Advanced Probability

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Reference Text: A First Look at Rigorous Probability Theory

Related modules: ST5214

Notation Throughout this document, $\mathbb{N}$ can vary in usage, sometimes $\mathbb{N} = \mathbb{Z}^+$, other times $\mathbb{N} = \mathbb{Z}_0^+$, depending on which is more convenient.

1.1 To see that $\exists z \in \mathbb{R}, P(Z = z) > 0$, we observe that

$$P(Z = 0) \geq P(Z = 0 \wedge X = 0) = P(Z = 0|X = 0) \cdot P(X = 0) \geq \frac{1}{2} \cdot P(X = 0) > 0$$

since $P(X = 0) > 0$.

1.2

Uncountable summation Given an uncountable non-negative set of numbers $\{r_a : a \in I\}$ indexed by I,

$$\sum_{a \in I} r_a := \sup\{\sum_{a \in J} r_a : J \subseteq I \wedge J \text{ finite}\}$$

R-shift (Equivalent definition) R-shift of $A \subseteq [0, 1]$. $A \oplus r = \{(a + r) \mod 1 : a \in A\}$

2.1 Notice that countability is used by 2 constructs. One, probability measure is countably additive (and not uncountably so). Two, the σ-algebra is closed under countable union and intersection (and not uncountably so).

Recall that the reason for disallowing uncountable operations in general is due to the fact that

$$\bigcup_{x \in A} \{x\} = A$$

for any set A, in particular $[0, 1]$ when discussing the uniform distribution on the unit interval.

Theorem 2.2.1 We provide a proof for this theorem.

First, we show that $\mathcal{F}$ is a σ-algebra. By definition, $\mathcal{F} = \mathcal{P}(\Omega)$. Hence, the unary complement operation is a mapping $\mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$ whose domain and codomain are just $\mathcal{F}$ as desired. Similarly, for the countable set operations of union and intersection, they are mappings with codomain as $\mathcal{P}(\Omega)$ and are also closed since $\mathcal{F} = \mathcal{P}(\Omega)$.
We also note that both $\emptyset, \Omega$ reside in $\mathcal{F}$

Next, we show that P is a probability measure. By definition of P, P is additive since $A \cap B = \emptyset \implies P(A \sqcup B) = \sum_{\omega \in A \sqcup B} p(\omega) = \sum_{\omega \in A} p(\omega) + \sum_{\omega \in B} p(\omega) = P(A) + P(B)$.

I am not quite sure about showing countable additivity however, perhaps using some form of diagonal summation argument it is possible to prove this.

$P(\Omega) = \sum_{\omega \in \Omega} p(\omega) = 1$. Furthermore, p is non-negative, hence P is indeed bounded between 0 and 1.

Ex 2.2.3 First, $\emptyset, \Omega = [0,1] \in \mathcal{J}$ by definition as they are intervals. Next, to show closure under finite intersection, it suffices to show closure under binary intersection. Consider cases: We only consider one endpoint, since we can "patch" together two endpoints.

- $[a$ and $[b$ intersect to give $[\max\{a,b\}$
- $(a$ and $(b$ intersect to give $(\max\{a,b\}$
- $[a$ and $(b$ intersect to give $[a$ if $a > b$ and $(b$ otherwise

We can do a similar case analysis for right endpoints. Given a stringified left endpoint $l \in \{"[a", "(a"\}$ and a stringified right endpoint $r \in \{"b]", "b)"\}$ we can form an interval via concatenation l, r. Hence, $\mathcal{J}$ is closed under finite intersection.

Consider the complement $J = [0,a) \cup (b,1]$ of an interval $[a,b] \in \mathcal{J}$, where depending on whether the left/right endpoint is closed or open we adjust J accordingly. Regardless, we see that J is a disjoint union of at most 2 intervals in $\mathcal{J}$.

Hence, $\mathcal{J}$ is a semialgebra of subsets of Ω.

Ex 2.2.5

a $\mathcal{B}_0 \subseteq \mathcal{P}(\Omega)$. Since $\mathcal{B}_0$ consists of all finite unions of elements of $\mathcal{J}$, in particular, $\mathcal{J} \subseteq \mathcal{B}_0$, so $, \Omega = [0,1] \in \mathcal{J} \subseteq \mathcal{B}_0$.

Next, the finite union and intersection of elements of $\mathcal{B}_0$ will give finite unions of elements of *semialg*, so that $\mathcal{B}_0$ is closed under finite union and intersection. (For intersection, we can argue using distributive law plus observe that the intersection of intervals gives another interval)

Let $B \in \mathcal{B}_0$, so that B is a finite union of the form $\bigcup_{1 \le i \le n} I_i$ for some intervals I_i in $[0,1]$. Then, $B^c = \bigcap_{1 \le i \le n} I_i^c$ by DeMorgan's Law, and we have already proven in Ex 2.2.3 that I_i^c is a disjoint union of intervals, i.e. $I_i^c \in \mathcal{B}_0$. Furthermore, we have proven that $\mathcal{B}_0$ is closed under finite intersection, so $B^c \in \mathcal{B}_0$.

Hence, $\mathcal{B}_0$ is an algebra.

b The difference between an algebra and a σ-algebra is that σ-algebras are closed under countable union and intersection but algebras are not necessarily so.

We consider Cantor's set C, which is a countable intersection of C_i, where each C_i is formed by removing from each interval in C_{i-1} the middle one-third.

Since each C_i is a union of (disjoint) intervals, by definition, $C_i \in \mathcal{B}_0$. If $\mathcal{B}_0$ is to be a σ-algebra, then we must have $C \in \mathcal{B}_0$, i.e. C can be formed from a finite union of intervals.

First of all, C does not contain any interval of non-zero length, so our options are reduced to forming C from a finite union of singletons, i.e. intervals of the form $[a,a] = \{a\}$. But we also know C to be uncountable, but a finite union of singletons is finite. Hence we have a contradiction.

Since $C \notin \mathcal{B}_0$, $\mathcal{B}_0$ is not closed under countable intersection, so it is not a σ-algebra.

We comment that $\mathcal{B}_1$ is similarly not a σ-algebra using the same counterexample. The countable union of singletons must be at most countable, so they cannot union to form an uncountable set like C.

Ex 2.3.16 Suppose $A \in \mathcal{M} \wedge P^*(A) = 0$ and $B \subseteq A$. To show that $B \in \mathcal{M}$, we show equivalently that:

For each $E \subseteq \Omega$, $P^*(E) \ge P^*(A \cap E) + P^*(A^c \cap E)$ (i.e. superadditivity. But by monotonicity, $P^*(A \cap E) = 0$, so that $P^*(E) \ge P^*(A^c \cap E)$ is automatically true by monotonicity. Hence, we have shown the completeness of the extension $(\Omega, \mathcal{M}, P^*)$.

See https://mathoverflow.net/questions/11554/whats-the-use-of-a-complete-measure
Remark A complete measure is good in the sense that it gives us more measurable sets, the more things we can measure, the better. For instance, one consequence of a complete measure is that when A is measurable, and B differs from A by a subset of a set of zero measure, then B is measurable as well.

Ex 2.4.3

a We make the following manipulations.

$$I \subseteq \bigcup_{1\leq j\leq n} I_j = \sqcup_{1\leq j\leq n} I'_j$$

where $I'_j = I_j - \bigcup_{1\leq l<j} I_l$. Hence,

$$I = I \cap \sqcup_{1\leq j\leq n} I'_j = \sqcup 1 \leq j \leq n(I \cap I'_j)$$

such that

$$\begin{aligned}
P(I) &= P(\sqcup 1 \leq j \leq n(I \cap I'_j)) \\
&= \sum_{1\leq j\leq n} |I \cap I'_j| \\
&\leq \sum_{1\leq j\leq n} |I'_j| \\
&\leq \sum_{1\leq j\leq n} |I_j| \\
&= \sum_{1\leq j\leq n} P(I_j)
\end{aligned}$$

where the first equality is proven in proposition 2.4.2 (disjoint union of intervals to form a single larger interval), and the latter inequalities are given by monotonicity of the length function $|\cdot|$.

b In $[0,1] \subseteq \mathbb{R}$, I is a closed and bounded interval, hence compact (Heine-Borel Theorem). Given countable cover $I_i, i \in \mathbb{N}$, there exists a finite subcover of I, i.e. $I \subseteq \bigcup_{1\leq j\leq n} I_{i_j}$. By part (a), we know that $P(I) \leq \sum_{1\leq j\leq n} P(I_{i_j}) \leq \sum_j P(I_j)$ since P is non-negative.

c We want to generalize our result in (b), i.e. I can be any interval, not just closed, and I_j can be any interval, not just open.

We extend each $I_j, j \geq 1$ to form $I'_j := (a_j - \epsilon 2^{-j-1}, b_j + \epsilon 2^{-j-1})$, such that $|I'_j| = |I_j| + \epsilon 2^{-j}$.

We compress I to form $I' := [a + \epsilon, b - \epsilon]$. This assumes $I \neq \emptyset$, since if I is empty, we trivially have $P(I) = 0 \leq \sum_j P(I_j)$.

We note that each I'_j may very well exceed the boundaries of $[0,1]$, such that P may not be defined for I'_j, but it doesn't matter, since the length function $|I'_j|$ is still well-defined.

Hence,

$$P(I) = |I| = |I'| + 2\epsilon$$

and

$$|I'| \leq \sum_j |I'_j| = \sum_j |I_j| + \epsilon = \sum_j P(I_j) + \epsilon$$

Combining gives

$$P(I) \leq \sum_j P(I_j) + 3\epsilon$$

which proves countable monotonicity in general for all sets $I, I_1, I_2, \dots \in \mathcal{J}$.

An unsuccessful attempt: I tried to use closure $I' := \overline{I} = [a,b]$, but it doesn't seem possible to show that the endpoints lie in some open interval.

Ex 2.4.5 Claim: $\sigma(\{(-\infty, b] : b \in \mathbb{R}\}) = \sigma(\{I \subseteq \mathbb{R} : I \text{ interval}\})$

We first note that for collections $\mathcal{A}_\infty, \mathcal{A}_\in$, $\mathcal{A}_\infty \subseteq \mathcal{A}_\in \implies \sigma(\mathcal{A}_\infty) \subseteq \sigma(\mathcal{A}_\in)$, since a Borel algebra that contains all elements of $\mathcal{A}_\in$ also contains all elements of $\mathcal{A}_\infty$.

Let $\mathcal{A}_\infty = \{(-\infty, b] : b \in \mathbb{R}\}$ and $\mathcal{A}_\in$ be the set of all intervals in $\mathbb{R}$. Let $\sigma_1 = \sigma(\mathcal{A}_\infty), \sigma_2 = \sigma(\mathcal{A}_\in)$. Clearly, $\sigma_1 \subseteq \sigma_2$.

Next, we make the following 4 steps

1. First, for each b, $(-\infty, b] \in \sigma_1$ by definition
2. By closure under complements, for each a, $(-\infty, a]^c = (a, \infty) \in \sigma_1$
3. By closure under countable union, $\bigcup_{n \in \mathbb{N}} (-\infty, b - \frac{1}{n}] = (-\infty, b) \in \sigma_1$
4. By closure under countable intersection $\bigcup_{n \in \mathbb{N}} (a - \frac{1}{n}, \infty) = [a, \infty) \in \sigma_1$

Now we observe that $\forall a, b \in \mathbb{R}, a \leq b$,

- $[a, b] = [a, \infty) \cap (-\infty, b]$
- $(a, b) = (a, \infty) \cap (-\infty, b)$
- $[a, b) = [a, \infty) \cap (-\infty, b)$
- $(a, b] = (a, \infty) \cap (-\infty, b]$

hence by closure, σ_1 contains all intervals, i.e. $A_2 \subseteq \sigma_1$. By minimality of Borel algebra σ_2, we have $\sigma_2 \subseteq \sigma_1$. Hence $\sigma_1 = \sigma_2$.

(We note that a Borel algebra of a set A is a subset of any σ-algebra containing the same set A)

A Borel algebra $\mathcal{B} = \sigma(A)$ reminds me of the subgroup $\langle S \rangle$ generated by a subset S of a group. $\langle S \rangle$ is equal to the set of all words formed by the (finite) product of elements of S, whereas $\mathcal{B}$ is the set of all countable unions, countable intersections, complements of elements of A.

Ex 2.4.7

a K is the countable intersection of intervals remaining at each step, whereas K^c is the countable union of the intervals removed at each step of the construction. Hence, by closure of σ-algebras under countable union and intersection, $K, K^c \in \mathcal{B}$.

b Since $\mathcal{B} \subseteq \mathcal{M}$, $K, K^c \in \mathcal{M}$

c Algebras are closed under countable union, so $K^c \in \mathcal{B}_1$

d K is an uncountable set. Furthermore, K contains no interval of non-zero length. It is not possible to obtain K with the countable union of singletons. Hence $K \notin \mathcal{B}_1$.

e Since $\mathcal{B}_1$ is not closed under complement, $\mathcal{B}_1$ is not a σ-algebra. (We can also say that $\mathcal{B}_1$ is not closed under countable intersection.)

2.5.4 To explicitly show the claim that $D_n = B_n \cap \bigcap_{1 \leq i \leq n-1} B_i^c$ is a disjoint union of elements of $\mathcal{J}$, we let each $B_n^c = \sqcup_{1 \leq j \leq l_n} F_{n,j}$ where each $F_{n,j} \in \mathcal{J}$.

Then

$$B_n \cap \bigcap_{1 \leq i \leq n-1} B_i^c = B_n \cap \bigcap_{1 \leq i \leq n-1} \sqcup_{1 \leq j \leq l_n} F_{n_j}$$

By distributive law, this resolves to a union of terms of the form $B_n \cap F_{1,\lambda_1} cap F_{2,\lambda_2} \cap \cdots \cap F_{n-1,\lambda_{n-1}}$ where for each $1 \leq i \leq n-1$, $1 \leq \lambda_i \leq l_i$. Since each term is a finite intersection, it resides in $\mathcal{J}$.

Any 2 distinct terms of the intersection, $B_n \cap F_{1,\lambda_1} \cap F_{2,\lambda_2} \cap \cdots \cap F_{n-1,\lambda_{n-1}}$, and $B_n \cap F_{1,\mu_1} \cap F_{2,\mu_2} \cap \cdots \cap F_{n-1,\mu_{n-1}}$ must have some i for which $\lambda_i \neq \mu_i$. This implies that $F_{i,\lambda_i} \cap F_{i,\mu_i} = \emptyset$ so the 2 terms must also be disjoint.

Ex 2.5.6 Suppose P satisfies finite additivity, and that given a monontonically decreasing set sequence (A_i), with each A_i being a finite disjoint union of elements of $\mathcal{J}$ and $\cap_n A_n = \emptyset$, that $\lim_{n\to\infty} P(A_n) = 0$.

Suppose we have a countably many pairwise disjoint sets $D_n, n \in \mathbb{N}$. Then, we note that we can write

$$\sqcup_i D_i = D_1 \sqcup D_2 \sqcup \cdots \sqcup D_n \sqcup (\sqcup_{i>n} D_i)$$

We now show some properties of the tail union $\sqcup_{i>n} D_i$. First, letting $A_n = \sqcup_{i>n} D_i = \sqcup_i D_i \cap D_1^c \cap D_2^c \cap \cdots \cap D_n^c$. Similar to **2.5.4**, we can show that A_n is a disjoint union of elements of $\mathcal{J}$. Furthermore, $A_n \supseteq A_{n+1}$ and $\cap_n A_n = \emptyset$ are easy to verify. Hence, by assumption, $\lim_{n\to\infty} P(A_n) = 0$.

Hence,

$$P(\sqcup_i D_i) = \lim_{n\to\infty} (P(D_1) + \cdots + P(D_n) + P(A_n)) = \lim_{n\to\infty} \sum_{1 \le i \le n} P(D_i) = \sum_i P(D_i)$$

as desired.

2.5.9 Let $S = \{(-\infty, x] : x \in \mathbb{R}\}$, $\mathcal{J}_1 = \{(-\infty, x] : x \in \mathbb{R}\} \cup \{(y, \infty) : y \in \mathbb{R}\} \cup \{(y, x] : y, x \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$, $\mathcal{J}_2$ be the set of all intervals on $\mathbb{R}$. Then $S \subseteq \mathcal{J}_1 \subseteq \mathcal{J}_2$. (Note that S is not a semialgebra but $\mathcal{J}_1$ is.)

Then $\sigma(S) \subseteq \sigma(\mathcal{J}_1) \subseteq \sigma(\mathcal{J}_2)$. But we have proven in **Ex 2.4.5** that $\sigma(S) = \sigma(\mathcal{J}_2)$, hence $\sigma(\mathcal{J}_1) = \sigma(\mathcal{J}_2)$.

Ex 2.6.1

a First, we check that indeed, $\emptyset, \Omega \in \mathcal{J}$.
Next, we check that $\mathcal{J}$ is closed under finite intersection.

- If $\emptyset$ is part of an intersection, then we just get $\emptyset$.
- If Ω is part of an intersection, then we can just ignore it.
- Now consider $B_1 \cap B_2$, where $B_1 = A_{a_1, a_2, \ldots, a_m}, B_2 = A_{a'_1, a'_2, \ldots, a'_n}$. We then consider cases. If WLOG, $a_1, \ldots, a_m$ is a prefix of $a'_1, \ldots, a'_n$, then $B_2 \subseteq B_1$ and $B_1 \cap B_2 = B_1 \in \mathcal{J}$. If neither is a prefix of the other, then $B_1 \cap B_2 = \emptyset \in \mathcal{J}$.

Finally, consider any $B \in \mathcal{J}$. The case where $B = \emptyset \vee B = \Omega$ is trivial. So suppose $B = A_{a_1, \ldots, a_n}$. Then B^c is the finite disjoint union of $A_{a'_1, \ldots, a'_n}$ where $\vee_{1 \le i \le n} a'_i \neq a_i$. Hence, the complement of any sets of $\mathcal{J}$ can be formed by a finite disjoint union of sets in $\mathcal{J}$.

Hence $\mathcal{J}$ is a semialgebra.

b (TODO) Suppose we have N disjoint sets $\{D_1, \ldots, D_n\}$.

Following the hint, letting $k \in \mathbb{N}$ be the number that the results of only coins 1 to k are specified by any $D_n, n \le N$, we partition Ω into 2^k subsets $A_{a_1, \ldots, a_k}$, $a_1, \ldots, a_k \in \{0, 1\}$.