Chapter 10

Oscillations

10.1 Simple Harmonic Motion

Simple harmonic motion is the periodic motion of an object whose acceleration along a certain direction, $\ddot{x}$, is proportional to its displacement from an equilibrium position in magnitude and opposite in direction to its displacement, x. We write the constant of proportionality as $-\omega^2$ for the sake of convenience.

$$\ddot{x} = -\omega^2 x$$

The general solution to this differential equation is

$$x = A\sin(\omega t + \phi) \tag{10.1}$$

where A is the amplitude of oscillation and ϕ is the initial phase angle or offset. Both constants are determined by the initial conditions on displacement and velocity. ω is termed the angular frequency of oscillation which is a characteristic of the oscillating system and is independent of the initial conditions.

Proof: Using the common trick that

$$\ddot{x} = \frac{d\dot{x}}{dt} = \frac{d\dot{x}}{dx} \cdot \frac{dx}{dt} = \dot{x}\frac{d\dot{x}}{dx}$$

We get

$$\int \dot{x} d\dot{x} = \int -\omega^2 x dx$$

$$\frac{\dot{x}^2}{2} = -\frac{\omega^2 x^2}{2} + \frac{c^2}{2}$$

$$\frac{dx}{dt} = \pm\sqrt{c^2 - \omega^2 x^2}$$

$$\int \frac{1}{\sqrt{c^2 - \omega^2 x^2}} dx = \pm \int dt$$

Using the substitution $x = \frac{c}{\omega}\sin\theta$, $dx = \frac{c}{\omega}\cos\theta d\theta$,

$$\int \frac{1}{c\cos\theta} \cdot \frac{c}{\omega}\cos\theta d\theta = \pm \int dt$$

$$\frac{\theta}{\omega} = \pm t + k$$

for some constant k. Substituting $\theta = \sin^{-1}\frac{\omega x}{c}$,

$$\sin^{-1}\frac{\omega x}{c} = \pm\omega t + \omega k$$

$$x = \frac{c}{\omega}\sin(\pm\omega t + \omega k) \implies x = A\sin(\omega t + \phi)$$

where the $\pm$ sign in front of the variable t has been absorbed into the initial phase angle ϕ as $\sin(-\omega t + \omega k) = \sin(\pi + \omega t - \omega k) = \sin(\omega t + \phi)$ where $\phi = \pi - \omega k$. Let us examine some of the terms in the above equation.

- A is the amplitude of the oscillation. It is the maximum magnitude of the displacement of an oscillating particle from an equilibrium position.
- ω is the angular frequency of oscillation which is the rate of change of the phase angle of oscillation. The period of an oscillation, T, refers to the time needed for one complete cycle while the frequency of an oscillation, f, refers to the number of complete cycles of oscillations per unit time. The angular frequency, ω is related to these quantities in the following manner:

$$\omega = 2\pi f = \frac{2\pi}{T} \tag{10.2}$$

- ϕ is the initial phase angle or phase offset. It is determined by the initial displacement and velocity. It gives a sense of where the oscillating particle is when $t = 0$ or when an observer starts his or her timer.
- The equilibrium point is the position where the object experiences no net acceleration or force. This occurs when $x = 0$.

10.1.1 Relationships Between Kinematic Quantities

In certain situations, we may be interested in other quantities describing simple harmonic motion such as the instantaneous velocity v and acceleration a of the object. We can solve for them by taking the time derivatives of its displacement.

$$v = \omega A \cos(\omega t + \phi) \tag{10.3}$$

$$a = -\omega^2 A \sin(\omega t + \phi) \tag{10.4}$$

We see that there is $\frac{\pi}{2}$ phase difference between the instantaneous velocity and the displacement of the object and a π phase difference between the instantaneous acceleration and the displacement of the object. If we let the initial phase angle be zero and plot the corresponding displacement, velocity and acceleration of the oscillating object against time, we obtain the following graphs.

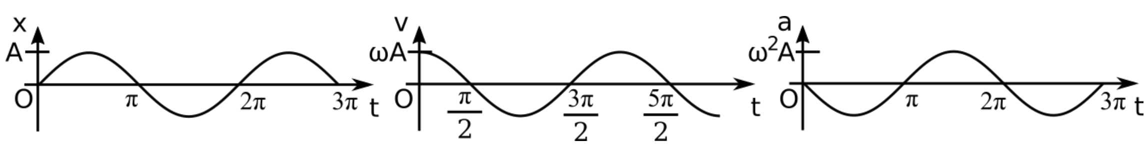

Figure 10.1: x, v and a against t graphs

Next, we might be interested in expressing the instantaneous velocity and acceleration of the object as functions of displacement instead. This is often more edifying as we can only physically observe the displacement of an object in a set-up most of the time.

$$v = \omega A \cos(\omega t + \phi) = \pm \omega A \sqrt{1 - \sin^2(\omega t + \phi)} = \pm \omega \sqrt{A^2 - x^2} \tag{10.5}$$

$$a = -\omega^2 x$$

If we plot the instantaneous velocity and acceleration of the oscillating body against its displacement, we see that we obtain an ellipse and a straight line respectively.

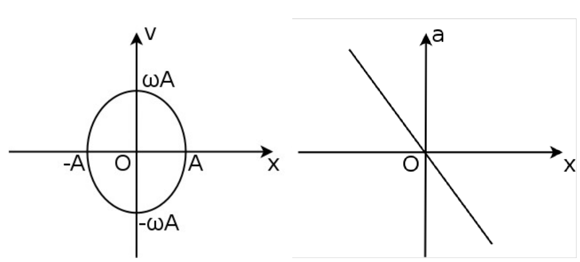

Figure 10.2: v and a against x graphs

To show that $v(x)$ is an ellipse, we can square Equation 10.5 to obtain

$$v^2 + \omega^2 x^2 = \omega^2 A^2$$

$$\implies \left(\frac{v}{\omega A}\right)^2 + \left(\frac{x}{A}\right)^2 = 1^2$$

which delineates an ellipse with axes length ωA and A along the v and x directions.

Problem: Determine the possible displacements of the oscillating particle from the equilibrium position when its instantaneous speed is half of its maximum speed.

Since the maximum speed of the particle is $A\omega$, its instantaneous velocity is

$$v = \pm\frac{1}{2}A\omega$$

$$\pm\frac{1}{2}A\omega = \pm\omega\sqrt{A^2 - x^2}$$

Evidently, we can only match the expressions of the same signs together. Then,

$$x^2 = \frac{3}{4}A^2$$

$$x = \pm\frac{\sqrt{3}}{2}A$$

10.1.2 Conservation of Energy

With regards to the dynamics of simple harmonic motion, the total mechanical energy of a body undergoing simple harmonic motion is conserved. This is because the simple harmonic differential equation implies that the force, which acts on the oscillating body, is conservative. Recall that in our derivation of the general solution to the simple harmonic differential equation,

$$\frac{\dot{x}^2}{2} + \frac{\omega^2 x^2}{2} = \frac{c^2}{2}$$

which is a constant. Observe that the left-hand-side is akin to the total mechanical energy of the particle divided by an inertial term—$\frac{\dot{x}^2}{2}$ is akin to a specific kinetic energy while $\frac{\omega^2 x^2}{2}$ is akin to a specific potential energy.

10.1.3 Examples of Simple Harmonic Set-ups

Now that we have studied the kinematics of simple harmonic motion, let us look at some realistic situations where this ubiquitous periodic motion arises. A standard procedure in deriving the simple harmonic differential equation would be to first write down the equation of motion of the system with respect to a particular coordinate, which we shall denote as x for now. Then, an x-coordinate x_0 that corresponds to a state of stable equilibrium is identified. For systems whose regime of simple harmonic motion is the immediate vicinity of the equilibrium position, small deviations of the system from x_0 is considered such that the x-coordinate of the system can be represented as $x = x_0 + \varepsilon$ where ε is a small deviation. Such systems are known to exhibit small oscillations. Maclaurin expansions are then performed while discarding second-order terms in ε to generate the required simple harmonic differential equation. In certain systems where simple harmonic motion is exact, we can consider a general displacement of the particle from its equilibrium position to reach the simple harmonic differential equation.

Consider the classic pendulum—a ball of negligible size and mass m is attached to a massless string of length l that is attached to the ceiling. We can show that at small angular displacements from the vertical (which is a stable equilibrium position), this pendulum exhibits simple harmonic motion.

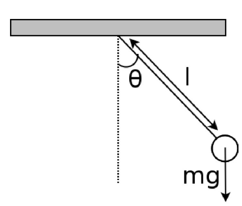

Figure 10.3: Simple Pendulum

Analysing forces in the tangential direction,

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = -mg\sin\theta$$

Since the length of the string remains constant, $r = l$.

$$ml\ddot{\theta} = -mg\sin\theta$$

For small angles, we can use the Maclaurin series to approximate $\sin\theta \approx \theta$ such that

$$ml\ddot{\theta} = -mg\theta \implies \ddot{\theta} = -\frac{g}{l}\theta$$

This is the simple harmonic differential equation. Solving it gives

$$\theta = A\sin\left(\sqrt{\frac{g}{l}}t + \phi\right)$$

Thus, we see that the angular frequency of a simple pendulum is

$$\omega = \sqrt{\frac{g}{l}}$$

which is in fact independent of the mass of the point mass attached to the string.

In the general case of an object attached to a pivot, the equation $\tau = I\ddot{\theta}$ is more useful in determining the angular frequency of oscillations and other related quantities. Both the moment of inertia I and the torque τ on the system should be computed with respect to the fixed pivot.

Problem: A uniform rod of mass m and length l is attached to a pivot at one of its ends. Determine the angular frequency of small oscillations of the rod about the vertical orientation.

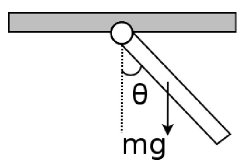

Figure 10.4: Rod Pendulum

Recall that the moment of inertia of a uniform rod about its center is $\frac{1}{12}ml^2$. Therefore, the moment of inertia of the rod about one of its ends is

$$I = \frac{1}{12}ml^2 + m\frac{l^2}{4} = \frac{1}{3}ml^2$$

by the parallel-axis theorem. The net torque on the system about the fixed pivot is due to that of the weight of the rod, acting at its center of mass. Therefore, the net torque on the rod is

$$\tau = -\frac{1}{2}mgl\sin\theta$$

Applying the equation $\tau = I\ddot{\theta}$ (as the pivot is an ICoR) and the small angle approximation $\sin\theta \approx \theta$,

$$\ddot{\theta} = -\frac{3g}{2l}\theta$$

The angular frequency is then

$$\omega = \sqrt{\frac{3g}{2l}}$$

Such systems involving extended dangling objects are known as physical pendulums. It is not difficult to generalize the above result to show that the angular frequency of a physical pendulum is

$$\omega = \sqrt{\frac{Mr_{CM}g}{I_{pivot}}} \tag{10.6}$$

where M is the total mass of the object, r_{CM} is the distance between the pivot and the center of mass of the object and I_{pivot} is the moment of inertia of the object about the fixed pivot.

Two-Dimensional Systems

The small oscillations of inherently two-dimensional systems whose equations of motion consist of two variables are usually considered in the context of perturbing a single coordinate. Then, variables in terms of the other coordinate should be eliminated to obtain a differential equation in terms of the single, relevant coordinate. This elimination is usually performed via the conservation of angular momentum in a certain direction.

Problem: A planet of mass m is currently undergoing circular motion about the Sun of mass M at a radius r_0. If the planet is somehow given a slight radial displacement, determine its angular frequency of small oscillations in the radial direction. What is the resultant trajectory of the planet?

The radial equation of motion of the planet in polar coordinates is

$$-\frac{GMm}{r^2} = m(\ddot{r} - r\dot{\theta}^2)$$

Since the push is only radial and because the gravitational force is central, the angular momentum of the planet relative to the Sun remains constant.

$$L = mr^2\dot{\theta}$$

The radial equation of motion becomes

$$-\frac{GM}{r^2} = \ddot{r} - \frac{L^2}{m^2r^3}$$

Furthermore, we know that when $r = r_0$, $\ddot{r} = 0$. Then,

$$\frac{GM}{r_0^2} = \frac{L^2}{m^2r_0^3}$$

This will be useful in cancelling some terms later—a common denominator in all small oscillation problems. Suppose that the radius now becomes[1] $r_0 + \varepsilon$. The equation of motion becomes

$$-\frac{GM}{r_0^2\left(1+\frac{\varepsilon}{r_0}\right)^2} = \ddot{\varepsilon} - \frac{L^2}{m^2r_0^3\left(1+\frac{\varepsilon}{r_0}\right)^3}$$

Performing a Maclaurin expansion and discarding second order terms in $\frac{\varepsilon}{r_0}$,

$$-\frac{GM}{r_0^2} + \frac{2GM}{r_0^3}\varepsilon = \ddot{\varepsilon} - \frac{L^2}{m^2r_0^3} + \frac{3L^2\varepsilon}{m^2r_0^4}$$

Observe that the first term on the left-hand-side cancels the second term on the right-hand-side. Then,

$$\begin{aligned}
\ddot{\varepsilon} &= \left(\frac{2GM}{r_0^3} - \frac{3L^2}{m^2r_0^4}\right)\varepsilon \\
&= \left(\frac{2GM}{r_0^3} - \frac{3GM}{r_0^3}\right)\varepsilon \\
&= -\frac{GM}{r_0^3}\varepsilon
\end{aligned}$$

Thus, the angular frequency of small oscillations in the radial direction is

$$\omega = \sqrt{\frac{GM}{r_0^3}}$$

At the first glance, one might expect the resultant trajectory of the planet to take the form of a "flower pattern" as the planet oscillates radially along an originally circular orbit. However, observe that the period of the original circular orbit is exactly ω! Therefore, the planet only attains the maximum and

[1] Note that in the previous set-ups, we did not have to explicitly state this as the equilibrium angles were $\theta = 0$.

minimum radial distance from the Sun once per complete revolution—indicating that the new orbit is an ellipse with semi-major and semi-minor axes $r_0 + |\varepsilon|$ and $r_0 - |\varepsilon|$! When $|\varepsilon| << r_0$, the eccentricity of the ellipse is virtually zero such that the new orbit is akin to a circle, with the Sun slightly displaced from the center of the circle.

This is intuitive from the perspective of Kepler's first law as the orbit of a planet is in general an ellipse, with the Sun as a focus. Note that the resultant orbit of the planet must still be bounded as a slight deviation imparts negligible mechanical energy to it. In retrospect, we could have also imposed the condition that the trajectory can only be an ellipse to conclude that the angular frequency of radial oscillations must match the angular frequency of the planet's original orbit!

10.2 Deriving Angular Frequency from Potential Energy

When a particle is solely under the influence of conservative forces, we can associate a potential energy function, that is strictly only a function of its position, to it. Let us consider an arbitrary potential energy function $U(x)$ for a one-dimensional system while keeping in mind that the net force on the particle is $F = -\frac{dU}{dx}$. The equilibrium positions correspond to the stationary points of the $U(x)$ graph as $F = -\frac{dU}{dx} = 0$ there.

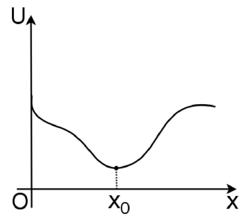

Figure 10.5: One-dimensional Potential Energy

In light of our goal of analyzing oscillations, let us observe the motion of the particle at positions in the vicinity of a minimum as the particle strives towards attaining a lower potential energy—implying that the minimum corresponds to a stable equilibrium state as any deviation from the minimum tends to be minimized. Assuming that the potential energy function has a minimum at x_0, we can expand $U(x)$ in a Taylor series about x_0.

$$U(x) = U(x_0) + U'(x_0)(x - x_0) + \frac{U''(x_0)}{2}(x - x_0)^2 + ...$$

where we will neglect third order terms as we assume that the particle is near x_0. Now let us consider the force on the particle in the x-direction.

$$F = -\frac{dU}{dx} = -U'(x_0) - U''(x_0)(x - x_0)$$

Since x_0 is a minimum, $U'(x_0) = 0$. Furthermore, if we use a change of variables $\varepsilon = x - x_0$ such that $F = m\ddot{x} = m\ddot{\varepsilon}$, we can simplify the equation above to

$$m\ddot{\varepsilon} = -U''(x_0)\varepsilon$$

Evidently, this describes a simple harmonic motion with an angular frequency given by

$$\omega = \sqrt{\frac{U''(x_0)}{m}} \tag{10.7}$$

Perhaps an intuitive and lucid explanation of why this should be the case is that if you zoom closer to the minimum, the regions around it will look like a parabola. Thus, the potential energy curve is approximately $U = A(x - x_0)^2 + c$ which gives $F = -2A(x - x_0)$—a simple harmonic force (like a spring).

Problem: As its name implies, a spring-mass system consists of a mass m connected by a massless spring of spring constant k to a fixed pivot. Determine the angular frequency of oscillations if the system lies on a horizontal table and if it hangs vertically.

In both cases, define the origin at the equilibrium position such that x denotes the displacement of m in the relevant direction, from the equilibrium position. In the horizontal case, the potential energy at a displacement x is

$$U = \frac{1}{2}kx^2$$

$$U''(0) = k$$

Therefore,

$$\omega = \sqrt{\frac{k}{m}}$$

In the vertical case, the equilibrium position of the mass is $\frac{mg}{k}$ below the relaxed length of the spring. At a displacement x below this equilibrium position, the potential energy associated with the particle is

$$U = \frac{1}{2}k\left(\frac{mg}{k} + x\right)^2 - mgx$$

$$U''\left(\frac{mg}{k}\right) = k$$

$$\omega = \sqrt{\frac{k}{m}}$$

Effective Potential for Central Force Systems

In the case of central force systems which are two-dimensional, the one-dimensional method above can be applied to radial oscillations through the introduction of an effective potential. In a central force system, the total mechanical energy is

$$\frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + U(r) = E$$

where $U(r)$ is the potential energy associated with the central force field and r is the radial position of the particle relative to the source of the central force field. Next, the angular momentum of the particle relative to the source is conserved.

$$mr^2\dot{\theta} = L$$

Therefore, the first equation can be expressed as

$$\frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} + U(r) = E$$

which is akin to a one-dimensional conservation of energy equation in r with an effective potential

$$U_{eff}(r) = \frac{L^2}{2mr^2} + U(r)$$

Then, the above results can be directly applied to conclude that the angular frequency of radial oscillations (if the particle actually oscillates) is

$$\omega = \sqrt{\frac{U''_{eff}(r_0)}{m}}$$

where r_0 is the equilibrium radial coordinate.

Problem: Redo the previous problem on a planet orbiting the Sun by the effective potential method.

The effective potential is

$$U_{eff} = \frac{L^2}{2m^2r^2} - \frac{GMm}{r}$$

Then,

$$U'_{eff} = -\frac{L^2}{m^2r^3} + \frac{GMm}{r^2}$$

We know that at the equilibrium radial coordinate r_0, $U'_{eff} = 0$. Therefore,

$$\frac{L^2}{m^2r_0^3} = \frac{GMm}{r_0^2}$$

Moving on, we calculate the second derivative of U_{eff}.

$$U''_{eff} = \frac{3L^2}{m^2r^4} - \frac{2GMm}{r^3}$$

$$U''_{eff}(r_0) = \frac{1}{r_0}\left(\frac{3L^2}{m^2r_0^3} - \frac{2GMm}{r_0^2}\right) = \frac{GMm}{r_0^3}$$

Therefore, the angular frequency of radial oscillations is

$$\omega = \sqrt{\frac{U''_{eff}(r_0)}{m}} = \sqrt{\frac{GM}{r_0^3}}$$

10.3 Damped Oscillations

In an ideal oscillatory system, there are no dissipative forces and the total mechanical energy of the system is conserved. The system then oscillates indefinitely with a constant amplitude. However, in real-world systems, there are often dissipative forces which cause the amplitude of oscillation to gradually decrease over time. The resultant oscillatory motion is known as damped oscillations.

10.3.1 Linear Differential Equations

Before we solve the equation for damped oscillations, it might be helpful to know some properties of linear differential equations. A nth degree linear differential equation (with respect to time) takes the form:

$$c_n\frac{d^nx}{dt^n} + c_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + ... + c_1\frac{dx}{dt} + c_0x = g(t)$$

where $g(t)$ may be a constant or a function of t. If $g = 0$, the equation is called a homogeneous equation. We shall first consider a homogeneous linear differential equation.

$$c_n\frac{d^nx}{dt^n} + c_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + ... + c_1\frac{dx}{dt} + c_0x = 0$$

It turns out that the previous method in solving the simple harmonic differential equation is not applicable to the general case of linear differential equations. However, we can invoke a general theorem of linear differential equations which states that a nth degree homogeneous linear differential equation has exactly n linearly independent solutions. Then, the next step is to determine these n solutions, by hook or by crook. The most general method is in fact guessing solutions of the form $x = e^{at}$.

$$c_na^ne^{at} + c_{n-1}a^{n-1}e^{at} + ... + c_1ae^{at} + c_0e^{at} = 0$$

Dividing the equation by e^{at} throughout and invoking the fundamental theorem of algebra, this n degree polynomial equation can be factorized into

$$(a - b_1)(a - b_2)...(a - b_n) = 0$$

which is also known as the characteristic equation. Thus, we have n roots for a which may be real, complex or even repeated roots. We can substitute any of these roots into the trial solution for x ($x = e^{at}$) and it would satisfy the differential equation. For example, $x = e^{b_1t}$ and $x = e^{b_2t}$ are solutions to the linear differential equation. Due to the linear nature of the differential equation, any linear combination of these solutions is also a solution. Thus, a general solution would take the form of

$$x = z_1e^{b_1t} + z_2e^{b_2t} + ... + z_ne^{b_nt}$$

where the z_i's are possibly complex constants. Since a nth degree homogeneous linear differential equation has n linearly independent solutions, we are done if there are no repeated roots and the expression for x above is the most general solution to the homogeneous linear differential equation. However, when there are repeated roots, we have to search for other linearly independent solutions. Usually, if a root b_j is repeated k times, we guess the following k solutions: $e^{b_jt}, te^{b_jt}, t^2e^{b_jt}, ..., t^{k-1}e^{b_jt}$.

Let us apply this guessing technique to the simple harmonic differential equation.

$$\ddot{x} + \omega^2x = 0$$

Substituting e^{at} into x,

$$a^2e^{at} + \omega^2e^{at} = 0$$

$$a = \pm i\omega$$

where $i = \sqrt{-1}$. Therefore, the general solution for x is

$$x = z_1e^{i\omega t} + z_2e^{-i\omega t}$$

where z_1 and z_2 are arbitrary constants which may be complex. Since x represents the displacement which is a physical quantity, it must be real at all times. Therefore, z_1 and z_2 must be complex conjugates.

$$z_1 = z_2^*$$

Then, we can represent z_1 and z_2 as

$$z_1 = \frac{A}{2}e^{i\phi}$$

$$z_2 = \frac{A}{2}e^{-i\phi}$$

where A and ϕ are arbitrary real constants. Correspondingly,

$$x = \frac{A}{2}\left(e^{i(\omega t+\phi)} + e^{-i(\omega t+\phi)}\right) = A\cos(\omega t + \phi)$$

by Euler's identity $e^{i\theta} = i\sin\theta + \cos\theta$. Observe that since z_1 and z_2 are complex conjugates and because x must be real, we have effectively taken the real component of either of them (up to a constant factor). Therefore, whenever we have a solution for a real x in terms of pairs of the form $z_1e^{i\omega t} + z_2e^{-i\omega t}$, x can be simply expressed in terms of the real component of one term from each pair, such as $\mathrm{Re}\left(z_1e^{i\omega t}\right)$.

This, in combination with the following final remark, can prove to be extremely useful. If all constants c_i in a homogeneous linear differential equation are real[2] and if y is a solution, y^* must also be a solution to the equation. This can be easily proven by taking the complex conjugate of the entire differential equation.

$$\left(c_n\frac{d^ny}{dt^n}\right)^* + \left(c_{n-1}\frac{d^{n-1}y}{dt^{n-1}}\right)^* + ... + \left(c_1\frac{dy}{dt}\right)^* + (c_0y)^* = 0^*$$

Since $(z_1z_2)^* = z_1^* \cdot z_2^*$ for two arbitrary complex numbers z_1 and z_2 and because the constants c_i are real,

$$c_n\left(\frac{d^ny}{dt^n}\right)^* + c_{n-1}\left(\frac{d^{n-1}y}{dt^{n-1}}\right)^* + ... + c_1\left(\frac{dy}{dt}\right)^* + c_0y^* = 0$$

The order of differentiation and complex conjugation does not matter. Therefore,

$$c_n\frac{d^ny^*}{dt^n} + c_{n-1}\frac{d^{n-1}y^*}{dt^{n-1}} + ... + c_1\frac{dy^*}{dt} + c_0y^* = 0$$

which proves that y^* is also a solution. In such cases, we can simply take the real component of y in writing the general solution for x—neglecting its complex conjugate y^*.

Lastly, we consider the non-homogeneous linear differential equation

$$c_n\frac{d^nx}{dt^n} + c_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + ... + c_1\frac{dx}{dt} + c_0x = g(t)$$

The general solution to this differential equation involves a particular solution and a homogenous solution. Firstly, we find a solution, x_p, which satisfies this equation. This is known as the particular solution. Then, we can add the particular solution to the homogeneous solution obtained by letting $g = 0$ in the equation above to obtain the general solution to the non-homogeneous linear differential equation. This is due to the fact that adding the homogeneous solution results in an additional value of zero on the right-hand side—leaving it unchanged. Lastly, note that the particular solution does not depend on the initial conditions. Instead, the initial conditions are still encoded in the homogeneous solution.

[2]This is usually the case in physical scenarios.

10.3.2 Equation of Motion

In most cases, the damping force on an oscillating body is proportional to its velocity and acts in the opposite direction. Thus, the differential equation for the displacement of the body takes the form

$$\ddot{x} + 2\gamma\dot{x} + \omega^2 x = 0$$

Guessing a solution of the form $x = e^{at}$, we obtain the characteristic equation.

$$a^2 + 2\gamma a + \omega^2 = 0$$

Solving, we obtain two possibly repeated roots.

$$a = \frac{-2\gamma \pm \sqrt{4\gamma^2 - 4\omega^2}}{2} = -\gamma \pm \sqrt{\gamma^2 - \omega^2}$$

Now we have to consider three cases—namely when the term inside the square root is negative, positive and zero.

10.3.3 Light Damping

When $\gamma < \omega$, the system experiences light damping or is underdamped. Thus, we can rewrite the two solutions for a as

$$a = -\gamma \pm i\sqrt{\omega^2 - \gamma^2}$$

Then, our general solution for the equation of motion is

$$x = c_1 e^{-\gamma t + i\sqrt{\omega^2-\gamma^2}t} + c_2 e^{-\gamma t - i\sqrt{\omega^2-\gamma^2}t} = e^{-\gamma t}\left(c_1 e^{i\sqrt{\omega^2-\gamma^2}t} + c_2 e^{-i\sqrt{\omega^2-\gamma^2}t}\right)$$

For our displacement to be strictly real at all instances, c_1 and c_2 has to be complex conjugates. Letting $c_1 = \frac{c_0}{2}e^{i\phi}$ and $c_2 = \frac{c_0}{2}e^{-i\phi}$,

$$x = e^{-\gamma t}\frac{c_0}{2}\left(e^{i\left(\sqrt{\omega^2-\gamma^2}t+\phi\right)} + e^{-i\left(\sqrt{\omega^2-\gamma^2}t+\phi\right)}\right)$$

$$x = e^{-\gamma t} c_0 \cos\left(\sqrt{\omega^2-\gamma^2}t + \phi\right) \tag{10.8}$$

where we have invoked the elegant Euler's identity. Again, we have effectively taken the real part of one of the solutions. We see that the amplitude of oscillation decreases exponentially over time and that the angular frequency of oscillation is smaller than that of an undamped system. The angular frequency of the underdamped system ω_d is given by

$$\omega_d = \sqrt{\omega^2 - \gamma^2}$$

The displacement of the object as a function of time is illustrated below.

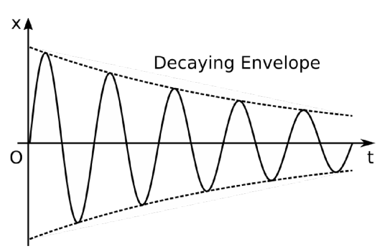

Figure 10.6: Light Damping

The graph represents an oscillation between an exponentially decreasing envelope which governs the amplitude of oscillation at every instant in time.

10.3.4 Heavy Damping

When $\gamma > \omega$, the system experiences heavy damping or is overdamped. There are two real solutions for a

$$a = -\gamma \pm \sqrt{\gamma^2 - \omega^2}$$

Thus, our general solution to the equation of motion is

$$x = c_1 e^{-\left(\gamma - \sqrt{\gamma^2-\omega^2}\right)t} + c_2 e^{-\left(\gamma + \sqrt{\gamma^2-\omega^2}\right)t} \tag{10.9}$$

where c_1 and c_2 are real constants. The system does not undergo oscillatory motion as the exponents are real. As time passes by, the displacement gradually tends to 0. This decay is indeed extremely gradual as the significant term in the long run is $e^{-\left(\gamma - \sqrt{\gamma^2-\omega^2}\right)t}$. A vivid example of an overdamped system would be a door that takes an incredibly long time to close. On another note, it is intriguing to show that the oscillating particle can cross the origin at most once. Substituting $x = 0$ and simplifying,

$$c_1 e^{\sqrt{\gamma^2-\omega^2}t} = -c_2 e^{-\sqrt{\gamma^2-\omega^2}t}$$

$$e^{2\sqrt{\gamma^2-\omega^2}t} = -\frac{c_2}{c_1}$$

which has one solution only if $\frac{c_2}{c_1} < 0$ (i.e. they are of opposite signs). In fact, the condition is much stricter—if the initial displacement is positive, $c_1 < 0$ and $c_2 > 0$ for the oscillating body to cross the origin. This is because, if the body really crossed the origin, it must tend to 0^- after a long time (i.e. approach 0 from below the t-axis) as it cannot cross the origin again. In the long term, $e^{-\left(\gamma - \sqrt{\gamma^2-\omega^2}\right)t}$ is more significant than the other exponential term. Therefore, c_1 must be negative—implying that c_2 is positive. If the initial displacement is negative, the converse occurs. Plotting the displacement against time graph of an overdamped system, we get the following possible curves.

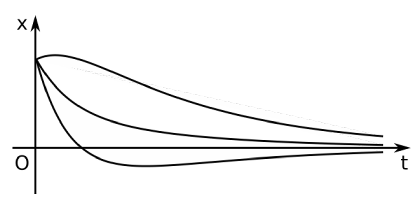

Figure 10.7: Heavy Damping

10.3.5 Critical Damping

When $\gamma = \omega$, we say that the system is critically damped. a has only one solution $a = -\gamma$. Thus, we have only found one independent solution from the characteristic equation.

$$x = e^{-\gamma t}$$

The other independent solution is $x = te^{-\gamma t}$ (you should verify this for yourself). Thus, the general solution for the equation of motion is

$$x = e^{-\gamma t}(c_1 + c_2 t) \tag{10.10}$$

where c_1 and c_2 are real constants. The exponential decay term in $c_2 te^{-\gamma t}$ dominates the $c_2 t$ term such that $x \to 0$ for very large t. The system does not oscillate at all and instead, returns to the equilibrium position in the shortest time possible, as compared to the other forms of damping. This can be proven by comparing the exponential decay constants. In the case of light damping, the exponential decay constant is γ in the regime $\gamma < \omega$ which is evidently smaller than $\gamma = \omega$ in the critical damping case. Furthermore, the dominant decay constant (the smaller one) in the case of overdamping is $\gamma - \sqrt{\gamma^2 - \omega^2}$ which can shown to be smaller than ω.

$$\gamma - \omega < \gamma + \omega$$

Since $\gamma > \omega$ in the regime of overdamping, we can multiply both sides by $\gamma - \omega > 0$ to obtain

$$(\gamma - \omega)^2 < \gamma^2 - \omega^2$$

$$\implies \gamma - \omega < \sqrt{\gamma^2 - \omega^2}$$

$$\gamma - \sqrt{\gamma^2 - \omega^2} < \omega$$

Thus, the particle returns to a state of equilibrium in the shortest time (still indefinitely long though) when the system is critically damped. Critical damping is paramount in many real systems, such as shock absorbers, in ensuring that a system stops immediately without oscillating about. On another note, the particle can, again, cross the origin at most once. When $x = 0$,

$$t = -\frac{c_1}{c_2}$$

which is only valid if $\frac{c_1}{c_2} < 0$. By a similar argument as above, if the initial displacement is positive $c_1 > 0$ and $c_2 < 0$ for the oscillating particle to cross the origin as the $e^{-\gamma t} c_2 t$ term is dominant over $e^{-\gamma t} c_1$ for large t. The possible displacement against time graphs of a critically damped system are depicted below. They look roughly the same as those in the heavy damping case with the exception that the graph for small t is essentially linear. This is because, for small t, the exponential term in x is approximately unity such that $x \approx c_1 + c_2 t$.

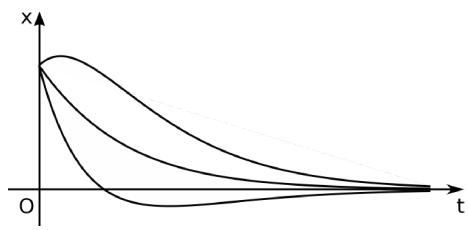

Figure 10.8: Critical Damping

10.3.6 Driven and Damped Oscillations

Considering the fact that most real-world systems experience dissipative forces, an external periodic driving force is often applied to sustain their motion. Before solving the equation of motion for real driven oscillations, we shall first tackle a differential equation that takes the following form:

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = c_0 e^{i\omega_e t}$$

We have added subscripts to the ω's to avoid confusion. To recapitulate, to solve a non-homogeneous linear differential equation, we need to determine a particular solution, before adding it to the homogeneous solution to procure the general solution. As there is a $e^{i\omega_e t}$ term on the right-hand side, it is wise to guess a particular solution $x_p = Ae^{i\omega_e t}$. In this case, we are solving for the constant A instead of the angular frequency. Substituting this trial solution into the expression, we get

$$-\omega_e^2 A + 2i\gamma\omega_e A + \omega_0^2 A = c_0 \implies A = \frac{c_0}{-\omega_e^2 + 2i\gamma\omega_e + \omega_0^2}$$

$$x_p = \frac{c_0}{-\omega_e^2 + 2i\gamma\omega_e + \omega_0^2} e^{i\omega_e t}$$

Lastly, we can obtain the general solution to this differential equation by adding the appropriate homogeneous solution, derived previously, to the particular solution.

Now, we can consider providing a driving force to our oscillatory system of the form $F = f \cos \omega_e t$. ω_e is known as the angular frequency of the external driving force. Then, the equation of motion of the oscillating body takes the form of

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = c_0 \cos \omega_e t$$

where c_0 is a real constant. ω_0 is known as the natural frequency of the system and is the angular frequency of oscillations when there are no damping and external driving forces. To solve this differential equation, consider the following auxiliary differential equation.

$$\ddot{y} + 2\gamma\dot{y} + \omega_0^2 y = c_0 e^{i\omega_e t}$$

Suppose that we have found a solution for y in the equation above. Then, we can take the real component of both sides.

$$\mathrm{Re}(\ddot{y}) + \mathrm{Re}(2\gamma\dot{y}) + \mathrm{Re}(\omega_0^2 y) = \mathrm{Re}\left(c_0 e^{i\omega_e t}\right)$$

Since the constants γ, ω_0 and c_0 are real and because the order of differentiating y and taking the real part of it does not matter,

$$\ddot{z} + 2\gamma\dot{z} + \omega_0^2 z = c_0 \cos\omega_e t$$

where $z = \mathrm{Re}(y)$ is the real component of y. Therefore, the solution for x that we desire is simply the real component of y! The particular solution for y was previously derived to be

$$y_p = \frac{c_0}{-\omega_e^2 + 2i\gamma\omega_e + \omega_0^2} e^{i\omega_e t}$$

This can be expressed in a more suggestive form by applying Euler's formula to the denominator.

$$\begin{aligned} y_p &= \frac{c_0}{\sqrt{(\omega_0^2 - \omega_e^2)^2 + 4\gamma^2\omega^2} e^{i\phi}} e^{i\omega_e t} \\ &= \frac{c_0}{\sqrt{(\omega_0^2 - \omega_e^2)^2 + 4\gamma^2\omega^2}} e^{i(\omega_e t - \phi)} \end{aligned}$$

where

$$\phi = \tan^{-1} \frac{2\gamma\omega_e}{\omega_0^2 - \omega_e^2} \tag{10.11}$$

Therefore, the particular solution x_p can be obtained by taking the real component of the above.

$$x_p = \frac{c_0}{\sqrt{(\omega_0^2 - \omega_e^2)^2 + 4\gamma^2\omega_e^2}} \cos(\omega_e t - \phi) \tag{10.12}$$

In the long run, the damping terms will cause the homogeneous solution to tend to zero. Therefore given different initial conditions, the system will eventually reach the same steady state, which is described by its particular solution. Furthermore, in light of the decaying amplitude of the homogeneous solution, the particular solution is often valued over the homogeneous solution as it is a more enlightening description of the behaviour of driven oscillations—we shall therefore not bother too much about the general solution which will tend to the particular solution in the long run.

Resonance

Consider a realistic oscillatory system. Because of damping, the amplitude of the system eventually diminishes to zero. Thus, we would like to sustain the motion of the system by applying a periodic force. However, what should the driving frequency ω_e (angular frequency of the periodic force) be such that the resultant amplitude of oscillation is the greatest? This frequency is known as the resonant frequency. Resonance is a phenomenon in which an oscillatory system responds with a maximum amplitude to an external periodic force. The condition for resonance can be derived from Equation 10.12. The amplitude of the particular solution of a driven damped oscillation, A, is

$$A = \frac{c_0}{\sqrt{(\omega_0^2 - \omega_e^2)^2 + 4\gamma^2\omega_e^2}} = \frac{c_0}{\sqrt{\left[\omega_e^2 - (\omega_0^2 - 2\gamma^2)\right]^2 + 4\omega_0^2\gamma^2 - 4\gamma^4}}$$

which attains the maximum value $\frac{c_0}{2\gamma\sqrt{\omega_0^2 - \gamma^2}}$ when

$$\omega_e = \sqrt{\omega_0^2 - 2\gamma^2}$$

Thus, $\sqrt{\omega_0^2 - 2\gamma^2}$ is the resonant frequency, ω_r. Observe that when $\gamma << \omega$ (no damping or light damping), $\omega_r \approx \omega_0$ is the condition for resonance. Thus, resonance occurs for underdamped and simple oscillations when the driving frequency is approximately equal to the natural frequency of the system.

When the system is underdamped and $\omega_e = \omega_r \approx \omega_0$, $\phi \to \frac{\pi}{2}$ as a result[3] of Equation 10.11. The displacement of the oscillating body lags the driving force by a quarter of a cycle while the velocity of the

[3] $\tan\phi$ tends to $+\infty$ as $\omega_e \to \omega_0^-$. Note that ϕ must be in the first quadrant $\left(\frac{\pi}{2}\right)$ and not the third $\left(\frac{3\pi}{2}\right)$ as both the real and complex components of $e^{i\phi}$ are positive. Refer to the specific juncture at which we substituted $e^{i\phi}$ for further clarifications.

body and the driving force are perfectly in phase. When the force is at its maximum, the displacement of the object is zero and it thus possess the greatest velocity (in the same direction as the external driving force). This is intuitive from the standpoint of energy as the force should act with the greatest magnitude on the object when it is travelling the fastest to maximize the work done by the driving force.

Lastly, we can plot the amplitude of driven oscillations, A, against the driving frequency ω_0 for different damping constants γ.

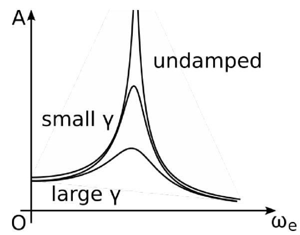

Figure 10.9: Maximum Amplitude against Driving Frequency

As the system experiences greater damping, the amplitude decreases, the resonance peak becomes broader and the resonant frequency slightly decreases. In the case of undamped oscillations, the amplitude tends to infinity when $\omega_e \to \omega_0$ from both sides.

Summary

- A lightly damped system oscillates with an amplitude that is exponentially decaying. Its angular frequency is slightly smaller than the natural frequency of oscillation.
- An overdamped or heavily damped system gradually returns to the equilibrium position.
- A critically damped system returns to the equilibrium position in the least possible time.
- Resonance is a phenomenon where an oscillatory system responds with the greatest amplitude to an external driving force. The angular frequency of the driving force is known as the driving frequency. The angular frequency at which resonance occurs is known as the resonant frequency.
- The angular frequency of a damped oscillation ω_d and the resonant frequency ω_r of a damped system are

$$\omega_d = \sqrt{\omega_0^2 - \gamma^2}$$
$$\omega_r = \sqrt{\omega_0^2 - 2\gamma^2}$$

  where ω_0 is the natural frequency of the system that refers to the angular frequency of the oscillation if there were no damping or driving forces.

10.4 Coupled Oscillations

In certain cases, we may have periodic systems which consist of many interdependent objects (we shall only deal with linear systems). Then, there may be multiple pure frequencies, known as its normal frequencies, that the system can oscillate at.

10.4.1 Decoupling

Consider the common example below. Two equal masses m are connected by three springs with equal spring constants k to each other and to the walls adjacent to them. Solve for the displacements of the masses as a function of time and the normal frequencies of oscillation if the masses were given a slight initial displacement.

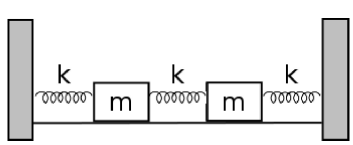

Figure 10.10: Coupled Masses

Let x_1 and x_2 be the displacements of the two masses from their respective equilibrium positions, with rightwards taken to be positive. Writing the equation of motion for each individual mass,

$$m\ddot{x}_1 = -kx_1 + k(x_2 - x_1)$$
$$m\ddot{x}_2 = -kx_2 - k(x_2 - x_1)$$

Be careful with the signs here. A reliable way to determine the signs would be to envision the physical scenario. Suppose that $x_2 > x_1$, this would physically mean that the middle spring has been stretched. Thus, the middle spring will pull the first mass towards the right ($+k(x_2 - x_1)$ in the first equation) and pull the second mass towards the left ($-k(x_2 - x_1)$ in the second equation).

Observing the equations of motion, we realise that they are "coupled" in the sense that the way in which the state of one object evolves depends on the state of the other object. To solve this pair of equations, we have to decouple them. Adding the two equations,

$$(\ddot{x}_1 + \ddot{x}_2) = -\frac{k}{m}(x_1 + x_2)$$
$$\implies x_1 + x_2 = A\sin\left(\sqrt{\frac{k}{m}}t + \phi_1\right)$$

Subtracting the second equation from the first,

$$(\ddot{x}_1 - \ddot{x}_2) = -\frac{3k}{m}(x_1 - x_2)$$
$$\implies x_1 - x_2 = B\sin\left(\sqrt{\frac{3k}{m}}t + \phi_2\right)$$

Thus,

$$x_1 = \frac{A}{2}\sin\left(\sqrt{\frac{k}{m}}t + \phi_1\right) + \frac{B}{2}\sin\left(\sqrt{\frac{3k}{m}}t + \phi_2\right)$$
$$x_2 = \frac{A}{2}\sin\left(\sqrt{\frac{k}{m}}t + \phi_1\right) - \frac{B}{2}\sin\left(\sqrt{\frac{3k}{m}}t + \phi_2\right)$$

We see that the two normal frequencies are $\sqrt{\frac{k}{m}}$ and $\sqrt{\frac{3k}{m}}$. The normal modes represent the possible forms of pure-frequency motions and are represented in terms of vectors. The above solution can be represented in matrices as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{A}{2}\sin\left(\sqrt{\frac{k}{m}}t + \phi_1\right) + \begin{pmatrix} 1 \\ -1 \end{pmatrix}\frac{B}{2}\sin\left(\sqrt{\frac{3k}{m}}t + \phi_2\right)$$

Then, $(1, 1)$ and $(1, -1)$ are the normal modes of this motion. Let us examine the physical meaning of these modes. With regards to the first normal mode $(1, 1)$, the displacements of the two masses from their equilibrium positions are identical. Then, the middle spring is not stretched or compressed during the entire motion and can be effectively removed—leading to the normal frequency $\sqrt{\frac{k}{m}}$. Next, the normal mode $(1, -1)$ corresponds to displacements of equal magnitude and opposite direction. Then, the middle spring is stretched or compressed twice as much as the displacements of the masses. This, in combination with the springs attached to the walls, causes each mass to be effectively attached to a spring of spring constant $3k$—implying that its corresponding normal frequency is $\sqrt{\frac{3k}{m}}$.

10.4.2 General Solution

In the general case of linear second-order, coupled simple-harmonic differential equations, the specific way of multiplying equations by constants and adding them in order to successfully decouple them is difficult to spot. Therefore, a general solution would be ideal. With n variables $(x_1, x_2, ..., x_n)$, we generally have the following set of equations.

$$\ddot{x}_1 = c_{11}x_1 + c_{12}x_2 + ... + c_{1n}x_n$$
$$\ddot{x}_2 = c_{21}x_1 + c_{22}x_2 + ... + c_{2n}x_n$$

$$\vdots$$

$$\ddot{x}_n = c_{n1}x_1 + c_{n2}x_2 + ... + c_{nn}x_n$$

where c_{ij} is the constant in the ith row and jth column. We let $\boldsymbol{X}$ be the $n \times 1$ matrix

$$\boldsymbol{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

and define A as the $n \times n$ matrix

$$\boldsymbol{A} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & & & \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix}$$

We can then represent the set of equations compactly by

$$\ddot{\boldsymbol{X}} = \boldsymbol{A}\boldsymbol{X}$$

Since they are coupled equations, we can try to guess that the solutions for the various x_i's all have the same angular frequency. As long as we can determine a general solution with $2n$ constants to accommodate the $2n$ initial conditions, we are done. Concretely, we guess

$$\boldsymbol{X} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} e^{i\omega t} = \boldsymbol{u}e^{i\omega t}$$

$$\implies \ddot{\boldsymbol{X}} = -\boldsymbol{u}\omega^2 e^{i\omega t}$$

Substituting these back into the matrix equation, we get

$$-\boldsymbol{u}\omega^2 e^{i\omega t} = \boldsymbol{A}\boldsymbol{u}e^{i\omega t}$$

$$\boldsymbol{A}\boldsymbol{u} = -\omega^2\boldsymbol{u} \tag{10.13}$$

Again, any linear combination of the solutions obtained by guessing is also a solution; the general solution is a linear combination of all the linearly independent solutions. Now, notice that if $\boldsymbol{u}e^{i\omega t}$ is a solution, $\boldsymbol{u}e^{-i\omega t}$ is also a solution (as they both result in the same Equation 10.13). Therefore, instead of including the complex conjugates such as $\boldsymbol{u}e^{-i\omega t}$ in the general solution, we can simply write the general solution as the real component of the linear combination of solutions with positive, imaginary exponents (i.e. $\boldsymbol{u}e^{i\omega t}$) as $\boldsymbol{X}$ must be real at all times. At this juncture, we proceed with a short linear algebra interlude.

Eigenvectors

Let $\boldsymbol{A}$ be a $n \times n$ matrix. A non-null vector $\boldsymbol{u}$ is known as an eigenvector of $\boldsymbol{A}$ if

$$\boldsymbol{A}\boldsymbol{u} = \lambda\boldsymbol{u}$$

for some scalar λ. λ is termed an eigenvalue of $\boldsymbol{A}$ and $\boldsymbol{u}$ is known as an eigenvector associated with the eigenvalue λ. The eigenvalues can be determined as follows.

$$\boldsymbol{A}\boldsymbol{u} - \lambda\boldsymbol{u} = \boldsymbol{0}$$

$$(\boldsymbol{A} - \lambda\boldsymbol{I})\,\boldsymbol{u} = \boldsymbol{0}$$

where $\boldsymbol{0}$ is a vector of zeroes with n rows and $\boldsymbol{I}$ is the identity matrix of order n. The identity matrix of order n is a $n \times n$ square[4] matrix whose top-left to bottom-right diagonal entries are one—all other entries are zero. As its nomenclature implies, multiplying a square matrix $\boldsymbol{X}$ by the identity matrix of

[4] A square matrix is simply one with an identical number of rows and columns.

the same dimensions simply returns $\boldsymbol{X}$ ($\boldsymbol{XI} = \boldsymbol{X}$ and $\boldsymbol{IX} = \boldsymbol{X}$). Therefore, we have simply expressed $\boldsymbol{u} = \boldsymbol{Iu}$ in writing the second equation.

Now, consider the following definition: an inverse $\boldsymbol{X}^{-1}$ of a square matrix $\boldsymbol{X}$ is defined as a matrix such that the matrix multiplications $\boldsymbol{X}\boldsymbol{X}^{-1} = \boldsymbol{I}$ and $\boldsymbol{X}^{-1}\boldsymbol{X} = \boldsymbol{I}$ (i.e. they yield the identity matrix of the same dimensions as $\boldsymbol{X}$). Suppose that an inverse of $(\boldsymbol{A} - \lambda\boldsymbol{I})$ exists. Then by multiplying this inverse to both sides of the previous equation, we obtain the trivial solution

$$\boldsymbol{u} = \boldsymbol{0}$$

which is contrary to what we want as an eigenvector is not a null vector by definition and because the null case is not physically meaningful in the case of coupled oscillations. Thus, in order for non-trivial solutions of $\boldsymbol{u}$ to exist, $(\boldsymbol{A} - \lambda\boldsymbol{I})$ must be non-invertible or singular. In linear algebra, this is equivalent to saying that the determinant of this term is zero.

Determinants

The determinant of a $n \times n$ square matrix, $\boldsymbol{X}$, is a quantity that can be computed recursively as follows. Given that

$$\boldsymbol{X} = \begin{pmatrix} x_{11} & x_{12} & ... & x_{1n} \\ x_{21} & x_{22} & ... & x_{2n} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ x_{n1} & x_{n2} & ... & x_{nn} \end{pmatrix}$$

$$\det(\boldsymbol{X}) = \begin{cases} x_{11} \quad \text{if} \quad n = 1 \\ x_{i1}Y_{i1} + x_{i2}Y_{i2} + ... + x_{in}Y_{in} \quad \text{for} \quad n \geq 2 \end{cases}$$

where i refers to that particular row. Y_{ij} is defined as

$$Y_{ij} = (-1)^{i+j}\det(\boldsymbol{Z}_{ij})$$

$\boldsymbol{Z}_{ij}$ is the matrix obtained by removing the ith row and jth column from $\boldsymbol{X}$.

$$\boldsymbol{Z}_{ij} = \begin{pmatrix} x_{11} & x_{12} & ... & x_{1(j-1)} & x_{1(j+1)} & ... & x_{1n} \\ x_{21} & x_{22} & ... & x_{2(j-1)} & x_{2(j+1)} & ... & x_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{(i-1)1} & x_{(i-1)2} & ... & x_{(i-1)(j-1)} & x_{(i-1)(j+1)} & ... & x_{(i-1)n} \\ x_{(i+1)1} & x_{(i+1)2} & ... & x_{(i+1)(j-1)} & x_{(i+1)(j+1)} & ... & x_{(i+1)n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & ... & x_{n(j-1)} & x_{n(j+1)} & ... & x_{nn} \end{pmatrix}$$

The recursive definition of $\det(\boldsymbol{X})$ for $n \geq 2$ above is known as the co-factor expansion along row i, where i is an arbitrary integer $1 \leq i \leq n$. In fact, the determinant can also be calculated via a co-factor expansion along any column j.

$$\det(\boldsymbol{X}) = \sum_{i=1}^{N} x_{ij}Y_{ij} \quad \text{for} \quad n \geq 2$$

Let us now evaluate the determinants of two concrete examples to clarify this esoteric definition. The most common form of matrices would be the 2×2 matrix. If we let $\boldsymbol{X}$ be

$$\boldsymbol{X} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then by performing a cofactor expansion along the first row,

$$\det(\boldsymbol{X}) = aY_{11} + bY_{12} = a \cdot (-1)^{1+1}|d| + b(-1)^{1+2}|c| = ad - bc$$

where the vertical lines denote taking the determinant of the matrix they enclose.

Problem: Determine the determinant of the following matrix.

$$\boldsymbol{X} = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

Performing a cofactor expansion along the first row,

$$\begin{aligned} \det(\boldsymbol{X}) &= 1 \cdot \begin{vmatrix} 3 & 0 \\ 1 & 2 \end{vmatrix} - 2 \cdot \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} \\ &= (3 \cdot 2 - 1 \cdot 0) - 2 \cdot (1 \cdot 2 - 0 \cdot 0) \\ &= 2 \end{aligned}$$

Incidentally, there is an efficient memorization scheme for the determinant of a 3x3 matrix known as Sarrus' rule.

$$\begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{vmatrix} = x_{11}x_{22}x_{33} + x_{12}x_{23}x_{31} + x_{13}x_{21}x_{32} - x_{31}x_{22}x_{13} - x_{32}x_{23}x_{11} - x_{33}x_{21}x_{12}$$

This sum can be visualized by replicating the first two columns on the right of the original block of numbers and taking the sum of the products along the bolded diagonals, minus the sum of the products along the dashed diagonals below.

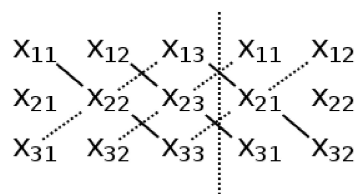

Figure 10.11: Sarrus' Rule

Evaluating Eigenvalues and Normal Frequencies

Returning to our main topic of eigenvectors, the eigenvalues associated with a matrix $\boldsymbol{A}$ can be computed by setting the determinant of $\boldsymbol{A} - \lambda \boldsymbol{I}$ to be zero.

$$\det\left(\boldsymbol{A} - \lambda \boldsymbol{I}\right) = 0$$

This will generate a n order polynomial for λ which has n roots by the fundamental theorem of algebra. In the case of coupled oscillators, by observing Equation 10.13, we have

$$\left(\boldsymbol{A} + \omega^2 \boldsymbol{I}\right) \boldsymbol{u} = \boldsymbol{0} \tag{10.14}$$

which has non-trivial solutions only if

$$\det\left(\boldsymbol{A} + \omega^2 \boldsymbol{I}\right) = 0$$

That is, the negative of the normal frequencies squared are the eigenvalues of the matrix $\boldsymbol{A}$. Let us consider the specific spring-mass oscillators in the previous section. The equations of motion produced

$$\ddot{x}_1 = -\frac{2k}{m}x_1 + \frac{k}{m}x_2$$

$$\ddot{x}_2 = \frac{k}{m}x_1 - \frac{2k}{m}x_2$$

Therefore, the matrix $\boldsymbol{A}$ is

$$\boldsymbol{A} = \begin{pmatrix} -\frac{2k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{2k}{m} \end{pmatrix}$$

Then, we require

$$\begin{vmatrix} -\frac{2k}{m}+\omega^2 & \frac{k}{m} \\ \frac{k}{m} & -\frac{2k}{m}+\omega^2 \end{vmatrix} = 0$$

$$\left(-\frac{2k}{m}+\omega^2\right)^2 - \left(\frac{k}{m}\right)^2 = \left(\omega^2 - \frac{3k}{m}\right)\left(\omega^2 - \frac{k}{m}\right) = 0$$

$$\omega^2 = \frac{k}{m} \quad \text{or} \quad \frac{3k}{m}$$

Thus, the eigenvalues of $\boldsymbol{A}$ are $-\frac{k}{m}$ and $-\frac{3k}{m}$ while the normal frequencies are $\sqrt{\frac{k}{m}}$ and $\sqrt{\frac{3k}{m}}$.

Evaluating Eigenvectors and Normal Modes

Now that we have computed the eigenvalues of $\boldsymbol{A}$, we can now determine the eigenvectors associated with an eigenvalue λ_i by substituting $\lambda = \lambda_i$ back into the equation $(\boldsymbol{A} - \lambda \boldsymbol{I})\,\boldsymbol{u} = \boldsymbol{0}$. Then, we can solve the resultant matrix equation (some variables will still be expressed in terms of the others) to obtain a general solution for $\boldsymbol{u}$ that satisfies the equation. This general solution, which is expressed in terms of a linear combination of independent vectors, is known as the eigenspace associated with the eigenvalue λ_i. The eigenspace is usually denoted as $\boldsymbol{E}_{\lambda_i}$ but we shall denote it as $\boldsymbol{E}_{-\lambda_i}$ (as $\lambda_i = -\omega_i^2$ where ω_i is the ith normal frequency) for our purposes. The independent vectors which appear in the linear combination are the basis eigenvectors associated with the eigenvalue λ_i as substituting any linear combination of them for $\boldsymbol{u}$ in $\boldsymbol{A}\boldsymbol{u}$ would result in $\lambda_i \boldsymbol{u}$. Furthermore, in the context of coupled oscillators, the basis eigenvectors associated with eigenvalue $\lambda_i = -\omega_i^2$ turn out to be the normal modes associated with the normal frequency ω_i. Do not worry too much about what these terms mean for now and consider the following specific example. In the case of the coupled spring-mass oscillators, we substitute the various values for ω^2, that we have found, into Equation 10.14.

When $\lambda_1 = -\omega_1^2 = -\frac{k}{m}$, we obtain

$$\begin{pmatrix} -\frac{k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{k}{m} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0$$

Solving gives

$$u_1 = u_2$$

Therefore, the eigenspace for $\boldsymbol{u}$ associated with λ_1 is

$$\boldsymbol{E}_{\frac{k}{m}} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

where C_1 is a scalar. That is, any vector $\boldsymbol{u}$ of this form would be an eigenvector associated with the eigenvalue λ_1. Evidently, the only basis eigenvector associated with the eigenvalue λ_1 is $(1, 1)$. Similarly, when $\lambda_2 = -\omega_2^2 = -\frac{3k}{m}$,

$$\begin{pmatrix} \frac{k}{m} & \frac{k}{m} \\ \frac{k}{m} & \frac{k}{m} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0$$

$$u_2 = -u_1$$

The eigenspace associated with λ_2 is

$$\boldsymbol{E}_{\frac{3k}{m}} = C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

for some scalar C_2. The only basis eigenvector associated with the eigenvalue λ_2 is $(1, -1)$. Finally, the general solution for the displacements of the masses, $\boldsymbol{X}$, is obtained by concatenating the various $\boldsymbol{E}_{\omega_i^2} e^{i\omega_i t}$'s. We will not include $\boldsymbol{E}_{\omega_i^2} e^{-i\omega_i t}$ as we will take the real component of the combination later to obtain the physical solution for $\boldsymbol{X}$ (see paragraph below Equation 10.13). The expression obtained from patching is

$$\boldsymbol{E}_{\frac{k}{m}} e^{i\sqrt{\frac{k}{m}}t} + \boldsymbol{E}_{\frac{3k}{m}} e^{i\sqrt{\frac{3k}{m}}t} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i\sqrt{\frac{k}{m}}t} + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{i\sqrt{\frac{3k}{m}}t}$$

If we let $C_1 = D_1 e^{i\phi_1}$ and $C_2 = D_2 e^{i\phi_2}$ where D_1, D_2, ϕ_1 and ϕ_2 are real constants, taking the real component of the above expression yields

$$\boldsymbol{X} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} D_1 \cos\left(\sqrt{\frac{k}{m}}t + \phi_1\right) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} D_2 \cos\left(\sqrt{\frac{3k}{m}}t + \phi_2\right)$$

As seen from above, the eigenvectors associated with eigenvalue λ_i now function as the normal modes of the normal frequency ω_i. The above expression is the most general solution for $\boldsymbol{X}$ as we have 4 constants to accommodate the 4 initial conditions (positions and velocities of both masses). As a last remark, this method of finding the eigenvalues is not foolproof. If there are repeated eigenvalues, we may not be able to find sufficient linearly independent solutions. Then, we would need to guess other forms of solutions. In the specific case where $\omega^2 = 0$ is a possibility, we should guess polynomials of degree one (i.e. $\boldsymbol{X} = \boldsymbol{u}(c_0 + c_1 t)$) as $\omega^2 = 0$ insinuates that the second derivative of $\boldsymbol{X}$ is zero.