Probability and Counting Interview Questions – Problem 13

Imagine 3 pairs of socks (6 total) labeled 1 1 2 2 3 3. Define a satisfactory pair as any pair of individual socks such that their numbers differ by a maximum of 1 (i.e. 2 3 is a satisfactory pair). In this game, drawing without replacement, what is the probability that a player will draw three satisfactory pairs of socks?

Solution 1

The only two types of socks that together form a pair which is not satisfactory is 1 and 3 so the problem is equivalent to finding the probability that each of the two 1-socks is paired with a sock which is not a 3-sock. This can happen in one of two mutually exclusive ways:

- The two 1-socks are paired with each other: the sequence of socks drawn without replacement can be viewed as a permutation of the six elements $\{a_1, a_2, b_1, b_2, c_1, c_2\}$, where the two 1-socks correspond to a_1 and a_2, the two 2-socks correspond to b_1 and b_2, etc. a_1 is just as likely to be paired with any of the other five socks so the probability that it is paired with a_2 is $\frac{1}{5}$.

- The two 1-socks are each paired with a 2-sock: using the notation of the first case, this is the probability that a_1 is paired with one of b_1 and b_2 and a_2 is paired with the other of b_1 and b_2.

$$\begin{aligned}
&\mathbb{P}[\{a_1 \text{ is paired with one of } b_1 \text{ and } b_2\} \cap \{a_2 \text{ is paired with the other of them}\}] \\
&= \mathbb{P}[\{a_1 \text{ is paired with one of } b_1 \text{ and } b_2\}] \\
&\times \mathbb{P}[\{a_2 \text{ is paired with the other of them}\} | \{a_1 \text{ is paired with one of } b_1 \text{ and } b_2\}] \\
&= \frac{2}{5} \times \frac{1}{3}
\end{aligned}$$

  where the last equality follows because a_1 is just as likely to be paired with any of the other socks, so that there is a $\frac{2}{5}$ chance that it is one of b_1 or b_2. Given that a_1 has been paired with one of those socks, among sequences of pairings for which that is true, that leaves 3 equally likely socks for a_2 to be paired with, only one of which is a b-sock (2-sock) so that happens with probability $\frac{1}{3}$.

The total probability of having three satisfactory pairs is then $\frac{1}{5} + \frac{2}{5} \cdot \frac{1}{3} = \boxed{\frac{1}{3}}$.

Note that this problem is an easier version of the last problem discussed in the Probability Theory module under Mutually Exclusive Events.

Solution 2

Since each arrangement of socks is equally likely, we can count the total number of possible arrangements and the number of ones with three satisfactory pairs of socks and then divide those two numbers.

If we consider all socks to be distinguishable (even the two socks of each type compared to each other), there are $6! = 720$ ways of drawing the socks without replacement.

As mentioned in the first solution, we get three satisfactory pairs exactly when the 1-socks are not paired with 3-socks. We can count the complement: when at least one 1-sock is paired with a 3-sock.

There are $2 \cdot 2 \cdot 2 \cdot 4! = 192$ arrangements where the first pair is a 1-sock and a 3-sock: 2 choices for the 1-sock in the pair, 2 choices for the 3-sock in the pair, 2 choices to arrange the two in the first pair, and 4! arrangements of the remaining socks in the other 4 positions. This is also true for the number of arrangements where the second pair is a 1-sock with a 3-sock, as well as the third pair, giving $3 \cdot 192 = 576$.

However, this overcounts cases when there are two pairs of a 1-sock and a 3-sock, for example, when the first pair and the second pair both are the pair, we count those arrangements twice in the last paragraph. To correct for this, we subtract this number of cases. When the two pairs are the first and second ones, the number of such arrangements is $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2! = 32$: there are 2 ways to select the 1-sock in the first pair, 2 ways to select the 3-sock in the first pair, 2 ways to arrange the 1-sock and 3-sock in the first pair, 2 ways to arrange the 1-sock and 3-sock in the second pair, and 2! ways of arranging the remaining 2 socks in the third pair. This is true for all $\binom{3}{2} = 3$ possible pairs of pairs of socks with a 1-sock and a 3-sock, giving $3 \cdot 32 = 96$ arrangements.

From the principle of inclusion-exclusion, there are $576 - 96 = 480$ arrangements with at least one pair of a 1-sock and a 3-sock. This leaves $720 - 480 = 240$ arrangements with only satisfactory pairs, so the probability is $\frac{240}{720} = \boxed{\frac{1}{3}}$.

Final answer: $\mathbf{\frac{1}{3}}$.

Python simulation:

```python
import random

def get_sock_arrangement():
    original_sock_list = [1, 1, 2, 2, 3, 3]
    sock_arrangement = []
    for step in range(6):
        index = random.randint(0, 5-step)
        sock_arrangement.append(original_sock_list[index])
```

```python
        del original_sock_list[index]
    return sock_arrangement


def get_proportion_with_three_satisfactory_pairs(num_trials):
    num_successes = 0
    for _ in range(num_trials):
        sock_arrangement = get_sock_arrangement()
        if has_three_satisfactory_pairs(sock_arrangement):
            num_successes += 1
    return num_successes / num_trials


def has_three_satisfactory_pairs(sock_arrangement):
    return (-1 <= sock_arrangement[0] - sock_arrangement[1] <= 1) and \
        (-1 <= sock_arrangement[2] - sock_arrangement[3] <= 1) and \
        (-1 <= sock_arrangement[4] - sock_arrangement[5] <= 1)


def run_simulation(trial_list):
    random.seed(10)
    trial_results = []
    for num_trials in trial_list:
        trial_results.append((num_trials, get_proportion_with_three_satisfactory_pairs(num_trials)))
    return trial_results


if __name__ == "__main__":
    trial_results = run_simulation([10**1, 10**2, 10**3, 10**4, 10**5, 10**6, 10**7])
    for num_trials, proportion in trial_results:
        print(f"Num trials: {num_trials}, proportion with success: {proportion}")
    print()
```

The output shows that with progressively more trials, the empirical proportion of cases resulting in three satisfactory pairs gets closer and closer to $\frac{1}{3}$:

```
Num trials: 10, proportion with success: 0.3
Num trials: 100, proportion with success: 0.35
Num trials: 1000, proportion with success: 0.338
Num trials: 10000, proportion with success: 0.3313
Num trials: 100000, proportion with success: 0.33472
Num trials: 1000000, proportion with success: 0.333424
Num trials: 10000000, proportion with success: 0.3331034
```