Closest pair problem

Input: A list of $n \geq 2$ points $S \subseteq \mathbb{R}^2$.
Output: Points $(a,b), (c,d) \in S$ that are the closest.

- Trivial or brute-force approach?

Go over all the possible pairs.
$\# \text{steps} \geq \binom{n}{2} = O(n^2), \Omega(n^2), \Theta(n^2)$.
(brush-up your asymptotics)
[Big-Oh, Big-Omega, Theta]

- If $n = 1$ billion $= 10^9$ then n vs. n^2 time is a Huge difference (10^{18} nanosecs > 300yrs is impossible!)

- Could you find a closest pair faster?

- Hint: Recurse.
But, how do we divide the points S?

- Sort by the x-coordinate & collect the lower $n/2$ points in the set L.
  The remaining $n/2$ points form R.

[Sorting takes $O(n \lg n)$ operations. Why?

Let us revise Merge Sort:

Given set $X = \{x_1, \dots, x_n\}$ we recursively sort the first $n/2$ numbers to get L & the last $n/2$ to get R.

Now we want to merge $L = \{l_1 \leq \dots \leq l_{n/2}\}$ & $R = \{r_1 \leq \dots \leq r_{n/2}\}$:

i) Find the earliest position of l_1 in R & insert it.

ii) From that position onwards, find the position of l_2 & insert it in R.

iii) Continue till you reach $l_{n/2}$ or $r_{n/2}$.

▷ The comparisons done in the Merge Step are only $O(n)$.

▷ The recurrence for time is:

$$T(n) = 2 \cdot T(n/2) + O(n)$$

$$\Rightarrow T(n) = O(n \lg n). \quad \square$$

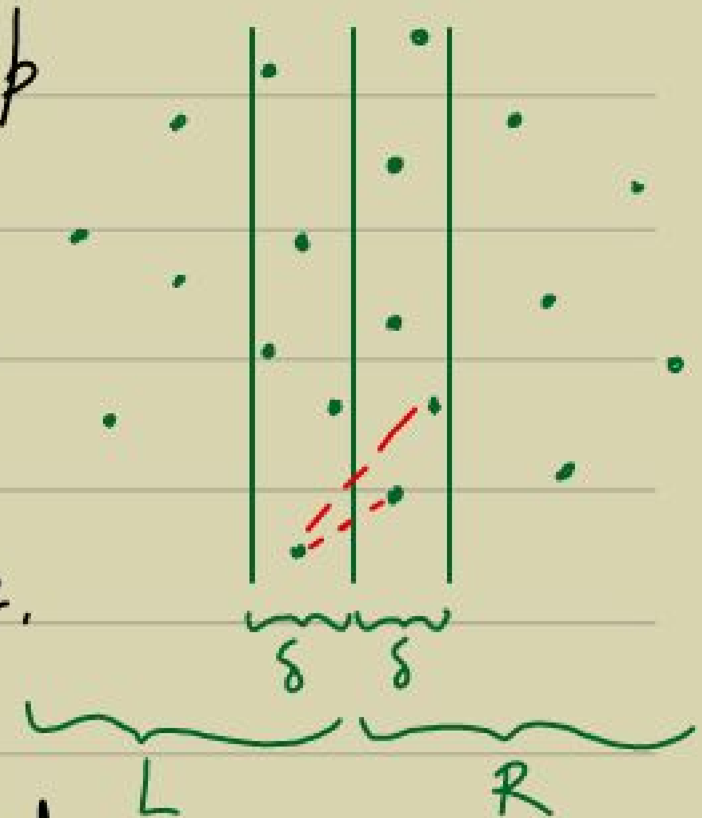
]

- Coming back to Closest Pair: We have left points L & right points R.

CP-dist(L) & CP-dist(R)

- Say, recursively we find closest pair distance δ_L in L & δ_R in R.
- Compute $\delta := \min(\delta_L, \delta_R)$.
- We can find the δ-strip on the left – S_L & the one on the right – S_R.

  How to combine?



distinct values →

- Sort S_R by y-coordinate.
- For each $p \in S_L$ {
  - let y_p be the y-coord.
  - Binary search for points $q \in S_R$ with y-coordinates $(y_p - \delta, y_p + \delta)$.

    [They all can be found in $O(\lg n)$ time.]
  - Compute the distance (p, q) & update δ if required. }
- Output δ.

▷ Time taken $T(n)$ is
- Divide step: $2T(n/2) + O(n \lg n)$
- Combine step: $O(n \lg n) + O(n)$.

$\Rightarrow T(n) = 2T(n/2) + O(n \lg n)$.

$\Rightarrow T(n) = O(n \lg^2 n) = \tilde{O}(n)$. ← soft-Oh

Theorem: There is an $O(n \lg^2 n)$ time algorithm to compute ClosestPair of n points in $\mathbb{R}^2$.

- Note: The time would also depend on the number of bits required to store a point.

- $O(n \lg^2 n)$ is an improvement over $O(n^2)$. Can we do better? Is this a lower bound?

- Suppose we want to reduce it to $O(n \lg n)$. Then, there are two places where we need to optimize:

x-sort S only once?

1) Don't sort S in Divide step.
2) " " S_R in Combine step.

- What are the alternatives?

Alternative Divide Step: (middle after sorting)

We find the x-median of S in $O(n)$ time.

- How can this be done without sorting?

Median of Medians Idea:

- Let Select (S, i) be the function that returns the element of rank i in S.

Generalization helps!

We will recursively define it!

Select(S, i) {

i) Divide the n elements S into $n/5$ groups each of size 5.

ii) Find the median in each group.

iii) Use Select( ) recursively to find the median x of these $n/5$ medians.

iv) Compute the rank k of x in S.

v) Divide S around x: $S_{<x}$ are elements $<x$ & $S_{>x}$ are those $>x$.

vi) If $i=k$ then return x.

Solve & Combine

If $i<k$ " " Select$(S_{<x}, i)$.

If $i>k$ " " Select$(S_{>x}, i-k)$.

- Proof of correctness requires you:
  - to check the base case,
  - to check the recursive calls, &
  - to check the returned value.

- Time complexity $T(n)$ has a recurrence,
  - First recursive call is on $n/5$ size.
  - Second " " " " $S_{<x}$ or $S_{>x}$

$\triangleright$ $\#S_{<x} \geq 3 \cdot \left(\frac{1}{2} \cdot \frac{n}{5} - 1\right) + 2 = \frac{3n}{10} - 1$

$\triangleright$ $\#S_{>x} \geq 3n/10 - 1$

$\triangleright$ $\#S_{<x} + \#S_{>x} = n-1$

- Thus, $\#S_{<x}, \#S_{>x} \leq 7n/10$.

$$\Rightarrow \quad T(n) \leq T(n/5) + T\left(\frac{7n}{10}\right) + O(n)$$

$$\Rightarrow \quad T(n) = T(n/5) + T(7n/10) + O(n)$$

Note: $1/5 + 7/10 = 0.9 < 1$

$$\Rightarrow \quad T(n) = O(n).$$

Lemma: Median is computable in $O(|S|)$ time.

Alternative Combine Step:

- We demand that CP-dist(L) & CP-dist(R) give us L & R each sorted by y-coordinate.

- Note that the combine step of CP-dist(S) could merge the two & sort S as well.

$\Rightarrow$ S_R is already sorted by y-coordinate.

$\triangleright$ The new implementation of CP-dist(S) has the recurrence: $T(n) = 2T(n/2) + O(n)$.

[Shamos-Hoey 1975]

Theorem: Closest pair in the plane is computable in $O(n \lg n)$ time.

Exercise! Write the full pseudocode for this algorithm. This will force you to deal with boundary cases & conditions.

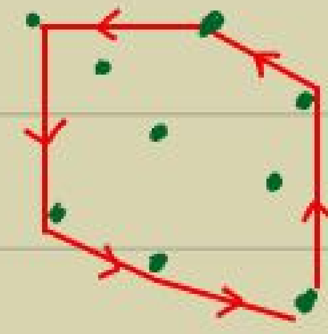
Convex Hull problem

CH(S):

Input: Given n points $S \subseteq \mathbb{R}^2$.

Output: Convex polygon of smallest area enclosing S.

(Think of a rubber band enclosing a set of pins!)



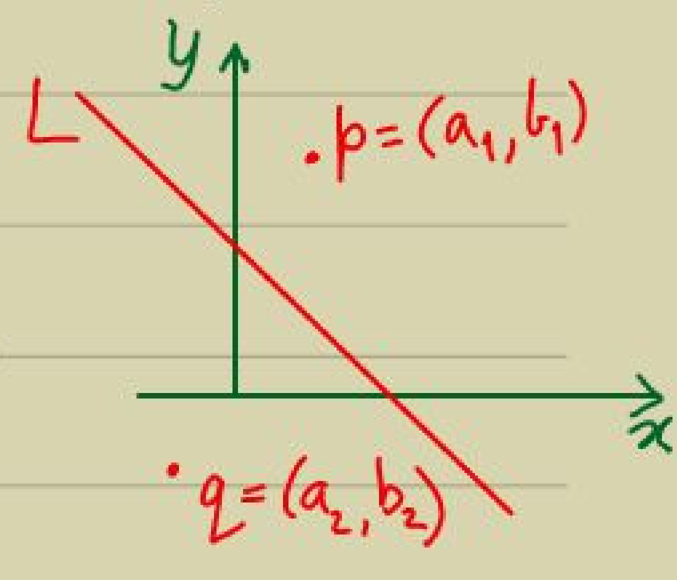
- Brute-force algorithm:

  Find edges such that all the other points in P lie on "one side" (say, left).

  Naively, it takes $O(n^2)$ time.

- Can we do better? (Exploit geometry!)

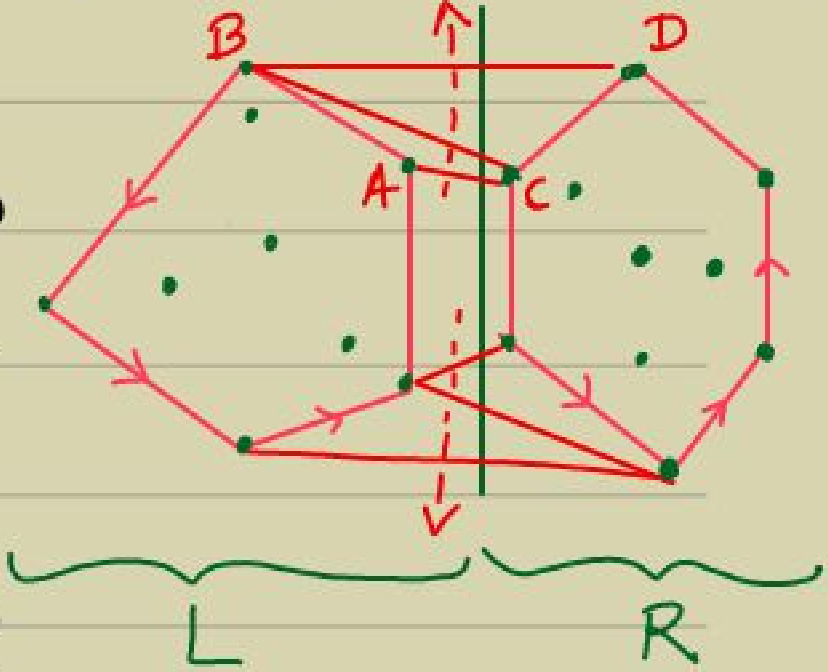
- Qn: Given a line $L: y = mx + c$ & points p, q how do you test whether they are on the same side of L?



▷ On the upper side of L $y > mx + c$ & on the lower side $y < mx + c$.

$\Rightarrow$ In $O(1)$ time we can test whether p, q are on the same side of L.

- Use this & Divide-Conquer to find $CH(S)$.



- Divide: Partition S into L & R by the x-median.

- Solve: $CH(L)$ & $CH(R)$.

Assume that the hull has vertices in the anticlockwise direction.

- Combine: Go over $CH(L)$, $CH(R)$ & do a new merge-like process to find the extremal edges.

Eg. for the cross-edge AC check whether B is above. If YES then pick BC.

For BC find the neighbour that is above (say, D). Pick BD.

As BD has no neighbour above, it

becomes an extremal edge.

Analogously, find the second extremal edge.

Exercise: Proof of correctness.

▷ The recurrence for time $T(n)$ is:

$$T(n) = 2T(n/2) + O(n).$$

$$\Rightarrow \quad T(n) = O(n \lg n).$$

Theorem: $CH(S)$ is computable in $O(|S| \lg |S|)$ time.

- Write the detailed pseudocode.

- Can the time be improved?

- The algorithm is by Preparata & Hong (1977).