MA3210 (Analysis II)

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1 Definitions

Sets

$$\mathbb{N} = \mathbb{Z}^+$$

2 Inequalities

- $|a + b| \leq |a| + |b|$
- $||a| - |b|| \leq |a - b|$
- $||a| - |b|| = ||a| - |-b|| \leq |a + b|$

3 Techniques

Infimum and supremum Let A, B be 2 sets of real numbers.
Given that $\forall a \in A, \exists b \in B, a \leq b$ we want to show $\sup A \leq \sup B$.
There are in general 2 ways to do this.

The direct way is to go from B to A. Take arbitrary $a \in A$, then $\exists b \in B, a \leq b \leq \sup B$. Then $\sup B$ is an upper bound of A, hence $\sup A \leq \sup B$. We call this going from B to A in the sense that we produce $\sup B$ before producing $\sup A$ in our equations.

The other way goes in the reverse direction. Choose arbitrary $\epsilon > 0$, and by definition of supremum, $\exists a \in A$, $\sup A - \epsilon < a \leq \sup A$. Again, there is a b such that $\sup A - \epsilon < a \leq b \leq \sup B$. Hence $\sup A - \epsilon < \sup B$. Since ϵ is arbitrary, $\sup A \leq \sup B$.

Perhaps a better mnemonic for these 2 ways is that the first goes from the *not pointy* bit of the inequality sign to the *pointy* bit.

4 Theorem Listing

4.1 Inf and sup

Scalar properties Given a bounded set $S \subset \mathbb{R}$

$$\inf(cS) = \begin{cases} c\inf(S) \text{ if } c > 0 \\ c\sup(S) \text{ if } c < 0 \end{cases}$$

$$\sup(cS) = \begin{cases} c\sup(S) \text{ if } c > 0 \\ c\inf(S) \text{ if } c < 0 \end{cases}$$

sup-inf condition Let S be a nonempty bounded subset of $\mathbb{R}$ and $K > 0$ such that $\forall s, t \in S, |s-t| \leq K$. Then $\sup(S) - \inf(S) \leq K$.

4.2 Continuity

Lipschitz property implies uniform continuity Lipschitz property: There is a constant K, such that for all x, y, $|f(x) - f(y)| \leq K|x - y|$.
It is then trivial to derive uniform continuity.

4.3 Differential Calculus

Caratheodory's Theorem Let $f : I \to \mathbb{R}$, $c \in I$. Then $f'(c)$ exists iff there is a function $\phi : I \to \mathbb{R}$ such that ϕ continuous at c and

$$\forall x \in I, f(x) - f(c) = \phi(x)(x - c)$$

When this is the case, $\phi(c) = f'(c)$.

Inverse Function Lemma Let $f : I \to \mathbb{R}$ be strictly monotone and continuous on I. Let $J = f(I)$ such that $f^{-1} : J \to \mathbb{R}$ inverts f (technically, we need to restrict the codomain of f and f^{-1} to just their range). Suppose f differentiable at c and $f'(c) \neq 0$. Then let $d = f(c)$ and

$$(f^{-1})'(d) = \frac{1}{f'(f^{-1}(d)} = \frac{1}{f'(c)}$$

Taylor's Theorem

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x - a)^k + \frac{f^{(n+1)}(c)}{(n + 1)!}(x - a)^{n+1}$$

4.4 Integral Calculus

Properties of Riemann Integral

- Linearity
- Order-preserving $f \leq g \implies \int_a^b f \leq \int_a^b g$
- f integrable implies $|f|$ integrable

- Triangle inequality
- Product of integrable functions is integrable
- Additive theorem: $\int_a^b f = \int_a^c f + \int_c^b f$

Fundamental Theorem of Calculus

FTC 2 If $f : [a, b] \to \mathbb{R}$ is integrable and f continuous at $c \in [a, b]$, then

$$\frac{d}{dx} \int_a^x f|_{x=c} = f(c)$$

FTC 1 If $g : [a, b] \to \mathbb{R}$ differentiable on $[a, b]$ and g' integrable on $[a, b]$, then

$$\int_a^b g' = g(b) - g(a)$$

Integration by parts Suppose functions $f, g : [a, b] \to \mathbb{R}$ are differentiable on $[a, b]$, and $f', g' \in R([a, b])$. Then

$$\int_a^b fg' = f(b)g(b) - f(a)g(a) - \int_a^b f'g$$

The antiderivative version: Given the same conditions, since $f', g' \in R([a, b])$, $fg', f'g$ both integrable, in particular, their antiderivative exists. This allows us to write

$$\int fg' = fg - \int f'g$$

Integration by substitution Suppose $\phi : [a, b] \to I$ is differentiable on $[a, b]$ and $\phi' \in R([a, b])$. Suppose $f : I \to \mathbb{R}$ continuous on I, then

$$\int_a^b f(\phi(t))\phi'(t)\, dt = \int_{\phi(a)}^{\phi(b)} f(x)\, dx$$

Note: To do "inverse substitution", we can start from the right side and find a suitable ϕ with the above mentioned characteristics. It doesn't need to be invertible, but we need to find a, b such that $\phi(a), \phi(b)$ are the lower and upper limits on the RHS.

Taylor's Theorem Integral Form Let $f : [a, b] \to \mathbb{R}$. Suppose $\forall x \in (a, b)$, $f^(n + 1)$ exists on $[a, x]$ and $f^{(n+1)} \in R([a, x])$. Then,

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x - a)^k + \frac{1}{n!} \int_a^x f^{(n+1)}(t)(x - t)^n\, dt$$

Equivalence Theorem Let $f : [a, b] \to \mathbb{R}$ be bounded. f is Darboux integrable iff f is Riemann integrable.

Infinite Series Suppose f is Riemann/Darboux integrable and we have a sequence of partitions (P_n) of $[a, b]$ as well as accompanying choice functions γ_n such that $\lim_{n\to\infty} ||P_n|| = 0$. Then

$$\lim_{n\to\infty} S(f, P_n)(\gamma_n) = \lim_{||P||\to 0} S(f, P)(\gamma) = \int_a^b f$$

Note that the γ_n are truly arbitrary, the important thing is that $||P_n|| \to 0$

6.11 Theorem *Suppose $f \in \mathscr{R}(\alpha)$ on $[a, b]$, $m \leq f \leq M$, ϕ is continuous on $[m, M]$, and $h(x) = \phi(f(x))$ on $[a, b]$. Then $h \in \mathscr{R}(\alpha)$ on $[a, b]$.*

Proof Choose $\varepsilon > 0$. Since ϕ is uniformly continuous on $[m, M]$, there exists $\delta > 0$ such that $\delta < \varepsilon$ and $|\phi(s) - \phi(t)| < \varepsilon$ if $|s - t| \leq \delta$ and $s, t \in [m, M]$.

Since $f \in \mathscr{R}(\alpha)$, there is a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \delta^2. \tag{18}$$

Let M_i, m_i have the same meaning as in Definition 6.1, and let M_i^*, m_i^* be the analogous numbers for h. Divide the numbers $1, \ldots, n$ into two classes: $i \in A$ if $M_i - m_i < \delta$, $i \in B$ if $M_i - m_i \geq \delta$.

For $i \in A$, our choice of δ shows that $M_i^* - m_i^* \leq \varepsilon$.

For $i \in B$, $M_i^* - m_i^* \leq 2K$, where $K = \sup|\phi(t)|$, $m \leq t \leq M$. By (18), we have

$$\delta \sum_{i\in B} \Delta\alpha_i \leq \sum_{i\in B} (M_i - m_i)\,\Delta\alpha_i < \delta^2 \tag{19}$$

so that $\sum_{i\in B} \Delta\alpha_i < \delta$. It follows that

$$\begin{aligned} U(P, h, \alpha) - L(P, h, \alpha) &= \sum_{i\in A} (M_i^* - m_i^*)\,\Delta\alpha_i + \sum_{i\in B} (M_i^* - m_i^*)\,\Delta\alpha_i \\ &\leq \varepsilon[\alpha(b) - \alpha(a)] + 2K\delta < \varepsilon[\alpha(b) - \alpha(a) + 2K]. \end{aligned}$$

4.5 Series of functions

4.6 Power series

Radius of convergence Lemma. Let $\sum_n a_n(x-x_0)^n$ be convergent at x_1. Then $|x-x_0| < |x_1-x_0|$ implies $\sum_n a_n(x-x_0)^n$ converges absolutely.

- It is due to this lemma that we can speak of radius of convergence. Because of this lemma, radius of convergence as a concept can exist independently of the root test.
- However, the root test is a convenient way to find the radius of convergence since lim sup is guaranteed to exist.
- Additionally, the root test itself can be used to prove the above lemma.

- Finally, we also note that the above lemma and the ratio test are related since both of their proofs make use of the geometric series.

Ratio test Given $\sum_n a_n$, consider $L_n = |\frac{a_{n+1}}{a_n}|$

- if $\lim_{n\to\infty} L_n < 1$, converges
- if $\lim_{n\to\infty} L_n > 1$, diverges
- if $\lim_{n\to\infty} L_n = 1$, indeterminate

Ratio test (variants)

- if $\limsup_{n\to\infty} L_n < 1$, converges
- if $\liminf_{n\to\infty} L_n > 1$, diverges
- if $L_n \geq 1$ for all but finitely many n, diverges

Note: If $L_n \geq 1$ for infinitely many n, the series can still converge.

Property of limsup Given that (a_n) converges,

$$\limsup_{n\to\infty} a_n b_n = \lim_{n\to\infty} a_n \cdot \limsup_{n\to\infty} b_n$$

Uniform convergence of power series Let $\sum_n a_n(x-x_0)^n$ have radius of convergence $R > 0$. Then for any $[a,b] \subseteq (x_0 - R, x_0 + R)$, the series converges uniformly on $[a,b]$.

This property of power series means we get some extent of uniform convergence for free.

Differentiability of power series Let $f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$ have radius of convergence $R > 0$. Then $f \in C^\infty(x_0 - R, x_0 + R)$. And $\forall k \in \mathbb{Z}_0^+$

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n^{\underline{k}}(x-x_0)^{n-k}$$

on $(x_0 - R, x_0 + R)$. In particular, the radius of convergence of $f^{(k)}$ is R, though the domain of convergence can be any of $[x_0 - R, x_0 + R], (x_0 - R, x_0 + R], [x_0 - R, x_0 + R), (x_0 - R, x_0 + R)$.

The proof is by taking the union over any $[x_0 - r, x_0 + r], r < R$ to give differentiability over the entire $(x_0 - R, x_0 + R)$.

Note: Properties like continuity and differentiability can be generalized by union.

Uniqueness of power series Suppose a function f, there is a power series such that $f(x) = \sum_n a_n(x-x_0)^n$ on $(x_0 - r, x_0 + r)$. Here, r must clearly be $\leq$ radius of convergence R. Then, $\forall k \in \mathbb{Z}_0^+$

$$f^{(k)}(x_0) = k^{\underline{k}} a_k = k! a_k$$

, that is,

$$a_k = \frac{f^{(k)}(x_0)}{k!}$$

Hence $\sum_n a_n(x-x_0)^n = \sum_n b_n(x-x_0)^n$ on some on $(x_0 - r, x_0 + r), r > 0$ implies $\forall n, a_n = b_n$.

Summation by parts Given sequences (b_n, c_n). Let $B_{n,m} = \sum_{m \le k \le n} b_k$. Then,

$$\sum_{m \le k \le n} b_k c_k = B_{n,m} c_n + \sum_{m \le k \le n-1} B_{k,m}(c_k - c_{k+1})$$

This can be used to prove both Dirichlet's and Abel's test.

Corollary to Abel's Theorem Suppose

$$f(x) = \sum_n a_n (x - x_0)^n$$

on $(x_0 - R, x_0 + R)$ and the power series converges at $x = x_0 + R$.

Here, we can view f as the closed form. Let g represent the power series. Abel's theorem says that g is defined on $(x_0 - R, x_0 + R]$ and the convergence to g is uniform on $[x_0, x_0 + R]$. In particular, g is continuous at $x_0 + R$. Hence, we have,

$$g(x_0 + R) = \lim_{x \to (x_0+R)^-} g(x) = \lim_{x \to (x_0+R)^-} f(x)$$

If we extend f to $(x_0 - R, x_0 + R]$ (assuming that the closed form f is defined at $x_0 + R$), and supposing f is also continuous at $x_0 + R$, then we have $g(x_0 + R) = f(x_0 + R)$. This allows us to equate

$$f(x_0 + R) = g(x_0 + R) = \sum_n a_n R^n$$

Merten's Theorem Given $(a_n), (b_n)$, suppose $\sum_n a_n$ converges absolutely and $\sum_n b_n$ converges. Let $c_n = \sum_{0 \le k \le n} a_k b_{n-k}$. Then,

$$\sum_{n \ge 0} c_n = \sum_{n \ge 0} a_n \sum_{n \ge 0} b_n$$

Corollary. We adjust the starting indices of $(a_n), (b_n)$. Suppose $\sum_{n \ge N_1} a_n$ converges absolutely and $\sum_{n \ge N_2} b_n$ converges. Then,

$$\sum_{n \ge N_1 + N_2} \sum_{N_1 \le k \le n - N_2} a_k b_{n-k} = \sum_{n \ge N_1} a_n \sum_{n \ge N_2} b_n$$

The proof is by defining $a_k, k < N_1$ and $b_k, k < N_2$ to be 0. Hence,

$$\sum_{n \ge N_1} a_n \sum_{n \ge N_2} b_n = \sum_{n \ge 0} \sum_{0 \le k \le n} a_k b_{n-k} a_k b_{n-k} [k \ge N_1][n - k \ge N_2]$$

and observing that $[k \ge N_1][n - k \ge N_2] = [k \ge N_1][n - k \ge N_2][k + n - k \ge N_1 + N_2] = [k \ge N_1][n - k \ge N_2][n \ge N_1 + N_2]$.

Application. Consider a power series $\sum_n a_n (x - x_0)^n$ with radius of convergence R. Then if $x \in (x_0 - R, x_0 + R)$, $\sum_n a_n (x - x_0)^n$ converges absolutely. This allows us to apply Merten's theorem when concerning products of power series.

Analytic functions A function f is analytic on (a, b) if

1. $f \in C^\infty(a, b)$
2. $\forall x_0 \in (a, b)$, f is equal to its Taylor series with basepoint at x_0 in some neighborhood of x_0.

Lemma. If for a certain $x_0 \in (a, b)$, f equals to its Taylor series with basepoint at x_0 over (a, b), then f is analytic on (a, b).

4.7 Special functions

Characterization of trig functions If $g : \mathbb{R} \to \mathbb{R}$ has the property, $\forall x \in \mathbb{R}$

$$g''(x) = -g(x)$$

, then $g(x) = g(0)\cos(x) + g'(0)\sin(x)$.

Characterization of exponential functions If E is a real function such that

$$E'(x) = E(x) \wedge E(0) = 1$$

on $\mathbb{R}$, then $E = \exp$.