Nonnegative formulas for counting skew tableaux

An honors thesis submitted in partial fulfillment of the requirements for the degree of Bachelor of Science with Honors in Mathematics

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NONNEGATIVE FORMULAS FOR COUNTING SKEW TABLEAUX

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Abstract. The classical hook-length formula from the 1950s, counting the number of standard young tableaux of straight shapes is one of the beautiful results in Enumerative Combinatorics. Unlike the straight shapes, there isn't an elegant product formula for skew shapes yet. Okounkov and Olshanski found a positive formula for enumerating the standard young tableaux of skew shapes in 1996. Only recently in 2014, Naruse introduced another formula to count the number of standard young tableaux of skew shapes as a positive sum over excited diagrams of products of hook-lengths. We study these formulas and hope to gain more insight into the graph representations and observed bijections with Young's Lattice.

Contents

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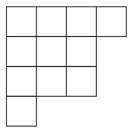
1. Introduction

Young Tableaux are fundamental in the study of algebraic and enumerative combinatorics. First developed by Frobenius in 1903, Young Tableaux were used to study representations of the symmetric group. In 1954, Frame, Robinson, and Thrall [2] discovered the classical hook length formula that counts the number of standard Young Tableaux. In 1996, Okounkov and Olshanski found a positive formula for the number of standard Young tableaux of a skew shape. In 2014, Naruse announced a more general formula for the number of standard Young tableaux of skew shapes as a positive-sum over excited diagrams of products of hook-lengths. The goal is to survey formulas to compute the number of standard tableaux of skew shapes. Additionally, we will look at three different determinantal formulas and study their properties and compare/analyze them, along with looking at one of the formulas that is yet to be studied combinatorially. We do indulge ourselves in studying of graph representations and posets in the later half of this paper.

2. Background and Preliminaries

A *partition* of an integer is given by $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_p)$ such that $\sum_{i=1}^{p} \lambda_i = n = \lambda$. For example, $\lambda = (2, 2, 1)$ where, $\lambda_1 = 2, \lambda_2 = 2, \lambda_3 = 1$. A *Ferrers diagram* is a visual representation of a given partition as demonstrated below for partition $(4, 3, 3, 1)$:



Note that the rows correspond to each λ_i We will use (i, j) to refer to cells with row i and column j or simply u for the rest of the paper to avoid confusions. For a cell, (i, j), define the *content* $c(u) = j - i$ and *arm length* $arm(u) = \lambda_i - i + 1$.

A *Young tableau* is a Ferrers diagram filled with integers $1, 2, \ldots, |\lambda|$. A *Standard Young Tableau* is a Ferrers diagram where all entries from 1 upto $|\lambda|$ in the rows and columns are increasing. An example of a standard tableau on the other hand is below.

A reverse plane partition (RPP) is a filling of boxes such that the boxes are filled with integers, $i \geqslant 0$ such that the ordering is weakly increasing along the

1	2	3	4
5	6	7	
8	9	10	
11			

,

1	2	2	3
4	4	5	
6	7	8	
9			

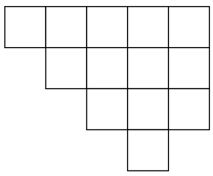
		1	2
	3	4	
5	6		
7			

		1	1
	3	3	
4	4		
5			

FIGURE 1. Examples of SYT And SSYT of straight and skew shape

rows and columns. A *semi-standard Young Tableau (SSYT)* is a RPP such that all columns are strictly increasing. Now we discuss what a *Skew-shaped Standard Young Tableau* is! Consider $[\mu]$ such that $[\mu] \subseteq [\lambda]$. See Figure 1 for an example of skew SYT of shape λ/μ for $\lambda = (4, 3, 2, 1)$ and $\mu = (2, 1)$.

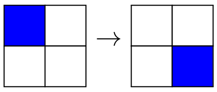
For a strict partition, the shifted Young diagram is obtained by shifting row i to position (i,i). A *Shifted skew shape*, denoted by λ^*/μ^* is $(\lambda_1 + d - 1, \lambda_2 + d - 2, \cdots, \lambda_d)/\mu_1 + d - 1, \mu_2 + d - 2, \cdots, \mu_d)$, where d is the usual length. An example of a shifted skew shape is given below.



3. EXCITED DIAGRAMS AND REVERSE EXCITED DIAGRAMS

Definition 3.1 (Excited Diagrams)**.** Excited cells are the ones in the shape μ. An excited move from $(i, j) \to (i + 1, j + 1)$ is allowed if and only if the cells at $(i, j + 1), (i + 1, j), \text{and}(i + 1, j + 1)$ are not excited.

We use $\mathcal{E}(\lambda/\mu)$ to denote the set of excited diagrams of λ/μ and $E(\lambda/\mu)$ denote the size of $\mathcal{RE}(\lambda/\mu)$. Excited diagram is a subset of a Young diagram of shape λ such that $\mu \subseteq [\lambda]$ and is constructed from the Young Diagram by a sequence of excited moves.

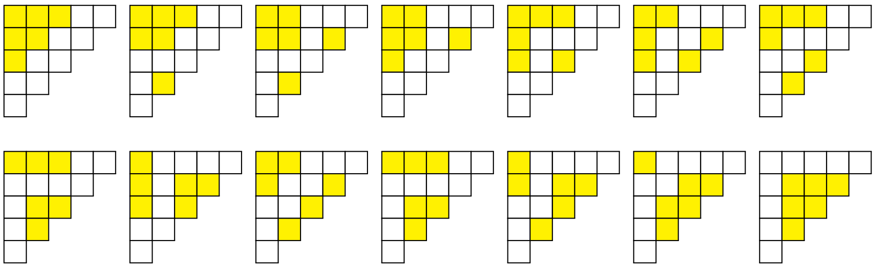


The excited diagrams are actually in correspondence with Dyck Paths given by

$$E(\lambda/\mu) = C_n = \frac{1}{n+1}\binom{2n}{n}$$

The n-th Catalan number counts the number of Dyck paths Dyck(n). We give a worked out example of the excited diagrams for shape 54321/321

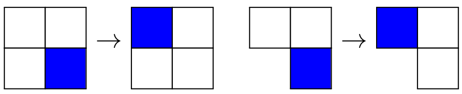
Example 3.1.



We can visualise the connections between excited diagrams such that we add an edge if from one excited diagram we can go to the other using some excited move. We can generate graphs that look like.

Definition 3.2 (Reverse Excited Diagrams)**.** A reverse excited diagram contains cells that can be excited to the upper North-west corner. A reverse excited move from $(i,j) \to (i-1, j-1)$ is allowed if and only if the cells at $(i, j-1), (i-1, j), \text{and} (i-1, j-1)$ are not excited.

We define them by $\mathcal{RE}(\lambda/\mu)$.

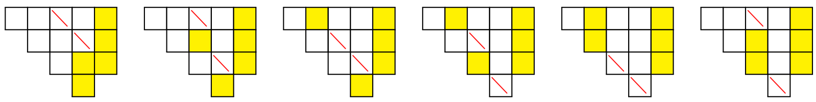


Definition 3.3 (Broken diagonal)**.** For an excited cell, the cells on the north, leading diagonally up to the first row and/or the cells on the SW corner leading

diagonally till an another excited cell or end of the tableau. We represent these by $B(D)$

These count the number of OOT of a given shape. They are obtained by applying the *reverse excited moves* on the shifted skew shape λ^*/μ^*

Example 3.2. Here, we obtain the reverse excited diagrams of shape 5431/4310



4. Number of Standard Young Tableau of shape λ

The study of the number of Standard Young Tableau has been a point of interest since the early 1900's. One of the most efficient and elegant ways of calculating the number of SYT is using the Hook Length formula discovered in 1953 by Mathematicians Frame, Robinson and Thrall. [2]

Theorem 4.1.

$$f^\lambda = \frac{\lambda!}{\prod\limits_{u \in \lambda} h(u)}$$

where f^λ is the number of standard tableaux of shape λ

Example 4.1. For the shape $\lambda = (2, 2, 1)$ we fill the the cells with their hook lengths are as follows $\begin{array}{|c|c|}\hline 4 & 2 \\ \hline 3 & 1 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$ So, we have $f^\lambda = \frac{5!}{4 \cdot 2 \cdot 3 \cdot 1 \cdot 1} = 5$

5. Number of Standard Young Tableau of shape λ/μ

There's no easy or efficient way to find out the number of SYT of skew-shape like the hook-length formula which only involves products. We denote the number of standard Young tableaux of shape λ/μ by $f^{\lambda/\mu}$. We are going to study the following three formulas.

5.1. **Determinant formula.** This is the oldest (1841) form of formula to calculate the number of standard young tableau of skew shape due to the *Jacobi-Trudi* identity [1]

Theorem 5.1.

$$f^{\lambda/\mu} = n! \cdot \det \left[\frac{1}{(\lambda_i - \mu_j - i + j)!} \right]_{i,j=1}^{l(\lambda)}$$

Example 5.1.

$$f^{(2,2,2,1)/(1,1)} = 4! \cdot \det \begin{bmatrix} \frac{1}{(\lambda_1-\mu_1-1+1)!} & \frac{1}{(\lambda_1-\mu_2-1+2)!} & \frac{1}{(\lambda_1-\mu_3-1+3)!} & \frac{1}{\lambda_1-\mu_4-1+4} \\ \frac{1}{(\lambda_2-\mu_1-2+1)!} & \frac{1}{(\lambda_2-\mu_2-2+2)!} & \frac{1}{(\lambda_2-\mu_3-2+3)!} & \frac{1}{(\lambda_2-\mu_4-2+4)!} \\ \frac{1}{(\lambda_3-\mu_1-3+1)!} & \frac{1}{(\lambda_3-\mu_2-3+2)!} & \frac{1}{(\lambda_3-\mu_3-3+3)!} & \frac{1}{(\lambda_3-\mu_4-3+4)!} \\ \frac{1}{(\lambda_4-\mu_1-3+1)!} & \frac{1}{(\lambda_4-\mu_2-3+2)!} & \frac{1}{(\lambda_4-\mu_3-3+3)!} & \frac{1}{(\lambda_4-\mu_4-3+4)!} \end{bmatrix}$$

$$= 24 \cdot \det \begin{bmatrix} \frac{1}{1!} & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{5!} \\ \frac{1}{0!} & \frac{1}{2!} & \frac{1}{2!} & \frac{1}{4!} \\ 0 & \frac{1}{0!} & \frac{1}{3!} & \frac{1}{3!} \\ 0 & 0 & \frac{1}{1!} & \frac{1}{2!} \end{bmatrix} = 9.$$

5.2. **Naruse Formula (2014).** The classical hook-length formula has been recently generalised by using what we call *Excited diagrams*. Now the Naruse hook length formula [9] is given by -

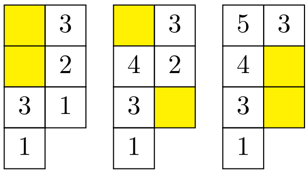
Theorem 5.2 (Naruse)**.**

$$f^{\lambda/\mu} = |\lambda/\mu|! \sum_{D \in E(\lambda/\mu)} \prod_{\mu \in [\lambda]/D} \frac{1}{h(u)}. \tag{5.1}$$

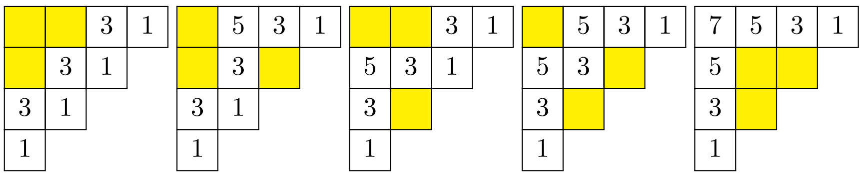
Example 5.2. For the skew shape $\lambda/\mu = (2,2,2,1)/(1,1)$ the Naruse hook length formula gives

$$\begin{aligned} f^{(2,2,2,1)/(1,1)} &= 5!\left(\frac{1}{3\cdot 3\cdot 2\cdot 1\cdot 1}+\frac{1}{4\cdot 3\cdot 3\cdot 2\cdot 1}+\frac{1}{5\cdot 4\cdot 3\cdot 3\cdot 1}\right)\\ &= 9 \end{aligned}$$

The corresponding excited diagrams are given as follows -



Example 5.3. Consider the skew-shape of $\lambda/\mu = (4,3,2,1)/(2,1)$. We give all possible excited diagrams for this shape along with their hook lengths in the non-excited cells.



For this particular case the Naruse hook length formula 5.1 gives

$$f^{(4,3,2,1)/(2,1)} =$$

$$7!\left(\frac{1}{3\cdot 3\cdot 3\cdot 1}+\frac{1}{5\cdot 3\cdot 3\cdot 3\cdot 1}+\frac{1}{5\cdot 5\cdot 3\cdot 3\cdot 3\cdot 1}+\frac{1}{7\cdot 5\cdot 5\cdot 3\cdot 3\cdot 1}+\frac{1}{5\cdot 3\cdot 3\cdot 1}\right)$$

$$= \frac{4080}{15} = 272.$$

5.3. **Okounkov-Olshanski Formula (1996).**

Theorem 5.3 (Okounkov-Olshanski)**.**

$$f^{\lambda/\mu} = \frac{|\lambda/\mu|!}{\prod_{u\in\lambda} h(u)} \sum_{T\in SSYT(\mu,d)} \prod_{u\in\mu} (\lambda_{d+1-T(u)} - c(u))$$

where $c(u) = j-i$, $d = l(\lambda)$, and $SSYT(\mu, d)$ is the set of SSYT of shape μ with entries $\leqslant d$. [10]

In the formula of Okounkov Olshanski, we denote the number of non-zero terms as $OOT(\lambda/\mu)$ that are counted by the determinants.

Example 5.4. For the shape $\lambda/\mu = (2,2,2,1)/(1,1)$, there are 6 different $SSYT(\mu,4)$ which give the following contributions

$$\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}\,(\lambda_4)(\lambda_1+1) = 1{\cdot}3 = 3 \quad \begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}\,(\lambda_3)(\lambda_1+1) = 2{\cdot}3 = 6 \quad \begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\,(\lambda_2)(\lambda_1+1) = 2{\cdot}3 = 6$$

$$\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\,(\lambda_3)(\lambda_2+1) = 2{\cdot}3 = 6 \quad \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}\,(\lambda_4)(\lambda_2+1) = 1{\cdot}3 = 3 \quad \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\,(\lambda_4)(\lambda_2+1) = 1{\cdot}3 = 3$$

$$f^{\lambda/\mu} = \frac{5!}{5\cdot 4\cdot 3\cdot 3\cdot 2\cdot 1\cdot 1}(3+6+6+6+3+3) = 9.$$

Example 5.5. For the shape $f^{(4,3,2,1)/(2,1)}$, there are 20 different $SSYT(\mu,4)$ which give the following contributions -

$$\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array} = 0 \quad \begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array} = 0 \quad \begin{array}{|c|c|}\hline 1&1\\\hline 4\\\cline{1-1}\end{array} = 0 \quad \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array} = 3 \quad \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} = 4$$

$$\begin{array}{|c|c|}\hline 1&2\\\hline 4\\\cline{1-1}\end{array} = 5 \quad \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} = 6 \quad \begin{array}{|c|c|}\hline 1&3\\\hline 3\\\cline{1-1}\end{array} = 8 \quad \begin{array}{|c|c|}\hline 1&4\\\hline 2\\\cline{1-1}\end{array} = 9 \quad \begin{array}{|c|c|}\hline 1&4\\\hline 3\\\cline{1-1}\end{array} = 12$$

$$\begin{array}{|c|c|}\hline 1&3\\\hline 4\\\cline{1-1}\end{array} = 10 \quad \begin{array}{|c|c|}\hline 1&4\\\hline 4\\\cline{1-1}\end{array} = 15 \quad \begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array} = 8 \quad \begin{array}{|c|c|}\hline 2&2\\\hline 4\\\cline{1-1}\end{array} = 10 \quad \begin{array}{|c|c|}\hline 2&3\\\hline 3\\\cline{1-1}\end{array} = 16$$

$$\begin{array}{|c|c|}\hline 2&4\\\hline 3\\\cline{1-1}\end{array} = 24 \quad \begin{array}{|c|c|}\hline 2&3\\\hline 4\\\cline{1-1}\end{array} = 20 \quad \begin{array}{|c|c|}\hline 2&4\\\hline 4\\\cline{1-1}\end{array} = 30 \quad \begin{array}{|c|c|}\hline 3&3\\\hline 4\\\cline{1-1}\end{array} = 30 \quad \begin{array}{|c|c|}\hline 3&4\\\hline 4\\\cline{1-1}\end{array} = 45$$

Sum of all the 20 contributions $= 255$

$$f^{\lambda/\mu} = \frac{7!}{7\cdot 5\cdot 5\cdot 3\cdot 3\cdot 3\cdot 1\cdot 1\cdot 1\cdot 1}(255) = 272$$

5.4. **Okounkov-Olshanski Formula - reverse excited diagram version.**

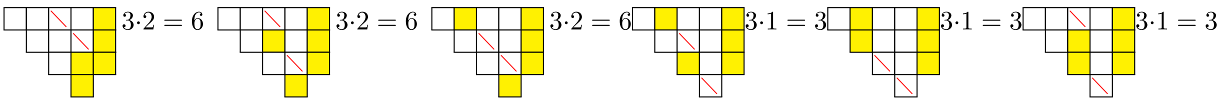
Theorem 5.4.

$$f^{\lambda/\mu} = \frac{|\lambda/\mu|!}{\prod_{u\in\lambda} h(u)} \sum_{D\in\mathcal{RE}(\lambda/\mu)} \prod_{i,j\in \mathrm{B(D)}} (\lambda_i + d - j)$$

where, $d = l(\lambda)$, B(D) are certain cells of $[\lambda/\mu]$ associated to D and $\lambda_i + d - j$ equals the arm-length of cell (i,j)

[8]

Example 5.6. For $(2,2,2,1)/(1,1)$ we have 6 Reverse excited diagrams with each reverse excited diagram giving the following contributions -



$$f^{\lambda/\mu} = \frac{5!}{5\cdot 4\cdot 3\cdot 3\cdot 2\cdot 1\cdot 1}(6+6+6+3+3+3) = 9$$

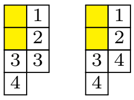
5.5. **Special Cases of OOT.**

- $a^b/(a-1)^{b-1}$ if $a = b$, it follows that there is a distinctive pattern of 1, 3, 6, 20, where the reccurence is given by $T_n = T_{n-1} + \sum_{i=1}^{n} i$

6. Number of terms of the Naruse and the OO formulas

6.1. **Flagged Tableaux.** Given a skew shape λ/μ, and a sequence of weakly increasing non-negative integers, $a = (a_1, a_2, \cdots, a_d)$ and $b = (b_1, b_2, \cdots, b_d)$, we a define a *flagged tableaux* $\mathcal{SF}$ to be a SSYT of shape λ/μ such that every entry in row i is between a_i and b_i. These were first studied by Lascoux and Schützenberger [4], Wachs [12], and Gessel–Viennot [3].

Example 6.1. Let $a = (1,2,3,4)$ and $b = (1,2,4,4)$ The flagged tableaux is given by $SSYT(2221/11, a, b)$



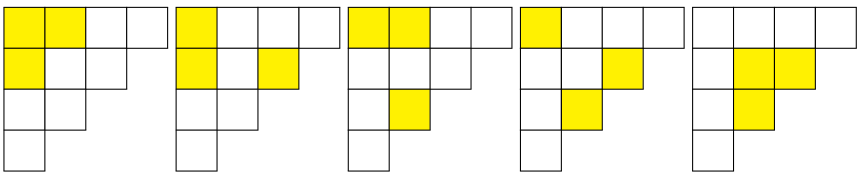
Theorem 6.1 (Wachs, Gessel-Viennot)**.**

$$|SSYT(\lambda/\mu, a, b)| = \det\left[\binom{\lambda_i - \mu_j + j - i + b_i - a_j}{b_i - a_j}\right]_{i,j=1}^{\ell(\lambda)} \tag{6.1}$$

Importantly, there's a correspondence between Excited diagrams $\mathcal{E}(\lambda/\mu)$ and Flagged Tableaux of shape μ and there's a correspondence between Reverse Excited diagrams and Flagged Tableaux of shape λ/μ

6.2. **Correspondence between Excited Diagrams and Flagged Tableaux.** We can describe the correspondence between the excited diagrams and the Flagged Tableaux such that for each excited diagram, a flagged tableaux keeps track of the row in the the excited cells. Let $\mathbf{a} = (\underbrace{1, \ldots, 1}_{\ell(\mu)})$ and $\mathbf{b} = (f_1, f_2, \ldots, f_{\ell(\mu)})$, where f_i is the last row of λ where the cell (i, μ_i) can be excited to. Define $\varphi : E(\lambda/\mu) \to SSYT(\mu, \mathbf{a}, \mathbf{b})$ where $D \mapsto T$, $T(i,j) = r$ where r is the row number in D where the cell (i,j) of μ was excited to. See Example 6.2. This map is a bijection. [7].

Example 6.2. Consider the excited diagrams of shape $(4,3,2,1)/(2,1)$ from Example 5.3. Here, we have $f_1 = 2$ and $f_2 = 3$ and their corresponding flag tableaux of shape (21) are:



1	1
2	

1	2
2	

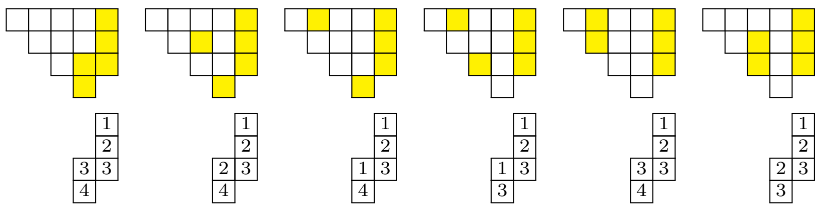
1	1
3	

1	2
3	

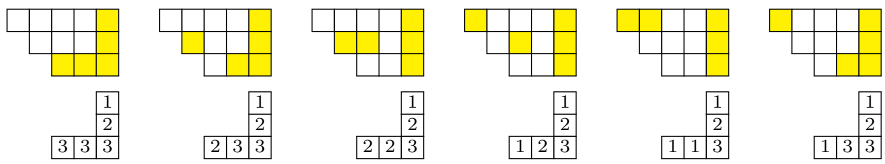
2	2
3	

6.3. **Correspondence between Reverse Excited Diagrams and Flagged Tableaux.** We now describe the correspondence between the reverse excited diagram and the flagged tableaux in a similar way. For each reverse excited diagram, the Flagged Tableaux keeps track of the rows of the reverse excited cells. Similarly, let $\mathbf{a} = (\underbrace{1, 1, 1, \ldots, 1}_{\ell(\lambda)})$ and $\mathbf{b} = (1, 2, \cdots, \ell(\lambda))$ and define $\phi : RE(\lambda/\mu) \to SSYT(\lambda/\mu, \mathbf{a}, \mathbf{b})$ where $R \mapsto T$, $T(i, j) = r$ where r is the row number in R where the cell (i, j) of λ/μ was reverse excited to. See Example 6.4.

Example 6.3. For the shape of $(5, 4, 3, 1)/(4, 3, 1, 0)$ from Example 3.2 we have the following reverse excited diagrams and the corresponding flag tableaux.



Example 6.4. For the shape of $(5, 4, 3)/(4, 3, 0)$ we have the following reverse excited diagrams and the corresponding flag tableaux.



6.4. **Number of terms of the Naruse formula.** We count the number of Excited Diagrams using the following corollary from [7]

Corollary 6.1 (Morales-Pak-Panova [7])**.**

$$|E(\lambda/\mu)| = \det \left[\binom{\mu_i + f_i + j - i - 1}{f_i - 1} \right]_{i,j=1}^{\ell(\mu)}$$

Proof. Since, the map φ from Section 6.2 is a bijection between the excited diagrams of λ/μ and the flagged tableaux of straight shape μ,

$$|E(\lambda/\mu)| = |SSYT(\mu, \mathbf{a}, \mathbf{b})|$$

where $\mathbf{a} = (1, 1, \cdots, 1)$ and $\mathbf{b} = (f_1, f_2, \cdots f_n)$. We then apply the Theorem 6.1 to the RHS above to obtain the desired formula. $\square$

Example 6.5. For the shape $\lambda/\mu = (4, 3, 2, 1)/(2, 1)$ we have

$$|E(4321/21)| = \det \begin{bmatrix} \binom{3}{1} & \binom{4}{1} \\ \binom{2}{2} & \binom{3}{2} \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 3 \end{bmatrix} = 5$$

6.5. **Number of terms of the Okounkov-Olshanski formula.** The following theorem counts the number of non-zero terms of the Okounkov-Olshanski formula given by OOT.

Theorem 6.2 (Morales-Zhu [8])**.**

$$OOT(\lambda/\mu) = det \left[\binom{\lambda_i - \mu_j + j - 1}{i - 1} \right]_{i,j=1}^{\ell(\lambda)}$$

Proof. Since, the map ϕ from Section 6.2 is a bijection between the reverse excited diagrams and the flagged tableaux of shape λ/μ,

$$|RE(\lambda/\mu)| = |SSYT(\lambda/\mu, a, b)|$$

Consider $a = (1, 1, \cdots, 1)$ and $b = (1, 2, 3, \cdots \ell(\lambda))$. We then apply the Theorem 6.1 to calculate the RHS above: for any i, j, we have $b_i = i$ and $a_j = 1$. Thus

$$\begin{aligned} OOT(\lambda/\mu) &= \det \left[\binom{\lambda_i - \mu_j + j - i + i - 1}{i - 1} \right]_{i,j=1}^{\ell(\lambda)} \\ &= \det \left[\binom{\lambda_i - \mu_j + j - 1}{i - 1} \right]_{i,j=1}^{\ell(\lambda)}. \end{aligned}$$

□

Example 6.6. For the shape $\lambda/\mu = (2,2,2,1)/(1,1)$ we have

$$|RE(\lambda/\mu)| = \det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 5 \\ 0 & 1 & 6 & 10 \\ 0 & 0 & 1 & 4 \end{bmatrix} = 6$$

This is exactly the number of reverse excited diagrams we just found in the above example.

Example 6.7. For the shape $\lambda/\mu = (3,3,3)/(2,2)$ we have

$$|RE(\lambda/\mu)| = \det \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \\ 0 & 1 & 10 \end{bmatrix} = 6$$

6.6. **Number of OOT terms for special cases.**

- Rectangle $OOT((a+1)^{b+1}/a^b) = \binom{a+b}{b}$

$$\binom{a+b}{b} = \binom{a+b-1}{b-1} + \binom{a-1+b}{b} = OOT(a^b/(a-1)^{b-1}) + OOT(a^{b+1}/(a-1)^b)$$

- Staircase $OOT(\lambda/\mu)$ for $\lambda/\mu = \delta_{k+2}/\delta_k = G_{2n}$ [8]

7. Graph of excited, reverse excited diagrams and Young's Lattice

In this section we look at graphs whose vertices are (reverse) excited diagrams of a given skew shape connected by excited moves.

7.1. **Graph of excited diagrams.**

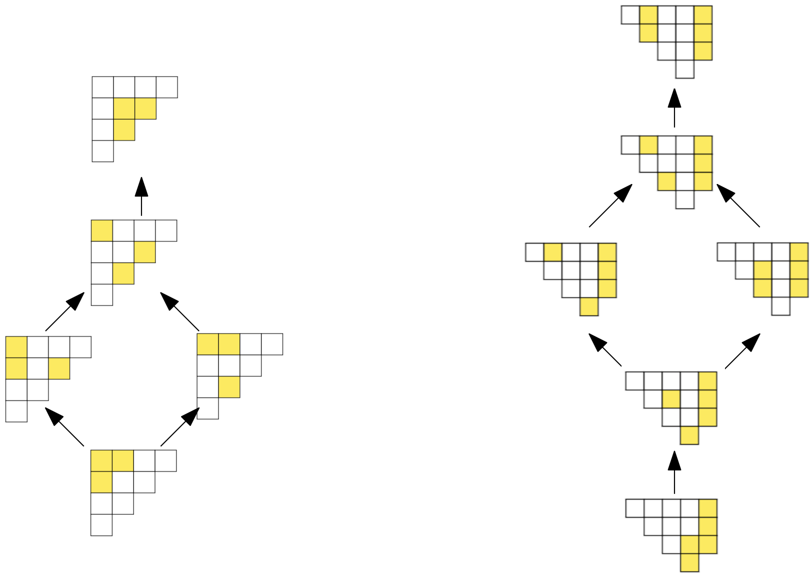
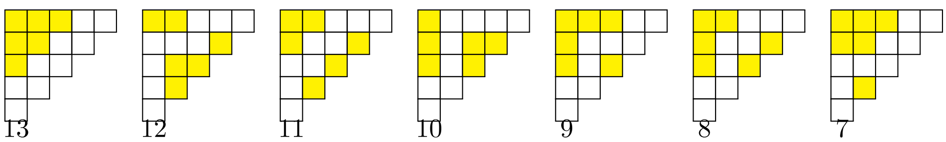


FIGURE 2. Excited Graph of shape 4321/21, Reverse Excited Graph of shape 5431/4310

Definition 7.1 (Graph of excited diagrams)**.** Given a shape λ/μ let $EG(\lambda/\mu)$ be the directed graph whose vertices $V = \mathcal{E}(\lambda/\mu)$ are the excited diagrams and whose edges are (D, D') if $D \to D'$ by doing an excited move.

Example 7.1. Considering the running example, of skew-shape $(4, 3, 2, 1)/(2, 1)$ the excited graph is illustrated in Figure 2.

Example 7.2. Considering the running example, of shape $(5, 4, 3, 2, 1)/(3, 2, 1)$ we have the graph in Figure 3 The numbers on the vertices represent the excited graphs given in the following order starting with excited graph 0 to 13. The algorithm generated this set of vertices for the associated graph.



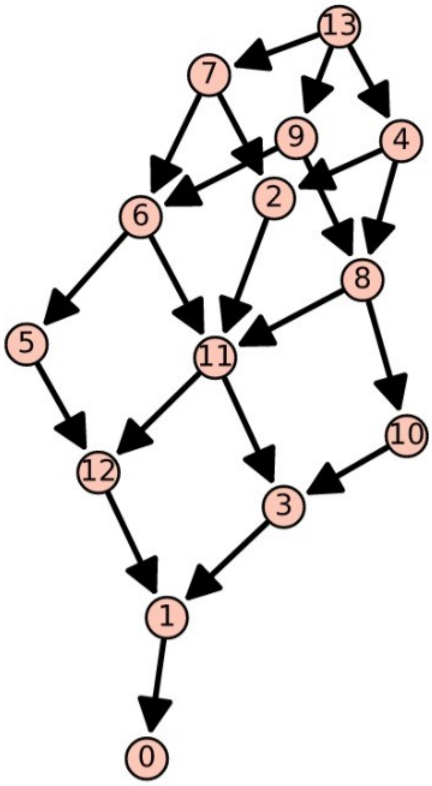
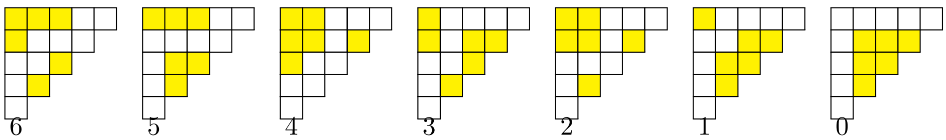


FIGURE 3. Graph of Excited Diagram



7.2. **Graph of reverse excited diagrams.**

Definition 7.2 (Graph of reverse excited diagrams)**.** Given a shape λ^*/μ^* let $REG(\lambda^*/\mu^*)$ be the directed graph whose vertices $V = \mathcal{RE}(\lambda^*/\mu^*)$ are the reverse excited diagrams and whose edges are (F, F') if $F \to F'$ by doing an reverse excited move.

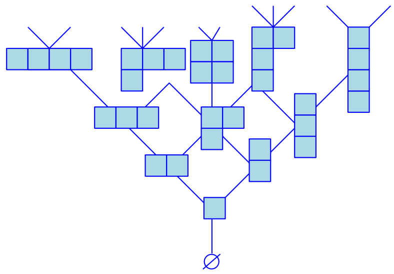


FIGURE 4. An example of Young's lattice.

Example 7.3. Considering the running example, of skew-shape $(5, 4, 3, 1)/(4, 3, 1, 0)$ the reverse excited graph is illustrated in Figure 2.

7.3. **Young's Lattice.**

Definition 7.3 (Young's Lattice)**.** Young's Lattice - Named after Alfred Young, Young's lattice is a partially ordered set and a lattice that is formed by all integer partitions. In the 4, as one moves up a level, a new box is added to the diagram. Each level represents the number of possible partitions of the corresponding level no. We take a starting diagram on say, level 2, and identify the no. of ways to add one more box to the diagram and add an edge to each new diagram. Hence, we finds young diagrams for the next level.

Definition 7.4 (Poset)**.** A partially ordered set or (poset) is a set P and a binary relation such that $\forall a, b, c \in P$

- $a \leqslant a$ (reflexivity)
- $a \leqslant b$ and $b \leqslant c$ implies $a \leqslant c$ (transitivity)
- $a \leqslant b$ and $b \leqslant a$ implies $a = b$ (anti-symmetry)

7.4. **Conjectures and questions.**

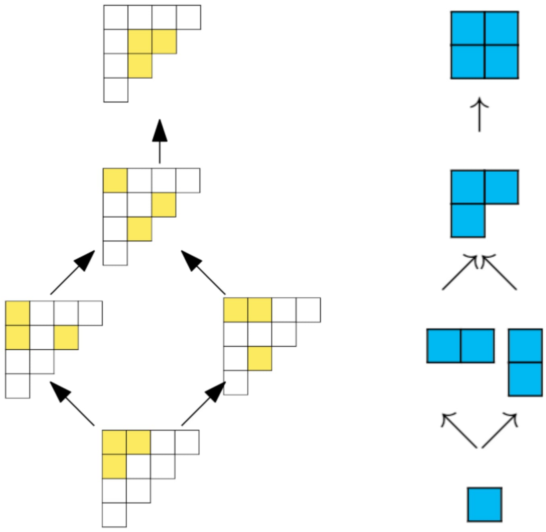


FIGURE 5. Conjectured Bijection

Conjecture 7.1 (Conjecture 1)**.** Given a graph of Excited diagrams or Reverse Diagrams, we can find a bijection into the Young's Lattice. Based on our understanding of this literature, the excited diagram graph, EDG has a link to the Young's Lattice. Taking a subset of Young's Lattice, we could possibly find a lattice that pertains to a structure similar to the excited graph from the previous sections. As an example, we shall look at the Excited Diagram Graph of shape 4321/21 corresponding to the following subset of Young's Lattice in Figure 5.

Conjecture 7.2 (Conjecture 2)**.** A border strip is a connected skew shape λ/μ containing no 2×2 box. An example is an inverted hook. The following border strips of rectangles exhibit symmetry since they have the equal number of excited cells. Hence the number of excited diagrams is equal leading to the equivalence of the excited graphs between the following shapes

$$EDG(n^{n+1}/(n-1)^n) = EDG((n+1)^n/n^{n-1})$$

Remark 7.1. It could be an interesting study to find the number of edges of the Excited and Reverse Excited graphs and if or not they follow any particular sequence.

Remark 7.2. It could be interesting to calculate the $OOT(\lambda/\mu)$ of a ribbon of the circle segment from [5].

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