Recurrences

Recursive Function - Function calling itself.

Recurrence - is an equation or inequality that describes a function in terms of its value on smaller inputs.

Example to understand how recurrence is created from algorithm?

```
Algorithm ABC (n)
{
    if (n>1)
        return (ABC (n-1))
}
```

$$T(n) = 1 + T(n-1) \quad \text{if } n>1$$
$$= O(1) \quad n=1$$

There are three methods to solve recurrences :-

1) The Substitution Method
2) The Recursion Tree Method
3) The Master Method.

[I] The Substitution Method

There are two steps

(i) Guess the form of the solution

(ii) Use mathematical induction to find constant & show that solution works.

Examples

1] $T(n) = 2T(n/2) + n$ for $n > 1$ and $T(1) = 1$

Determine upper bound.

Soln → Step 1: Guess the Solution

$T(n) = O(n \log n)$.

Step 2: Mathematical induction.

(a) To prove: $T(n) \le cn\log n$ ... (I) for some constant c ~~$n_0 > 0$~~

(b) Finding constant c:

Let us assume that bound holds for $T(n/2)$.

$\therefore T(n/2) \le c(n/2) \log (n/2)$ ...... (II)

Substituting equation II in recurrence relation.

$T(n) = 2T(n/2) + n$

$$T(n) = 2\left[c\, \tfrac{n}{2} \log(\tfrac{n}{2})\right] + n$$

$$= cn \log(n/2) + n$$

$$= cn\log n - cn\log 2 + n$$

$$T(n) = cn\log n - cn + n \quad \ldots \quad \underline{\text{III}}$$

We need to prove,

i.e. $T(n) \leq cn\log n$

$\cancel{cn\log n} - cn + n \leq cn\log n$

$-cn + n \leq 0$

Dividing by n,

$-c + 1 \leq 0$

$$\boxed{c \geq 1}$$

(c) <u>Prove the base condition</u>:

Suppose, put $n = 1$ in eqn I.

$T(1) \leq c(1)\log(1) \leq 0$

But $T(1) = 1$. Hence Not satisfied.

(d) <u>Reverse the induction</u>

Need to find lowest value of n_0 for which relation is satisfied.

Consider, $n_0 = 2$

$$T(2) = 2T(n/2) + n$$
$$= 2T(1) + 2$$
$$\boxed{T(2) = 4}$$

Now, for relation.

$$T(n) \le cn\log n$$
$$T(2) \le c(2\log 2)$$
$$T(2) \le 2c$$

For sufficient large value of c, $c \ge 2$ the condition holds the relation.

$\therefore$ $T(n) = O(~~log~~ n\log n)$ where $n_0 = 2$

$c = 2$

For, $n_0 = 3$,

$$T(n) = T(3) = 2T(3/2) + 3$$
$$= 2T(1) + 3$$
$$= 5$$

for relation,

$$T(n) \le cn\log n.$$
$$T(3) \le 2 \times 3 \log 3.$$
$$5 \le 2 \times 3 (1.5)$$
$$5 \le 9$$

$\therefore$ satisfied, $\therefore$ $T(n) = O(n\log n)$ $n_0 = 2$

$c = 2$

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2] $T(1) = 1$ and $T(n) = T(n/2) + 1$ for $n > 1$

determine the upper bound

Solution : step 1 : Guess the solution.

$T(n) = O(\log n)$

step 2 : Mathematical Induction

(a) To prove : $T(n) \le c \log n$ —— (I) for some constant c

(b) finding constant c :

Let us assume that bound holds for $T(n/2)$

$\therefore \quad T(n/2) \le c \log (n/2) \quad \cdots \text{(II)}$

using equation (II) in recurrence equation,

$$T(n) = T(n/2) + 1$$

$$= c \log(n/2) + 1$$

$$= c \log n - c + 1 \quad \text{—— (III)}.$$

We need to prove,

$$T(n) \le c \log n$$

$$\cancel{c \log n} - c + 1 \le \cancel{c \log n}.$$

$$-c + 1 \le 0$$

$$\boxed{c \ge 1}$$

ⓒ ~~Proof the~~ Prove the base condition :

Suppose, put $n=1$ in equation I

$T(1) \le c \log 1 \le 0$

But $T(1) = 1$, Hence Not satisfied.

ⓓ Revise the induction

Need to find the lowest value of n_0 for which relation is satisfied.

Consider $n_0 = 2$

$T(2) = T\left(\frac{2}{2}\right) + 1$

$= 2$

Now, for relation,

$T(n) \le c \log n$

$T(2) \le c \log 2$

$T(2) \le c$

Consider $n_0 = 3$

$T(3) = 2T\left(\frac{3}{2}\right) + 1$

$= T(1) + 1$

$= 2$

Now for relation

$T(n) \le c \log 3$

$T(3) \le 1.5\, c$

For sufficient larger value of c, $c \geq 2$ the condition holds the relation.

$\therefore$ $T(n) = O(\log n)$ for $n_0 = 2$

$c = 2$

3] Show that solution for $T(n) = T(n-1) + n$ is $O(n^2)$

Soln → step 1: Guess the solution & find exact form

$T(n) = O(n^2)$

Exact form :

$$T(n) = T(n-1) + n$$

$$T(1) = 1$$

$$T(2) = T(2-1) + 2 = 1 + 2$$

$$T(3) = T(3-1) + 3 = 1 + 2 + 3$$

$$T(4) = 1 + 2 + 3 + 4$$

$$\therefore T(n) = \frac{n(n+1)}{2}$$

step 2 : Mathematical Induction

(a) To prove :

$$T(n) \le c\left[\frac{n(n+1)}{2}\right] \quad \text{--- (I)}$$

for some constant c

(b) Finding constant c :

Let us assume that bound holds for $T(n-1)$.

$$\therefore T(n-1) \le c\left[\frac{(n-1)(n)}{2}\right] \quad \text{--- (II)}$$

using equation II, in recurrence equation,

$$T(n) = T(n-1) + n$$

$$= c\left[\frac{n(n-1)}{2}\right] + n$$

$$= \frac{c}{2}n^2 - \frac{c}{2}n + n$$

We need to prove,

$$T(n) \leq c\left[\frac{n(n+1)}{2}\right]$$

$$\cancel{\frac{cn^2}{2}} - \frac{c}{2}n + n \leq \cancel{\frac{cn^2}{2}} + \frac{c}{2}n$$

$$n \leq cn$$

Dividing by n,

$$\boxed{c \geq 1}$$

(c) Prove the base condition :

Suppose, put $n=1$ in equation I.

$$T(1) \leq c\left(\frac{n(n+1)}{2}\right)$$

$$\leq c\left(1\left(\frac{1+1}{2}\right)\right)$$

$$\leq c$$

$$\leq 1 \qquad [c=1]$$

which is satisfied.

$\therefore \quad T(n) = O(n^2)$ for $c=1$

$n_0 = 1$

4] Show that solution for $T(n) = 2T(\lceil n/2 \rceil + 17) + n$ is $O(n \log n)$.

Solution :

step 1 : Guess the solution

$\therefore \; T(n) = O(n \log n)$

Step 2 : Mathematical Induction

(a) To prove : $T(n) \leq cn \log n \quad \cdots \text{(I)}$

(b) Finding constant c :

Let us assume that the bound holds true for $T(n/2 + 17)$

$$\therefore \; T\left(\frac{n}{2} + 17\right) = c\left(\frac{n}{2} + 17\right) \log\left(\frac{n}{2} + 17\right) \quad \cdots \text{(II)}$$

using equation (II) in recurrence equation.

$$\therefore \; T(n) = 2\left[c\left(\frac{n}{2} + 17\right) \log\left(\frac{n}{2} + 17\right)\right] + n.$$

$$= cn \log\left(\frac{n}{2} + 17\right) + 34c \log\left(\frac{n}{2} + 17\right) + n$$

$$= cn \log\left(\frac{n+34}{2}\right) + 34c \log\left(\frac{n+34}{2}\right) + n$$

$$= cn \log(n+34) - cn + 34c \log(n+34) - 34c + n$$

$$= cn \log\left(n\left(1 + \frac{34}{n}\right)\right) + 34c \log\left(n\left(1 + \frac{34}{n}\right)\right) - c(n+34) + n$$

$$= \underline{cn \log n} + cn \log\left(1 + \frac{34}{n}\right) + \underline{34c \log n} + 34c \log\left(1 + \frac{34}{n}\right) - c(n+34) + n$$

$$= c(n+34)\log n + c(n+34)\log\left(1+\frac{34}{n}\right) - c(n+34) + n$$

When n is very large, then $34/n$ tends to zero and. $n+34 \cong n$.

$$= cn\log n + cn\log(1) - nc + n$$

$$T(n) = cn\log n - cn + n$$

We need to prove,

$$T(n) \le cn\log n$$

$$cn\cancel{\log}n - cn + n \le \cancel{c}n\cancel{\log}n$$

Dividing by n.

$$-c+1 \le 0$$

$$\therefore \boxed{c \geq 1}$$

(c) Prove the base condition

Suppose put $n=1$ in eqn (I)

$$T(1) \le c(1)\log(1) \le 0$$

But $T(1) = 1$. Hence not satisfied.

(d) Revise the Production

For larger values of n, $n/2 + 17 \cong n/2$.

$\therefore \quad T(n) = 2T(n/2 + 17) + n$

$\cong 2T\left(\frac{n}{2}\right) + n$

We need to find lowest value of n_0 for which relation is satisfied.

consider,

$n_0 = 2$

$T(2) = 2T(1) + 2$

$= 4$

For Relation,

$T(2) \leq c\,2 \log 2$

$\leq 2c$

$n_0 = 3$

$T(3) = 2T\left(\left[\frac{2}{3}\right]\right) + 3$

$= 2T(2) + 3$

$= 8 + 3$

$= 11$

For relation

$T(3) \leq c \cdot 3 \log 3$

$\leq 4.5\,c$

For sufficient larger value of c, $c \geq 3$ then condition holds the relation.

$\therefore \quad T(n) = O(n \log n)$ for $n_0 = 2$

$c = 3$

5] Solve the recurrence $T(n) = 2T(\lfloor\sqrt{n}\rfloor) + \log n$ by making change of variable

Solution : 1] Changing the Variable

Let $m = \log n$ — ①

$\therefore n = 2^m$ — ②

$\therefore \sqrt{n} = 2^{m/2}$ — ③

$\therefore$ substituting ①, ② ③ in recurrence equation,

$$T(2^m) = 2T(2^{m/2}) + m$$

Now Rename, $S(m) = T(2^m)$

$$\therefore S(m) = 2S(m/2) + m \quad \text{— ④}$$

2] Prove :

ⓐ Guess the Solution :

$$S(m) = O(m \log m) \quad \text{— ⑤}$$

ⓑ Mathematical Induction

To prove : $S(m) \leq cm \log m$ — ⑥

Finding c :

Let us assume bound holds for $S(m/2)$

$$\therefore S(m/2) \leq c\,m/2 \log m/2 \quad \text{— ⑦}$$

Substituting eqn (7) in (4)

$$S(m) = 2\left[c\frac{m}{2}\log m/_2\right] + m$$

$$= cm\log m - cm + m$$

We need to prove,

$$S(m) \le cm\log m$$

$$cm\log m - cm + m \le cm\log m$$

dividing by m,

$$-c + 1 \le 0$$

$$c \ge 1$$

Prove the base condition :

put $m = 1$, in equation (6)

$$S(1) \le c(1)\log(1) = 0$$

But $S(1) = 1$, Hence not satisfied.

Revise the induction

consider, $m_0 = 2$

$$S(2) = 2S(1) + 2$$

$$= 4$$

For Relation,

$$S(2) \le c \cdot 2\log 2$$

$$\le 2 \cdot c$$

For sufficient larger value of c,
$c \geq 2$, relation holds.

$\therefore \quad S(m) = O(m \log m)$ for $c = 2$, $m_0 = 2$

3] Changing Back from $S(m)$ to $T(n)$

$$T(n) = T(2^m)$$
$$= S(m)$$
$$= O(m \log m)$$

$$\boxed{T(n) = O(\log n \log \log n)}$$

$$n_0 = 2^{m_0}$$
$$= 2^2$$

$$\boxed{n_0 = 4}$$

$\therefore \quad T(n) = O(\log n \log \log n)$ for $c = 2$, $n_0 = 4$

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