MA3210 (Analysis II)

Jia Cheng

September 2021

1 Definitions

Sets

$$\mathbb{N} = \mathbb{Z}^+$$

2 Inequalities

- $|a + b| \leq |a| + |b|$
- $||a| - |b|| \leq |a - b|$
- $||a| - |b|| = ||a| - | - b|| \leq |a + b|$

3 Techniques

Infimum and supremum Let A, B be 2 sets of real numbers.
Given that $\forall a \in A, \exists b \in B, a \leq b$ we want to show $\sup A \leq \sup B$.
There are in general 2 ways to do this.

The direct way is to go from B to A. Take arbitrary $a \in A$, then $\exists b \in B, a \leq b \leq \sup B$. Then $\sup B$ is an upper bound of A, hence $\sup A \leq \sup B$. We call this going from B to A in the sense that we produce $\sup B$ before producing $\sup A$ in our equations.

The other way goes in the reverse direction. Choose arbitrary $\epsilon > 0$, and by definition of supremum, $\exists a \in A$, $\sup A - \epsilon < a \leq \sup A$. Again, there is a b such that $\sup A - \epsilon < a \leq b \leq \sup B$. Hence $\sup A - \epsilon < \sup B$. Since ϵ is arbitrary, $\sup A \leq \sup B$.

Perhaps a better mnemonic for these 2 ways is that the first goes from the *not pointy* bit of the inequality sign to the *pointy* bit.

4 Theorem Listing

4.1 Inf and sup

Scalar properties Given a bounded set $S \subset \mathbb{R}$

$$\inf(cS) = \begin{cases} c\inf(S) \text{ if } c > 0 \\ c\sup(S) \text{ if } c < 0 \end{cases}$$

$$\sup(cS) = \begin{cases} c\sup(S) \text{ if } c > 0 \\ c\inf(S) \text{ if } c < 0 \end{cases}$$

sup-inf condition Let S be a nonempty bounded subset of $\mathbb{R}$ and $K > 0$ such that $\forall s, t \in S, |s-t| \leq K$. Then $\sup(S) - \inf(S) \leq K$.

4.2 Continuity

Lipschitz property implies uniform continuity Lipschitz property: There is a constant K, such that for all x, y, $|f(x) - f(y)| \leq K|x - y|$.
It is then trivial to derive uniform continuity.

4.3 Differential Calculus

Caratheodory's Theorem Let $f : I \to \mathbb{R}$, $c \in I$. Then $f'(c)$ exists iff there is a function $\phi : I \to \mathbb{R}$ such that ϕ continuous at c and

$$\forall x \in I, f(x) - f(c) = \phi(x)(x - c)$$

When this is the case, $\phi(c) = f'(c)$.

Inverse Function Lemma Let $f : I \to \mathbb{R}$ be strictly monotone and continuous on I. Let $J = f(I)$ such that $f^{-1} : J \to \mathbb{R}$ inverts f (technically, we need to restrict the codomain of f and f^{-1} to just their range). Suppose f differentiable at c and $f'(c) \neq 0$. Then let $d = f(c)$ and

$$(f^{-1})'(d) = \frac{1}{f'(f^{-1}(d)} = \frac{1}{f'(c)}$$

Taylor's Theorem

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x - a)^k + \frac{f^{(n+1)}(c)}{(n + 1)!}(x - a)^{n+1}$$

4.4 Integral Calculus

Properties of Riemann Integral

- Linearity
- Order-preserving $f \leq g \implies \int_a^b f \leq \int_a^b g$
- f integrable implies $|f|$ integrable

- Triangle inequality
- Product of integrable functions is integrable
- Additive theorem: $\int_a^b f = \int_a^c f + \int_c^b f$

Fundamental Theorem of Calculus

FTC 2 If $f : [a, b] \to \mathbb{R}$ is integrable and f continuous at $c \in [a, b]$, then

$$\frac{d}{dx} \int_a^x f|_{x=c} = f(c)$$

FTC 1 If $g : [a, b] \to \mathbb{R}$ differentiable on $[a, b]$ and g' integrable on $[a, b]$, then

$$\int_a^b g' = g(b) - g(a)$$

Integration by parts Suppose functions $f, g : [a, b] \to \mathbb{R}$ are differentiable on $[a, b]$, and $f', g' \in R([a, b])$. Then

$$\int_a^b fg' = f(b)g(b) - f(a)g(a) - \int_a^b f'g$$

Integration by substitution Suppose $\phi : [a, b] \to I$ is differentiable on $[a, b]$ and $\phi' \in R([a, b])$. Suppose $f : I \to \mathbb{R}$ continuous on I, then

$$\int_a^b f(\phi(t))\phi'(t)\,dt = \int_{\phi(a)}^{\phi(b)} f(x)\,dx$$

Note: To do "inverse substitution", we can start from the right side and find a suitable ϕ with the above mentioned characteristics. It doesn't need to be invertible, but we need to find a, b such that $\phi(a), \phi(b)$ are the lower and upper limits on the RHS.

Taylor's Theorem Integral Form Let $f : [a, b] \to \mathbb{R}$. Suppose $\forall x \in (a, b)$, $f^(n + 1)$ exists on $[a, x]$ and $f^{(n+1)} \in R([a, x])$. Then,

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x - a)^k + \frac{1}{n!} \int_a^x f^{(n+1)}(t)(x - t)^n\,dt$$

Equivalence Theorem Let $f : [a, b] \to \mathbb{R}$ be bounded. f is Darboux integrable iff f is Riemann integrable.

Infinite Series Suppose f is Riemann/Darboux integrable and we have a sequence of partitions (P_n) of $[a, b]$ as well as accompanying choice functions γ_n such that $\lim_{n\to\infty} ||P_n|| = 0$. Then

$$\lim_{n\to\infty} S(f, P_n)(\gamma_n) = \lim_{||P||\to 0} S(f, P)(\gamma) = \int_a^b f$$

Note that the γ_n are truly arbitrary, the important thing is that $||P_n|| \to 0$

6.11 Theorem *Suppose $f \in \mathscr{R}(\alpha)$ on $[a, b]$, $m \le f \le M$, ϕ is continuous on $[m, M]$, and $h(x) = \phi(f(x))$ on $[a, b]$. Then $h \in \mathscr{R}(\alpha)$ on $[a, b]$.*

Proof Choose $\varepsilon > 0$. Since ϕ is uniformly continuous on $[m, M]$, there exists $\delta > 0$ such that $\delta < \varepsilon$ and $|\phi(s) - \phi(t)| < \varepsilon$ if $|s - t| \le \delta$ and $s, t \in [m, M]$.

Since $f \in \mathscr{R}(\alpha)$, there is a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \delta^2. \tag{18}$$

Let M_i, m_i have the same meaning as in Definition 6.1, and let M_i^*, m_i^* be the analogous numbers for h. Divide the numbers $1, \ldots, n$ into two classes: $i \in A$ if $M_i - m_i < \delta$, $i \in B$ if $M_i - m_i \ge \delta$.

For $i \in A$, our choice of δ shows that $M_i^* - m_i^* \le \varepsilon$.

For $i \in B$, $M_i^* - m_i^* \le 2K$, where $K = \sup|\phi(t)|$, $m \le t \le M$. By (18), we have

$$\delta \sum_{i \in B} \Delta\alpha_i \le \sum_{i \in B} (M_i - m_i)\, \Delta\alpha_i < \delta^2 \tag{19}$$

so that $\sum_{i \in B} \Delta\alpha_i < \delta$. It follows that

$$\begin{aligned} U(P, h, \alpha) - L(P, h, \alpha) &= \sum_{i \in A} (M_i^* - m_i^*)\, \Delta\alpha_i + \sum_{i \in B} (M_i^* - m_i^*)\, \Delta\alpha_i \\ &\le \varepsilon[\alpha(b) - \alpha(a)] + 2K\delta < \varepsilon[\alpha(b) - \alpha(a) + 2K]. \end{aligned}$$