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# MAGNETAR TIMING ANOMALIES FROM THERMOPLASTIC FAILURES AND HALL WAVES

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## ABSTRACT

*Subject headings:* dense matter — magnetic fields — stars: magnetars — stars: neutron — waves

## 1. HALL WAVES GENERATED BY THERMOPLASTIC WAVES

Beloborodov & Levin (2014) derived the analytical wave solution for thermoplastic waves assuming a constant heat conduction coefficient $\chi = \kappa/C_V$ and thermal softening of critical stress $\sigma_{\rm cr}(T) = \sigma_0 - \zeta(U_{\rm th} - U_0)$. Their model of plastic stress evolution is given by

$$\dot{s} = -\frac{\sigma - \sigma_{\rm cr}}{\eta}\Theta(\sigma - \sigma_{\rm cr}). \quad (1)$$

Using the conditions of frozen flux

$$b \equiv B/B_z = s \quad (2)$$

and stress balance

$$b = \sigma\frac{B_z^2}{4\pi} \equiv \sigma\mu_B, \quad (3)$$

we have

$$\dot{b} = -\frac{\mu_B}{\eta}(b - b_{\rm cr})\Theta(b - b_{\rm cr}). \quad (4)$$

For an initial state where $B_y = 0$ (i.e. $b$ is real), we can compare the above equation with the evolution equation of $B$ without Hall term. In the deep crust where $\mu_B \ll \mu$, the two equations are equivalent with the identification $\alpha = \mu_B/\eta$.

The thermoplastic wave has velocity $v \sim \sqrt{\alpha\chi}$ and wavefront thickness $l \sim \sqrt{\chi/\alpha}$.

Adding the Hall term, on the background with constant density thus constant $n_e$, and constant $D_{\rm H} = B_z c/(4\pi n_e)$, the equation reads

$$\dot{b} = -\alpha(b - b_{\rm cr})\Theta(b - b_{\rm cr}) - iD_{\rm H}\partial_z^2 b. \quad (5)$$

We model the softening of critical stress such that it drops to 10% of its initial value after the front. Hence $\zeta = 0$ and we have no softening of $b_{\rm cr}$ after the front. Eqn. (5) is a linear differential equation only about $b$. We seek quasi-steady propagating solutions after the front, where $b$ is a function of $u \equiv z - vt$. Initially, $b = b_0$ everywhere.

Defining $' = {\rm d}/{\rm d}u$, $\beta = b - b_{\rm cr}$ and turning all the spatial and temporal derivatives into derivatives with respect to $u$, we have

$$iD_{\rm H}\beta'' - v\beta' + \alpha\beta = 0. \quad (6)$$

When the Hall term is absent, the pure thermoplastic wave solutions gives

$$\beta = \beta_0 e^{\frac{\alpha}{v}u}. \quad (7)$$

$\beta_0 = b_0 - b_{\rm cr}$ and $\beta$ exponentially decays after the front. The front thickness is $l = v/\alpha = \sqrt{\chi/\alpha}$. The result agrees with (Beloborodov & Levin 2014).

We can estimate the importance of Hall term in the equation

$$\frac{D\beta''}{\alpha\beta} \sim \frac{D}{\alpha l^2} = \frac{D}{\chi} \equiv \epsilon. \quad (8)$$

In deep magnetar crust $\epsilon \ll 1$, so the Hall term can be treated as a perturbation. We are going to solve the linear differential equation and find solutions to linear order in $\epsilon$. Putting $\beta = e^{ku}$ into the equation, we have

$$iD_{\rm H}k^2 - vk + \alpha = 0. \quad (9)$$

$k$ has two roots in the quadratic equation. When $\epsilon \to 0$, the solution should reduce to Eqn.(7). Hence we only keep one root

$$k = -i\frac{\alpha}{v}\frac{1 - \sqrt{1 - 4i\epsilon}}{2\epsilon}. \quad (10)$$

To the linear order in $\epsilon$, we have $k = (1 + i\epsilon)\alpha/v$, and therefore

$$\beta = \beta_0 e^{\frac{\alpha}{v}u} e^{i\epsilon\frac{\alpha}{v}u}. \quad (11)$$

The Hall term will generate $B_y$ and spire the wave on a much longer length scale $l_H = l/\epsilon$.

## 2. MAGNETAR TIMING ANORMALIES

Magnetars show various timing anomalies (e.g. Dib and Kaspi 2013). In addition to glitches similar to those in pulsars, magnetars also have anti-glitches (Archibald et al. 2013). Radiative changes of magnetars are always accompanied by glitches (Dib and Kaspi 2013), while glitches may happen without radiative changes. The mechanism for such timing anomalies are still unclear.

### 2.1. *Difficulty of Vortex Line Unpinning*

The current common explanation for pulsar glitches are due to the unpinning of superfluid vortex lines in the crust (Anderson and Itoh 1975, Ruderman 1976).

Superfluid velocity is the gradient of its phase, and thus irrational $\nabla \times \boldsymbol{v}_s = 0$. In this picture, rotating superfluid create quantized vortex lines in the crust. The circulation around a vortex line is

$$\oint \boldsymbol{v}_s \cdot {\rm d}\boldsymbol{l} = \frac{2\pi\hbar}{m} = \kappa, \quad (12)$$

where $\kappa$ is the quantum of the vorticity and $m$ is the mass of the Cooper pair.

The vortex lines are pinned to nuclei through nuclear force and follow the motion of nuclei. The vortex line is also acted on by the Magnus force from the superfluid streaming past it. The force per unit length is

$$\boldsymbol{f}_{\rm mag} = -\rho_s \Delta \boldsymbol{v} \times \boldsymbol{\kappa}. \tag{13}$$

$\boldsymbol{v}_\delta = \boldsymbol{v}_v - \boldsymbol{v}_s$ is the difference between velocity of nuclei and superfluid.

The superfluid neither follows the motion of nuclei nor dissipates. When the star slows down, velocity difference $v_\delta$ increases and so does the Magnus force. When the Magnus force is larger than the nuclear pinning force, the vortex lines are unpinned. The Magnus force expels the vortex lines outward radially, decreases the superfluid velocity and transfers angular momentum to the star. Therefore the star spins up and produces a glitch. Theoretical calculations give the minimum $v_{\delta,\rm cr} \gtrsim 10^4$ cm/s at neutron drip point and increases with density (Link et al. 1993).

Magnetars are slow rotators with period $P \sim 10$ s and period derivative $\dot{P} \sim 10^{-10} - 10^{-12}$ (c.f. Fig.2 in McLaughlin et al. 2006). The waiting time between two glitches is estimated by

$$\Delta t = \frac{\Omega_{\rm cr}}{\dot{\Omega}} = \frac{v_{\delta,\rm cr}}{R}\frac{P^2}{2\pi\dot{P}} = 50\frac{v_4 P_2^2}{R_6 \dot{P}_{-10}}\ \text{yr}. \tag{14}$$

This optimal estimate does not agree with observations where glitches are observed every 2-5 years.

Another fact is that unpinning vortex lines can only produce glitches but never anti-glitches.

### 2.2. *Timing anomalies from plastic failure*

When a plastic failure takes places, the deformation of crust lattice also deform the vortex lines pinned to it. For constant vortex line density $\rho_v$ and vorticity density $\boldsymbol{\omega} = \rho_v \boldsymbol{\kappa}$, the solution of circular equation

$$\oint \boldsymbol{v}_s \cdot \mathrm{d}\boldsymbol{l} = \frac{2\pi\hbar}{m} = \rho_v \kappa \tag{15}$$

gives

$$v_s = \frac{\rho_v \kappa}{2} r = \frac{\omega}{2} r, \tag{16}$$

where $r$ is the distance to the center. The solution of $v_s$ is similar to a magnetic field inside a wire with constant current density. And the vorticity is determined from the angular velocity $\omega = 2\Omega_s$.

Consider the situation illustrated by Fig.1, initially $\boldsymbol{\omega}$ is in the $y$ direction, and velocity is in the $x$ direction (no $z$ component). The plastic failure deforms the crust and the vortex lines in the $x$ direction. The change of orientation of $\boldsymbol{\omega}$ perturbs the superfluid velocity. The perturbation of $v_s$ is estimated by

$$|\Delta v_z| \leq \frac{\omega h}{2} \sin\theta < \Omega h, \tag{17}$$

$$|\Delta v_{x,y}| \leq \frac{\omega h}{2} (1 - \cos\theta) < \Omega h. \tag{18}$$

And we have

$$\Omega h \sim 6 \times 10^3 \frac{h_4}{P_1}\ \text{cm/s}, \tag{19}$$

where the thickness $h$ of the plastic region is taken to be 100 m. The perturbation of superfluid velocity is not enough to unpin the vortex lines.

The change of vortex line orientations also change its angular momentum. The net change of superfluid angular velocity is in $y$ direction, since the perturbation in $x$ direction cancels,

$$\Delta\Omega_s = \Omega_s (1 - \cos\theta). \tag{20}$$

For a cylindrical superfluid shell at radius $R$ with thickness $l$ and height $h$ and density $\rho_s$, its angular momentum changes by

$$\Delta L_s = \rho_s 2\pi R^3 h l \Delta\Omega_s. \tag{21}$$

From the conservation of angular momentum, the star's angular momentum changes the same amount. Assuming $\Omega_s \sim \Omega$, the resulting variation on angular velocity is given by

$$\frac{\Delta\Omega}{\Omega} = \frac{2\pi R^3 \rho_s h l \Delta\Omega_s}{\frac{2}{5} M R^2 \Omega} \sim 10^{-5} \frac{R_6 \rho_{s,14} h_4 l_4}{1.4 M_\odot}, \tag{22}$$

where $\cos\theta$ from plastic failures can be of order unity. This variation is comparable to glitches observed from magnetars.

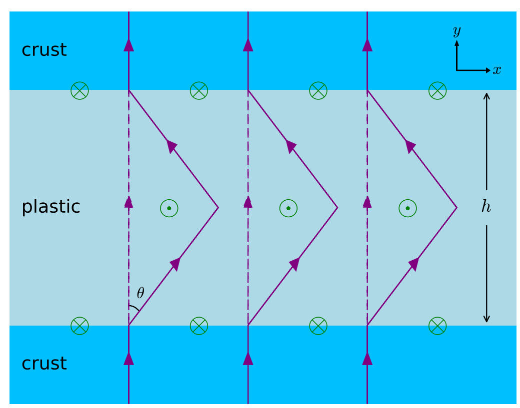


FIG. 1.— Illustration for the vortex lines in the crust in the x-y plane (z is the radial direction). Plastic flow is launched in the light blue region. Dashed and solid purple lines are the vortex lines before and after the plastic failure happens. The vortex lines are deformed in the plastic region. Green symbols denote the direction of z-component of $\boldsymbol{v}_s$ generated by the deformed vortex line.

REFERENCES

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