1 Full Navier Stokes and Continuity Equations

Starting off with the full Navier-Stokes and Continuity Equations, we have:

$$\rho(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}) = -\nabla p + \mu \nabla^2 \mathbf{u} \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{2}$$

We then expand the equations:

$$\rho(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}) = -\frac{\partial p}{\partial x} + \mu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}) \tag{3}$$

$$\rho(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}) = -\frac{\partial p}{\partial y} + \mu(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}) \tag{4}$$

$$\rho(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}) = -\frac{\partial p}{\partial z} + \mu(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}) \tag{5}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{6}$$

2 Nondimensionalization

Using the nondimensionalizations given by

$$u^* = \frac{u}{u_c}, \quad v^* = \frac{v}{u_c}, \quad w^* = \frac{w}{w_c}, \quad x^* = \frac{x}{\ell_1}, \quad y^* = \frac{y}{\ell_1}, \quad z^* = \frac{z}{\ell_2}, \quad p^* = \frac{p}{p_c}, \quad t^* = \frac{t}{t_c}, \tag{7}$$

We have for the continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{8}$$

$$\frac{u_c}{\ell_1}\frac{\partial u^*}{\partial x^*} + \frac{u_c}{\ell_1}\frac{\partial v^*}{\partial y^*} + \frac{w_c}{\ell_2}\frac{\partial w^*}{\partial z^*} = 0 \tag{9}$$

Which gives

$$w_c = \frac{\ell_2}{\ell_1} u_c \tag{10}$$

$$\frac{u_c}{\ell_1}\left(\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} + \frac{\partial w^*}{\partial z^*}\right) = 0 \tag{11}$$

And for the nondimensionalized continuity equation, we drop the stars to get:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{12}$$

3 Nondimensionalizing the x component

$$\rho(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}) = -\frac{\partial p}{\partial x} + \mu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}) \tag{13}$$

$$\rho(\frac{u_c}{t_c}\frac{\partial u^*}{\partial t^*} + \frac{u_c}{\ell_1}u_c u^*\frac{\partial u^*}{\partial x^*} + \frac{u_c}{\ell_1}u_c v^*\frac{\partial u^*}{\partial y^*} + \frac{u_c}{\ell_2}w_c w^*\frac{\partial u^*}{\partial z^*} = -\frac{p_c}{\ell_1}\frac{\partial p^*}{\partial x^*} + \mu(\frac{u_c}{\ell_1^2}\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{u_c}{\ell_1^2}\frac{\partial^2 u^*}{\partial y^{*2}} + \frac{u_c}{\ell_2^2}\frac{\partial^2 u^*}{\partial z^{*2}}) \tag{14}$$

$$\rho(\frac{u_c}{t_c}\frac{\partial u^*}{\partial t^*} + \frac{u_c^2}{\ell_1}u^*\frac{\partial u^*}{\partial x^*} + \frac{u_c^2}{\ell_1}v^*\frac{\partial u^*}{\partial y^*} + \frac{u_c^2}{\ell_1}w^*\frac{\partial u^*}{\partial z^*} = -\frac{p_c}{\ell_1}\frac{\partial p^*}{\partial x^*} + \mu(\frac{u_c}{\ell_1^2}\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{u_c}{\ell_1^2}\frac{\partial^2 u^*}{\partial y^{*2}} + \frac{u_c}{\ell_2^2}\frac{\partial^2 u^*}{\partial z^{*2}}) \tag{15}$$

$$\frac{\rho u_c^2}{\ell_1}(\frac{\ell_1}{u_c t_c}\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} + w^*\frac{\partial u^*}{\partial z^*}) = -\frac{p_c}{\ell_1}\frac{\partial p^*}{\partial x^*} + \mu(\frac{u_c}{\ell_1^2}\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{u_c}{\ell_1^2}\frac{\partial^2 u^*}{\partial y^{*2}} + \frac{u_c}{\ell_2^2}\frac{\partial^2 u^*}{\partial z^{*2}}) \tag{16}$$

$$\frac{\rho u_c^2}{\ell_1}\frac{\ell_2^2}{\mu u_c}(\frac{\ell_1}{u_c t_c}\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} + w^*\frac{\partial u^*}{\partial z^*}) = -\frac{p_c \ell_2^2}{\mu u_c \ell_1}\frac{\partial p^*}{\partial x^*} + \frac{\ell_2^2}{\ell_1^2}\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\ell_2^2}{\ell_1^2}\frac{\partial^2 u^*}{\partial y^{*2}} + \frac{\partial^2 u^*}{\partial z^{*2}} \tag{17}$$

Setting $t_c = \frac{\ell_1}{u_c}$, $\quad p_c = \frac{\mu u_c}{\ell_2^2/\ell_1}$, $\quad Re = \frac{\rho u_c(\ell_2^2/\ell_1)}{\mu}$,

$$Re(\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} + w^*\frac{\partial u^*}{\partial z^*}) = -\frac{\partial p^*}{\partial x^*} + (\frac{\ell_2}{\ell_1})^2(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}) + \frac{\partial^2 u^*}{\partial z^{*2}} \tag{18}$$

Finally, with the lubrication approximation of $\ell_2/\ell_1 << 1$, the terms with $(\frac{\ell_2}{\ell_1})^2$ become approximately 0, leading us to:

$$Re(\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} + w^*\frac{\partial u^*}{\partial z^*}) = -\frac{\partial p^*}{\partial x^*} + \frac{\partial^2 u^*}{\partial z^{*2}} \tag{19}$$

4 Nondimensionalizing the y component

$$\rho(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}) = -\frac{\partial p}{\partial y} + \mu(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}) \tag{20}$$

$$\rho(\frac{u_c}{t_c}\frac{\partial v^*}{\partial t^*} + \frac{u_c}{\ell_1}u_c u^*\frac{\partial v^*}{\partial x^*} + \frac{u_c}{\ell_1}u_c v^*\frac{\partial v^*}{\partial y^*} + \frac{u_c}{\ell_2}w_c w^*\frac{\partial v^*}{\partial z^*} = -\frac{p_c}{\ell_1}\frac{\partial p^*}{\partial y^*} + \mu(\frac{u_c}{\ell_1^2}\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{u_c}{\ell_1^2}\frac{\partial^2 v^*}{\partial y^{*2}} + \frac{u_c}{\ell_2^2}\frac{\partial^2 v^*}{\partial z^{*2}}) \tag{21}$$

$$\rho(\frac{u_c}{t_c}\frac{\partial v^*}{\partial t^*} + \frac{u_c^2}{\ell_1}u^*\frac{\partial v^*}{\partial x^*} + \frac{u_c^2}{\ell_1}v^*\frac{\partial v^*}{\partial y^*} + \frac{u_c^2}{\ell_1}w^*\frac{\partial v^*}{\partial z^*} = -\frac{p_c}{\ell_1}\frac{\partial p^*}{\partial y^*} + \mu(\frac{u_c}{\ell_1^2}\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{u_c}{\ell_1^2}\frac{\partial^2 v^*}{\partial y^{*2}} + \frac{u_c}{\ell_2^2}\frac{\partial^2 v^*}{\partial z^{*2}}) \tag{22}$$

$$\frac{\rho u_c^2}{\ell_1}(\frac{\ell_1}{u_c t_c}\frac{\partial v^*}{\partial t^*} + u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*} + w^*\frac{\partial v^*}{\partial z^*}) = -\frac{p_c}{\ell_1}\frac{\partial p^*}{\partial y^*} + \mu(\frac{u_c}{\ell_1^2}\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{u_c}{\ell_1^2}\frac{\partial^2 v^*}{\partial y^{*2}} + \frac{u_c}{\ell_2^2}\frac{\partial^2 v^*}{\partial z^{*2}}) \tag{23}$$

$$\frac{\rho u_c^2}{\ell_1}\frac{\ell_2^2}{\mu u_c}(\frac{\ell_1}{u_c t_c}\frac{\partial v^*}{\partial t^*} + u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*} + w^*\frac{\partial v^*}{\partial z^*}). = -\frac{p_c\ell_2^2}{\mu u_c\ell_1}\frac{\partial p^*}{\partial y^*} + \frac{\ell_2^2}{\ell_1^2}\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\ell_2^2}{\ell_1^2}\frac{\partial^2 v^*}{\partial y^{*2}} + \frac{\partial^2 v^*}{\partial z^{*2}} \tag{24}$$

Setting $t_c = \frac{\ell_1}{u_c}$, $\quad p_c = \frac{\mu u_c}{\ell_2^2/\ell_1}$, $\quad Re = \frac{\rho u_c(\ell_2^2/\ell_1)}{\mu}$,

$$Re(\frac{\partial v^*}{\partial t^*} + u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*} + w^*\frac{\partial v^*}{\partial z^*}) = -\frac{\partial p^*}{\partial y^*} + (\frac{\ell_2}{\ell_1})^2(\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial y^{*2}}) + \frac{\partial^2 v^*}{\partial z^{*2}} \tag{25}$$

With $\ell_2/\ell_1 << 1$,

$$Re(\frac{\partial v^*}{\partial t^*} + u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*} + w^*\frac{\partial v^*}{\partial z^*}) = -\frac{\partial p^*}{\partial y^*} + \frac{\partial^2 v^*}{\partial z^{*2}} \tag{26}$$

5 Nondimensionalizing the z component

$$\rho(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}) = -\frac{\partial p}{\partial z} + \mu(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}) \tag{27}$$

$$\rho(\frac{w_c}{t_c}\frac{\partial w^*}{\partial t^*} + \frac{w_c}{\ell_1}u_c u^*\frac{\partial w^*}{\partial x^*} + \frac{w_c}{\ell_1}u_c v^*\frac{\partial w^*}{\partial y^*} + \frac{w_c}{\ell_2}w_c w^*\frac{\partial w^*}{\partial z^*} = -\frac{p_c}{\ell_2}\frac{\partial p^*}{\partial z^*} + \mu(\frac{w_c}{\ell_1^2}\frac{\partial^2 w^*}{\partial x^{*2}} + \frac{w_c}{\ell_1^2}\frac{\partial^2 w^*}{\partial y^{*2}} + \frac{w_c}{\ell_2^2}\frac{\partial^2 w^*}{\partial z^{*2}}) \tag{28}$$

$$\rho(\frac{(\ell_2/\ell_1)u_c}{t_c}\frac{\partial w^*}{\partial t^*} + \frac{\ell_2 u_c^2}{\ell_1^2}u^*\frac{\partial w^*}{\partial x^*} + \frac{\ell_2 u_c^2}{\ell_1^2}v^*\frac{\partial w^*}{\partial y^*} + \frac{\ell_2 u_c^2}{\ell_1^2}w^*\frac{\partial w^*}{\partial z^*} = -\frac{p_c}{\ell_2}\frac{\partial p^*}{\partial z^*} + \mu(\frac{\ell_2 u_c}{\ell_1^3}\frac{\partial^2 w^*}{\partial x^{*2}} + \frac{\ell_2 u_c}{\ell_1^3}\frac{\partial^2 w^*}{\partial y^{*2}} + \frac{u_c}{\ell_1\ell_2}\frac{\partial^2 w^*}{\partial z^{*2}}) \tag{29}$$

$$\frac{\rho\ell_2 u_c^2}{\ell_1^2}(\frac{\ell_1}{u_c t_c}\frac{\partial w^*}{\partial t^*} + u^*\frac{\partial w^*}{\partial x^*} + v^*\frac{\partial w^*}{\partial y^*} + w^*\frac{\partial w^*}{\partial z^*}) = -\frac{p_c}{\ell_2}\frac{\partial p^*}{\partial z^*} + \mu(\frac{\ell_2 u_c}{\ell_1^3}\frac{\partial^2 w^*}{\partial x^{*2}} + \frac{\ell_2 u_c}{\ell_1^3}\frac{\partial^2 w^*}{\partial y^{*2}} + \frac{u_c}{\ell_1\ell_2}\frac{\partial^2 w^*}{\partial z^{*2}}) \tag{30}$$

$$\frac{\rho\ell_2 u_c^2}{\ell_1^2}\frac{\ell_1\ell_2}{\mu u_c}(\frac{\ell_1}{u_c t_c}\frac{\partial w^*}{\partial t^*} + u^*\frac{\partial w^*}{\partial x^*} + v^*\frac{\partial w^*}{\partial y^*} + w^*\frac{\partial w^*}{\partial z^*}) = -\frac{p_c\ell_1}{\mu u_c}\frac{\partial p^*}{\partial z^*} + \frac{\ell_2^2}{\ell_1^2}\frac{\partial^2 w^*}{\partial x^{*2}} + \frac{\ell_2^2}{\ell_1^2}\frac{\partial^2 w^*}{\partial y^{*2}} + \frac{\partial^2 w^*}{\partial z^{*2}} \tag{31}$$

Setting $t_c = \frac{\ell_1}{u_c}$, $\quad p_c = \frac{\mu u_c}{\ell_2^2/\ell_1}$, $\quad Re = \frac{\rho u_c(\ell_2^2/\ell_1)}{\mu}$,

$$Re(\frac{\partial w^*}{\partial t^*} + u^*\frac{\partial w^*}{\partial x^*} + v^*\frac{\partial w^*}{\partial y^*} + w^*\frac{\partial w^*}{\partial z^*}) = -\frac{\ell_1^2}{\ell_2^2}\frac{\partial p^*}{\partial z^*} + (\frac{\ell_2}{\ell_1})^2(\frac{\partial^2 w^*}{\partial x^{*2}} + \frac{\partial^2 w^*}{\partial y^{*2}}) + \frac{\partial^2 w^*}{\partial z^{*2}} \tag{32}$$

$$(\frac{\ell_2}{\ell_1})^2 Re(\frac{\partial w^*}{\partial t^*} + u^*\frac{\partial w^*}{\partial x^*} + v^*\frac{\partial w^*}{\partial y^*} + w^*\frac{\partial w^*}{\partial z^*}) = -\frac{\partial p^*}{\partial z^*} + (\frac{\ell_2}{\ell_1})^4(\frac{\partial^2 w^*}{\partial x^{*2}} + \frac{\partial^2 w^*}{\partial y^{*2}}) + (\frac{\ell_2}{\ell_1})^2\frac{\partial^2 w^*}{\partial z^{*2}} \tag{33}$$

With $\ell_2/\ell_1 << 1$, the $\frac{\partial^2 w^*}{\partial z^{*2}}$ goes away, but the left-hand side vanishes as well, giving us the same result for the z component as that for the previous analysis at low Reynolds numbers:

$$\frac{\partial p^*}{\partial z^*} = 0 \tag{34}$$

We drop the stars, and we have the following simplified equations:

$$Re(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}) = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial z^2} \tag{35}$$

$$Re(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}) = -\frac{\partial p}{\partial y} + \frac{\partial^2 v}{\partial z^2} \tag{36}$$

$$\frac{\partial p}{\partial z} = 0 \tag{37}$$

6 Nondimensionalizing the Solution Assumptions

Pressure p is a function of x and y only, and we can find u and v such that:

$$u(x,y,z) = f_1(x,y)\left(z^2 - \frac{\ell_2^2}{4}\right), \quad v(x,y,z) = f_2(x,y)\left(z^2 - \frac{\ell_2^2}{4}\right) \tag{38}$$

Using the nondimensionalizations given by

$$u^* = \frac{u}{u_c}, \quad v^* = \frac{v}{u_c}, \quad z^* = \frac{z}{\ell_2}, \quad f_1^* = \frac{f_1}{f_{1,c}}, \quad f_2^* = \frac{f_2}{f_{2,c}} \tag{39}$$

We can nondimensionalize u and v as such:

$$u_c u^* = f_{1,c} f_1^* \left(\ell_2^2 (z^*)^2 - \frac{\ell_2^2}{4}\right) \qquad u_c v^* = f_{2,c} f_2^* \left(\ell_2^2 (z^*)^2 - \frac{\ell_2^2}{4}\right) \tag{40}$$

$$u^* = \frac{\ell_2^2 f_{1,c}}{u_c} f_1^* \left((z^*)^2 - \frac{1}{4}\right) \qquad v^* = \frac{\ell_2^2 f_{2,c}}{u_c} f_2^* \left((z^*)^2 - \frac{1}{4}\right) \tag{41}$$

Setting $f_{1,c} = \frac{u_c}{\ell_2^2}$ and $f_{2,c} = \frac{u_c}{\ell_2^2}$,

$$u^* = f_1^* \left((z^*)^2 - \frac{1}{4}\right) \qquad v^* = f_2^* \left((z^*)^2 - \frac{1}{4}\right) \tag{42}$$

We now drop the stars to arrive at the nondimensionalized solution assumptions.

$$u(x,y,z) = f_1(x,y)\left(z^2 - \frac{1}{4}\right), \quad v(x,y,z) = f_2(x,y)\left(z^2 - \frac{1}{4}\right) \tag{43}$$

$$\frac{\partial u}{\partial z} = 2zf_1, \quad \frac{\partial v}{\partial z} = 2zf_2 \tag{44}$$

$$\frac{\partial^2 u}{\partial z^2} = 2f_1, \quad \frac{\partial^2 v}{\partial z^2} = 2f_2 \tag{45}$$

7 Plugging in the Solution Assumptions

Plugging these in,

$$Re(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + 2wzf_1) = -\frac{\partial p}{\partial x} + 2f_1 \tag{46}$$

$$Re(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + 2wzf_2) = -\frac{\partial p}{\partial y} + 2f_2 \tag{47}$$

Since u and v are not dependent on t, the time derivatives vanish, and we can continue to substitute in more derivatives:

$$\frac{\partial u}{\partial x} = (z^2 - \frac{1}{4})\frac{\partial f_1}{\partial x}, \quad \frac{\partial u}{\partial y} = (z^2 - \frac{1}{4})\frac{\partial f_1}{\partial y} \tag{48}$$

$$\frac{\partial v}{\partial x} = (z^2 - \frac{1}{4})\frac{\partial f_2}{\partial x}, \quad \frac{\partial v}{\partial y} = (z^2 - \frac{1}{4})\frac{\partial f_2}{\partial y} \tag{49}$$

$$Re((z^2 - \frac{1}{4})^2\frac{\partial f_1}{\partial x}f_1 + (z^2 - \frac{1}{4})^2\frac{\partial f_1}{\partial y}f_2 + 2wzf_1) = -\frac{\partial p}{\partial x} + 2f_1 \tag{50}$$

$$Re((z^2 - \frac{1}{4})^2\frac{\partial f_2}{\partial x}f_1 + (z^2 - \frac{1}{4})^2\frac{\partial f_2}{\partial y}f_2 + 2wzf_2) = -\frac{\partial p}{\partial y} + 2f_2 \tag{51}$$

For depth-averaging, we integrate these two equations across the z-direction, and divide by 1.

$$\int_{-1/2}^{1/2} \left(Re\left(\left(z^2 - \frac{1}{4}\right)^2 \frac{\partial f_1}{\partial x} f_1 + \left(z^2 - \frac{1}{4}\right)^2 \frac{\partial f_1}{\partial y} f_2 + 2wzf_1 \right)\right) dz = \int_{-1/2}^{1/2} \left(-\frac{\partial p}{\partial x} + 2f_1 \right) dz \tag{52}$$

$$\int_{-1/2}^{1/2} \left(Re\left(\left(z^2 - \frac{1}{4}\right)^2 \frac{\partial f_2}{\partial x} f_1 + \left(z^2 - \frac{1}{4}\right)^2 \frac{\partial f_2}{\partial y} f_2 + 2wzf_2 \right)\right) dz = \int_{-1/2}^{1/2} \left(-\frac{\partial p}{\partial y} + 2f_2 \right) dz \tag{53}$$

This becomes:

$$Re\int_{-1/2}^{1/2}(\frac{\partial f_1}{\partial x}f_1)(z^4 - \frac{1}{2}z^2 + \frac{1}{16})dz + Re\int_{-1/2}^{1/2}(\frac{\partial f_1}{\partial y}f_2)(z^4 - \frac{1}{2}z^2 + \frac{1}{16})dz + Re\int_{-1/2}^{1/2} 2wzf_1 dz = \int_{-1/2}^{1/2} -\frac{\partial p}{\partial x}dz + \int_{-1/2}^{1/2} 2f_1 dz \tag{54}$$

$$Re\int_{-1/2}^{1/2}(\frac{\partial f_2}{\partial x}f_1)(z^4 - \frac{1}{2}z^2 + \frac{1}{16})dz + Re\int_{-1/2}^{1/2}(\frac{\partial f_2}{\partial y}f_2)(z^4 - \frac{1}{2}z^2 + \frac{1}{16})dz + Re\int_{-1/2}^{1/2} 2wzf_2 dz = \int_{-1/2}^{1/2} -\frac{\partial p}{\partial y}dz + \int_{-1/2}^{1/2} 2f_2 dz \tag{55}$$

In order to solve for w, we use the continuity equation.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{56}$$

$$\left(z^2 - \frac{1}{4}\right)\frac{\partial f_1}{\partial x} + \left(z^2 - \frac{1}{4}\right)\frac{\partial f_2}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{57}$$

$$\frac{\partial w}{\partial z} = -\left(z^2 - \frac{1}{4}\right)\left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}\right) \tag{58}$$

$$\int dw = \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}\right)\int -\left(z^2 - \frac{1}{4}\right)dz \tag{59}$$

$$w = -\left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}\right)\left(\frac{1}{3}z^3 - \frac{1}{4}z + f_3\right) \tag{60}$$

In order for w to satisfy the boundary conditions of $w = 0$ at $+1/2$ or $-1/2$, $\left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}\right)$ must equal 0, which means $w = 0$. Now we can plug in $w = 0$, and solve the integrals to yield the depth-averaged equations.

$$Re(\frac{\partial f_1}{\partial x}f_1)(\frac{1}{30}) + Re(\frac{\partial f_1}{\partial y}f_2)(\frac{1}{30}) = -\frac{\partial p}{\partial x} + \int_{-1/2}^{1/2} 2f_1 dz \tag{61}$$

$$Re(\frac{\partial f_2}{\partial x}f_1)(\frac{1}{30}) + Re(\frac{\partial f_2}{\partial y}f_2)(\frac{1}{30}) = -\frac{\partial p}{\partial y} + \int_{-1/2}^{1/2} 2f_2 dz \tag{62}$$

This is equivalent to:

$$Re\left(\frac{1}{30}\right)\left(\frac{\partial f_1}{\partial x}f_1+\frac{\partial f_1}{\partial y}f_2\right)=-\frac{\partial p}{\partial x}+\int\limits_{-1/2}^{1/2}2f_1dz \tag{63}$$

$$Re\left(\frac{1}{30}\right)\left(\frac{\partial f_2}{\partial x}f_1+\frac{\partial f_2}{\partial y}f_2\right)=-\frac{\partial p}{\partial y}+\int\limits_{-1/2}^{1/2}2f_2dz \tag{64}$$

With the depth-averaged velocities, we can solve for f_1, f_2, and their derivatives.

$$\bar{u}(x,y)=\int\limits_{-1/2}^{1/2}u(x,y,z)\mathrm{d}z=-\frac{1}{6}f_1(x,y) \tag{65}$$

$$\bar{v}(x,y)=\int\limits_{-1/2}^{1/2}v(x,y,z)\mathrm{d}z=-\frac{1}{6}f_2(x,y). \tag{66}$$

$$f_1=-6\bar{u},\qquad f_2=-6\bar{v} \tag{67}$$

$$\frac{\partial f_1}{\partial x}=-6\frac{\partial\bar{u}}{\partial x},\quad \frac{\partial f_1}{\partial y}=-6\frac{\partial\bar{u}}{\partial y},\quad \frac{\partial f_2}{\partial x}=-6\frac{\partial\bar{v}}{\partial x},\quad \frac{\partial f_2}{\partial y}=-6\frac{\partial\bar{v}}{\partial y} \tag{68}$$

Substituting these in, we arrive at:

$$\frac{6}{5}Re\left(\bar{u}\frac{\partial\bar{u}}{\partial x}+\bar{v}\frac{\partial\bar{u}}{\partial y}\right)=-\frac{\partial p}{\partial x}-12\bar{u} \tag{69}$$

$$\frac{6}{5}Re\left(\bar{u}\frac{\partial\bar{v}}{\partial x}+\bar{v}\frac{\partial\bar{v}}{\partial y}\right)=-\frac{\partial p}{\partial y}-12\bar{v} \tag{70}$$

$$\frac{6}{5}Re\left(\bar{\boldsymbol{u}}\cdot\nabla_\perp\right)\bar{\boldsymbol{u}}=-\nabla_\perp p-12\bar{\boldsymbol{u}} \tag{71}$$

8 Rewriting into Streamfunction-Vorticity Formulation

We then take $\frac{\partial}{\partial y}$ of the x-component equation.

$$\frac{6}{5}Re\left(\frac{\partial\bar{u}}{\partial y}\frac{\partial\bar{u}}{\partial x}+\bar{u}\frac{\partial^2\bar{u}}{\partial x\partial y}+\frac{\partial\bar{v}}{\partial y}\frac{\partial\bar{u}}{\partial y}+\bar{v}\frac{\partial^2\bar{u}}{\partial y^2}\right)=-\frac{\partial^2 p}{\partial x\partial y}-12\frac{\partial\bar{u}}{\partial y} \tag{72}$$

Similarly, we take $\frac{\partial}{\partial x}$ of the y-component equation.

$$\frac{6}{5}Re\left(\frac{\partial\bar{u}}{\partial x}\frac{\partial\bar{v}}{\partial x}+\bar{u}\frac{\partial^2\bar{v}}{\partial x^2}+\frac{\partial\bar{v}}{\partial x}\frac{\partial\bar{v}}{\partial y}+\bar{v}\frac{\partial^2\bar{v}}{\partial x\partial y}\right)=-\frac{\partial^2 p}{\partial x\partial y}-12\frac{\partial\bar{v}}{\partial x} \tag{73}$$

Then we proceed to subtract the y-component equation from the x-component equation. (Eq. 72 - Eq. 73)

$$\frac{6}{5}Re\left(\frac{\partial\bar{u}}{\partial x}\left(\frac{\partial\bar{u}}{\partial y}-\frac{\partial\bar{v}}{\partial x}\right)+\bar{u}\frac{\partial}{\partial x}\left(\frac{\partial\bar{u}}{\partial y}-\frac{\partial\bar{v}}{\partial x}\right)+\frac{\partial\bar{v}}{\partial y}\left(\frac{\partial\bar{u}}{\partial y}-\frac{\partial\bar{v}}{\partial x}\right)+\bar{v}\frac{\partial}{\partial y}\left(\frac{\partial\bar{u}}{\partial y}-\frac{\partial\bar{v}}{\partial x}\right)\right)=-12\left(\frac{\partial\bar{u}}{\partial y}-\frac{\partial\bar{v}}{\partial x}\right) \tag{74}$$

We rearrange to get:

$$\frac{6}{5}Re\left(\left(\frac{\partial\bar{u}}{\partial y}-\frac{\partial\bar{v}}{\partial x}\right)\left(\frac{\partial\bar{u}}{\partial x}+\frac{\partial\bar{v}}{\partial y}\right)+\bar{u}\frac{\partial}{\partial x}\left(\frac{\partial\bar{u}}{\partial y}-\frac{\partial\bar{v}}{\partial x}\right)+\bar{v}\frac{\partial}{\partial y}\left(\frac{\partial\bar{u}}{\partial y}-\frac{\partial\bar{v}}{\partial x}\right)\right)=-12\left(\frac{\partial\bar{u}}{\partial y}-\frac{\partial\bar{v}}{\partial x}\right) \tag{75}$$

Plugging in the definitions of $u=\frac{\partial\psi}{\partial y}$, $v=-\frac{\partial\psi}{\partial x}$, and $\omega=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}$, the first term goes to zero, and we have:

$$\frac{6}{5}Re\left(\frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y}-\frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x}\right)=-12\omega \tag{76}$$

9 Next Steps

Poisson's Equation can be the second equation needed to solve for ψ and ω.

$$-\omega = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \tag{77}$$

Once we solve for ψ, we can get u and v.

10 Vorticity Equation

$$\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u} \tag{78}$$

Plugging in our assumptions of $u(x,y,z) = f_1(x,y)\left(z^2 - \frac{\ell_2^2}{4}\right)$, $v(x,y,z) = f_2(x,y)\left(z^2 - \frac{\ell_2^2}{4}\right)$, and $w = 0$,

$$\boldsymbol{\omega} = -\frac{\partial v(x,y,z)}{\partial z}\vec{i} + \frac{\partial u(x,y,z)}{\partial z}\vec{j} + \left(\frac{\partial v(x,y,z)}{\partial x} - \frac{\partial u(x,y,z)}{\partial y}\right)\vec{k} \tag{79}$$

$$\boldsymbol{\omega} = -2zf_2\vec{i} + 2zf_1\vec{j} + \left(\frac{\partial f_2}{\partial x}\left(z^2 - \frac{\ell_2^2}{4}\right) - \frac{\partial f_1}{\partial y}\left(z^2 - \frac{\ell_2^2}{4}\right)\right)\vec{k} \tag{80}$$

We use the vorticity-transport equation below:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\,\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\,\mathbf{u} + \nu \nabla^2 \boldsymbol{\omega} \tag{81}$$

$$\frac{\partial \omega_x}{\partial t} + u\frac{\partial \omega_x}{\partial x} + v\frac{\partial \omega_x}{\partial y} + w\frac{\partial \omega_x}{\partial z} = \left(\omega_x\frac{\partial u}{\partial x} + \omega_y\frac{\partial u}{\partial y} + \omega_z\frac{\partial u}{\partial z}\right) + \nu\left(\frac{\partial^2 \omega_x}{\partial x^2} + \frac{\partial^2 \omega_x}{\partial y^2} + \frac{\partial^2 \omega_x}{\partial z^2}\right) \tag{82}$$

$$\frac{\partial \omega_y}{\partial t} + u\frac{\partial \omega_y}{\partial x} + v\frac{\partial \omega_y}{\partial y} + w\frac{\partial \omega_y}{\partial z} = \left(\omega_x\frac{\partial v}{\partial x} + \omega_y\frac{\partial v}{\partial y} + \omega_z\frac{\partial v}{\partial z}\right) + \nu\left(\frac{\partial^2 \omega_y}{\partial x^2} + \frac{\partial^2 \omega_y}{\partial y^2} + \frac{\partial^2 \omega_y}{\partial z^2}\right) \tag{83}$$

$$\frac{\partial \omega_z}{\partial t} + u\frac{\partial \omega_z}{\partial x} + v\frac{\partial \omega_z}{\partial y} + w\frac{\partial \omega_z}{\partial z} = \left(\omega_x\frac{\partial w}{\partial x} + \omega_y\frac{\partial w}{\partial y} + \omega_z\frac{\partial w}{\partial z}\right) + \nu\left(\frac{\partial^2 \omega_z}{\partial x^2} + \frac{\partial^2 \omega_z}{\partial y^2} + \frac{\partial^2 \omega_z}{\partial z^2}\right) \tag{84}$$

Substituting ω into the vorticity-transport equation, setting $f_\Omega = \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}$, we have for the x-component:

$$\frac{\partial \omega_x}{\partial t} + \left(f_1\left(z^2 - \frac{\ell_2^2}{4}\right)\right)\left(-2z\frac{\partial f_2}{\partial x}\right) + \left(f_2\left(z^2 - \frac{\ell_2^2}{4}\right)\right)\left(-2z\frac{\partial f_2}{\partial y}\right) \tag{85}$$

$$= (-2zf_2)\left(\frac{\partial f_1}{\partial x}\left(z^2 - \frac{\ell_2^2}{4}\right)\right) + (2zf_1)\left(\frac{\partial f_1}{\partial y}\left(z^2 - \frac{\ell_2^2}{4}\right)\right) + f_\Omega\left(z^2 - \frac{\ell_2^2}{4}\right)(2zf_1) \tag{86}$$

$$+\nu\left(-2z\frac{\partial^2 f_2}{\partial x^2} - 2z\frac{\partial^2 f_2}{\partial y^2}\right) \tag{87}$$

The y-component:

$$\frac{\partial \omega_y}{\partial t} + \left(f_1\left(z^2 - \frac{\ell_2^2}{4}\right)\right)\left(2z\frac{\partial f_1}{\partial x}\right) + \left(f_2\left(z^2 - \frac{\ell_2^2}{4}\right)\right)\left(2z\frac{\partial f_1}{\partial y}\right) \tag{88}$$

$$= (-2zf_2)\left(\frac{\partial f_2}{\partial x}\left(z^2 - \frac{\ell_2^2}{4}\right)\right) + (2zf_1)\left(\frac{\partial f_2}{\partial y}\left(z^2 - \frac{\ell_2^2}{4}\right)\right) + f_\Omega\left(z^2 - \frac{\ell_2^2}{4}\right)(2zf_2) \tag{89}$$

$$+\nu\left(2z\frac{\partial^2 f_1}{\partial x^2} + 2z\frac{\partial^2 f_1}{\partial y^2}\right) \tag{90}$$

And the z-component,

$$\frac{\partial \omega_z}{\partial t} + \left(f_1\left(z^2 - \frac{\ell_2^2}{4}\right)\right)\left(\frac{\partial f_\Omega}{\partial x}\left(z^2 - \frac{\ell_2^2}{4}\right)\right) + \left(f_2\left(z^2 - \frac{\ell_2^2}{4}\right)\right)\left(\frac{\partial f_\Omega}{\partial y}\left(z^2 - \frac{\ell_2^2}{4}\right)\right) \tag{91}$$

$$= \nu\left(\left(z^2 - \frac{\ell_2^2}{4}\right)\left(\frac{\partial^2 f_\Omega}{\partial x^2} + \frac{\partial^2 f_\Omega}{\partial y^2}\right) + 2f_\Omega\right) \tag{92}$$

We simplify these equations to get for x:

$$\frac{\partial \omega_x}{\partial t} - 2z\left(z^2 - \frac{\ell_2^2}{4}\right)\left(f_1\frac{\partial f_2}{\partial x} + f_2\frac{\partial f_2}{\partial y}\right) = -2z\left(z^2 - \frac{\ell_2^2}{4}\right)\left(f_2\frac{\partial f_1}{\partial x} - f_1\frac{\partial f_1}{\partial y} - f_\Omega f_1\right) - 2z\nu\left(\frac{\partial^2 f_2}{\partial x^2} + \frac{\partial^2 f_2}{\partial y^2}\right) \tag{93}$$

y:

$$\frac{\partial \omega_y}{\partial t} + 2z\left(z^2 - \frac{\ell_2^2}{4}\right)\left(f_1\frac{\partial f_1}{\partial x} + f_2\frac{\partial f_1}{\partial y}\right) = -2z\left(z^2 - \frac{\ell_2^2}{4}\right)\left(f_2\frac{\partial f_2}{\partial x} - f_1\frac{\partial f_2}{\partial y} - f_\Omega f_2\right) + 2z\nu\left(\frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_1}{\partial y^2}\right) \tag{94}$$

z:

$$\frac{\partial \omega_z}{\partial t} + \left(z^4 - \frac{\ell_2^2}{2}z^2 + \frac{\ell_2^4}{16}\right)\left(f_1\frac{\partial f_\Omega}{\partial x} + f_2\frac{\partial f_\Omega}{\partial y}\right) = \nu\left(\left(z^2 - \frac{\ell_2^2}{4}\right)\left(\frac{\partial^2 f_\Omega}{\partial x^2} + \frac{\partial^2 f_\Omega}{\partial y^2}\right) + 2f_\Omega\right) \tag{95}$$

For depth-averaging, we integrate these equations across the z direction, and divide by ℓ_2.

$$\frac{1}{\ell_2}\int_{-\ell_2/2}^{\ell_2/2} \frac{\partial \omega_x}{\partial t} - 2z\left(z^2 - \frac{\ell_2^2}{4}\right)\left(f_1\frac{\partial f_2}{\partial x} + f_2\frac{\partial f_2}{\partial y}\right) dz \tag{96}$$

$$= \frac{1}{\ell_2}\int_{-\ell_2/2}^{\ell_2/2} -2z\left(z^2 - \frac{\ell_2^2}{4}\right)\left(f_2\frac{\partial f_1}{\partial x} - f_1\frac{\partial f_1}{\partial y} - f_\Omega f_1\right) - 2z\nu\left(\frac{\partial^2 f_2}{\partial x^2} + \frac{\partial^2 f_2}{\partial y^2}\right) dz \tag{97}$$

y:

$$\frac{1}{\ell_2}\int_{-\ell_2/2}^{\ell_2/2} \frac{\partial \omega_y}{\partial t} + 2z\left(z^2 - \frac{\ell_2^2}{4}\right)\left(f_1\frac{\partial f_1}{\partial x} + f_2\frac{\partial f_1}{\partial y}\right) dz \tag{98}$$

$$= \frac{1}{\ell_2}\int_{-\ell_2/2}^{\ell_2/2} -2z\left(z^2 - \frac{\ell_2^2}{4}\right)\left(f_2\frac{\partial f_2}{\partial x} - f_1\frac{\partial f_2}{\partial y} - f_\Omega f_1\right) + 2z\nu\left(\frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_1}{\partial y^2}\right) dz \tag{99}$$

z:

$$\frac{1}{\ell_2}\int_{-\ell_2/2}^{\ell_2/2} \frac{\partial \omega_z}{\partial t} + \left(z^4 - \frac{\ell_2^2}{2}z^2 + \frac{\ell_2^4}{16}\right)\left(f_1\frac{\partial f_\Omega}{\partial x} + f_2\frac{\partial f_\Omega}{\partial y}\right) dz = \frac{1}{\ell_2}\int_{-\ell_2/2}^{\ell_2/2} \nu\left(\left(z^2 - \frac{\ell_2^2}{4}\right)\left(\frac{\partial^2 f_\Omega}{\partial x^2} + \frac{\partial^2 f_\Omega}{\partial y^2}\right) + 2f_\Omega\right) dz \tag{100}$$

The definite integrals involving the functions $2z\left(z^2 - \frac{\ell_2^2}{4}\right)$ and $2z$ are odd functions, so they become 0, and we are left with the z component equation.

$$\frac{\partial \bar{\omega}_z}{\partial t} + \frac{\ell_2^4}{30}\left(f_1\frac{\partial f_\Omega}{\partial x} + f_2\frac{\partial f_\Omega}{\partial y}\right) = \nu\left(-\frac{\ell_2^2}{6}\left(\frac{\partial^2 f_\Omega}{\partial x^2} + \frac{\partial^2 f_\Omega}{\partial y^2}\right) + 2f_\Omega\right) \tag{101}$$

We can also depth-average vorticity by integrating across the z direction, and dividing by ℓ_2.

$$\bar{\boldsymbol{\omega}} = \frac{1}{\ell_2}\int_{-\ell_2/2}^{\ell_2/2} \boldsymbol{\omega} dz = \frac{1}{\ell_2}\int_{-\ell_2/2}^{\ell_2/2} -2zf_2\vec{i} + 2zf_1\vec{j} + f_\Omega\left(z^2 - \frac{\ell_2^2}{4}\right)\vec{k}dz \tag{102}$$

The x and y components reduce to 0 when the definite integrals are carried out, and we are only left with the z-component equation.

$$\bar{\omega}_z = -\frac{\ell_2^2}{6}f_\Omega \tag{103}$$

This can be rearranged to become:

$$f_\Omega = -\frac{6}{\ell_2^2}\bar{\boldsymbol{\omega}}_z \tag{104}$$

Now, if we substitute this into our streamfunction-vorticity equation, we get:

$$\frac{\partial \bar{\omega}_z}{\partial t} = \frac{1}{5}\ell_2^2\left(f_1\frac{\partial \bar{\omega}_z}{\partial x} + f_2\frac{\partial \bar{\omega}_z}{\partial y}\right) + \nu\left(\frac{\partial^2 \bar{\omega}_z}{\partial x^2} + \frac{\partial^2 \bar{\omega}_z}{\partial y^2}\right) - \frac{12\nu}{\ell_2^2}\bar{\omega}_z \tag{105}$$

To get another stream function equation, we plug in the definitions of $f_1 = \frac{\partial \psi}{\partial y}$ and $f_2 = -\frac{\partial \psi}{\partial x}$ into the depth-averaged vorticity equation.

$$\bar{\omega}_z = -\frac{\ell_2^2}{6}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) = -\frac{\ell_2^2}{6}\left(-\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2}\right) \tag{106}$$

Rearranging this, we arrive at:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \frac{6\bar{\omega}_z}{\ell_2^2} \tag{107}$$