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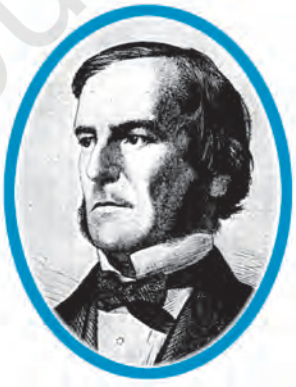
Chapter 14

MATHEMATICAL REASONING

❖*There are few things which we know which are not capable of mathematical reasoning and when these can not, it is a sign that our knowledge of them is very small and confused and where a mathematical reasoning can be had, it is as great a folly to make use of another, as to grope for a thing in the dark when you have a candle stick standing by you. – ARTHENBOT* ❖

14.1 Introduction

In this Chapter, we shall discuss about some basic ideas of Mathematical Reasoning. All of us know that human beings evolved from the lower species over many millennia. The main asset that made humans "*superior*" to other species was the ability to reason. How well this ability can be used depends on each person's power of reasoning. How to develop this power? Here, we shall discuss the process of reasoning especially in the context of mathematics.



George Boole
(1815 - 1864)

In mathematical language, there are two kinds of reasoning – inductive and deductive. We have already discussed the inductive reasoning in the context of mathematical induction. In this Chapter, we shall discuss some fundamentals of deductive reasoning.

14.2 Statements

The basic unit involved in mathematical reasoning is a *mathematical statement*.

Let us start with two sentences:

In 2003, the president of India was a woman.

An elephant weighs more than a human being.

When we read these sentences, we immediately decide that the first sentence is false and the second is correct. There is no confusion regarding these. In mathematics such sentences are called *statements.*

On the other hand, consider the sentence:

Women are more intelligent than men.

Some people may think it is true while others may disagree. Regarding this sentence we cannot say whether it is always true or false . That means this sentence is ambiguous. Such a sentence is not acceptable as a statement in mathematics.

A sentence is called a mathematically acceptable statement if it is either true or false but not both. Whenever we mention a statement here, it is a "*mathematically acceptable*" statement.

While studying mathematics, we come across many such sentences. Some examples are:

Two plus two equals four.

The sum of two positive numbers is positive.

All prime numbers are odd numbers.

Of these sentences, the first two are *true* and the third one is *false*. There is no ambiguity regarding these sentences. Therefore, they are statements.

Can you think of an example of a sentence which is vague or ambiguous? Consider the sentence:

The sum of x and y is greater than 0

Here, we are not in a position to determine whether it is true or false, unless we know what x and y are. For example, it is false where $x = 1$, $y = -3$ and true when $x = 1$ and $y = 0$. Therefore, this sentence is not a statement. But the sentence:

For any natural numbers x and y, the sum of x and y is greater than 0

is a statement.

Now, consider the following sentences :

How beautiful!

Open the door.

Where are you going?

Are they statements? No, because the first one is an exclamation, the second an order and the third a question. None of these is considered as a statement in mathematical language. Sentences involving variable time such as "today", "tomorrow" or "yesterday" are not statements. This is because it is not known what time is referred here. For example, the sentence

Tomorrow is Friday

is not a statement. The sentence is correct (true) on a Thursday but not on other days. The same argument holds for sentences with pronouns unless a particular person is referred to and for variable places such as "here", "there" etc., For example, the sentences

She is a mathematics graduate.

Kashmir is far from here.

are not statements.

Here is another sentence

There are 40 days in a month.

Would you call this a statement? Note that the period mentioned in the sentence above is a "variable time" that is any of 12 months. But we know that the sentence is always false (irrespective of the month) since the maximum number of days in a month can never exceed 31. Therefore, this sentence is a statement. So, what makes a sentence a statement is the fact that the sentence is either true or false but not both.

While dealing with statements, we usually denote them by small letters p, q, r,... For example, we denote the statement "*Fire is always hot*" by p. This is also written as

p: Fire is always hot.

Example 1 Check whether the following sentences are statements. Give reasons for your answer.

(i) 8 is less than 6.
(ii) Every set is a finite set.
(iii) The sun is a star.
(iv) Mathematics is fun.
(v) There is no rain without clouds.
(vi) How far is Chennai from here?

Solution (i) This sentence is false because 8 is greater than 6. Hence it is a statement.

(ii) This sentence is also false since there are sets which are not finite. Hence it is a statement.

(iii) It is a scientifically established fact that sun is a star and, therefore, this sentence is always true. Hence it is a statement.

(iv) This sentence is subjective in the sense that for those who like mathematics, it may be fun but for others it may not be. This means that this sentence is not always true. Hence it is not a statement.

(v) It is a scientifically established natural phenomenon that cloud is formed before it rains. Therefore, this sentence is always true. Hence it is a statement.

(vi) This is a question which also contains the word "Here". Hence it is not a statement.

The above examples show that whenever we say that a sentence is a statement we should always say why it is so. This "why" of it is more important than the answer.

EXERCISE 14.1

1. Which of the following sentences are statements? Give reasons for your answer.

(i) There are 35 days in a month.

(ii) Mathematics is difficult.

(iii) The sum of 5 and 7 is greater than 10.

(iv) The square of a number is an even number.

(v) The sides of a quadrilateral have equal length.

(vi) Answer this question.

(vii) The product of (–1) and 8 is 8.

(viii) The sum of all interior angles of a triangle is 180°.

(ix) Today is a windy day.

(x) All real numbers are complex numbers.

2. Give three examples of sentences which are not statements. Give reasons for the answers.

14.3 New Statements from Old

We now look into method for producing new statements from those that we already have. An English mathematician, "George Boole" discussed these methods in his book "The laws of Thought" in 1854. Here, we shall discuss two techniques.

As a first step in our study of statements, we look at an important technique that we may use in order to deepen our understanding of mathematical statements. This technique is to ask not only what it means to say that a given statement is true but also what it would mean to say that the given statement is not true.

14.3.1 *Negation of a statement* The denial of a statement is called the *negation* of the statement.

Let us consider the statement:

p: New Delhi is a city

The negation of this statement is