(v) It is a scientifically established natural phenomenon that cloud is formed before it rains. Therefore, this sentence is always true. Hence it is a statement.

(vi) This is a question which also contains the word "Here". Hence it is not a statement.

The above examples show that whenever we say that a sentence is a statement we should always say why it is so. This "why" of it is more important than the answer.

EXERCISE 14.1

1. Which of the following sentences are statements? Give reasons for your answer.
   - (i) There are 35 days in a month.
   - (ii) Mathematics is difficult.
   - (iii) The sum of 5 and 7 is greater than 10.
   - (iv) The square of a number is an even number.
   - (v) The sides of a quadrilateral have equal length.
   - (vi) Answer this question.
   - (vii) The product of (–1) and 8 is 8.
   - (viii) The sum of all interior angles of a triangle is 180°.
   - (ix) Today is a windy day.
   - (x) All real numbers are complex numbers.
2. Give three examples of sentences which are not statements. Give reasons for the answers.

14.3 New Statements from Old

We now look into method for producing new statements from those that we already have. An English mathematician, "George Boole" discussed these methods in his book "The laws of Thought" in 1854. Here, we shall discuss two techniques.

As a first step in our study of statements, we look at an important technique that we may use in order to deepen our understanding of mathematical statements. This technique is to ask not only what it means to say that a given statement is true but also what it would mean to say that the given statement is not true.

14.3.1 ***Negation of a statement*** The denial of a statement is called the *negation* of the statement.

Let us consider the statement:

$$p: \textit{New Delhi is a city}$$

The negation of this statement is

It is not the case that New Delhi is a city

This can also be written as

It is false that New Delhi is a city.

This can simply be expressed as

New Delhi is not a city.

Definition 1 If p is a statement, then the negation of p is also a statement and is denoted by $\sim p$, and read as 'not p'.

Note While forming the negation of a statement, phrases like, "It is not the case" or "It is false that" are also used.

Here is an example to illustrate how, by looking at the negation of a statement, we may improve our understanding of it.

Let us consider the statement

p: Everyone in Germany speaks German.

The denial of this sentence tells us that not everyone in Germany speaks German. This does not mean that no person in Germany speaks German. It says merely that at least one person in Germany does not speak German.

We shall consider more examples.

Example 2 Write the negation of the following statements.

(i) Both the diagonals of a rectangle have the same length.

(ii) $\sqrt{7}$ is rational.

Solution (i) This statement says that in a rectangle, both the diagonals have the same length. This means that if you take any rectangle, then both the diagonals have the same length. The negation of this statement is

It is false that both the diagonals in a rectangle have the same length

This means the statement

There is atleast one rectangle whose both diagonals do not have the same length.

(ii) The negation of the statement in (ii) may also be written as

It is not the case that $\sqrt{7}$ *is rational.*

This can also be rewritten as

$\sqrt{7}$ *is not rational.*

Example 3 Write the negation of the following statements and check whether the resulting statements are true,

(i) Australia is a continent.

(ii) There does not exist a quadrilateral which has all its sides equal.

(iii) Every natural number is greater than 0.

(iv) The sum of 3 and 4 is 9.

Solution (i) The negation of the statement is

It is false that Australia is a continent.

This can also be rewritten as

Australia is not a continent.

We know that this statement is false.

(ii) The negation of the statement is

It is not the case that there does not exist a quadrilateral which has all its sides equal.

This also means the following:

There exists a quadrilateral which has all its sides equal.

This statement is true because we know that square is a quadrilateral such that its four sides are equal.

(iii) The negation of the statement is

It is false that every natural number is greater than 0.

This can be rewritten as

There exists a natural number which is not greater than 0.

This is a false statement.

(iv) The negation is

It is false that the sum of 3 *and* 4 *is* 9.

This can be written as

The sum of 3 and 4 is not equal to 9.

This statement is true.

14.3.2 *Compound statements* Many mathematical statements are obtained by combining one or more statements using some connecting words like "and", "or", etc. Consider the following statement

p: There is something wrong with the bulb or with the wiring.

This statement tells us that there is something wrong with the bulb or there is something wrong with the wiring. That means the given statement is actually made up of two smaller statements:

q: There is something wrong with the bulb.

r: There is something wrong with the wiring.

connected by "or"

Now, suppose two statements are given as below:

p: 7 is an odd number.

q: 7 is a prime number.

These two statements can be combined with "and"

r: 7 is both odd and prime number.

This is a compound statement.

This leads us to the following definition:

Definition 2 **A Compound Statement** is a statement which is made up of two or more statements. In this case, each statement is called a ***component statement.***

Let us consider some examples.

Example 4 Find the component statements of the following compound statements.

(i) The sky is blue and the grass is green.

(ii) It is raining and it is cold.

(iii) All rational numbers are real and all real numbers are complex.

(iv) 0 is a positive number or a negative number.

Solution Let us consider one by one

(i) The component statements are

p: The sky is blue.

q: The grass is green.

The connecting word is 'and'.

(ii) The component statements are

p: It is raining.

q: It is cold.

The connecting word is 'and'.

(iii) The component statements are

p: All rational numbers are real.

q: All real numbers are complex.

The connecting word is 'and'.

(iv) The component statements are

p: 0 is a positive number.

q: 0 is a negative number.

The connecting word is 'or'.

Example 5 Find the component statements of the following and check whether they are true or not.

(i) A square is a quadrilateral and its four sides equal.

(ii) All prime numbers are either even or odd.

(iii) A person who has taken Mathematics or Computer Science can go for MCA.

(iv) Chandigarh is the capital of Haryana and UP.

(v) $\sqrt{2}$ is a rational number or an irrational number.

(vi) 24 is a multiple of 2, 4 and 8.

Solution (i) The component statements are

p: A square is a quadrilateral.

q: A square has all its sides equal.

We know that both these statements are true. Here the connecting word is 'and'.

(ii) The component statements are

p: All prime numbers are odd numbers.

q: All prime numbers are even numbers.

Both these statements are false and the connecting word is 'or'.

(iii) The component statements are

p: A person who has taken Mathematics can go for MCA.

q: A person who has taken computer science can go for MCA.

Both these statements are true. Here the connecting word is 'or'.

(iv) The component statements are

p: Chandigarh is the capital of Haryana.

q: Chandigarh is the capital of UP.

The first statement is true but the second is false. Here the connecting word is 'and'.

(v) The component statements are

$$p: \sqrt{2} \textit{ is a rational number.}$$

$$q: \sqrt{2} \textit{ is an irrational number.}$$

The first statement is false and second is true. Here the connecting word is 'or'.

(vi) The component statements are

$$p: \textit{24 is a multiple of } 2.$$

$$q: \textit{24 is a multiple of } 4.$$

$$r: \textit{24 is a multiple of } 8.$$

All the three statements are true. Here the connecting words are 'and'.

Thus, we observe that compound statements are actually made-up of two or more statements connected by the words like "and", "or", etc. These words have special meaning in mathematics. We shall discuss this mattter in the following section.

EXERCISE 14.2

1. Write the negation of the following statements:

(i) Chennai is the capital of Tamil Nadu.

(ii) $\sqrt{2}$ is not a complex number

(iii) All triangles are not equilateral triangle.

(iv) The number 2 is greater than 7.

(v) Every natural number is an integer.

2. Are the following pairs of statements negations of each other:

(i) The number x is not a rational number.
The number x is not an irrational number.

(ii) The number x is a rational number.
The number x is an irrational number.

3. Find the component statements of the following compound statements and check whether they are true or false.

(i) Number 3 is prime or it is odd.

(ii) All integers are positive or negative.

(iii) 100 is divisible by 3, 11 and 5.

14.4 Special Words/Phrases

Some of the connecting words which are found in compound statements like **"And",**

"Or", etc. are often used in Mathematical Statements. These are called connectives. When we use these compound statements, it is necessary to understand the role of these words. We discuss this below.

14.4.1 ***The word "And"*** Let us look at a compound statement with "And".

p: A point occupies a position and its location can be determined.

The statement can be broken into two component statements as

q: A point occupies a position.

r: Its location can be determined.

Here, we observe that both statements are true.

Let us look at another statement.

p: 42 *is divisible by* 5, 6 *and* 7.

This statement has following component statements

q: 42 is divisible by 5.

r: 42 is divisible by 6.

s: 42 is divisible by 7.

Here, we know that the first is false while the other two are true.

We have the following rules regarding the connective "And"

1. The compound statement with 'And' is true if all its component statements are true.
2. The component statement with 'And' is false if any of its component statements is false (this includes the case that some of its component statements are false or all of its component statements are false).

Example 6 Write the component statements of the following compound statements and check whether the compound statement is true or false.

(i) A line is straight and extends indefinitely in both directions.

(ii) 0 is less than every positive integer and every negative integer.

(iii) All living things have two legs and two eyes.

Solution (i) The component statements are

p: A line is straight.

q: A line extends indefinitely in both directions.

Both these statements are true, therefore, the compound statement is true.

(ii) The component statements are

p: 0 is less than every positive integer.

q: 0 is less than every negative integer.

The second statement is false. Therefore, the compound statement is false.

(iii) The two component statements are

p: All living things have two legs.

q: All living things have two eyes.

Both these statements are false. Therefore, the compound statement is false.

Now, consider the following statement.

p: A mixture of alcohol and water can be separated by chemical methods.

This sentence cannot be considered as a compound statement with "And". Here the word "And" refers to two things – alcohol and water.

This leads us to an important note.

Note Do not think that a statement with "And" is always a compound statement as shown in the above example. Therefore, the word "And" is not used as a connective.

14.4.2 ***The word "Or"*** Let us look at the following statement.

p: Two lines in a plane either intersect at one point or they are parallel.

We know that this is a true statement. What does this mean? This means that if two lines in a plane intersect, then they are not parallel. Alternatively, if the two lines are not parallel, then they intersect at a point. That is this statement is true in both the situations.

In order to understand statements with "Or" we first notice that the word "Or" is used in two ways in English language. Let us first look at the following statement.

p: An ice cream or pepsi is available with a Thali in a restaurant.

This means that a person who does not want ice cream can have a pepsi along with *Thali* or one does not want pepsi can have an ice cream along with *Thali*. That is, who do not want a pepsi can have an ice cream. A person cannot have both ice cream and pepsi. This is called an ***exclusive "Or".***

Here is another statement.

A student who has taken biology or chemistry can apply for M.Sc. microbiology programme.

Here we mean that the students who have taken both biology and chemistry can apply for the microbiology programme, as well as the students who have taken only one of these subjects. In this case, we are using **inclusive "Or".**

It is important to note the difference between these two ways because we require this when we check whether the statement is true or not.

Let us look at an example.

Example 7 For each of the following statements, determine whether an **inclusive "Or"** or **exclusive "Or"** is used. Give reasons for your answer.

(i) To enter a country, you need a passport or a voter registration card.

(ii) The school is closed if it is a holiday or a Sunday.

(iii) Two lines intersect at a point or are parallel.

(iv) Students can take French or Sanskrit as their third language.

Solution (i) Here "Or" is inclusive since a person can have both a passport and a voter registration card to enter a country.

(ii) Here also "Or" is inclusive since school is closed on holiday as well as on Sunday.

(iii) Here "Or" is exclusive because it is not possible for two lines to intersect and parallel together.

(iv) Here also "Or" is exclusive because a student cannot take both French and Sanskrit.

Rule for the compound statement with 'Or'

1. A compound statement with an 'Or' is true when one component statement is true or both the component statements are true.
2. A compound statement with an 'Or' is false when both the component statements are false.

For example, consider the following statement.

p: Two lines intersect at a point or they are parallel

The component statements are

q: Two lines intersect at a point.

r: Two lines are parallel.

Then, when q is true r is false and when r is true q is false. Therefore, the compound statement p is true.

Consider another statement.

p: 125 *is a multiple of* 7 *or* 8.

Its component statements are

q: 125 *is a multiple of* 7.

r: 125 *is a multiple of* 8.

Both q and r are false. Therefore, the compound statement p is false.

Again, consider the following statement:

p: The school is closed, if there is a holiday or Sunday.

The component statements are

q: School is closed if there is a holiday.

r: School is closed if there is a Sunday.

Both q and r are true, therefore, the compound statement is true.
Consider another statement.

p: Mumbai is the capital of Kolkata or Karnataka.

The component statements are

q: Mumbai is the capital of Kolkata.

r: Mumbai is the capital of Karnataka.

Both these statements are false. Therefore, the compound statement is false.
Let us consider some examples.

Example 8 Identify the type of "Or" used in the following statements and check whether the statements are true or false:

(i) $\sqrt{2}$ is a rational number or an irrational number.

(ii) To enter into a public library children need an identity card from the school or a letter from the school authorities.

(iii) A rectangle is a quadrilateral or a 5-sided polygon.

Solution (i) The component statements are

p: $\sqrt{2}$ *is a rational number.*

q: $\sqrt{2}$ *is an irrational number.*

Here, we know that the first statement is false and the second is true and "Or" is exclusive. Therefore, the compound statement is true.

(ii) The component statements are

p: To get into a public library children need an identity card.

q: To get into a public library children need a letter from the school authorities.

Children can enter the library if they have either of the two, an identity card or the letter, as well as when they have both. Therefore, it is inclusive "Or" the compound statement is also true when children have both the card and the letter.

(iii) Here "Or" is exclusive. When we look at the component statements, we get that the statement is true.

14.4.3 *Quantifiers* Quantifiers are phrases like, "There exists" and "For all". Another phrase which appears in mathematical statements is "there exists". For example, consider the statement. p: *There exists a rectangle whose all sides are equal.* This means that there is atleast one rectangle whose all sides are equal.

A word closely connected with "there exists" is "for every" (or for all). Consider a statement.

p: For every prime number p, $\sqrt{p}$ *is an irrational number.*

This means that if S denotes the set of all prime numbers, then for all the members p of the set S, $\sqrt{p}$ is an irrational number.

In general, a mathematical statement that says "for every" can be interpreted as saying that all the members of the given set S where the property applies must satisfy that property.

We should also observe that it is important to know precisely where in the sentence a given connecting word is introduced. For example, compare the following two sentences:

1. For every positive number x there exists a positive number y such that $y < x$.
2. There exists a positive number y such that for every positive number x, we have $y < x$.

Although these statements may look similar, they do not say the same thing. As a matter of fact, (1) is true and (2) is false. Thus, in order for a piece of mathematical writing to make sense, all of the symbols must be carefully introduced and each symbol must be introduced precisely at the right place – not too early and not too late.

The words "And" and "Or" are called *connectives* and "There exists" and "For all" are called *quantifiers.*

Thus, we have seen that many mathematical statements contain some special words and it is important to know the meaning attached to them, especially when we have to check the validity of different statements.

EXERCISE 14.3

1. For each of the following compound statements first identify the connecting words and then break it into component statements.

   (i) All rational numbers are real and all real numbers are not complex.

   (ii) Square of an integer is positive or negative.

   (iii) The sand heats up quickly in the Sun and does not cool down fast at night.

   (iv) $x = 2$ and $x = 3$ are the roots of the equation $3x^2 - x - 10 = 0$.

2. Identify the quantifier in the following statements and write the negation of the statements.

(i) There exists a number which is equal to its square.

(ii) For every real number x, x is less than $x + 1$.

(iii) There exists a capital for every state in India.

3. Check whether the following pair of statements are negation of each other. Give reasons for your answer.

(i) $x + y = y + x$ is true for every real numbers x and y.

(ii) There exists real numbers x and y for which $x + y = y + x$.

4. State whether the "Or" used in the following statements is "exclusive "or" inclusive. Give reasons for your answer.

(i) Sun rises or Moon sets.

(ii) To apply for a driving licence, you should have a ration card or a passport.

(iii) All integers are positive or negative.

14.5 Implications

In this Section, we shall discuss the implications of "if-then", "only if" and "if and only if ".

The statements with "if-then" are very common in mathematics. For example, consider the statement.

r: If you are born in some country, then you are a citizen of that country.

When we look at this statement, we observe that it corresponds to two statements p and q given by

p : *you are born in some country.*

q : *you are citizen of that country.*

Then the sentence "if p then q" says that in the event if p is true, then q must be true.

One of the most important facts about the sentence "if p then q" is that it does not say any thing (or places no demand) on q when p is false. For example, if you are not born in the country, then you cannot say anything about q. To put it in other words" not happening of p has no effect on happening of q.

Another point to be noted for the statement "if p then q" is that the statement does not imply that p happens.

There are several ways of understanding "if p then q" statements. We shall illustrate these ways in the context of the following statement.

r: If a number is a multiple of 9, then it is a multiple of 3.

Let p and q denote the statements

p : *a number is a multiple of* 9.

q: *a number is a multiple of* 3.