MA2108S (Mathematical Analysis I) Pointers

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1 Definitions and Formula

2 Point Set Topology

2.1 Definitions

- Metric Space M
- Open ball $B(p, r) = \{q \in M : d(p, q) < r\}$
- Boundary of S: $bd(S) = \{p \in S : \forall r > 0, \exists q \in S, \exists q' \in S^c, q, q' \in B(p, r)\}$
- Limit points of S: $lim(S) = \{p \in S : \forall r > 0, \exists q \in S, q \neq p, q \in B(p, r)\}$
- Interior of S: $int(S) = \{p \in S : \exists r > 0, B(p, r) \subseteq S\}$

2.2 Equivalence of definitions of closed set

The following definitions of a closed set are equivalent.

1. $bd(S) \subseteq S$
2. $lim(S) \subseteq S$

Proof: Suppose $bd(S) \subseteq S$. Let $x \in lim(S)$ and fix some arbitrary $r \in \mathbb{R}^+$. If $x \in S$, we are done. Otherwise, $x \notin S$. As x is a limit point, $\exists q \in S, q \neq x$ such that $q \in B(x, r)$. Now, notice that x is a boundary point of S, since x itself is not in S, and q is in S, and both $x, q \in B(x, r)$.
But by our initial assumption, we have $x \in bd(S) \subseteq S$. Contradiction.
Hence, $x \in S$ and $\lim(S) \subseteq S$.

Conversely, suppose $lim(S) \subseteq S$.
Let $x \in bd(S)$ and fix some arbitrary $r \in \mathbb{R}^+$. If $x \in S$, we are done.
Otherwise, $x \notin S$. As x is a boundary point, $\exists q \in S, q' \in S^c$ such that both are in $B(x, r)$. In particular, $q \neq x$ since $x \notin S$. This says that x is a limit point of S.
Our initial supposition says that $x \in lim(S) \subseteq S$. Again, we have a contradiction.

2.3 Sequential Compactness

The following 2 notions are equivalent.

1. Every infinite subset E of a set X has a limit point in X.

2. X is sequentially compact, that is, for every sequence $\{p_n\}$ in X, there exists a subsequence converging to some point of X.

3 Sequences

3.1 Infinite subsets vs sequences

There is a subtle difference between these 2 concepts. Of course, infinite sets are unordered, whereas sequences are ordered.
Additionally, it must be noted that sequences can have repeat elements (with different indices). The range of an infinite sequence may very well be finite. An example would be the sequence $\{i^n\}_{n\in\mathbb{N}}$, with range $\{\pm 1, \pm i\}$.

Also, with regard to infinite sets, we speak of limit points, and with regard to sequences, we speak of subsequence limits. Again, there is a difference between limit points and subsequential limits.

For example, a point can be a subsequential limit without being a limit point. Consider the earlier defined set $\{i^n\}$. Every element in the range $\{\pm 1, \pm i\}$ is a subsequential limit, but since the range is finite, none of these subsequential limits are limit points of the range.