15. Find the equation of a curve passing through origin and satisfying the differential equation $(1+x^2)\frac{dy}{dx}+2xy=4x^2$.

Sol. We have, $(1+x^2)\frac{dy}{dx}+2xy=4x^2$

$$\Rightarrow \quad \frac{dy}{dx}+\frac{2x}{1+x^2}\cdot y=\frac{4x^2}{1+x^2}$$

This is a linear differential equation.

On comparing it with $\frac{dy}{dx}+Py=Q$, we get

$$P=\frac{2x}{1+x^2}, Q=\frac{4x^2}{1+x^2}$$

$$\therefore \quad \text{I.F.}=e^{\int P dx}=e^{\int \frac{2x}{1+x^2}dx}=e^{\log(1+x^2)}=(1+x^2)$$

So, the general solution is:

$$y\cdot(1+x^2)=\int \frac{4x^2}{1+x^2}(1+x^2)dx+C$$

$$\Rightarrow \quad y\cdot(1+x^2)=\int 4x^2 dx+C$$

$$\Rightarrow \quad y\cdot(1+x^2)=\frac{4x^3}{3}+C \qquad \text{(i)}$$

Since, the curve passes through origin, then putting $x=0$ and $y=0$ in Eq. (i), we get $C=0$

So, the required equation of curve is: $y=\frac{4x^3}{3(1+x^2)}$

Recoganizing right form of the LDE is important in this question.
And it is not that tough to do. Just simple algebra of equations.

17. Find the general solution of the differential equation $(1 + y^2) + (x - e^{\tan^{-1} y})\frac{dy}{dx} = 0.$

Sol. Given, differential equation is

$$(1 + y^2) + (x - e^{\tan^{-1} y})\frac{dy}{dx} = 0$$

$$\Rightarrow \quad (1 + y^2)\frac{dx}{dy} + x - e^{\tan^{-1} y} = 0$$

$$\Rightarrow \quad \frac{dx}{dy} + \frac{x}{1 + y^2} = \frac{e^{\tan^{-1} y}}{1 + y^2}$$

This is a linear differential equation.

On comparing it with $\frac{dx}{dy} + Px = Q,$ we get

$$P = \frac{1}{1 + y^2}, Q = \frac{e^{\tan^{-1} y}}{1 + y^2}$$

$$\text{I.F.} = e^{\int P dx} = e^{\int \frac{1}{1+y^2} dx} = e^{\tan^{-1} y}$$

So, the general solution is:

$$x \cdot e^{\tan^{-1} y} = \int \frac{e^{\tan^{-1} y}}{1 + y^2} \cdot e^{\tan^{-1} y} dx + C$$

Put $\quad e^{\tan^{-1} y} = t$

$$\Rightarrow \quad \frac{e^{\tan^{-1} y}}{1 + y^2} dy = dt$$

$$\therefore \quad x \cdot e^{\tan^{-1} y} = \int t \, dt + C$$

$$\Rightarrow \quad xe^{\tan^{-1} y} = \frac{t^2}{2} + C$$

$$\Rightarrow \quad xe^{\tan^{-1} y} = \frac{e^{2\tan^{-1} y}}{2} + C$$

21. Solve the differential equation dy = cos x (2 - y cosec x) dx given that y=2 when x = π /2.

Sol. We have

$$dy = \cos x\,(2 - y \operatorname{cosec} x)dx$$

$$\Rightarrow \quad \frac{dy}{dx} = 2\cos x - y \operatorname{cosec} x \cdot \cos x$$

$$\Rightarrow \quad \frac{dy}{dx} + y \cot x = 2\cos x$$

This is a linear differential equation.

On comparing it with $\frac{dy}{dx} + Py = Q$, we get

$$P = \cot x,\; Q = 2\cos x$$

$$\text{I.F.} = e^{\int P dx} = e^{\int \cot x\, dx} = e^{\log \sin x} = \sin x$$

Thus, the general solution is:

$$y \cdot \sin x = \int 2\cos x \cdot \sin x\, dx + C$$

$$\Rightarrow \quad y \cdot \sin x = \int \sin 2x\, dx + C$$

$$\Rightarrow \quad y \cdot \sin x = -\frac{\cos 2x}{2} + C$$

Given that when $x = \frac{\pi}{2}$ and $y = 2$

$$\Rightarrow \quad 2 \cdot \sin\frac{\pi}{2} = -\frac{\cos \pi}{2} + C$$

$$\Rightarrow \quad 2 = \frac{1}{2} + C$$

$$\Rightarrow \quad C = \frac{3}{2}$$

On substituting the value of C in Eq. (i), we get

$$y \sin x = -\frac{1}{2}\cos 2x + \frac{3}{2}$$

55. The integrating factor of the differential equation $\frac{dy}{dx} + y = \frac{1+y}{x}$ is

(a) $\frac{x}{e^x}$ (b) $\frac{e^x}{x}$ (c) xe^x (d) e^x

Sol. **(b)** We have, $\frac{dy}{dx} + y = \frac{1+y}{x}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x} + \frac{y(1-x)}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} - \left(\frac{1-x}{x}\right)y = \frac{1}{x}$$

This is a linear differential equation.

On comparing it with $\frac{dy}{dx} + Py = Q$, we get

$$P = \frac{-(1-x)}{x}, Q = \frac{1}{x}$$

$$\text{I.F.} = {}^{\int P dx} = e^{-\int \frac{1-x}{x} dx}$$

Being able to recognize that it is a LDE, and its right form.
That is the main task here.

58. The solution of $x\dfrac{dy}{dx}+y=e^x$ is

(a) $y=\dfrac{e^x}{x}+\dfrac{k}{x}$ (b) $y=xe^x+cx$ (c) $y=xe^x+k$ (d) $x=\dfrac{e^y}{y}+\dfrac{k}{y}$

Sol. **(a)** We have, $x\dfrac{dy}{dx}+y=e^x$

$$\Rightarrow \quad \frac{dy}{dx}+\frac{y}{x}=\frac{e^x}{x}$$

This is a linear differential equation.

On comparing it with $\dfrac{dy}{dx}+Py=Q$, we get

$$P=\frac{1}{x} \text{ and } Q=\frac{e^x}{x}$$

$$\therefore \quad \text{I.F.}=e^{\int \frac{1}{x}dx}=e^{(\log x)}=x$$

So, the general solution is:

$$y\cdot x=\int \frac{e^x}{x}x\,dx$$

$$\Rightarrow \quad y\cdot x=\int e^x\,dx$$

$$\Rightarrow \quad y\cdot x=e^x+k$$

$$\Rightarrow \quad y=\frac{e^x}{x}+\frac{k}{x}$$

Basic textbook example of LDEs. And when you look at other problems related to these concepts. They have the same procedure as this one.

Concepts and Formulas

Linear Differential Equations:

Definition:

A first order differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

where P and Q are functions of x only. Here no product of y and its derivative dy/dx occur and the dependent variable y and its derivative with respect to independent variable x occurs only in the first degree.

General Solution:

Various forms of general solution are

$$y \cdot e^{\int P\,dx} = \int \left(Q.e^{\int P\,dx}\right) dx$$

OR

$$y = e^{-\int P\,dx} . \int \left(Q.e^{\int P\,dx}\right) dx + C$$

OR

$$y \times (I.F) = \int Q(I.F)dx + C$$

where $e^{\int Pdx}$ Integrating Factor(I.F)

Bernoulli Differential Equations:

Definition:

An ordinary differential equation is called a Bernoulli differential equation if it is of the form, with n being real number

$$y' + P(x)y = Q(x)y^n,$$

General Solution:

Trick is to first convert Bernoulli into Linear diff equation.

Replace $u = y^{1-n}$

We get:

$$\frac{du}{dx} - (n-1)P(x)u = -(n-1)Q(x).$$

Now general solution is same as linear diff equation.

Tips and Tricks:

1. I advice you to read NCERT for this topic. It is quite extensively explained with examples.

2. Read concepts and formulas section for general solutions of linear differential equations(LDE) and bernoulli differential equations(BDE).

3. As always try to do more and more problems to grasp the concepts completely. Mathematics can only be mastered through practice. Read your theory one or two times and make cheat sheet notes. And then start doing questions and first try to recall concepts on your own, but if you are not able to look into the cheat sheet.

4. Practice lots of questions in a test timed manner for efficiency.