Example 7 Differentiate $\sqrt{\tan\sqrt{x}}$ w.r.t. x

Solution Let $y = \sqrt{\tan\sqrt{x}}$. Using chain rule, we have

$$\frac{dy}{dx} = \frac{1}{2\sqrt{\tan\sqrt{x}}}.\frac{d}{dx}(\tan\sqrt{x})$$

$$= \frac{1}{2\sqrt{\tan\sqrt{x}}}.\sec^2\sqrt{x}\frac{d}{dx}(\sqrt{x})$$

$$= \frac{1}{2\sqrt{\tan\sqrt{x}}}(\sec^2\sqrt{x})\left(\frac{1}{2\sqrt{x}}\right)$$

$$= \frac{(\sec^2\sqrt{x})}{4\sqrt{x}\sqrt{\tan\sqrt{x}}}.$$

42. $\tan^{-1}\left(\dfrac{3a^2x-x^3}{a^3-3ax^2}\right), \dfrac{1}{\sqrt{3}}<\dfrac{x}{a}<\dfrac{1}{\sqrt{3}}$

Sol. Let $$y=\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right)=\tan^{-1}\left(\frac{3\frac{x}{a}-\left(\frac{x}{a}\right)^3}{1-3\left(\frac{x}{a}\right)^2}\right)$$

Put $\boxed{x=a\tan\theta \Rightarrow \theta=\tan^{-1}\frac{x}{a}}$

$\therefore$ $$y=\tan^{-1}\left[\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right]=\tan^{-1}(\tan 3\theta)=3\theta=3\tan^{-1}\frac{x}{a}$$

$\therefore$ $$\frac{dy}{dx}=3\frac{d}{dx}\tan^{-1}\frac{x}{a}$$

$$=3\left[\frac{1}{1+\frac{x^2}{a^2}}\right]\cdot\frac{d}{dx}\left(\frac{x}{a}\right)=\frac{3a^2}{a^2+x^2}\cdot\frac{1}{a}=\frac{3a}{a^2+x^2}$$

28. $\log [\log (\log x^5)]$

Sol. Let $y = \log [\log (\log x^5)]$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}[\log(\log \log x^5)]$$

$$= \frac{1}{\log \log x^5} \cdot \frac{d}{dx}(\log \cdot \log x^5)$$

$$= \frac{1}{\log \log x^5} \cdot \frac{1}{\log x^5} \cdot \frac{d}{dx} \log x^5$$

$$= \frac{1}{\log \log x^5} \cdot \frac{1}{\log x^5} \cdot \frac{d}{dx}(5 \log x)$$

$$= \frac{5}{x \cdot \log(\log x^5) \cdot \log(x^5)}$$

Try to solve exemplar problems specifically from 25-43 for this video's concepts.

31. $\cos(\tan\sqrt{x+1})$

Sol. Let $y = \cos(\tan\sqrt{x+1})$

$$\frac{dy}{dx} = \frac{d}{dx}\cos(\tan\sqrt{x+1})$$

$$= -\sin(\tan\sqrt{x+1})\frac{d}{dx}(\tan\sqrt{x+1})$$

$$= -\sin(\tan\sqrt{x+1})\sec^2\sqrt{x+1}\cdot\frac{d}{dx}(x+1)^{1/2}$$

$$= -\sin(\tan\sqrt{x+1})\sec^2\sqrt{x+1}\,\frac{1}{2}(x+1)^{-1/2}$$

$\therefore$
$$= \frac{-1}{2\sqrt{x+1}}\cdot\sin(\tan\sqrt{x+1})\cdot\sec^2(\sqrt{x+1})$$

33. $\sin^{-1}\dfrac{1}{\sqrt{x+1}}$

Sol. Let $\quad y = \sin^{-1}\dfrac{1}{\sqrt{x+1}}$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\left(\sin^{-1}\frac{1}{\sqrt{x+1}}\right)$$

$$= \frac{1}{\sqrt{1-\left(\dfrac{1}{\sqrt{x+1}}\right)^2}}\cdot\frac{d}{dx}\frac{1}{(x+1)^{1/2}}$$

$$= \frac{1}{\sqrt{\dfrac{x+1-1}{x+1}}}\cdot\frac{d}{dx}(x+1)^{-1/2}$$

$$= \sqrt{\frac{x+1}{x}}\cdot\frac{-1}{2}(x+1)^{-3/2} = \frac{-1}{2\sqrt{x}}\cdot\left(\frac{1}{x+1}\right)$$

Example of chain rule applied on an inverse trig function

Important Formulas:

DERIVATIVE OF STANDARD FUNCTIONS

$f(x)$	$\frac{d}{dx}(f(x))$	$f(x)$	$\frac{d}{dx}(f(x))$
x^n	nx^{n-1} ; $n \in R$	$\sec x$	$\sec x \tan x$, $x \neq (2n+1)\frac{\pi}{2}$
e^x	e^x	$\text{cosec}\, x$	$-\text{cosec}\, x \cot x$; $x \neq n\pi$
x^x	$x^x(1 + \ln x)$	$\cot x$	$-\text{cosec}^2 x$, $x \neq n\pi$
a^x	$a^x \log_e a$; $a > 0$, $a \neq 1$	$\sin^{-1}x$	$\frac{1}{\sqrt{1-x^2}}$; $-1 < x < 1$
$\log_e x$	$\frac{1}{x}$; $x > 0$	$\cos^{-1}x$	$-\frac{1}{\sqrt{1-x^2}}$; $-1 < x < 1$
$\log_a x$	$\frac{1}{x \log_e a}$; $x > 0$	$\tan^{-1}x$	$\frac{1}{1+x^2}$; $x \in R$
$\sin x$	$\cos x$	$\sec^{-1}x$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$; $\lvert x\rvert > 1$
$\cos x$	$-\sin x$	$\text{cosec}^{-1}x$	$\frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$; $\lvert x\rvert > 1$
$\tan x$	$\sec^2 x$; $x \neq (2n+1)\frac{\pi}{2}$ $n \in I$	$\cot^{-1}x$	$\frac{-1}{1+x^2}$; $x \in R$

RULES FOR DIFFERENTIATION

$\frac{d}{dx}(K(f(x)) = K \cdot \frac{d}{dx}(f(x))$, where K is constant

$\frac{d}{dx}\{f(x) \pm g(x)\} = \frac{d}{dx}(f(x)) \pm \frac{d}{dx}(g(x))$

Product Rule: $\frac{d}{dx}\{f(x).g(x)\} = f(x)\frac{d}{dx}(g(x)) + g(x)\frac{d}{dx}(f(x))$

Quotient Rule: $\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)\frac{d}{dx}(f(x)) - f(x)\frac{d}{dx}(g(x))}{(g(x))^2}$

Chain Rule: If y is a function of u, u is a function of v and v a function of x, then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dx}$$

Parametric differentiation: If x = P(t) , y = Q(t), where 't' is parameter then,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}(Q(t))}{\frac{d}{dt}(P(t))} = \frac{Q'(t)}{P'(t)}$$

Differentiation of one function w.r.t. other function

$$\frac{d(f(x))}{d(g(x))} = \frac{\frac{d}{dx}(f(x))}{\frac{d}{dx}(g(x))} = \frac{f'(x)}{g'(x)}$$

Logarithmic differentiation: It is applicable in following cases:

$\rightarrow y = \overbrace{f_1.f_2.f_3.....f_n}^{\text{All are functions of 'x'}}$ (product, divide or power form)

$\rightarrow y = (f(x))^{g(x)}$

* Take log on both sides and then differentiate.

Tips and Tricks

1. To be fast in exam one definitenly needs to remember derivative of standard and important functions.
   - See concepts pdf for nice compilation of such functions and their derivatives.

2. Another very important thing to keep in mind is the rules of differentition.
   - See concepts pdf for nice compilation of important theorems and their derivaitive results.

3. Practice and Practice these standard formulas and theorems to be able to recall them when required during an exam.

4. I strongly urge that you solve exemplar problems to get the hang of derivatives of standard functions.

Chain Rule: It is also called

- **Composite Function Rule or**
- **Function of a Function Rule**

Theorem:

Let $y = f(u)$ be a function of u and in turn let $u = g(x)$ be a function of x so that $y = f\left(g(x)\right) = (fog)(x)$.

Then $\frac{d}{dx}(f(g(x)) = f'(g(x))g'(x)$.

Proof:

In the above function $u = g(x)$ is known as the inner function and f is known as the outer function. Note that, ultimately, y is a function of x.

Now $\Delta u = g(x + \Delta x) - g(x)$

Therefore, $\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \times \frac{\Delta u}{\Delta x} = \frac{f(u + \Delta u) - f(u)}{\Delta u} \times \frac{g(x + \Delta x) - g(x)}{\Delta x}$.

Note that $\Delta u \to 0$ as $\Delta x \to 0$

$$
\begin{aligned}
\text{Therefore, } \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} &= \lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta u} \times \frac{\Delta y}{\Delta x} \right) \\
&= \lim_{\Delta u \to 0} \left(\frac{\Delta y}{\Delta u} \right) . \lim_{\Delta x \to 0} \left(\frac{\Delta y}{\Delta x} \right) \\
&= \lim_{\Delta u \to 0} \frac{f(u + \Delta u) - f(u)}{\Delta u} \times \lim_{\Delta x \to 0} \frac{g(x + \Delta x) - g(x)}{\Delta x} \\
&= f'(u) \times u'(x) \\
&= f'\left(g(x)\right) g'(x) \quad \text{or} \quad \boxed{\frac{d}{dx}\left(f\left(g(x)\right) = f'\left(g(x)\right) g'(x)\right.}.
\end{aligned}
$$

Remark:

Thus, to differentiate a function of a function $y = f\left(g(x)\right)$, we have to take the derivative of the outer function f regarding the argument $g(x) = u$, and multiply the derivative of the inner function $g(x)$ with respect to the independent variable x. The variable u is known as **intermediate argument**.

Examples:

Example-1:- Find the derivative of tan $(2x + 3)$.

Solution Let $f(x) = \tan(2x + 3)$, $u(x) = 2x + 3$ and $v(t) = \tan t$. Then

$$(v \circ u)(x) = v(u(x)) = v(2x + 3) = \tan(2x + 3) = f(x)$$

Thus f is a composite of two functions. Put $t = u(x) = 2x + 3$. Then $\frac{dv}{dt} = \sec^2 t$ and $\frac{dt}{dx} = 2$ exist. Hence, by chain rule

$$\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = 2\sec^2(2x+3)$$

Example-2: Find the derivative of tan $(2x + 3)$.

Sol: Let $f(x) = \tan(2x + 3)$, $u(x) = 2x + 3$ and $v(t) = \tan t$. Then

$$(v \circ u)(x) = v(u(x)) = v(2x + 3) = \tan(2x + 3) = f(x)$$

Thus f is a composite of two functions. Put $t = u(x) = 2x + 3$. Then $\frac{dv}{dt} = \sec^2 t$ and $\frac{dt}{dx} = 2$ exist. Hence, by chain rule

$$\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx} = 2\sec^2(2x+3)$$

Derivaitve of Inverse Function:

Inverse Function Theorem

Let $f(x)$ be a function that is both invertible and differentiable. Let $y = f^{-1}(x)$ be the inverse of $f(x)$. For all x satisfying $f'(f^{-1}(x)) \neq 0$,

$$\frac{dy}{dx} = \frac{d}{dx}(f^{-1}(x)) = (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}. \tag{eq-1}$$

Alternatively, if $y = g(x)$ is the inverse of $f(x)$, then

$$\boxed{g'(x) = \frac{1}{f'(g(x))}.} \tag{eq-2}$$

Important Formulas:

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\csc^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$$

`Examples from NCERT done by prof in class:`

Example 26 Find the derivative of f given by $f(x) = \sin^{-1} x$ assuming it exists.

Solution Let $y = \sin^{-1} x$. Then, $x = \sin y$.

Differentiating both sides w.r.t. x, we get

$$1 = \cos y \frac{dy}{dx}$$

which implies that $$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\cos(\sin^{-1} x)}$$

Observe that this is defined only for $\cos y \neq 0$, i.e., $\sin^{-1} x \neq -\frac{\pi}{2}, \frac{\pi}{2}$, i.e., $x \neq -1, 1$, i.e., $x \in (-1, 1)$.

To make this result a bit more attractive, we carry out the following manipulation. Recall that for $x \in (-1, 1)$, $\sin(\sin^{-1} x) = x$ and hence

$$\cos^2 y = 1 - (\sin y)^2 = 1 - (\sin(\sin^{-1} x))^2 = 1 - x^2$$

Also, since $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\cos y$ is positive and hence $\cos y = \sqrt{1-x^2}$

Thus, for $x \in (-1, 1)$,

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$$

Example 27 Find the derivative of f given by $f(x) = \tan^{-1} x$ assuming it exists.

Solution Let $y = \tan^{-1} x$. Then, $x = \tan y$.

Differentiating both sides w.r.t. x, we get

$$1 = \sec^2 y \frac{dy}{dx}$$

which implies that

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+(\tan(\tan^{-1} x))^2} = \frac{1}{1+x^2}$$