Charge-to-Mass Ratio Experiment

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***Abstract*—In the experiment, we accelerated electrons in a Fine Beam Tube with electric field and applied them a perpendicular magnetic field. We measured the radius of the circular trajectory and obtained a value for charge-to-mass ratio of $1.72 \times 10^{11} C/kg \pm 1.80 \times 10^{10} C/kg$, which is 0.15$\sigma$ away from the real value.**

I. Theory

The charge-to-mass ratio is a constant physical quantity relating the mass and the electric charge of a given particle, expressed in units of kilograms per coulomb (kg/C). The charge-to-mass ratio of the electron was first measured by J. J. Thomson in 1897[1]. Thomson measured the ratio by observing their motion in an applied magnetic field. He repeated his measurement of the ratio many times with different metals for cathodes and also different gases. Having reached the same value for q/m every time, it was concluded that a fundamental particle having a negative charge e and a mass 2000 times less than the lightest atom existed in all atoms. After his discovery, Thomson developed a new atomic model, called "plum pudding" model. Although this model has turned out to be wrong, the discovery of the charge-to-mass ratio was a crucial experiment. J. J. Thomson was awarded the Nobel Prize for physics in 1906 for this work[2].

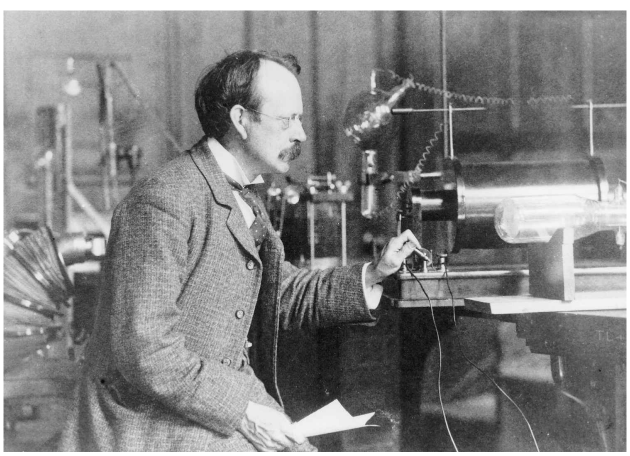

Fig. 1. J.J. Thomson

The way used to measure the charge-to-mass ratio is to accelerate electrons in a tube with electric field, apply a magnetic field perpendicular to its direction of motion and create a circular motion. Using the relationship between the radius of the circular path and the electric and magnetic field, one can obtain the charge-to-mass ratio of the electron. We can use the kinetic energy and force formulas to derive the relationship for q/m ratio.

The kinetic energy formula:

$$qV = \frac{1}{2}mv^2 \tag{1}$$

The Lorentz force formula:

$$\vec{F} = q(\vec{v} \times \vec{B}) \tag{2}$$

The force is perpendicular to the motion. Then,

$$qvB = \frac{mv^2}{r} \tag{3}$$

which is equal to

$$qB = \frac{mv}{r} \tag{4}$$

If we take square of both sides in (4), we get v^2 in both kinetic energy and force formulas. If we equate them, we get:

$$\frac{q}{m}B^2 = \frac{2V}{r^2} \tag{5}$$

In this experiment the q/m ratio can be extracted by measuring one of the quantities (applied voltage, magnetic field and the radius of the circular trajectory) as a function of another while keeping the third one fixed. The applied voltage and radius of curvature are measured directly. However, we also need a method for measuring the applied magnetic field. In the experiment, a pair of Helmholtz coils are used to produce the magnetic field. The magnetic field of the coils in the z-direction can be calculated using the Biot-Savart formula:

$$\vec{B}(z) = \int_C \frac{\mu_0}{4\pi} \frac{IRdl}{(R^2+z^2)^{\frac{3}{2}}} \hat{z} \tag{6}$$

$$B_z = \frac{\mu_0 I R^2}{\left(R^2 + \left(z - \frac{a}{2}\right)^2\right)^{3/2} + \left(R^2 + \left(z + \frac{a}{2}\right)^2\right)^{3/2}} \tag{7}$$

where I is the current going through the coils, R is the radius of the coils.

In our case, we have z = 0, a = R and also n turns at each coil. So,

$$B = \left(\frac{4}{5}\right)^{3/2} \mu_0 \frac{In}{R} \tag{8}$$

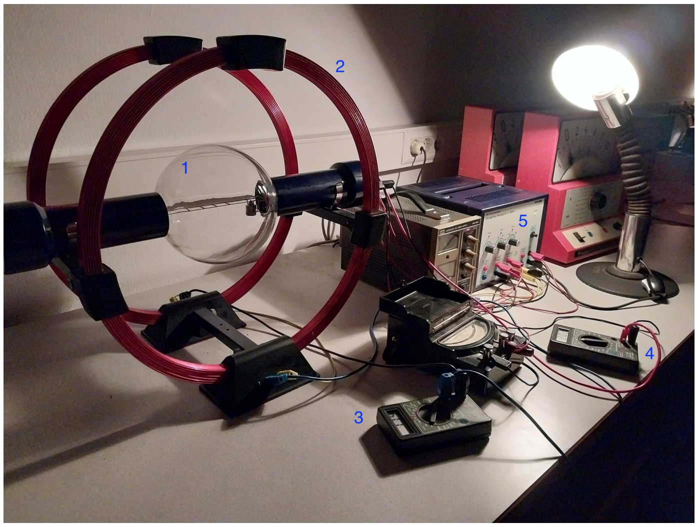

Fig. 2. The Experimental Setup

II. The Experimental Setup

1) Fine Beam Tube: It is filled with a very low pressure hydrogen gas. When the electrons travel through this gas, they excite the hydrogen atoms. The excited hydrogen atoms emit light along the electron's path when they de-excite. Blue light is dominant in this de-excitation and the result is a very thin beam.
2) Helmholtz Coils: It is used to produce magnetic field. The manufacturer supplied values for the coil radius and the number of turns are 15 cm and 130 turns, respectively.
3) Ammeter (0-5 Amperes)
4) Voltmeter (0-300 Volts)
5) Power Supply

III. Method

1) The circuit is connected as shown:

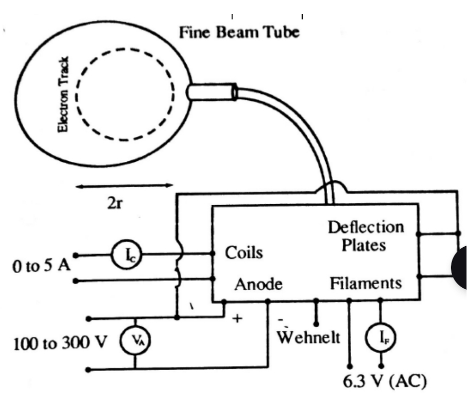

Fig. 3. The Fine Beam tube and the Helmholtz Coil connections [3]

2) Everything is turned off and all knobs are at 0 value.
3) The room is darkened to be able to observe the very thin blue light along the electron trajectory.
4) The electric field is turned on and electrons are accelerated. The path of the electrons should be observable.
5) The magnetic field is turned on.
6) The electric field and the current are adjusted such that a circular path is observed.
7) Find a current value such that when you change the voltage while the current is fixed, you can hit all lines in the ladder.
8) 4 data values are taken for different voltages.
9) The current is changed and 4 data values are taken again.
10) The same thing is done for 2 different constant voltages.
11) 4 sets of 4 data values are obtained.

IV. The Data

Our raw data looks like this:

TABLE I
RADIUS, VOLTAGE AND CURRENT VALUES

Radius (cm)	Voltage (V)	Current (A)
2.0	156	3.04
3.0	156	2.01
4.0	156	1.42
5.0	156	1.15
2.0	145	3.01
3.0	145	1.92
4.0	145	1.36
5.0	145	1.07
2.0	67	1.85
3.0	136	1.85
4.0	231	1.85
5.0	354	1.85
2.0	78	1.78
3.0	131	1.78
4.0	215	1.78
5.0	335	1.78

The uncertainties are as follows: 1V for voltage, 0.2 cm for radius and 0.01 A for current.

V. The Analysis

The analysis was made using CERN's framework ROOT, version v6.30.0. For the analysis, we have plotted $2V/r^2$ versus B^2 for constant voltages and 2V versus B^2r^2 for constant currents. After plotting these graphs, we have applied ROOT's built-in line fit to each plot. More details about the fits are given in the appendix. The slopes of the lines gives the charge-to-mass ratio.

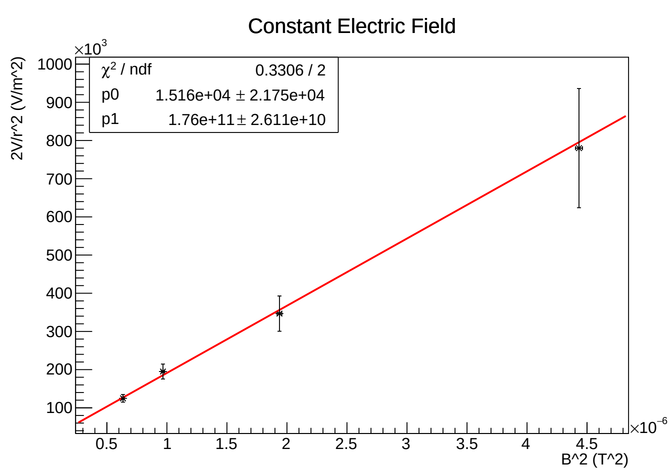

Fig. 4. Constant Electric Field Dataset 1

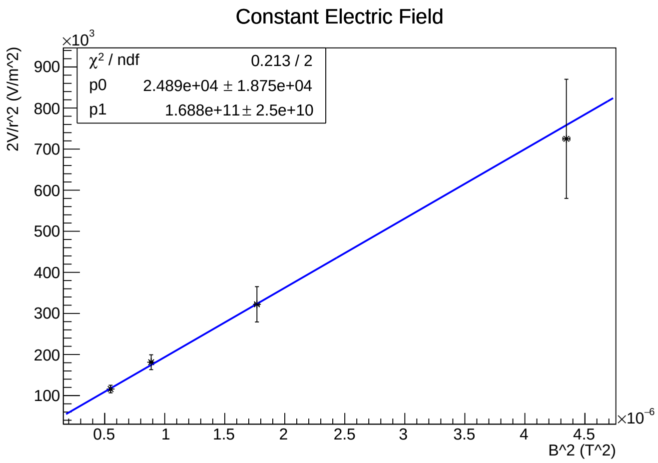

Fig. 5. Constant Electric Field Dataset 2

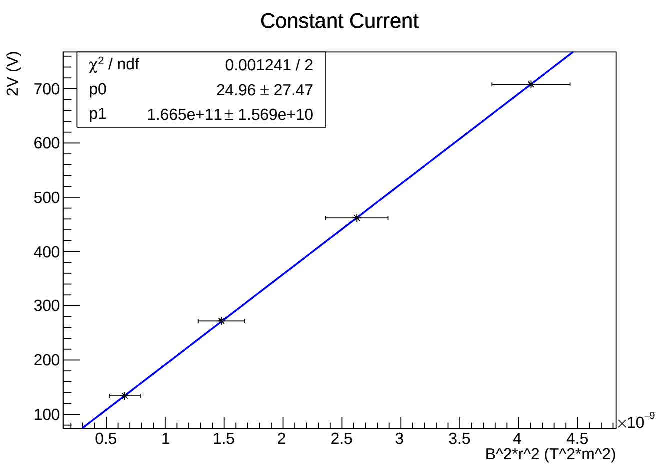

Fig. 6. Constant Current Dataset 1

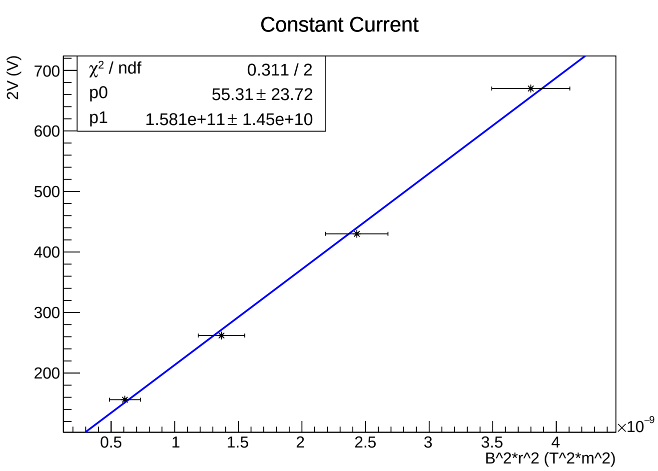

Fig. 7. Constant Current Dataset 2

The error propagation for the functions on the axis has been done using the error propagation formula[4], in the appendix.

We have calculated the uncertainty for each axis separately.

Error propagation for the magnetic field is:

$$\sigma_B = \sqrt{\left(\frac{\partial B}{\partial I}\right)^2 \sigma_I^2} \tag{9}$$

$$\sigma_B = \left(\frac{4}{5}\right)^{3/2} \mu_0 \frac{n}{R} \sigma_I \tag{10}$$

So, we have a constant value of $\sigma_B = 6.92 \times 10^{-6}$.

The uncertainty of B^2 is:

$$\sigma_{B^2} = 2B\sigma_B \tag{11}$$

Also, the uncertainty of $\frac{2V}{r^2}$ is:

$$\sigma_{\frac{2V}{r^2_{path}}} = \sqrt{\left(\frac{2}{r^2_{\text{path}}}\right)^2 (\sigma_V)^2 + \left(\frac{4V}{r^3_{\text{path}}}\right)^2 (\sigma_r)^2} \tag{12}$$

The uncertainty of B^2r^2 is:

$$\sigma_{B^2r^2} = 2Br\sqrt{r^2\sigma_B^2 + B^2\sigma_r^2} \tag{13}$$

And the uncertainty of 2V is simply equal to $2\sigma_V$.

After fitting the plots, we obtained 4 q/m ratios from 2 datasets. We used weighted average formula[4] for the final q/m ratios and their uncertainties, which is given in the appendix.

VI. The Result

After applying the weighted average formula, we have found the q/m ratios to be:

$$(q/m)_1 = 1.72 \times 10^{11} C/kg \pm 1.80 \times 10^{10} C/kg \tag{14}$$

$$(q/m)_2 = 1.62 \times 10^{11} C/kg \pm 1.06 \times 10^{10} C/kg \tag{15}$$

The accepted value of q/m for electron[1]:

$$q/m = 1.75 \times 10^{11} C/kg \tag{16}$$

Using the error calculation formula;

$$\text{Error} = \frac{\left|\left(\frac{q}{m}\right)_{\text{accp}} - \left(\frac{q}{m}\right)_{\text{weighted}}\right|}{\sigma_{\left(\frac{q}{m}\right)_{\text{weighted}}}} \tag{17}$$

Our first result is 0.15σ away from real value, while the second is 1.21σ away.

VII. THE CONCLUSION

In this experiment we have seen that the charge-to-mass ratio of the electron can be obtained by accelerating the electrons in an electric field, applying them a magnetic field perpendicular to its direction of motion and measuring the radius of the circular path the electrons travel on. We have used 2 different datasets to find the radius of the closed path, constant electric field and constant magnetic field. We observed that constant electric field yields better results. Both of our results are very close to the real value, and they deviate a low amount of σ. Although it is safe to say that the experiment was a success, there still can be errors. A source of error could be that the blue light was not very easy to observe. We tried our best to create an experiment environment as dark as possible, but the room could be a bit darker. Also, we could have used a precision of 0.1V for voltage instead of 1V.

REFERENCES

[1] *Mass-to-Charge ratio - Wikipedia*. URL: https://en.wikipedia.org/wiki/Mass-to-charge_ratio.

[2] *NobelPrize.org*. URL: https://www.nobelprize.org/prizes/lists/all-nobel-prizes-in-physics/.

[3] E. Gülmez. *Advanced Physics Experiments*. 1st. Boğaziçi University Publications, 1999.

[4] E. Gülmez. *Statistics Book*.

[5] *Regression Analysis - Wikipedia*. URL: https://en.wikipedia.org/wiki/Regression_analysis#Linear_regression.

VIII. APPENDIX

When you perform a ROOT line fit, you're essentially fitting a straight line to a set of data points in such a way that the sum of the squared differences between the observed values and the values predicted by the line (known as the residuals) is minimized. This is often done using the method of least squares. After performing a line fit in ROOT, we typically get several pieces of information as output, including the slope and intercept of the fitted line, as well as the χ^2 value of the fit.

The uncertainty propagation of the fits are calculated as:

$$\hat{\sigma}_{\beta_0} = \hat{\sigma}_\varepsilon \sqrt{\frac{1}{n} + \frac{\bar{x}^2}{\sum(x_i - \bar{x})^2}} \tag{18}$$

and,

$$\hat{\sigma}_{\beta_1} = \hat{\sigma}_\varepsilon \sqrt{\frac{1}{\sum(x_i - \bar{x})^2}} \tag{19}$$

where β_0 is the intercept and β_1 is the slope. Here, $\hat{\sigma}_\varepsilon$ is equal to:

$$\hat{\sigma}_\varepsilon^2 = \frac{SSR}{n-2} \tag{20}$$

Let's explain what "SSR" means. The residual, $e_i = y_i - \hat{y}_i$, is the difference between the value of the dependent variable predicted by the model, $\hat{y}_i$, and the true value of the dependent variable, y_i. One method of estimation is ordinary least squares. This method obtains parameter estimates that minimize the sum of squared residuals, SSR:

$$SSR = \sum_{i=1}^{n} e_i^2$$

Minimization of this function results in a set of normal equations, a set of simultaneous linear equations in the parameters, which are solved to yield the parameter estimators, $\hat{\beta}_0, \hat{\beta}_1$. For more details, refer to [5].

The error propagation formula used in analysis:

$$\sigma_f = \sqrt{\sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}\right)^2 \sigma_{x_i}^2} \tag{21}$$

The weighted average formula:

$$\left(\frac{q}{m}\right)_{\text{weighted}} = \frac{\sum_i \frac{\left(\frac{q}{m}\right)_i}{\sigma^2_{\left(\frac{q}{m}\right)_i}}}{\sum_i \frac{1}{\sigma^2_{\left(\frac{q}{m}\right)_i}}} \tag{22}$$

$$\sigma^2_{\left(\frac{q}{m}\right)_{weighted}} = \frac{1}{\sum_i \frac{1}{\sigma^2_{\left(\frac{q}{m}\right)_i}}} \tag{23}$$

```
#include <iostream>
#include <vector>
#include <string>
#include <cmath>
#include <utility>

{

double const rad = 0.2;
int const ncoil = 154;
float const mu0 = 1.257*pow(10, -6);
double const c = pow(0.8, 1.5);
float bc = (c*mu0*ncoil)/rad;
float qmtrue = 1.75*pow(10,11);

float rsigma = 0.002;
float vsigma = 1;
float isigma = 0.01;
float bsigma = 0.00000692;

TF1 *f1 = new TF1("f1", "[0] + [1]*x");
gStyle->SetOptFit(1);
gStyle->SetStatX(0.48);
gStyle->SetStatY(0.9);

TTree *t = new TTree("t", "t");
t->ReadFile("data.csv");
int n = t->GetEntries();
float * x, * y, * sx, * sy ;
x = new float[4];
y = new float[4];
sx = new float[4];
sy = new float[4];
float r,v,i;
t->SetBranchAddress("I", &i);
t->SetBranchAddress("V", &v);
```

```
std::vector<double> radius = {0.02, 0.03,
    0.04, 0.05};

std::vector<float> results;
std::vector<float> resultsigmas;


for (int j=0; j<2; j++){

for (int k = 0; k < 4; ++k) {
   t->GetEntry(k+j*4);
   x[k] = bc*bc*i*i;
   y[k] = (2*v)/(radius[k]*radius[k]);
   sx[k] = 2*bc*i*bsigma;
   sy[k] =
       sqrt(pow(((2*vsigma)/radius[k]*radius[k]),2)
       + pow((4*v*rsigma)/pow(radius[k],3),2));
}

TCanvas *c1 = new TCanvas();
TGraphErrors gr(4, x, y, sx, sy);
f1->SetParameter(1, 1.7*pow(10,11));
gr.Fit(f1,"S");
results.push_back(f1->GetParameter(1));
resultsigmas.push_back(f1->GetParError(1));
gr.Draw("A*");
f1->SetLineColor(kBlue);
gr.GetXaxis()->SetTitle("B^2 (T^2)");
gr.GetYaxis()->SetTitle("2V/r^2 (V/m^2)");
gr.SetTitle("Constant Electric Field");
std::string filename = "constE_" +
    std::to_string(j+1) + ".pdf";
c1->Print(filename.c_str());
}

for (int j=2; j<4; j++){

for (int k = 0; k < 4; ++k) {
   t->GetEntry(k+j*4);
   x[k] = bc*bc*i*i*(radius[k]*radius[k]);
   y[k] = (2*v);
   sx[k] =
       2*bc*i*radius[k]*sqrt(pow((radius[k]*bsigma),2)
       + pow((bc*i*rsigma),2));
   sy[k] = 2*vsigma;
}

TCanvas *c2 = new TCanvas();
gStyle->SetOptFit(1);
TGraphErrors gr(4, x, y, sx, sy);
f1->SetParameter(1, 1.7*pow(10,11));
gr.Fit(f1,"S");
results.push_back(f1->GetParameter(1));
resultsigmas.push_back(f1->GetParError(1));
gr.Draw("A*");
f1->SetLineColor(kBlue);
gr.GetXaxis()->SetTitle("B^2*r^2 (T^2*m^2)");
gr.GetYaxis()->SetTitle("2V (V)");
gr.SetTitle("Constant Current");
std::string filename = "constI_" +
    std::to_string(j-1) + ".pdf";
c2->Print(filename.c_str());
}

float uppersum1, uppersum2;
float lowersum1, lowersum2;

for (int i=0; i<2; i++){

uppersum1 += results[i]/pow(resultsigmas[i],
    2);
lowersum1 += 1/pow(resultsigmas[i], 2);

}

for (int i=2; i<4; i++){

uppersum2 += results[i]/pow(resultsigmas[i],
    2);
lowersum2 += 1/pow(resultsigmas[i], 2);

}

float finalqm1 = uppersum1/lowersum1;
float finalsigma1 = sqrt(1/lowersum1);

float finalqm2 = uppersum2/lowersum2;
float finalsigma2 = sqrt(1/lowersum2);
std::cout << "The Charge to Mass ratio with
    constant electric field is found to be: "
    << finalqm1 << " C/kg +- " << finalsigma1
    << " C/kg" <<std::endl;
std::cout << "It is " <<
    (qmtrue-finalqm1)/finalsigma1 << " away
    from true value." << std::endl;

std::cout << "The Charge to Mass ratio with
    constant current is found to be: " <<
    finalqm2 << " C/kg +- " << finalsigma2 <<
    " C/kg" <<std::endl;
std::cout << "It is " <<
    (qmtrue-finalqm2)/finalsigma2 << " sigma
    away from true value." << std::endl;


}

}
```