Exemplar Problem

Trigonometric Functions

4. If $\cos(\alpha + \beta) = \frac{4}{5}$ and $\sin(\alpha - \beta) = \frac{5}{13}$, where α lie between 0 and $\pi/4$, find value of $\tan 2\alpha$

[Hint: Express tan 2α as tan $(\alpha + \beta + \alpha - \beta]$

Solution:

According to the question,

$$cos(\alpha + \beta) = 4/5 ...(i)$$

We know that,

$$\sin x = \sqrt{1 - \cos^2 x}$$

Therefore,

$$\sin (\alpha + \beta) = \sqrt{1 - \cos^2 (\alpha + \beta)}$$

$$\Rightarrow$$
 sin $(\alpha + \beta) = \sqrt{(1 - (4/5)^2)} = 3/5 ...(ii)$

Also,

$$sin(\alpha - \beta) = 5/13 \{given\} ...(iii)$$

we know that,

$$\cos x = \sqrt{(1 - \sin^2 x)}$$

Therefore,

$$cos(\alpha - \beta) = \sqrt{1 - sin^2(\alpha - \beta)}$$

$$\Rightarrow$$
 cos $(\alpha - \beta) = \sqrt{1 - (5/13)^2} = 12/13 ...(iv)$

Therefore,

$$\tan 2\alpha = \tan (\alpha + \beta + \alpha - \beta)$$

We know that,

$$tan(x + y) = \frac{tan x + tan y}{1 - tan x tan y}$$

$$\frac{tan(\alpha + \beta) + tan(\alpha - \beta)}{1 - tan(\alpha + \beta) + tan(\alpha - \beta)}$$

$$\therefore tan 2\alpha = \frac{tan(\alpha + \beta) + tan(\alpha - \beta)}{tan(\alpha + \beta) + tan(\alpha - \beta)}$$

$$\frac{sin(\alpha + \beta) + tan(\alpha - \beta)}{tan(\alpha + \beta) + tan(\alpha - \beta)}$$

$$\Rightarrow tan 2\alpha = \frac{tan x + tan y}{1 - tan(\alpha + \beta) + tan(\alpha - \beta)}$$

From equation i, ii, iii and iv we have,

$$\Rightarrow \tan 2\alpha = \frac{\frac{3}{\frac{5}{4}} + \frac{5}{\frac{13}{12}}}{1 - \frac{5}{4} \times \frac{13}{12}}$$

$$= \frac{\frac{3}{4} + \frac{5}{12}}{1 - \frac{3}{4} \times \frac{5}{12}}$$

$$= \frac{\frac{9 + 5}{12}}{1 - \frac{15}{48}}$$

$$\Rightarrow \tan 2\alpha = \frac{14}{12\left(\frac{33}{48}\right)}$$

$$= \frac{56}{33}$$

Hence, $\tan 2\alpha = 56/33$