

# Advanced Probability Theory

Jia Cheng

December 2021

Reference Text: A First Look at Rigorous Probability Theory

Related modules: ST5214

**Notation** Throughout this document,  $\mathbb{N}$  can vary in usage, sometimes  $\mathbb{N} = \mathbb{Z}^+$ , other times  $\mathbb{N} = \mathbb{Z}_0^+$ , depending on which is more convenient.

**1.1** To see that  $\exists z \in \mathbb{R}, P(Z = z) > 0$ , we observe that

$$P(Z = 0) \geq P(Z = 0 \wedge X = 0) = P(Z = 0 | X = 0) \cdot P(X = 0) \geq \frac{1}{2} \cdot P(X = 0) > 0$$

since  $P(X = 0) > 0$ .

**1.2**

**Uncountable summation** Given an uncountable non-negative set of numbers  $\{r_a : a \in I\}$  indexed by  $I$ ,

$$\sum_{a \in I} r_a := \sup \left\{ \sum_{a \in J} r_a : J \subseteq I \wedge J \text{ finite} \right\}$$

**R-shift** (Equivalent definition) R-shift of  $A \subseteq [0, 1]$ .  $A \oplus r = \{(a + r) \bmod 1 : a \in A\}$

**2.1** Notice that countability is used by 2 constructs. One, probability measure is countably additive (and not uncountably so). Two, the  $\sigma$ -algebra is closed under countable union and intersection (and not uncountably so).

Recall that the reason for disallowing uncountable operations in general is due to the fact that

$$\bigcup_{x \in A} \{x\} = A$$

for any set  $A$ , in particular  $[0, 1]$  when discussing the uniform distribution on the unit interval.

**Theorem 2.2.1** We provide a proof for this theorem.

First, we show that  $\mathcal{F}$  is a  $\sigma$ -algebra. By definition,  $\mathcal{F} = \mathcal{P}(\Omega)$ . Hence, the unary complement operation is a mapping  $\mathcal{P}(\Omega) \rightarrow \mathcal{P}(\Omega)$  whose domain and codomain are just  $\mathcal{F}$  as desired. Similarly, for the countable set operations of union and intersection, they are mappings with codomain as  $\mathcal{P}(\Omega)$  and are also closed since  $\mathcal{F} = \mathcal{P}(\Omega)$ .

We also note that both  $\emptyset, \Omega$  reside in  $\mathcal{F}$

Next, we show that  $P$  is a probability measure. By definition of  $P$ ,  $P$  is additive since  $A \cap B = \emptyset \implies P(A \sqcup B) = \sum_{\omega \in A \sqcup B} p(\omega) = \sum_{\omega \in A} p(\omega) + \sum_{\omega \in B} p(\omega) = P(A) + P(B)$ .

I am not quite sure about showing countable additivity however, perhaps using some form of diagonal summation argument it is possible to prove this.

$P(\Omega) = \sum_{\omega \in \Omega} p(\omega) = 1$ . Furthermore,  $p$  is non-negative, hence  $P$  is indeed bounded between 0 and 1.

**Ex 2.2.3** First,  $\emptyset, \Omega = [0, 1] \in \mathcal{J}$  by definition as they are intervals. Next, to show closure under finite intersection, it suffices to show closure under binary intersection. Consider cases: We only consider one endpoint, since we can "patch" together two endpoints.

- $[a$  and  $b$  intersect to give  $[\max\{a, b\}$
- $(a$  and  $b$  intersect to give  $(\max\{a, b\}$
- $[a$  and  $(b$  intersect to give  $[a$  if  $a > b$  and  $(b$  otherwise

We can do a similar case analysis for right endpoints. Given a stringified left endpoint  $l \in \{ "[a", "(a" \}$  and a stringified right endpoint  $r \in \{ "b]", "b)" \}$  we can form an interval via concatenation  $l, r$ . Hence,  $\mathcal{J}$  is closed under finite intersection.

Consider the complement  $J = [0, a) \cup (b, 1]$  of an interval  $[a, b] \in \mathcal{J}$ , where depending on whether the left/right endpoint is closed or open we adjust  $J$  accordingly. Regardless, we see that  $J$  is a disjoint union of at most 2 intervals in  $\mathcal{J}$ .

Hence,  $\mathcal{J}$  is a semialgebra of subsets of  $\Omega$ .

### Ex 2.2.5

**a**  $\mathcal{B}_0 \subseteq \mathcal{P}(\Omega)$ . Since  $\mathcal{B}_0$  consists of all finite unions of elements of  $\mathcal{J}$ , in particular,  $\mathcal{J} \subseteq \mathcal{B}_0$ , so  $\Omega = [0, 1] \in \mathcal{J} \subseteq \mathcal{B}_0$ .

Next, the finite union and intersection of elements of  $\mathcal{B}_0$  will give finite unions of elements of *semialg*, so that  $\mathcal{B}_0$  is closed under finite union and intersection. (For intersection, we can argue using distributive law plus observe that the intersection of intervals gives another interval)

Let  $B \in \mathcal{B}_0$ , so that  $B$  is a finite union of the form  $\bigcup_{1 \leq i \leq n} I_i$  for some intervals  $I_i$  in  $[0, 1]$ . Then,  $B^c = \bigcap_{1 \leq i \leq n} I_i^c$  by DeMorgan's Law, and we have already proven in Ex 2.2.3 that  $I_i^c$  is a disjoint union of intervals, i.e.  $I_i^c \in \mathcal{B}_0$ . Furthermore, we have proven that  $\mathcal{B}_0$  is closed under finite intersection, so  $B^c \in \mathcal{B}_0$ .

Hence,  $\mathcal{B}_0$  is an algebra.

**b** The difference between an algebra and a  $\sigma$ -algebra is that  $\sigma$ -algebras are closed under countable union and intersection but algebras are not necessarily so.

We consider Cantor's set  $C$ , which is a countable intersection of  $C_i$ , where each  $C_i$  is formed by removing from each interval in  $C_{i-1}$  the middle one-third.

Since each  $C_i$  is a union of (disjoint) intervals, by definition,  $C_i \in \mathcal{B}_0$ . If  $\mathcal{B}_0$  is to be a  $\sigma$ -algebra, then we must have  $C \in \mathcal{B}_0$ , i.e.  $C$  can be formed from a finite union of intervals.

First of all,  $C$  does not contain any interval of non-zero length, so our options are reduced to forming  $C$  from a finite union of singletons, i.e. intervals of the form  $[a, a] = \{a\}$ . But we also know  $C$  to be uncountable, but a finite union of singletons is finite. Hence we have a contradiction.

Since  $C \notin \mathcal{B}_0$ ,  $\mathcal{B}_0$  is not closed under countable intersection, so it is not a  $\sigma$ -algebra.

We comment that  $\mathcal{B}_1$  is similarly not a  $\sigma$ -algebra using the same counterexample. The countable union of singletons must be at most countable, so they cannot union to form an uncountable set like  $C$ .

**Ex 2.3.16** Suppose  $A \in \mathcal{M} \wedge P^*(A) = 0$  and  $B \subseteq A$ . To show that  $B \in \mathcal{M}$ , we show equivalently that:

For each  $E \subseteq \Omega$ ,  $P^*(E) \geq P^*(A \cap E) + P^*(A^c \cap E)$  (i.e. superadditivity. But by monotonicity,  $P^*(A \cap E) = 0$ , so that  $P^*(E) \geq P^*(A^c \cap E)$  is automatically true by monotonicity. Hence, we have shown the completeness of the extension  $(\Omega, \mathcal{M}, P^*)$ .

See <https://mathoverflow.net/questions/11554/whats-the-use-of-a-complete-measure>

**Remark** A complete measure is good in the sense that it gives us more measurable sets, the more things we can measure, the better. For instance, one consequence of a complete measure is that when  $A$  is measurable, and  $B$  differs from  $A$  by a subset of a set of zero measure, then  $B$  is measurable as well.

**Ex 2.4.3**

**a** We make the following manipulations.

$$I \subseteq \bigcup_{1 \leq j \leq n} I_j = \sqcup_{1 \leq j \leq n} I'_j$$

where  $I'_j = I_j - \bigcup_{1 \leq l < j} I_l$ . Hence,

$$I = I \cap \sqcup_{1 \leq j \leq n} I'_j = \sqcup_{1 \leq j \leq n} (I \cap I'_j)$$

such that

$$\begin{aligned} P(I) &= P(\sqcup_{1 \leq j \leq n} (I \cap I'_j)) \\ &= \sum_{1 \leq j \leq n} |I \cap I'_j| \\ &\leq \sum_{1 \leq j \leq n} |I'_j| \\ &\leq \sum_{1 \leq j \leq n} |I_j| \\ &= \sum_{1 \leq j \leq n} P(I_j) \end{aligned}$$

where the first equality is proven in proposition 2.4.2 (disjoint union of intervals to form a single larger interval), and the latter inequalities are given by monotonicity of the length function  $|\cdot|$ .

**b** In  $[0, 1] \subseteq \mathbb{R}$ ,  $I$  is a closed and bounded interval, hence compact (Heine-Borel Theorem). Given countable cover  $I_i, i \in \mathbb{N}$ , there exists a finite subcover of  $I$ , i.e.  $I \subseteq \bigcup_{1 \leq j \leq n} I_{i_j}$ . By part (a), we know that  $P(I) \leq \sum_{1 \leq j \leq n} P(I_{i_j}) \leq \sum_j P(I_j)$  since  $P$  is non-negative.

**c** We want to generalize our result in (b), i.e.  $I$  can be any interval, not just closed, and  $I_j$  can be any interval, not just open.

We extend each  $I_j, j \geq 1$  to form  $I'_j := (a_j - \epsilon 2^{-j-1}, b_j + \epsilon 2^{-j-1})$ , such that  $|I'_j| = |I_j| + \epsilon 2^{-j}$ .

We compress  $I$  to form  $I' := [a + \epsilon, b - \epsilon]$ . This assumes  $I \neq \emptyset$ , since if  $I$  is empty, we trivially have  $P(I) = 0 \leq \sum_j P(I_j)$ .

We note that each  $I'_j$  may very well exceed the boundaries of  $[0, 1]$ , such that  $P$  may not be defined for  $I'_j$ , but it doesn't matter, since the length function  $|I'_j|$  is still well-defined.

Hence,

$$P(I) = |I| = |I'| + 2\epsilon$$

and

$$|I'| \leq \sum_j |I'_j| = \sum_j |I_j| + \epsilon = \sum_j P(I_j) + \epsilon$$

Combining gives

$$P(I) \leq \sum_j P(I_j) + 3\epsilon$$

which proves countable monotonicity in general for all sets  $I, I_1, I_2, \dots \in \mathcal{J}$ .

An unsuccessful attempt: I tried to use closure  $I' := \bar{I} = [a, b]$ , but it doesn't seem possible to show that the endpoints lie in some open interval.

**Ex 2.4.5** Claim:  $\sigma(\{(-\infty, b] : b \in \mathbb{R}\}) = \sigma(\{I \subseteq \mathbb{R} : I \text{ interval}\})$

We first note that for collections  $\mathcal{A}_\infty, \mathcal{A}_\infty \subseteq \mathcal{A}_\infty \implies \sigma(\mathcal{A}_\infty) \subseteq \sigma(\mathcal{A}_\infty)$ , since a Borel algebra that contains all elements of  $\mathcal{A}_\infty$  also contains all elements of  $\mathcal{A}_\infty$ .

Let  $\mathcal{A}_\infty = \{(-\infty, b] : b \in \mathbb{R}\}$  and  $\mathcal{A}_\infty$  be the set of all intervals in  $\mathbb{R}$ . Let  $\sigma_1 = \sigma(\mathcal{A}_\infty), \sigma_2 = \sigma(\mathcal{A}_\infty)$ . Clearly,  $\sigma_1 \subseteq \sigma_2$ .

Next, we make the following 4 steps

1. First, for each  $b$ ,  $(-\infty, b] \in \sigma_1$  by definition
2. By closure under complements, for each  $a$ ,  $(-\infty, a]^c = (a, \infty) \in \sigma_1$
3. By closure under countable union,  $\bigcup_{n \in \mathbb{N}} (-\infty, b - \frac{1}{n}] = (-\infty, b) \in \sigma_1$
4. By closure under countable intersection  $\bigcap_{n \in \mathbb{N}} (a - \frac{1}{n}, \infty) = [a, \infty) \in \sigma_1$

Now we observe that  $\forall a, b \in \mathbb{R}, a \leq b$ ,

- $[a, b] = [a, \infty) \cap (-\infty, b]$
- $(a, b) = (a, \infty) \cap (-\infty, b)$
- $[a, b) = [a, \infty) \cap (-\infty, b)$
- $(a, b] = (a, \infty) \cap (-\infty, b]$

hence by closure,  $\sigma_1$  contains all intervals, i.e.  $\mathcal{A}_2 \subseteq \sigma_1$ . By minimality of Borel algebra  $\sigma_2$ , we have  $\sigma_2 \subseteq \sigma_1$ . Hence  $\sigma_1 = \sigma_2$ .

(We note that a Borel algebra of a set  $A$  is a subset of any  $\sigma$ -algebra containing the same set  $A$ )

A Borel algebra  $\mathcal{B} = \sigma(A)$  reminds me of the subgroup  $\langle S \rangle$  generated by a subset  $S$  of a group.  $\langle S \rangle$  is equal to the set of all words formed by the (finite) product of elements of  $S$ , whereas  $\mathcal{B}$  is the set of all countable unions, countable intersections, complements of elements of  $A$ .

**Ex 2.4.7**

**a**  $K$  is the countable intersection of intervals remaining at each step, whereas  $K^c$  is the countable union of the intervals removed at each step of the construction. Hence, by closure of  $\sigma$ -algebras under countable union and intersection,  $K, K^c \in \mathcal{B}$ .

**b** Since  $\mathcal{B} \subseteq \mathcal{M}$ ,  $K, K^c \in \mathcal{M}$

**c** Algebras are closed under countable union, so  $K^c \in \mathcal{B}_1$

**d**  $K$  is an uncountable set. Furthermore,  $K$  contains no interval of non-zero length. It is not possible to obtain  $K$  with the countable union of singletons. Hence  $K \notin \mathcal{B}_1$ .

**e** Since  $\mathcal{B}_1$  is not closed under complement,  $\mathcal{B}_1$  is not a  $\sigma$ -algebra. (We can also say that  $\mathcal{B}_1$  is not closed under countable intersection.)

**2.5.4** To explicitly show the claim that  $D_n = B_n \cap \bigcap_{1 \leq i \leq n-1} B_i^c$  is a disjoint union of elements of  $\mathcal{J}$ , we let each  $B_n^c = \sqcup_{1 \leq j \leq l_n} F_{n,j}$  where each  $F_{n,j} \in \mathcal{J}$ .

Then

$$B_n \cap \bigcap_{1 \leq i \leq n-1} B_i^c = B_n \cap \bigcap_{1 \leq i \leq n-1} \sqcup_{1 \leq j \leq l_n} F_{n,j}$$

By distributive law, this resolves to a union of terms of the form  $B_n \cap F_{1,\lambda_1} \cap F_{2,\lambda_2} \cap \cdots \cap F_{n-1,\lambda_{n-1}}$  where for each  $1 \leq i \leq n-1$ ,  $1 \leq \lambda_i \leq l_i$ . Since each term is a finite intersection, it resides in  $\mathcal{J}$ .

Any 2 distinct terms of the intersection,  $B_n \cap F_{1,\lambda_1} \cap F_{2,\lambda_2} \cap \cdots \cap F_{n-1,\lambda_{n-1}}$ , and  $B_n \cap F_{1,\mu_1} \cap F_{2,\mu_2} \cap \cdots \cap F_{n-1,\mu_{n-1}}$  must have some  $i$  for which  $\lambda_i \neq \mu_i$ . This implies that  $F_{i,\lambda_i} \cap F_{i,\mu_i} = \emptyset$  so the 2 terms must also be disjoint.

**Ex 2.5.6** Suppose  $P$  satisfies finite additivity, and that given a monotonically decreasing set sequence  $(A_i)$ , with each  $A_i$  being a finite disjoint union of elements of  $\mathcal{J}$  and  $\cap_n A_n = \emptyset$ , that  $\lim_{n \rightarrow \infty} P(A_n) = 0$ . (Note that  $P$  is extended to sets  $A_n$  by finite additivity, mentioned in the errata document)

Suppose we have a countably many pairwise disjoint sets  $D_n, n \in \mathbb{N}$ . Then, we note that we can write

$$\sqcup_i D_i = D_1 \sqcup D_2 \sqcup \cdots \sqcup D_n \sqcup (\sqcup_{i>n} D_i)$$

We now show some properties of the tail union  $\sqcup_{i>n} D_i$ . First, letting  $A_n = \sqcup_{i>n} D_i = \sqcup_i D_i \cap D_1^c \cap D_2^c \cap \cdots \cap D_n^c$ . Similar to **2.5.4**, we can show that  $A_n$  is a disjoint union of elements of  $\mathcal{J}$ . Furthermore,  $A_n \supseteq A_{n+1}$  and  $\cap_n A_n = \emptyset$  are easy to verify. Hence, by assumption,  $\lim_{n \rightarrow \infty} P(A_n) = 0$ .

Hence,

$$P(\sqcup_i D_i) = \lim_{n \rightarrow \infty} (P(D_1) + \cdots + P(D_n) + P(A_n)) = \lim_{n \rightarrow \infty} \sum_{1 \leq i \leq n} P(D_i) = \sum_i P(D_i)$$

as desired.

**2.5.9** Let  $S = \{(-\infty, x] : x \in \mathbb{R}\}$ ,  $\mathcal{J}_1 = \{(-\infty, x] : x \in \mathbb{R}\} \cup \{(y, \infty) : y \in \mathbb{R}\} \cup \{(y, x] : y, x \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$ ,  $\mathcal{J}_2$  be the set of all intervals on  $\mathbb{R}$ . Then  $S \subseteq \mathcal{J}_1 \subseteq \mathcal{J}_2$ . (Note that  $S$  is not a semialgebra but  $\mathcal{J}_1$  is.)

Then  $\sigma(S) \subseteq \sigma(\mathcal{J}_1) \subseteq \sigma(\mathcal{J}_2)$ . But we have proven in **Ex 2.4.5** that  $\sigma(S) = \sigma(\mathcal{J}_2)$ , hence  $\sigma(\mathcal{J}_1) = \sigma(\mathcal{J}_2)$ .

### Ex 2.6.1

**a** First, we check that indeed,  $\emptyset, \Omega \in \mathcal{J}$ .

Next, we check that  $\mathcal{J}$  is closed under finite intersection.

- If  $\emptyset$  is part of an intersection, then we just get  $\emptyset$ .
- If  $\Omega$  is part of an intersection, then we can just ignore it.
- Now consider  $B_1 \cap B_2$ , where  $B_1 = A_{a_1, a_2, \dots, a_m}$ ,  $B_2 = A_{a'_1, a'_2, \dots, a'_n}$ . We then consider cases. If WLOG,  $a_1, \dots, a_m$  is a prefix of  $a'_1, \dots, a'_n$ , then  $B_2 \subseteq B_1$  and  $B_1 \cap B_2 = B_2 \in \mathcal{J}$ . If neither is a prefix of the other, then  $B_1 \cap B_2 = \emptyset \in \mathcal{J}$ .

Finally, consider any  $B \in \mathcal{J}$ . The case where  $B = \emptyset \vee B = \Omega$  is trivial. So suppose  $B = A_{a_1, \dots, a_n}$ . Then  $B^c$  is the finite disjoint union of  $A_{a'_1, \dots, a'_n}$  where  $\forall 1 \leq i \leq n, a'_i \neq a_i$ . Hence, the complement of any sets of  $\mathcal{J}$  can be formed by a finite disjoint union of sets in  $\mathcal{J}$ .

Hence  $\mathcal{J}$  is a semialgebra.

**b** Suppose we have  $N$  disjoint sets  $\{D_1, \dots, D_n\}$ .

Following the hint, letting  $k \in \mathbb{N}$  be the largest number such that  $a_1, a_2, \dots, a_k$  is a shared prefix amongst all sets  $D_1$  to  $D_n$ .

Let  $A = A_{a_1, \dots, a_k}$ . Then, we clearly have  $A \supseteq \bigcup_{1 \leq i \leq n} D_i$ . We claim that this is in fact an equality. Note that by assumption, the union of all  $D_i$  is a set in the semialgebra  $\mathcal{J}$ , so the union must be of the form  $A_{a_1, \dots, a_k, \dots, a_j}$  for some  $j \geq k$ . So we just want to show that  $j = k$ . Suppose not, such that  $j > k$ , then  $a_1, \dots, a_j$  is a shared prefix, but this contradicts the largest-ness of  $k$ .

Hence, we have proven  $\sqcup_{1 \leq i \leq n} D_i = A$ .

Next, we want to show finite additivity, i.e.  $P(A) = \sum_{1 \leq i \leq n} P(D_i)$ . Let  $K$  be the length of the longest sequence specified by  $D_i, 1 \leq i \leq n$ . For e.g. if  $D_1 = A_{1,0,1}, D_2 = A_{1,1}$ , then the longest sequence is of length  $K = 3$  (and  $k = 1$ ).

We shall prove this by (finite) induction.

Define for each  $m \in \{k, k+1, \dots, K\}$  the statement  $Q(m)$ :

Given any finite collection of disjoint sets  $D_i$  whose longest specified sequence is of length  $K$  and whose longest disjoint union is  $A^m := A_{a_1, \dots, a_m}$  for some length  $m$  sequence, then  $\sqcup_{1 \leq i \leq n} P(D_i) = P(A^m)$ .

The base case,  $Q(K)$  is trivial since the finite collection can only consist of 1 set,  $A^K$  itself. So we have  $P(D_1) = P(A^K)$ .

Next, suppose  $Q(j)$  holds for some  $k < j \leq K$ . Then, we want to show  $Q(j-1)$ . We partition the set  $D_i, 1 \leq i \leq n$  into 2 subcollections,  $B_1, \dots, B_{n_1}$  and  $C_1, \dots, C_{n_2}$ , such that each  $B_i \subseteq A_{a_1, \dots, a_j, 0}$  and each  $C_i \subseteq A_{a_1, \dots, a_j, 1}$ . It is not too hard to argue that  $\sqcup_{1 \leq i \leq n_1} B_i = A_{a_1, \dots, a_j, 0}$  and  $\sqcup_{1 \leq i \leq n_2} C_i = A_{a_1, \dots, a_j, 1}$ . Then, applying inductive assumption  $Q(j)$ , we have  $\sum_{1 \leq i \leq n_1} P(B_i) = P(A_{a_1, \dots, a_j, 0})$  and  $\sum_{1 \leq i \leq n_2} P(C_i) = P(A_{a_1, \dots, a_j, 1})$ . Hence,

$$P(A^j) = \frac{1}{2^j} = \frac{1}{2^{j+1}} + \frac{1}{2^{j+1}} = P(A_{a_1, \dots, a_j, 0}) + P(A_{a_1, \dots, a_j, 1}) = \sum_{1 \leq i \leq n} P(D_i)$$

This proves  $Q(j)$ .

By mathematical induction, we have  $\forall m \in \{k, k+1, \dots, K\}, Q(m)$ , in particular,  $Q(k)$  is true. This completes the proof.

**Ex 2.6.4** Clearly  $\Omega = \Omega_1 \times \Omega_2 \in \mathcal{J}$  and  $\emptyset = \emptyset \times \emptyset \in \mathcal{J}$ .

Next, if  $A_1 \times B_1, A_2 \times B_2 \in \mathcal{J}$ , then  $(A_1 \times B_1) \cap (A_2 \times B_2) = (A_1 \cap A_2) \times (B_1 \cap B_2) \in \mathcal{F}_1 \times \mathcal{F}_2 = \mathcal{J}$ , so  $\mathcal{J}$  is closed under finite intersection.

Finally, if  $A \times B \in \mathcal{J}$ , then  $(A \times B)^c = (A^c \times \Omega_2) \cup (\Omega_1 \times B^c) = (A^c \times B^c) \sqcup (A^c \times B) \sqcup (A \times B^c)$ .

Note that  $\mathcal{F}_1, \mathcal{F}_2$  are  $\sigma$ -algebra, so  $A^c \in \mathcal{F}_1 \wedge B^c \in \mathcal{F}_2$ . Hence  $\mathcal{J}$  is a semialgebra of  $\Omega_1 \times \Omega_2$ .

We also verify that  $P(\emptyset) = P_1(\emptyset)P_2(\emptyset) = 0$  and  $P(\Omega) = P_1(\Omega_1)P_2(\Omega_2) = 1$ .

#### Ex 3.1.4

Here, we use the property that  $[a, b] \cap [c, d] = [\max\{a, c\}, \min\{b, d\}]$ .

We write  $\omega$  as  $w$ .  $Z(w) > a \iff 3w + 4 > a \iff w > \frac{a-4}{3}$ . Hence,  $P(Z > a) = P(\{w \in [0, 1] : w > \frac{a-4}{3}\}) = P([0, 1] \cap (\frac{a-4}{3}, \infty)) = P([0, 1] \cap [\frac{a-4}{3}, \infty)) = P([\max\{0, \frac{a-4}{3}\}, 1])$ . We note that  $0 < \frac{a-4}{3} < 1 \iff 4 < a < 7$ , so

$$P([\max\{0, \frac{a-4}{3}\}, 1]) = \begin{cases} 1 & \text{if } a \leq 4 \\ 1 - \frac{a-4}{3} & \text{if } 4 < a < 7 \\ 0 & \text{if } a \geq 7 \end{cases}$$

Next,  $\forall w \in \Omega = [0, 1], X(w) < a \wedge Y(w) < b \iff w < a \wedge w < \frac{b}{2} \iff w < \min\{a, \frac{b}{2}\}$ . Hence,  $P(X < a \wedge Y < b) = P((-\infty, \min\{a, \frac{b}{2}\}) \cap [0, 1]) = P((-\infty, \min\{a, \frac{b}{2}\}] \cap [0, 1]) = P([0, \min\{1, a, \frac{b}{2}\}])$