4. $\left(\frac{x}{3}+\frac{1}{x}\right)^5$ **5.** $\left(x+\frac{1}{x}\right)^6$

Using binomial theorem, evaluate each of the following:

6. $(96)^3$ **7.** $(102)^5$ **8.** $(101)^4$

9. $(99)^5$

10. Using Binomial Theorem, indicate which number is larger $(1.1)^{10000}$ or 1000.

11. Find $(a + b)^4 – (a – b)^4$. Hence, evaluate $(\sqrt{3}+\sqrt{2})^4 - (\sqrt{3}-\sqrt{2})^4$.

12. Find $(x + 1)^6 + (x – 1)^6$. Hence or otherwise evaluate $(\sqrt{2} + 1)^6 + (\sqrt{2} – 1)^6$.

13. Show that $9^{n+1} – 8n – 9$ is divisible by 64, whenever n is a positive integer.

14. Prove that $\sum_{r=0}^{n} 3^r \, {}^nC_r = 4^n$.

8.3 General and Middle Terms

1. In the binomial expansion for $(a + b)^n$, we observe that the first term is ${}^nC_0 a^n$, the second term is ${}^nC_1 a^{n-1}b$, the third term is ${}^nC_2 a^{n-2}b^2$, and so on. Looking at the pattern of the successive terms we can say that the $(r + 1)^{th}$ term is ${}^nC_r a^{n-r}b^r$. The $(r + 1)^{th}$ term is also called the *general term* of the expansion $(a + b)^n$. It is denoted by T_{r+1}. Thus $T_{r+1} = {}^nC_r\, a^{n-r}b^r$.

2. Regarding the middle term in the expansion $(a + b)^n$, we have

(i) If n is even, then the number of terms in the expansion will be $n + 1$. Since n is even so $n + 1$ is odd. Therefore, the middle term is $\left(\frac{n+1+1}{2}\right)^{th}$, i.e., $\left(\frac{n}{2}+1\right)^{th}$ term.

For example, in the expansion of $(x + 2y)^8$, the middle term is $\left(\frac{8}{2}+1\right)^{th}$ i.e., 5^{th} term.

(ii) If n is odd, then $n + 1$ is even, so there will be two middle terms in the expansion, namely, $\left(\frac{n+1}{2}\right)^{th}$ term and $\left(\frac{n+1}{2}+1\right)^{th}$ term. So in the expansion

$(2x - y)^7$, the middle terms are $\left(\frac{7+1}{2}\right)^{th}$, i.e., 4th and $\left(\frac{7+1}{2}+1\right)^{th}$, i.e., 5th term.

3. In the expansion of $\left(x+\frac{1}{x}\right)^{2n}$, where $x \neq 0$, the middle term is $\left(\frac{2n+1+1}{2}\right)^{th}$, i.e., $(n + 1)^{th}$ term, as $2n$ is even.

It is given by ${}^{2n}C_n x^n \left(\frac{1}{x}\right)^n = {}^{2n}C_n$ (constant).

This term is called the *term independent* of x or the constant term.

Example 5 Find a if the 17th and 18th terms of the expansion $(2 + a)^{50}$ are equal.

Solution The $(r + 1)^{th}$ term of the expansion $(x + y)^n$ is given by $T_{r+1} = {}^nC_r x^{n-r} y^r$.

For the 17th term, we have, $r + 1 = 17$, i.e., $r = 16$

Therefore, $T_{17} = T_{16+1} = {}^{50}C_{16}\ (2)^{50-16}\ a^{16}$

$= {}^{50}C_{16}\ 2^{34}\ a^{16}.$

Similarly, $T_{18} = {}^{50}C_{17}\ 2^{33}\ a^{17}$

Given that $T_{17} = T_{18}$

So ${}^{50}C_{16}\ (2)^{34}\ a^{16} = {}^{50}C_{17}\ (2)^{33}\ a^{17}$

Therefore $$\frac{{}^{50}C_{16}\,.\,2^{34}}{{}^{50}C_{17}\,.\,2^{33}} = \frac{a^{17}}{a^{16}}$$

i.e., $$a = \frac{{}^{50}C_{16} \times 2}{{}^{50}C_{17}} = \frac{50!}{16!\,34!} \times \frac{17!\ .\ 33!}{50!} \times 2 = 1$$

Example 6 Show that the middle term in the expansion of $(1+x)^{2n}$ is $\frac{1.3.5...(2n-1)}{n!}\,2n\,x^n$, where n is a positive integer.

Solution As $2n$ is even, the middle term of the expansion $(1 + x)^{2n}$ is $\left(\frac{2n}{2}+1\right)^{\text{th}}$, i.e., $(n + 1)^{\text{th}}$ term which is given by,

$$T_{n+1} = {}^{2n}C_n(1)^{2n-n}(x)^n = {}^{2n}C_n x^n = \frac{(2n)!}{n!\ n!}x^n$$

$$= \frac{2n(2n-1)\ (2n-2)\ ...4.3.2.1}{n!\ n!}x^n$$

$$= \frac{1.2.3.4...(2n-2)(2n-1)(2n)}{n!n!}x^n$$

$$= \frac{[1.3.5...(2n-1)][2.4.6...(2n)]}{n!n!}x^n$$

$$= \frac{[1.3.5...(2n-1)]2^n[1.2.3...n]}{n!n!}x^n$$

$$= \frac{[1.3.5...(2n-1)]\,n!}{n!\ n!}2^n.x^n$$

$$= \frac{1.3.5...(2n-1)}{n!}2^n x^n$$

Example 7 Find the coefficient of x^6y^3 in the expansion of $(x + 2y)^9$.

Solution Suppose x^6y^3 occurs in the $(r + 1)^{\text{th}}$ term of the expansion $(x + 2y)^9$.

Now $\quad T_{r+1} = {}^9C_r\, x^{9-r}(2y)^r = {}^9C_r\, 2^r\,.\,x^{9-r}\,.\,y^r$.

Comparing the indices of x as well as y in x^6y^3 and in T_{r+1}, we get $r = 3$.

Thus, the coefficient of x^6y^3 is

$${}^9C_3\, 2^3 = \frac{9!}{3!\,6!}.2^3 = \frac{9.8.7}{3.2}.2^3 = 672.$$

Example 8 The second, third and fourth terms in the binomial expansion $(x + a)^n$ are 240, 720 and 1080, respectively. Find x, a and n.

Solution Given that second term $T_2 = 240$

We have $T_2 = {}^nC_1 x^{n-1} . a$

So ${}^nC_1 x^{n-1} . a = 240$ ... (1)

Similarly ${}^nC_2 x^{n-2} a^2 = 720$ ... (2)

and ${}^nC_3 x^{n-3} a^3 = 1080$ ... (3)

Dividing (2) by (1), we get

$$\frac{{}^nC_2 x^{n-2} a^2}{{}^nC_1 x^{n-1} a} = \frac{720}{240} \quad \text{i.e.,} \quad \frac{(n-1)!}{(n-2)!} . \frac{a}{x} = 6$$

or $$\frac{a}{x} = \frac{6}{(n-1)} \quad \text{... (4)}$$

Dividing (3) by (2), we have

$$\frac{a}{x} = \frac{9}{2(n-2)} \quad \text{... (5)}$$

From (4) and (5),

$$\frac{6}{n-1} = \frac{9}{2(n-2)} . \qquad \text{Thus, } n = 5$$

Hence, from (1), $5x^4 a = 240$, and from (4), $\frac{a}{x} = \frac{3}{2}$

Solving these equations for a and x, we get $x = 2$ and $a = 3$.

Example 9 The coefficients of three consecutive terms in the expansion of $(1 + a)^n$ are in the ratio1: 7 : 42. Find n.

Solution Suppose the three consecutive terms in the expansion of $(1 + a)^n$ are $(r - 1)^{th}$, r^{th} and $(r + 1)^{th}$ terms.

The $(r - 1)^{th}$ term is ${}^nC_{r-2}\, a^{r-2}$, and its coefficient is ${}^nC_{r-2}$. Similarly, the coefficients of r^{th} and $(r + 1)^{th}$ terms are ${}^nC_{r-1}$ and ${}^nC_r$, respectively.

Since the coefficients are in the ratio 1 : 7 : 42, so we have,

$$\frac{{}^nC_{r-2}}{{}^nC_{r-1}} = \frac{1}{7}, \text{ i.e., } n - 8r + 9 = 0 \quad \text{... (1)}$$

and $$\frac{{}^nC_{r-1}}{{}^nC_r} = \frac{7}{42}, \text{ i.e., } n - 7r + 1 = 0 \quad \text{... (2)}$$

Solving equations(1) and (2), we get, $n = 55$.

EXERCISE 8.2

Find the coefficient of

1. x^5 in $(x + 3)^8$ **2.** a^5b^7 in $(a - 2b)^{12}$.

Write the general term in the expansion of

3. $(x^2 - y)^6$ **4.** $(x^2 - yx)^{12}, x \neq 0$.

5. Find the 4^{th} term in the expansion of $(x - 2y)^{12}$.

6. Find the 13^{th} term in the expansion of $\left(9x - \frac{1}{3\sqrt{x}}\right)^{18}, x \neq 0$.

Find the middle terms in the expansions of

7. $\left(3 - \frac{x^3}{6}\right)^7$ **8.** $\left(\frac{x}{3} + 9y\right)^{10}$.

9. In the expansion of $(1 + a)^{m+n}$, prove that coefficients of a^m and a^n are equal.

10. The coefficients of the $(r - 1)^{th}$, r^{th} and $(r + 1)^{th}$ terms in the expansion of $(x + 1)^n$ are in the ratio 1 : 3 : 5. Find n and r.

11. Prove that the coefficient of x^n in the expansion of $(1 + x)^{2n}$ is twice the coefficient of x^n in the expansion of $(1 + x)^{2n-1}$.

12. Find a positive value of m for which the coefficient of x^2 in the expansion $(1 + x)^m$ is 6.

Miscellaneous Examples

Example 10 Find the term independent of x in the expansion of $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^6$.

Solution We have $T_{r+1} = {}^6C_r \left(\frac{3}{2}x^2\right)^{6-r} \left(-\frac{1}{3x}\right)^r$

$$= {}^6C_r \left(\frac{3}{2}\right)^{6-r} \left(x^2\right)^{6-r} (-1)^r \left(\frac{1}{x}\right)^r \left(\frac{1}{3^r}\right)$$