MTH215A − 2019-20 ⋆ ASSIGNMENT − 1

1. Let $a, b, c, d, x \in \mathbb{Z}$ such that $a \mid b$ and $c \mid d$. Show that
   (a) (1) $x \mid 0$ and $1 \mid x$ (2) $0 \mid x$ iff $x = 0$ (3) $a \mid bx$.
   (b) (1) $x \mid x$ (2) if $x \mid y$ and $y \mid z$ then $x \mid z$ (3) if $x \mid y$ and $y \mid x$ then $x = \pm y$.
   (c) (1) $ax \mid bx$ (2) $ac \mid bd$ (3) $a^n \mid b^n$ for all $n \in \mathbb{N}$.
2. Let $a, b, x, y, x_1, \ldots, x_k \in \mathbb{Z}$.
   (a) Let $d = \gcd(x_1, \ldots, x_k)$. Then $\gcd(x_1/d, \ldots, x_k/d) = 1$.
   (b) If $\gcd(x_1, x) = \cdots = \gcd(x_k, x) = 1$ then $\gcd(x_1 \cdots x_k, x) = 1$.
   (c) If $x_1, \ldots, x_k$ are pairwise coprime then $\gcd(x_1 \ldots x_{i-1}x_{i+1} \ldots x_k, x_i) = 1$ for all $i = 1, \ldots, k$.
   (d) If $\gcd(a, b) = 1$ and $x \mid a$ and $y \mid b$ then the $\gcd(x, y) = 1$.
3. $a, b, c, d, x, y \in \mathbb{Z}$ and let $n \geq 2$ be an integer such that $a \equiv b (\text{mod } n)$ and $a \equiv b (\text{mod } n)$. Show that
   (a) (1) $x \equiv x (\text{mod } n)$ (2) $a + x \equiv b + x (\text{mod } n)$ (3) $ax \equiv bx (\text{mod } n)$.
   (b) (1) $a + c \equiv b + d (\text{mod } n)$ (2) $ac \equiv bd (\text{mod } n)$ (3) $a^k \equiv b^k (\text{mod } n)$ for all $k \in \mathbb{N}$.
4. Let $p \geq 5$ be an odd prime. Then $p \equiv \pm 1 (\text{mod } n)$ for $n = 2, 3, 4$ & 6.
5. Let $n \in \mathbb{Z}$. Then $n^2 \equiv 0$ or $1 (\text{mod } 3)$.
6. For $n \in \mathbb{Z}$, prove that $3n^2 - 1$ can never be a perfect square.
7. For $n \geq 1$ prove that $6 \;\Big|\; n(7n^2 + 5)$.
8. For an odd integer n, show that $16 \;\Big|\; n^4 + 4n^2 + 11$.
9. Show that the product of n consecutive integers is divisible by $n!$.
10. Let $p \geq 5$ be a prime. Show that $p^2 + 2$ is composite. (Hint: $3 \nmid p$.)
11. Let $a, b \in \mathbb{Z}$ such that $\gcd(a, b) = 1$. Show that $\gcd(a^k, b^l) = 1$ for all $k, l \geq 1$.
12. For every $n \in \mathbb{Z}, n \neq \pm 1$, show that $n^4 + 4$ is composite.
13. For every composite integer $n > 4$, $n \;\Big|\; (n-1)!$.
14. **Bertrand's Postulate.** For every $n > 1$ there exists a prime p such that $n < p < 2n$. Verify this for $n \leq 4000$.
15. Using Bertrand's Postulate, show that $n!$ can not be a perfect square for $n > 1$.
16. Show that every integer $n > 11$ is a sum of two composites.

17. Let p_n denote the nth prime in increasing order, that is, $2 = p_1 < p_2 < \cdots < p_n$ and there is no prime between p_n and p_{n+1}. Show that $\sum_{i=1}^{n} \dfrac{1}{p_i}$ is not an integer.

18. Find all the pairs of primes p, q such that $p - q = 3$.

19. Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ be a nonzero polynomial, where a_i are integers and $n \geq 1$. Show that there are integers m such that $f(m)$ is composite.

20. Show that there are infinitely many primes of form $6n + 5$.

21. Given an integer $n > 0$, find an integer $m > 1$ such that $k \mid n^m - n$ for every $1 \leq k \leq n$ and $\gcd(k, n) = 1$.

22. Find all the solutions of $x^{13} + 12x \equiv 0 \pmod{13}$.

23. Let $p > 2$ be a prime. Show that $\dfrac{1}{1} - \dfrac{1}{2} + \dfrac{1}{3} - \cdots - \dfrac{1}{p-1} = \dfrac{a}{(p-1)!}$, where $a \equiv \dfrac{2 - 2^p}{p} \pmod{p}$.

24. Let p be a prime such that $p \equiv 1 (\mathrm{mod}\ 4)$. Show that there exist $a \in \mathbb{Z}$ such that $a^2 + 1 \equiv 0 (\mathrm{mod}\ p)$.

25. Let $n \geq 2, k$ be an integers, $1 \leq k < n$; and p be a prime. Let $\alpha \geq 0$ be such that $p \mid \binom{n}{k}$. Show that $\alpha = \sum_{i \geq 1} \Big([n/p^i] - [k/p^i] - [(n-k)/p^i] \Big)$ and that $p^\alpha \leq n$.

26. Let $n \in \mathbb{N}$. Show that $\binom{2n+1}{n} \leq 2^{2n}$ and $\binom{2n}{n} \geq 4^{n-1}/2n$.

27. Let $n \in \mathbb{N}$ and let p be a prime such that $n + 1 < p \leq 2n + 1$. Show that $p \mid \binom{2n+1}{n}$.

28. For every $x > 0$, let $f(x)$ denote the product of all prime less that or equal to x. If no such prime exists the put $f(x) = 1$.

    (a) Let $f(x) = f([x])$ and for $x \geq 2$, $f(x) = f(q)$ where q is the largest prime less than or equal to x.

    (b) For every $n \in \mathbb{N}$ show that $f(n) \leq 4^{n-1}$. Hence, show that $f(x) \leq 4^{x-1}$ for all $x > 0$.

29. For all $x \in \mathbb{R}, x \geq 1$ show that $x \leq 2^x$.

30. Let $n \geq 10$ be an integer such that there is no prime in $(n, 2n)$. Let $\binom{2n}{n} = p_1^{\alpha_1} \ldots p_r^{\alpha_r}$, where $p_1 < \cdots < p_r$ are primes and $\alpha_i \geq 1$. Show that

    (a) $\sqrt{2n} < 2n/3$.

    (b) For every i, $p_i \leq 2n/3$. Furthermore, $\alpha_i = 1$ whenever $\sqrt{2n} < p_i \leq 2n/3$.

    (c) $\prod_{p_i \leq \sqrt{2n}} p_i^{\alpha_i} \leq (2n)^{\sqrt{2n}}$.

    (d) $4^{n-1}/(2n) \leq \binom{2n}{n} \leq (2n)^{\sqrt{2n}} \times 4^{(2n/3)-1}$. Hence, conclude that $n \leq 4000$.