MA3238 (Stochastic Processes 1)

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Reference: Introduction to Probability Models (9th edition)

Note: The paragraphs in this document correspond to subsections in the textbook. For e.g. **4.2** corresponds to Chapman-Kolmogorov Equations.

4.1
We have $P_{i,j} = P(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0)$ for all $i_0, \ldots, i_{n-1}$ in the state space. Hence, $P(X_{n+1} = j | X_n = i) = \sum P(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) P(X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) = P_{i,j}$.

Hence, conditioning on $X_n = i$, the events $\{X_{n+1} = j\}$ and $\{X_{n-1} = i_{n-1}, \ldots, X_0 = i_0\}$ are independent.

4.2
Claim: Consider the case of a state space with absorbing states A. Suppose states $i, j \notin A, k \in A$. Then $P(X_{n+m} = j \cap X_n = k | X_0 = i) = 0$. i.e. once state k is reached, the probability of reaching any other state distinct from k is 0.

Since $P(X_{n+m} = j \cap X_n = k | X_0 = i) = P(X_{n+m} = j | X_n = k \wedge X_0 = i) P(X_n = k | X_0 = i) = P(X_m = j | X_0 = k) P(X_n = k | X_0 = i)$, it suffices to show that $P^m_{k,j} = 0$.
This can be verified inductively. We shall only show the inductive step. Suppose $P^{\lambda}_{k,j} = 0$ (in fact we need to assume the same for all j distinct from k). Then $P^{\lambda+1}_{k,j} = \sum_l P^{\lambda}_{k,l} \cdot P_{l,j} = P^{\lambda}_{k,k} \cdot P_{k,j} = 1 \cdot 0 = 0$.

4.3
Another way to interpret $P^n_{i,j}$ is the following. $P^n_{i,j} > 0$ iff there is a sequence $l_1, \ldots, l_{n-1}$ such that $P_{i,l_1}, P_{l_1,l_2}, \ldots, P_{l_{n-1},j} \neq 0$. This is equivalent to saying $i \to l_1 \to l_2 \to \cdots \to l_{n-1} \to j$. We can also let $l_0 := i, l_n := j$ and obtain a sequence $l_0, \ldots, l_n$.

To take example 4.12 in this section as an example: Suppose we want to show that states $0, 1$ cannot access state 2. One possible argument to make is to consider the candidates for states $l := l_{n-1}$ (for any n) such that $l_{n-1} \to 2$, or equivalently, $P_{l,2} > 0$. But looking at the 3rd column of the Markov matrix, we see that the only candidate for l is 2 itself. So it is impossible to have a series of state transition from 0 or 1 that ends up in 2. (A formal proof can involve induction on the number of state transitions.)

Characterization of recurrent and transient states For a state i, let f_i be the probability that state i is revisited, given that the current state is i.
Let $Y_i \sim Geom(p = (1 - f_i))$ be a r.v. counting the number of times state i was entered, given that the starting state is i. Transient states:

- Have a f_i value < 1
- Have a finite expectation $E[Y_i]$

Recurrent states:

- Have $f_i = 1$
- Have an infinite expectation $E[Y_i]$

Corollary 4.2 We provide a more succint derivation of this result.

- State i recurrent implies $\sum_n p_{i,i}^n = \infty$
- $i \leftrightarrow j$ implies $\exists m_1, m_2,\ p_{i,j}^{m_1} > 0 \wedge p_{j,i}^{m_2} > 0$

Hence,

$$\sum_n p_{j,j}^n \geq \sum_{n \geq m_1 + m_2} p_{j,j}^n \geq \sum_{n \geq m_1 + m_2} p_{j,i}^{m_2} p_{i,i}^{n-m_1-m_2} p_{i,j}^{m_1} = p_{j_i}^{m_2} p_{i,j}^{m_1} \sum_n p_{i,i}^n$$

The last sum is unbounded and we are done.