THE MASTER METHOD

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(FOR SOLVING RECURRENCES)

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- The master method is used for solving recurrences of the form :

$$T(n) = aT\left(\frac{n}{b}\right) + f(n) \quad \text{—— (I)}$$

where $a \geq 1$ and $b > 1$ are constants and $f(n)$ is an asymptotically positive function.

- The recurrence in equation (I) describes the running time of an algorithm that divides a problem of size n into a subproblems, each of size n/b, where a and b are positive constants.

- function $f(n)$ includes cost of dividing the problem and combining the results of subproblem.

→ MASTER THEOREM

Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be ~~defined~~ a recurrence defined as,

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

then $T(n)$ has the following asymptotic bounds

Sundaram

Teacher's Sign.:

case 1:

If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$

i.e $f(n) < n^{\log_b a}$

then,

$$\boxed{T(n) = \Theta(n^{\log_b a})}$$

case 2:

If $f(n) = \Theta(n^{\log_b a})$

i.e $f(n) = n^{\log_b a}$

then,

$$\boxed{T(n) = \Theta(n^{\log_b a} \log n)}$$

case 3:

i) If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$,

i.e $f(n) > n^{\log_b a}$

and

ii) if $a f(n/b) \leq c f(n)$ for some constant $c < 1$

then,

$$\boxed{T(n) = \Theta(f(n))}$$

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EXAMPLES

1. $T(n) = 9T\left(\frac{n}{3}\right) + n$

Step 1 : Comparing with standard form

$T(n) = a\,T\left(\frac{n}{b}\right) + f(n)$

$\therefore$ $a = 9$ $b = 3$ $f(n) = n$.

Step 2 : Calculate $n^{\log_b a}$

$n^{\log_b a} = n^{\log_3 9} = n^2$

Step 3 : Identifying the Case

$f(n) = n$ $n^{\log_b a} = n^2$

$\therefore$ $\boxed{f(n) < n^{\log_b a}}$

i.e $f(n) = O(n^{2-\epsilon})$ where $\epsilon = 1$

By case 1 of master theorem,

$T(n) = \Theta(n^{\log_b a})$

$\boxed{T(n) = \Theta(n^2)}$

Sundaram

Teacher's Sign.:

2. $T(n) = T\left(\frac{2n}{3}\right) + 1$

Step 1 : Comparing with standard form

i.e $T(n) = aT\left(\frac{n}{b}\right) + f(n)$

$a = 1$ $b = \frac{3}{2}$ $f(n) = 1$

Step 2 : Calculate $n^{\log_b a}$

$n^{\log_b a} = n^{\log_{3/2} 1} = n^0$

$$\boxed{n^{\log_b a} = 1}$$

Step 3 : Identifying the case

$f(n) = 1$ $n^{\log_b a} = 1$

$$\therefore \boxed{f(n) = n^{\log_b a}}$$

i.e $f(n) = \Theta(n^{\log_b a})$

By case 2 of master theorem,

$T(n) = \Theta(n^{\log_b a} \log n)$

$$\boxed{T(n) = \Theta(\log n)}$$

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3. $T(n) = 3T\left(\frac{n}{4}\right) + n \log n$

Step 1: Comparing with standard form

$T(n) = aT\left(\frac{n}{b}\right) + f(n)$

$\therefore \quad a = 3 \qquad b = 4 \qquad f(n) = n \log n$

Step 2: calculate $n^{\log_b a}$

$n^{\log_b a} = n^{\log_4 3} = n^{0.793}$

Step 3: Identifying the case

(i) $f(n) = n \log n \qquad n^{\log_b a} = n^{0.793}$

$\therefore \boxed{f(n) > n^{\log_b a}}$

i.e $f(n) = \Omega(n)$
$= \Omega(n^{0.793+\epsilon})$ where $\epsilon \cong 0.2$

(ii) $af(n/b) = 3\left[\frac{n}{4} \log \frac{n}{4}\right]$

$= \frac{3}{4} n \log n - \frac{3}{4} \log 4$

$\boxed{af(n/b) \le cn \log n}$ where $c = \frac{3}{4}$

By case 3 of Master theorem,
$\therefore \quad T(n) = \Theta(f(n)) = \Theta(n \log n)$

Teacher's Sign.:

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4. $T(n) = 4T(n/2) + n$

Step 1: Comparing with standard form

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

$\therefore$ $a = 4$ $b = 2$ $f(n) = n$

Step 2: Calculate $n^{\log_b a}$

$$n^{\log_b a} = n^{\log_2 4} = n^2$$

Step 3: Identifying the case

$f(n) = n$ $n^{\log_b a} = n^2$

$$\therefore \boxed{f(n) < n^{\log_b a}}$$

i.e $f(n) = O(n^{2-\epsilon})$ where $\epsilon = 1$

By case 1 of master theorem,

$$T(n) = \Theta(n^{\log_b a})$$

$$\boxed{T(n) = \Theta(n^2)}$$

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5. $T(n) = 2T(n/2) + n\log n$

Step 1 : Comparing with standard form

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

$\therefore$ $a = 2$ $\quad$ $b = 2$ $\quad$ $f(n) = n\log n$

Step 2 : Calculate $n^{\log_b^a}$

$$n^{\log_b^a} = n^{\log_2^2} = n$$

Step 3 : Identifying the case

$f(n) = n\log n$ $\quad$ $n^{\log_b^a} = n$

$\therefore$ $f(n)$ is asymptotically larger than $n^{\log_b^a}$, it is not polynomially larger.

$\because$ The ratio $f(n)/n^{\log_b^a} = \log n$ and $\log n$ is asymptotically less than n^{ϵ} for any positive constant ϵ.

$\therefore$ Hence, Not solvable using master method because recurrence falls in the gap between case 2 & case 3.

6. $T(n) = 2T(n/2) + \theta(n)$

$f(n) = n$ $\quad n^{\log_b a} = n$

– case 2

$\therefore T(n) = \theta(n \log n)$

7. $T(n) = 8T\left(\frac{n}{2}\right) + \theta(n^2)$

$f(n) = n^2$ $\quad n^{\log_b a} = n^{\log_2 8} = n^3$

– case 1

$\therefore T(n) = \theta(n^3)$

8. $T(n) = 7T(n/2) + \theta(n^2)$

$f(n) = n^2$ $\quad n^{\log_b a} = n^{\log_2 7} = n^{2.80}$

– case 1

$\therefore T(n) = \theta(n^{\log_2 7})$

9. $T(n) = 2T(n/4) + 1$

$f(n) = 1$ $\quad n^{\log_b a} = n^{\log_4 2} = n^{0.5} = n^{1/2}$

$= \sqrt{n}$

– case 1

$T(n) = \theta(n^{\log_4 2}) = \theta(n^{0.5})$

10. $T(n) = 2T(n/4) + \sqrt{n}$

$f(n) = \sqrt{n}$ $\quad n^{\log_b a} = \sqrt{n}.$

– case 2

$T(n) = \theta(\sqrt{n} \log n)$

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11. $T(n) = 2T(n/4) + n$

$f(n) = n$ $\quad n^{\log_b a} = n^{\log_4 2} = \sqrt{n}$

— case 3

$T(n) = \Theta(n)$

12. $T(n) = 2T(n/4) + n^2$

$f(n) = n^2$ $\quad n^{\log_b a} = \sqrt{n}$.

$f(n) > n^{\log_b a}$

$af(n/b) = 2\left[\frac{n^2}{16}\right] = \frac{2}{16} n^2 \leq cn^2$

$c = \frac{3}{16} < 1$

case 3

$\therefore T(n) = \Theta(n^2)$

12. $T(n) = 4T(n/2) + n^2 \log n$

Step 1 : Comparing with standard form

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

$a = 4$ $b = 2$ $f(n) = n^2 \log n$

Step 2 : Calculate $n^{\log_b a}$

$$n^{\log_b a} = n^{\log_2 4} = n^2$$

Step 3 : Identifying the case

$f(n) = n^2 \log n$ $n^{\log_b a} = n^2$

∴ $f(n)$ is asymptotically larger than $n^{\log_b a}$, but it is not polynomially larger

∴ The ratio $f(n)/n^{\log_b a} = \log n$ and $\log n$ is asymptotically less than n^{ϵ} for any positive constant ϵ.

∴ Hence, $\boxed{\text{Not Solvable}}$ using master method because recurrence falls in the gap between case 2 & case 3.

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THE MASTER METHOD (FOR SUBTRACT & CONQUER RECURRENCES)

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OR

(FOR DECREASING FUNCTION)

Not in Syllabus

MASTER THEOREM

Let. $a > 0$, $b > 0$ and $k >= 0$ be constants and $f(n)$ be a function such that $f(n) = O(n^k)$

let $T(n)$ be a recurrence defined as,

$$T(n) = aT(n-b) + f(n) \qquad n > 1$$

$$= c \qquad n \leq 1$$

where c is constant.

then $T(n)$ has following asymptotic bounds:

Case 1:

If $a < 1$ then
$T(n) = O(n^k)$ i.e $O(f(n))$

Case 2:

If $a = 1$ then
$T(n) = O(n^{k+1})$ i.e $O(n * f(n))$

Case 3:

If $a > 1$ then
$T(n) = O(n^k a^{n/b})$ i.e $O(f(n) \times a^{n/b})$

Ex ① $T(n) = 3T(n-2) + n$

Step 1: comparing with standard form

$T(n) = aT(n-b) + f(n)$

$\therefore$ $a = 3$ $b = 2$ $f(n) = n$ $K = 1$

Step 2: Identifying case.

$\therefore$ $a = 3 > 1$

By case 3 of master theorem,

$T(n) = O(f(n)\, a^{n/b})$ OR $O(n^K a^{n/b})$

$$\boxed{T(n) = O(n \times 3^{n/2})}$$

② $T(n) = T(n-1) + n$

Step 1: comparing with standard form

$T(n) = aT(n-b) + f(n)$

$\therefore$ $a = 1$ $b = 1$ $f(n) = n$ $K = 1$

Step 2: Identifying case.

$\therefore$ $a = 1$

By case 2 of master Theorem,

$T(n) = O(n \times f(n))$ OR $O(n \times n^K)$

$= O(n^2)$

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(3) $T(n) = 2T(n-2) + n^2$

Step 1 : Comparing with standard form

$T(n) \stackrel{?}{=} aT(n-b) + f(n)$

$\therefore$ $a = 2$ $b = 2$ $f(n) = n^2$ $K = 2$

Step 2 : Identifying case.

$\because$ $a = 2 > 1$

By case 3 of master theorem,

$T(n) = O(f(n)\, a^{n/b})$ OR $O(n^K a^{n/b})$

$= O(n^2\, 2^{n/2})$