Advanced Probability

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Reference Text: A First Look at Rigorous Probability Theory

Related modules: ST5214

1.1 To see that $\exists z \in \mathbb{R}, P(Z = z) > 0$, we observe that

$$P(Z = 0) \geq P(Z = 0 \land X = 0) = P(Z = 0|X = 0) \cdot P(X = 0) \geq \frac{1}{2} \cdot P(X = 0) > 0$$

since $P(X = 0) > 0$.

1.2

Uncountable summation Given an uncountable non-negative set of numbers $\{r_a : a \in I\}$ indexed by I,

$$\sum_{a \in I} r_a := \sup\{\sum_{a \in J} r_a : J \subseteq I \land J \text{ finite}\}$$

R-shift (Equivalent definition) R-shift of $A \subseteq [0, 1]$. $A \oplus r = \{(a + r) \mod 1 : a \in A\}$

2.1 Notice that countability is used by 2 constructs. One, probability measure is countably additive (and not uncountably so). Two, the σ-algebra is closed under countable union and intersection (and not uncountably so).

Recall that the reason for disallowing uncountable operations in general is due to the fact that

$$\bigcup_{x \in A} \{x\} = A$$

for any set A, in particular $[0, 1]$ when discussing the uniform distribution on the unit interval.

Theorem 2.2.1 We provide a proof for this theorem.

First, we show that $\mathcal{F}$ is a σ-algebra. By definition, $\mathcal{F} = \mathcal{P}(\Omega)$. Hence, the unary complement operation is a mapping $\mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$ whose domain and codomain are just $\mathcal{F}$ as desired. Similarly, for the countable set operations of union and intersection, they are mappings with codomain as $\mathcal{P}(\Omega)$ and are also closed since $\mathcal{F} = \mathcal{P}(\Omega)$.
We also note that both $\emptyset, \Omega$ reside in $\mathcal{F}$

Next, we show that P is a probability measure. By definition of P, P is additive since $A \cap B = \emptyset \implies P(A \sqcup B) = \sum_{\omega \in A \sqcup B} p(\omega) = \sum_{\omega \in A} p(\omega) + \sum_{\omega \in B} p(\omega) = P(A) + P(B)$.

I am not quite sure about showing countable additivity however, perhaps using some form of diagonal summation argument it is possible to prove this.

$P(\Omega) = \sum_{\omega \in \Omega} p(\omega) = 1$. Furthermore, p is non-negative, hence P is indeed bounded between 0 and 1.

Ex 2.2.3 First, $\emptyset, \Omega = [0,1] \in \mathcal{J}$ by definition as they are intervals. Next, to show closure under finite intersection, it suffices to show closure under binary intersection. Consider cases: We only consider one endpoint, since we can "patch" together two endpoints.

- $[a$ and $[b$ intersect to give $[\max\{a,b\}$
- $(a$ and $(b$ intersect to give $(\max\{a,b\}$
- $[a$ and $(b$ intersect to give $[a$ if $a > b$ and $(b$ otherwise

We can do a similar case analysis for right endpoints. Given a stringified left endpoint $l \in \{"[a", "(a"\}$ and a stringified right endpoint $r \in \{"b]", "b)"\}$ we can form an interval via concatenation l, r. Hence, $\mathcal{J}$ is closed under finite intersection.

Consider the complement $J = [0,a) \cup (b,1]$ of an interval $[a,b] \in \mathcal{J}$, where depending on whether the left/right endpoint is closed or open we adjust J accordingly. Regardless, we see that J is a disjoint union of at most 2 intervals in $\mathcal{J}$.

Hence, $\mathcal{J}$ is a semialgebra of subsets of Ω.

Ex 2.2.5

a $\mathcal{B}_0 \subseteq \mathcal{P}(\Omega)$. Since $\mathcal{B}_0$ consists of all finite unions of elements of $\mathcal{J}$, in particular, $\mathcal{J} \subseteq \mathcal{B}_0$, so $, \Omega = [0,1] \in \mathcal{J} \subseteq \mathcal{B}_0$.

Next, the finite union and intersection of elements of $\mathcal{B}_0$ will give finite unions of elements of *semialg*, so that $\mathcal{B}_0$ is closed under finite union and intersection. (For intersection, we can argue using distributive law plus observe that the intersection of intervals gives another interval)

Let $B \in \mathcal{B}_0$, so that B is a finite union of the form $\bigcup_{1 \leq i \leq n} I_i$ for some intervals I_i in $[0,1]$. Then, $B^c = \bigcap_{1 \leq i \leq n} I_i^c$ by DeMorgan's Law, and we have already proven in Ex 2.2.3 that I_i^c is a disjoint union of intervals, i.e. $I_i^c \in \mathcal{B}_0$. Furthermore, we have proven that $\mathcal{B}_0$ is closed under finite intersection, so $B^c \in \mathcal{B}_0$.

Hence, $\mathcal{B}_0$ is an algebra.

b The difference between an algebra and a σ-algebra is that σ-algebras are closed under countable union and intersection but algebras are not necessarily so.

We consider Cantor's set C, which is a countable intersection of C_i, where each C_i is formed by removing from each interval in C_{i-1} the middle one-third.

Since each C_i is a union of (disjoint) intervals, by definition, $C_i \in \mathcal{B}_0$. If $\mathcal{B}_0$ is to be a σ-algebra, then we must have $C \in \mathcal{B}_0$, i.e. C can be formed from a finite union of intervals.

First of all, C does not contain any interval of non-zero length, so our options are reduced to forming C from a finite union of singletons, i.e. intervals of the form $[a,a] = \{a\}$. But we also know C to be uncountable, but a finite union of singletons is finite. Hence we have a contradiction.

Since $C \notin \mathcal{B}_0$, $\mathcal{B}_0$ is not closed under countable intersection, so it is not a σ-algebra.

We comment that $\mathcal{B}_1$ is similarly not a σ-algebra using the same counterexample. The countable union of singletons must be at most countable, so they cannot union to form an uncountable set like C.

Ex 2.3.16 Suppose $A \in \mathcal{M} \wedge P^*(A) = 0$ and $B \subseteq A$. To show that $B \in \mathcal{M}$, we show equivalently that:

For each $E \subseteq \Omega$, $P^*(E) \geq P^*(A \cap E) + P^*(A^c \cap E)$ (i.e. superadditivity. But by monotonicity, $P^*(A \cap E) = 0$, so that $P^*(E) \geq P^*(A^c \cap E)$ is automatically true by monotonicity. Hence, we have shown the completeness of the extension $(\Omega, \mathcal{M}, P^*)$.