IF (J-I) 5,4,4
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  100 CONTINUE
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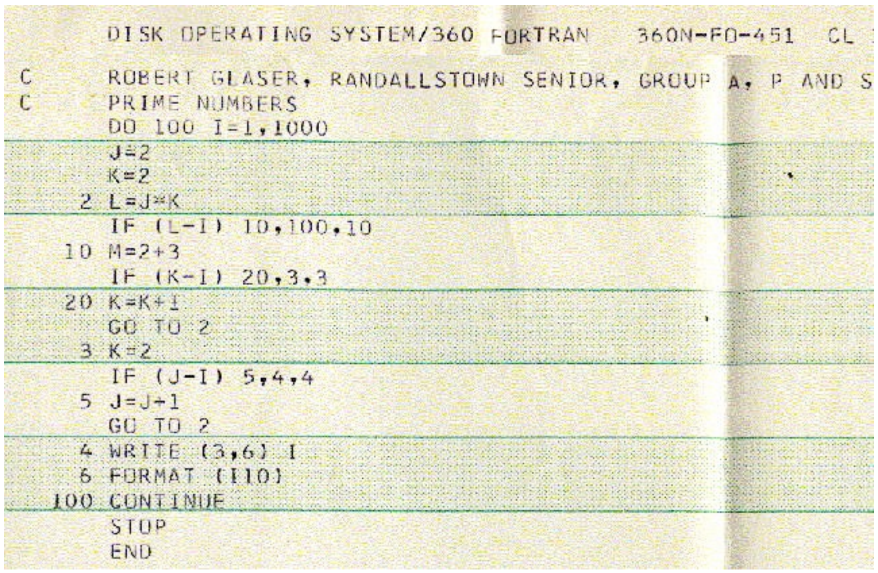

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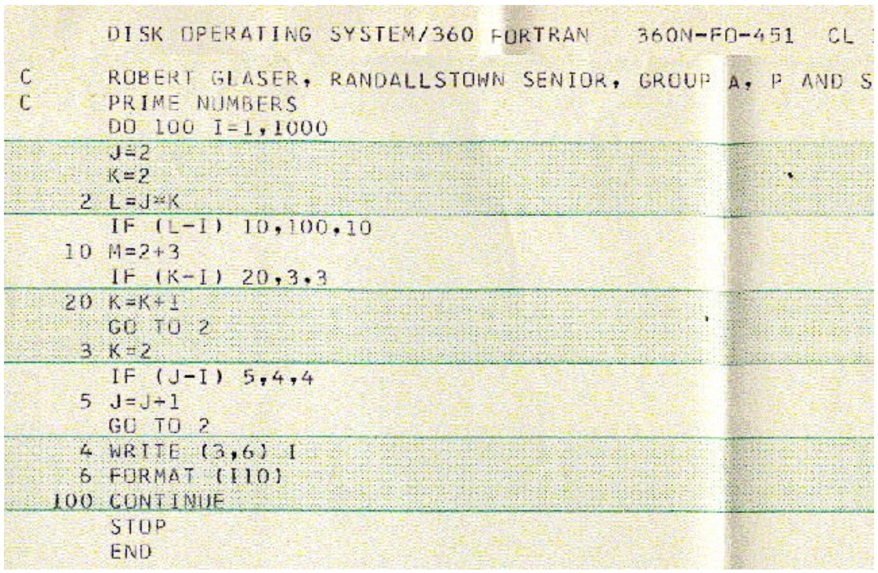

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Faster technology is for *better* prints,
not thousands of low-quality prints per second.
Why not do the same thing with computer arithmetic?

Big problems facing computing

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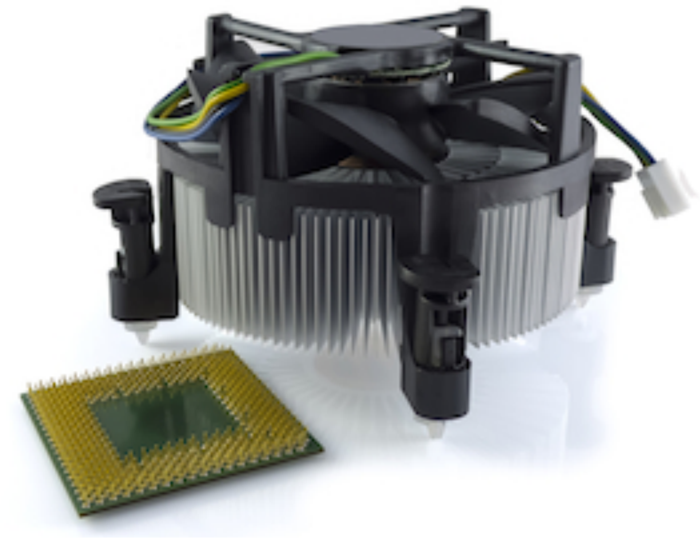

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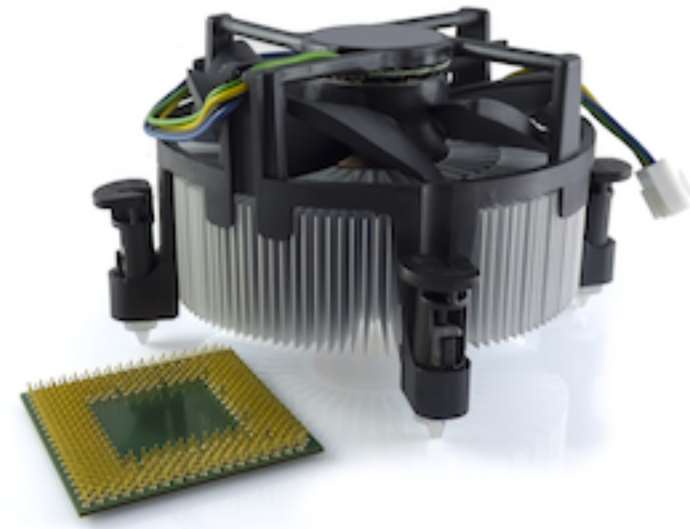

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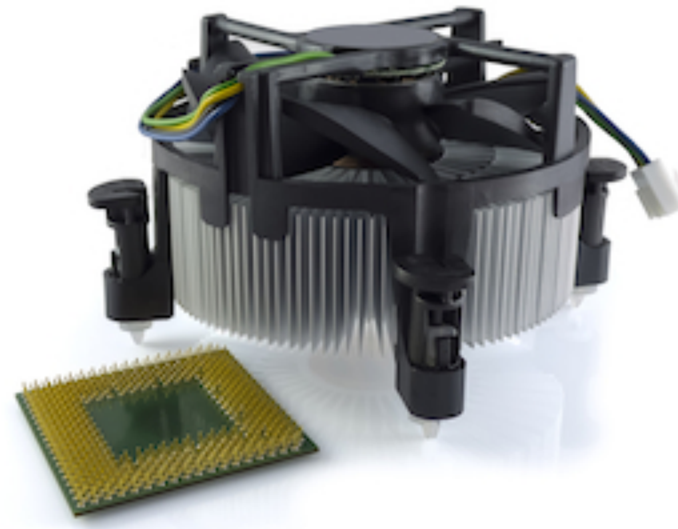

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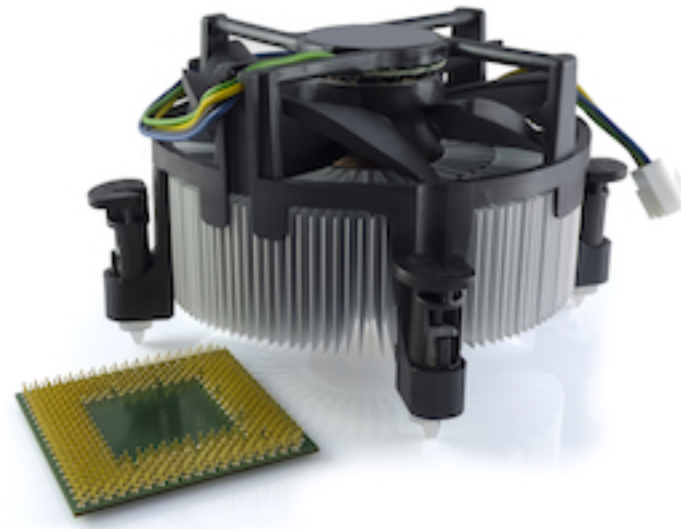

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- *Latency* limits *speed*

More parallel hardware than we can use

- Huge clusters usually partitioned into 10s, 100s of cores
- Few algorithms exploit millions of cores except LINPACK
- *Capacity* is not a substitute for *capability*!

Not enough bandwidth ("Memory wall")

Operation	Energy consumed	Time needed
64-bit multiply-add	200 pJ	1 nsec
Read 64 bits from cache	800 pJ	3 nsec
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One-size-fits-all overkill 64-bit precision
wastes energy, storage, bandwidth

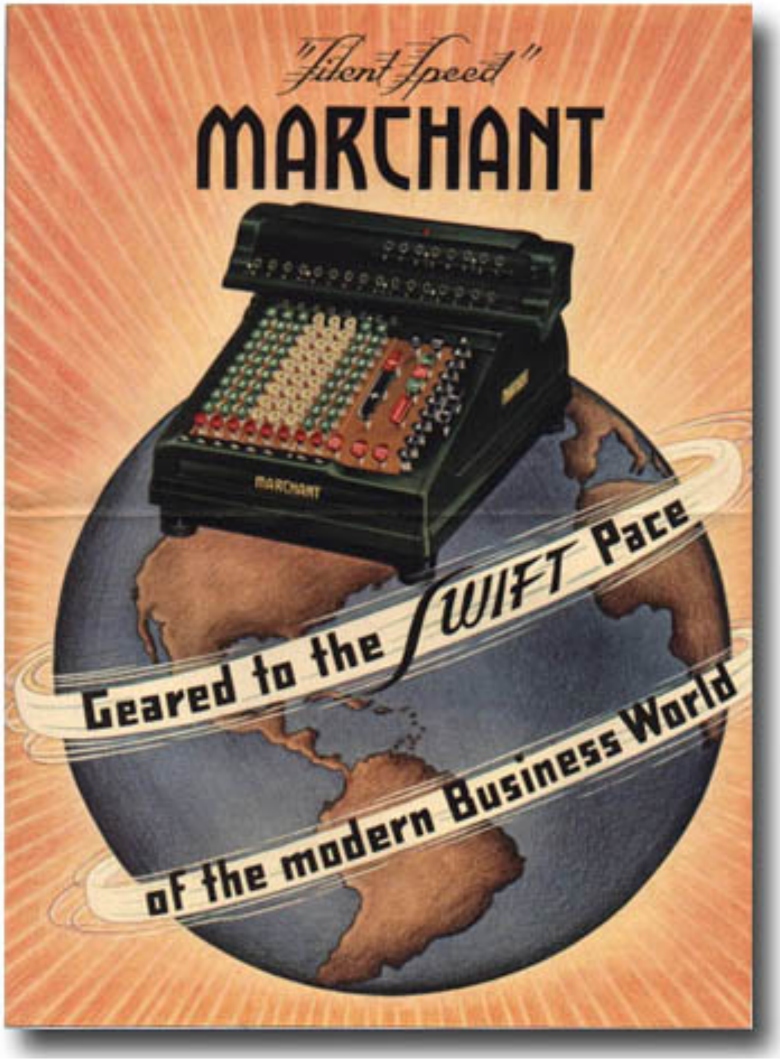
Happy 101st Birthday, Floating Point

1914: Torres y Quevedo proposes automatic computing with fraction & exponent.
2015: We still use a format designed for World War I hardware capabilities.

The “Original Sin” of Computer Math

“The computer cannot give you the exact value, sorry. Use *this* value instead. It’s close.”

Floats worked for *visible* scratch work



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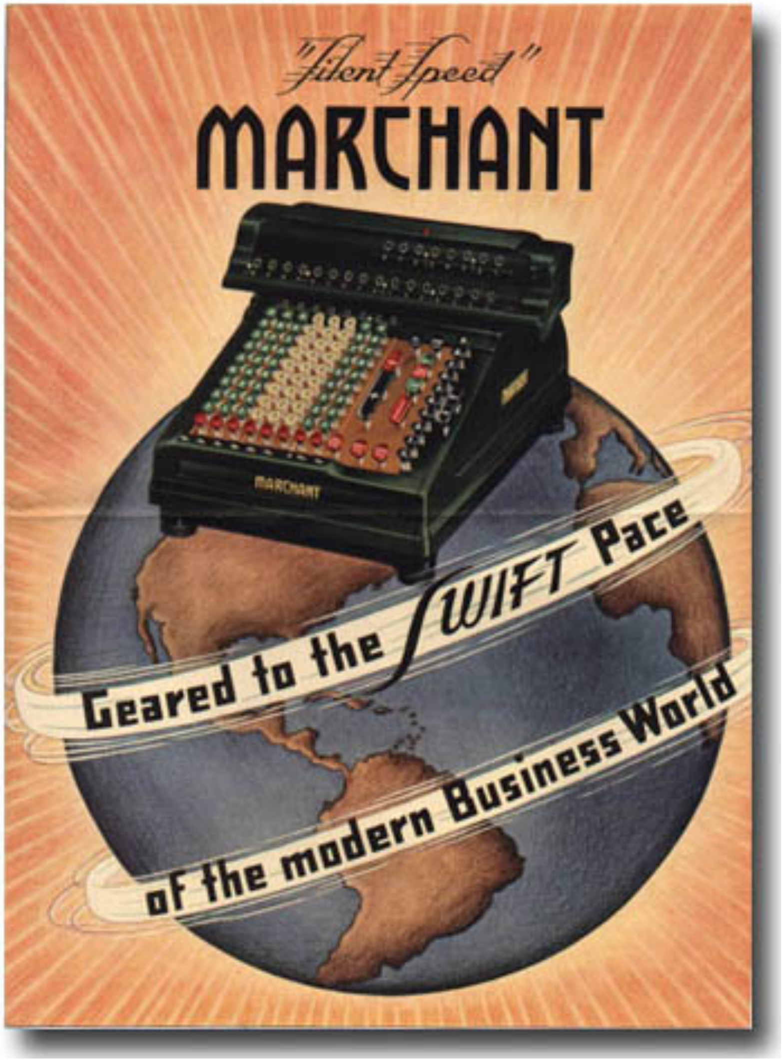

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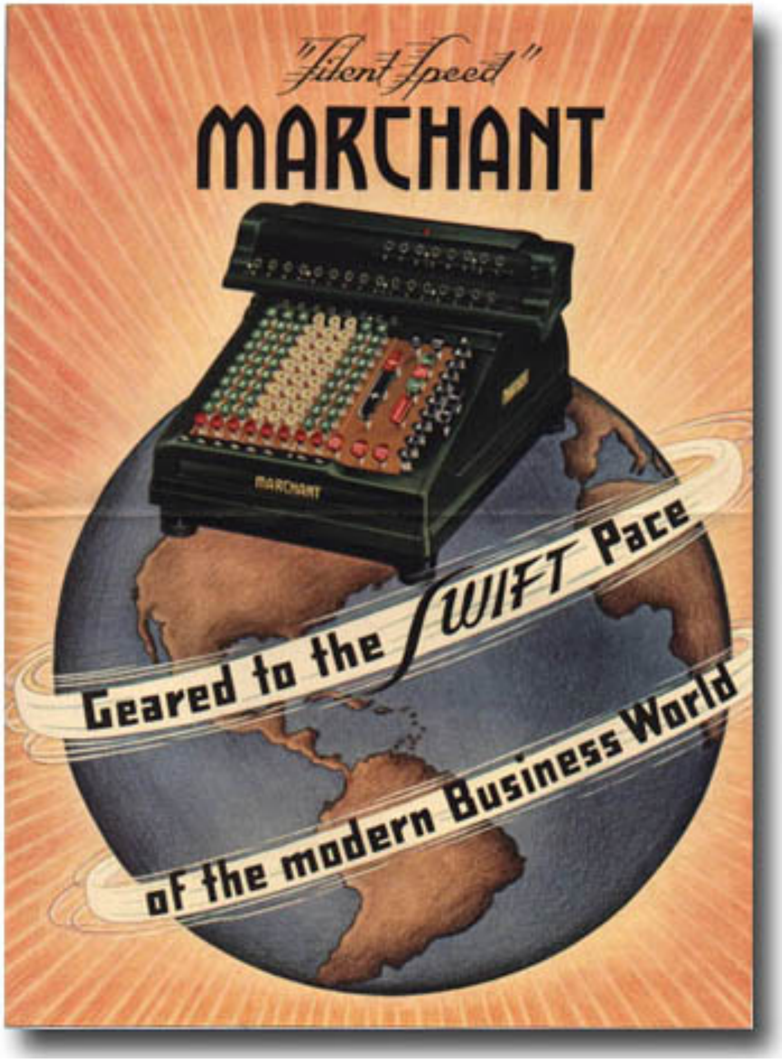

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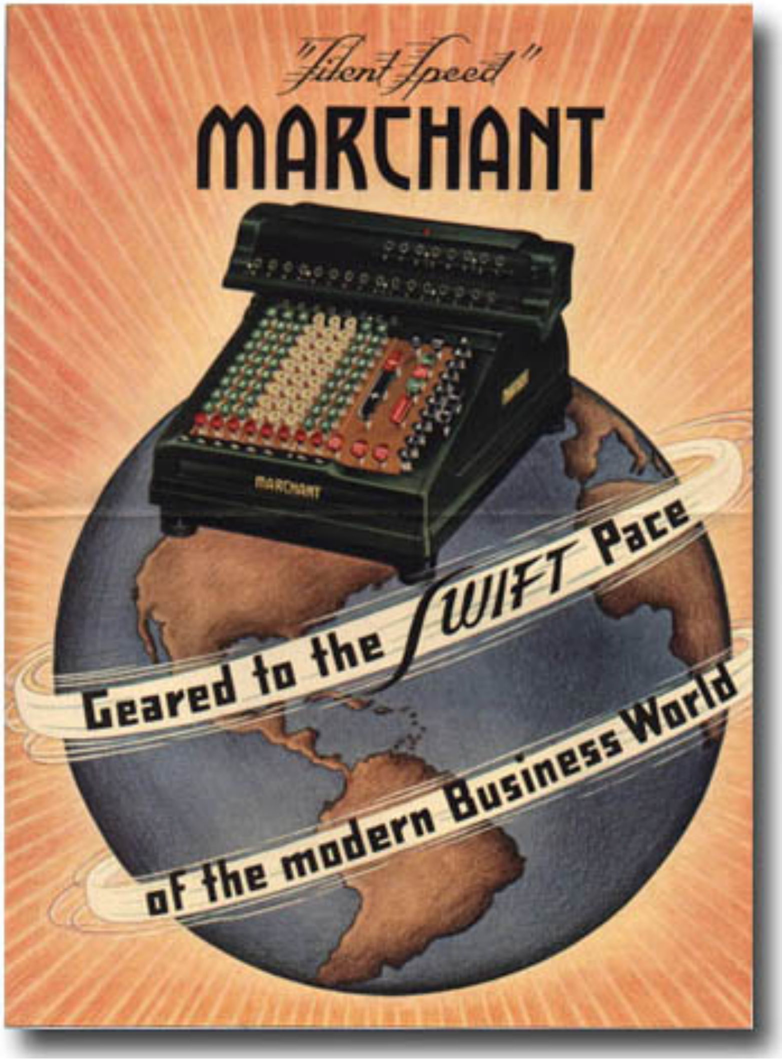

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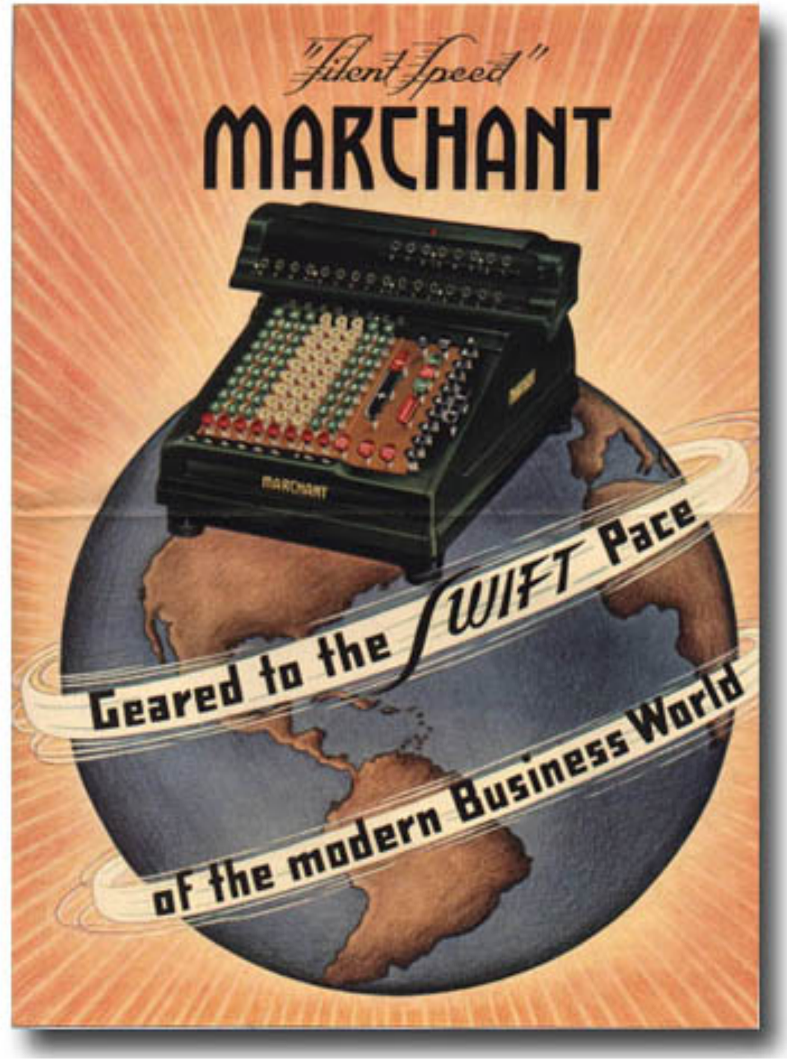

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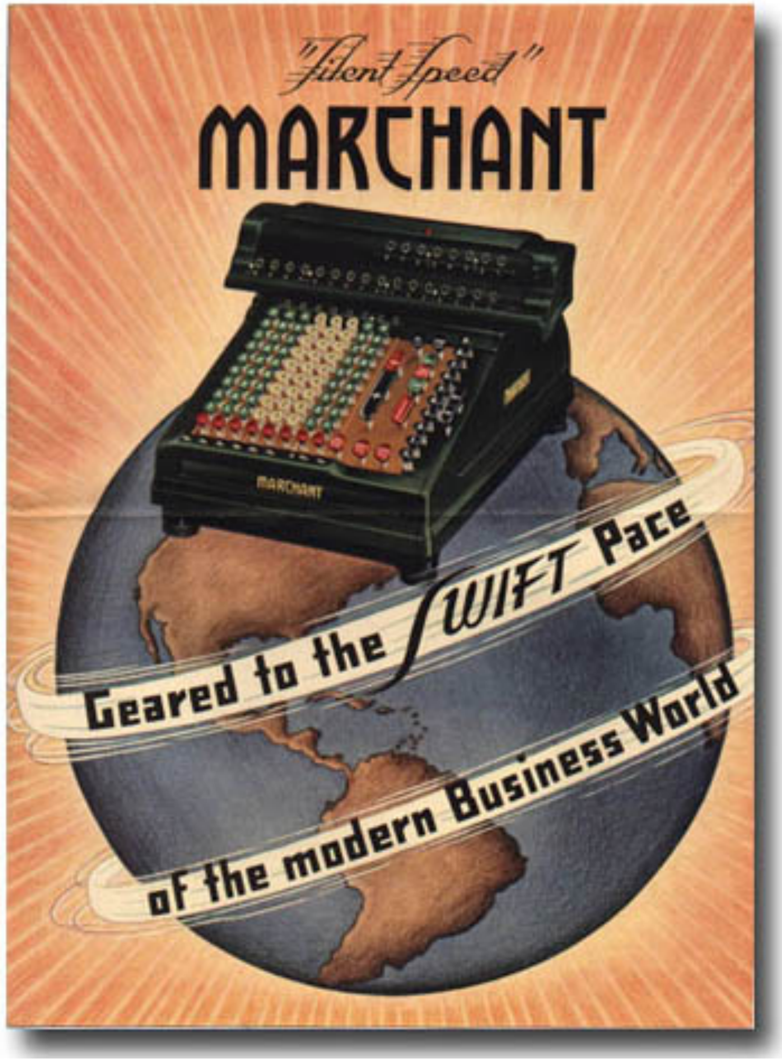

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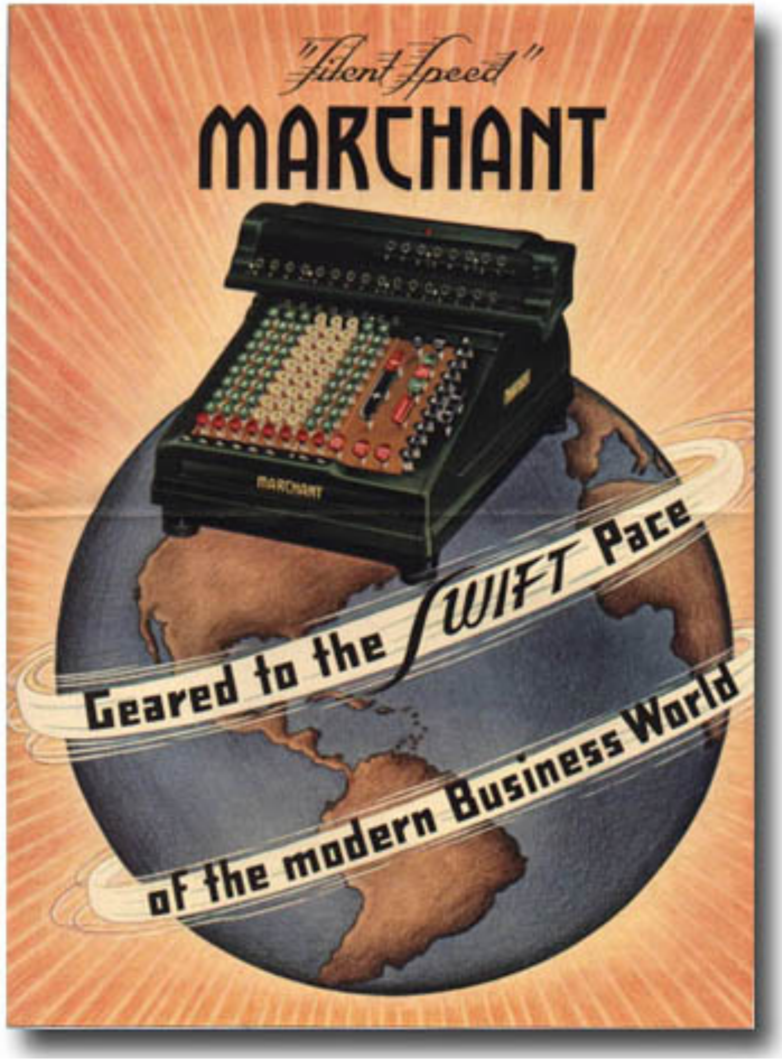
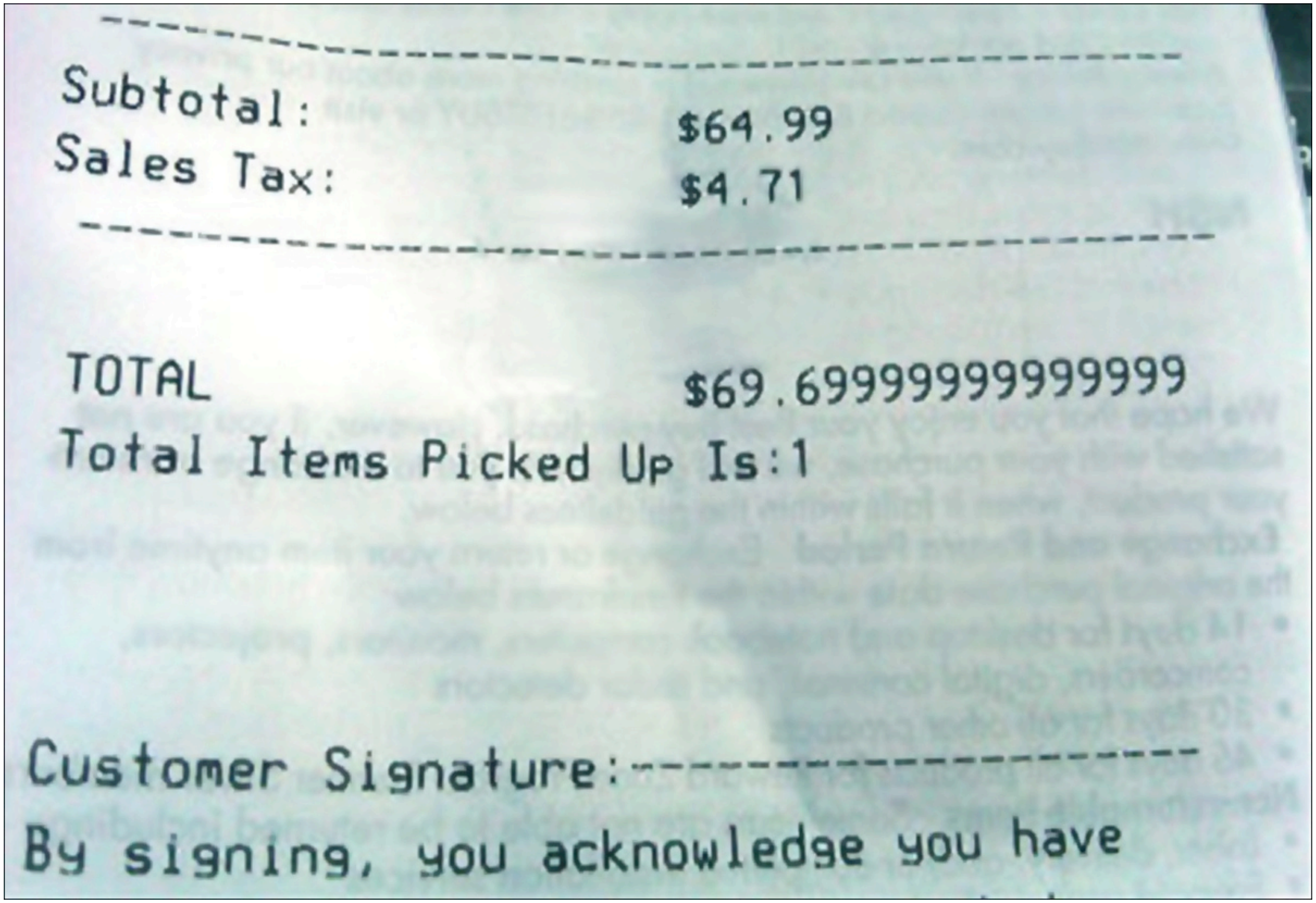


This is just… sad.



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- IEEE floats report rounding, overflow, underflow in *processor register bits that no one ever sees.*

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"You can't boil the ocean."

—*Former Intel exec, when shown the unum idea*

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Correct: $\pi = 3.14\cdots$

The latter means 3.14 < pi < 3.15, a **true statement**.

Presence or absence of the "…" is the *ubit*, just like a sign bit. It is 0 if exact, 1 if there are more bits after the last fraction bit, not all 0s and not all 1s.

Three ways to express a big number

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sign

Lots of digits

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sign	exponent (scale)	fraction

Unum (29 bits):

				utag	
0	11001101	111111100001	1	111	1011
sign	exp.	frac.	ubit	exp. size	frac. size

Self-descriptive "utag" bits track and manage uncertainty, exponent size, and fraction size

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- Double precision float range: 10^{632}

Why unums use fewer bits than floats

- Exponent smaller by about 5 – 10 bits, typically
- Trailing zeros in fraction compressed away, saves ~2 bits
- Shorter strings for more common values
- Cancellation removes bits and the need to store them

IEEE Standard Float (64 bits):

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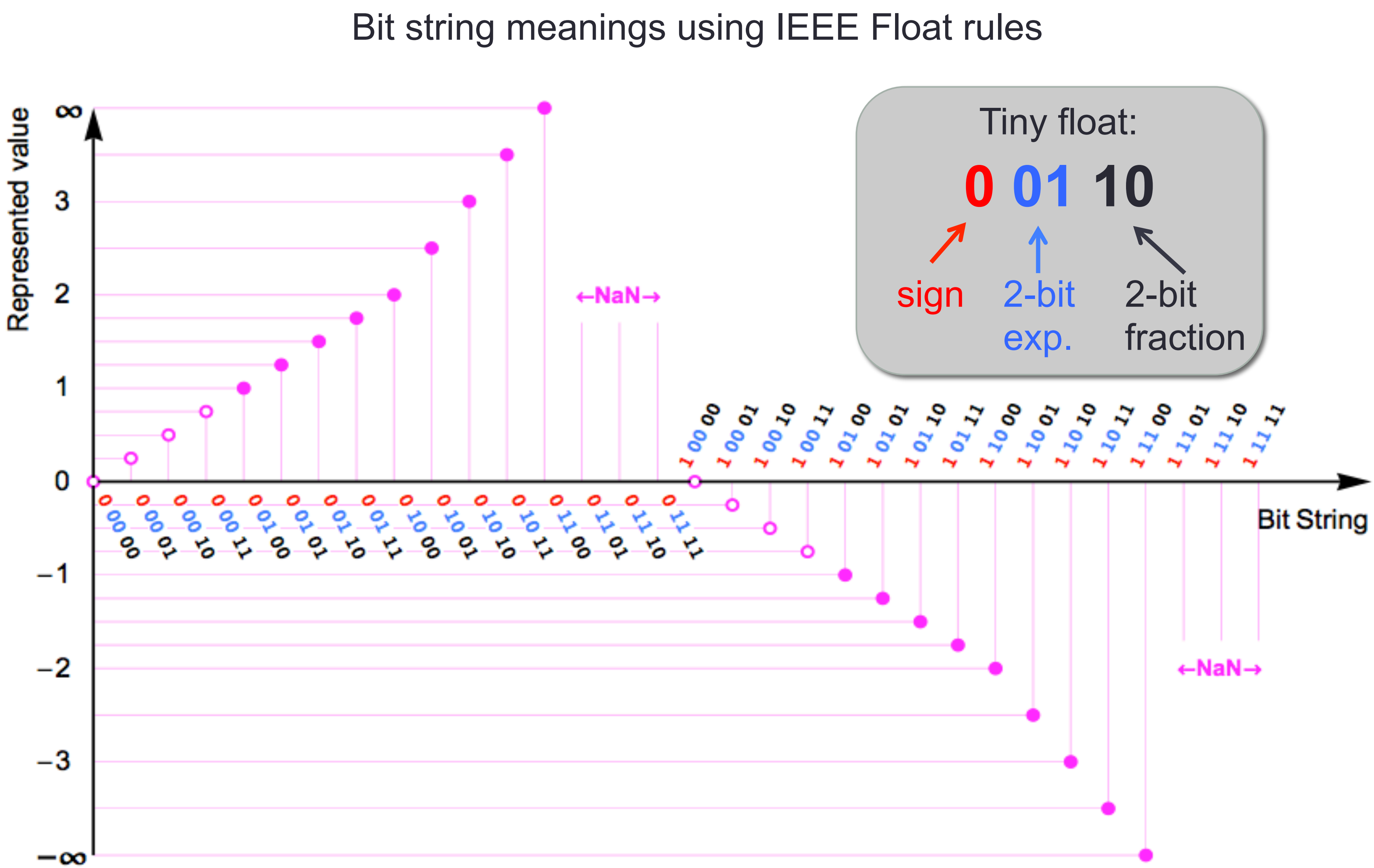
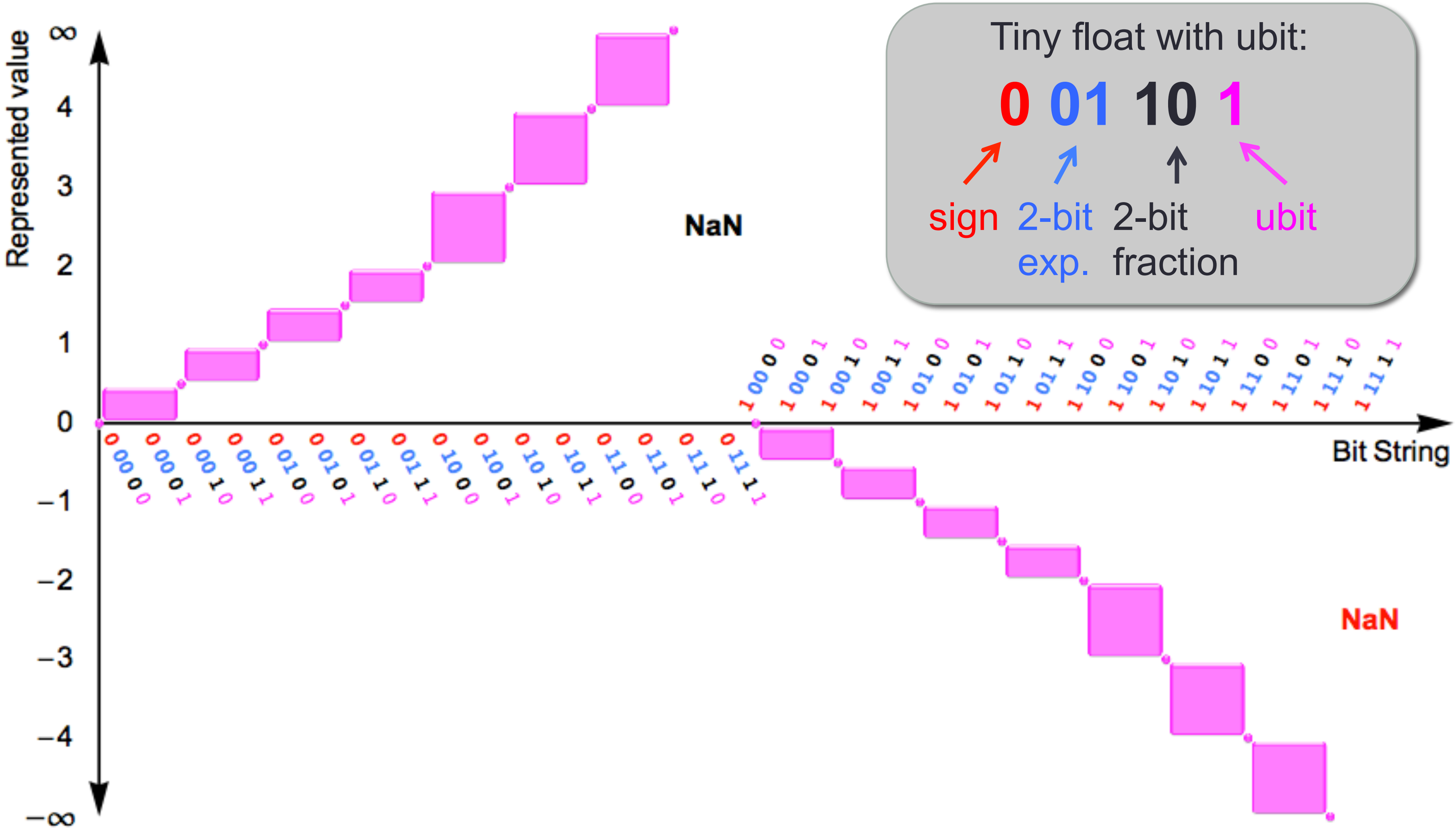
Value plot of tiny IEEE-style floats



Open *ranges*, as well as exact points

Complete representation of *all* real numbers using a finite number of bits



Cc

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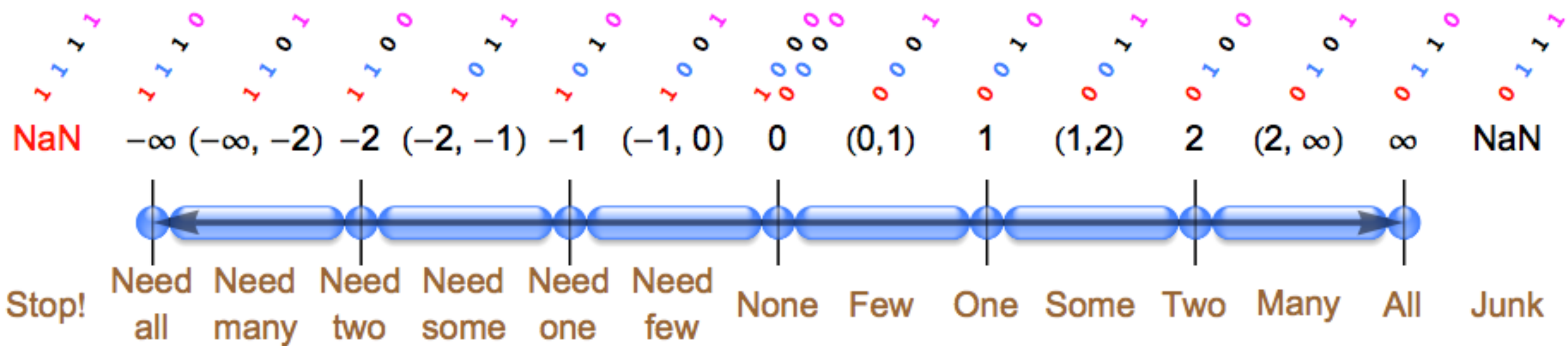

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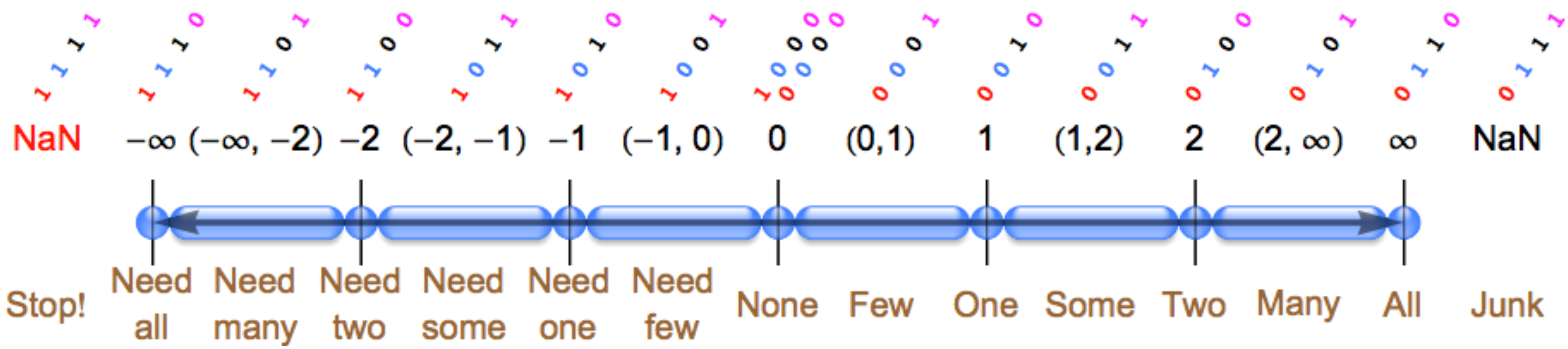

“Try everything” methods become *feasible*.

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- Warlpiri ubounds are one byte, but closed system for reals

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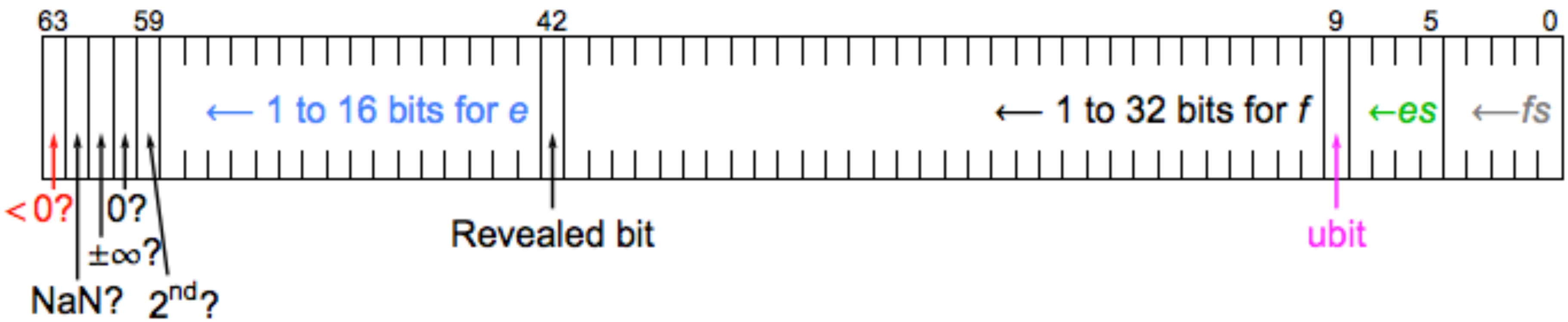

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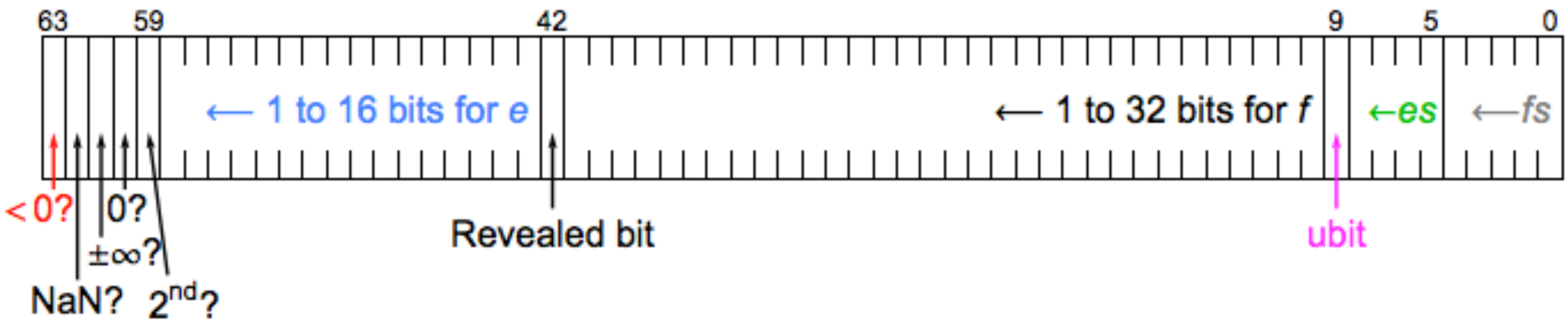

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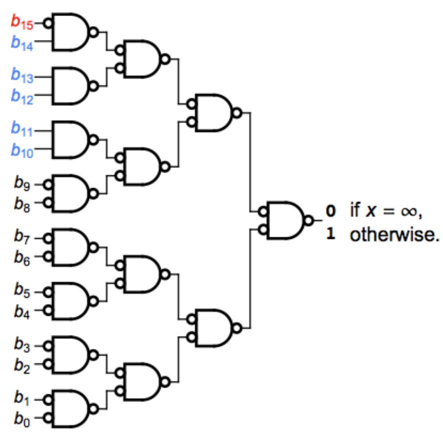

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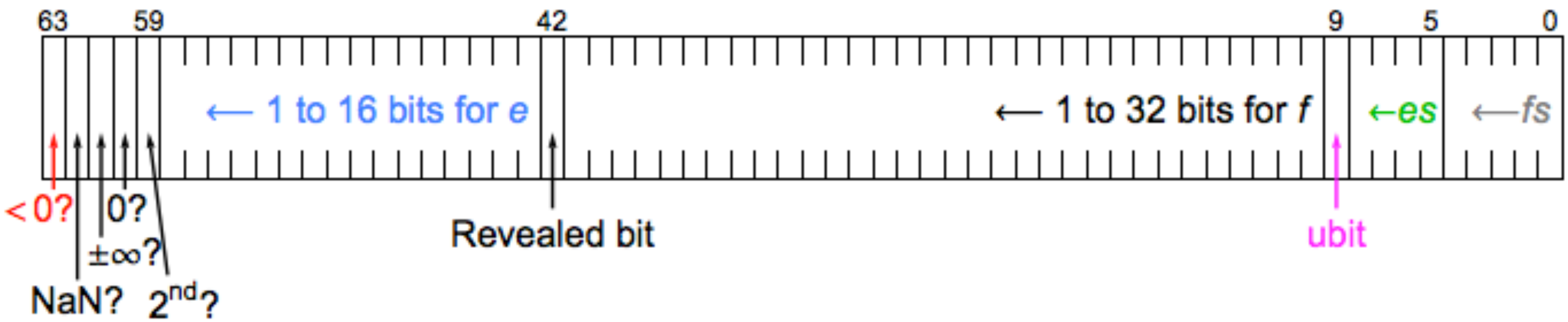

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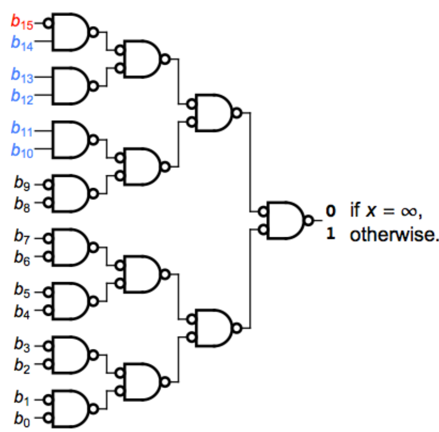

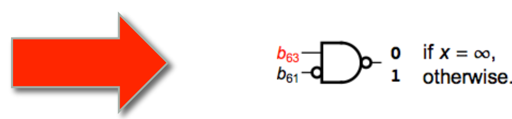

Circuit required for "unum = ∞?" (any precision)

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Working unum environment completed August 13, 2013.

Floating Point II: The Wrath of Kahan

- Berkeley professor William Kahan is the father of modern IEEE Standard floats
- Also the authority on their many dangers
- Every idea to fix floats faces his tests that expose how new idea is *even worse*

Working unum environment completed August 13, 2013.

Can unums survive the wrath of Kahan?

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I have been unable to find a problem that "breaks" unum math.

Kahan's "Smooth Surprise"

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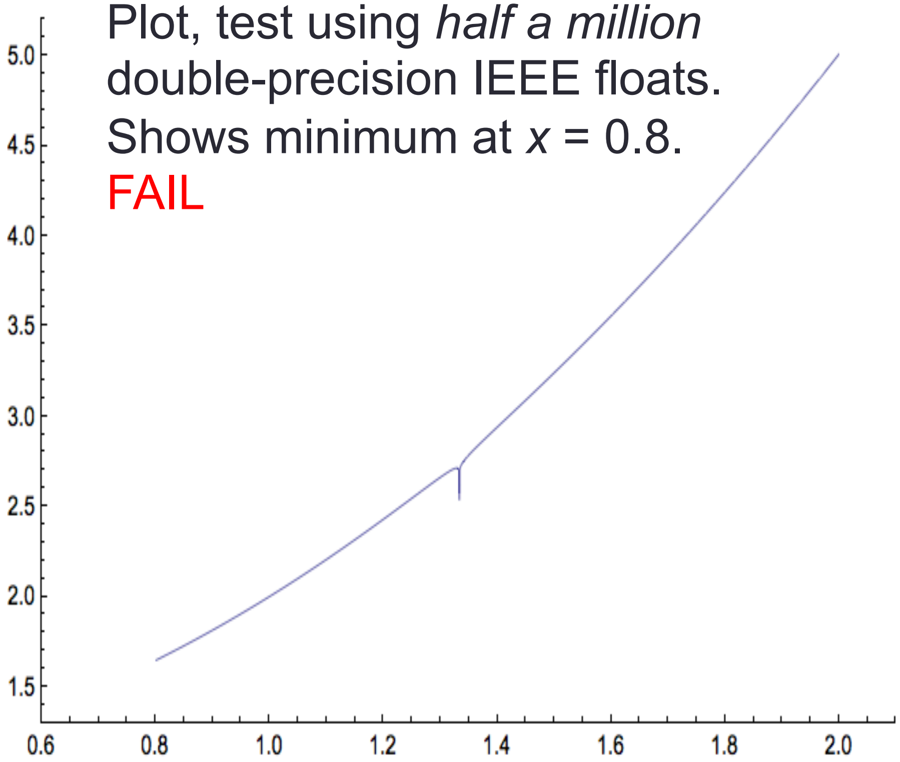

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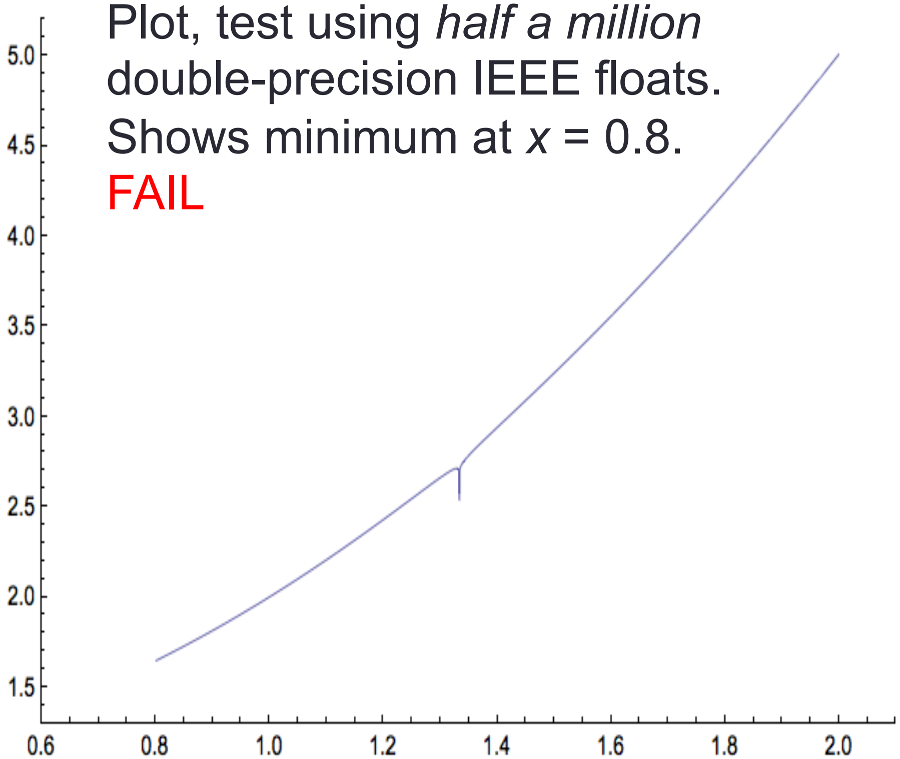

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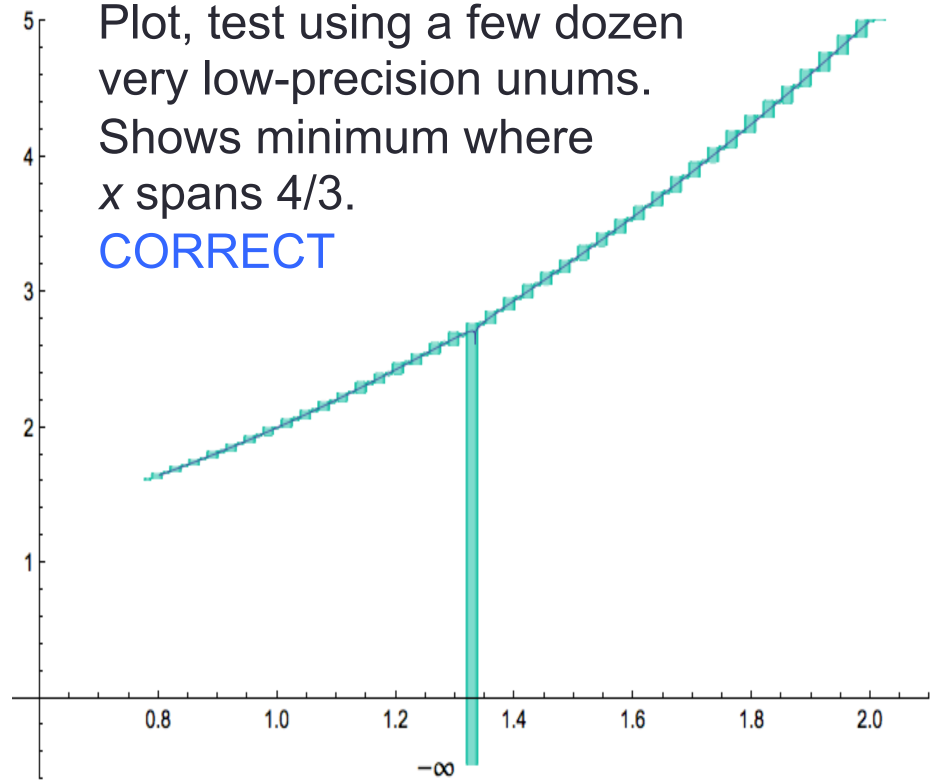

Plot, test using a few dozen very low-precision unums. Shows minimum where x spans 4/3.
CORRECT

Rump's Royal Pain

Compute $333.75y^6 + x^2(11x^2y^2 - y^6 - 121y^4 - 2) + 5.5y^8 + x/(2y)$
where $x = 77617$, $y = 33096$.

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Unums: **Correct answer** to 23 decimals using an average of only ***75*** bits per number. Not even IEEE 128-bit precision can do that. Precision, range adjust *automatically*.

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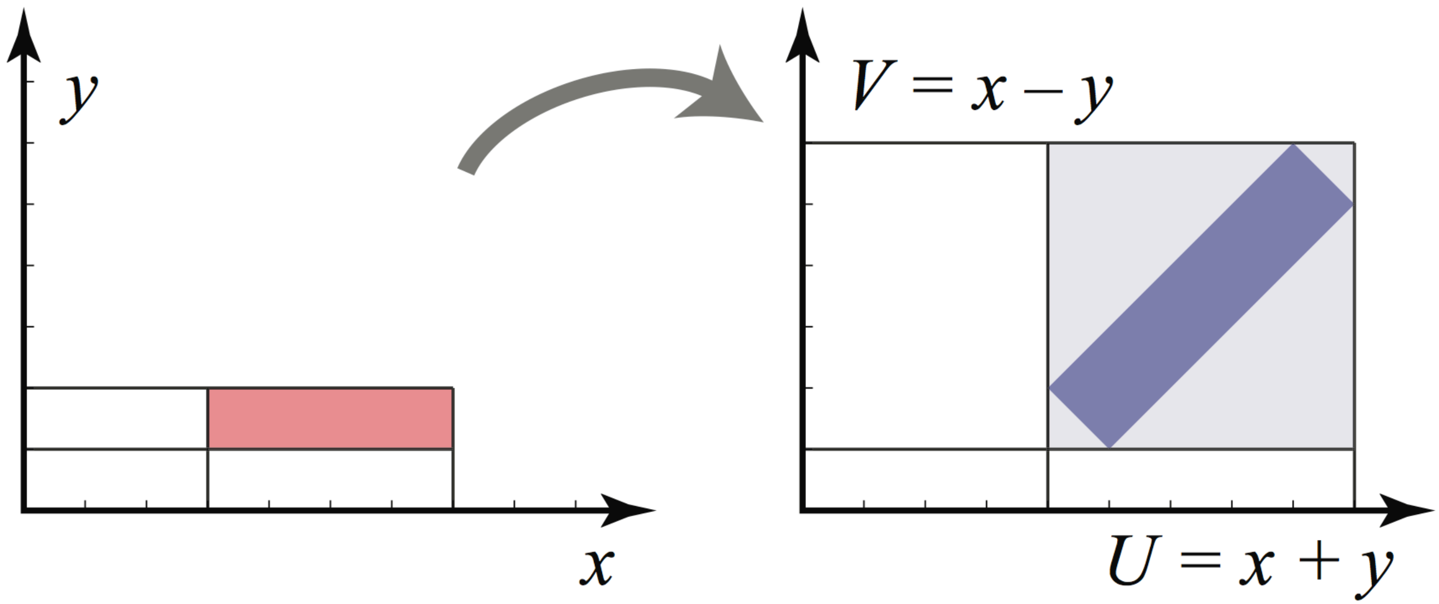
- Computer bears primary numerical analysis burden

Reason 1 why interval math hasn't displaced floats: The "Wrapping Problem"



Answer sets are complex shapes in general, but interval bounds are axis-aligned boxes, period.

No wonder interval bounds grow far too fast to be useful, in general!

Reason 2: The Dependency Problem

What wrecks interval arithmetic is simple things like

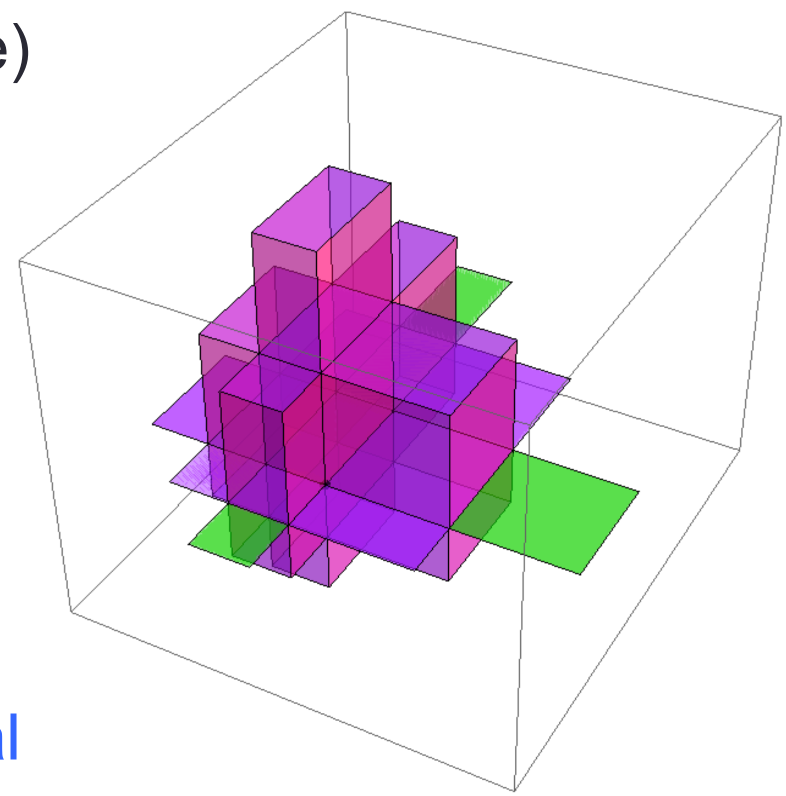
$$F(x) = x - x.$$

Should be 0, or maybe $[-\varepsilon, +\varepsilon]$. Say x is the interval [3, 4], then interval $x - x$ stupidly evaluates to [–1, +1], which doubles the uncertainty (interval width) and makes the interval solution far inferior to the point arithmetic method.

The unum architecture solves both drawbacks of traditional interval arithmetic.

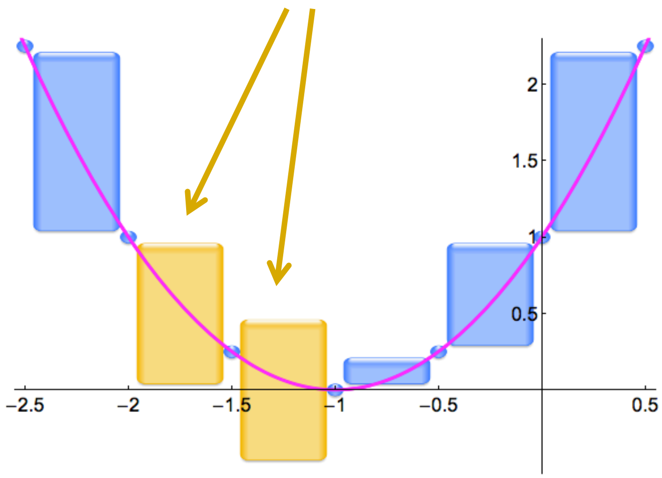
Uboxes and solution sets

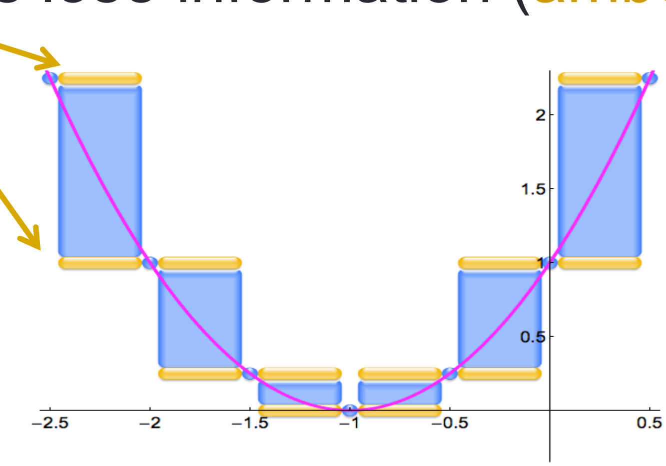
- A *ubox* is a multidimensional unum
- Exact or ULP-wide in each dimension (Unit in the Last Place)
- Sets of uboxes constitute a *solution set*
- One dimension per degree of freedom in solution
- Solves the main problems with interval arithmetic
- Super-economical for bit storage
- Massively data parallel in general



Polynomials: bane of classic intervals

Dependency and closed endpoints lose information (amber)





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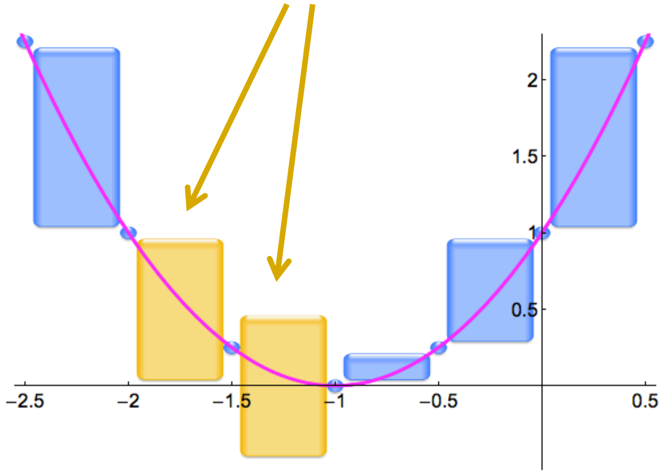
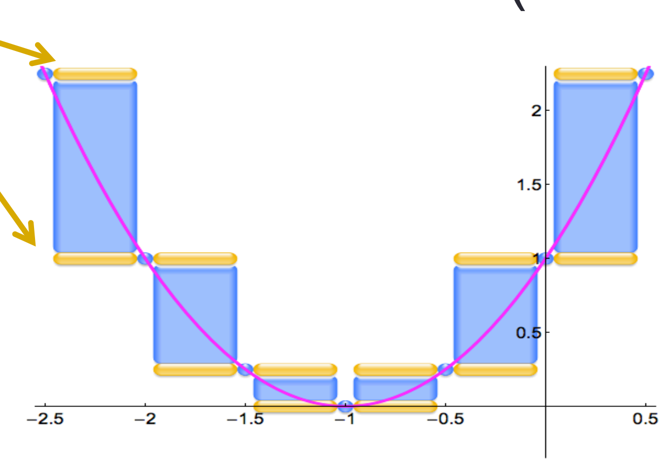
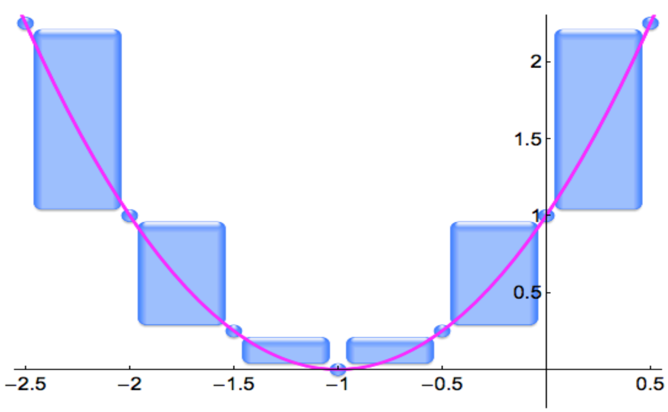


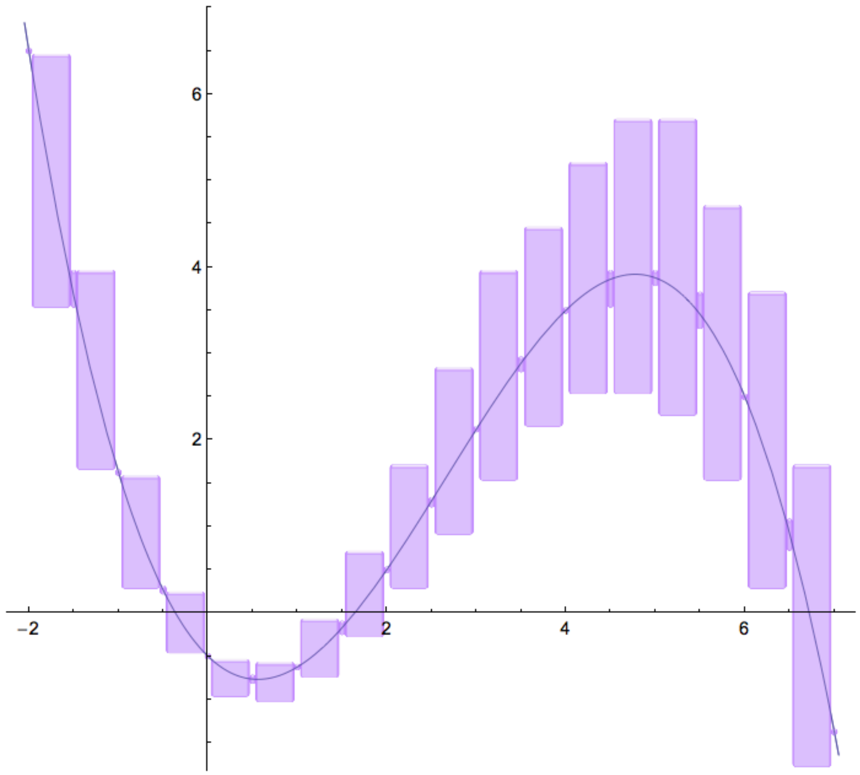




Unum polynomial evaluator loses *no* information.

Polynomial evaluation solved at last

Mathematicians have sought this for at least 60 years.



“Dependency Problem” creates sloppy range when input is an interval

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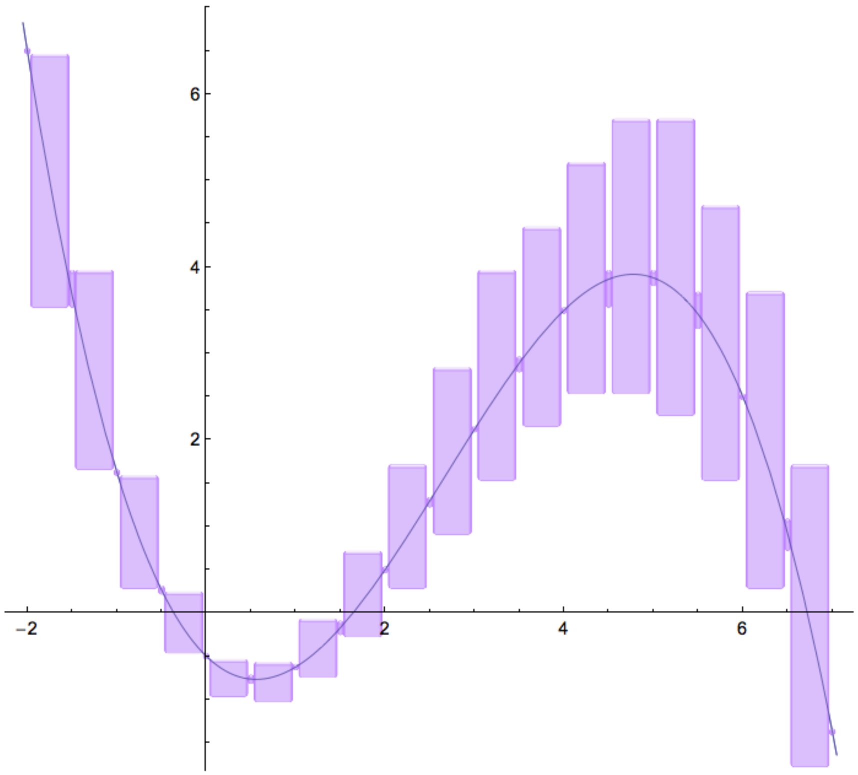

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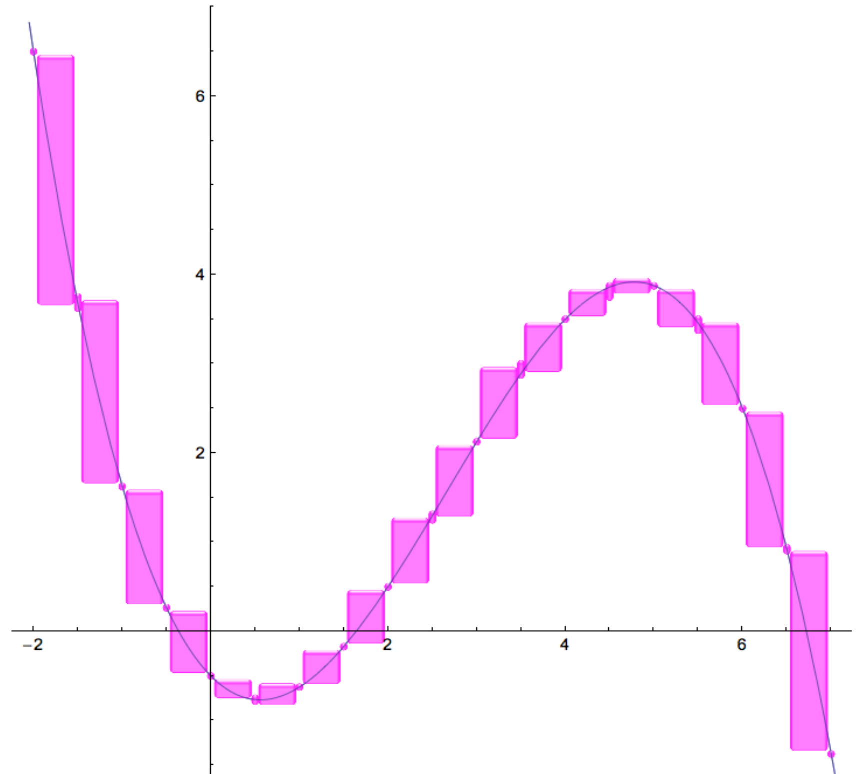
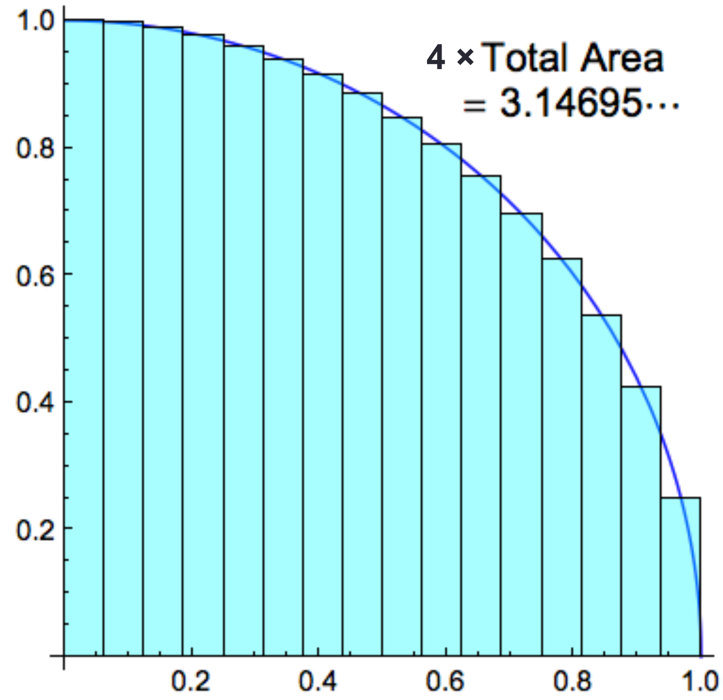


Unum evaluation refines answer to limits of the environment precision

The Deeply Unsatisfying Error Bounds of Classical Analysis

- Classical numerical texts teach this "error bound":

$$\text{Error} \le (b - a)\, h^2\, |f''(\xi)| / 24$$



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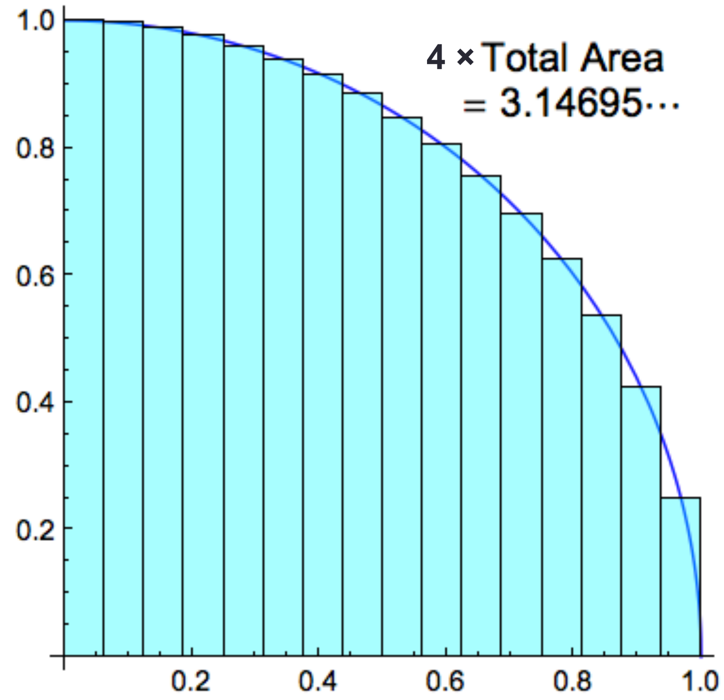

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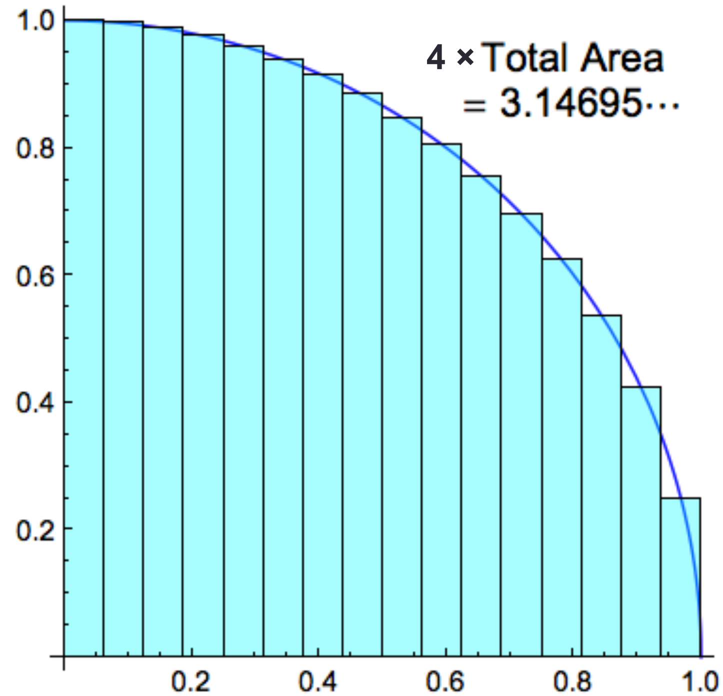

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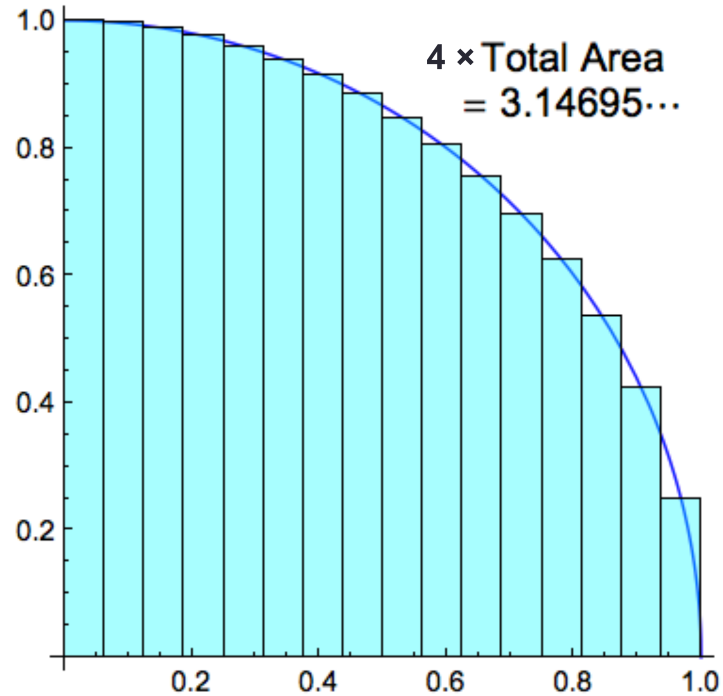

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- Bound is often *infinite*, which means no bound at all
- "Whatever it is, it's four times better if we make *h* half as big" creates supercomputing demand that *can never be satisfied*.

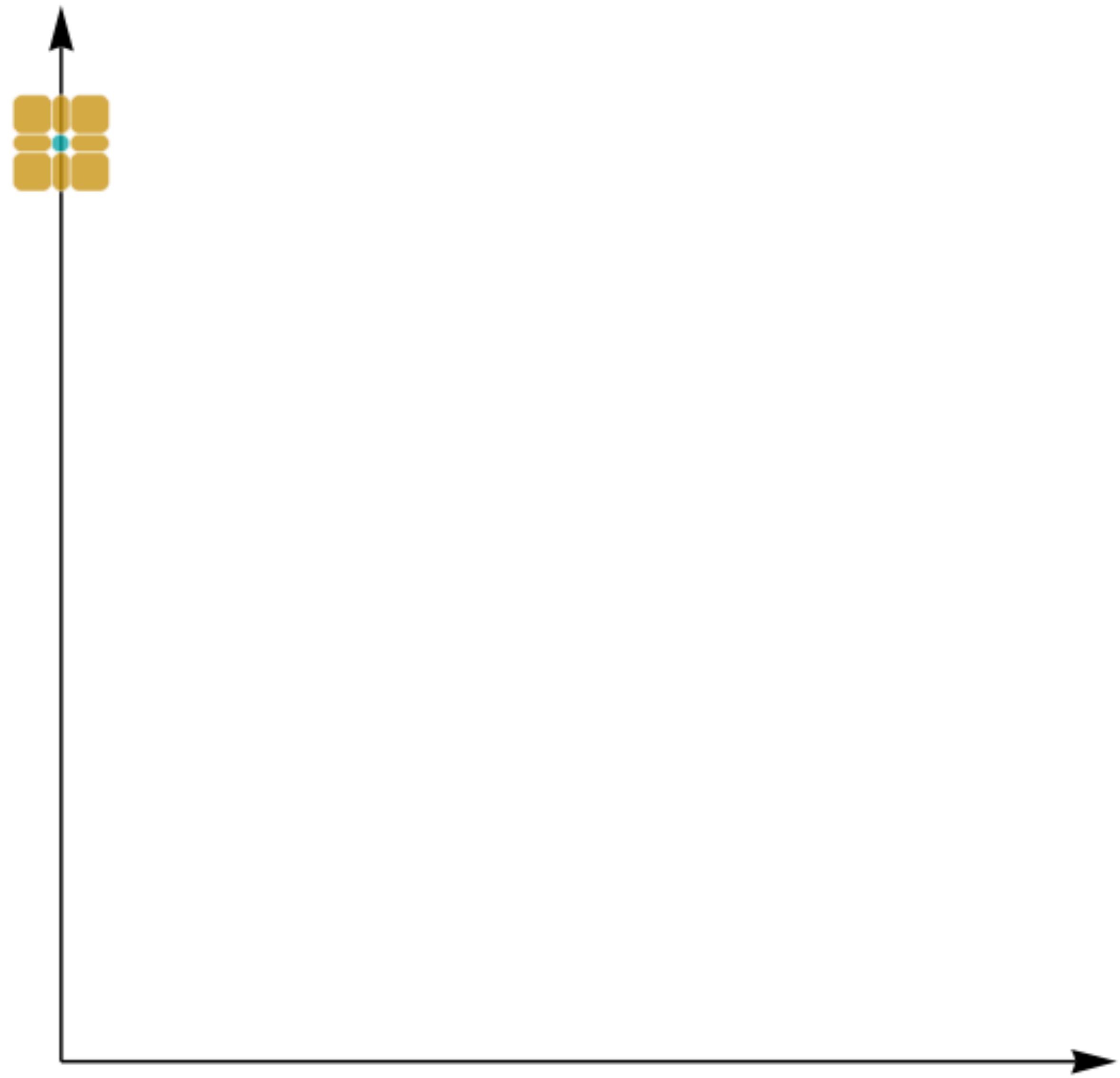
Quarter-circle example

- Suppose all we know is $x^2 + y^2 = 1$, and x and y are ≥ 0.
- Suppose we have at most ***2*** bits exponent, ***4*** bits fraction.

Task:
Bound the quarter circle area.
(i.e., bound the value of $\pi/4$)
Create the pixels for the shape.

Set is connected; need a seed

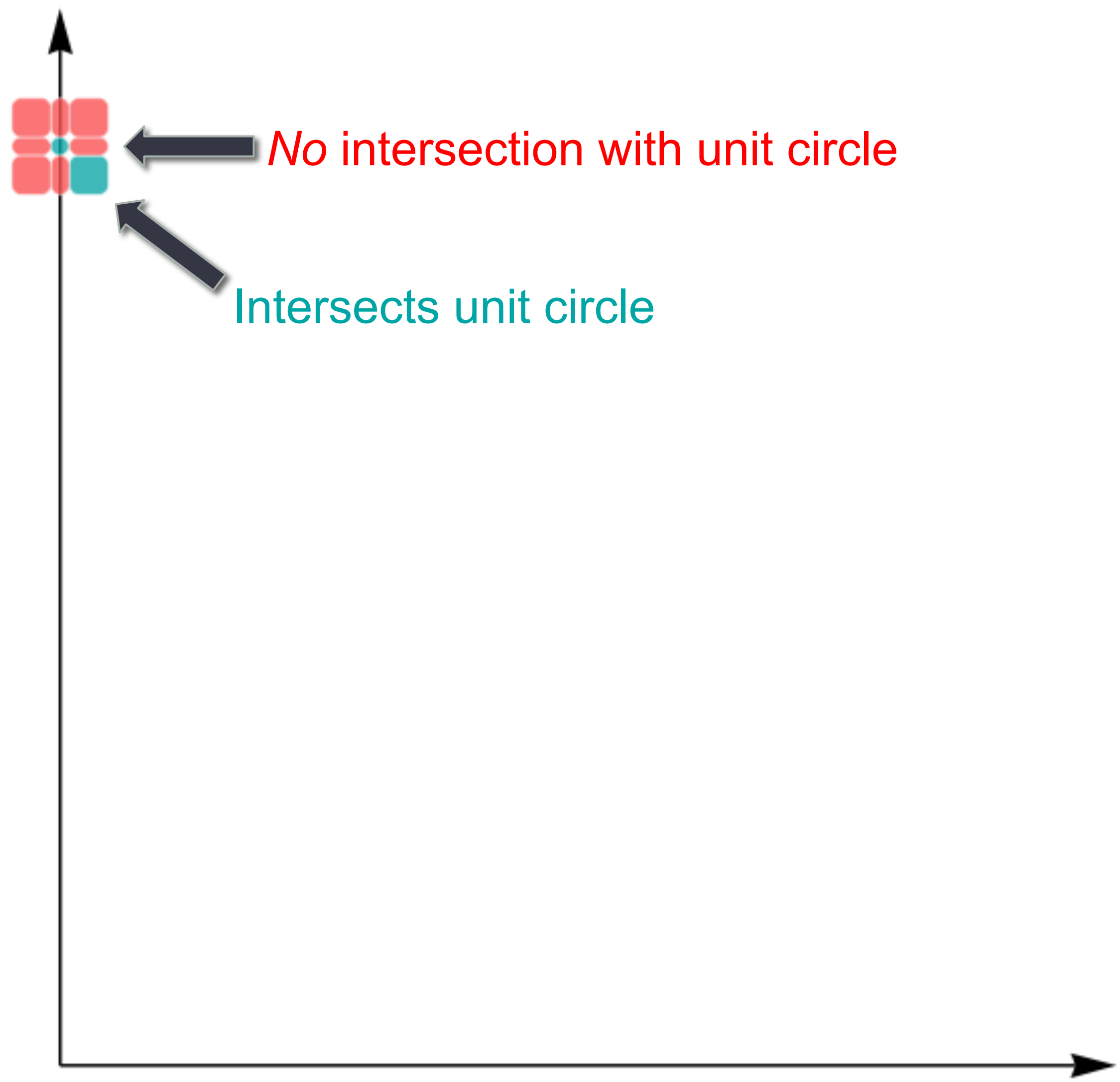
- We know $x = 0$, $y = 1$ works
- Find its 8 ubox neighbors in the plane
- Test $x^2 + y^2 = 1$, $x \geq 0$, $y \geq 0$
- Solution set is green
- Trial set is amber
- Failure set is red
- Stop when no more trials



Exactly one neighbor passes test

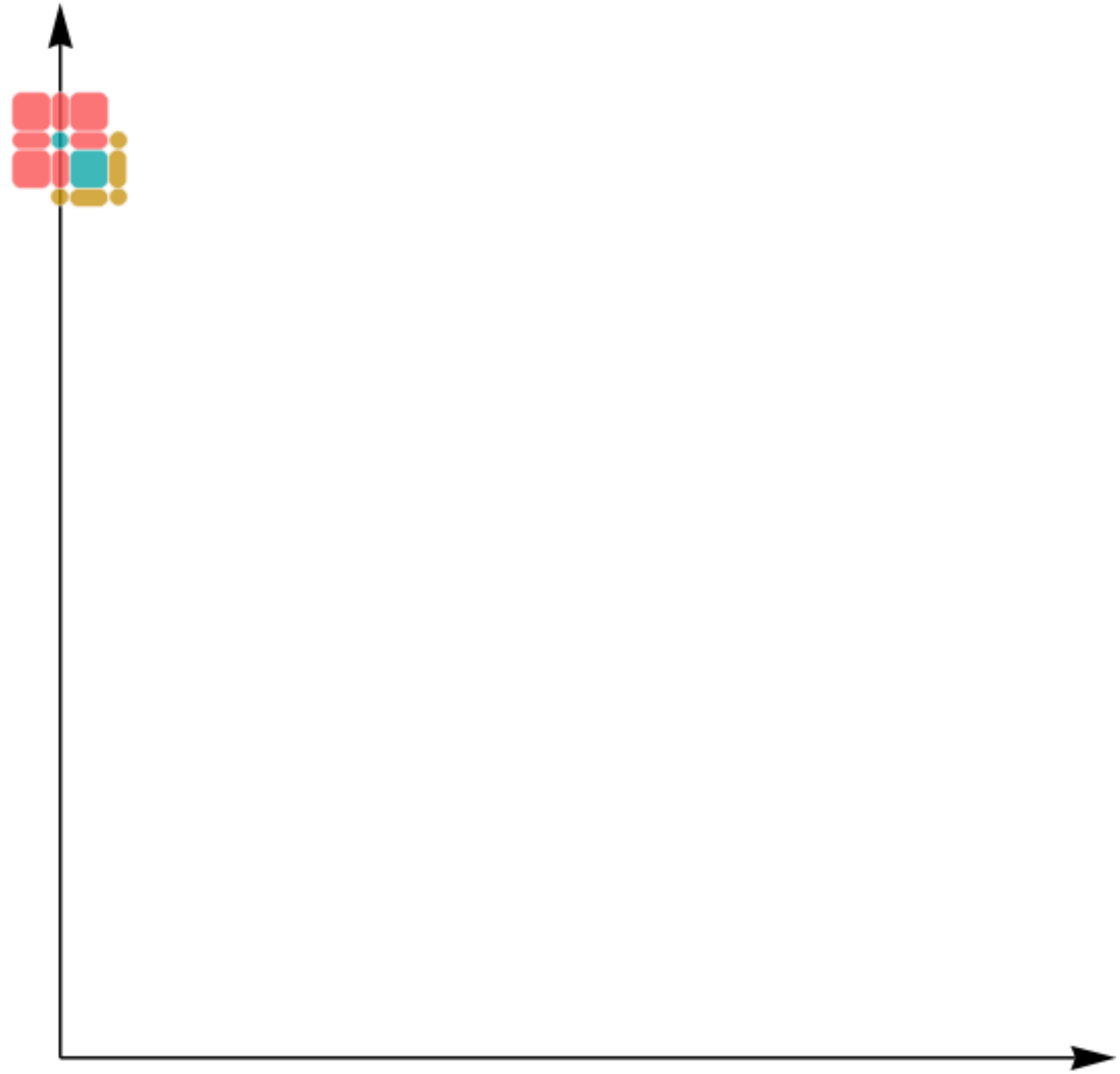
- Unum math automatically *excludes cases that floats would accept*
- Trials are neighbors of new solutions that
  - Are not already failures
  - Are not already solutions
- Note: *no* calculation of

$$y = \sqrt{1 - x^2}$$



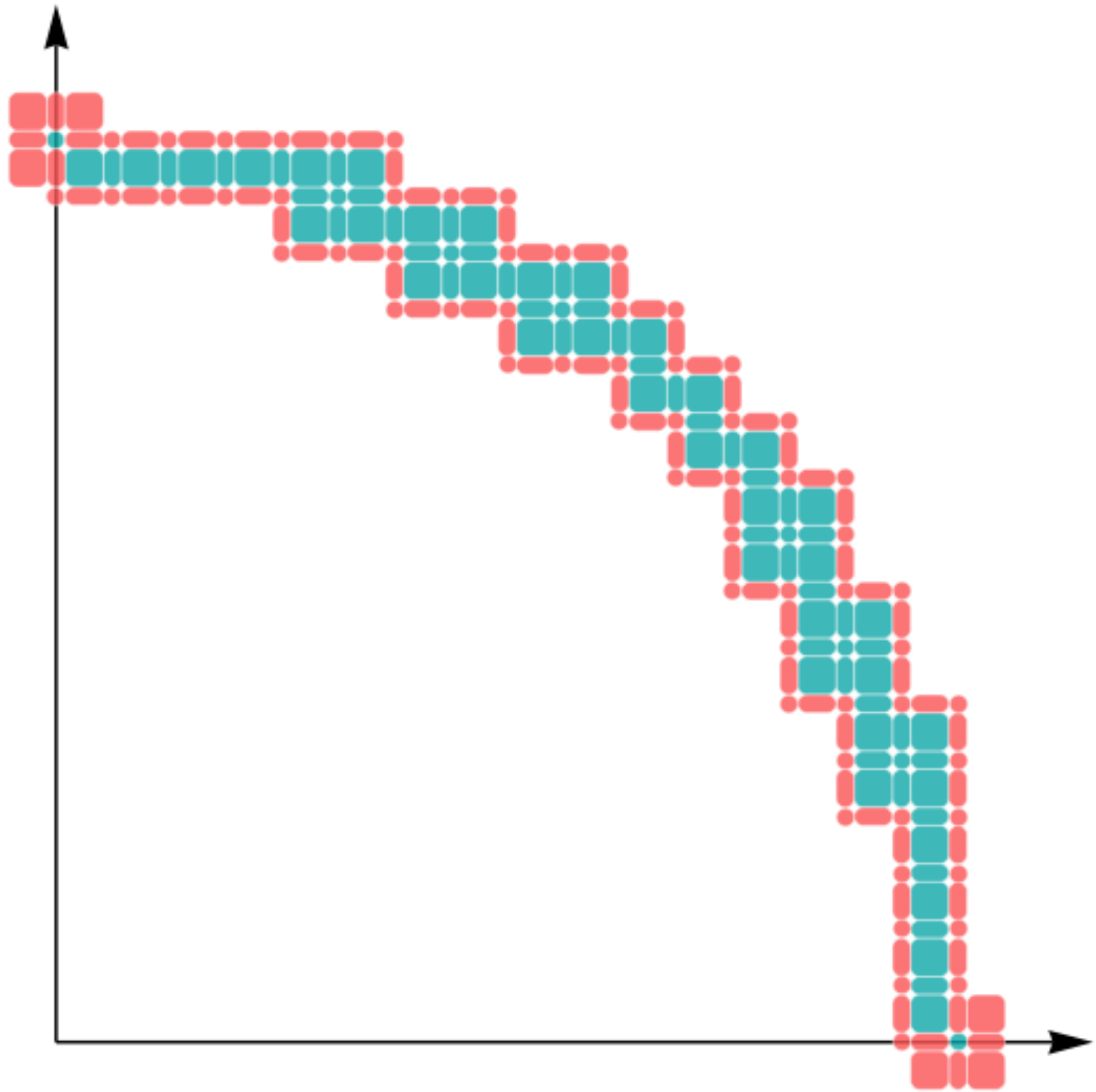
The new trial set

- Five trial uboxes to test
- Perfect, easy parallelism for multicore
- Each ubox takes only 15 to 23 bits
- Ultra-fast operations
- Skip to the final result…



The complete quarter circle

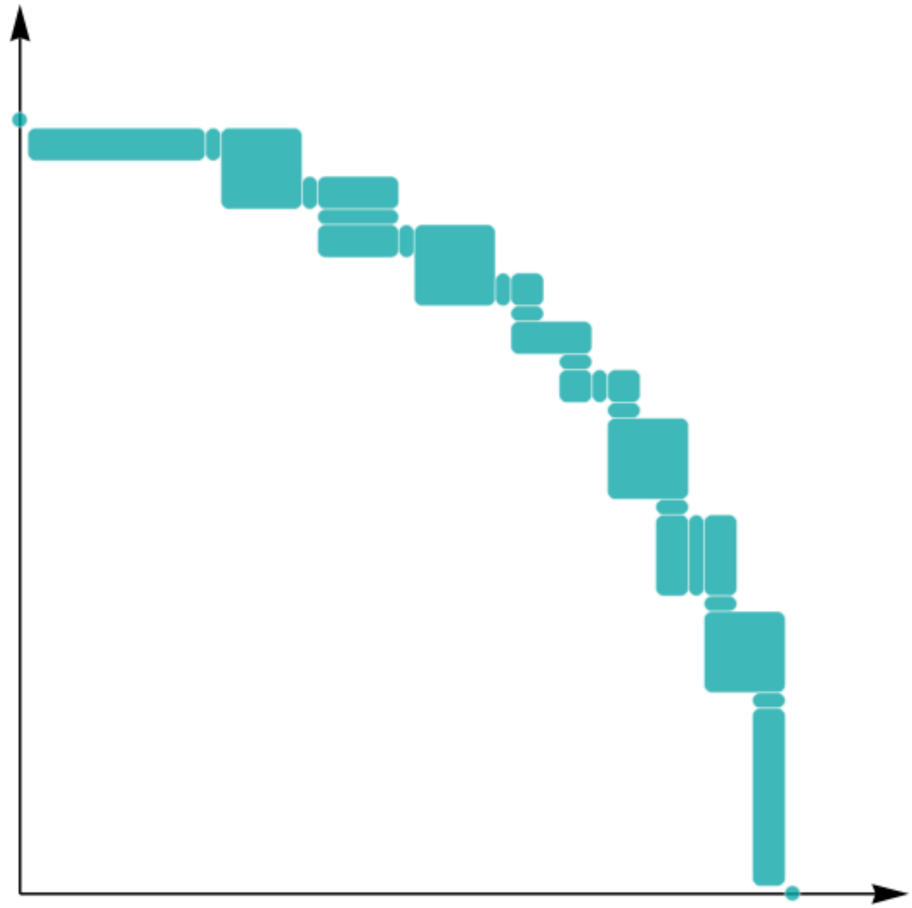
- Complete *solution*, to this finite precision
- *Information* = 1 / green area
- *Proves* value of π to 3% accuracy
- No calculus, no divides, and no square roots



Compressed Final Result

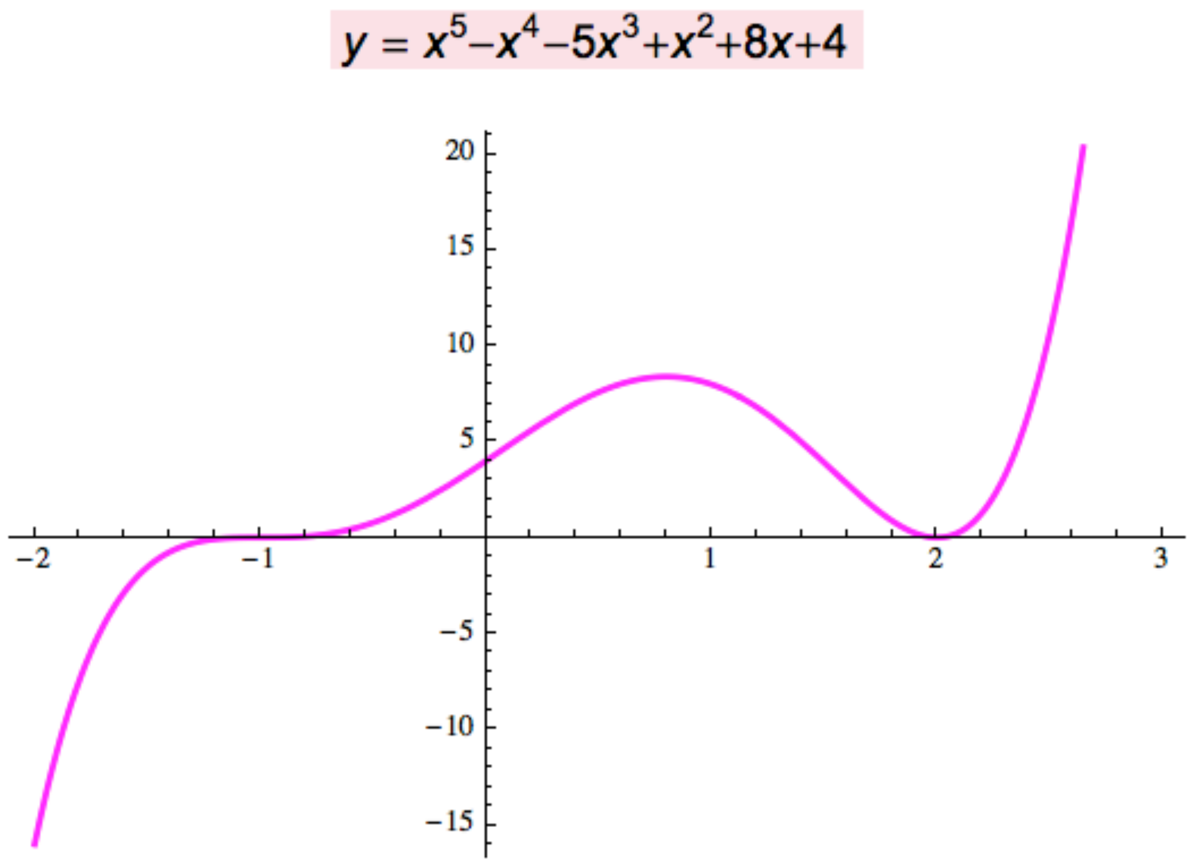
- Coalesce uboxes to largest possible ULP values
- *Lossless* compression
- Total data set: 603 bits!
- 6x faster graphics than current methods

Instead of ULPs being the source of error, they are the *atomic units of computation*



Fifth-degree polynomial roots

- Analytic solution: There isn't one.



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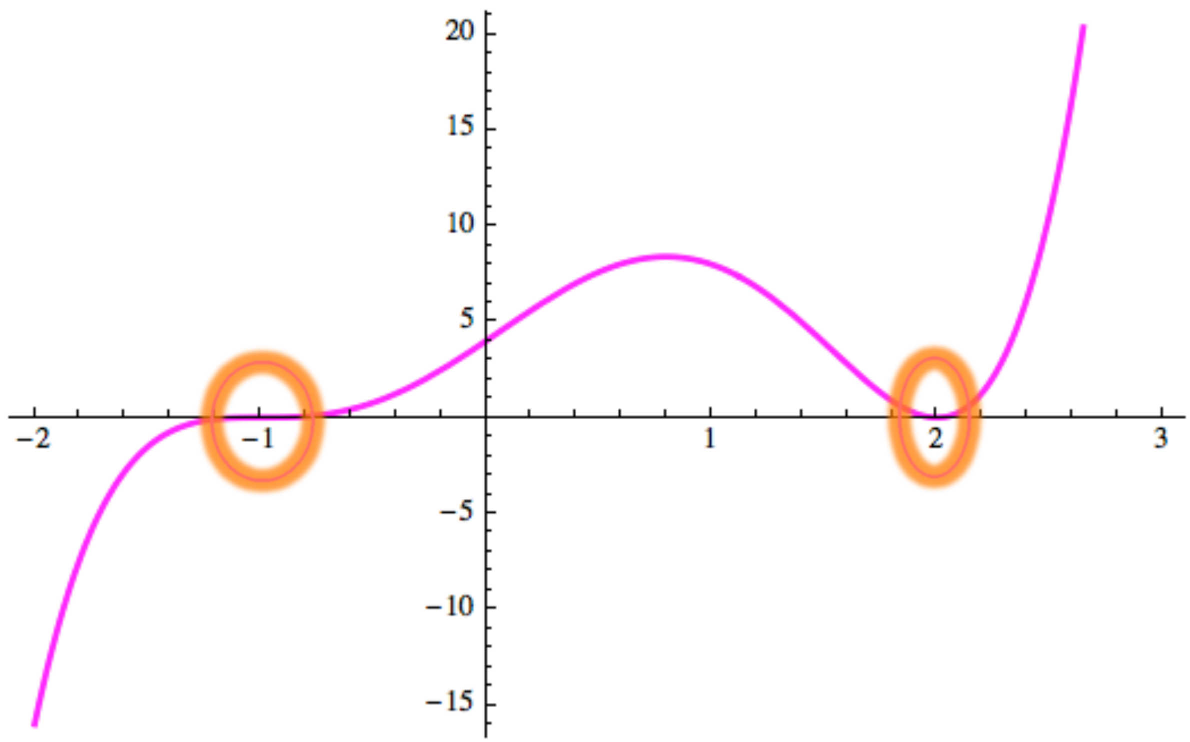
$$y = x^5 - x^4 - 5x^3 + x^2 + 8x + 4$$



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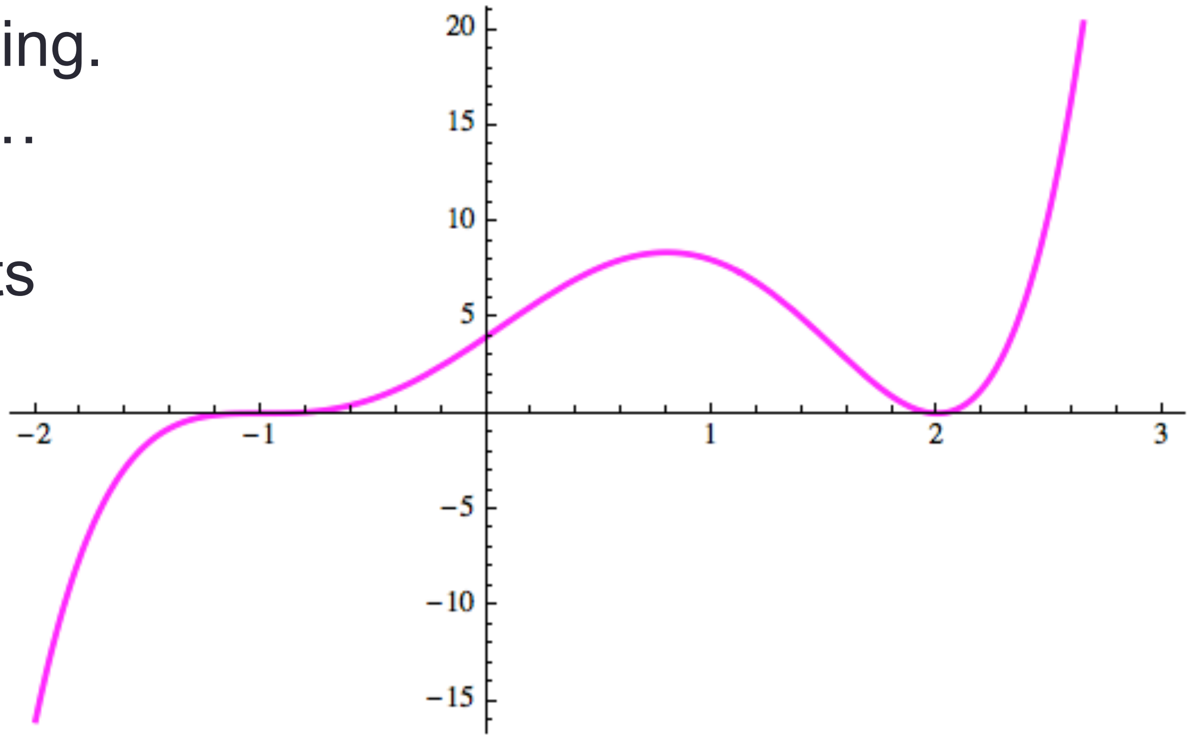
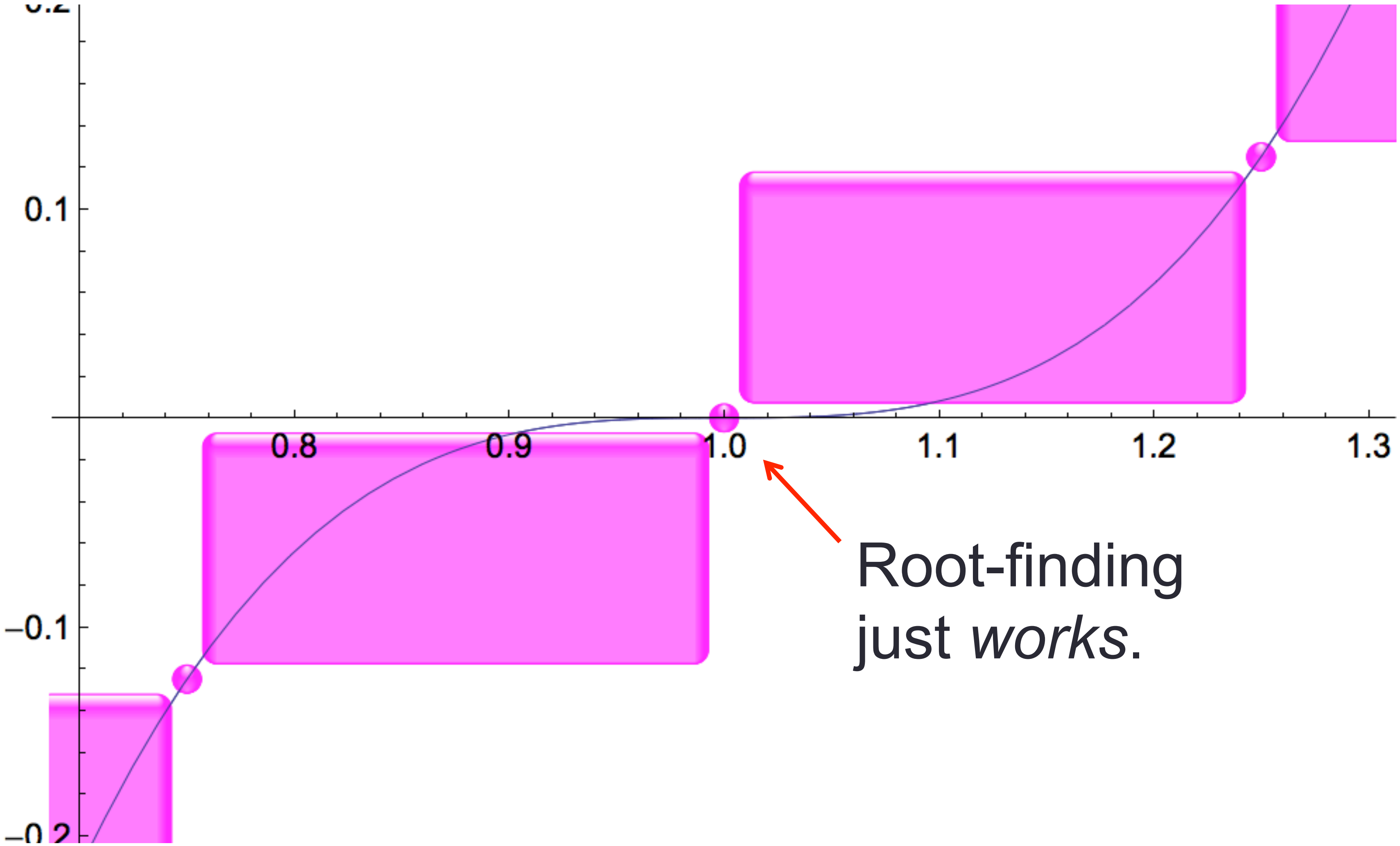
- Analytic solution: There isn't one.
- Numerical solution: Huge errors from *underflow to zero*
- Unums: quickly return $x = -1$, $x = 2$ as the exact solutions. No rounding. No underflow. Just… the *correct answer*. With as few as 4 bits for the operands!

$$y = x^5 - x^4 - 5x^3 + x^2 + 8x + 4$$



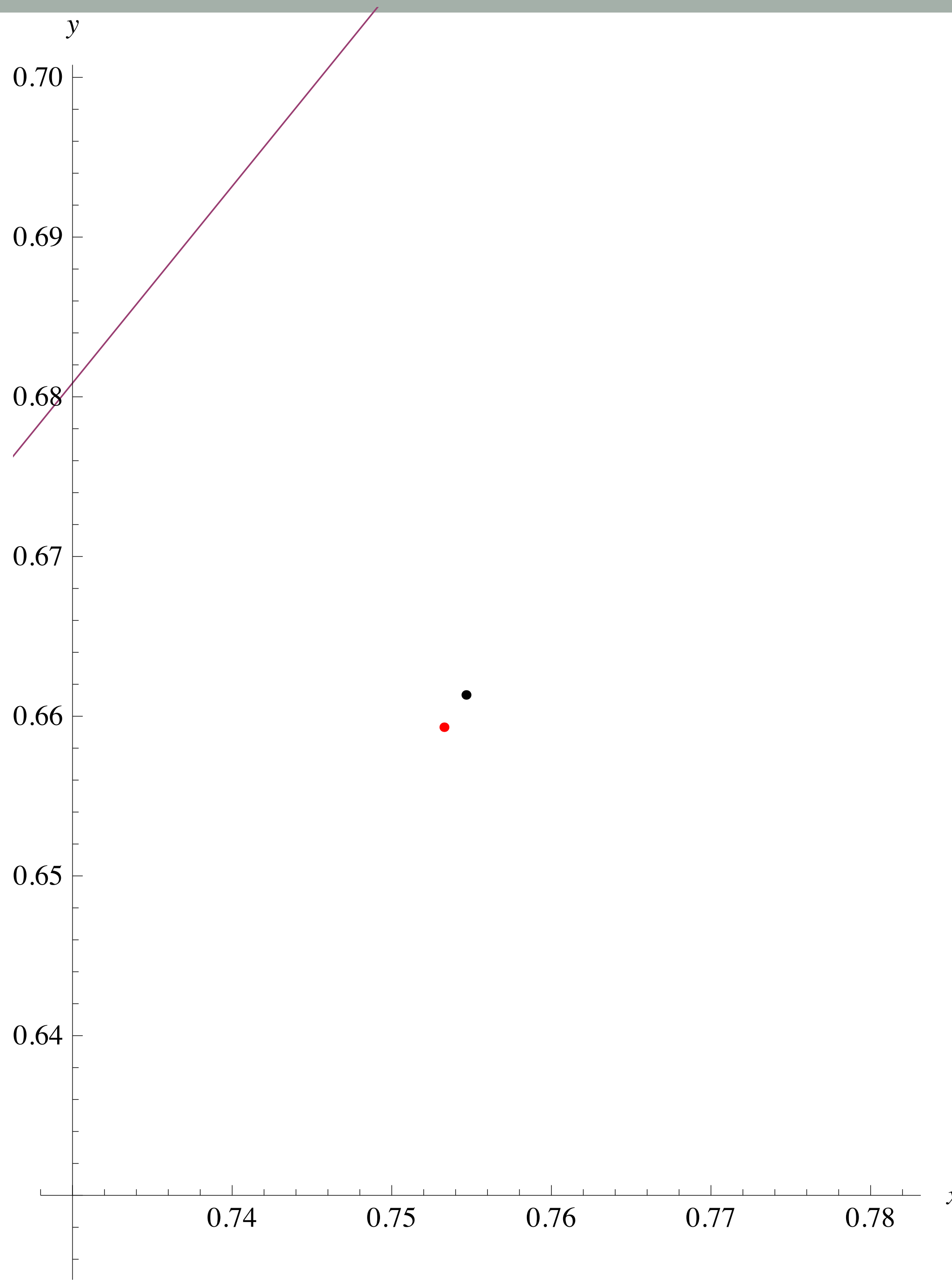
The power of open-closed endpoints



Root-finding just *works*.

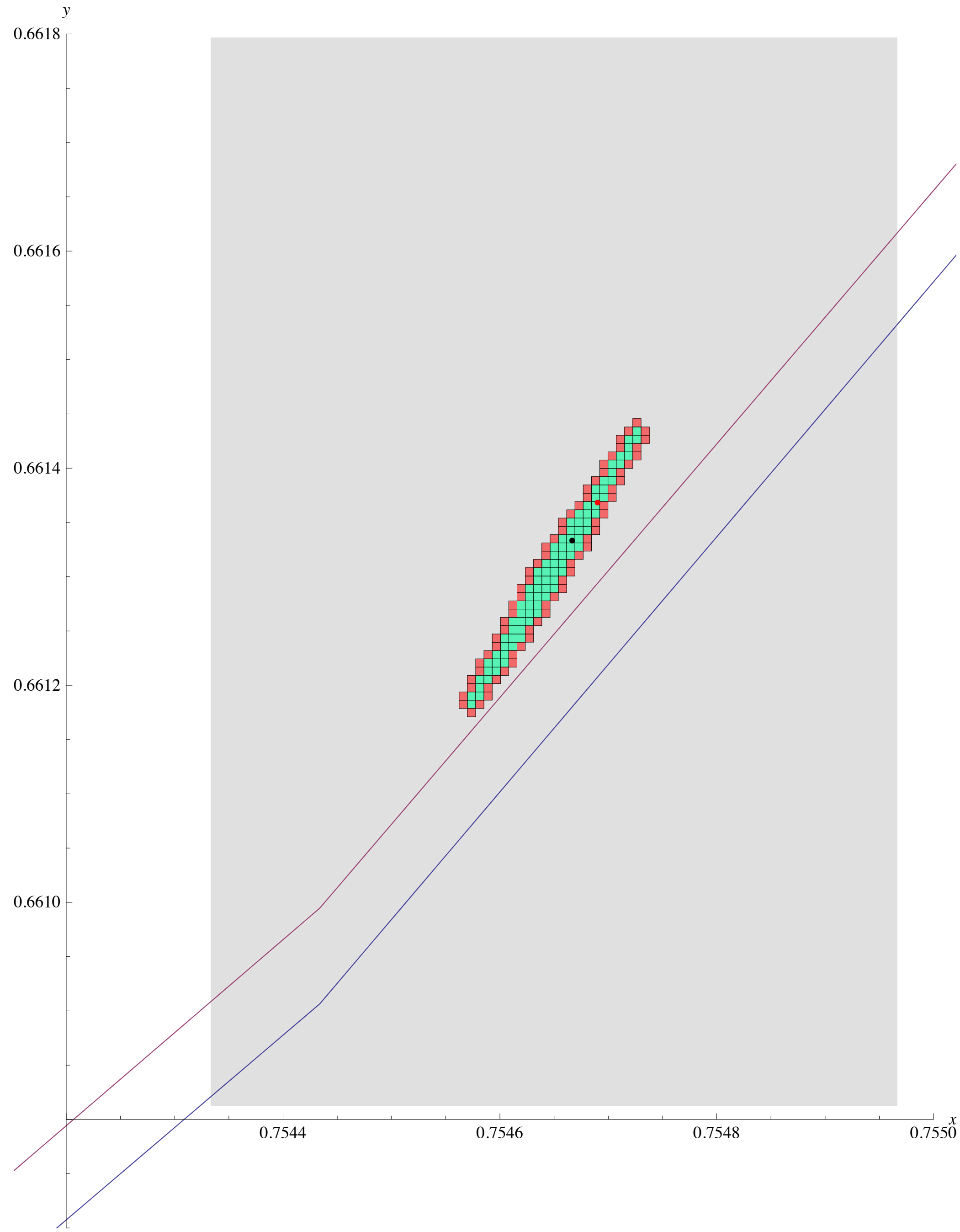
Linear solvers

- If the *A* and *b* values in *Ax=b* are rounded, the "lines" have width from uncertainty
- Apply a standard solver, and get the red dot as "the answer," *x*. A pair of floating-point numbers.
- Check it by computing *Ax* and see if it rigorously contains *b*. Yes, it does.
- Hmm… are there any *other* points that also work?



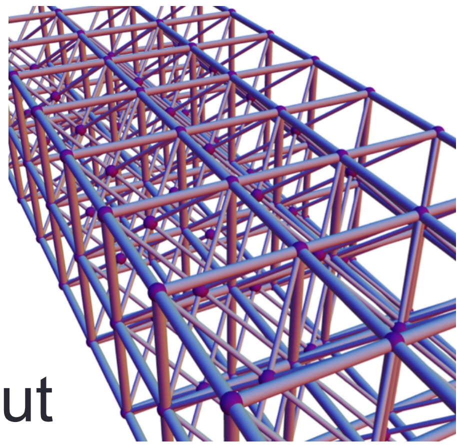
Float, Naïve Interval, and Ubox Solutions

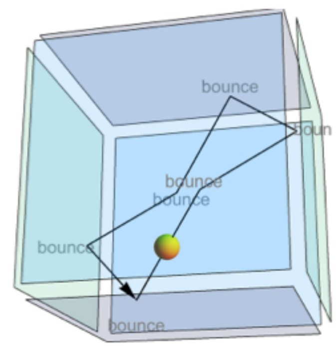
- Float solution (black dot) just gives *one of many* solutions; disguises instability
- Interval method (gray box) yields a bound too loose to be useful (naïve method)
- The ubox set (green) is the *best you can do for a given precision*
- Uboxes *reveal* ill-posed nature… yet provide solution anyway
- Works equally well on *nonlinear* problems!

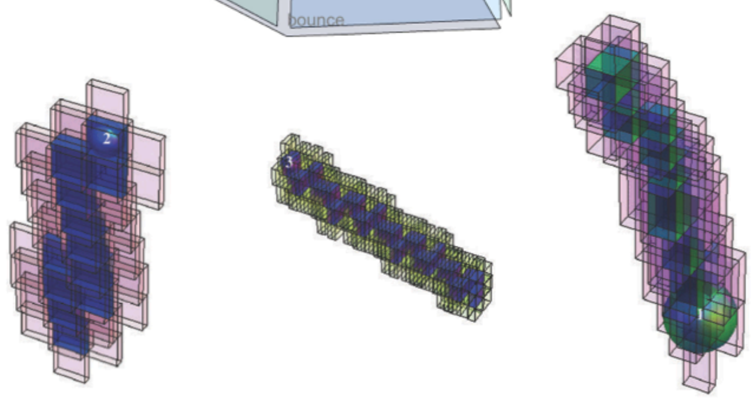


Other Apps with Ubox Solutions

- Photorealistic computer graphics
- *N*-body problems (!)
- Structural analysis
- Laplace's equation
- Perfect gas models without *statistical* mechanics







Imagine having **provable bounds** on answers for the first time, yet with easier programming, less storage, less bandwidth use, less energy/power demands, *and* abundant parallelism.

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  - "Paint bucket" and "Try everything" are brute force general methods that need no expertise… not even calculus

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- Custom VLSI processor
  - Initially with fixed-size data, plus lossless pack and unpack
  - Eventually, bit-addressed architecture with hardware memory management (similar to disc controller)

The End of Error

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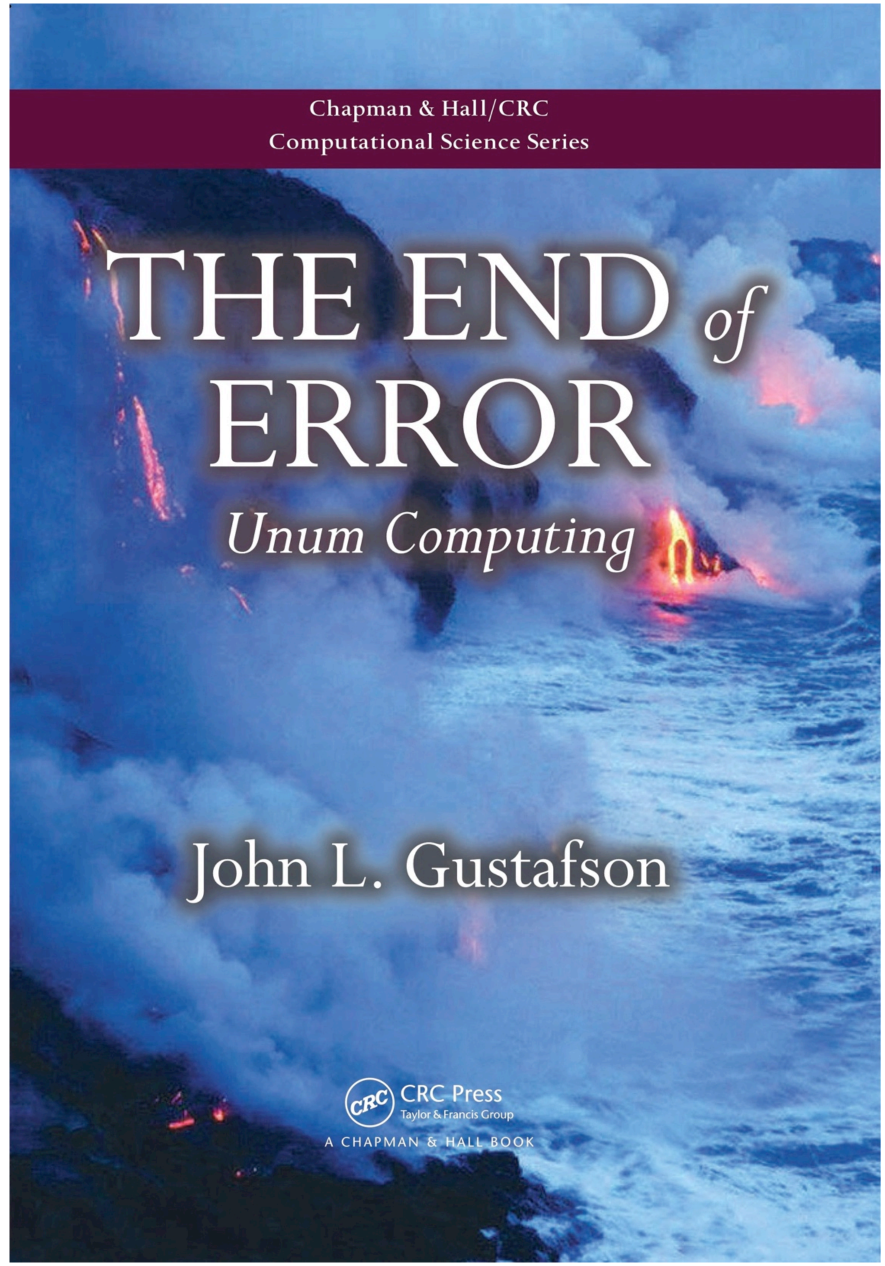

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