Chapter 8

Oscillations

A metronome, which is basically an upside down pendulum, is a device that produces regular ticks (beats). It dates back to the early 19th century. A metronome is used by some performing musicians for practice in maintaining a consistent tempo; it gives the composer an approximate way of specifying the tempo. From its inception, however, the metronome has been a highly controversial tool, and there are musicians who reject its use altogether.

- http://en.wikipedia.org/wiki/Metronome

Contents

Oscillations

Candidates should be able to:

(a) describe simple examples of free oscillations.

(b) investigate the motion of an oscillator, using experimental and graphical methods.

(c) understand and use the terms amplitude, period, frequency, angular frequency and phase difference and express the period in terms of both frequency and angular frequency.

(d) recall and use the equation $a = -\omega^2 x$ as the defining equation of simple harmonic motion.

(e) recognise and use $x = x_o \sin\omega t$ as a solution to the equation $a = -\omega^2 x$.

(f) recognise and use

$$v = v_o \cos \omega t$$

$$v = \pm\omega\sqrt{x_o^2 - x^2}$$

(g) describe, with graphical illustrations, the changes in displacement, velocity and acceleration during simple harmonic motion.

(h) describe the interchange between kinetic energy and potential energy during simple harmonic motion.

(i) describe practical examples of damped oscillations with particular reference to the effects of the degree of damping and the importance of critical damping in cases such as a car suspension system.

(j) describe practical examples of forced oscillations and resonance.

(k) describe graphically how the amplitude of a forced oscillation changes with frequency near to the natural frequency of the system, and understand qualitatively the factors which determine the frequency response and sharpness of the resonance.

(l) show an appreciation that there are some circumstances in which resonance is useful and other circumstances in which resonance should be avoided.

RELEVANT E-LEARNING WEBSITES

1) “Vibration and Waves” free downloadable textbook http://www.lightandmatter.com/area1book3.html
2) Wikipedia: http://en.wikipedia.org/wiki/Oscillation, http://en.wikipedia.org/wiki/Simple_harmonic_motion
3) University of Salford Online Lessons: http://www.acoustics.salford.ac.uk/feschools/waves/shm.htm#motion
4) PHYSCLIP Online Lessons
   a. Mechanics: SHM (http://www.animations.physics.unsw.edu.au/mechanics/chapter4_simpleharmonicmotion.html)
   b. Waves and Sound: Oscillations (http://www.animations.physics.unsw.edu.au/waves-sound/oscillations/)
5) Useful physics interactive applets http://ngsir.netfirms.com/englishVersion.htm#mechanics
6) Interactive tutorial http://www.physics.uoguelph.ca/tutorials/shm/Q.shm.html

8.0 Introduction

Oscillations occur when a system is disturbed from a position of stable equilibrium. This displacement from equilibrium changes periodically over time. Thus, oscillations are said to be periodic and display periodic motion. Oscillations are very common in everyday life with familiar examples such as the motion of a clock pendulum or the vibrations of strings on musical instruments. Even atoms in a lattice vibrate. Oscillations are also important in many mechanical systems in the real world such as car suspension. It is thus very important to be able to understand these mechanical systems in order to control them in critical situations.

What is a periodic motion?

- ***Periodic motion*** refers to any motion that repeats itself at equal intervals of time.

<u>Example 8.1:</u> Periodic Motion

Which of the following cases are periodic oscillations?

A

Vibration of a tuning fork prong

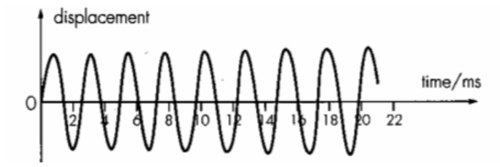


Fig. 8.1

B

Vibration of the body of the guitar

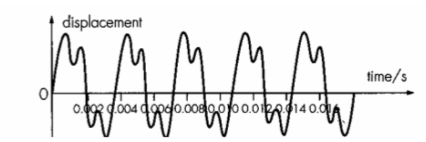


Fig. 8.2

C

Vibration of the cone of the loudspeaker

Fig. 8.3

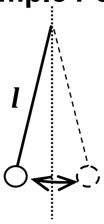
8.1 Simple Harmonic Motion

For this topic, we are interested in **periodic oscillations** – in which an object moves about a fixed point such that the time it takes to complete one to-and-fro motion is the same.

One of the simplest oscillations we shall concern ourselves with is known as the **simple harmonic motion (SHM).** Common examples of SHM include oscillations of a **simple pendulum** and a **spring-mass system** (assuming no loss of energy, so that the oscillations go on forever).

SHM provides a basis for the characterization of more complicated motions.

Simple Pendulum



$$T=2\pi\sqrt{\frac{l}{g}}$$

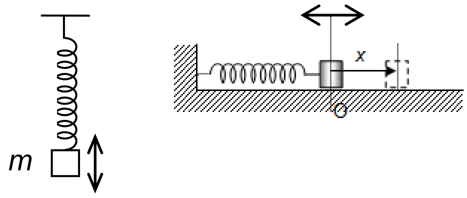
where T = period of oscillation

l = length of the string

g = gravitational acceleration

(Refer to Appendix 1 for proof of formula)

Spring-Mass System



$$T = 2\pi\sqrt{\frac{m}{k}}$$

where T = period of oscillation

m = mass of the oscillating mass

k = spring constant

In general, SHM is *isochronous* (the period and frequency are independent of the amplitude).

Let us take a closer look at a spring-mass system and learn the terms that describe the motion.

To study the motion we can attach a pen to the mass in such a way that it tracks the motion on a long vertical sheet of graph paper that is pulled to the left at a constant rate behind the system. This produces a trace on the graph paper which shows us how the displacement of the mass from its equilibrium position changes with time.

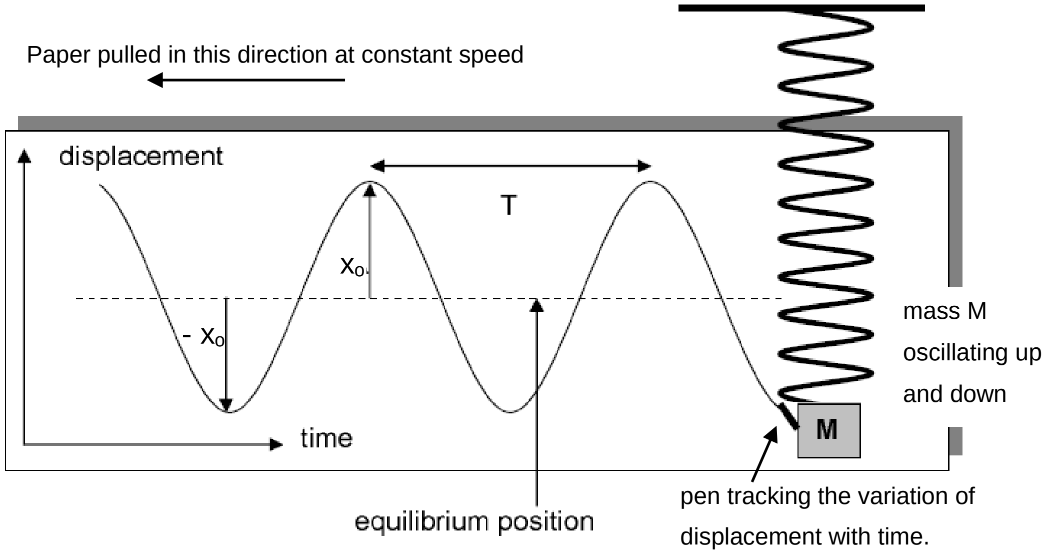


Fig. 8.4. Tracking the oscillation of a vertical spring-mass system

8.1.1 Terms used to describe SHM (with reference to Fig 8.4)

Terms	Definitions
Equilibrium **(x = 0)**	We often measure the displacement of the oscillator from the equilibrium position. Hence, the displacement $x = 0$ is designated at the equilibrium, i.e., the position at which the **net force on the oscillator is zero.**
Displacement, x	The displacement x is the linear distance of the oscillator from its equilibrium position (x = 0) in a specified direction. It is a vector; the + / – sign tells the direction of displacement from equilibrium. If you set the direction for displacement above equilibrium to be positive, then displacement below the equilibrium would be negative.
Amplitude, x_o	It is the magnitude of the maximum displacement of the oscillator from the equilibrium position. It is a scalar.
Period, T	It is the time taken for an oscillator to make one complete cycle of oscillation. S.I. unit: second.
Frequency, f	It is the number of cycles made by the oscillator per unit time: $$f = \frac{1}{T}$$ S.I. unit: hertz (Hz). 1 Hz = 1 s^{-1}.
Angular frequency, ω	It is the rate of change of the ***phase angle*** with respect to time: $\omega = \frac{d\theta}{dt}$ Since frequency is a fixed value for an SHM oscillator, and a complete cycle of oscillation is represented by a phase angle of 2π rad, $$\omega = \frac{2\pi}{T} = 2\pi f$$ S.I. unit: rad s^{-1}. Note: this is also the familiar equation that we encountered in circular motion where ω is the angular velocity of an object moving in a circle.

8.1.2 SHM - Displacement-time (*x*-*t*) relationship

From Fig. 8.4, we see that the variation of the oscillations of a vertical spring-mass system follows a *sinusoidal*[1] pattern. All SHMs follow a sinusoidal motion.

Consider the spring-mass system in Fig. 8.4. Suppose at t = 0, the mass is at equilibrium (x = 0) and will next move in the positive direction, the displacement-time graph of the oscillatory system will look like:

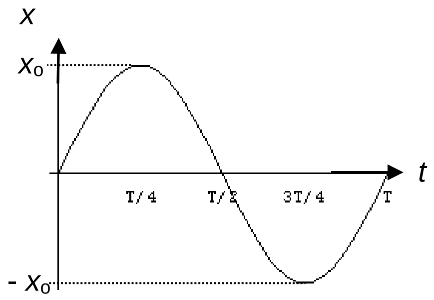


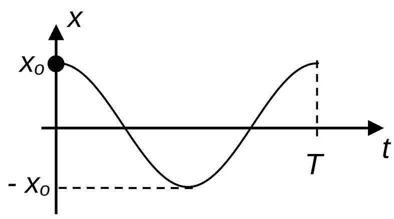
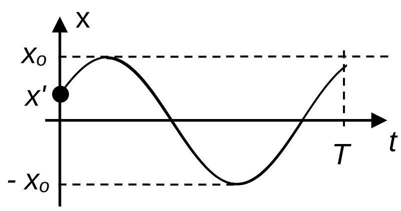
Fig. 8.5. A possible displacement-time graph of a simple harmonic oscillator

From the graph, we can see that the relationship between x and t is given by

$$\boxed{x = x_o \sin \omega t}$$

where $\omega = \dfrac{2\pi}{T}$ is the angular frequency and x_o is the amplitude of the oscillation.

Depending on the initial condition, i.e., when you start the timing, there are also other possible solutions. The following table lists some other possible solutions.

Initial conditions: (when t = 0)	**Displacement-time (*x*- *t*) Graph**	**Displacement-time Equation**
Oscillation starts from rest at amplitude (v = 0)		$x = x_o \cos \omega t$ where x_o is the amplitude.
Oscillation starts at positive displacement x' with initial positive velocity v		$x = x_0 \sin(\omega t + \phi_1)$ or $x = x_0 \cos(\omega t + \phi_2)$ where ϕ is the initial phase angle (not in syllabus)

Note: ωt is in radian. Remember to set your calculator to **RADIAN mode** before calculation!

[1] Sinusoidal is used to describe the shape of any sine or cosine curve regardless of whether it is shifted by some arbitrary horizontal amount.

8.1.3 SHM - Velocity-time (v-t) relationship and acceleration-time (a-t) relationship

From the topic of kinematics, we know that from the displacement-time relationship of an object, we can obtain its corresponding velocity-time and acceleration-time relationship by differentiating the equations or analysing the gradients of the tangents at the various points on the graphs accordingly (see Fig. 8.6).

We can similarly do so for SHM.

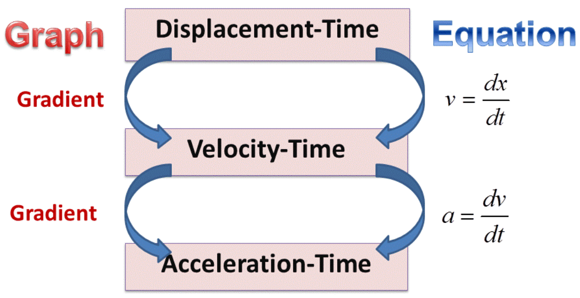
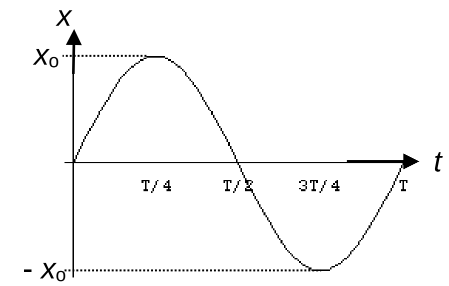
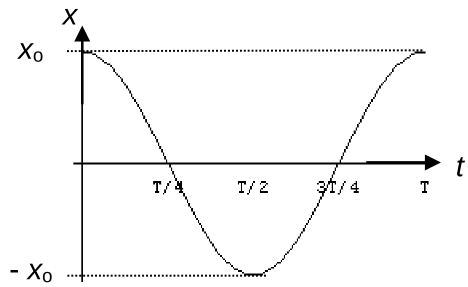
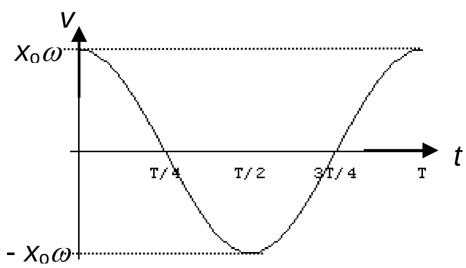
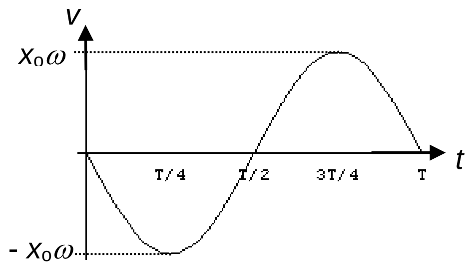
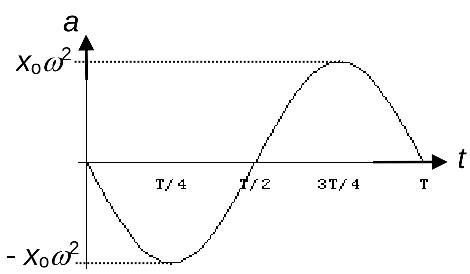
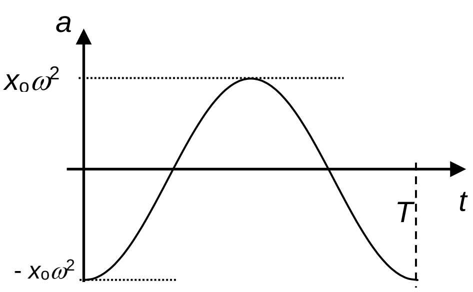


Fig. 8.6

The table below illustrates two examples.

t = 0 (initial conditions)	Example 1: $x = 0$, v is at maximum value	Example 2: $x = x_o$, $v = 0$
***x-t* graph**		
Equation	$x = x_0 \sin \omega t$	$x = x_0 \cos \omega t$
***v-t* graph**		
equation	$v = \omega x_0 \cos \omega t$	$v = -\omega x_0 \sin \omega t$
***a-t* graph**		
equations	$a = -\omega^2 x_0 \sin \omega t$	$a = -\omega^2 x_0 \cos \omega t$

Given that $-1 \le \sin\theta \le 1$ and $-1 \le \cos\theta \le 1$, we can also see that:

(1) Maximum value of velocity, v_o:

$$\boxed{v_o = \omega x_o}$$

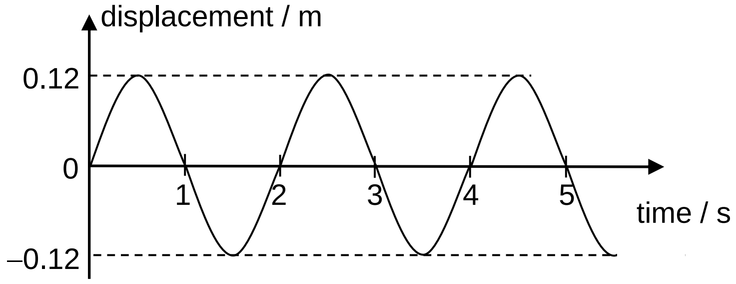
(2) Maximum value of acceleration, a_o:

$$\boxed{a_o = \omega^2 x_o}$$

where ω is the angular frequency and x_o is the amplitude of the oscillation.

Example 8.2 : Describing SHM in terms of time

The pendulum bob in a particular clock oscillates so that its displacement from a fixed point is as shown.



(a) By taking the necessary readings from the graph, determine

- (i) the amplitude x_o
- (ii) the angular frequency, ω
- (iii) the maximum velocity v_o and
- (iv) the maximum acceleration a_o

for this simple harmonic oscillator.

(b) Hence, write down the equation that describes the displacement-time relationship.

(c) Sketch labelled graphs, illustrating the corresponding variation of the oscillator's

- (i) velocity,
- (ii) acceleration

with respect to time.

<u>Example 8.3:</u> Displacement-time Relationship for SHM

The displacement *x* (in m) of a body moving in SHM is given by the equation

$$x = 3.5 \sin 4.0t$$

a) What is the amplitude of the motion?

b) What is the angular frequency?

c) What will be the displacement 0.20 s after oscillation has begun?

d) Determine the velocity at time t = 0.50 s.

8.1.4 Definition of SHM – relationship between acceleration and displacement

Returning to the table on page 8 and comparing the *x*-*t* and *a*-*t* relationship for both the examples, we see that $a = -\omega^2 x$. This is the defining equation for SHM!

<u>Definition of Simple Harmonic Motion</u>

Simple harmonic motion is a periodic motion in which the **acceleration** of the oscillator

- is **directly proportional** to the **displacement from its equilibrium point**, and
- is always in **opposite direction to the displacement.**

$$a = -\omega^2 x$$

where ω is the angular frequency and *x* is the displacement from the equilibrium point.

<u>Notes:</u>

(1) The **negative sign** in the equation indicates that **the acceleration in SHM is always in the direction opposite to the displacement from the equilibrium point**; in other words, the acceleration is always directed towards the equilibrium point.

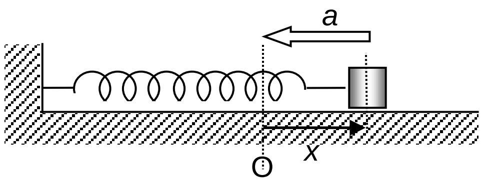
(2) This is a 2nd order differential equation. Since *a* = $\frac{d^2x}{dt^2}$, we can write $\frac{d^2x}{dt^2} = -\omega^2 x$.

$x = x_o \sin(\omega t + \phi)$ is a general solution of this differential equation, where ϕ depends on the starting position of the oscillator.

You are not required to know how to solve differential equations for the A-level Physics syllabus. However, by differentiating the expression $x = x_o \sin(\omega t + \phi)$ twice you can show that $\frac{d^2x}{dt^2} = -\omega^2 x$.

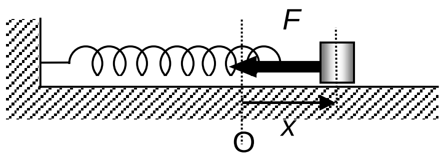
SHM of Spring-mass System Explained using Dynamics

Let us consider the oscillation of a mass oscillating at the end of a horizontal spring. Ideally, the mass of the spring is negligible, the mass slides without friction (undamped) on the horizontal surface and the spring obeys Hooke's Law. O is the equilibrium position, where the spring is unstretched.

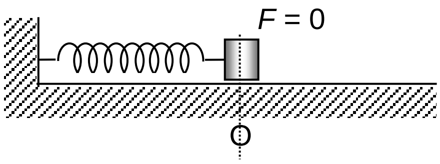


Let us look at how the restoring force of the spring on the mass varies with displacement *x*:

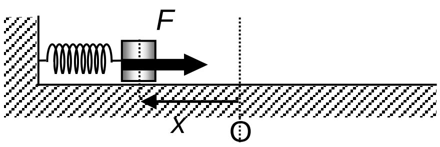
(a) when *x* is stretched to the right, the resultant restoring force is towards the left.



(b) when *x* is zero (unstretched), the resultant restoring force is zero.



(c) When *x* is compressed to the left, the resultant restoring force is towards the right.



Hence, we see that

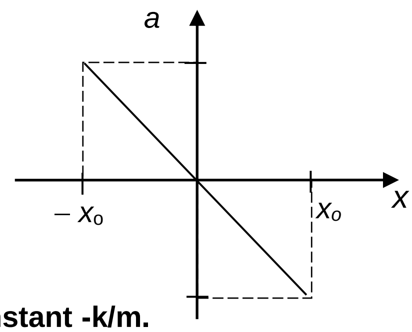
1. the **resultant restoring force is always towards the equilibrium position.**
2. The **direction of this resultant force is always opposite to that of its displacement** at all points, except at equilibrium position, where $a = x = 0$.
3. According to Hooke's Law, $F = -kx$, where k is the spring constant. The restoring force, F, is proportional to the displacement, x, but directed opposite to it (as indicated by the negative sign).

   By Newton's 2nd Law, $F_{net} = ma$

   Since F_{net} is provided by the restoring force of the spring,

   $$ma = -kx$$

   $$a = -\frac{k}{m}x$$



   Hence, the **acceleration is**

   ○ **opposite in direction to the displacement** and

   ○ **proportional to the displacement, with proportionality constant -k/m.**

   which is the definition of SHM! In short, for mechanical oscillation to be possible, there must be a restoring force opposite to the displacement.

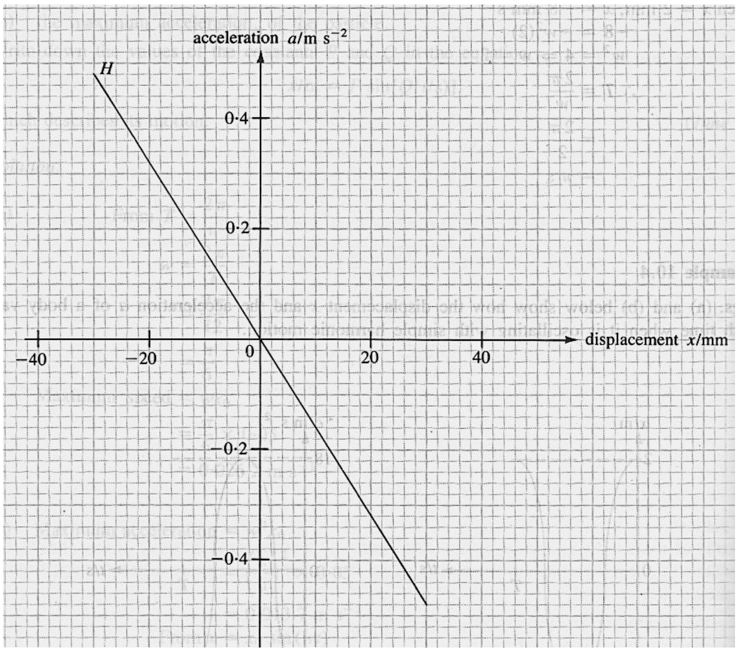
4. Comparing the expression above with the SHM equation $a = -\omega^2 x$: $\omega^2 = \frac{k}{m} \Rightarrow T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$

This is the expression we quoted for the period of the spring-mass system on page 5. Appendix 2 (pg. 29) provides the derivation of the period of a vertical spring-mass system, arriving at the same expression.

For a different SHM system, the angular frequency ω (and hence the natural frequency f and period T) will depend on different physical quantities, but the relationship $a = -\omega^2 x$ always holds.

Example 8.4: Relationship between acceleration and displacement for SHM

The graph represents the motion of a particle.



Find

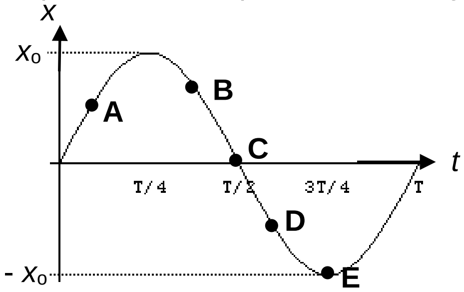
(a) the amplitude of oscillation,

(b) the angular frequency of oscillation,

(c) the period of oscillation.

Example 8.5: Relationship between acceleration and displacement for SHM

For each box in the table below, indicate whether the direction of the displacement x, velocity v and acceleration a of each point on the displacement time graph is "+", "-" or "0".



	A	B	C	D	E
x					
v					
a					

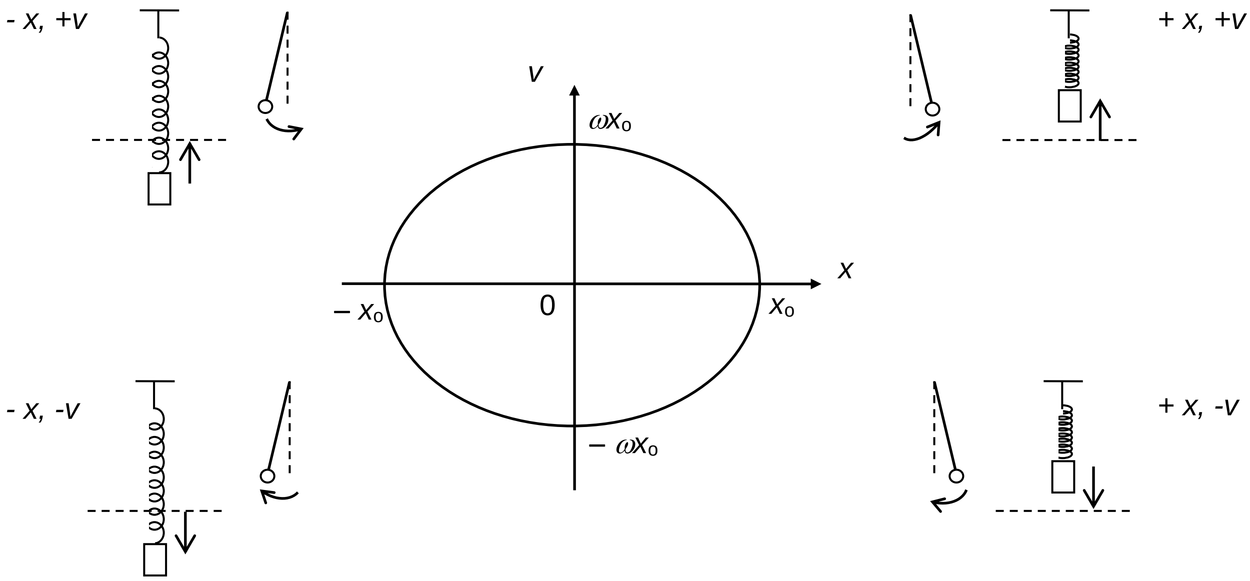
8.1.5 Simple Harmonic Motion: velocity-displacement (*v*-*x*) relationship

The velocity-displacement (*v*-*x*) relationship for SHM is given by

$$\boxed{v = \pm\,\omega\sqrt{\left(x_o^{\,2} - x^2\right)}}$$

<u>Note:</u>

1) The equation is provided in the A-level Physics formulae list. It is applicable whichever position the oscillation starts from.
2) The "±" sign implies that the body in SHM always passes through <u>each position</u> back and forth with the same speed.
3) At the amplitude positions, v = 0. At the equilibrium position (x = 0), the speed is maximum, $v_o = \omega x_o$.
4) The *v*-*x* graph is an ellipse as shown.



Derivation of the relationship between velocity and displacement (not in syllabus)

Consider $x = x_o \sin(\omega t + \phi)$ ....(1)

Differentiate (1) w.r.t. x: $v = \dfrac{dx}{dt} = \omega x_0 \cos(\omega t + \phi)$ .... (2)

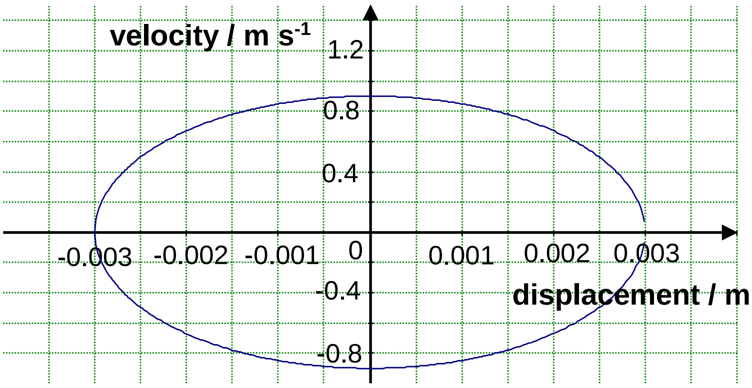
Given that $\sin^2\theta + \cos^2\theta = 1$, $\cos\theta = \pm\sqrt{1 - \sin^2\theta}$

Substitute into equation (2): $v = \omega x_o\left[\pm\sqrt{1 - \sin^2(\omega t + \phi)}\right]$

$\Rightarrow$ $v = \pm\omega\sqrt{x_o^{\,2} - x_o^{\,2}\sin^2(\omega t + \phi)}$

$\Rightarrow$ $v = \pm\omega\sqrt{x_0^{\,2} - x^2}$

Example 8.6: Relationship between velocity and displacement for SHM



(i) Explain why there are two values of velocity for zero displacement.
(ii) Explain why there are two values of displacement for zero velocity.
(iii) Write down the equation which describes the graph.

Example 8.7: Relationship between velocity and displacement for SHM

An object moving with simple harmonic motion has an amplitude of 0.020 m and a frequency of 20 Hz. Calculate

a) the period of oscillation.
b) the velocities at the equilibrium point and the position of maximum negative displacement.

Solution:

a) $T = \frac{1}{f} = \frac{1}{20} = 0.050 \text{ s}$

b) At equilibrium, $x = 0$ m

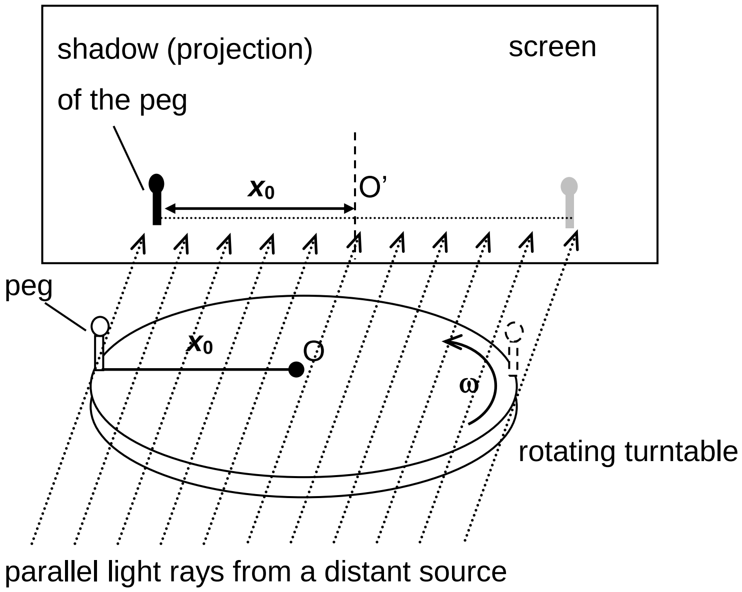
$$v = \pm\omega\sqrt{x_0^2 - x^2} = \pm\omega x_0 = \pm 2\pi f x_0$$
$$= \pm 2\pi(20)(0.020) = \pm 2.51 \text{ m s}^{-1}$$

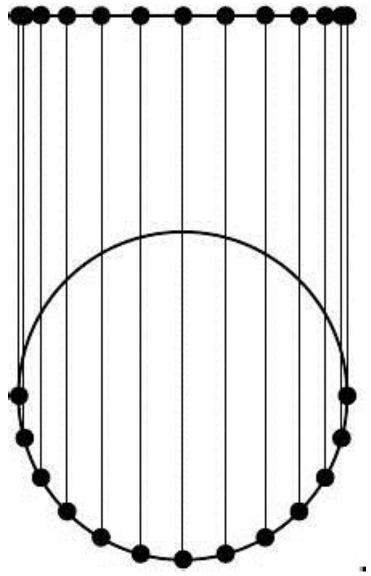
At negative amplitude, x = -0.020 m

$$v = \pm 2\pi(20)\sqrt{x_0^2 - x_0^2} = 0 \text{ m s}^{-1}$$

8.2 Relationship between Simple Harmonic Motion and Uniform Circular Motion

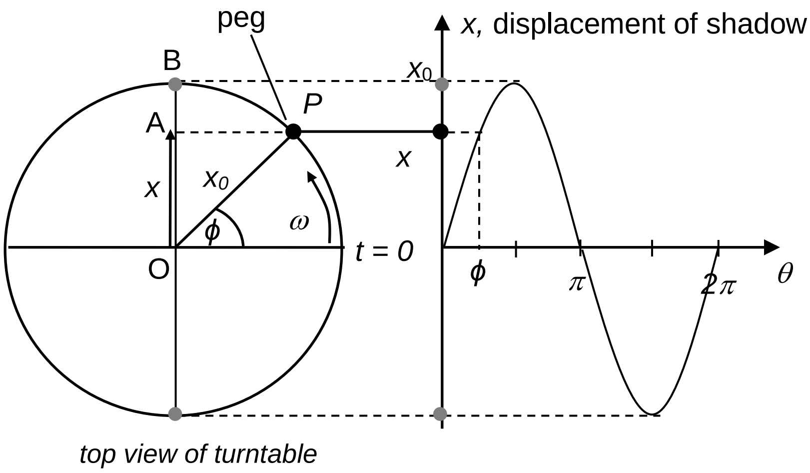
The experimental setup below illustrates how SHM is closely associated with circular motion. As the peg rotates on the turntable with constant angular velocity ω, its shadow on the screen moves back and forth in SHM.





Some observations

1) The motion is symmetrical about the fixed point, O' on the screen.
2) The time taken to complete one oscillation (period) is the same as the time taken to complete one cycle. Thus frequency of SHM = frequency of the circular motion.
3) The distance from the fixed point O' to the extremity (amplitude) is equal to the radius of the circle.
4) When the peg is shifted nearer to the centre of the turntable, the amplitude of the SHM decreases, but the period or frequency remains unchanged.
5) The shadow reaches maximum speed as it passes O' and is momentarily at rest at the extreme positions.
6) <u>Variation of displacement of peg shadow with time</u>



OA is the projection of OP (the displacement of the peg shadow) onto the vertical.

Applying trigonometry on ΔOAP:

$\Rightarrow \quad x = x_0 \sin \phi \quad (OP = x_0)$

Since $\phi = \omega t$, $\quad x = x_0 \sin \omega t \quad$ which is a typical SHM displacement-time equation.

Phase & Phase Difference

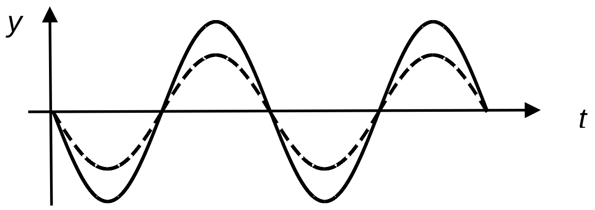
In SHM, the angle ϕ is called the **phase (angle)** of oscillation, analogous to the angular displacement in circular motion.

Thus the angular frequency ω in simple harmonic motion is interpreted as the rate of change of phase angle of a simple harmonic oscillation.

The idea of phase is useful for comparing two oscillators of the same frequency (and therefore period) that are not moving in step with each other due to a delay or advance at the start of the oscillations with respect to each other.

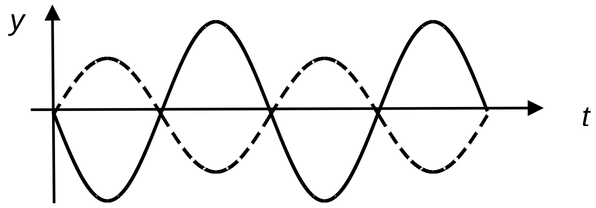
1. The difference in phase between two oscillations is called **phase difference $\Delta\phi$**.

2. Two oscillations are said to be ***in phase*** when they are in step, i.e., they reach their respective maximum displacements in the same direction at the same time. It does not matter what their respective amplitudes are.



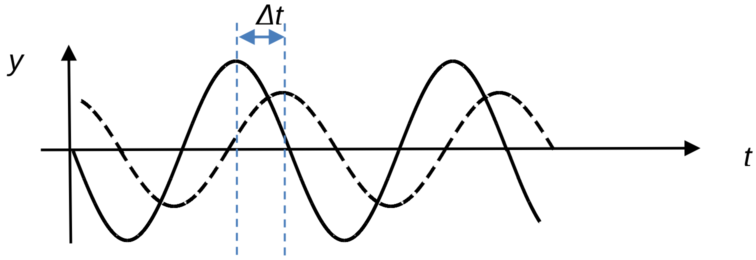
The phase difference $\Delta\phi$ = 0 or any integer multiple of 2π rad.

3. Two oscillations are said to be **in *anti-phase*** when they are completely out of step, i.e., one oscillation reaches maximum displacement in a given direction at the same instant as the other oscillation reaches maximum displacement in the opposite direction.



The phase difference $\Delta\phi$ = π rad or any odd integer multiple of π rad.

4. Of course the two oscillations may not be in phase or in anti-phase but at some intermediate phase difference. In such a case, the phase difference can be calculated by the following: $\Delta\phi = \frac{\Delta t}{T} \times 2\pi$.



Access the following applet to learn more about phase difference between 2 oscillators. http://ngsir.netfirms.com/englishhtm/phase.htm.

You will learn more about phase difference in the next chapter on Waves.

8.3 Energy in SHM

For a body moving in SHM, **the total mechanical energy E_T is conserved**. As the mass oscillates back and forth, the energy continuously transforms between potential energy E_p and kinetic energy E_k.

8.3.1 Variation of Energy-displacement (*E*-*x*) graphs for a simple harmonic oscillator

The sketch of the graphs can be deduced by observing the energy changes in a swinging pendulum or oscillating mass on a spring.

- **As the oscillator passes through its equilibrium position**, ***x* = 0:**

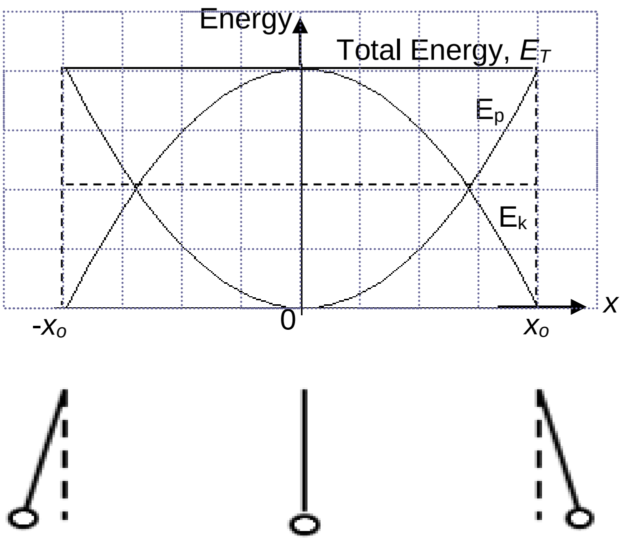
  ⇒ its speed and hence its **kinetic energy** are **maximum**.

- **At the extreme positions, $x = \pm x_o$,**

  ⇒ the speed and hence the **kinetic energy** are both **zero**.

  ⇒ Assuming that there is no energy loss (e.g. no friction or air drag), the **total energy E_T of the oscillator must remain constant.**

  ⇒ **The potential energy** will be **maximum**.



Note:

(1) **Variation of E_K with displacement *x*:**

For SHM, the velocity is given by $v = \pm\omega\sqrt{x_0^2 - x^2}$ ⇒ $E_k = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2(x_0^2 - x^2)$

$$\boxed{E_k = \frac{1}{2}m\omega^2(x_0^2 - x^2)}$$

⇒ Quadratic equation! Coefficient of x^2 is –ve.

⇒ Graph is an <u>inverted parabola</u> (see figure above).

(2) **Variation of E_T with displacement x:**

- Total energy is constant. $\Rightarrow$ Graph of E_T vs. x is a horizontal straight line graph.
- Now, E_T = max $E_K = \frac{1}{2}m\omega^2 x_0^{\ 2}$ (when x = 0)

$$\boxed{E_T = \frac{1}{2}m\omega^2 x_0^{\ 2}}$$

- The total energy E_T of SHM is proportional to the square of the amplitude, i.e., $E_T \propto x_0^{\ 2}$

(3) **Variation of E_p with displacement x:**

Total Energy $E_T = E_P + E_K$

$\Rightarrow$ $E_p = E_T - E_k = \frac{1}{2}m\omega^2 x_0^{\ 2} - \frac{1}{2}m\omega^2(x_0^{\ 2} - x^2) = \frac{1}{2}m\omega^2 x^2$

$$\boxed{E_p = \frac{1}{2}m\omega^2 x^2}$$

$\Rightarrow$ Quadratic equation! Coefficient of x^2 is +ve.

$\Rightarrow$ Graph is a parabola (see figure above).

Example 8.8: Relationship between Energies and Displacement

A mass, m = 0.50 kg, connected to a light spring with spring constant k = 20 N m^{-1} oscillates on a frictionless horizontal surface. Given that angular frequency $\omega = \sqrt{\frac{k}{m}}$, calculate the

(a) total energy of the system if the amplitude of the motion is 3.0 cm.

(b) (i) kinetic energy of the system when the displacement is 2.0 cm and

(ii) potential energy of the system when the displacement is 2.0 cm.

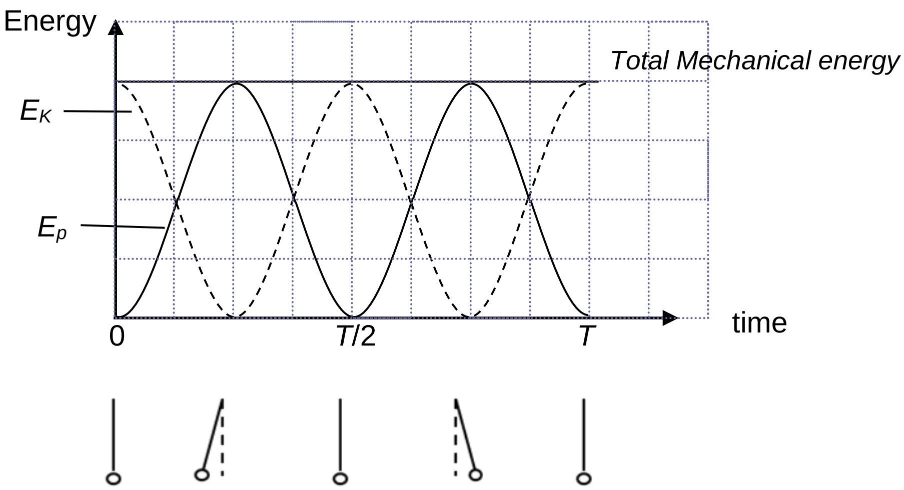
8.3.2 Variation of Energy with time graphs for a simple harmonic oscillator

The energy-time graphs depend on how the oscillation is started (initial conditions).

Again, without the actual equations, the shapes of the energy-time graphs can be deduced by observing a swinging pendulum or oscillating mass-spring.

Consider an oscillator started at its equilibrium position ($t = 0, x = 0, v = v_o$).

$\Rightarrow$ $x = x_o \sin(\omega t)$ and $v = v_o \cos(\omega t)$,



Note:

(1) When the oscillatory motion has completed one cycle in one period, both kinetic and potential energies have gone through two cycles.

(2) **Variation of total energy E_T with time t :**

- Total energy is constant. $\Rightarrow$ Graph is constant at $E_T = \frac{1}{2}m\omega^2 x_0{}^2$

(3) **Variation of kinetic energy E_K with time t :**

- $E_K = \frac{1}{2}mv^2 = (\frac{1}{2}mx_o{}^2\omega^2)\cos^2 \omega t$

$\Rightarrow$ Graph is a $\cos^2$ graph (it follows the shape of the v^2-t graph).

(4) **Variation of potential energy E_P with time t :**

- $E_p = \frac{1}{2}m\omega^2 x^2 = (\frac{1}{2}m\omega^2 x_o{}^2)\sin^2 \omega t$

$\Rightarrow$ Graph is a $\sin^2$ graph (it follows the shape of the x^2-t graph).

Example 8.9: Relationships between Energies and time

Sketch the E_T-t, E_K-t, E_P-t graphs for an oscillator released from rest from its extreme position (t = 0, $x = x_0$, v = 0).

8.4 Damped Oscillations, Forced Oscillations and Resonance

8.4.1 Free oscillations

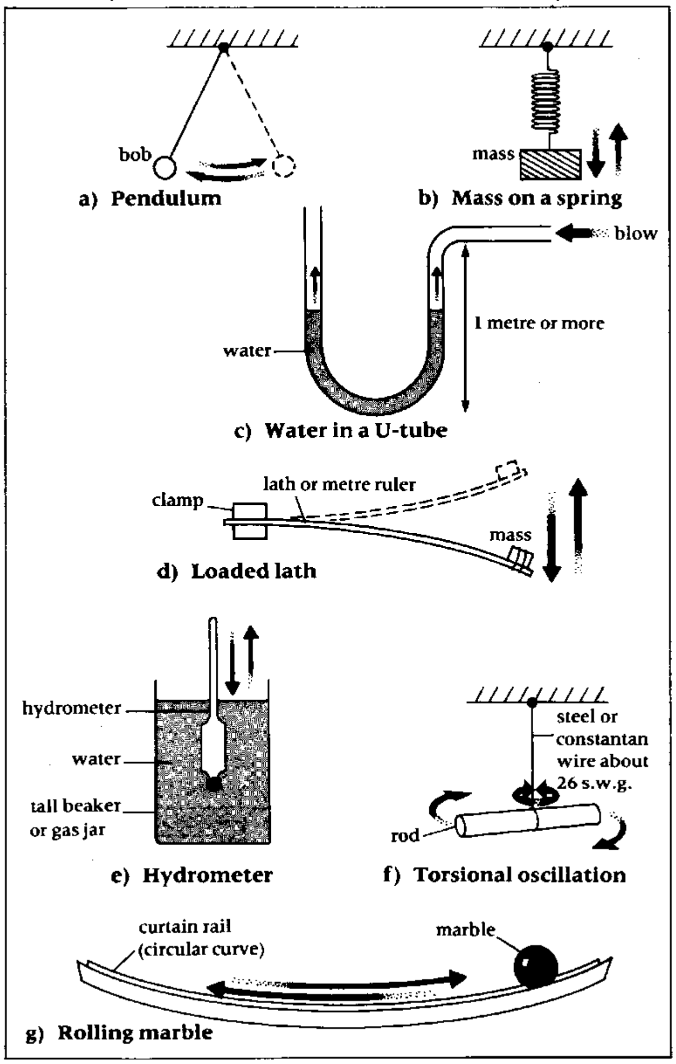
When a system is disturbed from its equilibrium position, such that it oscillates about this position with no resistive forces acting, the oscillation is said to be a **free oscillation**.

The frequency at which it oscillates is known as the **natural frequency** of the system. This frequency is characteristic of the system.

The simple harmonic motions that we have seen so far for the pendulum and spring-mass systems are ideal free oscillations in which no friction or any forms of resistance to motion exist. The oscillations would in theory continue forever.

Some examples of free oscillations

(the arrows indicate motion, not forces)



8.4.2 Damped oscillations

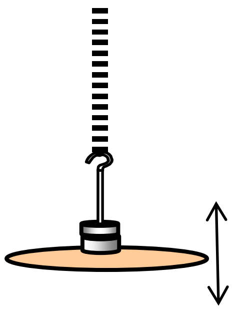
In an ideal oscillating system, there is no friction and the total mechanical energy is constant. The system in simple harmonic motion continues forever, with no decrease in amplitude.

Real-world systems experience inevitable friction and the mechanical energy of oscillations decreases with time due to the work done against friction. For instance, a simple pendulum gradually decreases in amplitude and stops eventually due to air resistance and friction at the support.

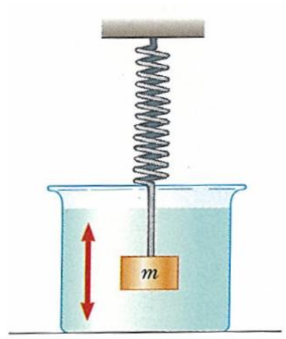
The progressive decrease in amplitude of any oscillatory motion caused by dissipative forces is called ***damping***. Such an oscillatory motion is known as a ***damped oscillation***. Due to damping, pendulum clocks require energy (wound up spring or batteries) to replenish energy loss in order for their amplitude to remain constant.

There are situations in which damping is introduced to reduce the amplitude of oscillation or bring the oscillation to rest.

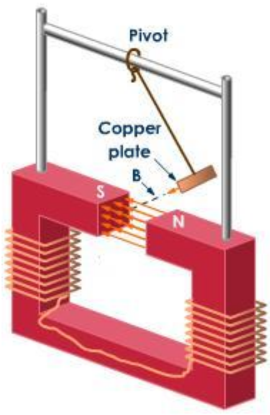
Commonly used damping methods:



(a)



(b)



(c)

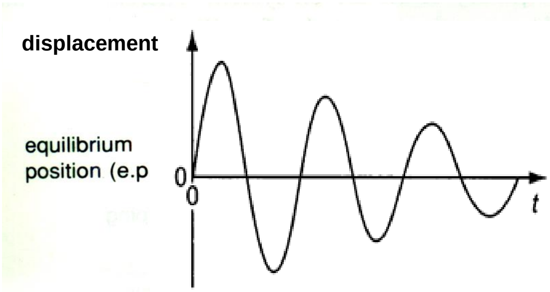
(a) Using air resistance, e.g., to bring an oscillating spring-mass system to rest faster, attach a piece of cardboard of negligible mass but large surface area to the base of the mass. The increase in air resistance increases damping during the oscillation.

(b) Using fluid viscosity, e.g., damping can be increased by submersing the mass of a spring-mass system in a fluid that is more viscous than air, such as oil.

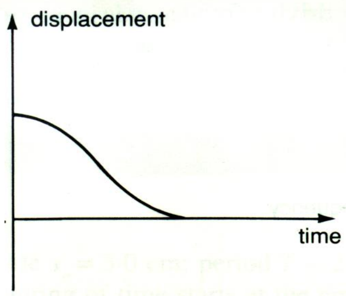
(c) Using eddy currents, e.g., when a copper plate swinging freely passes through a magnetic field, eddy currents induced in the plate will oppose the motion and bring the plate to a stop (further discussed in Electromagnetic Induction in year 2).

Extent of damping

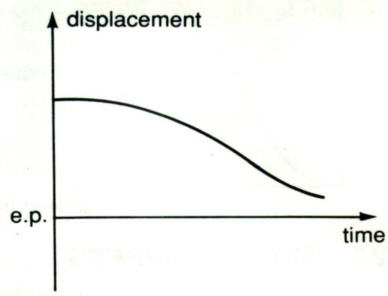
Damping changes the behaviour of SHM systems.



(1) light damping



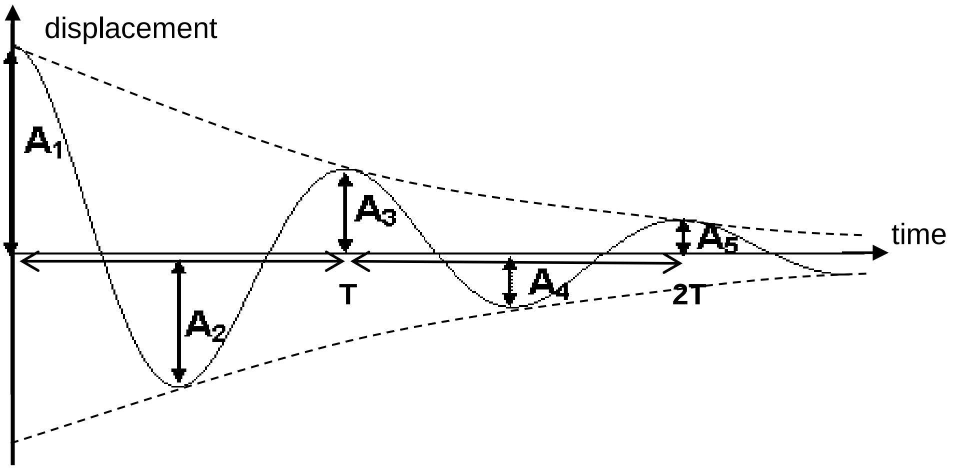
(2) critical damping



(3) heavy damping

***(1)	Light Damping* (*underdamped system*)**

- In a lightly damped system, the total energy of the system decreases with time as energy is dissipated when the system oscillates against resistive forces.
- The amplitude of the system decreases **exponentially with time.**

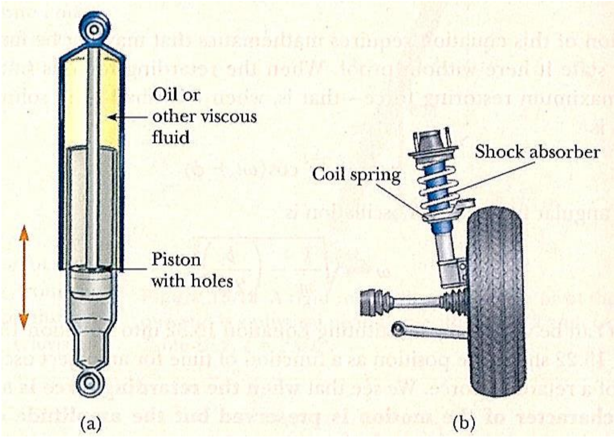


$A_1 > A_2 > A_3 > A_4 > A_5$

Note: It can be shown that the period in damped oscillations is slightly longer than the 'undamped' value by solving the differential equation involving the forces on the system. (This is not in the A-level syllabus – refer to Appendix 3 for details.)

***(2)	Critical Damping (critically damped system)***

- If we are able to gradually increase the resistive force acting on an oscillatory system, damping increases till a point such that when displaced, the system **returns to the equilibrium position in the shortest possible time without oscillating** at all. The system is said to be **critically damped**.
- Critical damping is deliberately introduced into some systems to prevent continuous oscillations. For example, in analogue instruments such as balances, ammeters and voltmeters, the pointer is critically damped so that it stops in the shortest possible time to indicate the reading, instead of oscillating about the reading (underdamped) or take an unnecessarily long time to crawl to the reading (overdamped).
- Critical damping is also employed in car suspension systems to ensure a smooth ride when the car moves on a bumpy road. The shock absorbers in the suspension system have hydraulic pistons to prevent the car from bouncing up and down excessively after hitting a bump. The damping of a car can be tested by pushing down the car with your weight and then releasing it; the car should rapidly return to its normal height with negligible overshoot.



(a) A shock absorber consists of a piston oscillating in a chamber filled with oil. As the piston oscillates, the oil is squeezed through holes between the piston and the chamber, causing a damping of the piston's oscillations.
(b) One type of car suspension system, in which a shock absorber is placed inside a coiled spring at each wheel.

(3) ***Heavy Damping*** **(*overdamped system*)**

- When the damping force is increased beyond the point of critical damping, we say that the system is **overdamped** or that it experiences heavy damping. Once displaced from equilibrium, the oscillator takes a very long time to return to its equilibrium position.
- An example would be of certain doors which are so heavily damped that they take forever to close.

8.4.3 Forced oscillations and resonance

Forced oscillations

As mentioned earlier, most real systems will experience some form of damping and will eventually stop oscillating. To keep them oscillating, one needs to periodically put in energy to the systems to make up for the energy lost. In other words, one needs to apply a periodic force to the systems.

- An oscillation under the influence of an external periodic force is called a ***forced oscillation***.
- The external force that is used to sustain (or drive) the oscillations is known as the **driving force.**
- The frequency with which the periodic force is applied is called the **forced frequency** or **driver frequency**.

One example of forced oscillation is pushing a child on a swing at the playground; one needs to continually give the swing a push at the right moment to sustain its oscillation.

Resonance

Consider the above example of a child on a swing. We have mentioned that because of damping, the oscillations will eventually die off. However, we can sustain the oscillations by periodically applying a driving force.

If the frequency of the driving force matches the natural frequency of the oscillatory motion of the child-swing system, an interesting phenomenon happens - the final amplitude of the oscillation would be the greatest compared to the amplitudes of oscillation for other frequencies of driving force. This phenomenon is known as ***resonance****.*

Resonance is a phenomenon in which an oscillatory system responds with <u>maximum amplitude</u> to an <u>external periodic driving force</u>, when the <u>frequency of the driving force (driver frequency) equals the natural frequency</u> of the driven system.

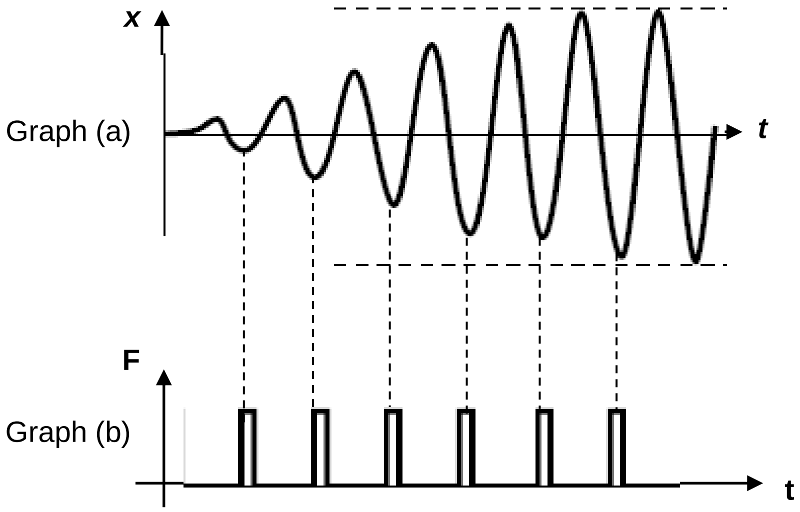
Note:

(1) The frequency at which resonance occurs is known as the **resonant frequency**.

(2) Under ideal conditions, i.e., if no resistive forces are present (there is no damping), the amplitude will build up to an infinitely large value.

(3) However, in real cases, damping occurs. So, whilst putting energy into the system in each cycle, energy is also taken away by damping and the amplitude will not build to infinity.

(4) Resonance represents the point where the (driving) energy is transferred most efficiently into the oscillatory system.

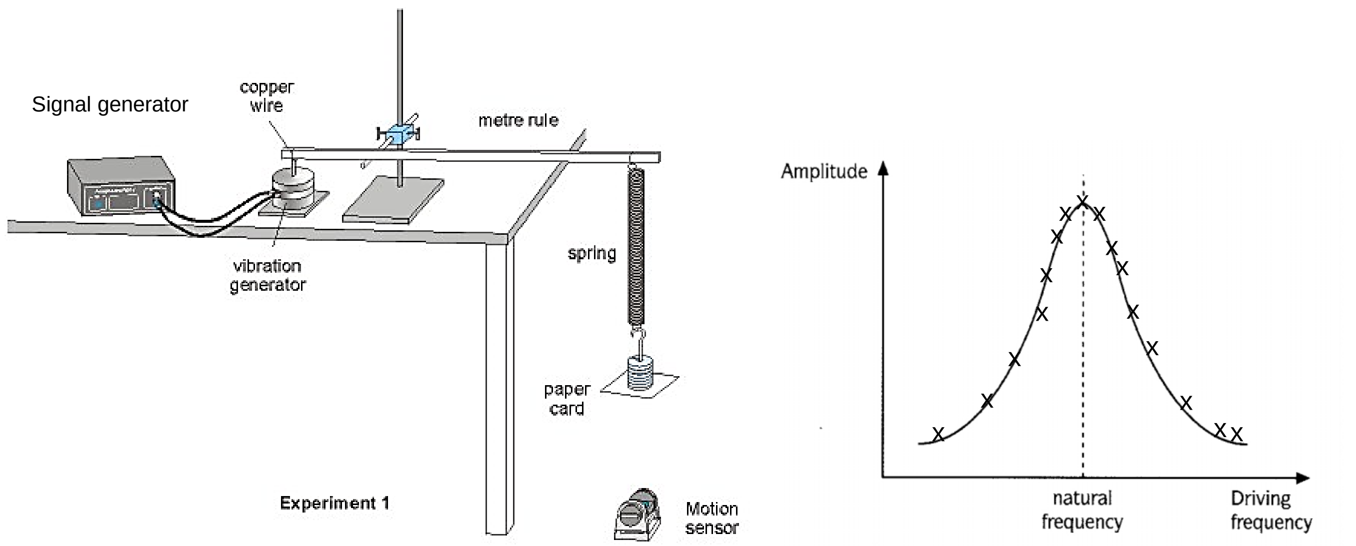


Graph (a) shows the child's *x*-*t* graph as one periodically transfers a fixed amount of energy into the swinging motion.

Graph (b) shows the variation of driving force with time.

Investigating Resonance in the Lab

Setup:



The above set-up may be used to investigate the phenomenon of resonance in the lab.

(1) The system under study is a spring-mass system that is driven to oscillate by a metre rule which is in turn driven to vibrate by a vibration generator. To vary the frequency of vibration, the vibration generator is connected to a signal generator.

(2) To vary the amount of damping experienced by the spring-mass system, a light paper card is attached to the mass. By varying the area of the paper card attached, the damping is varied. The larger the paper card, the greater the damping.

(3) To measure the amplitude of vibration, a motion sensor is placed directly below the paper card.

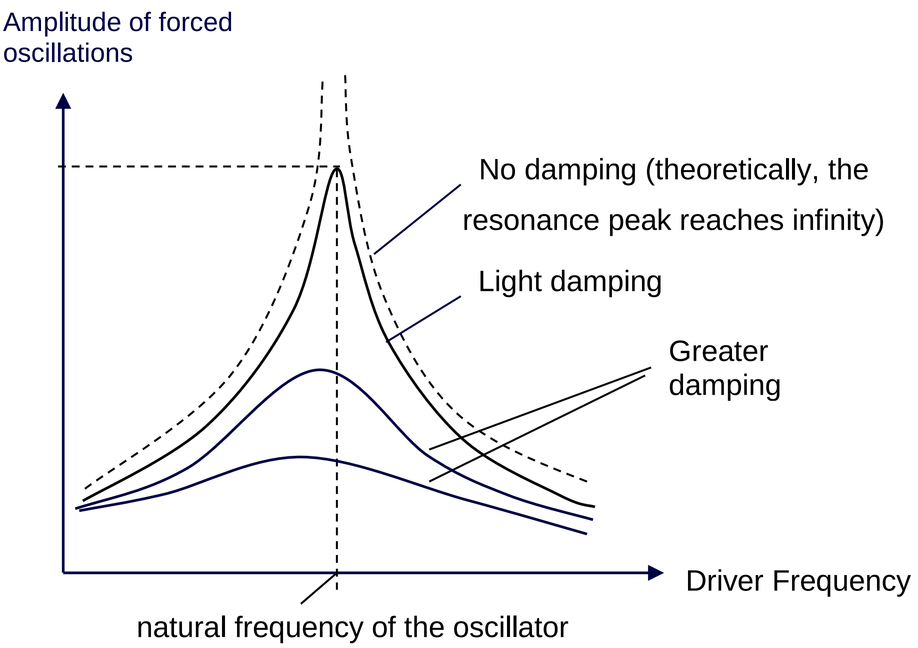
(4) For each frequency of vibration, we can then measure and record the amplitude of vibration of the spring-mass system. We start off with a driving frequency that is much lower than the natural frequency of the spring-mass system and slowly increase this frequency till beyond the natural frequency.

(5) The graph above shows the typical results obtained from the experiment for one area of card.

(6) The experiment is then repeated for different areas of cards – to produce different amounts of damping.

(7) The data points for each area can then be plotted on the same pair of axes for comparison.

The figure below shows the amplitude vs. driving frequency graphs that will be obtained.



Note:

1) The amplitude of a lightly damped system is very large at resonance and the peak is sharp. The amplitude drops off rapidly when the driver frequency differs from the natural frequency of the system.
2) When the driver frequency approaches zero, the amplitude of the forced oscillations will be the same as the amplitude of the driver.
3) As the system experiences greater damping,
   - the amplitude of the oscillation **decreases**.
   - the resonance peak becomes **broader**.
   - the resonance peak shifts slightly to the **left** to a slightly lower value of frequency.

From the graphs, it is evident that damping is a useful way to reduce the effects of resonance. Another way to avoid resonance is to ensure that the frequency of the periodic external force differs from the natural frequency of the driven system.

8.4.4 Examples of Destructive and Useful Resonance

Resonance is an important physical phenomenon that can appear in many different situations.

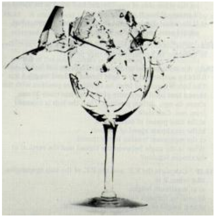
Examples of destructive resonance

1) Collapse of buildings in earthquakes

An earthquake consists of many low-frequency vibrations ranging from 1 to 10 Hz. During an earthquake, when the frequencies of the vibration match the natural frequencies of buildings, resonance may occur and result in serious damages. That also explains why some buildings collapse while others stand almost unaffected. In regions of the world where earthquakes happen regularly, buildings may be built on foundations that absorb the energy of the shock waves. In this way, the vibrations are damped and the amplitude of the oscillations cannot reach a dangerous level.

2) Shattering glass with voice

When an opera singer projects a high-pitched note whose frequency matches the natural frequency of a wine glass, the resonance may cause the glass to vibrate at so large an amplitude that it breaks.



3) Violent vibrations in vehicles and machines

If the panel in a bus rattles violently when the bus is travelling at a certain speed, it is likely that a resonant vibration is occurring. A washing machine with a load which has natural frequency matching the spinning frequency will get violent vibrations as resonance occurs.

Examples of useful resonance

1) Radio receptions

Resonance not only occurs in mechanical oscillations but also in electric circuits. A radio receiver works on the principle of resonance. Our air is filled with radio waves of many different frequencies which the aerial picks up. The tuner can be adjusted so that the frequency of the electrical oscillations in the circuits is the same as that of the radio wave transmitted from a particular station we desire. The effect of resonance amplifies the signals contained in this wave while the radio waves of other frequencies are diminished.

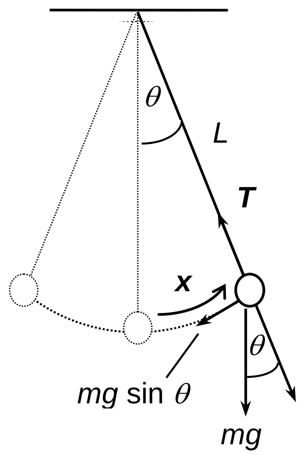
2) Magnetic Resonance Imaging

Magnetic Resonance Imaging (MRI) is increasingly used in medical diagnosis to produce images similar to those produced by X-rays. Strong, electromagnetic fields of varying radio frequencies are used to cause oscillations in atomic nuclei. When resonance occurs, energy is absorbed by the molecules. By analysing the pattern of energy absorption, a computer-generated image can be produced. The advantage of MRI scanners is that no ionising radiation (as in the process of producing X-ray images) is involved.

Appendix 1 Period of a Simple Pendulum

Restoring Force $F = -mg\sin\theta$

(–ve sign as it is in the opposite direction to the displacement)



Arc length $x = L\theta$

$\Rightarrow \quad \theta = x/L$

When the angular displacement θ is small,

$$\sin\theta \approx \theta$$

$$\Rightarrow \quad F = -mg\sin\theta \approx -mg\frac{x}{L}$$

From Newton's 2nd law: $ma \approx -mg\frac{x}{L}$

$$\Rightarrow \quad a \approx -\left(\frac{g}{L}\right)x$$

i.e., $a = -\omega^2 x$ where $\omega^2 = \frac{g}{L}$

Hence, motion of a simple pendulum bob attached to a light inextensible string oscillating with **a small angular displacement of ~ 10° or smaller** is, to a good approximation, a linear SHM with period, T, given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$$

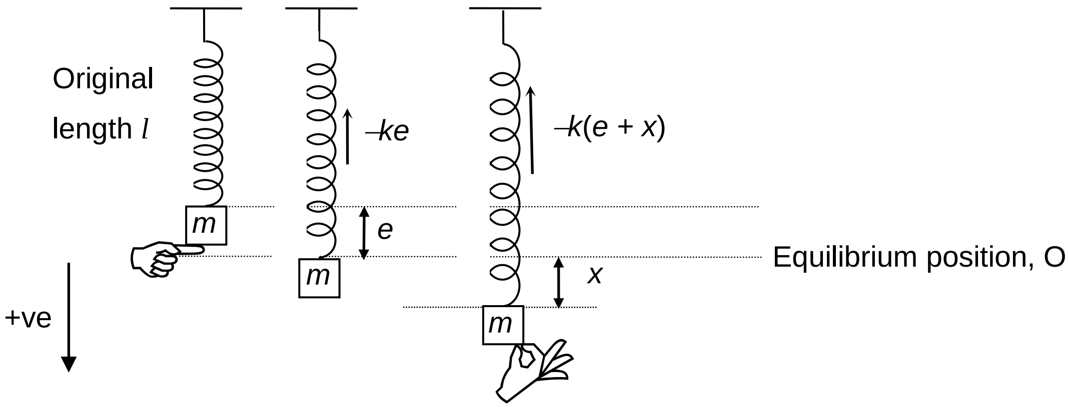
Note:

1) Other assumptions made are that the string is massless, the radius of the bob is small compared with the length of the string and air resistance is negligible.

2) The period T is independent of the amplitude of the oscillation.

Appendix 2 Period of Oscillating Mass on Vertical Spring

A mass m attached to the end of a spiral spring exerts a downward force on it and stretches it by an amount e as shown in the diagram.



If the spring obeys Hooke's law, then the extension e is directly proportional to the extending tension, F_s,

i.e., $F_s = -ke$ where k is a constant.

Since the mass is in equilibrium, the net force on it is zero ($\sum F = 0$).

Thus $mg + (-ke) = 0$

$\Rightarrow \quad mg = ke$

Suppose the mass is now pulled down a further distance x below its equilibrium position, then the tension in the spring is increased to $-k(e + x)$. Thus the resultant force on the mass is given by

$$F = mg - k(e + x)$$

$$= ke - k(e + x) \qquad \text{(since } mg = ke\text{)}$$

$$= -kx$$

i.e., the resultant force is acting upwards, with the direction opposite to that of the displacement.

By Newton's 2nd law, $ma = -kx$

$$\Rightarrow \quad a = -\frac{k}{m}x$$

Since k and m are fixed, the oscillation of a mass on spring is simple harmonic.

The angular frequency ω is given by $\omega^2 = \dfrac{k}{m}$

$$\Rightarrow \quad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

Appendix 3 Equations of motion for damped simple harmonic oscillations

In an oscillatory system, the decrease in amplitude caused by dissipative forces is called damping and the corresponding motion is called damped oscillation. The simplest case to analyse is a simple harmonic oscillator under damping, where its dissipative force is directly proportional to its *velocity*. This behaviour occurs in friction involving the flow of viscous fluid, such as in shock absorbers or sliding between oil-lubricated surfaces. The additional force on the body due to viscous drag is $f = -bv$, where *b* is a constant.

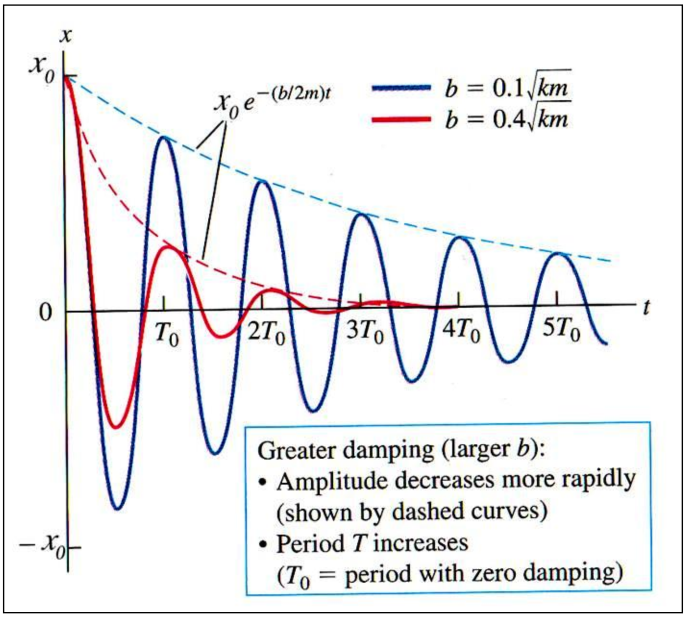
With a viscous drag, the net force on the body is ($-kx - bv$).

Applying Newton's 2nd second law,

$$m\frac{d^2x}{dt^2} = -kx - bv$$

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

This is a second order linear differential equation with constant coefficients.



If $b < 2\sqrt{km}$, the system is under light damping and the equation of motion is

$$x = \left(x_0 e^{-\left(\frac{b}{2m}t\right)}\right)\cos\left(\left(\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}\right)t + \phi\right)$$

Its amplitude ($x_0 e^{-\left(\frac{b}{2m}t\right)}$) is not a constant, but decreases with time because of the decaying exponential factor. In addition, the larger the value of *b*, the faster the decrease in the amplitude.

Its angular frequency $\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$, is smaller than that without damping, $\omega = \sqrt{\frac{k}{m}}$, and therefore the period is slightly larger.

If $b = 2\sqrt{km}$, then $x = x_0 e^{-\left(\frac{b}{2m}t\right)}$. The system is in critical damping, where it no longer oscillates but returns to its equilibrium position without overshooting the equilibrium position.

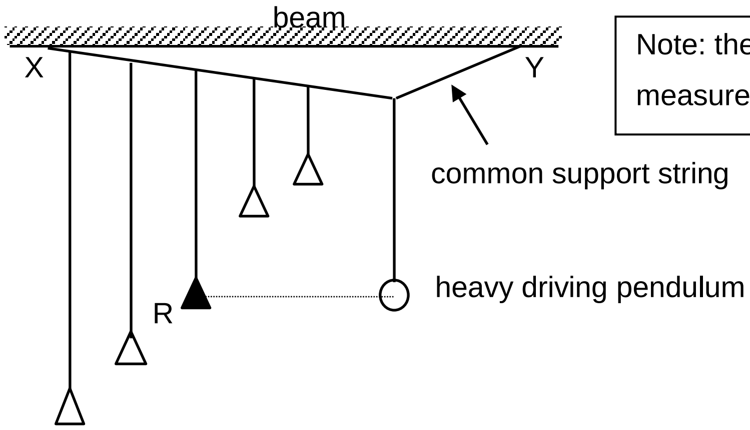
If $b > 2\sqrt{km}$, then the system is heavily damped. Again, there is no oscillation, but the return to its equilibrium takes longer than in the case of critical damping. Its solutions are in the form of $x = C_1 e^{-a_1 t} + C_2 e^{-a_2 t}$, where C_1 and C_2 are constants determined by initial conditions and a_1 and a_2 are constants determined by *m, k* and *b*.

Appendix 4 Demonstration of Forced Oscillation and Resonance - Barton's pendulums

Barton's pendulums consist of several pendulums of different lengths hung from a horizontal string. Each has its own natural frequency of oscillation. The 'driver' pendulum at the end has a large mass and its length is equal to that of pendulum R. When the driver is set swinging, its vibrations are transferred through the common support string to the other pendulums and start the forced oscillations.

Observations show that:

1. All the pendulums vibrate with the same frequency as the driver.
2. Pendulum R whose length matches that of the driver pendulum (therefore has the same natural frequency as the driver) has the largest amplitude. It is said to be resonating with the driver.



Note: the lengths of the pendulums are measured from the beam

Explanation:

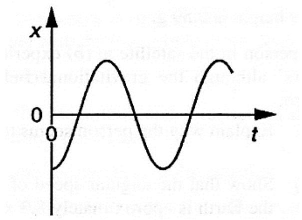
All the pendulums are coupled together by the suspension. As the driver swings, it moves the suspension, which in turn moves the other pendulums. The matching pendulum is being pushed slightly once each oscillation, and the amplitude gradually builds up to maximum. The other pendulums are being pushed at a frequency that does not match their natural frequency and they therefore oscillate with smaller amplitudes.

Tutorial 8 Oscillations

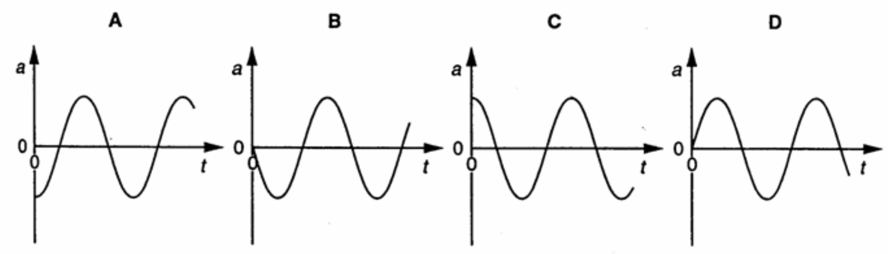
Self-Review Questions

Simple Harmonic Motion

S1 **N03/I/14:** A body moves with simple harmonic motion about point P. The graph shows the variation with time *t* of its displacement *x* from P.

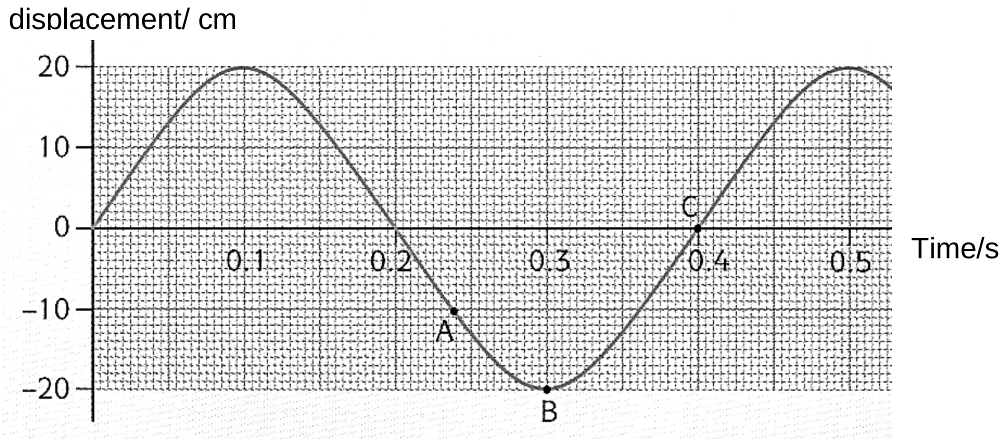


Which graph shows the variation with time of its acceleration?



S2 **J90/I/10**: A body performing simple harmonic motion has a displacement *x* given by the equation $x = 30\sin 50t$, where *t* is in seconds. What is the frequency of oscillation?

S3 The figure below shows the displacement-time graph of a simple harmonic oscillator.



(i) Deduce the following quantities:

(a) amplitude,

(b) period,

(c) frequency,

(d) angular frequency,

(e) displacement at A.

(ii) Using your answers to S3(i)(a) and (d), write down an equation that describes the variation of displacement, x, with time, t, as shown in the graph.

(iii) Substitute the displacement at A in the figure, into the equation to find the corresponding time. Check that it agrees with the value from the graph.

(iv) At which point (A,B,C) is (a) the velocity maximum? (b) the acceleration maximum?

State whether the direction is positive or negative for (a) and (b). Explain your answers.

(v) Deduce the following quantities:

(a) velocity at B

(b) velocity at C

(c) acceleration at B

(d) acceleration at C.

(iv) Sketch for this oscillator,
(a) the acceleration-time graph
(b) the velocity-time graph
(c) the velocity-displacement graph
(d) the acceleration-displacement graph.

S4 A 0.50 kg body performs simple harmonic motion with a frequency of 2.0 Hz and amplitude of 8.0 mm. Determine the maximum velocity and the maximum acceleration and the corresponding positions of the body.

Energy in simple harmonic motion

S5 Refer to the displacement-time graph in S3. The mass of the oscillator is 1.0 kg.

(i) At which point (A,B,C) is
(a) the potential energy maximum? (b) the kinetic energy maximum?

(ii) Find the kinetic energy and the potential energy of the oscillator at C and B.

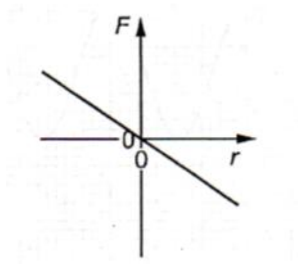
(iii) Sketch for the oscillator, labelled graphs on the same horizontal time axis,
(a) the variation of its kinetic energy,
(b) the variation of its elastic potential energy,
(c) the variation of its total mechanical energy.

(iv) Sketch for the oscillator, labelled graphs on the same horizontal displacement axis,
(a) the variation of its kinetic energy,
(b) the variation of its potential energy,
(c) the variation of its total mechanical energy.

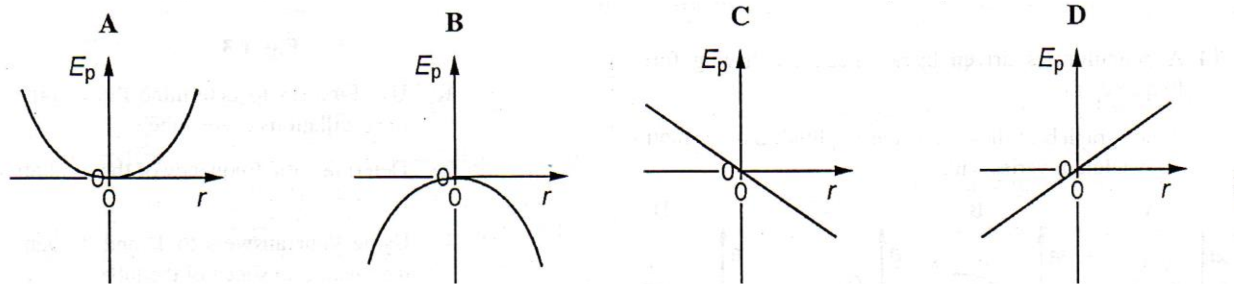
S6 A body oscillates vertically on the end of a light vertical spring with simple harmonic motion of frequency f and amplitude a. The total energy of the oscillations is proportional to

A $\sqrt{f}$ B $\sqrt{a}$ C a^2 D f

S7 N04/I/16: A particle is moving such that the force *F* on it changes with the distance *r* from a fixed point as shown.

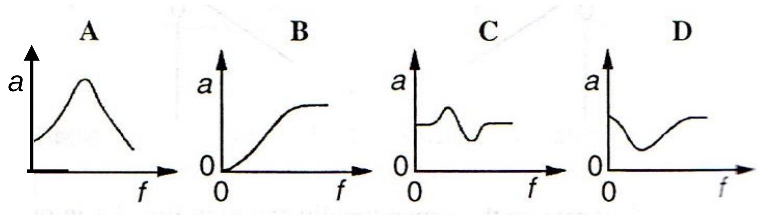


Which graph best shows the relationship between the potential energy E_p of the particle and the distance *r*?



Damped Oscillations, Forced Oscillations and Resonance

S8 N2000/I/9: A pendulum is driven by a sinusoidal driving force of frequency f. Which graph best shows how the amplitude a of the motion of the pendulum varies with f?



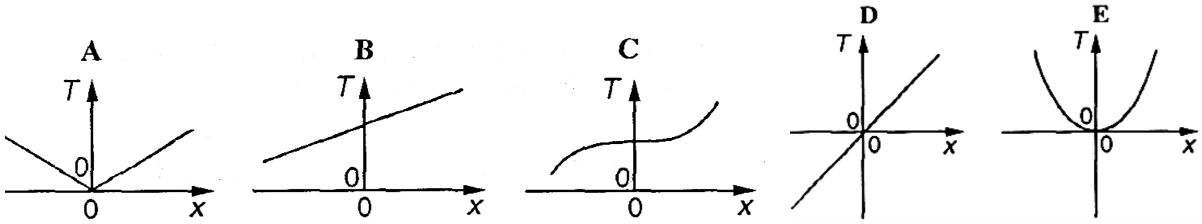
Discussion Questions

Simple Harmonic Motion

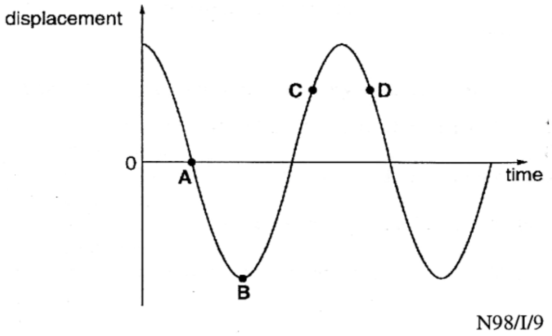
D1 Consider a situation of a ball free-falling and bouncing on the ground such that its collision is perfectly elastic.
(i) Is this motion a periodic motion?
(ii) Is this an example of simple harmonic motion?

D2 **J78/II/10:** In order to check the speed of a camera shutter, the camera was used to photograph the bob of a simple pendulum moving in front of a horizontal scale. The extreme positions of the bob were at the 600 mm and 700 mm marks. The photograph showed that while the shutter was open the bob moved from the 650 mm mark to 675 mm mark. If the period of the pendulum was 2 s, how long was the shutter remained open for?

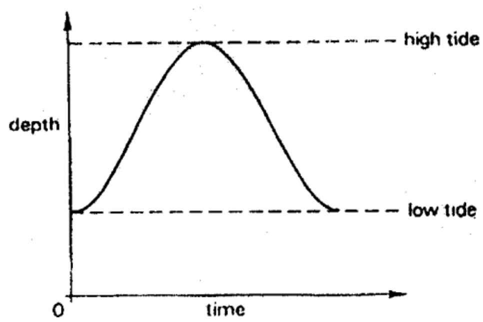
D3 **N88/I/7:** A mass is hung from the free end of a light helical spring and then given a small displacement vertically downwards. Which graph best represents how *T*, the tension in the spring, varies with *x*, the displacement of the mass from the equilibrium position during subsequent oscillations?



D4 **N98/I/9:** The diagram shows the graph of displacement against time for a body performing simple harmonic motion. At which point are the velocity and acceleration in opposite directions?



D5 **J91/I/9.** The rise and fall of water in a harbour is simple harmonic. The depth varies between 1.0 m at low tide and 3.0 m at high tide. The time between successive low tides is 12 hours. A boat, which requires a minimum depth of water of 1.5 m, approaches the harbour at low tide. How long will the boat have to wait before entering?



D6 **J99/III/4 (part):**
(b) Some sand is placed on a flat horizontal plate and the plate is made to oscillate with simple harmonic motion in a vertical direction, as shown.

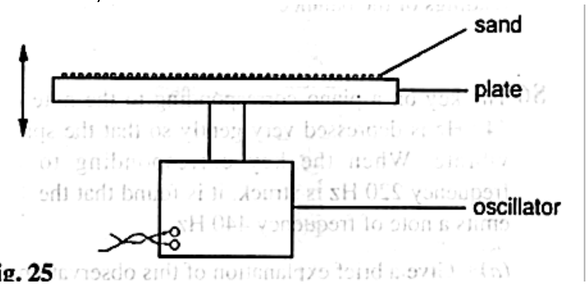


Fig. 25

The plate oscillates with a frequency of 13 Hz.
(i) Sketch a graph to show the variation with displacement *x* of the acceleration *a* of the plate.

(ii) The acceleration a is given by the expression

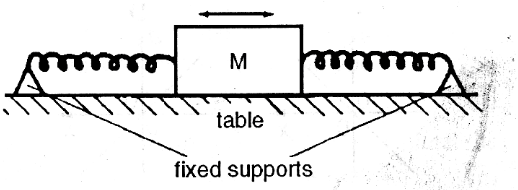
$$a = -\omega^2 x$$

where ω if the angular frequency. Calculate
1. the angular frequency ω
2. the amplitude of oscillation of the plate such that the maximum acceleration is equal to the acceleration of free fall.

(c) Suggest, with a reason, what happens to the sand on the plate in (b) when the amplitude of the oscillations of the plate exceeds the value calculated in (b)(ii)2.

Energy in Simple Harmonic Motion

D7 **J81/II/19:** A mass M on a smooth horizontal table is attached by two light springs to two fixed supports as shown on the right. The mass executes simple harmonic motion of amplitude *a* and period *T*.

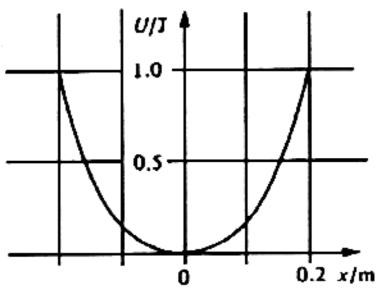


The energy associated with this simple harmonic motion is

A $2\pi ma^2 / T^2$
B $2\pi m^2 a^2 / T$
C $\pi^2 ma^2 / T^2$
D $2\pi^2 ma^2 / T^2$
E $4\pi^2 ma / T^2$

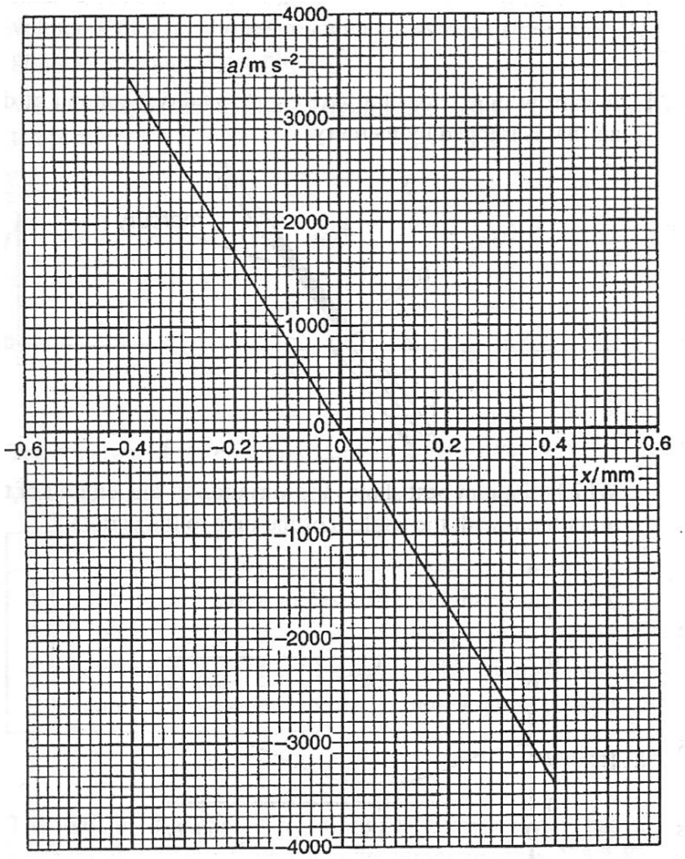
J81/II/16

D8 **N82/II/9:** A particle of mass 4 kg moves with simple harmonic motion and its potential energy *U* varies with position *x* as shown below.

Fig. 6


What is the period of oscillation of the mass?

D9 The figure on the right shows the variation with displacement x of the acceleration *a* of a particle P attached to the cone of a loudspeaker.



(a) Use the figure to

(i) explain why the motion of the particle P is simple harmonic. [2]

(ii) show that the frequency of oscillation of particle P is 460 Hz. [2]

(b) (i) The magnitude of the gradient of the line in the figure is *G*. Show that, for a particle of mass *m* oscillating with amplitude *A*, its maximum kinetic energy E_{MAX} is given by

$$E_{MAX} = \frac{1}{2} mGA^2$$ [3]

(ii) Determine E_{MAX} for a particle P of mass 2.5×10^{-3} kg. [2]

D10 **(N08/3/6)**

(a) Distinguish between *frequency* and *angular frequency* for a body undergoing simple harmonic motion. [2]

(b) A spring that has an unstretched length of 0.650 m is attached to a fixed point. A mass of 0.400 kg is attached to the spring and gently lowered until equilibrium is reached. The spring has then stretched elastically by a distance of 0.200 m.

Calculate, for the stretching of the spring,

(i) the loss in gravitational potential energy of the mass, [1]

(ii) the elastic potential energy gained by the spring. [2]

(c) Explain why the two answers to **(b)** are different. [2]

(d) The load on the spring is now set into simple harmonic motion of amplitude 0.200 m. Calculate

(i) the resultant force on the load at the lowest point of its movement, [2]

(ii) the angular frequency of the oscillation, [2]

(iii) the maximum speed of the mass. [1]

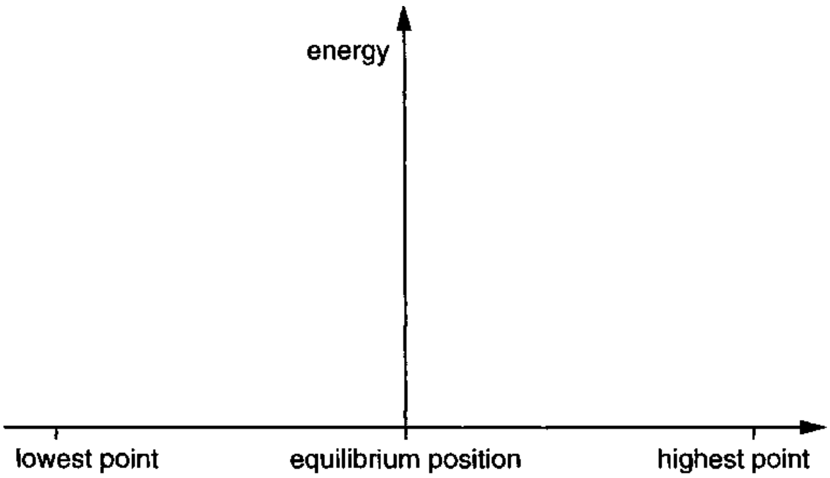
(e) Fig. 6.1 is a table of the energies of the simple harmonic motion. Complete the table.

	gravitational potential energy / J	elastic potential energy / J	kinetic energy / J	total energy / J
lowest point	0			
equilibrium position				
highest point				

Fig. 6.1

[5]

(f) On the axes of Fig. 6.2 below, sketch four graphs to show the shape of the variation with position of the four energies. Label each graph.



[3]

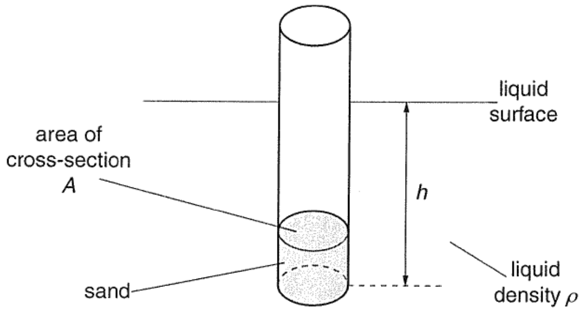
Fig. 6.2

Damped oscillations, forced oscillations and resonance

D11 N2013/III/7

a) State the origin of the upthrust acting on a body in a fluid.

b) A tube, sealed at one end, has a uniform area of cross-section *A*. Some sand is placed in the tube so that it floats upright in a liquid of density ρ, as shown in the figure below.



The total mass of the tube and the sand is *m*. The tube floats with its base a distance *h* below the surface of the liquid.
Derive an expression relating *m* to *h, A* and ρ. Explain your working.

c) The tube in (b) is displaced vertically and then released. For a displacement *x*, the acceleration *a* of the tube is given by the expression

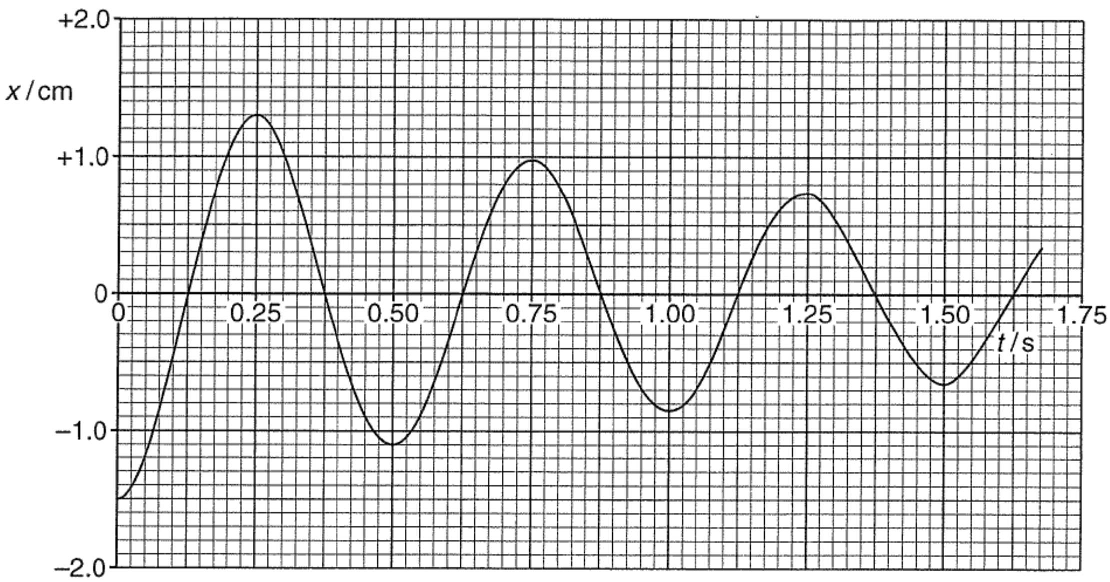
$$a = -\left(\frac{\rho A g}{m}\right)x$$

where *g* is the acceleration of free fall.

i) Explain why the expression leads to the conclusion that the tube is performing simple harmonic motion.

ii) The tube has total mass *m* of 32g and the area *A* of its cross-section is 4.2 cm^2. It is floating in liquid of density ρ of 1.0×10^3 kg m^{-3}. Show that the frequency of oscillation of the tube is 1.8 Hz.

d) The tube in (b) is now placed in a different liquid. The tube oscillates vertically. The variation with time *t* of the vertical displacement *x* of the tube is shown below.



i) Use the figure to

1. determine the frequency of oscillation of the tube,
2. calculate the density of the liquid.

ii) 1. Suggest two reasons why the amplitude of the oscillation decreases with time.

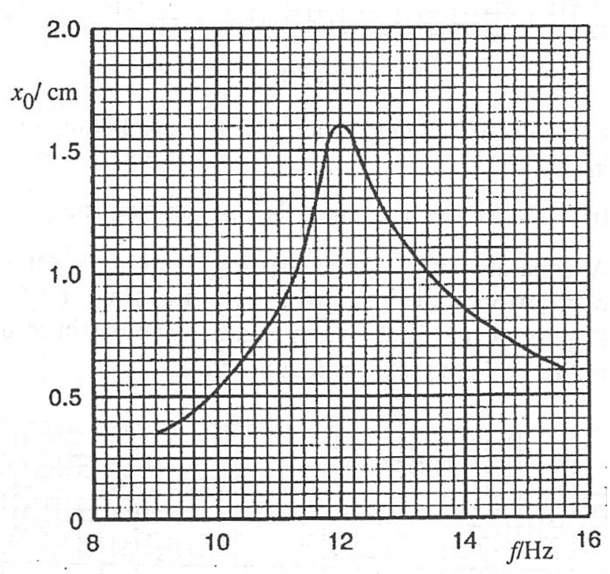
2. Calculate the decrease in energy of the oscillation during the first 1.0 s.

D12 N09/III/1c

In normal use, a loudspeaker produces a range of frequencies of sound. Suggest why it is important that the natural frequency of vibration of the cone of the loudspeaker is not within this range of frequencies.

D13 N05/II/4 (modified)

The figure shows the variation with frequency *f* of amplitude x_0 of the forced oscillations of a machine.



(a) State
(i) what is meant by a forced oscillation. [1]
(ii) the name of the effect illustrated in the figure. [1]

(b) At any value of frequency, the oscillations of the machine are simple harmonic.
(i) Calculate for the machine vibrating at maximum amplitude, the maximum magnitudes of
1. the linear speed [3]
2. the linear acceleration [2]
(ii) Determine the time interval between maximum linear speed and subsequent maximum linear acceleration [2]

(c) The mass of the machine is increased. Suggest what effect this increase mass will have on the shape of the the figure. You may draw in the figure if you wish. [3]

D14 N2010/III/6

a) Define force.
b) A light helical spring is suspended vertically from a fixed point, as shown in the figure on the right.

Different masses are suspended from the spring. The weight *W* of the mass and the length *L* of the spring are noted. The variation with weight *W* of the length *L* is shown in the figure below.

(i) On the figure, show clearly the area of the graph that represents the energy stored in the spring when the weight on the spring is increased from zero to 5.0 N.
(ii) For a spring undergoing an elastic change, the force per unit extension of the spring is known as the force constant *k.* Show that the energy *E* stored in the spring for an extension *x* of the spring is given by the expression

$$E = \frac{1}{2}kx^2$$

c) A mass of weight 4.0 N is suspended from the spring in (b). When the mass is stationary, it is then pulled downwards a distance of 0.80cm and held stationary.

(i) Determine the total length of the spring.

(ii) For an increase in extension of 0.80 cm, determine the magnitude of the change in
1. the gravitational potential energy of the mass,
2. the elastic potential energy of the spring.

(iii) Use your answers in (ii) to show that the work done to cause the additional extension of 0.80 cm is 4.0×10^{-3} J.

d) The mass in (c) is now released. The mass performs simple harmonic motion.
(i) State the total energy of oscillation of the mass.

(ii) Calculate, for the mass,
1. Its maximum speed,
2. the frequency of oscillation.

e) The spring in d) is assumed to be light. In practice, the spring will have some mass. Assuming that the spring constant *k* is unchanged, suggest and explain the effect on the frequency of oscillation of having a spring with mass.

Numerical Solutions

S1	C
S2	8.0 Hz
S3	(i)(a) 20.0 cm; (i)(b) 0.4000 s; (i)(c) 2.500 Hz; (i)(d) 15.71 rad s^{-1}; (i)(e) -10.0 cm (iii) 0.233 s; (v)(a) 0; (v)(b) 3.14 m s^{-1}; (v)(c) 49.3 m s^{-2}; (v)(d) 0
S4	v_{max} = 0.10 m s^{-1} at equilibrium position; a_{max} = 1.3 m s^{-2} at x = - 8.0 mm
S5	(i)(a) Point B; (i)(b) Point C; (ii) PE_C = 0, KE_C = 4.93 J, PE_B = 4.93 J, KE_B = 0
S6	C
S7	A
S8	A
D2	0.17 s
D5	2.0 h
D6	(b)(ii) 82 rad s^{-1}, 1.47 mm
D8	1.8 s
D9	(b)(ii) 1.7 x 10^{-3} J
D10	(b)(i) 0.785 J; (b)(ii) 0.392 J; (d)(i) 3.92 N; (d)(ii) 7.00 rad s^{-1}; (d)(iii) 1.40 m s^{-1}
D11	(d)(i)(1) 2.0 Hz, (2) 1200 kg m^{-3}, (d)(ii)(2) 3.86 x 10^{-4} J
D13	(b)(i)(1) 1.21 m s^{-1}, (2) 91.0 m s^{-2}; (b)(ii) 0.0208 s
D14	(c)(i) 0.10 m, (ii)(1) 0.032 J, (2) 0.036 J, (iii) 0.004 J, (d)(i) 0.004 J, (d)(ii)(1) 0.14m s^{-1}, (2) 2.8 Hz