MA1101R Pointers

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1 Sketchpad

$$\begin{pmatrix} \frac{3^n}{4} - \frac{(-1)^n}{4} \\ \frac{(-1)^n}{4} + \frac{3\,3^n}{4} \end{pmatrix}$$

$$Toinput$$

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2 Assumptions

Unless otherwise stated:

- For clarity, we shall assume all Euclidean spaces to be $\mathbb{R}^n$.
- All vectors will be treated as column vectors.

3 Linear Systems

4 Matrices

4.1 Common proof ideas

- Basic trace properties
  1. $tr(A + B) = tr(A) + tr(B)$
  2. $tr(cA) = ctr(A)$
  3. $tr(AB) = tr(BA)$
  4. Cyclic property (Generalisation of property 3)
     $tr(ABCD) = tr(BCDA) = tr(CDAB) = tr(DABC)$
- If matrix A is symmetric, then $A^T = A$.

5 Linear independence and Dimension

6 Row, column, null spaces

7 Orthogonality

7.1 Things to note

- Orthogonal sets can contain the zero vector. An orthogonal set that has no zero vector is linearly independent.
- An orthonormal set is always linearly independent since the norm of $\mathbf{0}$ is 0.

7.2 Common proof ideas

- A subspace has an orthogonal/orthonormal basis.
- Suppose W is a subspace and $W^\perp$ is its orthogonal complement. Then the union of the bases of W, $W^\perp$ is a basis for $\mathbb{R}^n$. To see this, take their orthogonal bases.
- The columns and rows of an orthogonal matrix form orthonormal sets.

8 Diagonalisation

8.1 Things to note

- Eigenvectors are non-zero vectors, but the zero vector is an element of every eigenspace.
  - Hence, suppose we given that $Ax = \lambda \mathbf{x}$. For $\mathbf{x}$ to be an eigenvector of A, we must first show that $\mathbf{x} \neq \mathbf{0}$
  - To show that $k\mathbf{x} \neq \mathbf{0}$, we need 2 conditions: $k \neq 0$ and $\mathbf{x} \neq \mathbf{0}$.
- The eigenspace E_0 of matrix A associated with eigenvalue 0, is also the nullspace of A.

8.2 Common proof ideas

- A diagonalisable matrix A has n linearly independent eigenvectors that **form a basis** for $\mathbb{R}^n$.
- A diagonalisable matrix A can be written as the product PDP^{-1}.
- A symmetric matrix is orthogonally diagonalisable. In particular, it is diagonalisable.
- Powers of matrices. If a matrix A is diagonalisable, and v_i is the eigenvector associated with eigenvalue λ_i, then $A\mathbf{v}_i = \lambda \mathbf{v}_i$
- Trace of a diagonal matrix:

$$\begin{aligned} tr(D) &= \sum_i \lambda_i \\ &= \sum_\lambda \lambda \dim(E_\lambda) \end{aligned}$$

- The intersection of distinct eigenspaces only contains the zero vector. i.e. $E_i \cap E_j = \{\mathbf{0}\}, i \neq j$. A consequence of this allows us to write:

$$\dim(E_i + E_j) = \dim(E_i) + \dim(E_j) - \dim(E_i \cap E_j) = \dim(E_i) + \dim(E_j)$$

9 Linear transformations