MTH215 $\star\star$ 2019-20 $\star\star$ ASSIGNMENT – 5

1. Let $\theta = \langle a_0, a_1, \ldots \rangle$ be an irrational number, where a_is are integers and $a_i \geq 1$ for $i \geq 1$. Show that $\mid \theta - h_n/k_n \mid < 1/k_n^2$.

2. Let θ be as in the previous question. Let $n \geq 0$ be an integer. Show that $\mid \theta - h_i/k_i \mid < 1/\sqrt{5}k_i^2$ for $i = n$ or $i = n+1$ or $i = n+2$.

3. A positive solution (a, b) of Pell's equation $x^2 - dy^2 = 1$, is said to be a fundamental solution if $b \leq \beta$ whenever (α, β) is a positive solutions of $x^2 - dy^2 = 1$. Find the fundamental solutions of $x^2 - dy^2 = 1$ for $d = 2, 3, 5, 6, 7, 8$.

4. Find all positive solutions of $x^2 - dy^2 = 1$ for $d = 2, 3, 5$ such that $y < 100$.

5. Let (a, b) be a solution of $x^2 - dy^2 = -1$. Show that $(db^2 + a^2, 2ab)$ is a solution of $x^2 - dy^2 = 1$.

6. Let (x_1, y_1) be the fundamental solution of $x^2 - dy^2 = 1$. Let $x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n$. Show that $x_{n+1} = x_1 x_n + d y_1 y_n$ and $y_{n+1} = x_1 y_n + x_n y_1$ for $n \geq 1$.

7. Let (x_1, y_1) be the fundamental solution of $x^2 - dy^2 = 1$. Let $x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n$. Show that $x_{n+1} = 2x_1 x_n - x_{n-1}$ and $y_{n+1} = 2x_1 y_n - y_{n-1}$ for $n \geq 2$.

8. Show that there are infinitely many primitive Pythagorean triples of the type $(a, a+1, c)$.

9. Show that there are infinitely many positive integers n such that both $2n+1$ and $n+1$ are squares.

10. Let $m \in \mathbb{Z}$. If $x^2 - dy^2 = m$ has a solution then show that it has infinitely many solutions.

11. Let $n \geq 2$ be an integer. Show that $x^2 - (n^2 - 1)y^2 = -1$ has no solution such that $x, y \in \mathbb{Z}$.

12. Let $d \in \mathbb{Z}$ such that $d \equiv 3 \pmod{4}$. Show that $x^2 - dy^2 = -1$ has no solution such that $x, y \in \mathbb{Z}$.

13. Let p be a prime such that $p \equiv 1 \pmod{4}$. Show that $x^2 - py^2 = -1$ has solutions such that $x, y \in \mathbb{Z}$.

14. Let (a, b) be a positive solution of $x^2 - dy^2 = -1$. Let $d = \langle a_0, a_1, a_2, \ldots \rangle$. Define h_i and k_i. Show that $(a, b) = (h_i, k_i)$ for some i.

15. Let (a, b) and (α, β) be positive solutions of Pell's equation $x^2 - dy^2 = -1$. Show that $b < \beta$ iff $a < \alpha$. (A positive solution (a, b) of Pell's equation $x^2 - dy^2 = -1$, is said to be a fundamental solution if $b \leq \beta$ whenever (α, β) is a positive solutions of $x^2 - dy^2 = 1$.)

16. Let (a, b) be the fundamental solution of $x^2 - dy^2 = -1$. Show that $(db^2 + a^2, 2ab)$ is the fundamental solution of $x^2 - dy^2 = 1$.

    In the questions below $\{a_n\}$ denotes the Fibonacci sequence. Let $n \in \mathbb{N}$

17. Let $2 \mid a_n$. Show that $4 \mid (a_{n+1}^2 - a_{n-1}^2)$.

18. Let $3 \mid a_n$. Show that $9 \mid (a_{n+1}^3 - a_{n-1}^3)$.

19. $a_{n+3} \equiv a_n \pmod{2}$ and $a_{n+5} \equiv 3a_n \pmod{5}$.

20. $\sum_{i=1}^{n} a_i^2 = a_n a_{n+1}$ and $a_{n+1}^2 = a_n^2 + 3a_{n-1}^2 + 2\sum_{i=1}^{n-2} a_i^2$.

21. Let $m, n \in \mathbb{N}$ with $\gcd(m, n) = 1$. Show that $\gcd(a_m, a_n) = 1$. Hence show that $a_m a_n \mid a_{mn}$.

22. $2^{n-1} a_n \equiv n \pmod{5}$ and $a_{2n} \equiv (-1)^{n+1} n \pmod{5}$.

23. Let $a, b \in \mathbb{N}$ such that $a_n < a < a_{n+1} < b < a_{n+2}$. Show that $a + b$ is not a Fibonacci number.

24. Prove that there exists no $n \in \mathbb{N}$ such that $\sum_{i=1}^{3n} a_i = 16!$.

25. Show that there exists n consecutive Fibonacci numbers.

26. Show that $9 \mid a_{n+24}$ iff $9 \mid a_n$.

27. Show that $\sum_{i=1}^{n} i a_i = (n+1)a_{n+2} - a_{n+4} + 2$.

28. Show that $\sum_{i=1}^{n} i a_{2i} = n a_{2n+1} - a_{2n}$.

29. Show that $\sum_{i=1}^{n} (-1)^{i-1} a_i = 1 + (-1)^{n-1} a_{n-1}$, where $n \geq 2$.

30. For $n \geq 2$, show that $a_{2n-1} = a_n^2 + a_{n-1}^2$ and $a_{2n} = a_{n+1}^2 - a_{n-1}^2$.

31. Let p be a prime of the form $4k + 3$. Show that there exists no $n \in \mathbb{N}$ such that $p \mid a_{2n-1}$.

32. (Binet Formula) $a_{2n+2} a_{2n-1} - a_{2n} a_{2n+1}$.

33. Show that $a_n = \sum_{i=0}^{k} \binom{n-1-i}{i}$ where $k = [(n-1)/2]$.

34. Show that $\sum_{i=1}^{n} \binom{n}{i} a_i = a_{2n}$.

35. Show that $\sum_{i=1}^{n} \binom{n}{i} (-1)^i a_i = -a_n$.