13. If A is the arithmetic mean and G_1, G_2 be two geometric means between any two numbers, then prove that

$$2A = \frac{G_1^2}{G_2} + \frac{G_2^2}{G_1}.$$

Solution:

Given A is the arithmetic mean and G_1, G_2 be two geometric means between any two numbers

Let the two numbers be 'a' and 'b'

The arithmetic mean is given by $A = \frac{a+b}{2}$ and the geometric mean is given by $G = \sqrt{ab}$

We have to insert two geometric means between a and b

Now that we have the terms a, G_1, G_2, b

G_1 will be the geometric mean of a and G_2 and G_2 will be the geometric mean of G_1 and b

Hence $G_1 = \sqrt{aG_2}$ and $G_2 = \sqrt{G_1 b}$

Square $G_1 = \sqrt{aG_2}$

$\Rightarrow G_1^2 = aG_2$

Put $G_2 = \sqrt{G_1 b}$

$\Rightarrow G_1^2 = a\sqrt{G_1 b}$

Squaring on both sides we get

$\Rightarrow G_1^4 = a^2 (G_1 b)$

$\Rightarrow G_1^3 = a^2 b$

$\Rightarrow G_1 = a^{\frac{2}{3}} b^{\frac{1}{3}}$ 1

Put value of G_1 in $G_2 = \sqrt{G_1 b}$

$\Rightarrow G_2 = \sqrt{a^{\frac{2}{3}} b^{\frac{1}{3}} b}$

$= \left(a^{\frac{2}{3}} b^{\frac{1}{3}+1}\right)^{\frac{1}{2}}$

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On simplification we get

$$= \left(a^{\frac{2}{3}} b^{\frac{4}{3}}\right)^{\frac{1}{2}}$$

$$= a^{\frac{1}{3}} b^{\frac{2}{3}} \ldots\ldots\ldots\ldots 2$$

Now we have to prove that $2A = \frac{G_1^2}{G_2} + \frac{G_2^2}{G_1}$

Consider RHS

$$\Rightarrow \text{RHS} = \frac{G_1^2}{G_2} + \frac{G_2^2}{G_1}$$

Substitute values of G_1 and G_2 from 1 and 2

$$\Rightarrow \text{RHS} = \frac{\left(a^{\frac{2}{3}} b^{\frac{1}{3}}\right)^2}{a^{\frac{1}{3}} b^{\frac{2}{3}}} + \frac{\left(a^{\frac{1}{3}} b^{\frac{2}{3}}\right)^2}{a^{\frac{2}{3}} b^{\frac{1}{3}}}$$

$$= \frac{a^{\frac{4}{3}} b^{\frac{2}{3}}}{a^{\frac{1}{3}} b^{\frac{2}{3}}} + \frac{a^{\frac{2}{3}} b^{\frac{4}{3}}}{a^{\frac{2}{3}} b^{\frac{1}{3}}}$$

Taking LCM and simplifying we get

$$= a^{\frac{4}{3}-\frac{1}{3}} b^{\frac{2}{3}-\frac{2}{3}} + a^{\frac{2}{3}-\frac{2}{3}} b^{\frac{4}{3}-\frac{1}{3}}$$

$\Rightarrow$ RHS = a + b

Divide and multiply by 2

$$\Rightarrow \text{RHS} = 2\frac{a+b}{2}$$

But $A = \frac{a+b}{2}$

Therefore

$\Rightarrow$ RHS = 2A

Hence RHS = LHS

Hence proved