MTH215 ⋆⋆ 2017-18 ⋆⋆ ASSIGNMENT – 2

1. Let $E_n = \{i \in \mathbb{N} \mid 1 \leq i \leq n, \gcd(i,n) = 1\}$. Then for $n \geq 2$, $\sum_{i \in E_n} i = n\varphi(n)/2$.
2. For $n \geq 3$, $\varphi(n)$ is even.
3. For every odd integer $n \in \mathbb{N}$, $\varphi(n) \geq \sqrt{n}$.
4. For every $n \in \mathbb{N}$, $\varphi(n) \geq \sqrt{n}/2$.
5. For $m, n \in \mathbb{N}$, $\varphi(m)|\varphi(n)$ whenever $m|n$.
6. Let $m, n \in \mathbb{N}$ such that every prime deviser of m is also a prime divisor of n. Then $\varphi(mn) = m\varphi(n)$.
7. For $m, n \in \mathbb{N}$, $\varphi(m)\varphi(n) = \varphi(mn)\varphi(d)/d$ where $d = \gcd(m,n)$. Moreover, $\varphi(m)\varphi(n) = \varphi(l)\varphi(d)$, where $l = \mathrm{lcm}(m,n)$.
8. For $n \in \mathbb{N}$ show that $\tau(n) < 2\sqrt{n}$.
9. Let $n \in \mathbb{N}$. Show that $\tau(n)$ is odd iff n is a perfect square.
10. Let $n \in \mathbb{N}$. Show that $\sigma(n)$ is odd iff n is a perfect square or 2 times a perfect square.
11. An integer n is called **square free** if n not divisible by a perfect square other than 1. For a square free integer n show that $\tau(n) = 2^r$, where r denotes the number of prime divisors of n.
12. Show that there are infinitely many pairs of positive integers m and n such that $\sigma(n) = \sigma(m)$. (Hint: $\sigma(16) = 31 = \sigma(25)$.)
13. Let $k \in \mathbb{N}$ such that $2^k - 1$ is prime. Show that $\sigma(2^{k-1}(2^k - 1)) = 2^k(2^k - 1)$.
14. Let f and g be multiplicative functions such that $f(p^r) = g(p^r)$ for every prime p and $r \in \mathbb{N}$ and $f(1) = g(1)$. Show that $f = g$.
15. If f and g are multiplicative then so is fg. Moreover, if $f(n)$ is never zero then $1/f$ is multiplicative.
16. Let $\omega(n)$ denote the number of distinct prime divisors of n for $n > 1$ and let $\omega(1) = 0$. Show that $2^{\omega(n)}$ is multiplicative. Further show that $\tau(n^2) = \sum_{d|n} 2^{\omega(d)}$.
17. Show that $\sum_{d|n} \tau(d)^3 = (\sum_{d|n} \tau(d))^2$.
18. Show that $\sum_{d|n} \sigma(d) = \sum_{d|n}(n/d)\tau(d)$ and $\sum_{d|n}(n/d)\sigma(d) = \sum_{d|n} d\tau(d)$ for all $n \in \mathbb{N}$.
19. Let $n \in \mathbb{N}$. Note that $(n+1)! + 2, (n+1)! + 3, \ldots, (n+1)! + n, (n+1)! + (n+1)$ are n successive conjugates.
20. Find p if p and $p^2 + 8$ are both primes.
21. Find all primes p such that $q = p + 2$ and $pq - 2$ are also primes.
22. Find all primes p which divide $n^2 + 3$ for two successive values of n.
23. Let x, y be real numbers and n be an integer. Show that

    (a) $[x + n] = [x] + n$.

(b) $[x+y] = [x]+[y]$ or $[x]+[y]+1$.

(c) $[x]-[-x] = 0$ or 1 according as x is an integer or not.

(d) $[x/n] = [[x]/n]$, where n is a positive integer.

(e) $[x]+[y]+[x+y] \leq [2x]+[2y]$.

24. For all nonnegative integers n, $[n/2]-[-n/2] = n$.

25. Show that $\binom{2n}{n}$ is an even integer for every $n \in \mathbb{N}$.

26. Find smallest integer n such that 12^{1000} divides $n!$.

27. Let f be a number theoretic function and let $\tilde{f}(n) = \sum_{d|n} f(d)$. Prove that $\sum_{k=1}^{n} \tilde{f}(k) = \sum_{k=1}^{n} f(k)[n/k]$.

28. Show that $\sum_{k=1}^{n} \mu(k)[n/k] = 1$.

29. Show that $\tau(n) = \sum_{k=1}^{n}\{[n/k]-[(n-1)/k]\}$.

30. Show that $\sum_{k=1}^{n} \gcd(k,n) = \sum_{d|n} d\varphi(n/d) = n\sum_{d|n} \varphi(d)/d$.

31. Show that there are infinitely values of n satisfying $\varphi(n) = n/3$.

32. Let $k \in \mathbb{N}$. If there exists a unique integer n such that $\varphi(n) = k$ then show that $4 \mid n$.

33. Let p be an odd prime. Show that there exist unique positive integers a and b such that $p = a^2 - b^2$.

34. Let $n \neq 1$ be an odd integer, which is neither a prime nor a square of a prime. Show that n can be written as a difference of two squares in two different ways.

35. Let $n \in \mathbb{N}$ be an odd integer such that $\tau(n) \geq 4$. Show that n can be written as a difference of two squares in two different ways.

36. Let (x,y,z) be a primitive Pythagorean triple. Show that $x \pm y \equiv \pm 1 \pmod{8}$. Further show that $xy \equiv 0 \pmod{12}$ and $xyz \equiv 0 \pmod{60}$.

37. Let (x,y,z) be a Pythagorean triple such that x,y,z are in arithmetic progression. Show that $(x,y,z) = (3k,4k,5k)$ for some positive integer n.

38. Find all $n \in \mathbb{N}$ such that $(n,y,n+1)$ form a Pythagorean triple for some y.

39. Determine all solutions of

    (a) $7x+9y=6$.

    (b) $4x+23y=3$.

    (c) $18x+5y=48$.

    (d) $10x-3y=17$.

40. Let p be a prime and let $n \in \mathbb{N}$. Let $a \in \mathbb{Z}$ be a primitive element for p^{n+1} then show that a is also a primitive element for p^n. Hence show that a is a primitive element for p^k for all $k \in \mathbb{N}$.