

Big / Omega Notation: prove that $g(n) = n^3 + 2n^2 + 4n$ is $\Omega(n^3)$

Sol)

$$g(n) \geq c \cdot n^3$$

$$g(n) = n^3 + 2n^2 + 4n$$

For finding constants c and n_0

$$n^3 + 2n^2 + 4n \geq c \cdot n^3$$

Divide both sides with n^3

$$1 + \frac{2n^2}{n^3} + \frac{4n}{n^3} \geq c$$

$$1 + \frac{2}{n} + \frac{4}{n^2} \geq c$$

Here $\frac{2}{n}$ and $\frac{4}{n^2}$ approaches 0

$$1 + \frac{2}{n} + \frac{4}{n^2} \gg 1$$

Example $c = \frac{1}{2}$

$$1 + \frac{2}{n} + \frac{4}{n^2} \geq \frac{1}{2}$$

$$1 + \frac{2}{n} + \frac{4}{n^2} \geq 1 \quad (1 \geq \frac{1}{2}, n \geq 1)$$

$$1 + \frac{2}{n} + \frac{4}{n^2} \geq \frac{1}{2} \quad (n \geq 1, n_0 = 1)$$

Thus $g(n) = n^3 + 2n^2 + 4n$ is indeed $\Omega(n^3)$

Big Theta Notation Determine whether $h(n)$
 $= 4n^2 + 3n$ is $\Theta(n^2)$ or not

$$C_1 n^2 \leq h(n) \leq C_2 n^2$$

In upper bound $h(n)$ is $O(n^2)$

In lower bound $h(n)$ is $\Omega(n^2)$

Upper Bound ($O(n^2)$):

$$h(n) = 4n^2 + 3n$$

$$h(n) \leq C_2 n^2$$

$$4n^2 + 3n \leq C_2 n^2 \Rightarrow 4n^2 + 3n \leq 5n^2$$

$$\text{let's } C_2 = 5$$

Divide both sides by n^2

$$4 + \frac{3}{n} \leq 5$$

$$h(n) = 4n^2 + 3n \text{ is } O(n^2) (C_2 = 5, n_0 = 1)$$

Lower bound:

$$h(n) = 4n^2 + 3n$$

$$h(n) \geq C_1 n^2$$

$$4n^2 + 3n \geq C_1 n^2$$

$$\text{let's } C_1 = 4 \Rightarrow 4n^2 + 3n \geq 4n^2$$

Divide both sides by n^2

$$4 + \frac{3}{n} \geq 4$$

$$h(n) = 4n^2 + 3n (C_1 = 4, n_0 = 1)$$

$$h(n) = 4n^2 + 3n \text{ is } \Theta(n^2)$$

(3)

Let $f(n) = n^3 - 2n^2 + n$ and $g(n) = n^2$. Show whether $f(n) = \Omega(g(n))$ is true or false and justify your answer.

$$f(n) \geq c \cdot g(n)$$

Substituting $f(n)$ and $g(n)$ into this inequality we get

$$n^3 - 2n^2 + n \geq c \cdot (n^2)$$

And c and n_0 holds $n \geq n_0$

$$n^3 - 2n^2 + n \geq -cn^2$$

$$n^3 - 2n^2 + n + cn^2 \geq 0$$

$$n^3 + (c-2)n^2 + n \geq 0$$

$$n^3 + (c-2)n^2 + n \geq 0 \quad (n^3 \geq 0)$$

$$n^3 + (1-2)n^2 + n = n^3 - n^2 + n \geq 0 \quad (c=2)$$

$$f(n) = n^3 - 2n^2 + n \text{ is } \Omega(g(n)) = \Omega(n^2)$$

Therefore the statement $f(n) = \Omega(g(n))$ is True

Determine whether $h(n) = n \log n + n$ is $\Theta(n \log n)$

Prove a rigorous proof for your conclusion

$$c_1 n \log n \leq h(n) \leq c_2 n \log n$$

Upper Bound:

$$h(n) \leq c_2 n \log n$$

$$h(n) = n \log n + n$$

$$n \log n + n \leq c_2 n \log n$$

Divide both sides by $n \log n$

$$1 + \frac{n}{n \log n} \leq 2$$

$$1 + \frac{1}{\log n} \leq C_2 \quad (\text{simplify})$$

$$1 + \frac{1}{\log n} \leq 2 \quad (C_2 = 2)$$

Then $h(n)$ is $O(n \log n)$ ($C_2 = 2, n_0 = 2$)

Lower Bound:

$$h(n) \geq C_1 n \log n$$

$$h(n) = n \log n + n$$

$$n \log n + n \geq C_1 n \log n$$

Divide both sides by $n \log n$

$$1 + \frac{n}{n \log n} \geq C_1$$

$$1 + \frac{1}{\log n} \geq C_1 \quad (\text{simplify})$$

$$1 + \frac{1}{\log n} \geq 1 \quad (C_1 = 1)$$

$$\frac{1}{\log n} \geq 0 \quad \text{For all } n > 1$$

$h(n)$ is $\Omega(n \log n)$ ($C_1 = 1, n_0 = 1$)

$h(n) = n \log n + n$ is $\Theta(n \log n)$

Solve the following recurrence relations and find the order of growth of solutions.

$$T(n) = 4T(n/2) + n^2, T(1) = 1$$

$$T(n) = 4T(n/2) + n^2, \quad T(1) = 1$$

$$T(n) = aT(n/b) + f(n)$$

$$a = 4, \quad b = 2, \quad f(n) = n^2$$

Applying master theorem

$$T(n) = aT(n/b) + f(n)$$

$$f(n) = O(n^{\log_b a - \epsilon}) \quad \left(\begin{array}{l} \epsilon > 0 \\ T(n) = \Theta(n^{\log_b a}) \end{array} \right)$$

$$f(n) = \Theta(n^{\log_b a}) \text{, then } T(n) = \Theta(n^{\log_b a} \log n)$$

$$f(n) = \Omega(n^{\log_b a + \epsilon}) \text{, then } T(n) = \Theta\left(\frac{n^{\log_b a} \log n}{f(n)}\right)$$

Calculating $\log_b a$:

$$\log_b a = \log_2 4 = 2$$

$$f(n) = n^2 = \Theta(n^2) \quad \left(\text{comparing } f(n) \text{ with } n^{\frac{\log a}{\log b}} \right)$$

$$f(n) = \Theta(n^2) = \Theta(n^{\log_b a}), \quad (\text{Case 2})$$

$$T(n) = 4T(n/2) + n^2$$

$$T(n) = \Theta(n^{\log_b a} \log n) = \Theta(n^2 \log n)$$

order of growth

$$T(n) = 4T(n/2) + n^2 \text{ with } T(1) = 1$$

$$\text{is } \Theta(n^2 \log n).$$