Simple 'log formulae' for pendulum motion valid for any amplitude

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Simple 'log formulae' for pendulum motion valid for any amplitude

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Abstract

By combining a logarithmic approximate formula for the pendulum period derived recently (valid for amplitudes below $\pi/2$ rad) with the Cromer asymptotic approximation (valid for amplitudes near to π rad), a new approximate formula accurate for all amplitudes between 0 and π rad is derived here. It is shown that this formula yields an error that tends to zero in both the small and large amplitude limits, a feature not found in any previous approximate formula. Some ways of refining this formula are also presented. Interestingly, when one of the improved expressions is taken for building a sinusoidal (harmonic) approximation to the solution of the pendulum equation of motion very good agreement is found. The simple log formulae derived here require only a few elementary function calls in a pocket calculator for accurate evaluations, being useful for analyzing pendulum experiments in introductory physics labs. They may also be of interest for those specialists working with nonlinear phenomena governed by pendulum-like differential equations, which arise in many fields of science and technology (e.g., analysis of acoustic vibrations, oscillations in small molecules, optically torqued nanorods, Josephson junctions, electronic filters, gravitational lensing in general relativity, advanced models in field theory, oscillations of buildings during earthquakes, and others).

(Some figures in this article are in colour only in the electronic version)

1. Introduction

The simple pendulum oscillatory motion is among the most investigated motions in physics [1]. In the *small*-angle regime, the approximation $\sin\theta \approx \theta$, where θ is the angular displacement from vertical, leads to a linear equation of motion which is easily solvable in terms of trigonometric functions of time, the simplicity of this approximation allowing its presentation in introductory physics courses, usually as a good example of a physical system in which harmonic oscillations take place [2]. A simple formula is then derived for the pendulum period which works well for angular amplitudes below, say, 10° [2, 3]. Since this formula yields the period as a function of ℓ and g only (the length of the pendulum and the local acceleration of gravity, respectively), it is the basis for one of the main undergraduate experiments for measuring g [4]. For increasing amplitudes, however, the nonlinear nature of the pendulum motion soon becomes apparent and the period changes with the amplitude, a fact that makes the pendulum motion a pedagogically rich theme [1, 5]. Though the period can be determined, for any given amplitude, by evaluating an elliptic integral, this is often avoided in introductory physics because it is not possible to express this integral in a closed form in terms of elementary functions [6, 7]. This has made way for research on simple approximate formulae for the increase of the period with amplitude, for applications in introductory physics labs [3, 8], some graduate courses (e.g., classical mechanics and electromagnetism) [1, 9], and even in physics research (acoustics, electronics, superconductivity, etc) [1, 10, 11]. The approximate formulae found by different authors can be classified as follows:

(i) 'Not so large-angle' formulae, i.e. those yielding good estimates for amplitudes below $\pi/2$ rad (a natural limit for a bob on the end of a flexible string), though the deviation with respect to the exact period increases monotonically with amplitude, being unsuitable for amplitudes near to π rad [3, 10, 12–14]; and
(ii) 'Very large-angle' formulae, i.e. those which approximate the exact period asymptotically for amplitudes near to π rad, with an error that increases monotonically for smaller amplitudes (unsuitable for small amplitudes) [15, 16].

Of course, the increase in the period is more apparent for amplitudes between $\pi/2$ and π rad, as has been observed in many experiments using either a rigid rod or a disc (instead of a string) [8, 17–19]. As accurate timers and sensors are currently available for students in introductory physics labs, the experimental errors found in such 'very large-angle' experiments are already small enough for a comparison with the exact period and a very good agreement between theory and those experiments in which friction effects are negligible has been found [8]. Since this activity has been encouraged by many instructors, a *simple* approximate formula for the pendulum period valid for all possible amplitudes, to which the experimental data can be compared, is sought[1]. In this paper, I investigate how to combine some of the type (i) and type (ii) formulae, above, into a single approximate formula valid for all amplitudes. Some possible refinements are also presented, as well as a simple harmonic approximation to the solution of the pendulum equation.

2. The exact pendulum period and some logarithmic approximations

A simple pendulum consists of a particle of mass m attached to one end of a weightless rod of length ℓ, the other end being attached to a fixed point C, around which the rod rotates without friction. When the bob is released, from rest, in a position in which the rod makes an initial angle θ_0 between 0 and π rad with the vertical, an oscillatory motion is observed. By neglecting dissipative forces, the symmetric restoring force leads to oscillations with constant amplitude and period. The equation of motion for this system is the following *nonlinear* differential equation [6]:

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{g}{\ell}\sin\theta = 0. \tag{1}$$

[1] A *simple formula*, here, means a finite mathematical expression that requires only a few operations on a pocket calculator, as found in, e.g., [3, 13] and references therein.

In the small-angle regime, the approximation $\sin\theta \approx \theta$ works, yielding the usual linearization for the above differential equation [2, 6], as given by $\mathrm{d}^2\theta/\mathrm{d}t^2 + \omega^2\theta(t) = 0$, where $\omega \equiv \sqrt{g/\ell}$. For the above-mentioned initial conditions, i.e., $\theta(0) = \theta_0$ and $\mathrm{d}\theta/\mathrm{d}t = 0$ at $t = 0$, the solution of this initial value problem is simply

$$\theta(t) = \theta_0 \cos(\omega t). \tag{2}$$

Hence, in this regime the motion is harmonic with a period $T_0 = 2\pi\sqrt{\ell/g}$ [2].

Beyond this regime, (1) can be solved numerically by applying, e.g., the Runge–Kutta method. However, as we are interested in analytical approximations, let us make use of an integral expression for the period which comes from the conservation of mechanical energy [3, 6]. By using the trigonometric rule $\cos\theta = 1 - 2\sin^2(\theta/2)$ and then substituting $\sin(\theta/2) = k\sin\varphi$, where $k \equiv \sin(\theta_0/2)$ (known as the *modulus* of the elliptic integral, below), one has

$$t = \frac{1}{\omega}\int_{\varphi}^{\pi/2} \frac{1}{\sqrt{1 - k^2\sin^2\phi}}\,\mathrm{d}\phi. \tag{3}$$

This integral can be written as $t = [K(k) - u(\varphi, k)]/\omega$, where $K(k) \equiv \int_0^{\pi/2} 1\big/\sqrt{1 - k^2\sin^2\phi}\,\mathrm{d}\phi$ is the complete elliptic integral of the first kind and $u(\varphi, k) \equiv \int_0^{\varphi} 1\big/\sqrt{1 - k^2\sin^2\phi}\,\mathrm{d}\phi$ is the incomplete one [21]. Though these integrals cannot be expressed in a closed form in terms of elementary functions [7], the exact solution $\varphi(t)$ can be expressed analytically in terms of the basic elliptic functions for any θ_0 between 0 and π rad (i.e., $0 < k < 1$), as will be discussed in the next section. Let T be the exact period for a given amplitude θ_0 (i.e., for a given k). By noting that $\theta(T/4) = 0$, one deduces from (3) that the exact period is simply

$$T = \frac{4}{\omega}\int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2\sin^2\phi}}\,\mathrm{d}\phi. \tag{4}$$

For simplicity, let us adopt T_0 as the standard for periods. Since this definite integral is just $K(k)$, one has

$$\frac{T}{T_0} = \frac{2}{\pi}K(k). \tag{5}$$

As a '$K(k)$-key' is not available in pocket calculators, the computation of the exact period requires the numerical evaluation of $K(k)$, which is usually developed with numerical integration or some recursive iterations in a computer [11, 21].

Alternatively, if the amplitude is below $\pi/2$ rad one can choose some of the practical approximate formulae available in the literature, which yield good estimates with only a few elementary function calls. In this sense, a simple but accurate approximate formula has been found by Lima and Arun in a recent work [3]. By taking the points $(0, 1)$ and $(\pi/2, a)$, where $a \equiv \sqrt{1 - k^2} = \cos(\theta_0/2)$, for a linear interpolation of the function $\sqrt{1 - k^2\sin^2\phi}$ in (4), we found that [3]

$$\frac{T_{\log}}{T_0} = \frac{\ln(1/a)}{1 - a} \tag{6}$$

yields a maximum error of only 0.25% (for $\theta_0 = \pi/2$ rad). The singularity at $\theta_0 = \pi$ rad obtained with this log formula is a characteristic of the exact period that is not found even in more elaborate approximations [14, 16, 22]. The fact that $T_{\log} > T$ for all θ_0 between 0 and π rad is another important feature that will be taken into account in our search for a formula

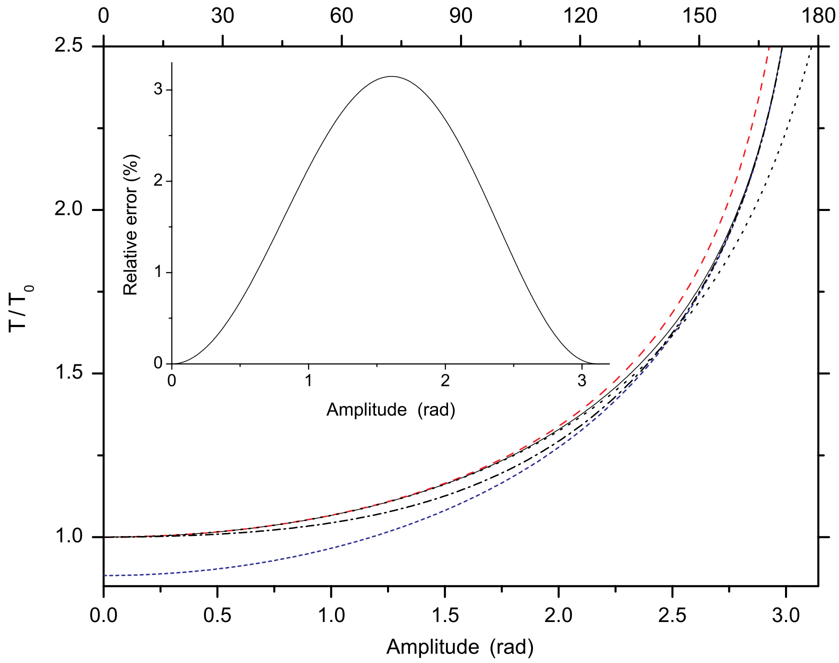

Figure 1. The increase of the pendulum period with θ_0. The solid line is for the exact period. The dashed line (red) is for the log formula of Lima and Arun. The dotted line is for Ganley's formula. The short-dashed line (blue) is for the Cromer formula. The dash-dotted line is for the new log-formula, found by averaging $T_{\log}$ and T_{Cromer}. Inset: the relative error found for T_{av} with respect to T. Note that the error tends to zero at both extremes.

valid for any amplitude. Among the existing simple formulae for $\theta_0 < \pi/2$ rad, the only competitor to $T_{\log}$ is Ganley's formula [12]:

$$\frac{T_{\mathrm{Ganley}}}{T_0} = \sqrt{\frac{\alpha}{\sin\alpha}}, \tag{7}$$

where $\alpha \equiv \sqrt{3}/2\theta_0$. From figure 1, it is clear that T_{Ganley} is more accurate than $T_{\log}$ for θ_0 up to 2.8 rad, but $T_{\log}$ is more accurate for larger amplitudes and, for θ_0 tending to π rad T_{Ganley} tends to 2.58 (in units of T_0), whereas $T_{\log}$ tends to infinity, as does the exact period[2]. This feature, together with the fact that T_{Ganley} *underestimates* T for any possible amplitude, led me to discard T_{Ganley} in favor of $T_{\log}$ in searching for a formula valid for all amplitudes.

For amplitudes between $\pi/2$ and π rad, which are also of interest for some applications [1, 11, 23], Cromer took into account the first term of a series expansion for $K(k)$ around $k = 1$ that converges for $0 < k < 1$ [21], finding that $K(k) \approx \ln(4/\sqrt{1-k^2})$ [16, 20]. By substituting this into (5), he found that

$$\frac{T_{\mathrm{Cromer}}}{T_0} = \frac{2}{\pi}\ln\left(\frac{4}{a}\right), \tag{8}$$

which is also a simple logarithmic formula. As seen in figure 1, T_{Cromer} always underestimates T, approximating it asymptotically when $\theta_0 \to \pi$ rad. Note, in this figure, that its deviation

[2] This is expected because the top vertical position is a point of (unstable) equilibrium.

from the exact period increases for decreasing amplitudes, reaching a maximum of 11.8% for $\theta_0 \to 0$.

The fact that $T_{\log}$ (T_{Cromer}) overestimates (underestimates) the exact period for all θ_0 between 0 and π rad aroused me to the possibility of averaging these two formulae in a manner to obtain a new log formula that could be valid for all amplitudes. By noting that the (trigonometric) relation $k^2+a^2 = 1$ remains for all θ_0, with k (a) tending to 0 (1) when $\theta_0 \to 0$ and k (a) tending to 1 (0) when $\theta_0 \to \pi$ rad, I deduced that a weighted average with statistical weights depending on θ_0 through k and a could be taken for approximating the behaviour of the exact period, as given by $T_{\text{av}} = a^2 T_{\log} + k^2 T_{\text{Cromer}}$. By taking (6) and (8) into account, I have found that

$$\frac{T_{\text{av}}}{T_0} = \frac{a^2}{1-a} \ln\left(\frac{1}{a}\right) + \frac{2k^2}{\pi} \ln\left(\frac{4}{a}\right). \tag{9}$$

This new log formula yields a very good approximation to the exact period, being practically indistinguishable from it in the ranges $\theta_0 < 0.5$ rad and $\theta_0 > 2.7$ rad, as seen in figure 1. In the inset, one sees that the relative error $(T_{\text{av}} - T)/T$ goes to zero at both extremes. The maximum error found is of 3%, for $\theta_0 = 1.61$ rad ($\approx 92°$).

Of course, this error can be reduced by refining the above averaging formula, though this affects its *simplicity*. Let me illustrate this by improving T_{Cromer} to $\widetilde{T}_{\text{Cromer}}$, as given by [21]

$$\frac{\widetilde{T}_{\text{Cromer}}}{T_0} = \frac{2}{\pi}\left[\ln\left(\frac{4}{a}\right) + \frac{a^2}{4}\left(\ln\left(\frac{4}{a}\right) - 1\right)\right]. \tag{10}$$

By putting this in (9), a curve similar to that for T_{av}, in figure 1, is obtained and the maximum error is reduced to 0.8% for $\theta_0 = 1.22$ rad ($\approx 70°$). Another easy way of refining the average formula in (9) is to search for the optimal numbers p and q such that, when multiplied by the weights a^2 and k^2, respectively, a minimum error is obtained. This changes (9) to $\widetilde{T}_{\text{av}} = (pa^2 T_{\log} + qk^2 T_{\text{Cromer}})/(pa^2 + qk^2)$, which is a difficult multivariate optimization problem. However, by noting that the two unknowns (p and q) can be combined into one by doing

$$\widetilde{T}_{\text{av}} = \frac{ra^2 T_{\log} + k^2 T_{\text{Cromer}}}{ra^2 + k^2}, \tag{11}$$

where $r = p/q$, the problem is reduced to a univariate optimization, for which the MAPLE subroutine NLPSolve (with the option *maximize*) is very efficient in finding, for each r, the maximum error for θ_0 between 0 and π rad. This yields $r = 7.17$, for which a maximum error of only 0.6% is found for $\theta_0 = 1.66$ rad ($\approx 95°$).

3. Solutions for the pendulum equation

Since the pendulum period increases with θ_0, the harmonic solution given in (2) is valid only within the small-angle regime. The exact solution for arbitrary amplitudes can be determined by solving the pendulum nonlinear differential equation, for a given set of initial conditions, either numerically or analytically. In the latter case, it can be shown mathematically that the solution cannot be expressed as a finite combination of elementary functions[3], though it can be written elegantly in terms of some special functions known as elliptic functions, as found independently by Abel and Jacobi in 1827 [9, 24]. The three basic Jacobi elliptic functions are defined as inverses of the incomplete elliptic integral $u(\varphi; k)$, for a given k ($k^2 < 1$) [21, 25]: $sn(u; k) \equiv \sin\varphi$, $cn(u; k) \equiv \cos\varphi$ and $dn(u; k) \equiv \sqrt{1 - k^2 \sin^2\varphi}$. All

[3] This is, in fact, a difficult mathematical problem that was first treated by Euler in 1736.

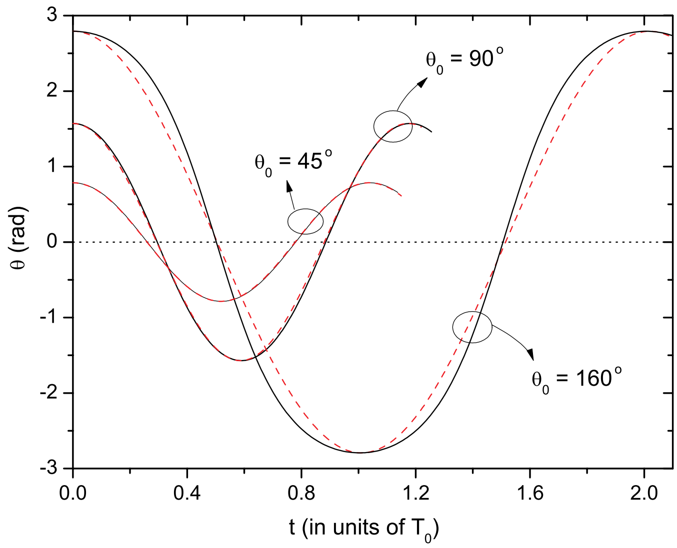

Figure 2. The sinusoidal approximate solution for the pendulum equation of motion (red, dashed lines) for three different initial angles: $\theta_0 = 45°, 90°$ and $160°$. The solid lines are for the corresponding exact solutions found in terms of the elliptic function $sn(u; k)$. The horizontal dotted line is just to mark the equilibrium position ($\theta = 0$).

these functions are periodic with period $4K(k)$. From the definition of $u(\varphi; k)$, one promptly finds that $\mathrm{d}u/\mathrm{d}\varphi = 1/dn(u; k)$. This allows us to apply the chain rule for calculating the first derivatives with respect to u. The result is: $sn' = cn(u; k)dn(u; k)$, $cn' = -sn(u; k)dn(u; k)$ and $dn' = -k^2 sn(u; k)cn(u; k)$ [21, 25]. Taking these derivatives into consideration, it is easy to show that [25]

$$\frac{\mathrm{d}^2 sn}{\mathrm{d}u^2} = 2k^2 sn^3 - (1 + k^2) sn. \tag{12}$$

The connection with the pendulum equation is found with the change of variable $z = \sin\varphi$. In this variable, equation (1) becomes

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} + \omega^2[(1 + k^2)z - 2k^2 z^3] = 0, \tag{13}$$

which is analogous to (12). In this new variable, the initial conditions stated for $\theta(t)$ just above (2) become $z(0) = 1$ and $\mathrm{d}z/\mathrm{d}t = 0$ at $t = 0$. By comparing equations (12) and (13), one finds that $z(t) = C sn(\omega t + \varphi_0; k)$ is, for some C and φ_0, the desired solution. By putting this function in (13) and imposing the initial conditions, it is easy to find that $C = 1$ and $\varphi_0 = K(k)$, which yields

$$z(t) = sn(\omega t + K(k); k). \tag{14}$$

Since $z = \sin(\theta/2)/k$, the exact solution for the pendulum initial value problem treated here is simply

$$\theta(t) = 2 \arcsin[k\, sn(\omega t + K(k); k)]. \tag{15}$$

With the above log formulae for the period in mind, I have noted that they could be taken into account for developing a harmonic approximate solution for the pendulum equation in the form

$$\theta(t) \approx \theta_0 \cos(\widetilde{\omega} t), \tag{16}$$

where $\widetilde{\omega} \equiv 2\pi/\widetilde{T}_{\rm av}$. Of course, this sinusoidal solution has a constant amplitude θ_0 and is periodic, with period $\widetilde{T}_{\rm av}$, as given by (11). The graph in figure 2 shows a period of my approximate solution for three distinct values of θ_0, namely 45°, 90° and 160° (dashed lines). The corresponding exact solutions, found by solving (1) numerically, are also shown for comparison (solid lines). For $\theta_0 = 45°$ ($k = 0.383$), the approximate and the exact solutions are indistinguishable. For $\theta_0 = 90°$ ($k = 0.707$), a very small dephasing is perceptible and the sinusoidal approximation works well. For $\theta_0 = 160°$ ($k = 0.985$), the harmonic solution shows small deviations from the exact solution, though their periods are practically the same.

4. Conclusion

In this work, I have used an weighted average for a log formula proposed recently for the pendulum period and the Cromer asymptotic approximation for deriving a new approximate formula accurate for all amplitudes between 0 and π rad. The error associated with this new formula tends to zero in both the small and large amplitude limits, a feature not found in any previous approximation. Some ways of refining this formula were also presented and one of the improved approximations was taken for composing a sinusoidal approximate solution for the nonlinear pendulum equation. This simple approximation exhibits only a small dephasing per period in comparison to the exact solution and is valid except for very large amplitudes, for which the exact solution stretches much in comparison to my harmonic approximation, certainly due to the increasing contributions of higher harmonics, as may be confirmed via Fourier analysis[4]. In carrying out large-angle pendulum experiments, the 'log formulae' presented here will certainly be useful since they require only a pocket calculator for accurate evaluations. Apart from being useful in both introductory and graduate physics courses, they can also be of interest for those specialists working with nonlinear phenomena governed by pendulum-like differential equations in many different fields of research (e.g., analysis of acoustic vibrations [10], oscillations in small molecules [1], optically torqued nanorods [26], Josephson superconducting junctions [1, 23], elliptic filters for electronic devices [11], gravitational lensing in general relativity [28], analysis of smectic *C* liquid crystals [27], advanced models in field theory [29], oscillations of buildings during earthquakes [30], etc).

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