Proof of Euler's Theorem w/ Maclaurin's

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^r}{r!} + \dots$$

$$e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \dots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} + \dots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right)$$

$$= \cos\theta + i\sin\theta$$

$\boxed{\text{START}}$

Finding Sq rt using a definite arithmetic procedure

$\sqrt{2}$, we choose 1.

$a' = \frac{1}{2}\left[a + \left(\frac{N}{a}\right)\right] \Rightarrow a' = \frac{1}{2}\left[1 + \frac{2}{1}\right] = 1.5$

$a'' = \frac{1}{2}\left[1.5 + \frac{2}{1.5}\right] = 1.4166$

$a''' = 1.4142$

if $a = \sqrt{2}$,

$a' = \frac{1}{2}\left[\sqrt{2} + \frac{2}{\sqrt{2}}\right] = \left(\sqrt{2} + \sqrt{2}\right)\frac{1}{2} = \sqrt{2}$

Forced Oscillation with damping

For $R = cv = m\gamma v$.

$R = ma = cv \Rightarrow \frac{ma}{v} \Rightarrow \frac{a}{v} = \gamma$??

↳ if γ small, friction small

γ big, friction big

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F \quad (ma = F - cv - kx)$$

$$m\left[\hat{x}e^{i\omega t}\right]'' + c\left[\hat{x}e^{i\omega t}\right]' + k\hat{x}e^{i\omega t} = \hat{F}e^{i\omega t}$$

where $\hat{F} = F_0 e^{i\phi}$

$$m(i\omega)^2\hat{x} + c(i\omega)\hat{x} + k\hat{x} = \hat{F}$$

$$\hat{x}(-m\omega^2 + ci\omega + k) = \hat{F}$$

$$\hat{x} = \frac{\hat{F}}{k - m\omega^2 + ci\omega} = \frac{\hat{F}}{m\omega_0^2 - m\omega^2 + ci\omega}$$

$$= \frac{\hat{F}}{m(\omega_0^2 - \omega^2 + \gamma i\omega)}$$

independent of t: $\hat{B}$

$$\hat{x} = \hat{F}\hat{B} \quad \text{where } \hat{B} = \frac{1}{m(\omega_0^2 - \omega^2 + \gamma i\omega)}$$

Expressing $\hat{F}$ & $\hat{B}$ in complex exp form;

$$\hat{x} = F_0 e^{i\phi} \times \rho e^{i\mu} = F_0\rho e^{i(\phi+\mu)}$$

$\times\, e^{i\omega t}$ throughout

↳ $x = \mathrm{Re}\left[\hat{x}e^{i\omega t}\right]$

Taking the real part,

$$x = \mathrm{Re}\left[F_0\rho e^{i(\omega t + \phi + \mu)}\right]$$

$$= F_0\rho\cos(\omega t + \phi + \mu)$$

ρ ↑ size of response

μ ↑ phase shift of response

→ ① ρ frictional term

② x does not oscillate in phase with the force.

↑ phase ϕ $\quad x = x_0\cos(\omega t + \phi)$ $\boxed{+\ \mu}$

$r^2 = zz^* = (x + yi)(x - yi)$

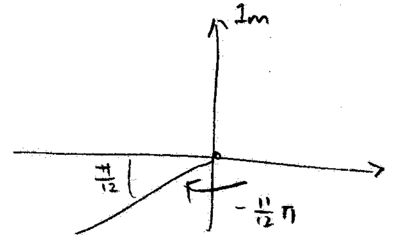
$a \cdot z^{-1}$

$z^5 = 5 = 5e^{i(2\pi k)}$

$z = 5^{\frac{1}{5}} e^{i\left(\frac{2k}{5}\pi\right)} = \sqrt[5]{5}\, e^{i\left(\frac{2}{5}\pi\right)}$ .....

$$\left(2^{\frac{1}{4}} e^{\left(\frac{13}{12}\pi\right)}\right)^4 = 2e^{\frac{13}{3}\pi} = 2e$$

eliminate ans more than π



$$\frac{e^{i\theta}}{1 - e^{i\theta}} \times \frac{e^{i\frac{\theta}{2}}}{e^{-i\frac{\theta}{2}}}$$

$$= \frac{e^{i\frac{\theta}{2}}}{e^{-i\frac{\theta}{2}} - e}$$

$(e^{is})(e^{-is}) = e^0 = 1 = (x + iy)(x - iy)$

$e^{is} = x + iy$, $\quad e^{-is} = x - iy$

$1 = x^2 - (-1)y^2 = x^2 + y^2$

$\hat{F} = F_0\cos\omega t \Rightarrow F_0 e^{i\omega t}$.

$\boxed{\text{FORCED OSCILLATIONS}}$ 23-2

For a force with a certain delay

$\hat{F} = F_0\cos(\omega t + \phi) \Rightarrow F_0 e^{i(\omega t + \phi)} = F_0 e^{i\omega t} \times e^{i\phi} = \hat{F}e^{i\omega t}$

$$\frac{d^2x}{dt^2} + \frac{kx}{m} = \frac{F}{m} = \frac{F_0}{m}\cos\omega t$$

$$\frac{d}{dt^2}\left(\hat{x}e^{i\omega t}\right) + \frac{k}{m}\left(\hat{x}e^{i\omega t}\right) = \frac{1}{m}\hat{F}e^{i\omega t}$$

$$(i\omega)^2\hat{x}\cancel{e^{i\omega t}} + \frac{k\hat{x}}{m}\cancel{e^{i\omega t}} = \frac{1}{m}\hat{F}\cancel{e^{i\omega t}}$$

$$(i\omega)^2\hat{x} + \frac{k}{m}\hat{x} = \frac{\hat{F}}{m}$$

$$\hat{x}\left[(i\omega)^2 + \frac{k}{m}\right] = \frac{\hat{F}}{m}$$

$$\hat{x}\left[-\omega^2 + \omega_0^2\right] = \frac{\hat{F}}{m}$$

$$\hat{x} = \frac{\hat{F}}{m(\omega_0^2 - \omega^2)}$$

ω_0: natural freq ↑ $\quad\omega$: drive freq

21-5 FORCED OSCILLATIONS

$$m\frac{d^2x}{dt^2} = -kx + F(t)$$

↳ $= F\cos(\omega t)$

Suppose $x = C\cos\omega t$,

$$-m(\omega)^2 C\cos\omega t = -kC\cos(\omega t) + F_0\cos\omega t.$$

$\because ma = -kx = -m\omega_0^2 x \Rightarrow k = m\omega_0^2$

$\therefore -C\,m(\omega^2) = -m\omega_0^2 C + F_0$

$mC(\omega_0^2 - \omega^2) = F_0$

$$\Rightarrow C = \frac{F_0}{m(\omega_0^2 - \omega^2)}$$

↓ Amplitude of resultant motion

START HERE

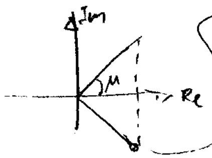
To find ρ : we employ $zz^* = R^2$. $z^* \Rightarrow$ change sign of i

$$\rho^2 = \hat{B}\hat{B}^* = \frac{1}{m(\underbrace{\omega_0^2 - \omega^2 + \gamma i\omega}_{a\ +\ b}) \cdot m(\underbrace{\omega_0^2 - \omega^2 - \gamma i\omega}_{a\ -\ b})} = \frac{1}{m^2\left[(\omega_0^2-\omega^2)^2 - (\gamma i\omega)^2\right]}$$

$$\boxed{\rho^2 = \frac{1}{m^2\left[(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2\right]}} \Rightarrow \rho = \frac{1}{m\sqrt{(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2}}$$

$\frac{1}{\hat{B}} \approx$

$\frac{1}{\rho} e^{i\mu}$ $\quad m(\omega_0^2 - \omega^2 + i\gamma\omega)$ from $\hat{B} = \frac{1}{\ldots}$



$\because \tan(-\theta) = -\tan\theta$

$\therefore \boxed{\tan(-\mu) = -\tan\mu = -\frac{\gamma\omega}{\omega_0^2 - \omega^2}}$

Amplitude of the response $= F \times \rho$ ← multiplier

Response shifted by μ.

$\rho^2 \approx$ Amplitude$^2 \approx$ Energy.

Some interesting things about the graph (ρ^2 vs ω)

- The smaller your ω_0, the greater the amplitude as $\omega \to \omega_0$
- when γ is small, $\frac{1}{(\omega_0^2-\omega^2)^2}$

  when $\omega \to \omega_0$, $\frac{1}{\gamma\omega^2}$

} Plot on the spot..?

If $\gamma \ll \omega_0$ and $\omega \approx \omega_0$:

$$\hat{x} = \frac{\hat{F}}{m(\omega_0^2-\omega^2+i\gamma\omega)} = \frac{\hat{F}}{m[(\omega_0+\omega)(\omega_0-\omega) + i\gamma\omega]}$$

$$(\text{approx}) \approx \frac{\hat{F}}{m[2\omega_0(\omega_0-\omega) + i\gamma\omega_0]}$$

$$= \frac{\hat{F}}{2m\omega_0[\omega_0 - \omega + \frac{1}{2}i\gamma]}$$

$$\Rightarrow \rho^2 = \frac{1}{4m^2\omega_0^2[(\omega_0-\omega)^2 + \frac{1}{4}\gamma^2]} \quad \Rightarrow \text{let } y = \rho^2, \ \omega = x$$

At max, $\omega = \omega_0$, $\therefore$ max height $= \frac{1}{m^2\omega_0^2\gamma^2}$

At half max height, $\Rightarrow$ half height$_{max}$ $= \frac{1}{2m^2\omega_0^2\gamma^2}$

$$\cancel{2m^2\omega_0^2}\gamma^2 = \cancel{4m^2\omega_0^2}\,{}^2[(\omega_0-\omega)^2 + \tfrac{1}{4}\gamma^2]$$

$$\tfrac{1}{2}\gamma^2 = 2(\omega_0-\omega)^2$$

$$\pm\sqrt{\tfrac{1}{4}\gamma^2} = \omega_0 - \omega \Rightarrow \omega = \omega_0 \mp \tfrac{1}{2}\gamma$$

$$\text{FWHM} = 2 \times \tfrac{1}{2}\gamma = \gamma$$

Resonance sharper if γ smaller

Thus, we try to write oscillation with damping,

$$m\frac{d^2x}{dt^2} = -c\frac{dx}{dt} - kx$$

$$\cancel{m}\frac{d^2x}{dt^2} = -\cancel{m}\gamma\frac{dx}{dt} - \cancel{m}\omega_0^2 x$$

Let x be the real part of $\hat{x}e^{i\omega t}$

$$(i\omega)^2\hat{x}e^{i\omega t} = -\gamma(i\omega)\hat{x}e^{i\omega t} - \omega_0^2\hat{x}e^{i\omega t}$$

$$\hat{x}e^{i\omega t}\left[(i\omega)^2 + \gamma(i\omega) + \omega_0^2\right] = 0$$

If $\hat{x}e^{i\omega t} = 0$, there is no oscillation, thus

$$(i\omega)^2 + \gamma(i\omega) + \omega_0^2 = 0$$

$$-\omega^2 + \gamma i\omega + \omega_0^2 = 0$$

which is a quadratic eqn in ω.

$$\omega = \frac{-\gamma i \pm \sqrt{(\gamma i)^2 - 4(-1)(\omega_0^2)}}{2(-1)}$$

$$= \frac{\gamma}{2}i \mp \frac{\sqrt{4\omega_0^2 - \gamma^2}}{2}$$

$$= \frac{\gamma}{2}i \mp \sqrt{\omega_0^2 - \tfrac{1}{4}\gamma^2}$$

$$\Rightarrow x = Ae^{\pm i\phi t}e^{i\left(\frac{\gamma}{2}i \pm \sqrt{\omega_0^2 - \frac{1}{4}\gamma^2}\right)t}$$

$$= Ae^{\pm i\phi t}e^{-\frac{\gamma}{2}t}e^{i\left(\pm\sqrt{\omega_0^2-\frac{1}{4}\gamma^2}\right)t}$$

$$\Rightarrow x = Ae^{-\frac{\gamma}{2}t}\cos\left(\pm\sqrt{\omega_0^2 - \tfrac{1}{4}\gamma^2}\,t + \phi\right)$$