# Chapter 14

# Appendices

## 14.1 Appendix A: Useful formulas

### 14.1.1 Taylor series

$$f(x_0+\epsilon) = f(x_0) + f'(x_0)\epsilon + \frac{f''(x_0)}{2!}\epsilon^2 + \frac{f'''(x_0)}{3!}\epsilon^3 + \cdots \tag{14.1}$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \tag{14.2}$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots \tag{14.3}$$

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots \tag{14.4}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{14.5}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \tag{14.6}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \tag{14.7}$$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \cdots \tag{14.8}$$

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{x}{2} + \frac{3x^2}{8} + \cdots \tag{14.9}$$

$$(1+x)^n = 1 + nx + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots \tag{14.10}$$

### 14.1.2 Nice formulas

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{14.11}$$

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}), \qquad \sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}) \tag{14.12}$$

$$\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}}, \qquad \sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}} \tag{14.13}$$

$$\tan\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{1-\cos\theta}{\sin\theta} = \frac{\sin\theta}{1+\cos\theta} \tag{14.14}$$

$$\sin 2\theta = 2\sin\theta\cos\theta, \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta \tag{14.15}$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \tag{14.16}$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{14.17}$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \tag{14.18}$$

$$\cosh x = \frac{1}{2}(e^x + e^{-x}), \qquad \sinh x = \frac{1}{2}(e^x - e^{-x}) \tag{14.19}$$

$$\cosh^2 x - \sinh^2 x = 1 \tag{14.20}$$

$$\frac{d}{dx}\cosh x = \sinh x, \qquad \frac{d}{dx}\sinh x = \cosh x \tag{14.21}$$

### 14.1.3 Integrals

$$\int \ln x\, dx = x\ln x - x \tag{14.22}$$

$$\int x\ln x\, dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} \tag{14.23}$$

$$\int xe^x = e^x(x-1) \tag{14.24}$$

$$\int \frac{dx}{1+x^2} = \tan^{-1} x \qquad \text{or} \qquad -\cot^{-1} x \tag{14.25}$$



$$\int \frac{dx}{x(1+x^2)} = \frac{1}{2}\ln\left(\frac{x^2}{1+x^2}\right) \tag{14.26}$$

$$\int \frac{dx}{1-x^2} = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad \text{or} \qquad \tanh^{-1} x \qquad (x^2 < 1) \tag{14.27}$$

$$\int \frac{dx}{1-x^2} = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right) \qquad \text{or} \qquad \coth^{-1} x \qquad (x^2 > 1) \tag{14.28}$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x \qquad \text{or} \qquad -\cos^{-1} x \tag{14.29}$$

$$\int \frac{dx}{\sqrt{x^2+1}} = \ln(x + \sqrt{x^2+1}\,) \qquad \text{or} \qquad \sinh^{-1} x \tag{14.30}$$

$$\int \frac{dx}{\sqrt{x^2-1}} = \ln(x + \sqrt{x^2-1}\,) \qquad \text{or} \qquad \cosh^{-1} x \tag{14.31}$$

$$\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x \qquad \text{or} \qquad -\csc^{-1} x \tag{14.32}$$

$$\int \frac{dx}{x\sqrt{1+x^2}} = -\ln\left(\frac{1+\sqrt{1+x^2}}{x}\right) \qquad \text{or} \qquad -\operatorname{csch}^{-1} x \tag{14.33}$$

$$\int \frac{dx}{x\sqrt{1-x^2}} = -\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) \qquad \text{or} \qquad -\operatorname{sech}^{-1} x \tag{14.34}$$

$$\int \frac{dx}{\cos x} = \ln\left(\frac{1+\sin x}{\cos x}\right) \tag{14.35}$$

$$\int \frac{dx}{\sin x} = \ln\left(\frac{1-\cos x}{\sin x}\right) \tag{14.36}$$

## 14.2 Appendix B: Units, dimensional analysis

There are two strategies (at least) that you should invoke without hesitation when solving problems. One is the consideration of units (that is, dimensions), which is the subject of this Appendix. The other is the consideration of limiting cases, which is the subject of the next Appendix.

The consideration of units offers two main benefits. First, looking at units before beginning a calculation can tell you roughly what the answer has to look like, up to numerical factors. (In some problems, you can determine the numerical factors by considering a limiting case of a certain parameter. So you may not have to do *any* calculations to solve a problem!) Second, checking units at the end of a calculation (which is something you should *always* do) tells you if your answer has a chance at being correct. It won't tell you that your answer is definitely correct, but it might tell you that your answer is definitely incorrect.

> "Your units are wrong!" cried the teacher.
> "Your church weighs six joules — what a feature!
> And the people inside
> Are four hours wide,
> And eight gauss away from the preacher!"

In practice, the second of the above two benefits is what you will generally make use of. But let's do a few examples relating to the first benefit, since these can be a little more exciting. To solve the following problems exactly, we would need to invoke results derived in earlier chapters in the text. But let's just see how far we can get by using only dimensional analysis. We'll use the "[ ]" notation for units, and we'll let $M$ stand for mass, $L$ for length, and $T$ for time. For example, we will write a speed as $[v] = L/T$, and we will write the gravitational constant as $[G] = L^3/MT^2$ (you can figure this out by noting that $Gm_1m_2/r^2$ has the dimensions of force).

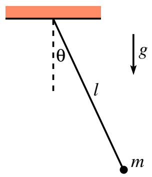


**Figure 14.1**

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**Example 1 (The pendulum):** A mass $m$ hangs from a massless string of length $\ell$ (see Fig. 14.1). The acceleration due to gravity is $g$. What can we say about the frequency of oscillations?

The only dimensionful quantities given in the problem are $[m] = M$, $[\ell] = L$, and $[g] = L/T^2$. There is one more quantity, the maximum angle $\theta_0$, which is dimensionless (this one is easy to forget). We want to find the frequency, which has dimensions $1/T$. Clearly, the only combination of our given dimensionful quantities with the units of $1/T$ is $\sqrt{g/\ell}$. We can't rule out any $\theta_0$ dependence, so the most general possible form for the frequency (in radians per second) is

$$\omega = f(\theta_0)\sqrt{\frac{g}{\ell}}, \tag{14.37}$$

where $f$ is a dimensionless function of the dimensionless variable $\theta_0$.

Remarks: It just so happens that for small oscillations, $f(\theta_0)$ is essentially equal to 1, and so the frequency is essentially $\sqrt{g/\ell}$. But there is no way to show this using only dimensional analysis. For larger values of $\theta_0$, the higher-order terms in the expansion of $f$ become important. Exercise 3.1 deals with the leading correction. (The answer is $f(\theta_0) = 1 - \theta_0^2/16 + \cdots$.)

There is only one mass scale in the problem, so there is no way that the frequency, with units $1/T$, can depend on $[m] = M$. ♣

What can we say about the total energy (relative to the lowest point) of the pendulum? Energy has units $ML^2/T^2$, and the only combination of the given dimensionful constants of this form is $mg\ell$. Therefore, the energy must be of the form $f(\theta_0)mg\ell$, where $f$ is some function. We know, in fact, that the total energy equals the potential energy at the highest point, which is $mg\ell(1-\cos\theta_0)$. Using the Taylor expansion for $\cos\theta$, we see that $f(\theta_0) = \theta_0^2/2 - \theta_0^4/24 + \cdots$. Unlike in the above case for the frequency, the maximum angle $\theta_0$ plays a critical role in the energy.



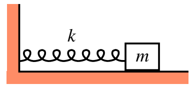


**Figure 14.2**

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**Example 2 (The spring):** A spring with spring-constant $k$ has a mass $m$ on its end (see Fig. 14.2). The force is $F(x) = -kx$, where $x$ is the displacement from equilibrium. What can we say about the frequency of oscillations?

The only dimensionful quantities in the problem are $[m] = M$, $[k] = M/T^2$ (obtained by noting that $kx$ has the dimensions of force), and the maximum displacement from equilibrium, $[x_0] = L$. (There is also the equilibrium length, but the force doesn't depend on this, so there is no way it can come into the answer.) We want to find the frequency, which has dimensions $1/T$. It is easy to see that the only combination of our given dimensionful quantities with these units is

$$\omega = C\sqrt{\frac{k}{m}}, \tag{14.38}$$

where $C$ is a dimensionless number. Note that, in contrast with the case of the pendulum, the frequency cannot have any dependence on the maximum displacement. It just so happens that $C$ is equal to 1, but there is no way to show this using only dimensional analysis.

What can we say about the total energy of the spring? Energy has units $ML^2/T^2$, and the only combination of the given dimensionful constants of this form is $Bkx_0^2$, where $B$ is a dimensionless number. It turns out that $B = 1/2$, so the total energy is given by $kx_0^2/2$.

Remark: Real springs don't have perfect parabolic potentials (that is, linear forces), so the force looks more like $F(x) = -kx + bx^2 + \cdots$. If we truncate the series at the second term, then we have one more dimensionful quantity to work with, $[b] = M/LT^2$. To form a quantity with dimensions of frequency, $1/T$, we clearly need the $x_0$ and $b$ to appear in the combination $(x_0 b)$ (to get rid of the $L$). It is then easy to see that the frequency must be of the form $f(x_0b/k)\sqrt{k/m}$. So we can have $x_0$ dependence in this case. Note that this answer must reduce to $C\sqrt{k/m}$ for $b = 0$. Hence, $f$ must be of the form $f(y) = C + c_1y + c_2y^2 + \cdots$. ♣

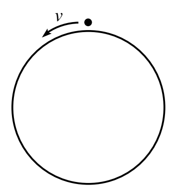


**Figure 14.3**

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**Example 3 (Speed of low-orbit satellite):** A satellite of mass $m$ travels in an orbit just above the earth's surface (see Fig. 14.3). What can we say about its speed?

**Solution:** The only dimensionful quantities in the problem are $[m] = M$, $[g] = L/T^2$, and the radius of the earth $[R] = L$. We want to find the speed, which has dimensions $L/T$. Clearly, the only combination of our dimensionful quantities with these units is

$$v = C\sqrt{gR}. \tag{14.39}$$

It turns out that $C = 1$.

### 14.2.1 Problems

1. **Escape velocity**

   Find the escape velocity from the earth (that is, the speed above which a particle escapes to infinity), up to factors of order 1.

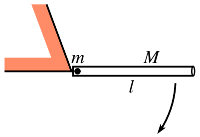


**Figure 14.4**

2. **Mass in tube**

   A tube of mass $M$ and length $\ell$ is free to swing on a pivot at one end. A mass $m$ is positioned inside the tube at this end. The tube is held horizontal and then released (see Fig. 14.4).

   Let $\eta$ be the fraction of the tube the mass has traversed by the time the tube becomes vertical. Does $\eta$ depend on $\ell$?

3. **Damping** $**$

   A particle with mass $M$ and initial speed $V$ is subject to a velocity-dependent damping force of the form $bv^n$.

   (a) For $n = 0, 1, 2, \ldots$, determine how the stopping time depends on $M$, $V$, and $b$.

   (b) For $n = 0, 1, 2, \ldots$, determine how the stopping distance depends on $M$, $V$, and $b$.

   (Careful! See if your answers make sense. Dimensional analysis only gives the answer up to a numerical factor. This is a tricky problem.)

### 14.2.2 Solutions

1. **Escape velocity**

   The reasoning is exactly the same as in the satellite example above. Therefore, the answer is $v = C\sqrt{gR} = C\sqrt{GM/R}$. It turns out that $C = 2$.

2. **Mass in tube**

   The relevant dimensionful constants are $[g] = L/T^2$, $[\ell] = L$, $[m] = M$, and $[M] = M$. We want to produce a dimensionless number $\eta$. Since $g$ is the only constant involving time, $\eta$ cannot depend on $g$. This then implies that $\eta$ cannot depend on $\ell$, the only length remaining. Therefore, $\eta$ depends only on the ratio $m/M$. So the answer to the stated problem is, "No."

   It turns out that you have to solve the problem numerically to find $\eta$. Some results are: If $m = M$, then $\eta \approx 0.341$. If $m \ll M$, then $\eta \approx 0.349$. And if $m \gg M$, then $\eta \approx 0$.



3. **Damping**

   (a) The constant $b$ has units $[b] = [\text{Force}][v^{-n}] = (ML/T^2)(T^n/L^n)$. The other constants are $[M] = M$ and $[V] = L/T$. There is also $n$, which is dimensionless. The only combination with units of $T$ is

   $$t = f(n)\frac{M}{bV^{n-1}}, \tag{14.40}$$

   where $f(n)$ is a dimensionless function of $n$.

   For $n = 0$, we have $t = f(0)MV/b$. (This increases with $M$ and $V$, and decreases with $b$, as it should.)

   For $n = 1$ we have $t = f(1)M/b$. So we *seem* to have $t \sim M/b$. This, however, cannot be correct, because $t$ should definitely grow with $V$. (A large initial speed $V_1$ requires some non-zero time to slow down to a smaller speed $V_2$, after which point we just have the same scenario with initial speed $V_2$.) Where did we go wrong? After all, the answer *does* have to look like $t = f(1)M/b$, where $f(1)$ is a numerical factor.

   The resolution to this puzzle is that $f(1)$ is infinite. (If we wanted to work out the problem exactly, we would encounter an integral that diverges.) So for any $V$, $t$ is infinite.[1]

   Similarly, for $n \geq 2$, there is at least one power of $V$ in the denominator of $t$. This certainly cannot be correct, because $t$ should not decrease with $V$. So $f(n)$ must likewise be infinite for all of these cases.

   The moral of this exercise is that you have to be careful when using dimensional analysis. The numerical factor in front of your answer generally turns out to be of order 1, but sometimes it turns out to be 0 or $\infty$.

   REMARK: For $n \geq 1$, the expression in eq. (14.40) still has relevance. For example, for $n = 2$, the $M/(Vb)$ expression is relevant if you want to know how long it takes to go from $V$ to some final speed $V_f$. The answer involves $M/(V_f b)$, which diverges as $V_f \to 0$. ♣

   (b) The only combination with units of $L$ is

   $$\ell = g(n)\frac{M}{bV^{n-2}}, \tag{14.41}$$

   where $g(n)$ is a dimensionless function of $n$.

   For $n = 0$, we have $\ell = g(0)MV^2/b$.

   For $n = 1$, we have $\ell = g(1)MV/b$.

   For $n = 2$ we have $\ell = g(2)M/b$. So we *seem* to have $\ell \sim M/b$. But as in part (a), this cannot be correct, because $\ell$ should definitely depend on $V$. (A large initial speed $V_1$ requires some non-zero distance to slow down to a smaller speed $V_2$, after which point we just have the same scenario with initial speed $V_2$.) So, from the reasoning in part (a), the total distance is infinite for $n \geq 2$, because the function $g$ is infinite in these cases.

   REMARK: Note that for $n \neq 1$, $t$ and $\ell$ are either both finite or both infinite. For $n = 1$, however, the total time is infinite, whereas the total distance is finite. (This situation holds for $1 \leq n < 2$, if we want to consider fractional $n$.) ♣

[1] The total time $t$ is ill-defined, of course, since the particle never comes to rest. But $t$ does grow with $V$, in the sense that if $t$ is defined to be the time to attain some given small speed, then $t$ grows with $V$.

## 14.3 Appendix C: Approximations, limiting cases

Along with checking units, checking limiting cases is something you should always do at the end of a calculation. As in the case with checking dimensions, this won't tell you that your answer is definitely correct, but it might tell you that your answer is definitely incorrect. It is generally true that your intuition about limiting cases is much better than your intuition about generic values of the parameters. You should use this fact to your advantage.

A main ingredient in checking limiting cases is the Taylor series approximations. The series for many functions are given in Appendix A.

The examples presented below have been taken from various problems throughout the book. For the most part, we'll just repeat here what we've said in the remarks given in the solutions earlier in the text.

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**Example 1 (Dropped ball):** Consider a dropped ball that is subject to an air-resistance drag force of the form $F_d = -m\alpha v$, in addition to the usual $mg$ force downward. Let the initial height be $h$. We found in Section 2.3 that the ball's speed and position are given by

$$v(t) = -\frac{g}{\alpha}\left(1 - e^{-\alpha t}\right), \qquad \text{and} \qquad y(t) = h - \frac{g}{\alpha}\left(t - \frac{1}{\alpha}\left(1 - e^{-\alpha t}\right)\right). \tag{14.42}$$

Let's look at some limiting cases. If $t$ is very small (more precisely,[2] if $\alpha t \ll 1$), then we can use the Taylor series $e^{-x} \approx 1 - x + x^2/2$ to make approximations to leading order in $\alpha t$. Eq. (14.42) gives

$$\begin{aligned} v(t) &= -\frac{g}{\alpha}\left(1 - \left(1 - \alpha t + \frac{(\alpha t)^2}{2} - \cdots\right)\right) \\ &\approx -gt, \end{aligned} \tag{14.43}$$

plus terms of higher order in $\alpha t$. This answer is expected, because the drag force is negligible at the start, so we almost have a freely falling body. We also have

$$\begin{aligned} y(t) &= h - \frac{g}{\alpha}\left[t - \frac{1}{\alpha}\left(1 - \left(1 - \alpha t + \frac{(\alpha t)^2}{2} - \cdots\right)\right)\right] \\ &\approx h - \frac{gt^2}{2}, \end{aligned} \tag{14.44}$$

plus terms of higher order in $\alpha t$. Again, this answer is expected.

We may also look at large $t$ (or rather, large $\alpha t$). In this case, $e^{-\alpha t}$ is essentially 0, so eq. (14.42) gives

$$v(t) \approx -\frac{g}{\alpha}. \tag{14.45}$$

This is the "terminal velocity". This value makes sense, because it is the velocity for which the force $-mg - m\alpha v$ vanishes. Also, eq. (14.42) gives

$$y(t) \approx h - \frac{gt}{\alpha} + \frac{g}{\alpha^2}. \tag{14.46}$$

[2]See the "Remark" following this example.



Apparently, for large $t$, $g/\alpha^2$ is the distance our ball lags behind another ball which started out already at the terminal speed, $g/\alpha$.

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Whenever you derive approximate answers as we did above, you gain something and you lose something. You lose some truth, of course, because your new answer is technically not correct. But you gain some aesthetics. Your new answer is invariably much cleaner (sometimes involving only one term), and this makes it a lot easier to see what's going on.

Remark: In the above example, it makes no sense to look at the limit where $t$ is small (or large), because $t$ has dimensions. Is a year a large or small time? How about a hundredth of a second? There is no way to answer this without knowing what problem you're dealing with. A year is short on the time scale of galactic evolution, but a hundredth of a second is long on the time scale of nuclear processes.

It only makes sense to look at the limit of a small (or large) *dimensionless* quantity. In the above example, this quantity is $\alpha t$. The given constant $\alpha$ has units of $[T^{-1}]$; hence, $1/\alpha$ sets a typical time scale for the system. It therefore makes sense to look at the limit where $t \ll 1/\alpha$ (that is, $\alpha t \ll 1$), or likewise $t \gg 1/\alpha$ (that is, $\alpha t \gg 1$).

In the limit of a small dimensionless quantity, a Taylor series can be used to expand your answer in powers of the small quantity.

We will sometimes get sloppy and say things like, "In the limit of small $t$." But you know that we really mean, "In the limit of some small dimensionless quantity that has a $t$ in the numerator," or, "In the limit where $t$ is much smaller that a certain quantity that has the dimensions of time." ♣

The results of the limits you check generally fall into two categories. Most of the time you know what the result should be, so this provides a double-check for your answer. But sometimes an interesting limit pops up that you might not expect. Such is the case in the following examples.

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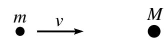


**Figure 14.5**

**Example 2 (Two masses in 1-D):** A mass $m$ with speed $v$ approaches a stationary mass $M$ (see Fig. 14.5). The masses bounce off each other elastically. Assume all motion takes place in one dimension. We found in Section 4.6.1 that the final speeds of the particles are

$$v_m = \frac{(m-M)v}{m+M}, \qquad \text{and} \qquad v_M = \frac{2mv}{m+M}. \tag{14.47}$$

Some obvious special cases to check are the following.

- If $m = M$, then $m$ stops, and $M$ picks up a speed of $v$. This is fairly believable. And it becomes quite obvious once you realize that these final speeds clearly satisfy conservation of energy and momentum with the initial conditions.
- If $M \gg m$, then $m$ bounces backward with speed $\approx v$, and $M$ hardly moves. This is clear, since $M$ basically acts like a brick wall.
- If $m \gg M$, then $m$ keeps plowing along at speed $\approx v$, and $M$ picks up a speed of $\approx 2v$. This $2v$ is an interesting result (it is clear if you consider what is happening in the reference frame of the heavy mass $m$), and it leads to some neat effects, as in Problem 4.25.

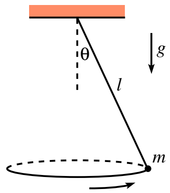


**Figure 14.6**

**Example 3 (Circular pendulum):** A mass hangs from a string of length $\ell$. Conditions have been set up so that the mass swings around in a horizontal circle, with the string making a constant angle of $\theta$ with the vertical (see Fig. 14.6). We found in Section 2.5 that the angular frequency, $\omega$, of this motion is

$$\omega = \sqrt{\frac{g}{\ell \cos\theta}}\,. \tag{14.48}$$

There are two obvious limits to look at.

- If $\theta \approx 90^\circ$, then $\omega \to \infty$. This is clear. The mass has to spin quickly to avoid flopping down.
- If $\theta \approx 0$, then $\omega \approx \sqrt{g/\ell}$, which is the same as the frequency of a plane pendulum of length $\ell$. (Convince yourself why this should be true.)

In the above examples, we checked limiting cases of answers that are correct (I hope). This whole process is more useful (and actually a bit more fun) when you check the limits of an answer that is *incorrect*. In this case, you gain the unequivocal information that your answer is wrong. This information is something you should be happy about, considering that the alternative is to carry on in a state of blissful ignorance. Personally, if there's any way I'd want to discover that my answer is garbage, this is it. At any rate, checking limits can often save you a lot of trouble in the long run . . .

The lemmings get set for their race.
With one step and two steps they pace.
They take three and four,
And then head on for more,
Without checking the limiting case.