Outline :

① General cost function $c(x_t, y_t)$

② 2-period problem under $c(x_t, y_t)$
- Substitution
- Perturbation method

③ Interpretation of Hotelling's / No-Arbitrage Condition

④ When stock constraint is not binding (do not exhaust stock)

EEP/ECON 102, Section 4: Blank

Friday Section Sept 15, 2023 - Wednesday Section Sept 20, 2023

This section continues our work with the two-period non-renewable resource problem. In Chapter 3, we introduced a simple two-period problem, assuming constant marginal costs, and solved it using the substitution method. This week, we assume a more general cost-function, that may depend on both the stock and flow of extraction in each period. Furthermore, we move beyond the "substitution" method to learn about the perturbation method, which will be used in solving a multi-period problem later in this course. After discussing the economic interpretation of the F.O.C., we will introduce the concept of rent and rewrite the F.O.C. as the Hotelling rule. We conclude this section with a summary of the steps in solving a two-period problem.

1. Setup of the two-period problem

Consider the two-period problem we learned in Chapter 4. A firm's problem is to allocate a fixed stock of a resource between two periods, $t = 0$ and $t = 1$. Denote the price in period t by p_t, the stock at the beginning of period t by x_t, the extraction in period t by y_t. The extraction cost $c(x, y)$ depends on both the stock and the extraction. Assume the firm is a price-taker. The firm's objective is to maximize its present value of profits in two periods.

$$\max_{y_0, y_1} \pi = (p_0 y_0 - c(x_0, y_0)) + \rho\,(p_1 y_1 - c(x_1, y_1))$$
$$\text{s.t.} \quad 0 \leqslant y_t \leqslant x_t,\ t = 0, 1$$
$$x_1 = x_0 - y_0,$$

where ρ is the discount factor.

Stock x_t = stock at beginning of period - t

flow y_t = extraction during period - t

$c(x_t, y_t) \longrightarrow$ stock dependent cost function

Generally, we will assume $c(x_t, y_t) = G\,(\sigma + x_t)^{-\alpha}\, y_t^{1+\beta}$, $\sigma, \alpha, \beta \geq 0$ and $0 \leq y_t \leq x_t$

$\dfrac{d\,c(x_t, y_t)}{d\,y_t}$ = Try Deriving ≥ 0 The more you extract in a given period, the higher the total cost.

$\dfrac{d\,c(x_t, y_t)}{d\,x_t}$ = Try Deriving ≤ 0 The higher your starting stock is in a given period, the lower your total cost will be for the same y_t.

2 period & p.c

Constant MC = C

$$\max_{y_0, y_1} \; (\underbrace{\underbrace{p_0 y_0}_{TR_0} - \underbrace{c\, y_0}_{TC_0}}_{\pi_0}) + \underbrace{\rho\,(p_1 y_1 - c\, y_1)}_{\text{present value of } \pi_1} \quad s.t. \quad y_0 + y_1 = x_0$$

2 period & p.c

$c(x_t, y_t) \to$ Total cost

$$\max_{y_0, y_1} \; [p_0 y_0 - c(x_0, y_0)] + \rho\,[p_1 y_1 - c(x_1, y_1)]$$

s.t. $0 \le y_t \le x_t$

$x_1 = x_0 - y_0 \to x_1(y_0)$

assume $y_1 = x_1 = x_0 - y_0 \to y_1(y_0)$

$$\max_{y_0} \; [p_0 y_0 - c(x_0, y_0)] + \rho\,[p_1\, y_1(y_0) - c(x_1(y_0), y_1(y_0))]$$

2. Solving the problem

To obtain the first-order condition that an optimal extraction schedule must satisfy, we can take two approaches, or two perspectives. One is to maximize the profit over the extraction directly (which we did last week using the substitution method). From this perspective, the decision variable is the extraction. The other perspective, in contrast, takes the perturbation as a decision variable, the perturbation to a candidate extraction schedule. If the candidate is optimal, the optimal perturbation to this candidate must be zero. In other words, if a candidate solution is optimal, we don't want to deviate from this solution, because we are already maximizing profits.

2.1 Substitution method

Let's look at how we maximize profit over extraction. Focusing on the case where it is optimal to extract all the resource (i.e., $y_1 = x_1$), we have $y_1 = x_0 - y_0$. Substituting the constraint into the profit equation above, the present value of the firm's profits in two periods is:

$$\max_{y_0} \; [p_0 y_0 - c(x_0, y_0)] + \rho\,[p_1 y_1(y_0) - c(x_1(y_0), y_1(y_0))]$$

The decision variable is the extraction in period 0, y_0. The first-order condition (FOC) is

$$\frac{d\pi}{dy_0} = \left[p_0 - \frac{dc(x_0,y_0)}{dy_0}\right] + \rho\left[p_1 \frac{dy_1}{dy_0} - \left(\frac{dc(x_1,y_1)}{dx_1}\frac{dx_1}{dy_0} + \frac{dc(x_1,y_1)}{dy_1}\frac{dy_1}{dy_0}\right)\right]$$

Recall $y_1 = x_1 = x_0 - y_0 \Rightarrow \frac{dy_1}{dy_0} = \frac{dx_1}{dy_0} = -1$

$$= \left[p_0 - \frac{dc(x_0,y_0)}{dy_0}\right] + \rho\left[p_1 (-1) - \left(\frac{dc(x_1,y_1)}{dx_1}(-1) + \frac{dc(x_1,y_1)}{dy_1}(-1)\right)\right]$$

$$= \left[p_0 - \frac{dc(x_0,y_0)}{dy_0}\right] - \rho\left[p_1 - \left(\frac{dc(x_1,y_1)}{dx_1} + \frac{dc(x_1,y_1)}{dy_1}\right)\right] \overset{set}{=} 0$$

$$p_0 - \frac{dc(x_0,y_0)}{dy_0} = \rho\left[p_1 - \frac{dc(x_1,y_1)}{dy_1} - \frac{dc(x_1,y_1)}{dx_1}\right]$$

No arbitrage condition for P.C.
(Hotelling's condition for P.C)

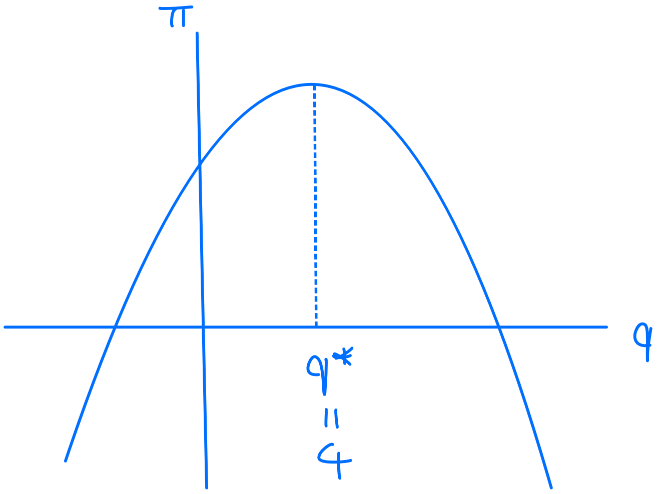
Here, the marginal revenue less the marginal cost in period 0 must be equal to the discounted marginal revenue less the marginal cost (and the change in costs with respect to the stock) in period 1. If we decided to extract one more unit of oil today, we get the additional revenue from selling that unit today, less the costs of extracting it today. However, by extracting that unit today, we are no longer able to extract or sell that unit in period one. In addition, the smaller stock that we face in period one changes the costs of extraction. We call the marginal revenue less the marginal costs "rent". This term will show up regularly as we move beyond the two period problem.

Toy Example

Usual maximization method

$\pi(q) = -[q-4]^2 + 30$ $\Rightarrow$ profit at each level of q

Q: At what level of q is profit maximized



Perturbation method

- Assume $q = \underline{q}$, a candidate level of production
- Q: To maximize $\pi(\,)$, now $g(\,)$, how much should we perturb (ε) our original choice of $\underline{q}$

write profit function by $q \Leftrightarrow \underline{q} + \varepsilon$

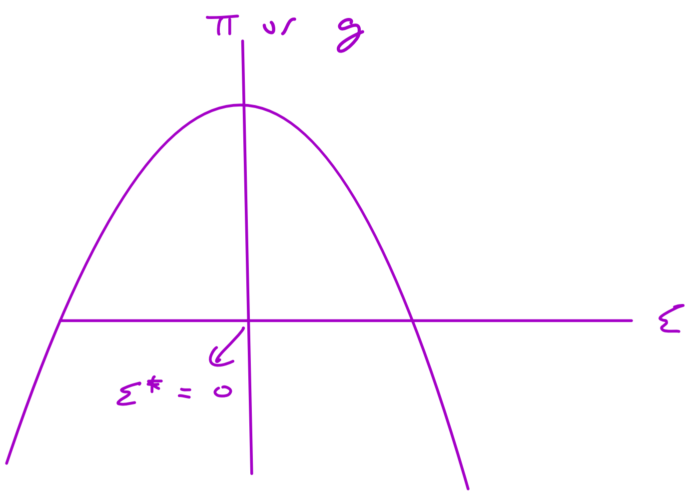
$g(\varepsilon;\underline{q}) = -[(\underline{q}+\varepsilon)-4]^2 + 30$ $\rightarrow$ Now, ε is the choice variable; $\underline{q}$ is just a number

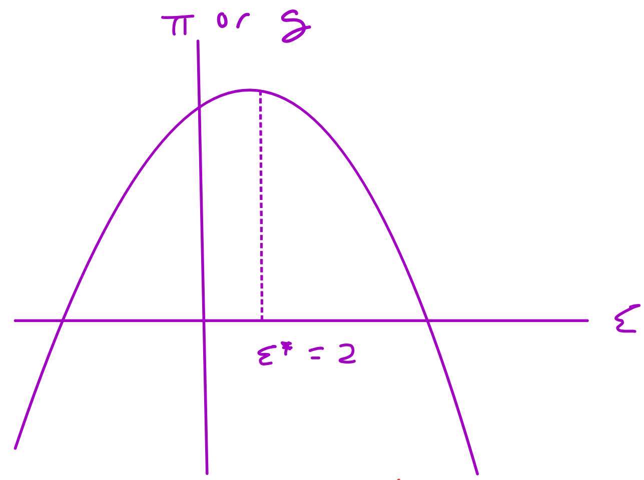
If $\underline{q} = q^* = 4$,

$g(\varepsilon;\underline{q}=4) = -\varepsilon^2 + 30$

If $\underline{q} = 2$,

$g(\varepsilon;\underline{q}=2) = -[\varepsilon-2]^2 + 30$





To find ε^* at each level of $\underline{q}$, we would $\dfrac{dg(\varepsilon;\underline{q})}{d\varepsilon} = 0$

Key: But, if $\underline{q}$ already at optimal q^*, then $\left.\dfrac{dg(\varepsilon;q^*)}{d\varepsilon}\right|_{\varepsilon=0} = 0$

2-period problem w/ perturbation method

Original : $\pi(\,) = [p_0 y_0 - c(x_0, y_0)] + \rho[p_1 y_1 - c(x_1, y_1)]$

Perturbation p.o.v. : Assume $\underline{y_0}$ (also predetermine $\underline{y_1}, \underline{x_1}$ under full exhaustion)

How much should we perturb $\underline{y_0}$ by, ε^*, to maximize $g(\,)$.

Before	After perturbation by ε
$\underline{y_0}$ →	$\underline{y_0} + \varepsilon$
$\underline{y_1}$ →	$\underline{y_1} - \varepsilon$
x_0 →	No change x_0
$\underline{x_1}$ →	$\underline{x_1} - \varepsilon$

assuming binding stock constraint (full exhaustion)

$$g(\varepsilon) = [p_0(\underline{y_0} + \varepsilon) - c(x_0, \underline{y_0} + \varepsilon)] + \rho[p_1(\underline{y_1} - \varepsilon) - c(\underline{x_1} - \varepsilon, \underline{y_1} - \varepsilon)]$$

profit after perturbation by ε

To find ε^* : $\dfrac{dg(\varepsilon; \underline{y_0}, \underline{y_1}, \underline{x_1})}{d\varepsilon} = 0$ ⟹ True at any $\underline{q}$

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} g(\varepsilon; y_0, x_1, y_1) = \left(p_0 - \frac{\partial c(x_0, y_0 + \varepsilon)}{\partial (y_0 + \varepsilon)}\right) - \rho\left(p_1 - \frac{\partial c(x_1 - \varepsilon, y_1 - \varepsilon)}{\partial (x_1 - \varepsilon)} - \frac{\partial c(x_1 - \varepsilon, y_1 - \varepsilon)}{\partial (y_1 - \varepsilon)}\right) = 0$$

But if we started at optimal $\underline{y_0} = y^*$, $\underline{y_1} = y_1^*$, we would find $\varepsilon^* = 0$

Hence, $\left.\dfrac{dg(\,)}{d\varepsilon}\right|_{\varepsilon^* = 0} = 0$ ⟹ True at $\underline{q} = q^*$

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} g(\varepsilon; y_0, x_1, y_1) = \left(p_0 - \frac{\partial c(x_0, y_0 \cancel{+ \varepsilon})}{\partial (y_0 \cancel{+ \varepsilon})}\right) - \rho\left(p_1 - \frac{\partial c(x_1 \cancel{- \varepsilon}, y_1 \cancel{- \varepsilon})}{\partial (x_1 \cancel{- \varepsilon})} - \frac{\partial c(x_1 \cancel{- \varepsilon}, y_1 \cancel{- \varepsilon})}{\partial (y_1 \cancel{- \varepsilon})}\right) = 0$$

2.2 Perturbation method

Scenario	Flow at t=0	Stock at t=0	Flow at t=1	Stock at t=1
Original	y_0	x_0	y_1	$x_1 = x_0 - y_0$
Perturbed	$y_0' = y_0 + \epsilon$	$x_0' = x_0$	$y_1' = y_1 - \epsilon$	$x_1' = x_0 - (y_0 + \epsilon)$
				$= x_1 - \epsilon$

Table 1: Original and perturbed stocks and flows in the two-period problem

Let's look at the perturbation to a candidate extraction schedule. Start with a candidate solution y_0 and y_1, and associated x_1. According to the constraint, $x_1 = x_0 - y_0$. As we did last week, we assume that it is optimal to extract all the resource. By this assumption, we have $y_1 = x_1 = x_0 - y_0$. Now consider a perturbation ε to this candidate. The extraction in period 0 becomes $y_0 + \varepsilon$ after perturbation. As a result, the stock at the beginning of period 1 becomes $x_1 - \varepsilon$ and the extraction in period 1 becomes $y_1 - \varepsilon$. Table 1 represents the changes due to the perturbation. The payoff (present value of profits in periods 0 and 1) after perturbation is now

$$g(\varepsilon; y_0, x_1, y_1) = \Big(p_0(y_0 + \varepsilon) - c(x_0, y_0 + \varepsilon)\Big) + \rho\Big(p_1(y_1 - \varepsilon) - c(x_1 - \varepsilon, y_1 - \varepsilon)\Big). \tag{1}$$

Still, $g(\varepsilon; y_0, x_1, y_1)$ is the present value of profits in two periods. But we now regard it as a function of perturbation ε (instead of y_0, y_1), and call it a gain function. If the candidate extraction schedule y_0 and y_1 is optimal, then, as we pointed out above, the optimal perturbation ε (to this candidate) must be zero. Since ε can be positive or negative, $\varepsilon = 0$ marks an interior optimum. Therefore, the necessary condition for optimal extraction is:

$$\left.\frac{\mathrm{d}g}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} = 0. \tag{2}$$

Total differentiating (1) with respect to ε, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\varepsilon} g(\varepsilon; y_0, x_1, y_1) = &\left(p_0 - \frac{\partial c(x_0, y_0 + \varepsilon)}{\partial (y_0 + \varepsilon)}\right) \\ &- \rho\left(p_1 - \frac{\partial c(x_1 - \varepsilon, y_1 - \varepsilon)}{\partial (x_1 - \varepsilon)} - \frac{\partial c(x_1 - \varepsilon, y_1 - \varepsilon)}{\partial (y_1 - \varepsilon)}\right).\end{aligned} \tag{3}$$

Evaluating (3) at $\varepsilon = 0$, we have the FOC

$$\left.\frac{\mathrm{d}g}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} = \left(p_0 - \frac{\partial c(x_0, y_0)}{\partial y_0}\right) - \rho\left(p_1 - \frac{\partial c(x_1, y_1)}{\partial x_1} - \frac{\partial c(x_1, y_1)}{\partial y_1}\right) = 0. \tag{4}$$

This is the same as the FOC we obtained through the first approach.

3. Interpretation

Inspired by the perturbation method, we can interpret the equilibrium condition (the FOC) in another way. Rearranging (4), we have

$$p_0 - \frac{\partial c(x_0, y_0)}{\partial y_0} = \rho\left(p_1 - \frac{\partial c(x_1, y_1)}{\partial y_1} - \frac{\partial c(x_1, y_1)}{\partial x_1}\right) \tag{5}$$

From the discussion above on perturbation method, we know that the left-hand side (LHS) of (8) is the impact of a marginal change in y_0 on the firm's profit in period 0. Specifically, if the extraction in period 0 increase by $d\varepsilon$, the firm's profit in period 0 will increase by the LHS of (8) times $d\varepsilon$. But the increase in period-0 extraction will decrease the stock at the beginning of period 1 and hence the extraction in period 1. So the firm's profit in period 1 is also affected. In addition to the "direct" impact of decreased extraction, the firm's profit in period 1 is also decreased "indirectly" by the higher extraction cost due to the decreased stock. The total decrease in the present value of the firm's profit in period 1 is the RHS of (8) times $d\varepsilon$. The FOC says that at the optimum, the marginal increase in the current (period 0) profit due to an increase in the current production, equals the present value of the decrease in next period (period 1) profit, due to this increase in current production. In other words, a perturbation to the optimal extraction in period 0 causes zero first-order change to the payoff.

At some y_0, y_1, consider perturbing by ε

shifting ε units from period-1 sales

to period-0 sales

$$\underbrace{\varepsilon\left(p_0 - \frac{\partial c(x_0, y_0)}{\partial y_0}\right)}_{\text{marginal Gains from Selling } \varepsilon \text{ more in period-0}} = \underbrace{\rho\left(p_1 - \frac{\partial c(x_1, y_1)}{\partial y_1} - \frac{\partial c(x_1, y_1)}{\partial x_1}\right)\varepsilon}_{\text{P.V. marginal Loss from Selling } \varepsilon \text{ less in period-1}}$$

At optimal y_0^*, y_1^*, both sides (marginal gains and marginal loss) from perturbation should be equal, meaning there is no incentives to arbitrage.

meaning of each individual term.

4. When the Resource Isn't Exhausted

If it is not optimal to extract all of the resource, then the assumption we made earlier, where $x_1 = y_1$ does not hold. Instead, $y_1 < x_1$. This means that the resource constraint does not bind. If the firm does not exhaust the resource, it will continue to extract up until the point where the marginal revenue less the marginal costs are equal to zero, or

$$p_1 - \frac{\partial c(x_1, y_1)}{\partial y_1} = 0$$

in the case of a competitive firm in a two-period setting. Substituting this equation into the FOC we derived earlier yields the equilibrium condition for the firm:

$$p_0 - \frac{\partial c(x_0, y_0)}{\partial y_0} = -\rho\left[\frac{\partial c(x_1, y_1)}{\partial x_1}\right].$$

Previously, we assumed stock constraint binds (full extraction).
In this world, decision about y_0^* completely determined period-1's outcomes.

Assume stock constraint does not bind, (when it is suboptimal to extract all) $x_1 = x_0 - y_0$ (same) $y_1 < x_1$ (different)

You now have to determine y_1^* and it is no longer completely dictated by y_0^*.

Period 1 decision (knowing $y_1 < x_1$) $\Rightarrow$ Back to Intro / Intermediate micro
maximize period-1 profit by producing up-to where $MR = MC$

$\Rightarrow MR(y_1) - MC(y_1; x_1) = 0$ monopoly

$p_1 - MC(y_1; x_1) = 0$ P.C.

P.C. $\quad p_0 - MC_0 = \rho\left(\underbrace{p_1 - MC_1}_{0 \text{ when } y_1 < x_1} - \frac{dc(x_1, y_1)}{dx_1}\right)$

$$p_0 - MC_0 = \rho\left(-\frac{dc(x_1, y_1)}{dx_1}\right)$$

Even when stock constraint is not binding

> 0 if stock dependent cost function since $\frac{dc(x_1, y_1)}{dx_1} > 0$

$= 0$ if stock independent cost function

5. Steps in solving a two-period model in perfect competition

The general steps in solving a two-period problem in perfect competition are as follows:

Step 1. Solve for y_0 and y_1 assuming the resource is extracted (i.e., $y_1 = x_1$) using either the substitution or perturbation method. $\Longleftrightarrow$ start w/ Hotelling's condition (no-arbitrage condition) $\Rightarrow$ solve $\hat{y}_0, \hat{y}_1$

Step 2. Evaluate

$$p_1 - \frac{\partial c(x_1, y_1)}{\partial y_1} \begin{cases} \geq 0 & \Longrightarrow x_1 = y_1. \text{ The resource constraint binds. The problem is done.} \\ < 0 & \Longrightarrow x_1 > y_1. \text{ The resource constraint does not bind. Go to step 3.} \end{cases}$$

Step 3. Follow the procedure in part 4 to solve for the new y_0 and y_1. That is the final solution to the problem.

check if, evaluating at $\hat{y}_1$,

$MR(y_1^*) - mc(y_1^*)$ ≥ 0 if positive, done. Why?

< 0 if negative, firm has incentives to decrease y_1 from $\hat{y}_1$.

step 3) Solve for y_1^* w/ $p_1 - \dfrac{\partial c(x_1, y_1)}{\partial y_1} = 0$ or $MR_1 - mc_1 = 0$

Solve for y_0^* $p_0 - \dfrac{\partial c(x_0, y_0)}{\partial y_0} = -\rho \left[\dfrac{\partial c(x_1, y_1)}{\partial x_1} \right]$

or $MR_0 - mc_0 = -\rho \left[\dfrac{dc(x_1, y_1)}{dx_1} \right]$

6. Exercise

Consider the cost function $c(x_t, y_t) = 3x_t^{-1}y_t^2$ and the inverse demand function $p_t = 30 - 3y_t$. Let the total number of units available to extract be 9, and let $\rho = 0.9$. Assume Perfect Competition. Find the equilibrium solution using the substitution and perturbation methods. Make sure to check to see if the resource constraint is binding.

Using the substitution method:

(Step 1 - Assuming binding constraint)

$$\max_{y_0} \pi = \max_{y_0} \left[p_0 y_0 - 3(9)^{-1} y_0^2 \right] + 0.9 \left[p_1 \overset{y_1}{(9 - y_0)} - 3 \underbrace{\overset{x_1}{(9-y_0)}^{-1} \overset{y_0}{(9-y_0)}^2}_{3(9-y_0)} \right]$$

$$= \max_{y_0} \left[p_0 y_0 - \frac{1}{3} y_0^2 \right] + 0.9 \left[9p_1 - p_1 y_0 - 27 + 3y_0 \right]$$

$$[y_0] \quad \left[p_0 - \frac{2}{3} y_0 \right] + 0.9 \left[-p_1 + 3 \right] \overset{\text{set}}{=} 0$$

$$\Leftrightarrow \quad p_0 - \frac{2}{3} y_0 = -0.9 \left[-p_1 + 3 \right]$$

$$30 - 3y_0 - \frac{2}{3} y_0 = -0.9 \left[-30 + 3y_1 + 3 \right]$$

$$30 - \frac{11}{3} y_0 = -0.9 \left[-27 + 3(9 - y_0) \right]$$

$$= -0.9 \left[-3y_0 \right]$$

$$= 0.9 \left[3y_0 \right]$$

$$30 = \frac{110 + 81}{30} y_0 \quad \Rightarrow \quad y_0^* = \frac{30 \cdot 30}{191} = 4.71 \qquad p_0 = 15.87$$

$$y_1^* = 9 - 4.71 = 4.29 \qquad p_1 = 17.13$$

We could have started w/ Hotelling's condition

Given P.C. $\quad p_0 - mc_0 = \rho \left(p_1 - mc_1 - \dfrac{dc(x_1, y_1)}{dx_1} \right)$

$$\Leftrightarrow \quad p_0 - 3\frac{(2)y_0}{x_0} = \rho \left(p_1 - 3\frac{(2)y_1}{x_1} - \frac{3y_1^2(-1)}{x_1^2} \right)$$

We reached same result.

$$\Leftrightarrow \quad p_0 - \frac{2}{3} y_0 = 0.9 \left(p_1 - 6 - (-3) \right)$$

$$= 0.9 \left(p_1 - 3 \right)$$

(step 2)

In period - 1, if stock constraint binds (optimal to exhaust),

$$MR_1 - MC_1 = p_1 - 3 x_1^{-1} y_1$$

since p.c. → when optimal to exhaust, $x_1 = y_1$

$$= 17.13 - 3$$

$$= 14.13$$

$$> 0$$

In period - 2, we see that, when we exhaust resources, firm still has incentives to increase extraction, since $MR_1 - MC_1 > 0$, evaluated at $y_1^* = x_1^*$.

But it cannot, since we are out of resources.

Hence, there is no reason to not exhaust resources in period - 2.

Using the perturbation method: $c(x_\epsilon, y_\epsilon) = 3x_\epsilon^{-1} y_\epsilon^2$

(Step 1 - Assuming binding Constraint)

$$g(\varepsilon) = \left[p_0 \cdot (y_0 + \varepsilon) - c(x_0 = 9, y_0 + \varepsilon) \right]$$
$$+ 0.9 \left[p_1 \cdot (y_1 - \varepsilon) - c(x_1 - \varepsilon, y_1 - \varepsilon) \right]$$

$$\left. \frac{dg(\varepsilon)}{d\varepsilon} \right|_{\varepsilon^* = 0} = 0$$ should give you

$$p_0 - \frac{2}{3} y_0 = 0.9 (p_1 - 3)$$

Rest of steps are same as substitution case.

7. Rent and Hotelling Rule

If we define

$$R_t = p_t - \frac{\partial c(x_t, y_t)}{\partial y_t}, \tag{6}$$

then we can rewrite the FOC for an optimal extraction schedule (8) in a neater way:

$$R_0 = \rho \left(R_1 - \frac{\partial c(x_1, y_1)}{\partial x_1} \right), \tag{7}$$

which is called the Hotelling rule. From this rule, we see that if the extraction cost depends on (usually decreases in) the stock of resource, then the rent in period 0 (i.e., the difference between price and marginal cost) can be positive even if the rent in period 1 is zero. This positive rent occurs in a competitive market due to the dependence of extraction costs on the stock which is a feature of natural resources, and hence cannot be eliminated by competition. A more general definition of rent is $MR - MC$, of which (6) is a special case. This is important when we define the Hotelling rule for monopoly, where we replace the price (which is the MR for competitive firms) with the MR of the monopoly.