

# ST2132

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## 1 Limit Theorems

**Properties of weak convergence** (Also known as convergence in probability)

**Additivity** Suppose  $A_n \rightarrow_P \alpha, B_n \rightarrow_P \beta$  for sequences of random variables  $(A_n), (B_n)$ . Then,

$$P(|(A_n + B_n) - (\alpha + \beta)| > \epsilon) \leq P(|A_n - \alpha| + |B_n - \beta| > \epsilon) \leq P(\{|A_n - \alpha| > \frac{\epsilon}{3}\} \cup \{|B_n - \beta| > \frac{\epsilon}{3}\}) \rightarrow 0 + 0 = 0$$

Therefore,  $A_n + B_n \rightarrow_P \alpha + \beta$ .

Here, we use the inequalities  $|A_n - \alpha| + |B_n - \beta| \geq |A_n - \alpha + B_n - \beta|$  and  $\epsilon > \frac{\epsilon}{3} + \frac{\epsilon}{3}$ .

**Closure under Continuity** Suppose  $X_n \rightarrow_P \alpha$ . Let  $g$  be a continuous function. Then in particular,  $g$  is continuous at  $\alpha$ .

For a fixed  $\epsilon$ , let  $\delta > 0$  be such that  $|x - \alpha| \leq \delta \implies |g(x) - g(\alpha)| \leq \epsilon$ . The converse then says that  $|g(x) - g(\alpha)| > \epsilon \implies |x - \alpha| > \delta$ , which is what we will use below.

$$P(|g(X_n) - g(\alpha)| > \epsilon) \leq P(|X_n - \alpha| > \delta) \rightarrow 0$$

Hence,  $g(X_n) \rightarrow_P g(\alpha)$ .

## 2 Distributions derived from Normal distribution

**t-distribution pdf derivation** First, note that  $T = \frac{Z}{\sqrt{\frac{U}{n}}} \sim t_n$ .

Let  $Y = Z, X = \sqrt{\frac{U}{n}}$ .

$$F_X(u) = P\left(\sqrt{\frac{U}{n}} \leq u\right) = P(U \leq nu^2) = F_U(nu^2)$$

Hence,

$$\begin{aligned} f_X(u) &= 2nu f_U(u) = 2nu \cdot \frac{\left(\frac{1}{2}\right)^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} (nu^2)^{\frac{n}{2}-1} e^{-\frac{nu^2}{2}} [nu^2 \geq 0] \\ &= \frac{n^{\frac{n}{2}}}{2^{\frac{n}{2}-1} \Gamma\left(\frac{n}{2}\right)} u^{n-1} e^{-\frac{nu^2}{2}} [u \geq 0] \end{aligned}$$

Note the formula for the pdf of a quotient.

$$f_{\frac{Y}{X}}(t) = \int_{\mathbb{R}} |x| f_X(x) f_Y(tx) dx$$

Hence,

$$\begin{aligned}
f_T(t) &= \int_{\mathbb{R}} |x| \frac{n^{\frac{n}{2}}}{2^{\frac{n}{2}-1} \Gamma\left(\frac{n}{2}\right)} x^{n-1} e^{-\frac{nt^2}{2}} [t \geq 0] \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(tx)^2}{2}} dx \\
&= \frac{n^{\frac{n+1}{2}}}{2^{\frac{n-1}{2}} \sqrt{n\pi} \Gamma\left(\frac{n}{2}\right)} \int_{\mathbb{R}^+} x^n e^{-\frac{(n+t^2)x^2}{2}} dx \\
&= \frac{n^{\frac{n+1}{2}}}{2^{\frac{n-1}{2}} \sqrt{n\pi} \Gamma\left(\frac{n}{2}\right)} \int_{\mathbb{R}^+} \left(\frac{2}{n+t^2} u\right)^{\frac{n}{2}} e^{-u} \frac{1}{\sqrt{n+t^2} \sqrt{2u}} du \text{ reverse substitution: } x = \sqrt{\frac{2}{n+t^2}} u \\
&= \frac{n^{\frac{n+1}{2}}}{2^{\frac{n-1}{2}} \sqrt{n\pi} \Gamma\left(\frac{n}{2}\right)} \cdot \frac{2^{\frac{n-1}{2}}}{(n+t^2)^{\frac{n+1}{2}}} \int_{\mathbb{R}^+} u^{\frac{n-1}{2}} e^{-u} du \\
&= \frac{1}{\Gamma\left(\frac{n}{2}\right) \sqrt{n\pi}} \left(\frac{n+t^2}{n}\right)^{-\frac{n+1}{2}} \int_{\mathbb{R}^+} u^{\frac{n+1}{2}-1} e^{-u} du \\
&= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right) \sqrt{n\pi}} \left(\frac{n+t^2}{n}\right)^{-\frac{n+1}{2}}
\end{aligned}$$

### Square of t-distribution

$$T^2 = \frac{Z^2}{\frac{U}{n}}$$

where  $Z^2 \sim \chi_1^2, U \sim \chi_n^2$ , hence  $T^2 \sim F_{1,n}$

### F-distribution pdf derivation Note that

$$W = \frac{\frac{U}{m}}{\frac{V}{n}}$$

where  $U \sim \chi_m^2, V \sim \chi_n^2$ .

We can derive that

$$\begin{aligned}
F_{\frac{U}{m}}(x) = F_U(mx) &\implies f_{\frac{U}{m}}(x) = m f_U(mx) \\
F_{\frac{V}{n}}(x) = F_V(nx) &\implies f_{\frac{V}{n}}(x) = n f_V(nx)
\end{aligned}$$

$$\begin{aligned}
f_W(w) &= \int_{\mathbb{R}} |x| f_X(x) f_Y(wx) dx \\
&= \int_{\mathbb{R}} |x| \cdot n f_V(nx) \cdot m f_U(mwx) dx \\
&= \int_{\mathbb{R}} |x| \cdot mn \cdot [nx \geq 0] \frac{\left(\frac{1}{2}\right)^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} (nx)^{\frac{n}{2}-1} e^{-\frac{nx}{2}} \cdot [mwx \geq 0] \frac{\left(\frac{1}{2}\right)^{\frac{m}{2}}}{\Gamma\left(\frac{m}{2}\right)} (mwx)^{\frac{m}{2}-1} e^{-\frac{mwx}{2}}
\end{aligned}$$

We consider cases.

- If  $w \leq 0$ , then  $[nx \geq 0][mwx \geq 0] = [x \geq 0][x \leq 0] = [x = 0]$ , so the integral is over a single point, hence evaluates to 0.
- Otherwise, if  $w > 0$ , then  $[nx \geq 0][mwx \geq 0] = [x \geq 0]$ , so the integral can then be restricted to  $\mathbb{R}^+$ .

Hence, in the case where  $w > 0$ , we need to evaluate

$$\begin{aligned}
f_W(w) &= \int_{\mathbb{R}^+} x \cdot mn \cdot \frac{\left(\frac{1}{2}\right)^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} (nx)^{\frac{n}{2}-1} e^{-\frac{nx}{2}} \cdot \frac{\left(\frac{1}{2}\right)^{\frac{m}{2}}}{\Gamma\left(\frac{m}{2}\right)} (mw x)^{\frac{m}{2}-1} e^{-\frac{mw x}{2}} \\
&= \frac{\left(\frac{1}{2}\right)^{\frac{m+n}{2}} (m)^{\frac{m}{2}} w^{\frac{m}{2}-1} n^{\frac{n}{2}}}{\Gamma\left(\frac{m}{2}\right) \Gamma\left(\frac{n}{2}\right)} \int_{\mathbb{R}^+} x^{\frac{m+n}{2}-1} e^{-\frac{n+mw}{2}x} dx \\
&= \frac{\left(\frac{1}{2}\right)^{\frac{m+n}{2}} \left(\frac{m}{n}\right)^{\frac{m}{2}} w^{\frac{m}{2}-1} n^{\frac{m+n}{2}}}{\Gamma\left(\frac{m}{2}\right) \Gamma\left(\frac{n}{2}\right)} \int_{\mathbb{R}^+} x^{\frac{m+n}{2}-1} e^{-\frac{n+mw}{2}x} dx \\
&= \frac{\left(\frac{1}{2}\right)^{\frac{m+n}{2}} \left(\frac{m}{n}\right)^{\frac{m}{2}} w^{\frac{m}{2}-1} n^{\frac{m+n}{2}}}{\Gamma\left(\frac{m}{2}\right) \Gamma\left(\frac{n}{2}\right)} \int_{\mathbb{R}^+} \left(\frac{2}{n+mw}u\right)^{\frac{m+n}{2}-1} e^{-u} \frac{2}{n+mw} dx \text{ Substitution: } u = \frac{n+mw}{2}x \\
&= \frac{\left(\frac{1}{2}\right)^{\frac{m+n}{2}} \left(\frac{m}{n}\right)^{\frac{m}{2}} w^{\frac{m}{2}-1} n^{\frac{m+n}{2}}}{\Gamma\left(\frac{m}{2}\right) \Gamma\left(\frac{n}{2}\right)} \int_{\mathbb{R}^+} \left(\frac{2}{n+mw}u\right)^{\frac{m+n}{2}-1} e^{-u} \frac{2}{n+mw} dx \\
&= \frac{\left(\frac{m}{n}\right)^{\frac{m}{2}} w^{\frac{m}{2}-1} n^{\frac{m+n}{2}}}{\Gamma\left(\frac{m}{2}\right) \Gamma\left(\frac{n}{2}\right)} \int_{\mathbb{R}^+} \left(\frac{1}{n+mw}u\right)^{\frac{m+n}{2}-1} e^{-u} \frac{1}{n+mw} dx \\
&= \frac{\left(\frac{m}{n}\right)^{\frac{m}{2}} w^{\frac{m}{2}-1}}{\Gamma\left(\frac{m}{2}\right) \Gamma\left(\frac{n}{2}\right)} \left(\frac{n+mw}{n}\right)^{-\frac{m+n}{2}} \int_{\mathbb{R}^+} u^{\frac{m+n}{2}-1} e^{-u} du \\
&= \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right) \Gamma\left(\frac{n}{2}\right)} \left(\frac{m}{n}\right)^{\frac{m}{2}} w^{\frac{m}{2}-1} \left(1 + \frac{m}{n}w\right)^{-\frac{m+n}{2}}
\end{aligned}$$

### 3 Cheatsheet

Copy this on paper for the finals.

#### Double exponential distribution facts

- If  $X \sim DoubleExp(\lambda)$ , then  $|X| \sim Exp(\lambda)$

#### Gamma distribution facts

- Sum: Given  $X \sim Gamma(\alpha_1, \lambda)$ ,  $Y \sim Gamma(\alpha_2, \lambda)$ ,  $X + Y \sim Gamma(\alpha_1 + \alpha_2, \lambda)$
- Scaling: Given  $X \sim Gamma(\alpha, \lambda)$ ,  $c \in \mathbb{R}^+$ ,  $cX \sim Gamma(\alpha, \frac{\lambda}{c})$

Density of quotient of random variables TODO