# ST2132

Jia Cheng

Jan 2022

## 1 Limit Theorems

**Properties of weak convergence** (Also known as convergence in probability)

**Additivity** Suppose $A_n \to_P \alpha, B_n \to_P \beta$ for sequences of random variables $(A_n), (B_n)$. Then,

$$P(|(A_n + B_n) - (\alpha + \beta)| > \epsilon) \leq P(|A_n - \alpha| + |B_n + \beta| > \epsilon) \leq P(\{|A_n - \alpha| > \frac{\epsilon}{3}\} \cup \{|B_n - \beta| > \frac{\epsilon}{3}\}) \to 0 + 0 = 0$$

Therefore, $A_n + B_n \to_P \alpha + \beta$.
Here, we use the inequalities $|A_n - \alpha| + |B_n - \beta| \geq |A_n - \alpha + B_n - \beta|$ and $\epsilon > \frac{\epsilon}{3} + \frac{\epsilon}{3}$.

**Closure under Continuity** Suppose $X_n \to_P \alpha$. Let $g$ be a continuous function. Then in particular, $g$ is continuous at $\alpha$.

For a fixed $\epsilon$, let $\delta > 0$ be such that $|x - \alpha| \leq \delta \implies |g(x) - g(\alpha)| \leq \epsilon$. The converse then says that $|g(x) - g(\alpha)| > \epsilon \implies |x - \alpha| > \delta$, which is what we will use below.

$$P(|g(X_n) - g(\alpha)| > \epsilon) \leq P(|X_n - \alpha| > \delta) \to 0$$

Hence, $g(X_n) \to_P g(\alpha)$.

## 2 Distributions derived from Normal distribution

**t-distribution pdf derivation** First, note that $T = \frac{Z}{\sqrt{\frac{U}{n}}} \sim t_n$.
Let $Y = Z, X = \sqrt{\frac{U}{n}}$.

$$F_X(u) = P(\sqrt{\frac{U}{n}} \leq u) = P(U \leq nu^2) = F_U(nu^2)$$

Hence,

$$\begin{aligned} f_X(u) = 2nuf_U(u) &= 2nu \cdot \frac{\left(\frac{1}{2}\right)^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} (nu^2)^{\frac{n}{2}-1} e^{-\frac{nu^2}{2}} [nu^2 \geq 0] \\ &= \frac{n^{\frac{n}{2}}}{2^{\frac{n}{2}-1}\Gamma\left(\frac{n}{2}\right)} u^{n-1} e^{-\frac{nu^2}{2}} [u \geq 0] \end{aligned}$$

Note the formula for the pdf of a quotient.

$$f_{\frac{Y}{X}}(t) = \int_{\mathbb{R}} |x| f_X(x) f_Y(tx) dx$$

Hence,

$$
\begin{aligned}
f_T(t) &= \int_{\mathbb{R}} |x| \frac{n^{\frac{n}{2}}}{2^{\frac{n}{2}-1}\Gamma\left(\frac{n}{2}\right)} x^{n-1} e^{-\frac{nt^2}{2}} [t \geq 0] \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(tx)^2}{2}} dx \\
&= \frac{n^{\frac{n+1}{2}}}{2^{\frac{n-1}{2}}\sqrt{n\pi}\Gamma\left(\frac{n}{2}\right)} \int_{\mathbb{R}^+} x^n e^{\frac{-(n+t^2)x^2}{2}} dx \\
&= \frac{n^{\frac{n+1}{2}}}{2^{\frac{n-1}{2}}\sqrt{n\pi}\Gamma\left(\frac{n}{2}\right)} \int_{\mathbb{R}^+} \left(\frac{2}{n+t^2}u\right)^{\frac{n}{2}} e^{-u} \frac{1}{\sqrt{n+t^2}\sqrt{2u}} du \text{ reverse substitution: } x = \sqrt{\frac{2}{n+t^2}u} \\
&= \frac{n^{\frac{n+1}{2}}}{2^{\frac{n-1}{2}}\sqrt{n\pi}\Gamma\left(\frac{n}{2}\right)} \cdot \frac{2^{\frac{n-1}{2}}}{(n+t^2)^{\frac{n+1}{2}}} \int_{\mathbb{R}^+} u^{\frac{n-1}{2}} e^{-u} du \\
&= \frac{1}{\Gamma\left(\frac{n}{2}\right)\sqrt{n\pi}} \left(\frac{n+t^2}{n}\right)^{-\frac{n+1}{2}} \int_{\mathbb{R}^+} u^{\frac{n+1}{2}-1} e^{-u} du \\
&= \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\sqrt{n\pi}} \left(\frac{n+t^2}{n}\right)^{-\frac{n+1}{2}}
\end{aligned}
$$

**Square of t-distribution**

$$T^2 = \frac{Z^2}{\frac{U}{n}}$$

where $Z^2 \sim \chi_1^2, U \sim \chi_n^2$, hence $T^2 \sim F_{1,n}$

**F-distribution pdf derivation** Note that

$$W = \frac{\frac{U}{m}}{\frac{V}{n}}$$

where $U \sim \chi_m^2, V \sim \chi_n^2$.

We can derive that

$$
\begin{aligned}
F_{\frac{U}{m}}(x) = F_U(mx) &\implies f_{\frac{U}{m}}(x) = mf_U(mx) \\
F_{\frac{V}{n}}(x) = F_V(nx) &\implies f_{\frac{V}{n}}(x) = nf_V(nx)
\end{aligned}
$$

$$
\begin{aligned}
f_W(w) &= \int_{\mathbb{R}} |x| f_X(x) f_Y(wx) dx \\
&= \int_{\mathbb{R}} |x| \cdot nf_V(nx) \cdot mf_U(mwx) dx \\
&= \int_{\mathbb{R}} |x| \cdot mn \cdot [nx \geq 0] \frac{\left(\frac{1}{2}\right)^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} (nx)^{\frac{n}{2}-1} e^{-\frac{nx}{2}} \cdot [mwx \geq 0] \frac{\left(\frac{1}{2}\right)^{\frac{m}{2}}}{\Gamma\left(\frac{m}{2}\right)} (mwx)^{\frac{m}{2}-1} e^{-\frac{mwx}{2}}
\end{aligned}
$$

We consider cases.

- If $w \leq 0$, then $[nx \geq 0][mwx \geq 0] = [x \geq 0][x \leq 0] = [x = 0]$, so the integral is over a single point, hence evaluates to 0.
- Otherwise, if $w > 0$, then $[nx \geq 0][mwx \geq 0] = [x \geq 0]$, so the integral can then be restricted to $\mathbb{R}^+$.

Hence, in the case where $w > 0$, we need to evaluate

$$\begin{aligned}
f_W(w) &= \int_{\mathbb{R}^+} x \cdot mn \cdot \frac{\left(\frac{1}{2}\right)^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} (nx)^{\frac{n}{2}-1} e^{-\frac{nx}{2}} \cdot \frac{\left(\frac{1}{2}\right)^{\frac{m}{2}}}{\Gamma\left(\frac{m}{2}\right)} (mwx)^{\frac{m}{2}-1} e^{-\frac{mwx}{2}} \\
&= \frac{\left(\frac{1}{2}\right)^{\frac{m+n}{2}} (m)^{\frac{m}{2}} w^{\frac{m}{2}-1} n^{\frac{n}{2}}}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} \int_{\mathbb{R}^+} x^{\frac{m+n}{2}-1} e^{-\frac{n+mw}{2}x} dx \\
&= \frac{\left(\frac{1}{2}\right)^{\frac{m+n}{2}} \left(\frac{m}{n}\right)^{\frac{m}{2}} w^{\frac{m}{2}-1} n^{\frac{m+n}{2}}}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} \int_{\mathbb{R}^+} x^{\frac{m+n}{2}-1} e^{-\frac{n+mw}{2}x} dx \\
&= \frac{\left(\frac{1}{2}\right)^{\frac{m+n}{2}} \left(\frac{m}{n}\right)^{\frac{m}{2}} w^{\frac{m}{2}-1} n^{\frac{m+n}{2}}}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} \int_{\mathbb{R}^+} \left(\frac{2}{n+mw}u\right)^{\frac{m+n}{2}-1} e^{-u} \frac{2}{n+mw} dx \text{ Substitution: } u = \frac{n+mw}{2}x \\
&= \frac{\left(\frac{1}{2}\right)^{\frac{m+n}{2}} \left(\frac{m}{n}\right)^{\frac{m}{2}} w^{\frac{m}{2}-1} n^{\frac{m+n}{2}}}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} \int_{\mathbb{R}^+} \left(\frac{2}{n+mw}u\right)^{\frac{m+n}{2}-1} e^{-u} \frac{2}{n+mw} dx \\
&= \frac{\left(\frac{m}{n}\right)^{\frac{m}{2}} w^{\frac{m}{2}-1} n^{\frac{m+n}{2}}}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} \int_{\mathbb{R}^+} \left(\frac{1}{n+mw}u\right)^{\frac{m+n}{2}-1} e^{-u} \frac{1}{n+mw} dx \\
&= \frac{\left(\frac{m}{n}\right)^{\frac{m}{2}} w^{\frac{m}{2}-1}}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} \left(\frac{n+mw}{n}\right)^{-\frac{m+n}{2}} \int_{\mathbb{R}^+} u^{\frac{m+n}{2}-1} e^{-u} dx \\
&= \frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} \left(\frac{m}{n}\right)^{\frac{m}{2}} w^{\frac{m}{2}-1} \left(1 + \frac{m}{n}w\right)^{-\frac{m+n}{2}}
\end{aligned}$$