theory problem on matrix inverse

Example 5 If A is 3×3 invertible matrix, then show that for any scalar k (non-zero), kA is invertible and $(kA)^{-1} = \frac{1}{k}A^{-1}$

Solution We have

$$(kA)\left(\frac{1}{k}A^{-1}\right) = \left(k.\frac{1}{k}\right)(A.\ A^{-1}) = 1\ (I) = I$$

Hence (kA) is inverse of $\left(\frac{1}{k}A^{-1}\right)$ or $(kA)^{-1} = \frac{1}{k}A^{-1}$

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practice problem on row operations

67. On using elementary row operation $R_1 \to R_1 - 3R_2$ in the following matrix equation $\begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$, we have

(a) $\begin{bmatrix} -5 & -7 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -7 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} -5 & -7 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} -1 & -3 \\ 1 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} -5 & -7 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & -7 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 4 & 2 \\ -5 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & -3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

Sol. (a) We have, $\begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

using elementary row operation $R_1 \to R_1 - 3R_2$

$$\begin{bmatrix} -5 & -7 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -7 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

Since, on using elementary row operation on $X = AB$, we apply these operation simultaneously on X and on the first matrix A of the product AB on RHS.

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40. If $A = \begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$, then find $A^2 - 5A - 14I$. Hence, obtain A^3.

Sol. We have, $A = \begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$ ...(i)

$\therefore$ $A^2 = A \cdot A = \begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 29 & -25 \\ -20 & 24 \end{bmatrix}$

$\therefore$ $A^2 - 5A - 14I = \begin{bmatrix} 29 & -25 \\ -20 & 24 \end{bmatrix} - \begin{bmatrix} 15 & -25 \\ -20 & 10 \end{bmatrix} - \begin{bmatrix} 14 & 0 \\ 0 & 14 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Now, $A^2 - 5A - 14I = O$

$\Rightarrow$ $A \cdot A^2 - 5A \cdot A = 14A\,I = O$

$\Rightarrow$ $A^3 - 5A^2 - 14A = O$

$\Rightarrow$ $A^3 = 5A^2 + 14A$

$= 5\begin{bmatrix} 29 & -25 \\ -20 & 24 \end{bmatrix} + 14\begin{bmatrix} 3 & -5 \\ -4 & -2 \end{bmatrix}$

$= \begin{bmatrix} 145 & -125 \\ -100 & 120 \end{bmatrix} + \begin{bmatrix} 42 & -70 \\ -56 & 28 \end{bmatrix} = \begin{bmatrix} 187 & -195 \\ -156 & 148 \end{bmatrix}$

NOTE: for topics covered in this lecture, there is very few problems in ncert. to practice concepts covered here, please solve problems from similar and past year ques pdfs.

Useful formulas and concepts

How to find the Rank of a Matrix?
To find the rank of a matrix, we will transform that matrix into its echelon form.
Then determine the rank by the number of non zero rows.

I have demoed a process in next page with an example.

The rank of a unit matrix of order m is m.

If A matrix is of order $m \times n$, then $\rho(A) \leq \min\{m, n\}$ = minimum of m, n.

If A is of order $n \times n$ and $|A| \neq 0$, then the rank of A = n.

If A is of order $n \times n$ and $|A| = 0$, then the rank of A will be less than n.

There are three cases for system of linear equations:

Case-1

Consider $Ax=b$

rank(A)=rank(A|b)=n	unique solution

Case-2

rank(A)=rank(A|b)=m<n	infinte solutions

Case-3

rank(A)≠rank(A|b)	no solution

Rank of a Matrix by Row – Echelon Form

We can transform a given non-zero matrix to a simplified form called a Row-echelon form, using the row elementary operations . In this form, we may have rows all of whose entries are zero. Such rows are called zero rows. A non-zero row is one in which at least one of the elements is not zero.

Example 3:

Find the rank of the matrix.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Solution:

Given

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$R_2 \rightarrow R_2 - R_1$

$R_3 \rightarrow R_3 - R_1$

We get

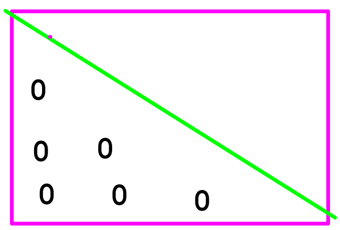
$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Here number of non zero rows = 1

Hence the rank of the matrix = 1

TRICK:

If a matrix is in row-echelon form, then all elements below the leading diagonal are zeros.



Important matrix tip for inverse of matrix:

SHORT TRICK TO FIND INVERSE OF 3 × 3 MATRIX

To find inverse of $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

STEP 1

$$\begin{matrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{matrix}$$

Copy I^{st} column and II^{nd} column

STEP 2

$$\begin{matrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \\ a & b & c & a & b \\ d & e & f & d & e \end{matrix}$$

Copy I^{st} row and II^{nd} row form step 1

STEP 3

$$\begin{matrix} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \\ a & b & c & a & b \\ d & e & f & d & e \end{matrix}$$

Neglect first row

From up to down arrow take positive sign

From down to up arrow take negative sign

Neglect first column

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} ei-hf & fg-id & dh-eg \\ hc-bi & ai-cg & bg-ah \\ bf-ec & cd-af & ae-bd \end{bmatrix}^T = \begin{bmatrix} ei-hf & hc-bi & bf-ec \\ fg-id & ai-cg & cd-af \\ dh-eg & bg-ah & ae-bd \end{bmatrix}$$