MA3238 (Stochastic Processes 1)

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Reference: Introduction to Probability Models (9th edition)

1 Revision

Limit of the binomial distribution is the poisson distribution Let $X_n \sim (n, p = \frac{\lambda}{n})$ and consider the distribution of X_n as $n \to \infty$. **Note** that as we increase n, we also shrink p since that $p = \frac{\lambda}{n}$.

Observe that

$$\begin{aligned}\binom{n}{k} p^k (1-p)^{n-k} &= \frac{n^{\underline{k}}}{k!} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \\ &= \frac{\lambda^k}{k!} \cdot \frac{n^{\underline{k}}}{n^k} \left(1 - \frac{\lambda}{n}\right)^{n-k}\end{aligned}$$

Since $\frac{\lambda^k}{k!}$ is independent of n, it suffices to take the limit of the latter expression. We have

$$\begin{aligned}\lim_{n\to\infty} \frac{n^{\underline{k}}}{n^k} \left(1 - \frac{\lambda}{n}\right)^{n-k} &= \lim_{n\to\infty} \frac{n^{\underline{k}}}{n^k} \left(1 - \frac{\lambda}{n}\right)^{n} \left(1 - \frac{\lambda}{n}\right)^{-k} \\ &= \lim_{n\to\infty} \frac{n^{\underline{k}}}{n^k} \lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^{n} \lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^{-k} \\ &= 1 \cdot e^{-\lambda} \cdot 1 = e^{-\lambda}\end{aligned}$$

A r.v. cannot be both continuous (not necessarily absolutely continuous) and discrete.

- A discrete r.v. takes probability 1 on a countable set C. Equivalently, there is a countable set C such that $\sum_{c\in C} p(c) = 1$ for some function p, such that $P(X = c) := p(c)$.
- A continuous r.v. has a continuous cdf.

Suppose X is discrete. Let C be a countable set for which $P(X \in C) = \sum_{c\in C} p(c) = 1$. Clearly, it is not possible for all $p(c) = 0$, so $\exists c \in C, p(c) > 0$. Let F be the cdf of X. Then we claim that F is left-discontinuous at c. Because $\forall \epsilon, F(c - \epsilon) < P(X < c) = F(c) - p(c)$. $\blacksquare$

Non-independent but covariance 0. Consider $Y = 1, -1$ with probability 1/2 each. Then,

- Given $Y = 1$, $X = 1$ with probability 1/2 and 0 otherwise
- Given $Y = -1$, $X = 2$ with probability 1/4 and 0 otherwise

The idea of Y is clear, the positive half and negative half cancel out. The idea of X is to vary its behavior given differing value of Y, but not too much, so that the positive part of XY cancels out the negative parts, with their respective probability weights.

Note: The paragraphs in this document correspond to subsections in the textbook. For e.g. **4.2** corresponds to Chapman-Kolmogorov Equations.

4.1
We have $P_{i,j} = P(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0)$ for all $i_0, \ldots, i_{n-1}$ in the state space. Hence, $P(X_{n+1} = j | X_n = i) = \sum P(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) P(X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) = P_{i,j}$.

Hence, conditioning on $X_n = i$, the events $\{X_{n+1} = j\}$ and $\{X_{n-1} = i_{n-1}, \ldots, X_0 = i_0\}$ are independent.

4.2
Claim: Consider the case of a state space with absorbing states A. Suppose states $i, j \notin A, k \in A$. Then $P(X_{n+m} = j \cap X_n = k | X_0 = i) = 0$. i.e. once state k is reached, the probability of reaching any other state distinct from k is 0.

Since $P(X_{n+m} = j \cap X_n = k | X_0 = i) = P(X_{n+m} = j | X_n = k \wedge X_0 = i) P(X_n = k | X_0 = i) = P(X_m = j | X_0 = k) P(X_n = k | X_0 = i)$, it suffices to show that $P^m_{k,j} = 0$.
This can be verified inductively. We shall only show the inductive step. Suppose $P^\lambda_{k,j} = 0$ (in fact we need to assume the same for all j distinct from k). Then $P^{\lambda+1}_{k,j} = \sum_l P^\lambda_{k,l} \cdot P_{l,j} = P^\lambda_{k,k} \cdot P_{k,j} = 1 \cdot 0 = 0$.

4.3
Another way to interpret $P^n_{i,j}$ is the following. $P^n_{i,j} > 0$ iff there is a sequence $l_1, \ldots, l_{n-1}$ such that $P_{i,l_1}, P_{l_1,l_2}, \ldots, P_{l_{n-1},j} \neq 0$. This is equivalent to saying $i \to l_1 \to l_2 \to \cdots \to l_{n-1} \to j$. We can also let $l_0 := i, l_n := j$ and obtain a sequence $l_0, \ldots, l_n$.

To take example 4.12 in this section as an example: Suppose we want to show that states $0, 1$ cannot access state 2. One possible argument to make is to consider the candidates for states $l := l_{n-1}$ (for any n) such that $l_{n-1} \to 2$, or equivalently, $P_{l,2} > 0$. But looking at the 3rd column of the Markov matrix, we see that the only candidate for l is 2 itself. So it is impossible to have a series of state transition from 0 or 1 that ends up in 2. (A formal proof can involve induction on the number of state transitions.)

Characterization of recurrent and transient states For a state i, let f_i be the probability that state i is revisited, given that the current state is i.
Let $Y_i \sim Geom(p = (1 - f_i))$ be a r.v. counting the number of times state i was entered, given that the starting state is i. Transient states:

- Have a f_i value < 1
- Have a finite expectation $E[Y_i]$

Recurrent states:

- Have $f_i = 1$
- Have an infinite expectation $E[Y_i]$

Hence to argue that in a finite state space, there must be at least 1 recurrent state, we argue that the sum of expectation of all Y_i must be infinite, so there must be some i for which $E[Y_i] > 0$.

Corollary 4.2 We provide a more succint derivation of this result.

- State i recurrent implies $\sum_n p^n_{i,i} = \infty$
- $i \leftrightarrow j$ implies $\exists m_1, m_2$, $p^{m_1}_{i,j} > 0 \wedge p^{m_2}_{j,i} > 0$

Hence,

$$\sum_n p^n_{j,j} \geq \sum_{n \geq m_1 + m_2} p^n_{j,j} \geq \sum_{n \geq m_1 + m_2} p^{m_2}_{j,i} p^{n-m_1-m_2}_{i,i} p^{m_1}_{i,j} = p^{m_2}_{ji} p^{m_1}_{i,j} \sum_n p^n_{i,i}$$

The last sum is unbounded and we are done.