4.

$$\begin{vmatrix} 3x & -x+y & -x+z \\ x-y & 3y & z-y \\ x-z & y-z & 3z \end{vmatrix}$$

Solution:

Given, $\begin{vmatrix} 3x & -x+y & -x+z \\ x-y & 3y & z-y \\ x-z & y-z & 3z \end{vmatrix}$

[Applying $C_1 \to C_1 + C_2 + C_3$]

$$= \begin{vmatrix} x+y+z & -x+y & -x+z \\ x+y+z & 3y & z-y \\ x+y+z & y-z & 3z \end{vmatrix}$$

[Taking $(x+y+z)$ common from column C_1]

$$= (x+y+z)\begin{vmatrix} 1 & -x+y & -x+z \\ 1 & 3y & z-y \\ 1 & y-z & 3z \end{vmatrix}$$

[Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$]

$$= (x+y+z)\begin{vmatrix} 1 & -x+y & -x+z \\ 0 & 2y+x & x-y \\ 0 & x-z & 2z+x \end{vmatrix}$$

[Applying $C_2 \to C_2 - C_3$]

$$= (x+y+z)\begin{vmatrix} 1 & -x+y & -x+z \\ 0 & 3y & x-y \\ 0 & -3z & 2z+x \end{vmatrix}$$

[Expanding along first column]

$= (x + y + z) . 1[3y(3z + x) + (3z)(x − y)]$

$= (x + y + z)(3yz + 3yx + 3xz)$

$= 3(x + y + z)(xy + yz + zx)$