ST2132

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1 Limit Theorems

Properties of weak convergence (Also known as convergence in probability)

Additivity Suppose $A_n \to_P \alpha, B_n \to_P \beta$ for sequences of random variables $(A_n), (B_n)$. Then,

$$P(|(A_n + B_n) - (\alpha + \beta)| > \epsilon) \leq P(|A_n - \alpha| + |B_n + \beta| > \epsilon) \leq P(\{|A_n - \alpha| > \frac{\epsilon}{3}\} \cup \{|B_n - \beta| > \frac{\epsilon}{3}\}) \to 0 + 0 = 0$$

Therefore, $A_n + B_n \to_P \alpha + \beta$.
Here, we use the inequalities $|A_n - \alpha| + |B_n - \beta| \geq |A_n - \alpha + B_n - \beta|$ and $\epsilon > \frac{\epsilon}{3} + \frac{\epsilon}{3}$.

Closure under Continuity Suppose $X_n \to_P \alpha$. Let g be a continuous function. Then in particular, g is continuous at α.

For a fixed ϵ, let $\delta > 0$ be such that $|x - \alpha| \leq \delta \implies |g(x) - g(\alpha)| \leq \epsilon$. The converse then says that $|g(x) - g(\alpha)| > \epsilon \implies |x - \alpha| > \delta$, which is what we will use below.

$$P(|g(X_n) - g(\alpha)| > \epsilon) \leq P(|X_n - \alpha| > \delta) \to 0$$

Hence, $g(X_n) \to_P g(\alpha)$.