16. If p th, q th, and r th terms of an A.P. and G.P. are both a, b and c respectively, show that

$a^{b-c} . b^{c-a} . c^{a-b} = 1$

Solution:

Let the first term of AP be m and common difference as d
Let the GP first term as l and common ratio as s
The n^{th} term of an AP is given as $t_n = a + (n - 1) d$ where a is the first term and d is the common difference
The n^{th} term of a GP is given by $t_n = ar^{n-1}$ where a is the first term and r is the common ratio
The p^{th} term (t_p) of both AP and GP is a
For AP
$\Rightarrow t_p = m + (p - 1) d$
$\Rightarrow a = m + (p - 1) d$ 1
For GP
$\Rightarrow t_p = ls^{p-1}$
$\Rightarrow a = ls^{p-1}$ 2
The q^{th} term (t_q) of both AP and GP is b
For AP
$\Rightarrow t_q = m + (q - 1) d$
$\Rightarrow b = m + (q - 1) d$ 3
For GP
$\Rightarrow t_q = ls^{q-1}$
$\Rightarrow b = ls^{q-1}$ 4
The r^{th} term (t_r) of both AP and GP is c
For AP
$\Rightarrow t_r = m + (r - 1) d$
$\Rightarrow c = m + (r - 1) d$ 5
For GP
$\Rightarrow t_r = ls^{r-1}$

$\Rightarrow c = ls^{r-1}$ 6
Let us find b – c, c – a and a – b
Using 3 and 5
$\Rightarrow b - c = (q - r) d$ ... (i)
Using 5 and 1
$\Rightarrow c - a = (r - p) d$ ... (ii)
Using 1 and 3
$\Rightarrow a - b = (p - q) d$ ... (iii)
We have to prove that $a^{b-c}.b^{c-a}.c^{a-b} = 1$
LHS = $a^{b-c}.b^{c-a}.c^{a-b})$

$\Rightarrow c - a = (r - p)\ d$ ... (ii)

Using 1 and 3

$\Rightarrow a - b = (p - q)\ d$ ... (iii)

We have to prove that $a^{b-c}.b^{c-a}.c^{a-b} = 1$

LHS = $a^{b-c}.b^{c-a}.c^{a-b}$)

Using 2, 4 and 6

$\Rightarrow$ LHS = $(ls^{p-1})^{b-c}.(ls^{q-1})^{c-a}.(ls^{r-1})^{a-b}$

$= \left(\frac{ls^p}{s}\right)^{b-c}.\left(\frac{ls^q}{s}\right)^{c-a}.\left(\frac{ls^r}{s}\right)^{a-b}$

$= \frac{l^{b-c}s^{p(b-c)}}{s^{b-c}}.\frac{l^{c-a}s^{q(c-a)}}{s^{c-a}}.\frac{l^{a-b}s^{r(a-b)}}{s^{a-b}}$

$= \frac{l^{b-c+c-a+a-b}}{s^{b-c+c-a+a-b}}.s^{p(b-c)}.s^{q(c-a)}.s^{r(a-b)}$

$= s^{p(b-c)}.s^{q(c-a)}.s^{r(a-b)}$

Substituting values of a – b, c – a and b – c from (iii), (ii) and (i)

$= s^{p(q-r)d}.s^{q(r-p)d}.s^{r(p-q)d}$

$= s^{pqd-prd}.s^{qrd-pqd}.s^{prd-qrd}$

$= s^{pqd-prd+qrd-pqd+prd-qrd}$

$= s^0 = 1$

$\Rightarrow$ LHS = RHS

Hence proved