MTH215 ⋆⋆ 2019-20 ⋆⋆ ASSIGNMENT – 4

1. Show that $\left| \sum_{d=1}^{n} \dfrac{\mu(d)}{d} \right| \leq 1$.
2. Let x, y be positive real numbers. Show that $[x-y] \leq [x]-[y] \leq [x-y]-1$.
3. Show that $\displaystyle\sum_{k\geq 1} \left[\frac{n}{2^k}+\frac{1}{2}\right] = n$.
4. Let x be a positive real number. Show that $\displaystyle\sum_{k=0}^{n-1} \left[x+\frac{k}{n}\right] = [nx]$.
5. Let x be a positive real number. Show that $\displaystyle\sum_{k=1}^{n} \frac{[kx]}{k} \leq [nx]$.
6. Let $m, n \in \mathbb{N}$ such that $\gcd(m,n)=1$. Show that $\displaystyle\sum_{k=1}^{n-1} \left[\frac{km}{n}\right] = \frac{(m-1)(n-1)}{2}$.
7. Show that $[\sqrt{n}+\sqrt{n+1}] = [\sqrt{n}+\sqrt{n+2}]$.
8. Let p be a prime of type $4k+1$. Write $p = a^2+b^2$, where $a, b \in \mathbb{N}$. Assume that a is odd. Show that $(a/p) = 1$.
9. In F_n, show that (1) 0 and $1/n$ are consecutive (2) $(n-1)/n$ and 1 are consecutive.
10. Let a/b, $1/2$ and α/β be consecutive terms in F_n in increasing order, where $\gcd(a,b) = 1 = \gcd(\alpha,\beta)$. Show that $b = \beta = 1+2[(n-1)/2]$ and that $a/b+\alpha/\beta = 1$.
11. Let a/b, c/d and e/f be three consecutive terns in F_n, in increasing order, where $\gcd(a,b) = \gcd(c,d) = \gcd(e,f) = 1$. Show that $c/d = (a+e)/(b+f)$.
12. Let $n > 1$. Prove that $\min\{y-x \mid x, y \in F_n \text{ are consecutive}, x < y\} = 1/n(n-1)$ and $\max\{y-x \mid x, y \in F_n \text{ are consecutive}, x < y\} = 1/n$.
13. Let a/b and α/β be rational numbers such that $b > 0, \beta > 0$, $a/b < \alpha/\beta$ and $b\alpha - a\beta = 1$. Let $n = \max\{b, \beta\}$. Show that a/b and α/β are consecutive terms in F_n but need not be consecutive terms in F_{n+1}.
14. Suppose that F_n has k elements, say $a_1/b_1, \ldots, a_k/b_k$ arranged in increasing order such that $\gcd(a_i, b_i) = 1$ for every $i = 1, \ldots, k$. Show that $\sum_{i=1}^{k-1} 1/b_i b_{i+1} = 1$.
15. Let $x \in (0,\ 1)$ be an irrational number. Show that there exists $a/b \in F_n$ such that $|x - a/b| < 1/2b^2$.
16. Let $x = (\sqrt{5}+1)/2$. Let $\lambda > 0$ and $\alpha > 2$ be real numbers. Show that there are only finitely many rational numbers a/b such that $|x - a/b| < 1/\lambda b^{\alpha}$.
17. Let x and $\alpha > 1$ be a real numbers. If there exist infinitely many rational numbers a/b such that $|x - a/b| < b^{-\alpha}$ then show that x is irrational.
18. Find a rational number a/b such that $|\sqrt{2} - a/b| < 1/(\sqrt{5}b^2)$.
19. Let $x \in [0,1]$. Show that there exists $a/b \in F_n$ such that $\gcd(a,b) = 1$ and $|x - a/b| \leq 1/(b(n+1))$.

20. Find a rational a/b such that $|\sqrt{2} - a/b| \leq 1/10b$.

21. Find $x/y, u/v \in F_{1000}$ such that $\gcd(x, y) = 1 = \gcd(u, v)$ and $x/y, 3/10, u/v$ are consecutive in F_{1000}.

22. Let $n \geq 2$ and $a_0, a_1, \ldots, a_n$ be real numbers such that $a_1, \ldots, a_n$ are positive. Show that
$\langle a_1, a_2, \ldots, a_n \rangle$ is positive. Hence show that $a_0 < \langle a_0, a_1, \ldots, a_n \rangle < a_0 + 1/a_1$.

23. Let $a_0, a_1, \ldots, a_n$ be a sequence of real numbers such that $a_1, \ldots, a_n$ are positive. Let $x = \langle a_0, a_1, \ldots, a_n \rangle$ and $y = \langle a_k, a_{k+1}, \ldots, a_n \rangle$ for some k. Show that $x = \langle a_0, a_1, \ldots, a_{k-1}, y \rangle$.

24. Let $a_0, a_1, a_2 \ldots$ be a sequence of real numbers such that $a_i \geq 1$ for $i \geq 1$. Let $x = \langle a_0, a_1, a_2, \ldots \rangle$ and $y = \langle a_k, a_{k+1}, \ldots \rangle$ for some k. Show that $x = \langle a_0, a_1, \ldots, a_{k-1}, y \rangle$.

25. Find the values of $\langle 1, 1, 2, 2, 1, 1, 1 \ldots \rangle$ and $\langle 1, 2, 3, 1, 2, 3, \ldots \rangle$.

26. Let $a_0, a_1, \ldots, a_n$ and $b_0, b_1, \ldots, b_n$ be positive integers. Find a condition so that $\langle a_0, a_1, a_2, \ldots, a_n \rangle < \langle b_0, b_1, b_2, \ldots, b_n \rangle$.

27. Let $a_0, a_1, \ldots, a_n, c$ be real numbers such that $a_1, \ldots, a_n, c$ are positive. Which one is larger between $\langle a_0, a_1, a_2, \ldots, a_n \rangle$ and $\langle a_0, a_1, a_2, \ldots, a_n + c \rangle$?

28. Let x be an irrational number such that $x = \langle a_0, a_1, a_2, \ldots \rangle$, where $a_i \geq 1$ are integers. Suppose that there exists a positive integer n such that $a_i = a_{i+kn}$ for all $i = 0, 1, \ldots, n-1$ and $k \geq 0$. Show that x satisfies a quadratic equation with rational coefficients.

29. Let x be an irrational number such that $x = \langle a_0, a_1, a_2, \ldots \rangle$, where a_i's are integers and $a_i \geq 1$ for $i \geq 1$. Suppose that there exist positive integers n and m such that $a_{n+i} = a_{n+i+km}$ for all $i = 0, 1, 2, \ldots, m-1$ and $k \geq 0$. Show that x satisfies a quadratic equation with rational coefficients.

30. Let $a_0, a_1, \ldots, a_n$ be real numbers such that $a_1, \ldots, a_n$ are positive. Define $h_{-2} = 0, h_{-1} = 1, h_i = a_i h_{i-1} + h_{i-2}$ and $k_{-2} = 1, k_{-1} = 0, k_i = a_i k_{i-1} + k_{i-2}$ for $i = 0, \ldots, n$. Show that $k_0, \ldots, k_n > 0$ and $\langle a_0, a_1, \ldots, a_i \rangle = h_i/k_i$ for every $i = 0, 1, \ldots, n$.

31. In previous problem show that $k_i/k_{i-1} = \langle a_i, a_{i-1}, \ldots, a_1 \rangle$ for every $i = 0, 1, \ldots, n$. Furthermore, if $a_0 > 0$ then show that $h_i/h_{i-1} = \langle a_i, a_{i-1}, \ldots, a_0 \rangle$ for every $i = 0, 1, \ldots, n$.

32. Show that $\sqrt{n^2+1} = \langle n, 2n, 2n, \ldots \rangle$, $\sqrt{n^2+2} = \langle n, n, 2n, n, 2n, \ldots \rangle$ and $\sqrt{n^2+2n} = \langle n, 1, 2n, 1, 2n, \ldots \rangle$.