CS3231 Tut Qns

Jia Cheng

August 2021

Ex 2.18 Assume that (N, Σ, P, S) is a regular grammar and h is a constant such that $|N| < h$ and $\forall A \to wB, A \to w \in P$ where $A, B \in N, w \in \Sigma^*$, we have $|w| < h$. Claim: theorem 2.15(a) holds with pumping constant $k := h^2$.

Definition Suppose $A_{i_1} \to A_{i_2}, A_{i_2} \to A_{i_3}, \ldots, A_{i_l} \to A_{i_1}$ are some rules in P with $i_1, i_2, \ldots, i_l$ distinct. Then these rules form an **atomic cycle** starting at A_{i_1}. Since $i_1, i_2, \ldots, i_l$ are distinct there is no sub-cycle.

Lemma 1 The longest atomic cycle in A is of length $|N|$. This follows since $i_1, i_2, \ldots, i_l$ as mentioned above cannot be such that $l > |N|$.

Lemma 2 The longest word u without a cycle is $< h^2$ in length. This can be proven from the observation that u could only have been generated by at most $|N| - 1$ rules of the form $A \to wB$ and 1 rule of the form $A \to w$. Altogether, there are $|N|$ rules, each rule adding $|w|$ (which may vary for each rule) to the length of u. In total, $|u| \le |w|h < h^2$.

Let $u \in L, |u| > h^2$. By Lemma 2, $u[1..h^2]$, i.e. the first h^2 elements of u must have at least 1 cycle. Hence suppose this cycle occurs at $u[i..j], i < j \le h^2$, in other words, the substring $u[i], u[i+1], \ldots u[j]$ was generated by a cycle $A_{i_1} \to A_{i_2}, A_{i_2} \to A_{i_3}, \ldots, A_{i_l} \to A_{i_1}$.

Split u into xyz, where $x := u[1..i-1]$, $y := u[i..j]$, $z := u[j+1..|u|]$, then $xy^*z \subseteq L$. Furthermore, $|xy| = j \le h^2$. This proves the claim.

Ex 2.22

For L observe that $\{0^n 1^m 2^k : n < m \vee m < k\} = \{0^n 1^m 2^k : n < m\} \cup \{0^n 1^m 2^k : m < k\} = L_1 \cup L_2$.

$$L_1(\{S, T\}, \Sigma, \{S \to S2|T1, T \to 0T1|T1|\epsilon\}, S)$$
$$L_2(\{S, T\}, \Sigma, \{S \to 0S|T2, T \to 1T2|T2|\epsilon\}, S)$$

The generator for L can then formed from the generators of L_1, L_2 respectively, using the format $(\{S\} \cup N_1 \cup N_2, \Sigma, \{S \to S1|S2\} \cup P_1 \cup P_2, S)$.

For H notice that we can pair the 0s and the outer 2s, then the 1s and the inner 2s.

$$H(\{S, T\}, \Sigma \to 0S2|T, T \to 1T2|\epsilon, S)$$

For K notice that we are free to do whatever we want with the part of K that isn't the subword $20^n 1^n 2$. So we generate the other parts of the word of K first, then finally we generate the subword

20^n1^n2.

$$K(\{S,T\},\Sigma,\{S\to 0S|S0|1S|S1|2S|S2|2T2, T\to 0T1|01\},S)$$

L fails 2.15(a). Suppose λ is a pumping constant, then we can set

- $n := \lambda+1$
- $m := \lambda+2$
- $k := \lambda+1$

In particular, $n < m, m \geq k$. Since $n > \lambda$, we can only pump the 0s, but then this will eventually cause $n \geq m, m \geq k$, which no longer belongs in L.

L satisfies 2.16. As long as the word $w \in L$ is non-empty, there are 2 cases.
$w \in L_1$. Then we can just pump the 1s.
If $w \in L_2$, then we can pump the 2s.

H fails 2.15(a). For a non-empty word w with 0s, pumping the 0 would lead to no. of 0s + no. of 1s > no. of 2s.

H satisfies 2.16. Let $w \in K$, w non-empty, then there are 2 cases.
If $m > 0$, then we can pump the subword 12.
If $m = 0$, then $n > 0$ and w is of the form 0^n2^n hence we can pump $m = 0$.

K satisfies 2.15(a) and hence 2.16 as well. Since $w \in K$ implies w must have an occurrence of the subword 20^n1^n2, it is always safe to pump the 2.

2.23

Suppose k is the pumping constant. Then considering

$$\text{word } u := 0^k1^k2^k$$

it is easy to see that however we split u into the form $vwxyz$, where $|wxy| \leq k$, we cannot have $vw^nxy^nz \in L$, since wxy cannot contain both 0s and 2s at the same time.

3.22

Assume the alphabet Σ has 5000 elements. Define a language, $L \subseteq \Sigma^*$ such that Jaffe's Matching Pumping Lemma is satisfied with constant $k = 3$ while every dfa recognising L has more than 5000 states.

Let $\Sigma = \{a_i : i \in \{1,\ldots,5000\}\}$
Let $S_i = a_{2i-1}a_{2i}^+, i = 1,\ldots,2500$.
Let $L = \{S_1,\ldots,S_{2500}\}^*$. Hence L is regular.

Claim Every dfa recognising L has > 5000 states, i.e. $Q > 5000$.

Denote $S := \{\epsilon, S_1,\ldots,S_{2500}\}^*$

$$\begin{aligned} L_\epsilon = SL_{a_{2i-1}} &= a_{2i}^+\cdot S \\ L_{a_{2i-1}a_{2i}} &= a_{2i}^*\cdot S \end{aligned}$$

Clearly, we have shown that there are at least 5001 pairwise distinct derivatives.

$Q \geq |\{L_x : x \in \Sigma\}| \geq 5001 > 5000$ The first inequality is a consequence of Nerode's theorem.

Claim Jaffe's Matching Pumping Lemma is satisfied with constant $k = 3$.

Let $x \in \Sigma^*, y \in \Sigma^k = \Sigma^3$, then $y = y_1 y_2 y_3$, where $|y_i| = 1$.
Consider 2 cases.

Case 1: Suppose each adjacent pair $y_i, y_{i+1}, i = 1, 2$ are distinct. If xy is valid, i.e. $L_{xy} \neq \emptyset$, then $\exists i \in \{2, 3\}, y_i = a_{2j}$ for some $j \in \{1, \ldots, 2500\}$. Then, $\forall h \in \mathbb{N}, L_{xy} = L_{xy_1(y_2y_3)^h}$.

If $L_{xy} = \emptyset$, then there must be a violation of the property that for some 2 adjacent alphabets a, b in xy.

- If this pair lies fully in y, then $L_{xy^h} = \emptyset = L_{xy}$.
- If this pair lies fully in y, i.e. $a = y_i, b = y_{i+1}, i \in \{1, 2\}$. Then we have $a = a_{2i-1}, b \neq a_{2i}$ or $b = a_{2i}, a \neq a_{2l-1}$ for any l. Then let $j \in \{1, 2, 3\} \setminus \{i, i+1\}$, then we can pump y_j. i.e. pump the entry that isn't part of this offending pair.
- If only b lies in y, then we must have $y_1 = b$. Pumping y_2 or y_3 or both would do the job.

Case 2: Suppose there is at least one adjacent pair that are equal, i.e. $y_i = y_{i+1}, i \in \{1, 2\}$.

If $L_{xy} \neq \emptyset$, then we can pump y_i or y_{i+1} since $y_i = a_{2j}$ for some j.

If $L_{xy} = \emptyset$, then there must be a violation of the property that for some 2 adjacent alphabets a, b in xy. See case 1 as the argument is the same.

Hence we have proven that $k = 3$ is a pumping constant.

4.8

Find a characterisation when a regular language L is recognised by an nfa only having accepting states.

Proposed characterisation: $\forall w \in L$, where $w = a_1 \ldots a_n, n \in \mathbb{N}$, (note that $a_1..a_0 = \epsilon$), we have $\forall i \in \{1, \ldots, n-1\} a_1..a_i \in L$.

Suppose a regular language L is recognized by nfa M with only accepting states. Let $w = a_1 \ldots a_n \in L, n \in \mathbb{N}$. Then let $q_0 \ldots q_n$ be any accepting run of w, where $\delta(q_i, a_{i+1}) = q_{i+1}$. By assumption of M, all the q_i are accepting. This means that the word $a_1 \ldots a_i$ is accepted by L since $\delta(s, a_1 \ldots a_i) = \delta(q_0, a_1 \ldots a_i) = q_i \in F$. Note that $\delta(s, \epsilon) = \delta(q_0, \epsilon) = q_0 \in F$.

Conversely, suppose L has the proposed characterisation. Since L is regular, the set of derivatives of L is finite. And each non-empty derivative also has the element ϵ. Consider the minimal dfa $M = (Q, \Sigma, \delta, s, F)$ recognizing L with states as the derivatives. We can trivially create a corresponding nfa M' as follows:

$$\begin{aligned} Q' &= Q \setminus \emptyset \\ s' &= s \\ F' &= F = Q' \\ \delta'(q, a) &= \{\delta(q, a)\} \quad \forall \delta(q, a) \in F \end{aligned}$$

With this construction, we have an nfa with only accepting states recognizing L.

Addendum $s \neq \emptyset$, such that $s' \in Q'$. As a result, $\epsilon \in M'$.

5.21 Given a context-free grammar for a language L, is there also one for $L \cap L^{\mathrm{mi}}$?

Let $L := L_1 \cup L_2$, where $L_1 := \{0^m 1^n 2^n : m, n \in \mathbb{N}\}, L_2 := \{2^m 1^n 0^n : m, n \in \mathbb{N}\}$. We claim this is a counterexample.

Claim $L \cap L^{mi} = \{0^n 1^n 2^n, 2^n 1^n 0^n : n \in \mathbb{N}\}$

We first note that $w^{mi} \in L \iff w \in L \cap L^{mi}$.
Suppose $w \in L$, there are 2 cases.

- Case 1: $w \in L_1$, i.e. $w = 0^m 1^n 2^n$ for some m, n. Then $w^{mi} = 2^n 1^n 0^m \in L \iff w^{mi} \in L_2 \iff m = n$.
- Case 2: $w \in L_2$, i.e. $w = 2^m 1^n 0^n$ for some m, n. Then $w^{mi} \in L \iff w^{mi} \in L_1 \iff m = n$.

Claim L is context free.

Consider the grammar $G_1 = (\{S, T\}, \{0, 1, 2\}, \{S \to 0S|T, T \to 1T2|\epsilon\}, S)$. This is a context free grammar G_2 that generates L_1. We can similarly create a context free grammar generating L_2. L being the union of the 2 context free languages, is also context free.

Finally, $L \cap L^{mi}$ is not context free as it fails pumping lemma $2.15(b)$. Hence, we have established a counterexample.

Alternatively, $L \cap L^{mi}$ is not context-free: Consider the intersection with $0, 1, 2^*$. This must be then also context-free, but it is exactly the set of all $0^n 1^n 2^n$.

5.35 Construct a context-sensitive language L and a homomorphism h such that L has polynomial growth and $h(L)$ has exponential growth.

Intuition: It is well known that $\{u, v\}^*$ has exponential growth, if $uv \neq vu$. Now, we want $h(L)$ to be something of this form. But how do we make L itself polynomial? The idea here is that the homomorphism h helps to cut down on certain characters, e.g. mapping them to ϵ.

Hence we have the following construction:

$$L = \bigcup_{n \in \mathbb{N}} \{0, 1\}^n \cdot 2^{2^n}$$

This has polynomial growth as the 2^{2^n} is "limiting the potential" of $\{0, 1\}^n$. If we let $h(0) = 0, h(1) = 1, h(2) = \epsilon$, then we have

$$h(L) = \bigcup_{n \in \mathbb{N}} \{0, 1\}^n = \{0, 1\}^*$$

Related question

Self-test 8.35 Construct a homomorphism h and a context-free set L of exponential growth such that h(L) has polynomial growth and is not regular.

Intuition: We are going the opposite way. Now we want to reduce the growth.

$$L = \{2^n 3^n : n \in \mathbb{N}\} \cdot \{0, 1\}^*$$

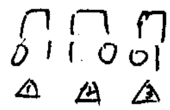
Note that L is the right concatenation of a deterministic context free language with a regular language, so L is DCFL, in particular, L is context free.

Since $\{0, 1\}^* \subseteq L$, L has exponential growth.

Let $h(0) = h(1) = \epsilon, h(2) = 2, h(3) = 3$, and we have $h(L) = \{2^n 3^n : n \in \mathbb{N}\}$, which has polynomial growth.

6.25 

Having 4 ~~Sae~~ Subcase of 6.19.

6.19. 

Consider the following cases:

① $\triangle_1 \geq \triangle_2 \geq \triangle_3$

(By symmetry $\triangle_3 \geq \triangle_2 \geq \triangle_1$

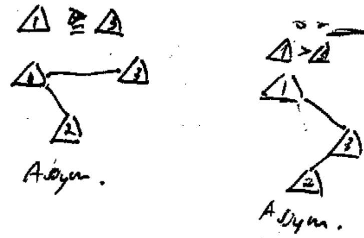
$\triangle_2 > \triangle_3$

$3 \times 2 = 6$

② $\triangle_1 \geq \triangle_3 \geq \triangle_2$

(By sym, ~~$\triangle_1 \geq \triangle_2 \geq \triangle_3$~~ or $\triangle_3 \geq \triangle_1 \geq \triangle_2$)

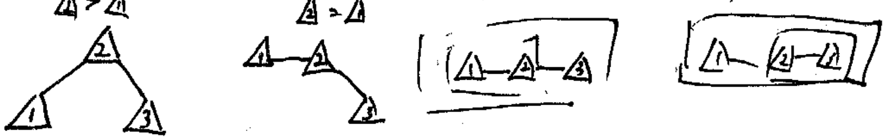
$\triangle_1 \geq \triangle_3$ or $\triangle_1 > \triangle_2$



$2 \times 2 = 4$

③ $\triangle_2 \geq \triangle_1 \geq \triangle_3$ and (By symmetry $\triangle_2 \geq \triangle_3 \geq \triangle_1$)

$\triangle_2 > \triangle_1$ $\triangle_3 = \triangle_1$

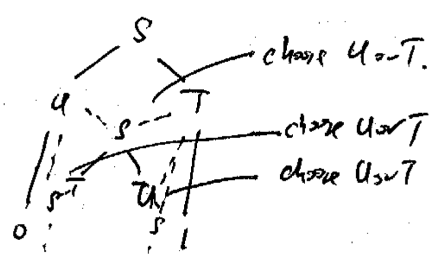


A-Symmetry. 

5 cases

$6+4+5 = 15$ cases.

 4 cases.



~~2 choice~~

$2 \times 2 = 4$.

6.4

$$
\begin{aligned}
&(\{S, S_1, S_2, S_3, T, U, V, W\}, \{0,1\}, P, S)\\
&T \to UV|UW\\
&V \to TW\\
&U \to 0\\
&W \to 1\\
&S \to \epsilon|TS_1|UV|UW\\
&S_1 \to TS_2|UV|UW\\
&S_2 \to TS_3|UV|UW\\
&S_3 \to UV
\end{aligned}
$$

This can be simplified for e.g. by replacing S_3 with T.

Additional note: $T \to \epsilon$ cannot be added since it goes against Chomsky NF's requirements, instead, adding $T \to UW$ is actually equivalent, since the equivalent of T generating ϵ, is to simply generate fewer T's.

6.5

$$
\begin{aligned}
U &\to 0\\
W &\to 1\\
V &\to TW\\
T &\to UW|UV\\
S &\to UW|UV|ST\\
S' &\to UW|UV|ST
\end{aligned}
$$

6.6

$$
\begin{aligned}
U &\to 0\\
V &\to 1\\
S &\to US_1\\
S_1 &\to VS_2\\
S_2 &\to VU\\
S &\to UT_1\\
T_1 &\to ST_2\\
T_2 &\to ST_3\\
T_3 &\to VT_4\\
T_4 &\to VT_5\\
T_5 &\to ST_6\\
T_6 &\to SU\\
S' &\to US_1|UT_1
\end{aligned}
$$

7.11 L is not linear. By elimination, H, K must be linear.

A linear grammar for H is as follows.

$$S \to 0S2|1S2|\epsilon$$

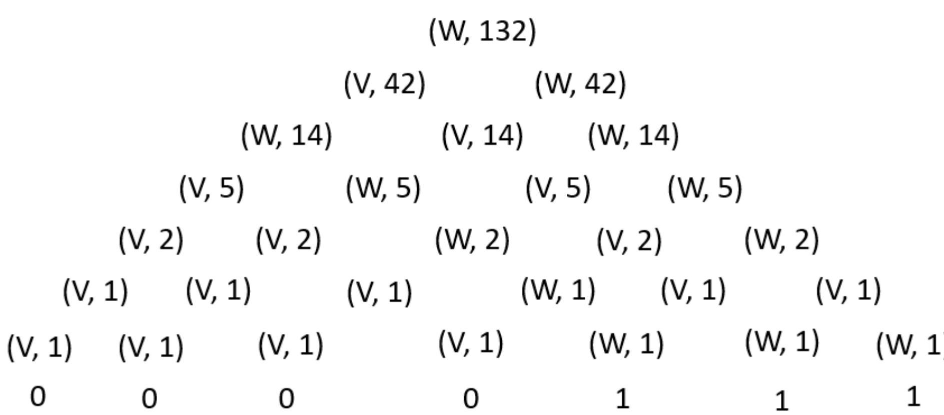
A linear grammar for K is as follows

$$\begin{aligned} S &\to S_1|S_2 \\ S_1 &\to S_1 2|T_1|T_2 \\ S_2 &\to 0S_2|T_3|T_4 \\ T_1 &\to 1|T_1 1|0T_1 1 \\ T_2 &\to 0|0T_2|0T_2 1 \\ T_3 &\to 2|T_3 2|1T_3 2 \\ T_4 &\to 1|1T_4|1T_4 2 \end{aligned}$$

Reasoning for K,

- $S_1 : n \neq m$
- $S_2 : m \neq k$
- $T_1 : n < m$. To be precise, T_1 generates $\{0^n 1^m : n < m\}$.
- $T_2 : n > m$
- $T_3 : m < k$
- $T_4 : m > k$

7.18 We count the derivation tree as follows. Since $(W, 132)$ is non-zero at the tree root, we can have 132 derivation trees of 0000111.



7.23 Proof by induction on n.

We need to show $(k-1)^{n+1} - 1 \leq k^n(k-2)$.

For instance, the base case where $n = 1$, is shown by $(k-1)^2 - 1 = k^2 - 2k + 1 - 1 = k^2 - 2k = k(k-2) \leq k(k-2)$.

Alternatively by binomial expansion, we get

$$k^n = (k-1+1)^n = \sum_{0 \leq i \leq n} \binom{n}{i} (k-1)^i \geq \sum_{0 \leq i \leq n} (k-1)^i = (k-1)^{n+1} - 1$$

7.24 The algorithm is very similar, but instead of returning 1 or 0, the algo returns the length of the shortest derivation.
Note that Min refers to the algo, whereas min is the mathematical set operation.

$$Min(u, v, t) := \min(\{Min(u, u', t') + Min(u', v, t') : u' \in (N \cup \Sigma)^*, |u| \leq |u'| \leq |v|\} \cup \{\infty\})$$

where t' is similarly defined as in $Check(u, v, t)$

Example 7.28 Let L be an infinite language satisfying lemma 2.15(b) with pumping constant k. We let $c = 2k$ and claim that for any $t \in \mathbb{N}$, we can find a word $u \in L$ such that $|u| \in \{t, t+1, \ldots, t+c\}$

For any word u, $|u| \geq k$, we can always pump it down until it's length l is $k \leq l < 2k$. Hence, WLOG, we assume that $k \leq |u| < 2k$. This also implies that for each $t \in \{0, \ldots, |u|\}, |u| \in \{t, t+1, \ldots, t+c\}$.

Now since $|u| \geq k$, there is a splitting $u = vwxyz$ such that $\forall h \in \mathbb{N}, vw^hxy^hz \in L$. We note that the difference between adjacent words, $|vw^{h+1}xy^{h+1}z| - |vw^hxy^hz| = |wy| \leq k$, such that $\forall \lambda \in \mathbb{N}$, there is a word in L of length $|w| + \lambda|bd|$.

Now let $t > |w|$. We can find λ such that $|w| + \lambda|bd| \leq t < |w| + (\lambda+1)|bd| \leq t + k < t + c$. In particular, $|w| + (\lambda+1)|bd| \in \{t, t+1, \ldots, t+c\}$.

And we are done with the proof.

7.29 Let L be such a context free language and G a context free grammar generating L.

Convert G to G' in Chomsky Normal Form, where $G' = (N', \Sigma, P', S')$. We now construct $G'' = (N'', \Sigma, P'', S'')$ in the following manner.

$S'' = S'$
Define rules P'' as follows. (We add elements into N'' as required by the rules).
We first add all rules $A \to BC \in P'$ into P''.
For all rules of form $A \to BC$, if $B \to b$, add the rule $A \to bC$ into P''. If $C \to c \in P'$, add the rule $A \to Bc$ into P''.
If $B \to b \in P' \wedge C \to c \in P'$, add the rule $A \to bc$ into P''.

Then G'' generates all words with length > 1 in L.

Hence $L = H_1 \cup H_2$, where H_1 is the language generated by G'' and
$H_2 = \{a \in \Sigma : a \in L\}$ if $\epsilon \notin L$,
$H_2 = \{a \in \Sigma : a \in L\} \cup \{\epsilon\}$ if $\epsilon \in L$.

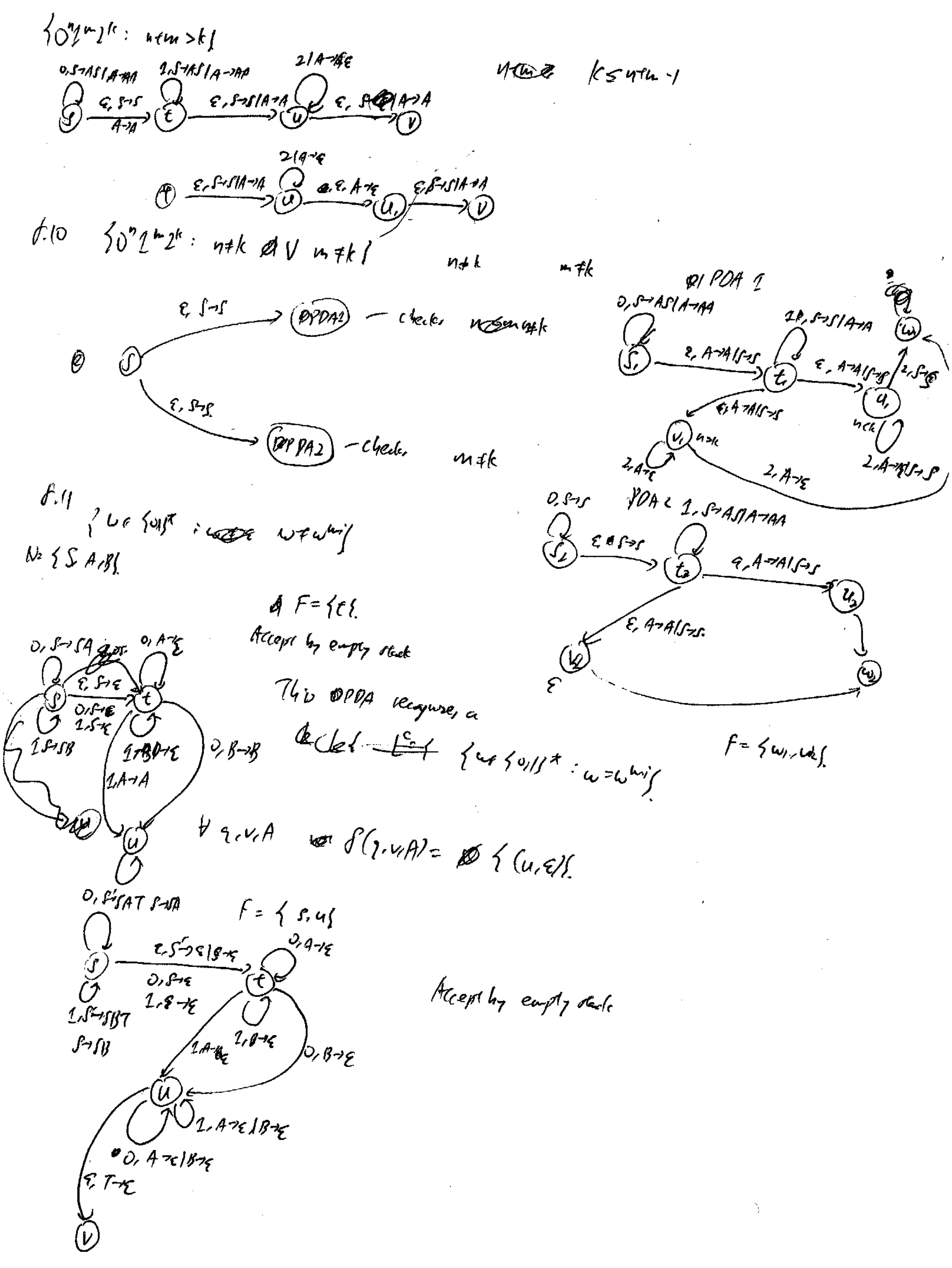
7.30 Let $G = (N, \Sigma, P, S)$ be a grammar. We will construct a corresponding growing context sensitive grammar $G' = (N', \Sigma', P', S')$

- $S' = S$
- $N' = N$
- $\Sigma' = \Sigma \cup \{0\}$, where 0 is not a symbol in Σ
- For each rule $l \to r \in P$, add the rule $l \to r0^{|l|+1}$. This is so that $|RHS| = |r|+|l|+1 \geq |l|+1 > |l|$, hence the new grammar is growing.
- Homomorphism h such that $\forall i \in \Sigma \cup N, h(i) = i$ and $h(0) = \epsilon$.

Designing PDAs There are a few common tricks to make designing PDAs easier.

- Make generous use of ϵ-transitions, especially if the PDA is not required to be deterministic. An epsilon transition is a transition in which no input is read. e.g. $\delta(q, v, A)$ where $v = \epsilon$. This can greatly reduce the number of cases to consider.
- There are 2 general ways to enforce the following inequality. To be precise, I use this example: We want to check for $0^n 1^m$, where $n \leq m + k$ for some fixed $k > 0$.
  - One way is to push k extra **non-terminals** in a single shot.
  - The other way is to have k extra intermediate **states** that consume the non-terminals generated by parsing 0's.

8.9



Example 8.17 We will prove that $\forall w \in L, ds(w) < |w|$, where $ds(w)$ denotes the digit sum of w.

We will conduct induction on derivation length. For brevity, we skip the base case. Our induction hypothesis is that for all words w that can be derived via a derivation of length $\leq n$, we must have $ds(w) < |w|$.

Now let $w \in L$ be a word for which there exists a derivation of length $n+1$. We consider the first step of such a derivation.

- Case 1: $S \Rightarrow 0$. Then $w = 0$ and we clearly have $ds(w) = 0 < 1 = |w|$
- Case 2: $S \Rightarrow 1S \Rightarrow^* 1v$, where $S \Rightarrow^* v$ takes n steps and $w = 1v$. By the induction hypothesis, $ds(v) < v$. Hence $ds(w) = 1 + ds(v) < 1 + |v| = |w|$
- For the other 2 cases, we note that the induction hypothesis, $ds(v) < |v| \implies ds(v) \leq |v| - 1$.
  Case 3: $S \Rightarrow 2SS \Rightarrow^* 2v_1v_2$, where $SS \Rightarrow^* v_1v_2$ in n steps. By our induction hypothesis, we have $ds(w) = 2 + ds(v_1) + ds(v_2) \leq |v_1| + |v_2| = |w| - 1 < |w|$
  The final case where $S \Rightarrow 3SSS \Rightarrow^* 3v_1v_2v_3$ is similarly proven.

Proposition 8.20 Note that another consequence of this theorem is that given a DPDA, we can construct an equivalent DPDA recognizing the same language that does not get stuck.

8.24 We first construct a context free grammar for the language $L = \{0^n10^m : n \geq m\}$.

- $S \to 0SV_0|1$
- $V_0 \to 0|\epsilon$

The deterministic pushdown automaton is defined as follows. Note that a dpa is allowed to get stuck. Also, a dpa must accept by state.

- $Q = \{s, t\}$
- $F = \{t\}$
- $N = \{S, V_0\}$
- $\Sigma = \{0, 1\}$

$$\begin{aligned}
\delta(s, 0, S) &= \{(s, SV_0)\} \\
\delta(s, 1, S) &= \{(t, \epsilon)\} \\
\delta(s, 0, V_0) &= \{(t, \epsilon)\} \\
\delta(s, \epsilon, V_0) &= \{(t, \epsilon)\}
\end{aligned}$$

The dpa gets stuck when all the V_0 in the stack are consumed, but the state at this point must be t.

We first suppose that L^* is DCFL. Recall the DCFLs are closed under complement and closed under union and intersection with regular languages.

Hence, $H := (L^* \cap 0^*10^*) \cup (L^* \cap 0^*10^*10^*) = \{0^a10^b : a \geq b\} \cup \{0^a10^b10^c : b \geq c\}$ must be DCFL.

Using a rewiring argument, we can show that H is not DCFL. Let M_1 be a deterministic pushdown automatons recognizing H. We conduct the rewiring as follows:

M_2 is a DPDA recognizing the language H' formed by replacing all 0s with 2s.

We connect the accepting states of M_1 for $\{0^a10^b : a \geq b\}$ (and change them to rejecting) to the states of M_2 that are "halfway" to the accepting states of $\{2^a12^b12^c : b \geq c\}$. The resulting DPDA then accepts a certain language L that has $\{0^a10^b12^c : a \geq b \geq c\}$ as a subset. Note that L does not have any other elements of the form $0^a10^b12^c$.

L is not context free as it fails the pumping lemma 2.15(b). Hence we arrive at a contradiction and L^* is not deterministic context free.

8.25 We are given that L is DCFL and H is a regular language. Claim: $L \cdot H$ is DCFL.

Let $(Q, \Sigma, N, \delta, s, S, F)$ be a DPDA for L.
Let $(Q', \Sigma, \delta', s', F')$ be a DFA for H.

Consider the following DPDA.

$$(Q \times \mathcal{P}(Q'), \Sigma, N, \delta'', s'', S, F'')$$

where

$$s'' = \begin{cases} (s, \emptyset) \text{ if } \epsilon \notin L \\ (s, \{s'\} \text{ if } \epsilon \in L \end{cases}$$

$$\delta''((q, q''), \epsilon, A) = \begin{cases} \{((p, q''), w) : (p, w) \in \delta(q, \epsilon, A)\} \text{ if } p \notin F \\ \{((p, q'' \cup \{s'\}), w) : (p, w) \in \delta(q, \epsilon, A)\} \text{ if } p \in F \end{cases}$$

$$\delta''((q, q''), a, A) = \begin{cases} \{((p, \{\delta'(q') : q' \in q''\}), w) : (p, w) \in \delta(q, a, A)\} \text{ if } p \notin F \\ \{((p, \{\delta'(q') : q' \in q''\} \cup \{s'\}), w) : (p, w) \in \delta(q, a, A)\} \text{ if } p \in F \end{cases}$$

$$F'' = \{(q, q'') : q'' \cap F' \neq \emptyset\}$$

This DPDA recognizes $L \cdot H$. Hence $L \cdot H$ is a DCFL.

8.26 Given that L is DCFL and H is regular, is $H \cdot L$ DCFL?

The general idea of the counterexample is as follows. Let's say we have 2 DCFL languages L_1, L_2, such that we know very well that $L_1 \cup L_2$ is not DCFL. Now how do we make use of this?

Suppose $L_1, L_2 \subseteq \Sigma^*$. We find some letter $a \notin \Sigma^*$. And it is quite easy to see that $a \cdot L_1 \cup L_2$ is DCFL. Why? Because we know that there exists a DPDA M_1 recognizing L_1, and a DPDA M_2 recognizing L_2. We can combine these two DPDA to form a DPDA M that does the following: Check the first letter of the input. If the first letter is a, since we know that $a \notin \Sigma$, we must be dealing with L_1, otherwise if the first letter is in Σ, then we must be dealing with L_2. So, if the first letter is a, M delegates the work to M_1, and otherwise, M delegates the work to M_2.

Here, we suppose that left concatenation of a regular language to a DCFL results in a DCFL (in order to arrive at a contradiction).
Then in particular, $a^* \cdot (a \cdot L_1 \cup L_2)$ is a DCFL.
Since DCFLs are closed under intersection with regular languages,

$$(a^* \cdot (a \cdot L_1 \cup L_2)) \bigcap (a \cdot \Sigma^*) = a \cdot L_1 \cup a \cdot L_2$$

must be DCFL. But now the first a is meaningless. Intuitively, it tells nothing about whether a word is about to be in L_1 or L_2. Hence, $L_1 \cup L_2$ must be DCFL, a contradiction.

An example of such L_1, L_2 would be the following:

$$L_1 = \{0^n 1^n : n \in \mathbb{N}\}$$
$$L_2 = \{0^n 1^{2n} : n \in \mathbb{N}\}$$
$$L_1 \cup L_2 = \{0^n 1^n, 0^n 1^{2n} : n \in \mathbb{N}\}$$

A rewiring argument can be used to show that $\{0^n 1^n, 0^n 1^{2n} : n \in \mathbb{N}\}$ is not a DCFL.

This also works when we know of 2 DCFL whose intersection is not a DCFL. Suppose we have L_1, L_2 DCFL, but $L_1 \cap L_2$ not a DCFL. Then since DCFL are closed under complement, $(L_1 \cap L_2)^c = L_1^c \cup L_2^c$ is not a DCFL, even though L_1^c, L_2^c are DCFLs. An example of such languages would be $L_{0,1}$ and $L_{1,2}$ introduced in the textbook. $L_{0,1}$ is the set of all words with as many 0s as 1s. $L_{1,2}$ is the set of all words with as many 1s as 2s.

8.29 The answer is YES. If one has a language H_k, one looks at the derivative which comes along b. So for each H_k, one makes a nonterminal X_k. Now if H_k derived by b gives $H_i H_j H_h$ (or any other product) then one makes the rule $X_k- > bX_i X_j X_h$. Here are some specific cases. If H_k derived b gives epsilong then the rule is just $X_k- > b$. If H_k derived b gives emptyset then there is no rule (or a rule $X_k- > bX_3$ where H_3 is the emptyset. Start symbol is X_1 which represents $H_1 = L$.

This condition to be added is sufficient and Exercise 8.30 shows that without this condition, one cannot make the grammar. Indeed, it is an "if and only if" condition.

8.32

$$\begin{aligned}(L \cup H)_x &= \{w \in \Sigma^* : xw \in L \cup H\} \\ &= \{w \in \Sigma^* : xw \in L \vee xw \in H\} \\ &= \{w \in \Sigma^* : xw \in L\} \cup \{w \in \Sigma^* : xw \in H\} \\ &= L_x \cup H_x\end{aligned}$$

9.3 Construct a Turing machine representing the function $x \mapsto 3x$. We consider the following machine with state $Q = \{s, t, u, v, w\}$.

- s is the start state when the Turing machine scans the word from left to right.
- t, u, v are the states where the Turing machine actually does processing
- We note that $3x = x + 2x$. With this in mind, if we let $x = b_n b_{n-1} \ldots b_0$, to get $3x$, we need to add $0 b_n b_{n-1} \ldots b_0$(x) and $b_n b_{n-1} \ldots b_0 0$(2x). At any point in time, we are adding three bits, the carry bit C, the "previous bit" b_{i-1} and the current bit b_i. To account for the edge cases, we take $b_{-1} = 0$ and $b_{n+1} = 0$.
  - t represents having $C + b_i = 0$
  - u represents having $C + b_i = 1$
  - v represents having $C + b_i = 2$
- w is the halting state, i.e. $F = \{w\}$

The rules are as follows:

$$
\begin{aligned}
\delta(s,0) &= (s,0,\rightarrow)\\
\delta(s,1) &= (s,1,\rightarrow)\\
\delta(s,\sqcup) &= (t,\sqcup,\leftarrow)\\
\delta(t,0) &= (t,0,\leftarrow)\\
\delta(t,1) &= (u,1,\leftarrow)\\
\delta(u,0) &= (t,1,\leftarrow)\\
\delta(u,1) &= (v,1,\leftarrow)\\
\delta(v,0) &= (u,0,\leftarrow)\\
\delta(v,1) &= (v,1,\leftarrow)\\
\delta(t,\sqcup) &= (w,\sqcup,\leftarrow)\\
\delta(u,\sqcup) &= (w,1,\leftarrow)\\
\delta(v,\sqcup) &= (u,0,\leftarrow)
\end{aligned}
$$

9.4 Construct a Turing machine representing the function $x \mapsto x+5$. We consider the following machine with state $Q = \{s, t_0, t_1, t_2, t_1', t_2', t, u\}$.

We are trying to add x to $(101)_2$, the binary representation of 5. In order to track the position of the word we are reading, we need the extra states t_0 to t_2. The primed states t_1' and t_2' representing having a carry bit of 1.

u represents having a carry bit of 1, t represents having a carry bit of 0 (i.e. no carry).

$$
\begin{aligned}
\delta(s,0) &= (s,0,\rightarrow)\\
\delta(s,1) &= (s,1,\rightarrow)\\
\delta(s,\sqcup) &= (t_0,\sqcup,\leftarrow)\\
\delta(t_0,0) &= (t_1,1,\leftarrow)\\
\delta(t_0,1) &= (t_1',0,\leftarrow)\\
\delta(t_1,0) &= (t_2,0,\leftarrow)\\
\delta(t_1,1) &= (t_2,1,\leftarrow)\\
\delta(t_1',0) &= (t_2,1,\leftarrow)\\
\delta(t_1',1) &= (t_2',0,\leftarrow)\\
\delta(t_2,0) &= (t,1,\leftarrow)\\
\delta(t_2,1) &= (u,0,\leftarrow)\\
\delta(t_2',0) &= (u,0,\leftarrow)\\
\delta(t_2',1) &= (u,1,\leftarrow)\\
\delta(u,0) &= (t,1,\leftarrow)\\
\delta(u,1) &= (u,0,\rightarrow)\\
\delta(u,\sqcup) &= (t,1,\leftarrow)
\end{aligned}
$$

We let $F = \{t\}$. Notice that when state t is reached, we no longer need to modify any more bits of the word on the tape.

Description 9.11

Turing Machine R_1 $|\Gamma|$ R_5 Current tape position P R_7 Current state q

R_2 $|Q|$ R_6 Current symbol a R_8 Current instruction

R_3 Turing Table $(Q-F)\times\Gamma \to \Gamma \times Q \times \{left, right\}$

R_4 Turing tape T

Starting state $s = 0$

Halting state $F = \{1\}$. $2|Q||\Gamma|$

Read $(|\Gamma|$, Tape, Position$)$

Write $(|\Gamma|$, Tape, Position, New Symbol$)$ To write to position P on Tape T

$R_9 \leftarrow 2|Q||\Gamma|$ $R_6 \leftarrow \text{Read}(|\Gamma|, T, P)$

$R_8 \leftarrow$ Read $(2|Q||\Gamma|$, Turing Table, $|\Gamma|\cdot R_7 q + a)$ // Current entry

$R_{10} \leftarrow R_8 / 2R_2$

$$= \left\lfloor \frac{\gamma|Q|\cdot 2 + p\cdot 2 + \text{movement}}{2|Q|} \right\rfloor$$

$= \gamma \in \Gamma$ // new symbol

R_8 is of form

$\gamma\cdot|Q|\cdot 2 + p\cdot 2$ + movement

where $\gamma \in \Gamma$, $p \in Q$, movement $\in \{left, right\}$.

$R_4 \leftarrow$ Write $(|\Gamma|, T, P, \gamma)$ // Write New Symbol and return New ~~State~~ Tape.

$R_7 \leftarrow \left\lfloor \frac{R_8}{2} \right\rfloor \bmod R_2 = (\gamma|Q| + p) \bmod |Q|$

$= p$ // New State

If $R_7 = 1 \in F$, (Halting state) then Stop.

Else ~~$R_5 \leftarrow R_5 + 2\left\lfloor \frac{R_8}{2} \right\rfloor - 1 = R_5$~~ $+$

$$R_5 \leftarrow R_5 + \underbrace{2(R_8 \bmod 2) - 1}_{\text{movement}} = \begin{cases} R_5 + 2(0) - 1 = R_5 - 1 & (\text{if left}) \\ R_5 + 2(1) - 1 = R_5 + 1 & (\text{if right}) \end{cases}$$

Example 9.17 We define the following notations.

- $Id : x \mapsto x$ is the identity function.
- $\phi_i : (x_1, x_2, \ldots, x_n) \mapsto x_i$ is the projection function that selects the i-th entry. To be fully rigorous, we actually need to denote ϕ_i as ϕ_i^n, since technically $\phi_i^n : (x_1, x_2, \ldots, x_n) \mapsto x_i$ and $\phi_i^m : (x_1, x_2, \ldots, x_m) \mapsto x_i$ are different functions accepting different inputs. However, we will be lazy and use ϕ_i when things are unambiguous. Also note that $Id = \phi_1^1$
- $\mathbf{0}$ as the constant zero function. Sometimes we will be lazy and not use boldface.
- $S : x \mapsto x + 1$ or $succ : x \mapsto x + 1$ as the successor function
- $P : x \mapsto x - 1$ or $pred : x \mapsto x - 1$ as the predecessor function. Note that $P(0) = 0 - 1 = 0$.

Of the above function, S, ϕ_i are by definition, primitive recursive functions. Example 9.17 provides a proof that P is a primitive recursive function as well.

Now we rewrite the examples in 9.17 in a more readable manner.

Claim P is primitive recursive.
We use primitive recursion to define P.
$P(0) := f() = \mathbf{0}() = 0$, where $f = \mathbf{0}$
$P(x + 1) = x := g(x, P(x))$, where $g(x, y) = \phi_1(x, y) = x$
Hence, P is primitive recursive.

Claim Addition, denoted by $h(x, y) = x + y$ is primitive recursive.
Like before, we use primitive recursion to define h.
$h(0, y) := f(y) = y$, where $f = Id$
$h(S(x), y) := g(x, h(x, y), y) = S(h(x, y))$, where $g(x, y, z) = S(y) = S(\phi_2(x, y, z))$. Note that g is primitive recursive by composition.
Hence, h is primitive recursive.

Claim Subtraction, denoted by $h'(y, x) = x - y$ is primitve recursive.
Using primitive recursion to define h'.
$h'(0, x) := f(x) = x$, where $f = Id$
$h'(S(y), x) := g(y, h'(y, x), x) = P(h'(y, x))$, where $g(x, y, z) = P(\phi_2(x, y, z))$.

Corollary $h(x, y) = x - y$ is primitive recursive.
Observe that $h(x, y) = h'(\phi_2(x, y), \phi_1(x, y))$, hence h is primitive recursive by composition.

Claim Equality checking, denoted by $eq(x, y) = 1 - (x - y) - (y - x)$ is primitive recursive. (As before, $x - y = 0$ if $x \leq y$)

Notice that $eq(x, y) = h(h(1, h(x, y)), h(y, x))$.
We first prove that $(x, y) \mapsto h(1, h(x, y))$ is primitive recursive. Note that the constant $\mathbf{1}$ function is primitive recursive. In fact, all constant functions are primitive recursive. Hence $(x, y) \mapsto h(1, h(x, y))$ can be viewed as a composition as follows: $h(\mathbf{1}(x, y), h(x, y))$. Hence $(x, y) \mapsto h(1, h(x, y))$ is primitive recursive.
We can use a similar composition argument for eq itself, therefore eq is primitive recursive.

9.18 Claim: Every function of the form $h(x_1, x_2, \ldots, x_n) = \sum_{1 \leq i \leq n} a_i x_i + b$ for some fixed $a_1, a_2, \ldots, a_n, b \in \mathbb{N}$ is primitive recursive.

We will proceed by induction. For each $n \in \mathbb{N}$, let $P(n)$ be the statement that every function of the form $h(x_1, x_2, \ldots, x_n) = \sum_{1 \leq i \leq n} a_i x_i + b$ is primitive recursive.

We will show $P(0)$ as a base case. $P(0)$ essentially states that every constant function (with the constant being a natural number) is primitive recursive. We again proceed by induction and define for each $b \in \mathbb{N}$, the statement $Q(b)$ that the constant function $\mathbf{b} : any \mapsto b$ is primitive recursive. Clearly $Q(0)$ is true since $\mathbf{0}$ is a base case of the structural definition of primitive recursive functions. Now suppose $Q(b)$ true for some $b \in \mathbb{N}$. Then $\mathbf{b+1} = \mathbf{b} + \mathbf{1} = S(\mathbf{b})$, hence $\mathbf{b+1}$ is primitive recursive. This show $Q(b+1)$. Hence by induction, all constant functions are primitive recursive.

It is also easy to show that for any constant $a \in \mathbb{N}$, the function $x \mapsto x + a$ is primitive recursive. Because, for $a > 1$, such a function involves repeated applications of the successor function S. So we take this fact for granted.

Let $n \in \mathbb{N}$ and suppose $P(n)$ true. Then

$$h(x_1, x_2, \ldots, x_{n+1}) = \sum_{1 \leq i \leq n+1} a_i x_i + b = a_1 x_1 + \sum_{2 \leq i \leq n+1} a_i x_i + b$$

We now attempt to define h using the recursive strategy.

When $x_1 = 0$, we have $h(0, x_2, \ldots, x_{n+1}) = h'(x_2, \ldots, x_{n+1})$, where $h'(y_1, y_2, \ldots, y_n) := \sum_{1 \leq i \leq i} a_{i+1} y_i + b$. By induction hypothesis, h' is primitive recursive.

For any $x_1 \in \mathbb{N}$, $h(x_1+1, x_2, \ldots, x_{n+1}) = g(x_1, h(x_1, x_2, \ldots, x_{n+1}), x_2, \ldots, x_{n+1}) = a_1 + h(x_1, x_2, \ldots, x_{n+1})$. Here, $g(y_1, y_2, \ldots, y_{n+2}) = y_2 + a_1 = \phi_2(y_1, y_2, \ldots, y_{n+2}) + a_1$. Hence g is primitive recursive.

By the recursive strategy, we have shown h to be primitive recursive. Hence $P(n+1)$ holds.

By induction, $\forall n \in \mathbb{N}, P(n)$.

9.19 Claim: $h(n) = \sum_{1 \leq i \leq n} i$ is primitive recursive.

We will use the recursive strategy.

$h(0) = \mathbf{0}() = 0$

$h(x+1) = g(x, h(x)) = x + 1 + h(x)$, where $g(x, y) = x + 1 + y = S(x + y)$. Since addition is primitive recursive, as is S, g is primitive recursive.

Hence h is primitive recursive.

9.20 Claim: Multiplication $h(x, y) = x \cdot y$ is primitive recursive.

We will use the recursive strategy.

$h(0, y) = \mathbf{0}(y) = 0$

$h(x+1, y) = g(x, h(x, y), y) = h(x, y) + y$ where $g(x, y, z) = y + z = \phi_2(x, y, z) + \phi_3(x, y, z)$

Hence, h is primitive recursive.

Theorem 9.25, 9.26 In these two theorems, n is the input size of **both** the Turing machine and the register machine. It is very important to realize this.

For instance, let word (or rather, tape) $w = a_1 a_2 \ldots a_n \in \Gamma^*$ be the input to the Turing machine, and $|w| = n$. Since the size of input to a Turing machine is defined as the number of symbols, the input size is indeed the length of the word w, which is n.

To convert w to an input to the register machine. We view w as a base-$|\Gamma|$ number. Consider the base-b encoding of w, where $b > 1$. Denote the encoding as $e(w)$. Then the word-length of $e(w)$ is proportional to n. If $b = |\Gamma|$, then the length of $e(w)$ is simply n. We define the input size to a register machine by $\log(x)$ where x is the input. So $\log(e(w))$, where $e(w)$ is viewed as a base-b number, is proportional to n.

Going back to the theorems.

Theorem 9.25 says that the time complexity of the Turing machine is the same as the time complexity of the register machine.

Note Theorem 9.26 does **not** say that the space complexity of the register machine is $O(2^{q(n)})$. It only says that the **value** taken by certain registers is $O(2^{q(n)})$, which is around $O(x)$, where x is the input.

Example 9.27 These 2 register programs can be understood as follows.

PolyMult To multiply $x := R_1$ and $y := R_2$, `polymult` takes the following high-level idea.

1. Initialize result $z = 0$. Register R_4 has a similar role as z.
2. Find the largest n such that $\sum_{0 \le i \le n} 2^i \le x < \sum_{0 \le i \le n+1} 2^i$
3. Add this value $\sum_{0 \le i \le n} 2^i$ to z, i.e. $z \leftarrow z + \sum_{0 \le i \le n} 2^i$.
4. Subtract $x \leftarrow x - \sum_{0 \le i \le n} 2^i$. Note that while no subtraction takes place in the program, we can see this is what register R_3 is essentially doing.
5. Go back to step 2, rinse and repeat.

We can picture x as the number

$$\begin{aligned} x =& 1 + 2 + 2^2 + \cdots + 2^{n_1} \\ &+ 1 + 2 + 2^2 + \cdots + 2^{n_2} \\ &+ 1 + 2 + 2^2 + \cdots + 2^{n_3} \\ &+ \text{and so on...} \end{aligned}$$

To see the runtime of `polymult`, we can place an upper bound on x. Let $n = n_1$. Then $n_3 \le n_1 - 1$, $n_5 \le n_3 - 1$ and so on. The reasoning is as follows.
We know that $1 + 2 + 2^2 + \cdots + 2^{n_1} \le x < 1 + 2 + 2^2 + \cdots + 2^{n_1+1}$, so $1 + 2 + 2^2 + \cdots + 2^{n_2} + 1 + 2 + 2^2 + \cdots + 2^{n_3} = 2^{n_2+1} + 2^{n_3+1} - 2 < 2^{n_1+1}$. This implies that it is not possible for both $n_2, n_3 = n_1$. Since we must have $n_1 \ge n_2 \ge n_3$, we can safely conclude that $n_3 \le n_1 - 1$.
So an upper bound on x is for $n_1 = n_2 = n, n_3 = n_4 = n - 1, \ldots n_{2n-1} = n_{2n} = 1$. Then $x \le 2 \cdot \sum_{0 \le i \le n+1} (2^i - 1) = 2^{O(n)}$.

The runtime of `polymult` is proportional to the sum of all the n_i, that is, $T = O(\sum_i n_i) = O(n^2)$. The size of the input (encoding) is $O(\log 2^{O(n^2)}) = O(n)$.

In summary,

- Input is x, such that input size is bounded above by $O(n)$.
- Runtime is $O(n^2)$.
- If we assume that the input size is actually $O(n)$, then the runtime is quadratic relative to input size. In particular, the runtime is polynomial.

BinaryMult `binarymult` works like this: Let $x := R_1, y := R_2$.

We represent y as $(b_1 b_2 \dots b_n)_2$, where b_1 is the most significant bit.

1. Let $z \leftarrow 0$ be the result.
2. Take the most significant bit of y, b_1. Multiply it with x and add to z, i.e. $z \leftarrow b_1 x = x$. At this point, z is the product of $(b_1)_2$ and x.
3. We now consider $b_1 b_2$.
4. $z \leftarrow 2z + b_2 x$. At this point, z is the product of $(b_1 b_2)_2$ and x.
5. We now consider $b_1 b_2 b_3$.
6. $z \leftarrow 2z + b_3 x$. At this point, z is the product of $(b_1 b_2 b_3)_2$ and x.
7. And so on... until z is the product of $(b_1 b_2 \dots b_n)_2$ and x.

The analysis of the algo is a lot simpler. In each iteration, we go from $z = b_1 b_2 \dots b_k \cdot x$ to $z = b_1 b_2 \dots b_{k+1} \cdot x$, that is, we process one more digit of $y = R_2$. Assuming that addition is $O(1)$, this algorithm is $O(|y|)$, i.e. length of word y. Hence the algorithm has linear runtime. (Note that the input sizes are $|x|, |y|$.)

9.28 We can make use of the polynomial time divide macro in 9.29.

1. Function Remainder(R_1, R_2):
2. $R_3 \leftarrow$ Divide(R_1, R_2)
3. $R_4 \leftarrow$ BinaryMult(R_3, R_2)
4. $R_5 \leftarrow R_1 - R_4$
5. Return(R_5)

Runtime: $O(n^2)$, where $n = |R_1|$ (length of R_1), due to Divide macro.

9.29

1. Function Divide(R_1,R_2):
2. $R_3 \leftarrow R_1$, $R_5 \leftarrow 0$ // R_5 stores the final result
3. $R_6 \leftarrow 1$, $R_4 \leftarrow R_2$ // R_4 acts as a holder of $R_2 \cdot 2^k$ for some $k \geq 1$, R_6 holds the value that we add to R_5
4. $R_7 \leftarrow R_4 + R_4$ // R_7 acts as a temporary holder of $R_4 \cdot 2$
5. if $R_7 > R_3$ GOTO line 8
6. $R_4 \leftarrow R_7$, $R_6 \leftarrow R_6 + R_6$
7. GOTO line 4
8. if $R_3 = 0$ GOTO line 12
9. if $R_3 < R_4$ GOTO line 3
10. $R_3 \leftarrow R_3 - R_4$, $R_5 \leftarrow R_5 + R_6$
11. GOTO line 8
12. Return(R_5)

Runtime: $O(n^2)$, where $n = |R_1|$ (length of R_1)

9.30 We claim the power function is one such function that can be computed in polynomial time by the extended register machine, but not by a normal register machine.

We first show the existence of a polynomial time algorithm for the extended register machine.

1. Function Pow(R_1, R_2): // Returns ${R_1}^{R_2}$ where R_2 is a always a power of 2.
2. $R_3 \leftarrow R_1$, $R_4 \leftarrow 1$
3. if $R_4 = R_2$ GOTO line 5
4. $R_3 \leftarrow R_3 \cdot R_3$, $R_4 \leftarrow R_4 + R_4$ // $O(1)$
5. GOTO line 2
6. Return(R_3)

Line 4 runs $O(log(R_2)) = O(|R_2|)$ times, hence Pow's runtime is linear.

Consider an algo by a normal register machine.

10.17 Let M be a deterministic register program that has $m = O(1)$ lines of instructions and $k = O(1)$ distinct registers. The overall state of a register program can be quantified in terms of the values of each register and the line of execution within the program M.
We are given that each register's size is bounded by a polynomial $p(n)$. Then the state for all the k registers is $k2^{O(p(n))} = \Theta(2^{O(p(n))})$. The state of the program is then $2^{O(p(n))} \cdot m = \Theta(2^{O(p(n))})$ where the m accounts for the line of execution.

Hence we know that M has $2^{O(p(n))}$ states of execution. If the program terminates, no state can repeat, so the runtime is also bounded by $2^{O(p(n))}$.

10.18 Suppose $G(V, E)$ is a YES-instance of the Connected Halves problem. Then a certificate for G would be a partition of V into U, W such that $|U| \leq |W| \leq |U| + 1$ and there is an edge between any two nodes one in U , one in W.

To verify this in polynomial time, we first let $n = |V|$. We can count the number of nodes in U, W respectively in $O(|U| + |W|) = O(n)$ time. Comparing $|U| \leq |W| \leq |U| + 1$ clearly take $O(1)$ time. Finally, for each $u \in U, w \in W$, we need to check for membership of edge $\{u, w\}$ in E. There are $O(n)$ choices of u and $O(n)$ choices of w, hence this takes $O(n^2 q(n))$ time, where $q(n)$ is the time taken to check set membership in E. Assuming q is a polynomial, we then have the overall verification time of $O(n + n^2 q(n))$ which is polynomial.

Hence Connected Halves is in NP.

10.20

1. Function SimulateExpo(R_1, R_2) // simulates R_1 for R_2 steps
2. $LN = 2$
3. For $T = 0$ to R_2
4. if $LN = 2$ do $R_3 = 1$; $LN = 3$; GOTO Line 9 end
5. if $LN = 3$ do if $R_1 = 0$ do $LN = 7$ else $LN = 4$ end; GOTO Line 9 end
6. if $LN = 4$ do $R_3 = R_3 + R_3$; $LN = 5$; GOTO Line 9 end

7. if $LN = 5$ do $R_1 = R_1 - 1$; $LN = 6$; GOTO Line 9 end
8. if $LN = 6$ do $LN = 3$; GOTO Line 9 end
9. Next T
10. if $LN = 7$ Return($R_3 + 1$) else Return(0) end

10.21

1. Function SimulateRepeatAdd(R_1, R_2):
2. $LN = 2$
3. For $T = 0$ to R_2
4. If $LN = 2$ do $R_3 = 3$; $LN = 3$; GOTO Line 9 end
5. If $LN = 3$ do if $R_1 = 0$ do $LN = 7$ else $LN = 4$ end; GOTO Line 9 end
6. If $LN = 4$ do $R_3 = R_3 + R_3 + R_3 + 3$; $LN = 5$; GOTO Line 9 end
7. If $LN = 5$ do $R_1 = R_1 - 1$; $LN = 6$; GOTO Line 9 end
8. If $LN = 6$ do $LN = 3$; GOTO Line 9 end
9. Next T
10. If $LN = 7$ do Return(R_3) else Return(0) end

Theorem 10.24 This theorem has a few interpretations.

- If a set A is the range of a partial recursive function, then by going over all inputs, we can enumerate A, as mentioned in definition 10.25.
- If a set A is the domain of a partial recursive function, then members of A can be computed in finite time by a register machine. Hence A can accepted by a register machine. (but not necessarily decided)

Theorem 10.24 states that these 2 notions are equivalent, enumeration and acceptance. Such languages having this property are then called recursively enumerable.

Theorem 10.28 Note that in the proof of this theorem, Halt computes the characteristic function of the diagonal halting problem K.

10.30 Claim: A set L is recursive iff both $L, L^c := \mathbb{N} - L$ are recursively enumerable.

By definition, L is recursive
iff
there exists total recursive function f that acts as the characteristic function of L, i.e. $f = \chi_L$. (In particular $\forall x \in \mathbb{N}$, there is a register/Turing machine simulating f that halts on x and produces the correct output (YES-NO to membership))

Such a function f can also be used to define a partial recursive function. For instance we define g as follows

1. Function g(R_1):
2. $R_2 \leftarrow$ f(R_1)

3. If $R_2 = 1$ do Return(1) else Infinite Loop end.

We have defined a register computable function g to halt on members of L and infinitely loop on non-members of L. Thus L is the domain of partial recursive function g.

Similarly, we can define a function h to halt only on members of L^c (where $R_2 = 0$). The domain of h would then be L^c.

Hence both L and L^c are recursively enumerable.

Conversely, suppose L and L^c are recursively enumerable. We take partial recursive functions g, h that accept L, L^c respectively. In other words, the domain of g is L and the domain of h is L^c.

Then we can define a function f deciding L as follows.

1. Function f(R_1)
2. While True, alternate between steps of g and h

If $w \in L$, then $\exists n \in \mathbb{N}$, g accepts w in n steps. Then f accepts w in $O(n)$ steps.

Similarly for $w \in L^c$.

10.31-10.33 The main idea of how to approach these questions comes by considering a general case instead.

Let $f : \mathbb{N} \to \mathbb{N}$ be a function such that, for each $y \in \mathbb{N}$, the number of elements in the setwise preimage $f^{-1}(\{y\})$ is **finite**, that is at most finitely many elements in the domain maps to the same element in the codomain. This also means that the range of f is infinite.

Claim: The set $L = \{e \in \mathbb{N} : \varphi_e(f(e)) \text{ defined } \} \subseteq \mathbb{N}$ is undecidable.

Suppose not, such that membership in this set L is decidable by some total function. Then there is a corresponding (total) register machine function **Halt** that outputs

$$\textbf{Halt}(e) = \begin{cases} 1, \text{ if } \varphi_e(f(e)) \text{ defined} \\ 0, \text{ otherwise} \end{cases}$$

We then "diagonalize" as follows, like in the proof of undecidability of the diagonal halting problem.

We define a function $g : \mathbb{N} \to \mathbb{N}$. For each $y \in \mathbb{N}$, we consider $f^{-1}(\{y\})$, the setwise preimage of y.

Let $S = \{e : e \in f^{-1}(\{y\}) \wedge \textbf{Halt}(e) = 1\}$ be the set of elements $e \in \mathbb{N}$ where $f(e) = y$ and $\phi_e(y)$ is defined. Then, define

$$g(y) = \begin{cases} \max\{\varphi_e(f(e)) : e \in f^{-1}(\{y\}) \wedge \textbf{Halt}(e) = 1\} + 1, \text{ if } S \neq \emptyset \\ 0, \text{ if } S = \emptyset \end{cases}$$

Note that g can be computed by a register machine since **Halt** is assumed to be a register machine function. Note that we can take maximum since we assume $S \subseteq f^{-1}(\{y\})$ to be finite.

We observe that for any $e \in \mathbb{N}$, $g(f(e)) \neq \varphi_e(f(e))$ and hence $g \neq \varphi_e$. Hence $g \notin \{\varphi_e : e \in \mathbb{N}\}$ and we arrive at a contradiction.

Lemma Claim: The set $\{e : \varphi_e(0) \text{ defined}\}$ is not recursive.

Suppose not, then there is a recursive register function **Halt** such that

- **Halt**$(e) = 1$ if $\varphi_e(0)$ defined

- **Halt**$(e) = 0$ otherwise

More generally, let $Halt'(g) = 1$ iff $g(0)$ defined.

We then define the following function f to obtain a contradiction.

1. Function f(e):
2. Define a function $g = x \mapsto \varphi_e(e)$
3. Return Halt'(g)

Note that $x \mapsto \varphi_e(e)$ is a constant function (or a function that is everywhere undefined), so it is partial recursive.

Halt(m) returns 1 iff $\varphi_m(0) = \varphi_e(e)$ is defined, i.e. iff $e \in K$ where K is the diagonal halting problem.

Hence we have reduced computing f to the membership problem of K, which we know is undecidable.

11.4 $S = \{x \in \mathbb{N} : x \equiv 1(\mod 2) \wedge 97|x\}$

We note that $\forall x \in \mathbb{N}$,

$$\begin{aligned} x \in S &\iff \exists y_1, y_2 \in \mathbb{N}, x = 2y_1 + 1 \wedge x = 97y_2 \\ &\iff \exists y_1, y_2 \in \mathbb{N}, (x - 1 - 2y_1)^2 = 0 \wedge (x - 97y_2)^2 = 0 \\ &\iff \exists y_1, y_2 \in \mathbb{N}, (x - 1 - 2y_1)^2 + (x - 97y_2)^2 = 0 \end{aligned}$$

Using definition (b), we see that S is diophantine.

11.5 $S = \{x \in \mathbb{N} : 5|x \wedge 7 \not| x\}$

We note that $\forall x \in \mathbb{N}$,
$5|x \iff \exists y_1 \in \mathbb{N}, x = 5y_1$
$7 \not| x \iff \exists y_2, y_3, y_4 \in \mathbb{N}, x = 7y_2 + 1 + y_3 \wedge x + 1 + y_4 = 7(y_2 + 1)$. These 2 clauses combined is equivalent to stating that x is strictly between $7y_2$ and $7y_2 + 7$.

Hence, we have the equivalent condition

$$\exists y_1, y_2, y_3, y_4 \in \mathbb{N}, (x - 5y_1)^2 + (x - 7y_2 - 1 - y_3)^2 + (x + 1 + y_4 - 7(y_2 + 1))^2 = 0$$

Using definition (b), we see that S is diophantine.

11.6 We are given the diophantine set $S = \{x \in \mathbb{N} : \exists y_1, y_2 \in \mathbb{N}, (2y_1 + 3) \cdot y_2 - x = 0\}$.

We can see that running across all y_1, $2y_1 + 3$ gives all the odd numbers greater than 1. Since y_2 is any natural number, we see that any number that has a non-trivial odd factor is an element of this set. Since $2y_1 + 3 > 1$, any element of S must have a non-trivial odd factor. Hence S is the setwise complement of $\{2^k : k \in \mathbb{N}\}$.

That is, $S = \mathbb{N} - \{2^k : k \in \mathbb{N}\}$.

Proposition 11.7 Note that this construction is distinct from the simulation construction we see in theorem 10.24.

The register function R defined here is: Let $a \in A$, in the case where $A \neq \emptyset$.

$$R(x, y_1, \dots, y_n) = \begin{cases} x, p(x, y_1, y_2, \dots, y_n) = 0 \\ a, p(x, y_1, y_2, \dots, y_n) \neq 0 \end{cases}$$

It is clear that R is total recursive, since $R(x, y_1, \dots, y_n)$ terminates for all inputs.

11.9 We are given that A is a Diophantine set. Then $\exists p \in P(\mathbb{Z}), \forall x \in \mathbb{N}, (x \in A \iff (\exists y_1, \dots, y_n \in \mathbb{N}, p(x, y_1, \dots, y_n) = 0))$. Let $B = \{x \in \mathbb{N} : \exists x' \in \mathbb{N}, (x + x')^2 + x \in A\}$.

Consider the polynomial $q \in P(\mathbb{Z})$, where $q(x, x,', y_1, \dots, y_n) = p((x + x')^2 + x, y_1, \dots, y_n)$.

Then $\forall x \in \mathbb{N}$,

$$\begin{aligned} &\exists x', y_1, \dots, y_n \in \mathbb{N}, q(x, x', y_1, \dots, y_n) = 0 \\ &\iff \exists x', y_1, \dots, y_n \in \mathbb{N}, p((x + x')^2 + x, y_1, \dots, y_n) = 0 \\ &\iff \exists x' \in \mathbb{N}, (x + x')^2 + x \in A \\ &\iff x \in B \end{aligned}$$

Hence membership in B is decided by q and using definition (b), B is Diophantine.

Note: The second logical equivalence is due to the equivalence

$$\exists y_1, \dots, y_n, p((x + x')^2 + x, y_1, \dots, y_n) = 0 \iff (x + x')^2 + x \in A$$

Example 11.14 To supplement my notes in the textbook, this summarizes what is meant by the arithmetic characterization of a run with input x and output y.

Note: We will use $\mathbf{LN}, \mathbf{R}_i$ to denote the base-p numbers representing the values taken by line number LN, registers R_i over the course of the computation.
That is, for e.g., if $\mathbf{LN} = b_n b_{n-1} \dots b_1$, then at step $1 \leq t \leq n$, the value of line number LN is b_t.

- Point 1 just says that there is an (prime) upper bound p of the values taken by line number LN and registers $R_i, i = 1, 2, 3$ at any step of the computation.
- Point 2 says that the least significant digit of $\mathbf{R}_1$ is the input x, i.e the initial value of R_1 is x and the first line number is Line 1.
- Point 3 says that the most significant digit of R_2 is y and the final value of LN is 4. This makes sense, since the Sum function returns at line 4.
- Point 4 says that for every adjacent pair of digits in each of $\mathbf{LN}, \mathbf{R}_i$, there is a legitimate transition. i.e. The register machine can actually undergo such a transition in configuration. (Note, see Example 11.13 for a definition of configuration.)

Example 11.14 We can make the following conclusions from this example. Given a partial recursive/register function f, let R be defined as in this example, i.e. $R = \{(x, f(x)) : f(x) \text{ defined}\}$ is the graph of f. Let the domain of definition of f be $D = \{x \in \mathbb{N} : \exists y, (x, y) \in R\}$.

We see from example 11.14 that R can be defined using arithmetic formula, together with quantifiers $\exists, \forall$ and similarly the same can be said for D.

This discussion relates to the generalization suggested at the end of this section. It is claimed that the above can be generalized to any register machine computation.

Claim: Let $T = (e,x) \mapsto \phi_e(x)$ be the Universal Turing machine on 1 input. Then the set $H = \{(e,x) : \phi_e(x) \text{ halts/defined}\}$ is definable in arithmetic. Indeed, T is a register machine, hence by example 11.14, its graph R and domain of definition D are both definable in arithmetic.
In particular, D is definable in arithmetic. But D is precisely H. Hence, we have the desired result.

Relations between sets From this chapter, we can draw the following relations between sets

$$\textbf{Diophantine} \subseteq \textbf{RecursivelyEnumerable} \subseteq \textbf{Arithmetical}$$

Definition 11.16-11.17 Some notes on definitions:

A numbering of a set S in computability theory is a surjective mapping with $\mathbb{N}$ as the domain and S as the range. For instance for the set of partial recursive functions, Turing's theorem states that there is an enumeration from the natural numbers to the set of all partial recursive functions. Hence, Turing's Universal Turing machine-function is a numbering $e \mapsto \phi_e$. to the set of partial recursive functions.

In CS3231, a numbering is defined slightly differently, it is a mapping $(e,x) \mapsto \phi_e(x)$, so it includes the value x to feed into the function ϕ_e as well.
Furthermore,in CS3231 a numbering of partial recursive functions need not cover all partial recursive functions. (i.e. no need to be surjective wrt to the set of partial recursive functions)

But we can see the intuitive idea is the same. A numbering is a surjective listing of a set.

Corollary 11.20 Some things to take note: In order to make use of Rice's Theorem, we need to show that the following pre-conditions hold

- The numbering $e \mapsto \phi_e$ (or rather $(e,x) \mapsto \phi_e(x)$) related to the Universal Turing machine is acceptable. This is known to be true.
- The set $I = \{e : \forall x, (e,x) \in H\}$ is an index. This is easy to show. $\forall d, e, \phi_d = \phi_e$ if and only if ϕ_d, ϕ_e are both total functions if and only ig $d, e \in I$ by definition of I. Hence I is an index.

Observation 11.21 A short proof of this observation is as follows. Since B is recursively enumerable, there exists a partial function h for which B is its domain. That is, we can write $h : B \subseteq \mathbb{N} \to \mathbb{N}$.

Since g is recursive, $g : \mathbb{N} \to \mathbb{N}$ and we can then form the "composition" $l = h \circ g$. We claim that l is a partial function whose domain is A.

Reason: If $x \in A$, then $g(x) \in B$ and $h(g(x))$ is defined. If $x \notin A$, then $g(x) \notin B$ and $h(g(x))$ is undefined. Hence l is defined over precisely A.

Another more intuitive way to reason this is that B can be accepted by a register program. So we can then accept A by writing a register program that computes for each $a \in A$, the value $g(a)$, then check membership of $g(a)$ in B. Clearly, such a register program accepts A, hence A is recursively enumerable.

11.26 Let $F = \{e : \phi_e(x) \text{ defined on at least 1 } x\}$. Let $A = \{e : \phi_e(x) \text{ defined for exactly one } x\}$.

We define the following functions.

$$\mu(e) = \min\{x : \phi_e(x) \text{ defined}\}$$

Clearly μ is a partial recursive function on e.

Note There is a intricacy here. There is a problem with conducting μ-minimization over partial functions. For e.g. suppose $\phi_e(1)$ is defined, but $\phi_e(0)$ is not. But μ-minimization checks if $\phi_e(0)$ is

defined first, but this wouldn't terminate, and we are stuck.
The way to correct this is to consider $\{(e, x) : \phi_e(x) \textit{ defined}\}$ as the range of a total function h. And define the function $\mu(e) = \min\{y : h(y) = (e', x) \wedge e' = e\}$. This new μ is guaranteed to terminate in the case where $\{x : \phi_e(x) \textit{ defined}\} \neq \emptyset$.
This idea can also be applied to **Self-test 12.35**. We can find the k-th smallest y such that $h(y) = (e, x)$.

Define

$$f(e, x) = \begin{cases} \phi_e(x) \text{ if } \mu(e) \text{ defined and } x = \mu_e \\ \text{undefined otherwise} \end{cases}$$

Then f is partial recursive function on e, x.

There exists a recursive function g such that $\forall e, x, \phi_{g(e)}(x) = f(e, x)$. Then we can see that $e \in F$ iff $\exists x, f(e, x)$ defined and $\forall x' \neq x$, $f(e, x')$ undefined, iff $\phi_{g(e)}$ defined at exactly 1 value, iff $g(e) \in A$.

Thus F has a many-one reduction to A.

11.27 Let A be as defined in the question.

We first discuss the applicability of Rice's Theorem. We already know that the Universal Turing machine numbering $e \mapsto \phi_e$ is acceptable. Next, A is also clearly an index. Hence we can use Rice's Theorem here.

Since $A \neq \emptyset$ and $A \neq \mathbb{N}$, A is not recursive by Rice's Theorem.
To show $A \neq \mathbb{N}$, we can (trivially) find some function ϕ_x such that $\phi_x(x + 1)$ is undefined and ϕ_e such that $\phi_e(x)$ is also undefined.
To show $A \neq \emptyset$, we can define a partial recursive function f such that $f_e(x) = 1$ when $\phi_x(x + 1)$ is undefined and $f_e(x)$ undefined when $\phi_x(x + 1)$ is defined. Such a function has a corresponding e for which $f = \phi_e$ and e is not a member of A.

Next, we claim that A is not recursively enumerable. The idea is that to check the validity of an index $e \in A$, equivalently ϕ_e meets the conditions that $\forall x, \phi_e(x)$ defined iff $\phi_x(x + 1)$ undefined, we need to check infinitely many values of x. So we cannot decide membership of $e \in A$ just by using a finite list., i.e. if $e \in A$ is indeed decided by a finite list, then we can construct a counterexample $\phi_{e'}$ which passes the finite tuple's checks but fails the condition that $\forall x, \phi_{e'}(x)$ defined iff $\phi_x(x + 1)$ undefined.

There is a small but important detail here that needs to be mentioned: We need to show that there are infinitely many x such that $\varphi_x(x + 1)$ is defined. This is true because there are infinitely many total recursive functions, so in particular for any total function φ_y, it is defined at $y + 1$.

Hence A is not recursively enumerable.

11.28 Let B be as defined in the question.

We first note that Rice's Theorem is applicable to this question since B is an index set.

We claim that part (b) of Rice's Theorem can be applied here. That is, there exists a recursive enumeration of lists that can decide membership of $e \in B$. The lists are all 10-tuples of the form $(x_1, y_1, \ldots, x_5, y_5)$ where $x_1, x_2, \ldots, x_5$ are pairwise distinct and $y_1, \ldots, y_5$ can take any values in $\mathbb{N}$. We can see that these lists themselves can be checked for validity by a register function. A register function only needs to check 2 criteria:

- The tuple is of length 10.
- All the elements with odd indices are pairwise distinct.

Furthermore, such an enumeration of lists decides membership in B. Because, $e \in B$ iff $\exists x_1, x_2, x_3, x_4, x_5 \in \mathbb{N}, \phi_e$ is defined at those 5 values, iff, there is a 10-tuple of the form $(x_1, y_1, \ldots, x_5, y_5$ such that $\phi_e(x_i) = y_i$ for each $i = 1, 2, 3, 4, 5$ (in particular, ϕ_e is defined on those 5 values).

Hence B is recursively enumerable. Finally, B is not recursive since B is neither empty (the zero function $\mathbf{0}$ is in B) or equal to $\mathbb{N}$ (the everywhere undefined partial function is not in B).

11.29 By a similar argument to exercise 11.27, we can see that no recursive enumeration of finitely long lists can decide membership in C. Hence C is not recursively enumerable. In particular, C is not recursive.

11.30 We claim that ψ enumerates all partial recursive functions. Since φ is itself a numbering, it suffices to show that there exists a surjective map from the functions numbered by ψ to the functions numbered by φ. Any partial recursive function is of the form ϕ_e. We then consider cases. Define function $\phi(d, e) = \frac{(d+e)\cdot(d+e+1)}{2} + e$, this is a bijection between $\mathbb{N}^2$ and $\mathbb{N}$.

- If $\varphi_e(0)$ is undefined then $\psi_{\phi(0,e)} = \varphi_e$
- If $\varphi_e(0)$ is defined then let $d = \varphi_e(0) - 1$ and $\psi_{\phi(d,e)} = \varphi_e$

Hence ψ enumerates all partial recursive functions.

I'm not so sure about the next part, but this is my intuitive argument. I claim that ψ is not acceptable. Suppose not, such that ψ is acceptable. Then there exists a recursive function g such that $\forall e \in \mathbb{N}, \varphi_e = \psi_{g(e)}$.

In particular, we can fix some $e' \in \mathbb{N}$. Let $g(e') = \phi(d, e)$ for some d, e.

We must have $e \in I := \{e \in \mathbb{N} : \varphi_e = \varphi_{e'}\}$ and $d = 0$ if $\varphi_{e'}(0)$ undefined and $d = \varphi_{e'}(0) + 1$ otherwise.

If $\varphi_{e'}(0)$ is undefined, then there is no total register machine that can halt on $\varphi_{e'}(0)$, so a total register machine would not be able to tell whether $\varphi_{e'}(0)$ is defined in any finite period of steps. Hence g cannot exist because a register machine is unable to decide if d should be 0.

An answer to this on math.stackexchange

The solution on math.stackexchange is a more refined version of my intuitive argument. That is, the existence of such a recursive function g suggests that the Halting Problem is computable, which is a contradiction. This is because, deciding the value of d (as mentioned above) is somewhat equivalent to deciding whether a partial recursive function halts on a certain input.

11.31

12.4 Note that here $a - b$ is defined as 0 if $a < b$. We assume that $R_4 = 0$ initially.

1. Operation $R_1 = R_2 - R_3$
2. If $R_1 = 0$ GOTO Line 4
3. $R_1 \leftarrow R_1 - 1$; GOTO Line 2
4. If $R_2 = 0$ GOTO Line 6
5. $R_4 \leftarrow R_4 + 1$; $R_2 \leftarrow R_2 - 1$; GOTO Line 4;
6. If $R_4 = 0$ GOTO Line 8
7. $R_4 \leftarrow R_4 - 1$; $R_2 \leftarrow R_2 + 1$; $R_1 \leftarrow R_1 + 1$; GOTO Line 6; // evaluate $R_1 \leftarrow R_2$

8. If $R_3 = 0$ GOTO Line 10
9. $R_4 \leftarrow R_4 + 1$; $R_3 \leftarrow R_3 - 1$; GOTO Line 8;
10. If $R_4 = 0$ GOTO Line 15
11. If $R_1 = 0$ GOTO Line 13
12. $R_4 \leftarrow R_4 - 1$; $R_1 \leftarrow R_1 - 1$; $R_3 \leftarrow R_3 + 1$; GOTO Line 10; // evaluate $R_1 \leftarrow R_2 - R_3$
13. If $R_4 = 0$ GOTO Line 15;
14. $R_4 \leftarrow R_4 - 1$; $R_3 \leftarrow R_3 + 1$; GOTO Line 13; // transfer the rest of R_4 back to R_3
15. Continue to next operation

In general, for such questions, it is easier to do the conditional check before looping.

12.5 We assume that $R_4 = 0$ initially.

1. Operation If $R_1 \leq R_2$ GOTO Line 200
2. If $R_1 = 0$ GOTO Line 5
3. If $R_2 = 0$ GOTO Line 7
4. $R_4 \leftarrow R_4 + 1$; $R_1 \leftarrow R_1 - 1$; $R_2 \leftarrow R_2 - 1$; GOTO Line 2;
5. If $R_4 = 0$ GOTO Line 200 // $R_1 \leq R_2$
6. $R_4 \leftarrow R_4 - 1$; $R_1 \leftarrow R_1 + 1$; $R_2 \leftarrow R_2 + 1$; GOTO Line 5;
7. If $R_4 = 0$ GOTO Line 9 // $R_1 > R_2$
8. $R_4 \leftarrow R_4 - 1$; $R_1 \leftarrow R_1 + 1$; $R_2 \leftarrow R_2 + 1$; GOTO Line 7;
9. Continue to next operation

12.8 On input 001, we trace the algorithm using the encoding. The separators | are only there for clarity. We understand exponentiation of 4 in this way. 4^x is 4 concatenated x times and 4^{a^b} is 4 concatenated a^b times, with a^b being arithmetic exponentiation. In abstract algebra, one way to differentiate different notions of exponentiation is to place the multiplicative operation next to the exponent. E.g. if we let $\cdot$ denote concatenation and $\times$ denote multiplication in $\mathbb{R}^\times = \mathbb{R} - \{0\}$, then we have 4^{a^b} written as $4^{\cdot, a^{\times, b}}$.

$$34|03^24|3^34|03^24^{2^1}|3^34^{2^1}|13^24^{2^2}|3^44^{2^2}|3^54^{2^2}|23^24^{2^1}|3^74^{2^1}\,\text{(accept)}$$

12.9 For input 001111000, the trace is

$$34|03^24|3^34|03^24^2|3^34^2|13^24^4|3^44^4|3^54^4|3^24^2|3^44^2|3^54^2|13^24|3^44|3^64\text{(reject)}$$

Theorem 12.10 A detail to note in the proof is that the set of all $S \Rightarrow v_1 \Rightarrow \cdots \Rightarrow v_n$, $n \in \mathbb{N}$ is countable.

Because for a fixed n, $S \Rightarrow v_1 \Rightarrow \cdots \Rightarrow v_n$ is countable (in fact finite since there are finitely many substitutions for each v_i). Since $\mathbb{N}$ is countable, we have a countable union of countable sets, hence we still have an at most countable domain.

We need to consider this detail since partial recursive functions need to have an at most countable domain.

Corollary 12.11 Point 2 of the corollary says this: There exists 2 DCF languages L_1, L_2 such that it is not possible to determine empty-ness of $L_1 \cap L_2$. The proof goes: Take an **undecideable recursively enumerable language** K, and let L, H be the two DCFL described in Theorem 12.7.

Note that $L \cap H$ encode all valid computations of a counter machine computing K. Then for any word $w \in \{0,1\}^*$, we consider the set R_w as described in the proof of Corollary 12.11. Define H_w the same way as well.

Then, we have the existence of two DCF languages $L_1 = L$, $L_2 = H_w$, such that it is undecideable in general whether $L_1 \cap L_2 = \emptyset$. Because, $L \cap H_w \neq \emptyset$ iff $w \in K$, and we know that it is always possible to decide membership of w in K by our choice of K.

12.13 There is no solution. We claim that no pair can go first. The reason is that we can observe that for any pair (x_i, y_i), neither is the prefix of the other.
For instance, $(123, 1258)$ cannot go first since there will be a mismatch in the third letter.

12.14 Yes, there is a solution with indices 6,1,3. The corresponding word would be 89125.

12.15 First, we show that there exists a DPDA that accepts L. (H is shown similarly)

$$Q = \{s, t, r\}, F = \{t\}, N = \{S\} \cup \{i' : i \in \{1, 2, \ldots, n\}\}$$

Starting state s, Starting stack S.

The set of transitions is defined below:

$$\begin{aligned}
\forall i \in \{1, 2, \ldots, n\}, \delta(s, i, A) &= \{(s, i'A)\} \\
\delta(s, n+1, A) &= \{(t, A)\} \\
\forall j \in \{1, 2, \ldots, n\}, x_j = x_i \implies \delta(t, x_i, i') &= \{(t, \epsilon)\} \\
\forall j \in \{1, 2, \ldots, n\}, x_j \neq x_i \implies \delta(t, x_i, i') &= \{(r, \epsilon)\} \\
\delta(t, x_i, S) &= \{(r, \epsilon)\}
\end{aligned}$$

where A can be any symbol on the stack.

If we allow ϵ transitions, we can have a slightly clearer presentation

$$\begin{aligned}
\forall i \in \{1, 2, \ldots, n\}, \delta(s, i, \epsilon) &= \{(s, i')\} \\
\delta(s, n+1, \epsilon) &= \{(t, \epsilon)\} \\
\forall j \in \{1, 2, \ldots, n\}, x_j = x_i \implies \delta(t, x_i, i') &= \{(t, \epsilon)\} \\
\forall j \in \{1, 2, \ldots, n\}, x_j \neq x_i \implies \delta(t, x_i, i') &= \{(r, \epsilon)\} \\
\delta(t, x_i, S) &= \{(r, \epsilon)\}
\end{aligned}$$

Note that the automata is deterministic since each set only has 1 tuple.

It is easy to see the equivalence between $L \cap H \neq \emptyset$ and $\exists k_1, k_2, \ldots, k_m \in \{1, 2, \ldots, n\}$, $m \geq 1$ such that $x_{k_1} x_{k_2} \ldots x_{k_m} = y_{k_1} y_{k_2} \ldots y_{k_m}$.

12.20 For the conditions to hold, we must have $\lfloor \frac{w}{5} \rfloor = 0^*10^*10^*$.

Next, we note that $3^5 = 243$.

We consider cases, in terms of overlap.

- Overlap of 2 letters: $2430 + 243 = 2673$
- Overlap of 1 letter: $24300 + 243 = 24543$
- No overlap

Hence, the corresponding set for w is

$$0^*26730^* \cup 0^*245430^* \cup 0^*2430^*2430^*$$

12.21 We consider cases.

$$w = 7v_1 + r_1$$
$$v_1 = 7v_2 + r_2$$

where $r_1, r_2 \in \{0, 1, \dots, 6\}$.

We then consider cases.

- If $r_1 = r_2 = 5$, then we have $w = 49v_1 + 40$
- If $r_1 = 4 \wedge r_2 = 6$, then we have $w = 49v_1 + 46$
- If $r_1 = 6 \wedge r_2 = 4$, then we have $w = 49v_1 + 34$

Hence $w \in L \iff w \mod 49 \in \{34, 40, 46\}$. It then suffices to construct a dfa that recognizes all natural numbers that are $34, 40, 46$ modulo 49.

For revision purposes, here is the dfa $M = (Q, \Sigma, \delta, s, F)$

- $Q = \{q_i : i \in \mathbb{Z}/49\mathbb{Z}\}$
- $\Sigma = \{0, 1, 2, \dots, 9\}$
- $s = q_0$
- $F = \{q_{34}, q_{40}, q_{46}\}$
- $\delta : Q \times \Sigma \to Q, \delta = (q_i, j) \mapsto q_{(i+j) \mod 49}$

12.22 Claim: The Turing machine, referred to as T, recognizes the language $L = \{2\dot{(}2n + 1) : n \in \mathbb{N}\} = \{2(1), 2(3), 2(5), \dots\}$, i.e. all natural numbers that have precisely 1 factor of 2.

Let $w = b_k b_{k-1} \dots b_1$ be any word in $\{0, 1, 2\}^*$ to be processed.

First, we observe that $\sum_k b_k 3^k \equiv \sum_k b_k \equiv \sum_k [b_k = 1] (\mod 2)$. Hence, the natural number interpretation w is even iff $\sum_k [b_k = 1]$ is even, that is, the number of 1s in w is even.

Next, we observe that the single pass of T is effectively doing division by 2, using the elementary school algorithm.
To see this, we define two variables, a being the variable keeping track of the remainder-carry, b being the variable represents the current letter of w being read by T.
We note that for any value of a, the next value of a, which we will denote as a' after processing b, will be $\frac{3a+b}{2} - \lfloor \frac{3a+b}{2} \rfloor$. The value written by T to the tape will be denoted $b' := \lfloor \frac{3a+b}{2} \rfloor$. This is all based on elementary school division.
Initially, $a = 0$. We then consider cases.

- If $b = 0$, then $a' = 0, b' = 0$
- If $b = 2$, then $a' = 0, b' = 1$

- If $b = 1$, then $a' = 1, b' = 0$

We see that when T encounters a 1, $a \leftarrow a' = 1$.

When $a = 1$, we also consider cases.

- If $b = 0$, then $a' = 1, b' = 1$
- If $b = 2$, then $a' = 1, b' = 2$
- If $b = 1$, then $a' = 0, b' = 2$

We see that when T encounters a 1, $a \leftarrow a' = 0$

This is precisely the behavior of T.

Hence, we have shown that the language decided by T are all ternary words (possibly with leading zeroes) that represent natural numbers that have only 1 factor of 2 in their prime factorization.

Selftest 12.25 An alternative solution. This takes inspiration from a PDA. We keep a primed nonterminal $0'$ to indicate the last 0.

$N = \{S, T, 0'\}, \Sigma = \{0\}$

- $S \to ST|0'$
- $0'T \to 0'0$
- $0T \to 00T$
- $0' \to 0$

Sample derivation for 0^4.
$S \Rightarrow ST \Rightarrow STT \Rightarrow 0'TT \Rightarrow 0'0T \Rightarrow 0'T00 \Rightarrow 0'000 \Rightarrow 0000$