MA2108S (Mathematical Analysis I) Pointers

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1 Definitions and Formula

2 Point Set Topology

2.1 Definitions

- Metric Space M
- Open ball $B(p,r) = \{q \in M : d(p,q) < r\}$
- Boundary of S: $bd(S) = \{p \in S : \forall r > 0, \exists q \in S, \exists q' \in S^c, q, q' \in B(p,r)\}$
- Limit points of S: $lim(S) = \{p \in S : \forall r > 0, \exists q \in S, q \neq p, q \in B(p,r)\}$
- Interior of S: $int(S) = \{p \in S : \exists r > 0, B(p,r) \subseteq S\}$

2.2 Equivalence of definitions of closed set

The following definitions of a closed set are equivalent.

1. $bd(S) \subseteq S$
2. $lim(S) \subseteq S$

Proof: Suppose $bd(S) \subseteq S$. Let $x \in lim(S)$ and fix some arbitrary $r \in \mathbb{R}^+$. If $x \in S$, we are done. Otherwise, $x \notin S$. As x is a limit point, $\exists q \in S, q \neq x$ such that $q \in B(x,r)$. Now, notice that x is a boundary point of S, since x itself is not in S, and q is in S, and both $x, q \in B(x,r)$.
But by our initial assumption, we have $x \in bd(S) \subseteq S$. Contradiction.
Hence, $x \in S$ and $\lim(S) \subseteq S$.

Conversely, suppose $lim(S) \subseteq S$.
Let $x \in bd(S)$ and fix some arbitrary $r \in \mathbb{R}^+$. If $x \in S$, we are done.
Otherwise, $x \notin S$. As x is a boundary point, $\exists q \in S, q' \in S^c$ such that both are in $B(x,r)$. In particular, $q \neq x$ since $x \notin S$. This says that x is a limit point of S.
Our initial supposition says that $x \in lim(S) \subseteq S$. Again, we have a contradiction.