

MA1101R

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Uniqueness of row-reduced echelon forms We provide a proof for the uniqueness of RREFs.

Let A be an augmented $m \times (n+1)$ matrix representing a linear system S in n variables. Let R_1, R_2 be 2 matrices in RREF equivalent to A . (Here equivalence is known as row equivalence.) By symmetry and transitivity of equivalence, we have $R_1 \equiv R_2$.

Claim: The pivot (and non-pivot) columns of R_1, R_2 are the same.

Suppose not. WLOG (if not we simply swap the roles of R_1 and R_2), there exists an index $i \in \{1, 2, \dots, n\}$ such that the i -th column is a pivot column in R_2 but not a pivot in R_1 . We then form a new $2m \times (n+1)$ matrix by stacking R_2 below R_1 .

We note that the linear system S' represented by this stacked matrix is equivalent to the original linear system S because: Let S_1 be the linear system described by R_1 , S_2 be the linear system described by R_2 . Then $S' = S_1 \wedge S_2 = S \wedge S = S$.

By performing Gauss-Jordan elimination again, we see that the resulting matrix consists of strictly more pivot columns than R_1 originally had. In particular, all the pivot columns of R_1 are still pivot columns, plus column i and possibly some more columns. This is a contradiction since the general set of solutions had a strictly smaller dimension. Even if we ignore dimension, we can make the following argument. Let $C \subseteq \{1, 2, \dots, n\}$ be the indices of non-pivot columns in R_1 , C' be the number of non-pivot columns in the RREF of the stacked matrix. Then $C' \subsetneq C$.

For any fixed tuple corresponding to the non-pivot columns of C' , there are infinitely many solutions to be found by S_1 , since $C - C' \neq \emptyset$ and we can vary any variable corresponding to the non-pivot columns in $C - C'$. Whereas in system S' , fixed the variables corresponding to C' forces a single tuple of solutions. This suggests that S_1 and S are not equivalent, which is a contradiction.

Hence R_1, R_2 have the same pivot columns. Now we consider the first row of R_1, R_2 .

Claim: The first row of these 2 matrices must be the same.

We already know that cells in other pivot columns must be zero for both matrices, by definition of RREF. Now consider the cells corresponding to the non-pivot columns. We see that the variable corresponding to the pivot cell can be expressed in terms of the variables corresponding to the non-pivot cells. In other words, for both R_1, R_2 , the coefficients w.r.t. those non-pivot variables must be the same. This says that all cells in the first row of R_1, R_2 must be the same.

By repeating this argument for every row, we prove that all rows of R_1, R_2 are equal and hence $R_1 = R_2$.