Hwa Chong Institution

Singapore Junior Physics Olympiad 2018

Mechanics Diagnostic Test (Secondary 3)

Answer Key

(Created by Christopher Ong)

A Short-Answer Questions (16 marks)

1. $\boxed{(\sqrt{2}+1)v/g}$ The motion of the object can be split into two components: from the start to its highest point, and from the highest point back to its original position. In the first component, let t_1 be the time taken for it to reach its highest point, and h be the height it reaches. From the basic kinematics equations, we can easily find t_1 and h.

$$v = u + at \Rightarrow 0 = v - gt_1 \Rightarrow t_1 = \frac{v}{g}$$
$$v^2 = u^2 + 2as \Rightarrow 0 = v^2 - 2gh \Rightarrow h = \frac{v^2}{2g}$$

In the second component, the mass of the Earth is halved. This means that the acceleration of the object due to gravity is also halved (since $g = GM/R^2$). Let t_2 be the time taken for it to return to its original position.

$$s = ut + \frac{1}{2}at^2$$
$$h = \frac{1}{2}(\frac{g}{2})t_2^2$$
$$t_2^2 = \frac{4h}{g} = \frac{4}{g}\frac{v^2}{2g}$$
$$t_2^2 = \frac{2v^2}{g^2}$$
$$t_2 = \sqrt{2}\frac{v}{g}$$

To get the total time of its journey, sum up t_1 and t_2, to get:

$$t = t_1 + t_2 = \frac{v}{g}(\sqrt{2}+1)$$

2. $\boxed{N_1 = 5F/6,\ N_2 = F/2}$ The acceleration of the three boxes are equal. Let this acceleration be a. Considering the net force on the entire system,

$$\begin{aligned} F &= (m + 2m + 3m)a \\ a &= \frac{F}{6m} \end{aligned}$$

The force acting on the rightmost box and causing it to accelerate is N_2, whereas the net force causing the middle box to accelerate is $N_1 - N_2$. Writing the $F = ma$ equations for each of these boxes:

$$\begin{aligned} N_2 &= 3ma = 3m\frac{F}{6m} = \frac{F}{2} \\ N_1 - N_2 &= 2ma = 2m\frac{F}{6m} = \frac{F}{3} \Rightarrow N_1 = \frac{F}{2} + \frac{F}{3} = \frac{5F}{6} \end{aligned}$$

(Adapted from Baby Morin MCQ 4.3.)

3. $\boxed{\text{Higher}}$ When a single spring is cut into two equal springs, its spring constant is doubled (an exercise for the reader). Let the spring constant of the original setup be k. The extension of the spring in the original setup, x, can be easily found since the system is in static equilibrium.

$$kx = 2mg \Rightarrow x = \frac{2mg}{k}$$

The new setup is more tricky. The spring constant of each spring is now $2k$. Let x_1 be the extension of the top spring, and x_2 be the extension of the bottom spring. Balancing the vertical forces for the bottom mass:

$$2kx_2 = mg \Rightarrow x_2 = \frac{mg}{2k}$$

And for the top mass:

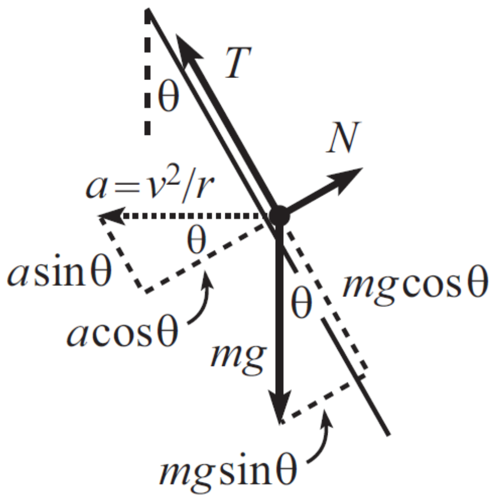
$$\begin{aligned} 2kx_1 &= mg + 2kx_2 \\ 2kx_1 &= 2mg \\ x_1 &= \frac{mg}{k} \end{aligned}$$

So the total extension is $x = x_1 + x_2 = \frac{3mg}{2k}$. Since $\frac{3mg}{2k} < \frac{2mg}{k}$, the extension for the new setup is smaller than the extension for the original setup, so the bottom mass in the new setup will be higher than in the originl setup.

(Stolen from Baby Morin MCQ 4.21 - a detailed qualitative explanation can be found there.)

4. $\boxed{v/2 \text{ rightwards}}$ Let the required velocity of the paddle be u, taking rightwards as positive. From the frame of the paddle, the velocity of the ping-pong ball approaching the paddle is $v - u$. Since the paddle is massive compared to the ping-pong ball, the paddle will remain travelling at velocity u after the collision. Furthermore, given that the collision is elastic, the relative speed between the ping-pong ball and the paddle remains equal in magnitude but opposite in direction, so that the ping-pong ball begins travelling at $v - u$ away from the paddle, relative to the paddle. Returning to the lab frame, the ping-pong ball travels $v - 2u$ leftwards. For the ping-pong ball to become stationary, $v - 2u = 0$, so $u = v/2$. Therefore, the paddle should move at $v/2$ rightwards.

5. It may be helpful to draw the free-body diagram of this scenario, as shown below. (The diagram does not include friction, as friction acts along the plane of the paper.)



Weight increases the vertical component of tension, so a higher weight would increase the magnitude of the tension force and the normal force. It does not directly contribute to the centripetal acceleration of the mass.

The horizontal component of tension provides centripetal acceleration of the mass towards the center of the circle, enabling the mass to travel in circular motion.

The horizontal component of the normal force opposes the horizontal component of tension, lowering the centripetal acceleration of the mass.

Friction acts tangentially and does not contribute to the mass' centripetal force. It merely provides tangential deceleration to the mass, slowing the mass as time progresses.

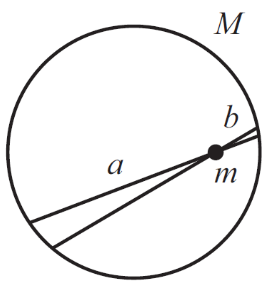
(For a quantitative treatment of this scenario, see Baby Morin OEQ 4.24, which considers the maximum possible speed of the mass for it to remain on the cone.)

6. (a) $\boxed{L \text{ only}}$ L is conserved because the force from the string is always radial, it always points directly towards the center of the hole. Hence, the torque on the system is zero, and angular momentum is conserved. E is not conserved, because due to slight motion of the mass in the radial direction as it spirals in, there is a component of the velocity that points in the same direction as the radial tension in the string, causing a non-zero amount of work to be done on the mass.

   (b) $\boxed{E \text{ only}}$ E is conserved because the force from the string is always perpendicular to the velocity of the mass, so no work is done on the mass. This is because the mass is always instantaneously moving in a circle for which the string points along the radius, where the center of the circle is the string's instantaneous point of contact with the pole. L is not conserved because the tension in the string doesn't point towards the center of the pole (which has a non-zero radius), causing some torque on the mass relative to the center of the pole.

   (Stolen from Baby Morin MCQ 8.10 and 8.11)

7. (a) $\boxed{\text{Rightwards}}$ A rigorous geometrical proof of this involves drawing two lines through the mass m, with a very small angle between them, and look at the forces between the two short arcs they define on the ring. Let the distances a and b be as shown in the diagram below.



The arc on the left is longer by a factor a/b, so there is a/b more mass there. But the force from a small bit of mass in the left arc is smaller by a factor $1/(a/b)^2$ than the force from an equal bit in the right arc, due to the inverse-square nature of the gravitational force. The force from the whole left arc is therefore smaller by a factor $(a/b)/(a/b)^2 = b/a$ than the force from the whole right arc. Therefore, the net force has a rightward component.

(Stolen from Baby Morin MCQ 11.5)

(b) $\boxed{\text{Leftwards}}$ This part, being an original question set by us, is much simpler because we are kind! The force at radius r inside a solid sphere is effectively due only to the mass of the sphere that lies inside, that is, the part of the sphere from 0 to r. Yes, the force from the outer mass of the sphere (outside of mass m, beyond r) is zero. This is because the gravitational force inside a uniform hollow spherical shell is zero, and since the mass of the sphere outside r may be considered to be built up from many concentric spherical shells, the outer mass therefore does not exert any gravitational force on the object. Hence, we only need to consider the mass of the sphere inside mass m, and thus the gravitational force is directed towards the center of the sphere, which is to the left.

8. The main issue with Chris' statement is that the normal force may or may not rotate the block in the anti-clockwise direction. In the case where the center of mass of the block is to the left of its point of contact with the floor, the normal force causes an anti-clockwise torque. However, in the case where the center of mass of the block is to the right of its point of contact with the floor, the normal force causes a clockwise torque. You would be able to see this by drawing the lever arm of the force. (And in the same way, the normal force contributes no torque if the center of mass is horizontally aligned with the point of contact of the block). Therefore, the setup can indeed be in static equilibrium, in the case where the normal force provides a clockwise torque.

B Structured Questions (104 marks)

1. (a) $\boxed{\rho_1 = \rho_2/2}$ Consider forces acting on cube A in the vertical direction. Weight acts downwards, while buoyancy force acts upwards.

At equilibrium, these forces are equal in magnitude. Therefore:

$$
\begin{aligned}
mg &= \rho_2(\frac{V}{2})g \\
(\rho_1 l^3)g &= \rho_2(\frac{l^3}{2})g \\
\rho_1 &= \frac{\rho_2}{2}
\end{aligned}
$$

(b) The height of the two cubes above the surface of the liquid will be equivalent to l, i.e. the surface of the liquid is in between the two cubes. This is due to the density of the system of the two cubes remaining the same at ρ_1. Hence, half of the system will still float above the surface of the liquid. You may work this out quantitatively if you are not convinced.

(c) Essentially, cube B now has a density of $\rho_3 = 2\rho_1 = \rho_2$, while cube A still has a density of ρ_1. Once again, we consider vertical forces acting on the entire system. Now, the total downward weight force is $(\rho_1 + \rho_2)l^3 g$, and the upward buoyancy force is $\rho_2 l^3 g + \rho_2 l^2(l-h)g$. Hence, we can derive:

$$
\begin{aligned}
(\rho_1 + \rho_2)l^3 g &= \rho_2 l^3 g + \rho_2 l^2(l-h)g \\
(\rho_1 + \rho_2)l &= \rho_2 l + \rho_2(l-h) \\
\rho_1 l &= \rho_2 l - \rho_2 h \\
h &= \frac{\rho_2 l - \rho_1 l}{\rho_2} \\
h &= \frac{\rho_2 l - \frac{\rho_2 l}{2}}{\rho_2} \\
h &= \frac{l}{2}
\end{aligned}
$$

(d) In our calculations here, we will assume that there are no damping forces resisting the motion of the cube, and ignoring any induced motion of the water.

(i) $\boxed{2\pi\sqrt{\frac{l}{2g}}}$ Let the height of the cube above the surface of the water be x. Note that you can use any reference point for your variable x - your method will be slightly different but the final answer you derive would be the same. The upward buoyancy force will then be equivalent to $\rho_2 l^2(l-x)$, whereas the downward weight force is $\rho_1 l^3 g$. Writing the $F = ma$ equation for its motion:

$$\begin{aligned}
\rho_2 l^2(l-x)g - \rho_1 l^3 g &= \rho_1 l^3 a \\
\rho_2(l-x) - \rho_1 l &= \frac{\rho_1 l a}{g} \\
2\rho_1(l-x) - \rho_1 l &= \frac{\rho_1 l a}{g} \\
\frac{l}{2} - x &= \frac{la}{2g}
\end{aligned}$$

Here, we have nearly obtained the general equation of simple harmonic motion. However, there is still the constant term bothering us. We can easily eliminate this by substituting a variable.

$$\begin{aligned}
y &= \frac{l}{2} - x \Rightarrow \ddot{y} = -\ddot{x} = -a \\
y &= -\frac{l\ddot{y}}{2g} \\
\ddot{y} &= -\frac{2g}{l} y
\end{aligned}$$

We have obtained the familiar general equation for simple harmonic motion of the cube in terms of y. We can thus derive the period of oscillations:

$$\begin{aligned}
\omega &= \sqrt{\frac{2g}{l}} \\
T &= 2\pi\sqrt{\frac{l}{2g}}
\end{aligned}$$

(Note: This substitution works because the additional term in the equation is constant, so that you can end up with a clean relation between $\ddot{y}$ and $\ddot{x}$ and easily transform the equation from an equation in x to an equation in y. However, if you happened to define your reference point for your variable to be $l/2 - x$, then you're in luck, as no substitution is necessary. The same substitution method is used for solving for the period of oscillations in a vertical spring, which entails an additional mg term.)

(ii) $\boxed{\frac{l}{2}\cos(\sqrt{\frac{2g}{l}}t) + \frac{l}{2}}$ Writing the general equation for the position of an object undergoing simple harmonic motion:

$$\begin{aligned} x(t) &= A\cos(\omega t) + c \\ x(t) &= \frac{l}{2}\cos(\sqrt{\frac{2g}{l}}t) + c \end{aligned}$$

We have an additional constant c here, where the value of c would depend on the initial conditions of the motion. As given in the question, at $t = 0$, $x = l$ (since the top of the cube is at a distance $l/2$ at equilibrium, and pushing it upwards gives $l/2 + l/2 = l$). Therefore,

$$x(t) = \frac{l}{2}\cos(\sqrt{\frac{2g}{l}}t) + \frac{l}{2}$$

2. We can write the $x(t)$ and $y(t)$ functions describing the position of the object.

$$\begin{aligned} x(t) &= v\cos\theta t \\ y(t) &= v\sin\theta t - \frac{1}{2}gt^2 \end{aligned}$$

(a) (i) $\boxed{\sqrt{\frac{gx}{2\sin\theta\cos\theta}}}$ When the object reaches the ground, $y(t) = 0$. To get the time taken for this to occur:

$$\begin{aligned} v\sin\theta t - \frac{1}{2}gt^2 &= 0 \\ v\sin\theta &= \frac{1}{2}gt \\ t &= \frac{2v\sin\theta}{g} \end{aligned}$$

For the object to reach the point on the ground, $x(t) = x$.

$$\begin{aligned} x &= v\cos\theta(\frac{2v\sin\theta}{g}) \\ x &= \frac{2v^2\sin\theta\cos\theta}{g} \\ v &= \sqrt{\frac{gx}{2\sin\theta\cos\theta}} \end{aligned}$$

(ii) $\boxed{\theta = 45°,\ v = \sqrt{gx}}$ To minimise the value of v, the denominator of the fraction should be maximised. That is, we are trying to maximise the value of $2\sin\theta\cos\theta$. Using the double angle formula, $2\sin\theta\cos\theta = \sin 2\theta$. The maximum value of $\sin 2\theta$ is 1, when $\theta = 45°$. Hence, the minimum value of v is $\sqrt{gx}$ when $\theta = 45°$.

(b) (i) $\boxed{\sqrt{\frac{gx^2}{2x\sin\theta\cos\theta - 2y\cos^2\theta}}}$ When the object reaches the point in air, $x(t) = x$ and $y(t) = y$ simultaneously.

$$v\cos\theta t = x \Rightarrow t = \frac{x}{v\cos\theta}$$

$$\begin{aligned}
v\sin\theta t - \frac{1}{2}gt^2 &= y \\
v\sin\theta\frac{x}{v\cos\theta} - \frac{g}{2}\frac{x^2}{v^2\cos^2\theta} &= y \\
x\frac{\sin\theta}{\cos\theta} - \frac{gx^2}{2v^2\cos^2\theta} &= y \\
2x\sin\theta\cos\theta - \frac{gx^2}{v^2} &= 2y\cos^2\theta \\
\frac{gx^2}{v^2} &= 2x\sin\theta\cos\theta - 2y\cos^2\theta \\
v &= \sqrt{\frac{gx^2}{2x\sin\theta\cos\theta - 2y\cos^2\theta}}
\end{aligned}$$

This answer is a generalisation of the scenario in (a)(i). You may verify your answer by letting $y = 0$, and you will get the same result as in (a)(i).

(ii) $\boxed{\arctan y/x < \theta < 90°}$ For v to be defined, the fraction inside the square root must be positive. Since gx^2 is always positive, the denominator must always be positive in order for v to be defined. That is,

$$\begin{aligned}
2x\sin\theta\cos\theta - 2y\cos^2\theta &> 0 \\
\cos\theta(x\sin\theta - y\cos\theta) &> 0
\end{aligned}$$

This yields two conditions. For $\cos\theta > 0$, $0° < \theta < 90°$, which is a trivial solution. For $x\sin\theta - y\cos\theta > 0$, the solution is $\theta > \arctan y/x$. Hence, for v to be defined, $\arctan y/x < \theta < 90°$.

The condition for $\theta > \arctan y/x$ makes sense; if you wish to throw an object to a point in the air, you must aim slightly above the straight line directly connecting to the point in the air to be able to reach this point.

3. (a) If there is no relative motion between the masses, the net vertical force on m_2 must be zero. The tension in the string connecting m_1 to m_2 must therefore be $T = m_2 g$. The horizontal $F = ma$ equation on m_1 is then $T = m_1 a$. Therefore $m_1 a = m_2 g$, which yields $a = (m_2/m_1)g$. This is the desired acceleration of the system.

   (Stolen from Baby Morin OEQ 4.14)

   (b) In this part, since the masses are stationary relative to the block, we are only concerned with static friction, and not kinetic friction.

   (i) Static friction can take on a range of values, which allows a range of values of a to be acceptable for (b).

   (ii) When $a = a_1$, m_2 is just about to slide downwards. When $a = a_2$, m_2 is just about to slide upwards.
   In the case of $a = a_1$, friction acts on m_1 in the leftwards direction, while friction acts on m_2 in the upwards direction.

$$\mu_s m_1 g \longleftarrow \mathbf{m_1} \longrightarrow T \text{ (horizontal)}$$
$$\mu_s m_2 a \longleftarrow T \longleftarrow \mathbf{m_2} \longrightarrow m_2 g \text{ (vertical)}$$

   Note that friction acting on m_2 takes a magnitude of $\mu_s m_2 a$ since the horizontal normal force acting on the mass is $m_2 a$, whereas the friction acting on m_1 takes a magnitude of $\mu_s m_1 g$ since the vertical normal force acting on the mass is $m_1 g$.
   In the case of $a = a_2$, friction acts on m_1 in the rightwards direction, while friction acts on m_2 in the downwards direction.

$$\mathbf{m_1} \longrightarrow T \longrightarrow \mu_s m_1 g \text{ (horizontal)}$$
$$T \longleftarrow \mathbf{m_2} \longrightarrow m_2 g \longrightarrow \mu_s m_2 a \text{ (vertical)}$$

   (iii) $\boxed{a_1 = \frac{m_2 g - \mu_s m_1 g}{m_1 + \mu_s m_2},\ a_2 = \frac{m_2 g + \mu_s m_1 g}{m_1 - \mu_s m_2}}$ Firstly, let's consider the case of $a = a_1$.

$$m_1 a = T - \mu_s m_1 g$$
$$m_2 g = T + \mu_s m_2 a_1$$

Solving simultaneously,

$$m_2 g = \mu_s m_1 g + m_1 a + \mu_s m_2 a$$
$$m_2 g - \mu_s m_1 g = a(m_1 + \mu_s m_2)$$
$$a_1 = a = \frac{m_2 g - \mu_s m_1 g}{m_1 + \mu_s m_2}$$

Consider the case of $a = a_2$.

$$m_1 a = T + \mu_s m_1 g$$
$$m_2 g + \mu_s m_2 a = T$$

Solving simultaneously,

$$m_1 a = m_2 g + \mu_s m_2 a + \mu_s m_1 g$$
$$m_1 a - \mu_s m_2 a = m_2 g + \mu_s m_1 g$$
$$a_2 = a = \frac{m_2 g + \mu_s m_1 g}{m_1 - \mu_s m_2}$$

(iv) When $\mu_s = \mu_k = 0$, no friction exists. We will return to scenario in (a).
Substituting $\mu_s = 0$ into our results in (iii), $a_1 = a_2 = m_2 g/m_1$. This is the same as our answer in (a), hence we have verified our results.

4. (a) $\boxed{\alpha = 5}$ Consider the speed of the car at the top of the loop. Let this speed be v_f. In the case of minimum possible speed, the normal force acting on the car is 0, such that the weight force is the only force acting on the car. In this case, gravity is the only centripetal force, such that $\frac{v_f^2}{R} = g$, i.e. $v_f = \sqrt{gR}$. Now, to link v_f with v_0, conserve energy:

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv_f^2 + mgh$$
$$v_0^2 = v_f^2 + 2gh$$
$$v_0^2 = gR + 4gR$$
$$v_0 = \sqrt{5gR}$$

Hence $\alpha = 5$.

(b) $\boxed{v = \sqrt{v_0^2 - 2gR(1-\cos\theta)}}$ Using conservation of energy, we can relate the height of the car above the bottom point with θ as $R(1-\cos\theta)$.

$$\begin{aligned}\frac{1}{2}mv_0^2 &= \frac{1}{2}mv^2 + mgh \\ v_0^2 &= v^2 + 2gh \\ v^2 &= v_0^2 - 2gR(1-\cos\theta) \\ v &= \sqrt{v_0^2 - 2gR(1-\cos\theta)}\end{aligned}$$

(c) $\boxed{N = \frac{mv_0^2}{R} - 2mg + 3mg\cos\theta}$ Centripetal force is $N - mg\cos\theta$. Equating this to mv^2/R, where we have found v in our previous part, we can derive an expression for N.

$$\begin{aligned}N - mg\cos\theta &= \frac{mv^2}{R} \\ N &= mg\cos\theta + \frac{m}{R}\left(v_0^2 - 2gR(1-\cos\theta)\right) \\ N &= \frac{mv_0^2}{R} - 2mg(1-\cos\theta) + mg\cos\theta \\ N &= \frac{mv_0^2}{R} - 2mg + 3mg\cos\theta\end{aligned}$$

(d) (i) The car falls off the loop at the second quadrant, between $\theta = 90°$ and $\theta = 180°$.

(ii) $\boxed{\arccos\left(\frac{2}{3} - \frac{v_0^2}{3gR}\right)}$ When the car loses contact with the loop, $N = 0$. Therefore:

$$\begin{aligned}\frac{mv_0^2}{R} - 2mg + 3mg\cos\phi &= 0 \\ 3g\cos\phi &= 2g - \frac{v_0^2}{R} \\ \cos\phi &= \frac{2}{3} - \frac{v_0^2}{3gR} \\ \phi &= \arccos\left(\frac{2}{3} - \frac{v_0^2}{3gR}\right)\end{aligned}$$

(iii) ϕ should lie between $90°$ and $180°$, which lies in the second quadrant. In this range, $-1 < \cos\phi < 0$.

$$\begin{aligned}
\frac{2}{3} - \frac{v_0^2}{3gR} &> -1 \\
2 - \frac{v_0^2}{gR} &> -3 \\
v_0^2/gR &< 5 \\
v_0 &< \sqrt{5gR}
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{2}{3} - \frac{v_0^2}{3gR} &< 0 \\
2 - \frac{v_0^2}{gR} &< 0 \\
\frac{v_0^2}{gR} &> 2 \\
v_0 &> \sqrt{2gR}
\end{aligned}$$

From this, we have verified that this angle is defined for $\sqrt{2gR} < v_0 < \sqrt{5gR}$.

(iv) Substituting this angle into our results in (b),

$$\begin{aligned}
v &= \sqrt{v_0^2 - 2gR(1 - \cos\theta)} \\
v &= \sqrt{v_0^2 - 2gR\left(1 - \frac{2}{3} + \frac{v_0^2}{3gR}\right)} \\
v &= \sqrt{\frac{v_0^2}{3} - \frac{2}{3}gR} \\
v &= \sqrt{\frac{1}{3}\left(v_0^2 - 2gR\right)}
\end{aligned}$$

5. (a) (i) $\boxed{\text{Distance between particles} = 2 * \text{Radius of orbit}}$ The two masses are orbiting around their center of mass. Let the radius of the orbit be r. Since the two masses travel in the same circular orbit, the distance between the two masses is equivalent to $2r$.

(ii) $\boxed{\frac{Gm}{4v^2}}$ Since the gravitational force is the centripetal force for the circular orbit,

$$\begin{aligned}
\frac{mv^2}{r} &= \frac{Gm^2}{(2r)^2} \\
\frac{v^2}{r} &= \frac{Gm}{4r^2} \\
v^2 &= \frac{Gm}{4r} \\
r &= \frac{Gm}{4v^2}
\end{aligned}$$

(iii) The particles are placed a distance $\frac{Gm}{2v^2}$ apart, and each imparted velocity v in opposite directions.

(b) (i) The free particle reaches a maximum distance from the stationary particle, before returning to the stationary particle due to the gravitational attraction.
Alternatively, if the free particle was imparted sufficient initial kinetic energy, it would escape from the stationary particle and travel to infinity.

(ii) $\boxed{\frac{2Gml}{2Gm-lv^2}}$ Let r be the largest distance reached between the free particle and the stationary particle. From conservation of energy,

$$\begin{aligned}
\frac{1}{2}mv^2 - \frac{Gm^2}{l} &= -\frac{Gm^2}{r} \\
v^2 - \frac{2Gm}{l} &= -\frac{2Gm}{r} \\
\frac{2Gm}{r} &= \frac{2Gm}{l} - v^2 \\
\frac{r}{2Gm} &= \frac{1}{\frac{2Gm}{l} - v^2} \\
r &= \frac{2Gml}{2Gm - lv^2}
\end{aligned}$$

(iii) $\boxed{v \geq \sqrt{\frac{2Gm}{l}}}$ When this distance is infinite, $2Gm - lv^2 \leq 0$.

$$
\begin{aligned}
lv^2 &\geq 2Gm \\
v^2 &\geq \frac{2Gm}{l} \\
v &\geq \sqrt{\frac{2Gm}{l}}
\end{aligned}
$$

When v takes on this range of values (which is the escape velocity), the free particle will be able to escape the gravitational attraction of the stationary particle, and travel to infinity.

(c) (i) $\boxed{\frac{\pi}{4}\sqrt{\frac{l^3}{Gm}}}$ This is a tricky question, possibly the hardest question in the paper. In an orbit involving two masses M and m, Kepler's 3rd law states that $n^2a^3 = \mu$, where $\mu = G(M+m)$, $n = 2\pi/T$ (where T is the orbital period), and a is the length of the semi-major axis.

In the case of two colliding masses, we have reached a special case of Kepler's 3rd law, where the semi-minor axis is zero, the apocenter is at an initial distance l, and the pericenter is zero. Also, $l = 2a$, so the masses will only complete half an orbit. This takes time t:

$$
\begin{aligned}
t &= T/2 = \frac{\pi a^{3/2}}{\sqrt{\mu}} = \pi\sqrt{\frac{l^3}{8G(m+m)}} \\
t &= \pi\sqrt{\frac{l^3}{16Gm}} \\
t &= \frac{\pi}{4}\sqrt{\frac{l^3}{Gm}}
\end{aligned}
$$

(ii) A. The value of x will be smaller than the answer in (b)(ii). In this case, since both masses are in motion, the initial velocity imparted to one of the masses will cause the other mass to also move towards the first mass, causing the largest distance reached between both masses to be smaller.

B. $\boxed{x = \frac{4Gml}{4Gm-lv^2}}$ Energy and momentum of the system of the two masses are both conserved. When the masses have

reached their largest distance, they are travelling at the same speed. Let this speed be v_f.
From conservation of momentum, $2mv_f = mv$, which gives $v_f = v/2$. That is, the final speed of both masses at maximum distance is $v/2$.
From conservation of energy,

$$\begin{aligned}
\frac{1}{2}mv^2 - \frac{Gm^2}{l} &= 2\left(\frac{1}{2}m(\frac{v}{2})^2\right) - \frac{Gm^2}{x} \\
\frac{1}{2}v^2 - \frac{Gm}{l} &= \frac{1}{4}v^2 - \frac{Gm}{x} \\
\frac{Gm}{x} &= \frac{Gm}{l} - \frac{1}{4}v^2 \\
x &= \frac{Gm}{\frac{Gm}{l} - \frac{1}{4}v^2} \\
x &= \frac{4Gml}{4Gm - lv^2}
\end{aligned}$$

To check whether this value of x is larger or smaller than the answer in (b)(ii), take the difference between our results in both parts, and check whether it is positive or negative.

$$\begin{aligned}
&\frac{2Gml}{2Gm - lv^2} - \frac{4Gml}{4Gm - lv^2} \\
&= \frac{2Gml(4Gm - lv^2) - 4Gml(2Gm - lv^2)}{(2Gm - lv^2)(4Gm - lv^2)} \\
&= \frac{8G^2m^2l - 2Gml^2v^2 - 8G^2m^2l + 4Gml^2v^2}{(2Gm - lv^2)(4Gm - lv^2)} \\
&= \frac{2Gml^2v^2}{(2Gm - lv^2)(4Gm - lv^2)}
\end{aligned}$$

This result is clearly positive. Hence the result in (b)(ii) is higher than our answer here in (c)(ii), i.e. the value of x is indeed smaller than the answer in (b)(ii).

6. (a) In this part, since the cylinder is slipping with respect to the ground, we will consider kinetic friction instead of static friction.
   (i) The cylinder's angular velocity will increase, whereas its translational velocity will decrease, up till the point when $v = R\omega$ and it no longer slips.

(ii) $\boxed{\frac{v_0}{3\mu_k g}}$ Write the $F = ma$ and $\tau = I\alpha$ equations for the cylinder's motion:

$$\begin{aligned} \text{Translational: } -\mu_k Mg &= Ma \\ a &= -\mu_k g \\ \text{Rotational: } \mu_k MgR &= \frac{1}{2}MR^2\alpha \\ \alpha &= \frac{2\mu_k g}{R} \end{aligned}$$

Using the kinematics equations (for both translational and rotational motion), we can derive expressions for v and ω after time t:

$$\begin{aligned} v &= v_0 - \mu_k gt \\ \omega &= \frac{2\mu_k g}{R}t \end{aligned}$$

When the cylinder is not slipping with respect to the ground, $v = R\omega$.

$$\begin{aligned} v_0 - \mu_k gt &= R * \frac{2\mu_k g}{R}t \\ v_0 - \mu_k gt &= 2\mu_k gt \\ v_0 &= 3\mu_k gt \\ t &= \frac{v_0}{3\mu_k g} \end{aligned}$$

(iii) $\boxed{\frac{2v_0}{3}}$ Substituting this value of t into the kinematics equation for v,

$$\begin{aligned} v &= v_0 - \mu_k gt = v_0 - \mu_k g\frac{v_0}{3\mu_k g} \\ v &= \frac{2v_0}{3} \end{aligned}$$

(iv) In theory, the cylinder will stay in motion indefinitely, because once the cylinder has stopped slipping, its point of contact with the ground is not moving, hence no friction acts on the cylinder.
However, in practice, rolling resistance causes the cylinder to eventually stop. This rolling resistance arises from the deformation of the surface of the cylinder.

(b) $\boxed{\theta \leq \arctan 3\mu_s}$ Consider the forces along the incline. $Mg\sin\theta$ acts downwards, whereas the frictional force $\mu_s Mg\cos\theta$ (we use the coefficient of static friction here, since the cylinder is not slipping) acts upwards along the incline. Writing the $F = ma$ and $\tau = I\alpha$ equations for this scenario:

$$\begin{aligned}
\text{Translational: } Mg\sin\theta - \mu_s Mg\cos\theta &= Ma \\
a &= g\sin\theta - \mu_s g\cos\theta \\
\text{Rotational: } \mu_s Mg\cos\theta R &= \frac{1}{2}MR^2\alpha \\
\alpha &= \frac{2\mu_s g\cos\theta}{R}
\end{aligned}$$

Note that this is for the maximum value of θ for which the cylinder does not slip, where the static friction takes on its maximum value. Now, invoking the non-slipping condition $a = R\alpha$:

$$\begin{aligned}
g\sin\theta - \mu_s g\cos\theta &= 2\mu_s g\cos\theta \\
\sin\theta &= 3\mu_s\cos\theta \\
\tan\theta &= 3\mu_s
\end{aligned}$$

The cylinder can achieve a state of motion with no slipping with respect to the ground for $\theta \leq \arctan 3\mu_s$.

(c) (i) No friction acts on the cylinder. No net force acting on the cylinder, causing it to move at constant velocity.

(ii) $\boxed{\frac{v_0(2L^2-3R^2)}{4L^2+3R^2} \text{ forward, } \frac{12v_0L}{4L^2+3R^2} \text{ anti-clockwise}}$ Let the final linear speed of the cylinder's CM be v, and let the final angular speed be ω (with anticlockwise positive). Momentum of the cylinder isn't conserved because there is an external force from the wall. But energy is conserved since the collision is elastic. Angular momentum around the corner of the wall is conserved, because the external force at the corner produces no torque around that point.

Conservation of energy gives:

$$\begin{aligned}
\frac{1}{2}Mv_0^2 &= \frac{1}{2}Mv^2 + \frac{1}{2}\left(\frac{1}{4}MR^2 + \frac{1}{12}ML^2\right)\omega^2 \\
v_0^2 &= v^2 + \left(\frac{1}{4}R^2 + \frac{1}{12}L^2\right)\omega^2
\end{aligned}$$

Conservation of angular momentum around the corner gives:

$$\begin{aligned}
Mv_0\frac{L}{2} &= Mv\frac{L}{2} + \left(\frac{1}{4}MR^2 + \frac{1}{12}ML^2\right)\omega \\
v_0L &= vL + 2\left(\frac{1}{4}R^2 + \frac{1}{12}L^2\right)\omega \\
v_0 &= v + \frac{2}{L}\left(\frac{1}{4}R^2 + \frac{1}{12}L^2\right)\omega
\end{aligned}$$

We have two equations in v and ω. Solving simultaneously, we can derive:

$$\begin{aligned}
\left(v + \frac{2}{L}\left(\frac{1}{4}R^2 + \frac{1}{12}L^2\right)\omega\right)^2 &= v^2 + \left(\frac{1}{4}R^2 + \frac{1}{12}L^2\right)\omega^2 \\
v^2 + \frac{4}{L^2}\left(\frac{1}{4}R^2 + \frac{1}{12}L^2\right)^2\omega^2 + \frac{4v}{L}\left(\frac{1}{4}R^2 + \frac{1}{12}L^2\right)\omega &= v^2 + \left(\frac{1}{4}R^2 + \frac{1}{12}L^2\right)\omega^2 \\
\frac{4}{L^2}\left(\frac{1}{4}R^2 + \frac{1}{12}L^2\right)\omega + \frac{4v}{L} &= \omega \\
\omega\left(1 - \frac{4}{L^2}\left(\frac{1}{4}R^2 + \frac{1}{12}L^2\right)\right) &= \frac{4v}{L} \\
\omega\left(\frac{2}{3} - \frac{R^2}{L^2}\right) &= \frac{4v}{L} \\
\omega = \frac{4v}{L\left(\frac{2}{3} - \frac{R^2}{L^2}\right)} &= \frac{12vL}{2L^2 - 3R^2}
\end{aligned}$$

We have derived a relationship between v and ω. Now substituting this back into our previous equations:

$$\begin{aligned}
v &= v_0 - \frac{2}{L}\left(\frac{1}{4}R^2 + \frac{1}{12}L^2\right)\frac{12vL}{2L^2 - 3R^2} \\
v &= v_0 - \frac{24v}{2L^2 - 3R^2}\left(\frac{1}{4}R^2 + \frac{1}{12}L^2\right) \\
v &= v_0 - \frac{v}{2L^2 - 3R^2}\left(6R^2 + 2L^2\right) \\
v &= \frac{v_0}{1 + \frac{6R^2 + 2L^2}{2L^2 - 3R^2}} \\
v &= \frac{v_0(2L^2 - 3R^2)}{4L^2 + 3R^2}
\end{aligned}$$

And to find ω,

$$\omega = \frac{12L}{2L^2 - 3R^2} \frac{v_0(2L^2 - 3R^2)}{4L^2 + 3R^2}$$
$$\omega = \frac{12v_0 L}{4L^2 + 3R^2}$$

Therefore, the cylinder moves forward with a velocity $\frac{v_0(2L^2-3R^2)}{4L^2+3R^2}$, and rotates in the anti-clockwise direction with a rotational velocity $\frac{12v_0L}{4L^2+3R^2}$.
(Adapted from Baby Morin OEQ 8.10)
Setter's note: We did not foresee that the process of solving this question would be so tedious. In the future, questions like these would carry higher weightage, since the math involved is very lengthy.

7. (a) Firstly, the volume flow rate of water across the entire stream must be constant, otherwise it would imply that water is magically being lost somewhere, which is not true assuming the flow of water is steady. Second, the speed of the water flow increases at the bottom (due to Bernoulli's principle). As such, by the continuity equation, the radius of the water stream must decrease.

$$\text{Continuity: } \pi r_0^2 v_0 = \pi r^2 v$$
$$r = r_0 \sqrt{\frac{v_0}{v}}$$
$$\text{Bernoulli: } \frac{1}{2}\rho v^2 = \frac{1}{2}\rho v_0^2 + \rho g h$$
$$v^2 = v_0^2 + 2gh$$

Solving these simultaneously yields

$$r = r_0 \sqrt{\frac{v_0}{\sqrt{v_0^2 + 2gh}}}$$

(b) (i) We can relate the impulse from the stream of water to the weight of the disc by the equation $F = dp/dt$:

$$
\begin{aligned}
\frac{dp}{dt} &= mg = \sigma\pi r^2 g \\
\rho v_0 \frac{dV}{dt} &= \sigma\pi r^2 g \\
\rho v_0(\pi r_0^2 v_0) &= \sigma\pi r^2 g \text{ (continuity equation)} \\
v_0 &= \sqrt{\frac{\sigma\pi r^2 g}{\rho\pi r_0^2}} \\
v_0 &= \frac{r}{r_0}\sqrt{\frac{g\sigma}{\rho}}
\end{aligned}
$$

(Adapted from Baby Morin OEQ 6.23)

(ii) This part requires us to only consider the radius of the water stream, and ignore any momentum considerations.
Using Bernoulli's principle to find the velocity of the water stream at the height h:

$$
\begin{aligned}
\rho gh + \frac{1}{2}\rho v^2 &= \frac{1}{2}v_0^2 \\
2gh + v^2 &= v_0^2 \\
v^2 &= v_0^2 - 2gh
\end{aligned}
$$

Using the continuity principle (due to the fact that the volume flow rate of water at all points throughout the stream is the same),

$$
\begin{aligned}
\pi r_0^2 v_0 &= \pi r^2 v \\
r_0^4 v_0^2 &= r^4 v^2 \\
r_0^4 v_0^2 &= r^4(v_0^2 - 2gh) \\
r_0^4 v_0^2 &= r^4 v_0^2 = 2r^4 gh \\
2r^4 gh &= r^4 v_0^2 - r_0^4 v_0^2 \\
h &= \frac{r^4 v_0^2 - r_0^4 v_0^2}{2r^4 g} \\
h &= \frac{v_0^2}{2g} - \frac{r_0^4 v_0^2}{2r^4 g} \\
h &= \frac{v_0^2}{2g}\left(1 - \frac{r_0^4}{r^4}\right)
\end{aligned}
$$

(iii) A. At this height, the impulse delivered by the water stream is insufficient to balance its weight, causing the disc to have a net downwards acceleration.

B. $h_0 = 0$, that is, right above the mouth of the hose.

(iv) $\boxed{v_0 = \frac{r^2}{r_0^2}\sqrt{\frac{\sigma g}{\rho}},\ h = \frac{\sigma}{2\rho}\left(\frac{r^4}{r_0^4} - 1\right)}$ With the disc at a height h above the mouth of the hose, let v be the speed of the water stream at this height. We can write the $F = dp/dt$ equation:

$$\begin{aligned}
\rho v \frac{dV}{dt} &= \sigma \pi r^2 g \\
\rho v (\pi r_0^2 v_0) &= \sigma \pi r^2 g \\
\rho r_0^2 v_0 v &= \sigma r^2 g \\
\rho^2 r_0^4 v_0^2 v^2 &= \sigma^2 r^4 g^2 \\
\rho^2 r_0^4 v_0^2 \left(v_0^2 - 2g\left(\frac{v_0^2}{2g}\left(1 - \frac{r_0^4}{r^4}\right)\right)\right) &= \sigma^2 r^4 g^2 \\
\rho^2 r_0^4 v_0^2 \left(v_0^2 - v_0^2\left(1 - \frac{r_0^4}{r^4}\right)\right) &= \sigma^2 r^4 g^2 \\
\rho^2 r_0^4 v_0^2 \left(\frac{v_0^2 r_0^4}{r^4}\right) &= \sigma^2 r^4 g^2 \\
\rho^2 r_0^8 v_0^4 &= \sigma^2 r^8 g^2 \\
\rho r_0^4 v_0^2 &= \sigma r^4 g \\
v_0^2 &= \frac{\sigma r^4 g}{\rho r_0^4} \\
v_0 &= \frac{r^2}{r_0^2}\sqrt{\frac{\sigma g}{\rho}}
\end{aligned}$$

Substituting to find h:

$$\begin{aligned}
h &= \frac{\frac{r^4}{r_0^4}\frac{\sigma g}{\rho}}{2g}\left(1 - \frac{r_0^4}{r^4}\right) \\
h &= \frac{r^4 \sigma}{2 r_0^4 \rho}\left(1 - \frac{r_0^4}{r^4}\right) \\
h &= \frac{r^4 \sigma}{2 r_0^4 \rho} - \frac{\sigma}{2\rho} \\
h &= \frac{\sigma}{2\rho}\left(\frac{r^4}{r_0^4} - 1\right)
\end{aligned}$$

End of Answer Key