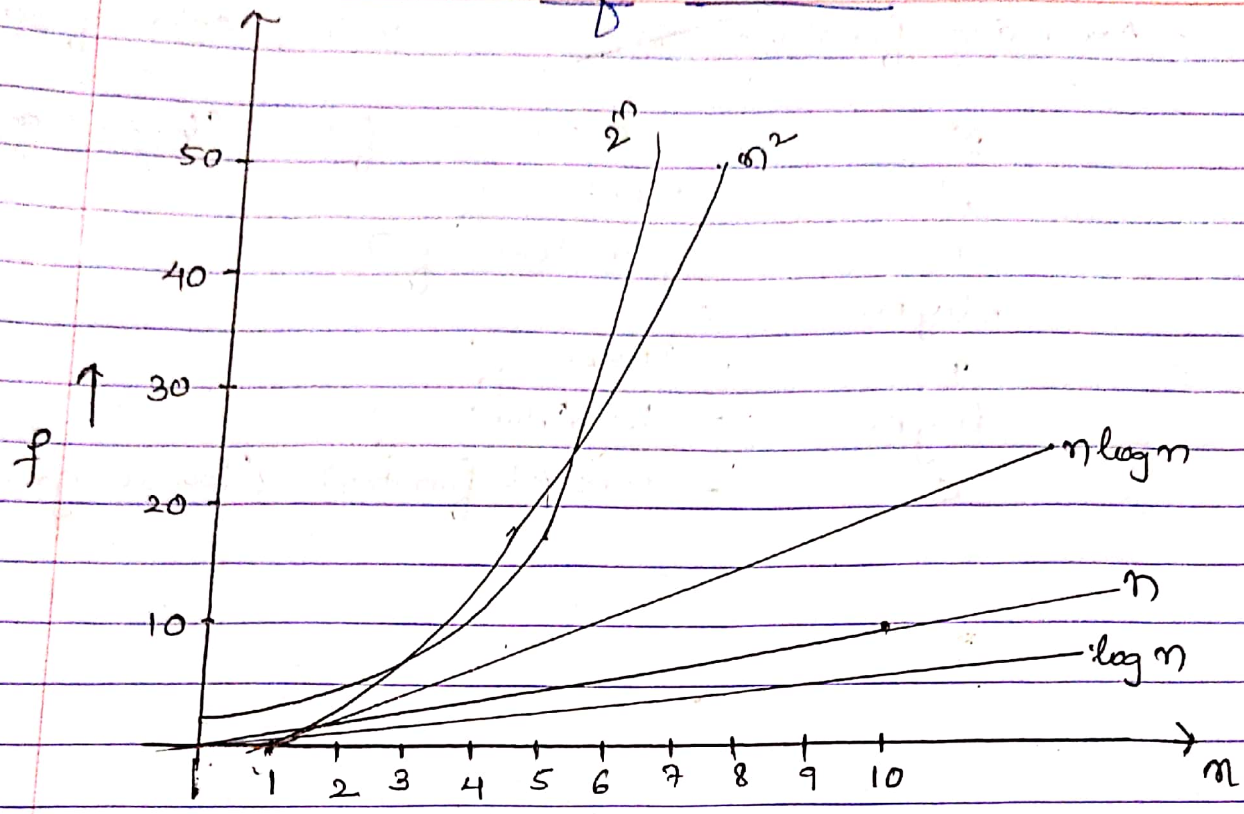
Growth of Function



- Growth rate of function can be determined by plotting the graph of the function.
- The function,

  $f(n) = c$ has lowest growth where c is any constant.

- In above case, the value 'c' remains same although value of n is going to change.

- Function 2^n is exponential function, which changes exponentially even small change in n.

- Some standard functions to compare their rate of growth & can be utilized in asymptotic notation

- Arranging functions in their descending order of grow.

2^n	- exponential f^n	(highest growth)
n^3	- Cubic f^n	
n^2	- Quadratic f^n	
$n \log n$	-	
n	- linear f^n	
$\log n$	- logarithmic f^n	
c	- constant function	(lowest growth)

Asymptotic Notations

Asymptote – line whose distance to given curve tends to zero.

Asymptotic notations helps to represent the relation of given function with standard functions.

3 asymptotic notations

(i) Big - Oh (O)
(ii) Big - Omega (Ω)
(iii) Theta (Θ)

] → loose bounds.

1] Big - Oh (O) notation

- known as Asymptotic Upper Bound.
- written as $f(n) = O(g(n))$

The above statement is read as "f(n) is big-oh of g(n)".

- **Definition:** $f(n)$ & $g(n)$ are non negative function then, we can say that;

$$f(n) = O(g(n))$$

if and only if there exists positive constants c and n_0 such that:

$$f(n) \leq c * g(n) \quad \text{for all } n \geq n_0$$

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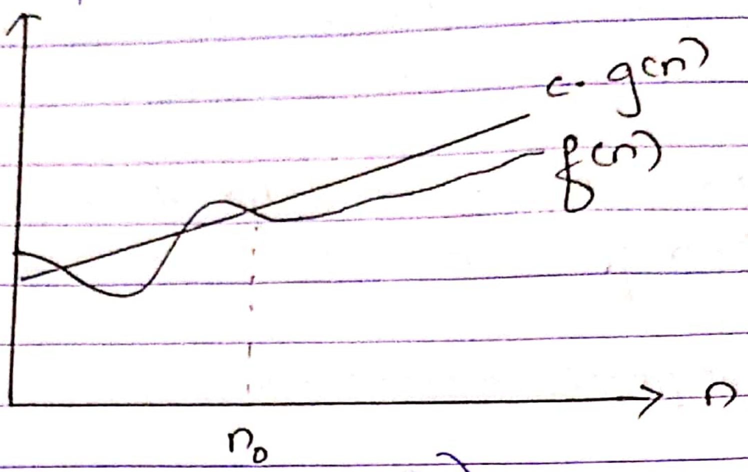
- Graphical Representation:



$$f(n) = O(g(n))$$

- Examples:

(i) $3n+2 \leq 4n$ for all $n \geq 2$

Here, $c = 4$, $n_0 = 2$

$f(n) = 3n+2$

$g(n) = n$

$\therefore \quad 3n+2 = O(n)$

(ii) $3n+3 \leq 4n$ for all $n \geq 3$

Comparing with standard form,

i.e $f(n) \leq c.g(n) \quad \forall\, n \geq n_0$

$c = 4$, $n_0 = 3$

$f(n) = 3n+3 \qquad g(n) = n$

$\therefore \quad f(n) = O(n)$

$3n+2 = O(n)$

(iii) $10n^2 + 4n + 2 \leq 11n^2$ for $n \geq 5$

Comparing with standard form, i.e

$f(n) \leq c * g(n)$ $\forall n \geq n_0$.

$f(n) = 10n^2 + 4n + 2$

$g(n) = n^2$

$n_0 = 5$

$c = 11$

$\therefore$ $f(n) = O(n^2)$

i.e $10n^2 + 4n + 2 = O(n^2)$

(iv) $\overset{f(n)}{6 * 2^n + n^2} \leq 7 * 2^n$ for $n \geq 4$

Comparing with standard form

$f(n) = 6 * 2^n + n^2$

$g(n) = 7 * 2^n$

$c = 7$.

$n_0 = 4$

$\therefore$ $f(n) = O(2^n)$

i.e. $6 * 2^n + n^2 = O(2^n)$

Note: $g(n)$ should be as small function of n for which $f(n) = O(g(n))$ should be true

2] Big - Omega (Ω) Notation

- Known as Asymptotic lower bound
- written as $f(n) = \Omega(g(n))$
- Read as "f(n) is big omega of g(n)".

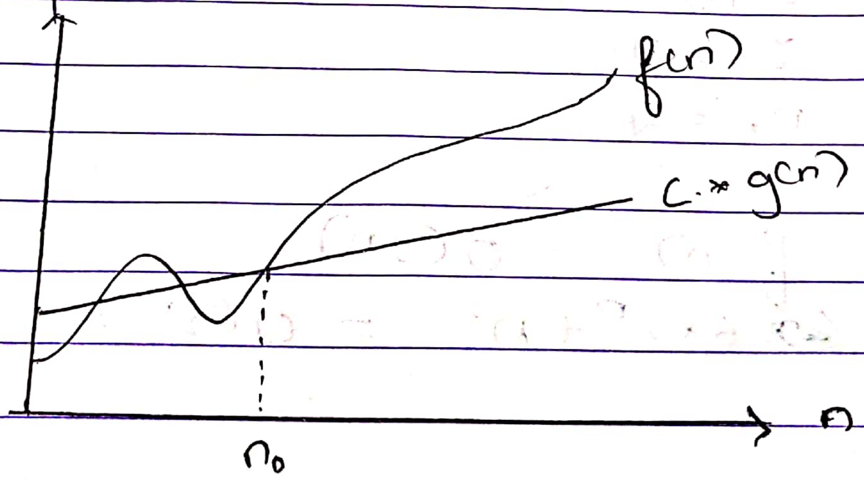
- **Definition:** f(n) and g(n) are non negative function then, we can say that

$$f(n) = \Omega(g(n))$$

if and only if there exist positive constants C and n_0 such that

$$f(n) \geq C * g(n) \quad \text{for all } n \geq n_0$$

- Graphical Representation,



$$f(n) = \Omega(g(n))$$

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Examples

(i) $3n+2 \geq 3n$ for $n \geq 1, 0$

Comparing with standard form i.e.

$f(n) \geq \cancel{\Omega(g(n))}$ $\forall\ n \geq n_0$

$c * g(n)$

$f(n) = 3n+2$, $g(n) = n$

$c = 3$, $n_0 = 1$

$\therefore$ $3n+2 = \Omega(n)$

(ii) $10n^2+4n+2 \geq n^2$ for $n \geq 1, 0$

comparing with standard form i.e.

$f(n) \geq c * g(n)$ $\forall\ n \geq n_0$

$f(n) = 10n^2+4n+2$, $g(n) = n^2$

$c = 1$, $n_0 = 1$

$\therefore$ $10n^2+4n+2 = \Omega(n^2)$

(iii) $6 * 2^n + n^2 \geq 2^n$ for $n \geq 1, 0$

Comparing with standard form.

$f(n) = 6 * 2^n + n^2$ $c = 1$

$g(n) = 2^n$ $n_0 = 1$

$\therefore$ $6 * 2^n + n^2 = \Omega(2^n)$

Note: $g(n)$ should be as large function of n for which $f(n) = \Omega(g(n))$ is true.

3] Theta Notation (Θ)

- Known as Asymptotic tight bound.
- It represents both upper and lower bound for a given function $f(n)$.

- written as : $f(n) = \Theta(g(n))$
- Read as : "f(n) is theta of g(n)"

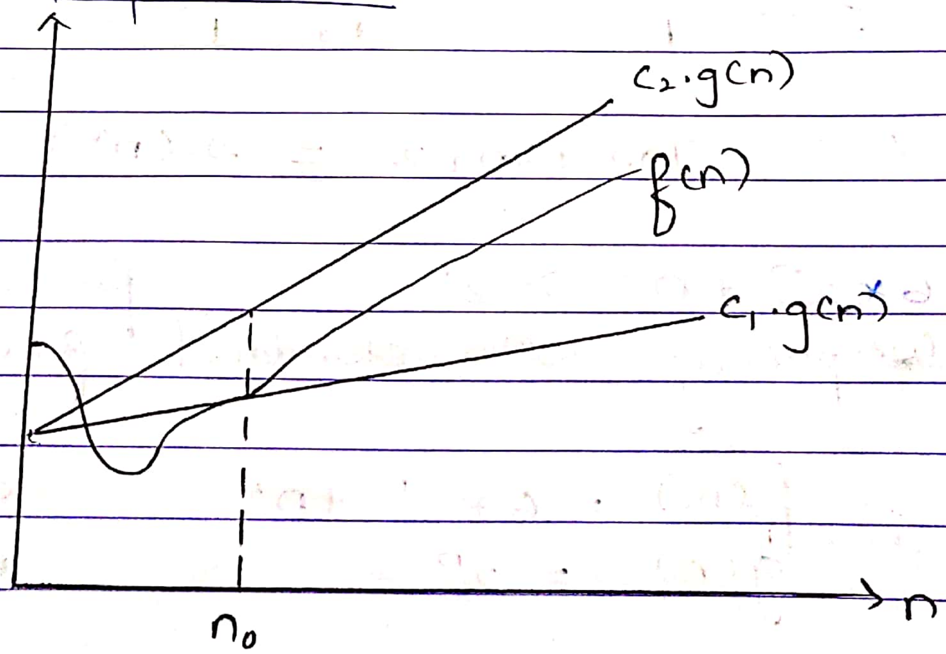
- Definition: f(n) and g(n) are non-negative function then we can say that

$$f(n) = \Theta(g(n))$$

If and only if there exists positive constant c_1, c_2 and n_0 such that

$$c_1 g(n) \leq f(n) \leq c_2 g(n) \quad \text{for } n \geq n_0$$

- Graphical Representation



$$f(n) = \Theta(g(n))$$

Examples

(i) $3n+2 \geq 3n$ for all $n \geq 2$

$3n+2 \leq 4n$ for all $n \geq 2$

Comparing with standard form,

$f(n) = 3n+2$ $\quad c_1 = 3$ $\quad n_0 = 2$

$g(n) = n$ $\quad c_2 = 4$

$\therefore \boxed{3n+2 = \Theta(n)}$

$f(n) \neq \Theta(n^2)$ X

Similarly it can be proved for the following relations

$10n^2+4n+2 = \Theta(n^2)$

$6*2^n + n^2 = \Theta(2^n)$

$10 * \log n + 4 = \Theta(\log n)$

(ii) $f(n) = n!$

$f(n) = n! = 1 \times 2 \times 3 \ldots \times n$

$1 \times 1 \times 1 \times 1 \ldots 1 \leq 1 \times 2 \times 3 \ldots \times n \leq n \times n \times \ldots n$

$1 \leq 1 \times 2 \times 3 \ldots \times n \leq n^n$

$f(n) = O(n^n)$ $\quad f(n) = \Omega(1)$

But cannot be represented in Θ.

Incorrect Bounds

1) Big-Oh (O) – upper Bound.

$6n+3 \neq O(1) \neq O(\log n)$

$= O(n) = O(n^2) = O(n\log n) = O(2^n)$

$2n^3+2n^2+3 \neq O(n^2) \neq O(n) \neq O(\log n) \neq O(1)$

$= O(n^3) = O(2^n)$

2) Big-omega (Ω) – lower Bound.

$6n+3 = \Omega(n) = \Omega(1) = \Omega(\log n)$

$\neq \Omega(n^2) \neq \Omega(n^3) \neq \Omega(2^n)$

$2n^3+2n^2+3 = \Omega(1) = \Omega(n) = \Omega(n^2) = \Omega(n^3)$

$\neq \Omega(n^4) \neq \Omega(2^n)$

(iii) $f(n) = \log n!$

$f(n) = \log(1\times2\times3\cdots n)$

$\log(1\times1\times1\times\cdots1) \le \log(1\times2\cdots n) \le \log(n\times n\times\cdots n)$

$\log 1 \le \log n! \le \log n^n$

1 → $0 \le \log n! \le n\log n$

$\therefore O(n\log n) \quad \Omega(1)$

- Hence, when we know exact time complexity then use Θ - notation. [exact time]
- O - notatn & Ω notation are used when we are not sure of exact time complexity.