Chapter 2

Kinematics

Carl Lewis flying through the air during the long jump finals at the 1992 Olympic Games in Barcelona, Spain.

(AFP/ Eric Feferberg)

"The most important factor for the distance travelled by an object is its velocity at takeoff - both the speed and angle. Elite jumpers usually leave the ground at an angle of twenty degrees or less; therefore, it is more beneficial for a jumper to focus on the speed component of the jump. The greater the speed at takeoff, the longer the trajectory of the centre of mass will be. The importance of a takeoff speed is a factor in the success of sprinters in this event."

(http://en.wikipedia.org/wiki/Long_jump)

H2 Physics Syllabus 9646

Topic 2: Kinematics

Kinematics	**Learning Outcomes** Students should be able to:
Rectilinear motion	(a) define and use displacement, speed, velocity and acceleration. (b) use graphical methods to represent distance, displacement, speed, velocity and acceleration. (c) identify and use the physical quantities from the gradients of displacement-time graphs and areas under and gradients of velocity-time graphs, including cases of non-uniform acceleration. (d) derive, from the definitions of velocity and acceleration, equations which represent uniformly accelerated motion in a straight line. (e) solve problems using equations which represent uniformly accelerated motion in a straight line, including the motion of bodies falling in a uniform gravitational field without air resistance. (f) describe qualitatively the motion of bodies falling in a uniform gravitational field with air resistance.
Projectile motion	(g) describe and explain motion due to a uniform velocity in one direction and a uniform acceleration in a perpendicular direction.

2.0 Introduction

Kinematics (from Greek κινεῖν, *kinein*, to move) is the branch of mechanics that describes the motion of objects without consideration of the causes leading to the motion. The study of why objects move falls under the branch of *Dynamics*.

2.1 Terminology

2.1.1 Distance vs. Displacement

Distance, *x*

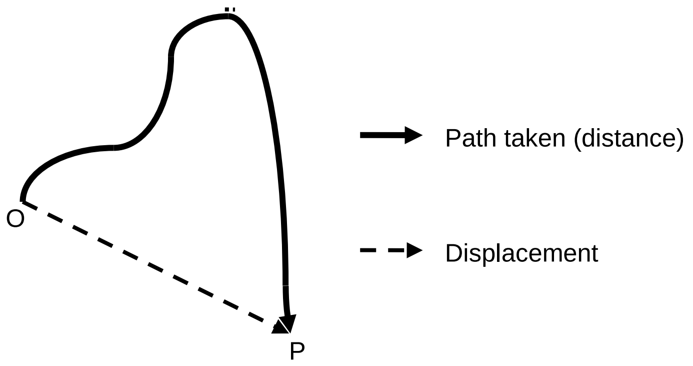
Distance is the total length travelled by a moving object irrespective of the direction of motion.

Displacement, s

The **displacement** of an object from a reference point O is the **linear distance** and **direction** of the object from O.

Note that distance is a **scalar** quantity whereas displacement is a **vector**[i] quantity.

For illustration, in the diagram below, an object travels from point O to point P along the path represented by the solid line. The distance travelled is represented by the length of the solid line. The displacement is represented by the **length and direction** of the dashed line.



2.1.2 Speed vs. Velocity

Speed, *v*

The speed of an object is defined as the **rate of change of distance** travelled by an object with respect to **time**.

Velocity, v

The velocity of an object is its **rate of change of displacement** with respect to **time.**

Note that speed is a **scalar** quantity whereas velocity is a **vector** quantity.

---

[i] Generally, scalar quantities in textbooks are italicised and non-bold, e.g. *x*, while vector quantities in textbooks are either represented by an arrow above the quantity, a twiddle (~) below the quantity or italicised and bold, e.g. $\vec{A}$, $\underset{\sim}{a}$, or simply ***A***.

In past examinations, students were given zero marks for defining speed as distance travelled in one second. Quantities should always be defined in terms of other quantities and units should always be defined in terms of other units.

Mathematically, average and instantaneous speed[ii] and velocity can be calculated as follow:

$$\text{Average speed, } \langle v \rangle = \frac{\text{Total distance travelled}}{\text{Total time taken}} = \frac{\Delta x}{\Delta t}$$

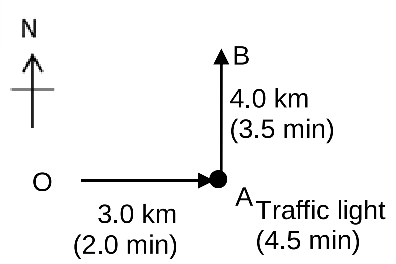
$$\text{Instantaneous speed, } v = \frac{dx}{dt}$$

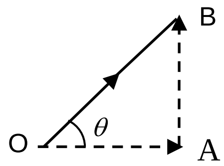
$$\text{Average velocity, } \langle \boldsymbol{v} \rangle = \frac{\text{Total change in displacement}}{\text{Total time taken}} = \frac{\Delta \boldsymbol{s}}{\Delta t}$$

$$\text{Instantaneous velocity, } \boldsymbol{v} = \frac{d\boldsymbol{s}}{dt}$$

Example 1

A car travels 3.0 km due east from a point O to a point A for 2.0 minutes where it stops at a traffic light at A for 4.5 minutes. It then continues 4.0 km due north of A to point B for another 3.5 minutes. Find the average speed and average velocity of the car.





$$\text{Average speed} = \frac{\text{total distance travelled}}{\text{time taken}} = \frac{3.0+4.0}{2.0+4.5+3.5} = 0.70 \text{ km min}^{-1} = 12 \text{ m s}^{-1}$$

$$\text{Average velocity} = \frac{\text{total change in displacement OB}}{\text{time taken}} = \frac{\sqrt{3.0^2+4.0^2}}{2.0+4.5+3.5} = 0.50 \text{ km min}^{-1} = 8.3 \text{ m s}^{-1}$$

$$\tan\theta = \frac{4}{3}$$

$$\theta = 53^{\circ}$$

Note that as velocity is a *vector* quantity, it is necessary to find the direction for average velocity.

[ii] It is rare that the speed of an object is constant. When we refer to the speed of the object, we are usually referring to the **instantaneous speed**. Similarly when we refer to *velocity* and *acceleration*, it is the *instantaneous velocity* and *instantaneous acceleration* that is referred to, unless stated otherwise.

Example 2

A particle moves with an initial velocity of 10 m s^{-1} in the *x*-direction for 5.0 s. It then changes its velocity to 30 m s^{-1} at an angle of 60^o to the horizontal and moves for a further 2.0 s. (Figure 2.1)

(i) Calculate the change in speed and velocity of the particle in this interval.

(ii) Find the average speed and average velocity of the particle.

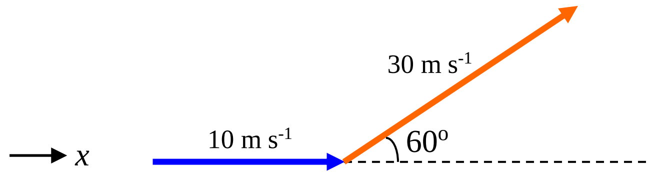


Figure 2.1

Solution

(i) Change in speed = final speed – initial speed

= (30 – 10) m s^{-1}

= 20 m s^{-1}

Change in velocity = $v_{\text{final}} - v_{\text{initial}} = v_{\text{final}} + (-v_{\text{initial}})$

Magnitude of the change in velocity

Using cosine rule,

$\Delta v = \sqrt{(10^2 + 30^2 - 2(10)(30)\cos 60^o)}$

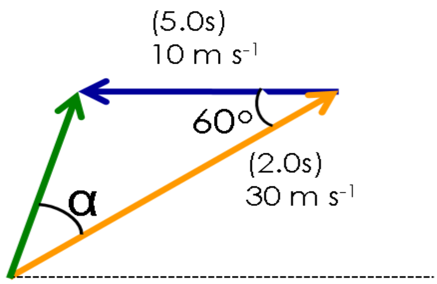
= 26.5 m s^{-1}

Direction of the change in velocity

Using sine rule,

$$\frac{\sin\alpha}{10} = \frac{\sin 60^o}{26.5}$$

$\alpha = 19.1^o$

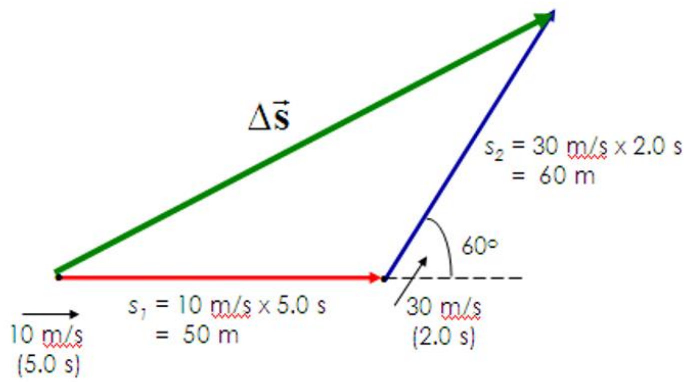


The change in the velocity is **26.5 m s^{-1}** at an angle of **79.1^o** anticlockwise from the horizontal

(ii) Average speed $= \dfrac{\text{total distance travelled}}{\text{total time taken}}$

$= \dfrac{(10 \times 5.0) + (30 \times 2.0)}{(5.0 + 2.0)}$

= 15.7 m s^{-1}



$$<\vec{v}> = \frac{\Delta\vec{s}}{\Delta t}$$

Using cosine rule, $\Delta s = \sqrt{(60^2 + 50^2 - 2(60)(50)\cos 120^o)}$ = 95.4 m

Magnitude of the average velocity, $|v|$= 95.4 / 7.0 = 13.6 m s^{-1}

Using sine rule, $\beta = 33.0^o$

The average velocity is **13.6** m s^{-1} at **33.0^o** to the initial velocity.

2.1.3 Acceleration

Acceleration, *a*

The **acceleration** of an object is its **rate of change of velocity** with respect to time.

Mathematically, average and instantaneous acceleration can be calculated as follow:

Average acceleration, $\langle \boldsymbol{a} \rangle = \dfrac{\text{Total change in velocity}}{\text{Total time taken}} = \dfrac{\Delta \boldsymbol{v}}{\Delta t}$

$= \dfrac{\boldsymbol{v}_{final} - \boldsymbol{v}_{initial}}{\Delta t}$

Instantaneous acceleration, $\boldsymbol{a} = \dfrac{d\boldsymbol{v}}{dt}$

- S. I. units of acceleration: m s^{-2}
- Acceleration is a vector quantity. The direction of acceleration is in the same direction as the vector representing the **change** in velocity[iii].

TIPS **Sign Convention**

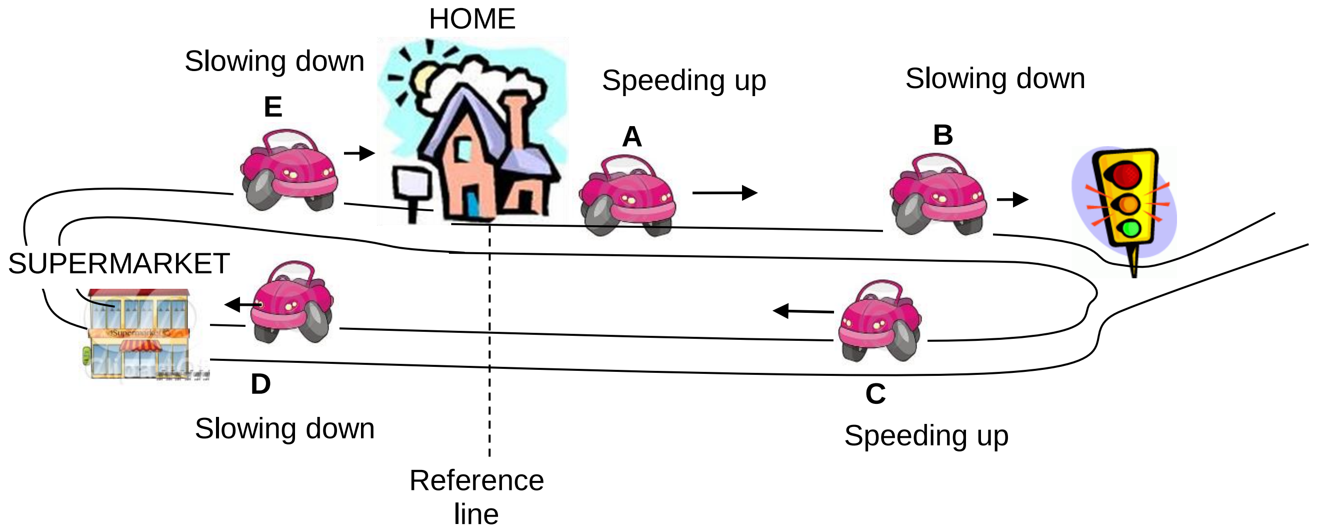
1. For an object moving in one-dimension (rectilinear), there are only two possible directions in which the object can move. One direction is opposite the other. Hence, for convenience, we define one direction as positive and the other negative.
2. For vertical motion, we usually define upward direction as positive*: +ve $\uparrow$
3. For horizontal motion, we usually take direction to the right as positive*: $\longrightarrow$ +ve

*But this depends on the problem given, if it is more convenient to define downwards as positive or leftwards as positive, you may do so. In any case, **you should always state clearly the convention you have adopted.**

[iii] You will learn in Dynamics that the direction of any object's acceleration is determined by the direction of the resultant force acting on the object (Newton's 2nd Law of Motion).

Example 3

John drove a car from his home towards the traffic junction. As the traffic light turned red, he stopped and waited for his turn. He then made a U-turn and drove to a supermarket to buy some groceries. John drove back to his home after making the stop at the supermarket. The journey is illustrated in the figure below, where points A, B, C, D and E are located along his journey.



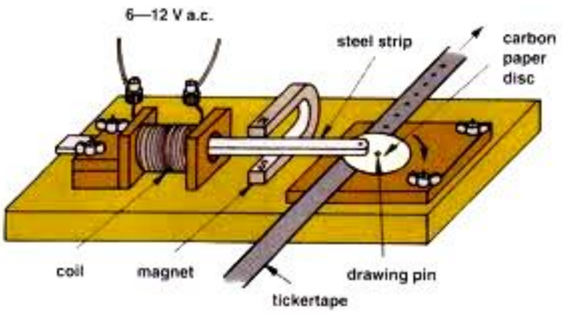
The reference line passes through John's home, and displacement to the right of it is defined as positive. Indicate the sign for John's displacement, velocity and acceleration at A, B, C, D and E.

	Displacement	Velocity	Acceleration
A (speeding up)			
B (slowing down)			
C (speeding up)			
D (slowing down)			
E (slowing down)			

2.2 Describing Motion using Diagrams

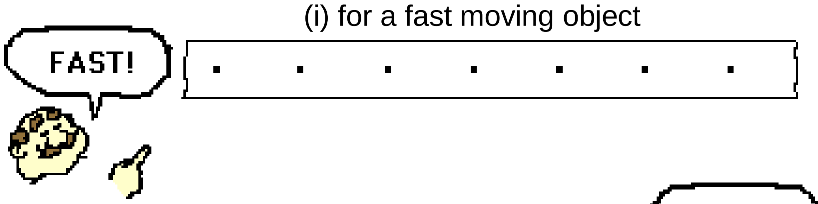
2.2.1 Ticker tape diagrams

A common way of analyzing the motion of objects in physics labs is to perform a ticker tape analysis. A long tape is attached to a moving object and threaded through a device that places a tick upon the tape at regular intervals of time – say every 0.1 second. As the object moves, it drags the tape through the "ticker," thus leaving a trail of dots. The trail of dots provides a history of the object's motion and is therefore a representation of the object's motion.

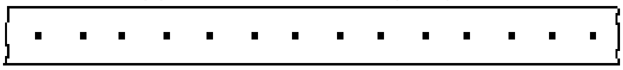


The **distance between dots** on a ticker tape represents the object's position change during that time interval.

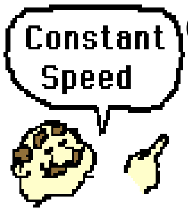
(i) for a fast moving object



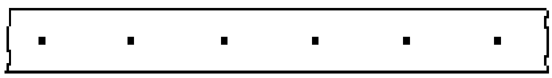
(ii) for a slow moving object

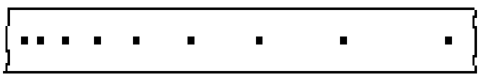


The analysis of a ticker tape diagram will also reveal if the object is moving with a constant velocity or with a changing velocity (accelerating).



(iii) for an object moving at constant speed

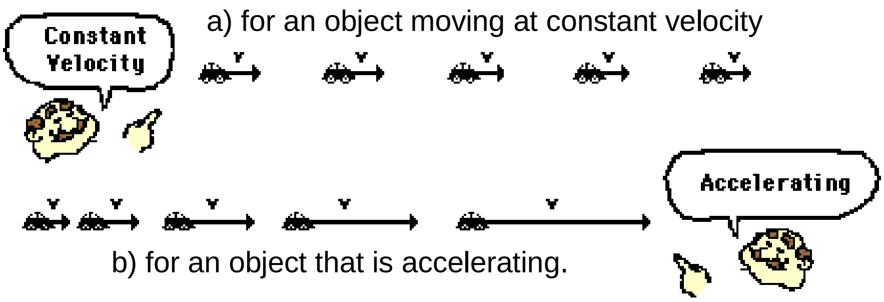




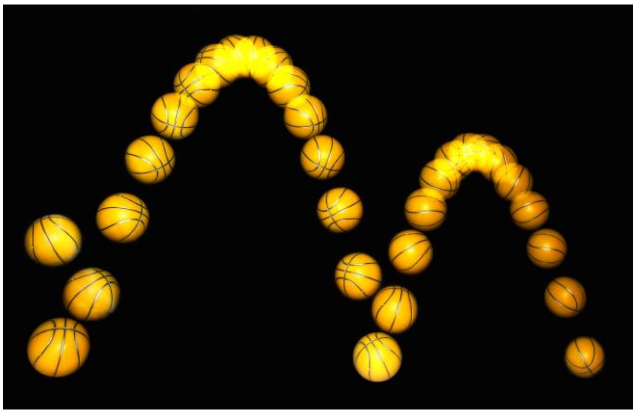
(iv) for an object that is accelerating

2.2.2 Vector diagrams

Vector diagrams use vector arrows to depict the direction and relative magnitude of a vector quantity. Vector diagrams can be used to describe the velocity of a moving object during its motion. For example, the velocity of a car moving down the road could be represented by a vector diagram. In a vector diagram, the **magnitude** of the vector is represented by the length of the **vector** arrow.



2.2.3 Recording motion using stroboscope[iv]



Multi-flash photo of a bouncing basketball

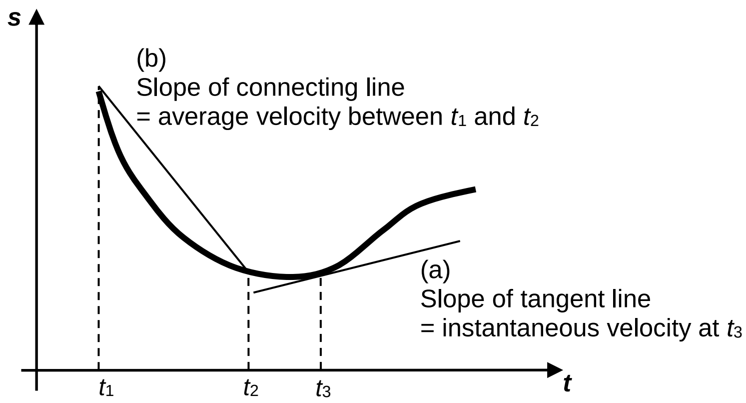
Multi-flash photographs or stroboscopic photographs are photographs of a subject illuminated with a flashing light, taken by a camera with an open shutter. By keeping the subject in the dark and then illuminating the subject only at certain times, the resultant image will show a series of different positions of the subject.

Today, by using digital video camera and digital image compositing, a multi-flash photograph can be easily re-constructed from several video frames. The video camera enables us to record motion conveniently, even at high speed. A single video camera captures 2-dimensional motion on a plane parallel to the image plane of the camera. To record complex 3-dimensional motion, many video cameras are used. This technique is useful for sports biomechanical analysis and character analysis in animated film production.

[iv] http://edgerton-digital-collections.org/techniques/multiflash

2.3 Describing Motion using Graphs

2.3.1 Displacement vs. Time Graph (*s*-*t* graph)

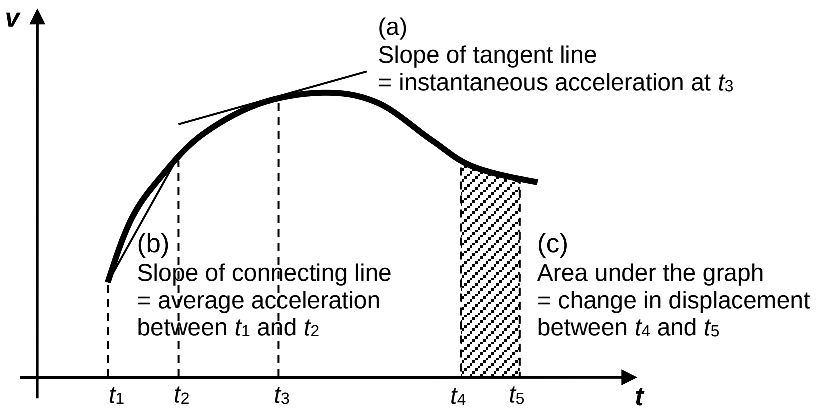


On an ***s*-*t*** graph,

The **instantaneous velocity** is the **slope** (or **gradient**) of the **tangent line** at a given **instant** of time.

- Average velocity is the **slope** of the **straight line connecting two points** corresponding to a given time **interval**.
- If an object moves with **constant** velocity, its ***s***-*t* graph will be a **straight** line graph, and the instantaneous velocity at any instant is equal to the average velocity over the entire time interval.

2.3.2 Velocity vs. Time Graph (*v*-*t* graph)



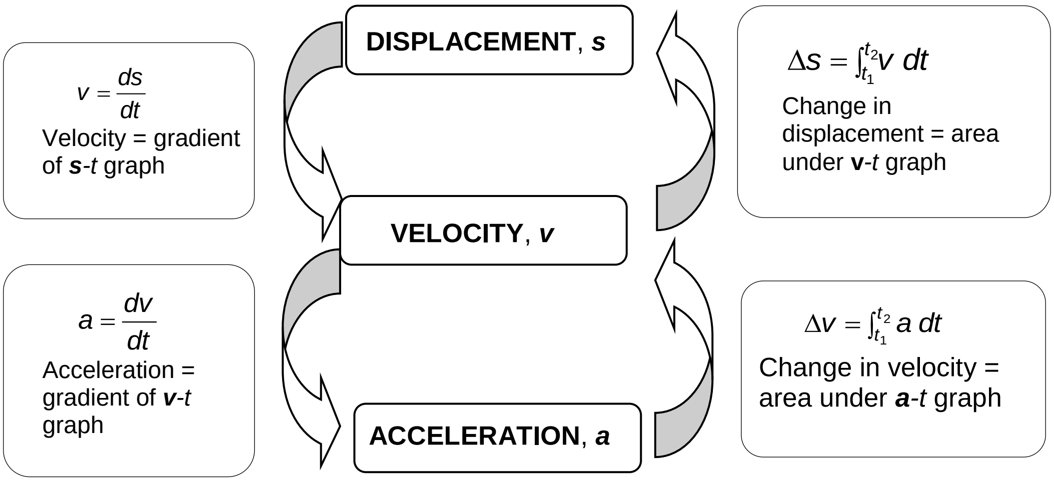
On an ***v*-*t*** graph,

The **instantaneous acceleration** is the **slope** (or **gradient)** of the **tangent line** at a given instant of time.

- Average acceleration is the **slope** of the **straight line connecting two points** corresponding to a given time **interval**.
- If an object moves with **constant** acceleration, its **v**-*t* graph will be a **straight** line graph, and the instantaneous acceleration at any instant is equal to the average acceleration over the entire time interval.

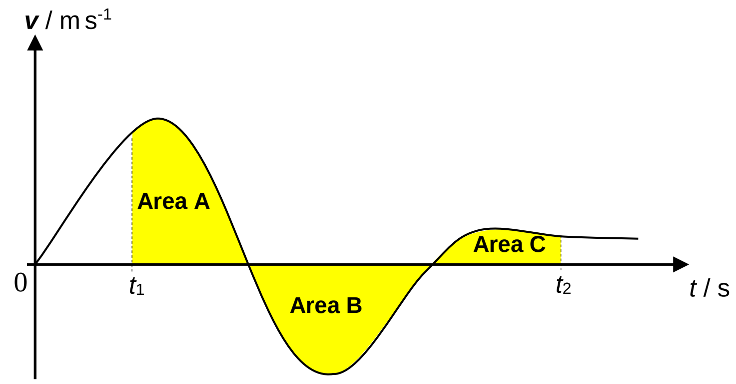
The **area under a *v*-*t* graph** for a time interval of Δt gives the **change in displacement** of the object in that time interval.

2.3.3 Summary of relations among *s*-*t*, *v*-*t* and *a*-*t* graphs



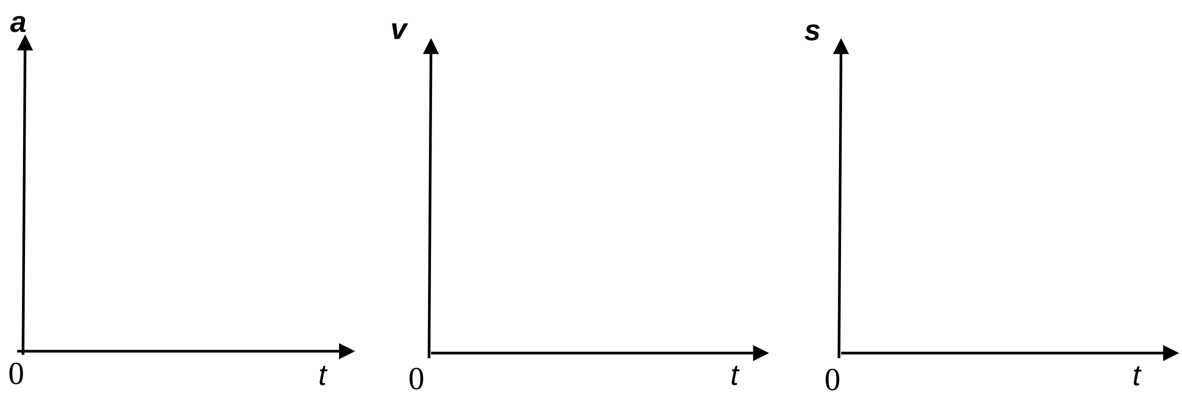
Example 4

What is the expression of the change in displacement, $\Delta\boldsymbol{s}$, of the object from t_1 to t_2?
If you need to find distance travelled by the object in the same time interval, what would be your answer?



Example 5

An object is at rest at the origin. If the object undergoes constant acceleration, sketch its ***a***-*t*, ***v***-*t* and ***s***-*t* graphs.

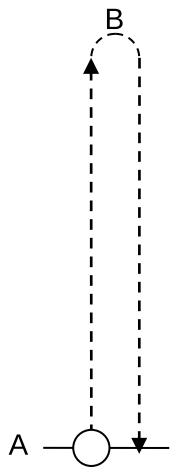


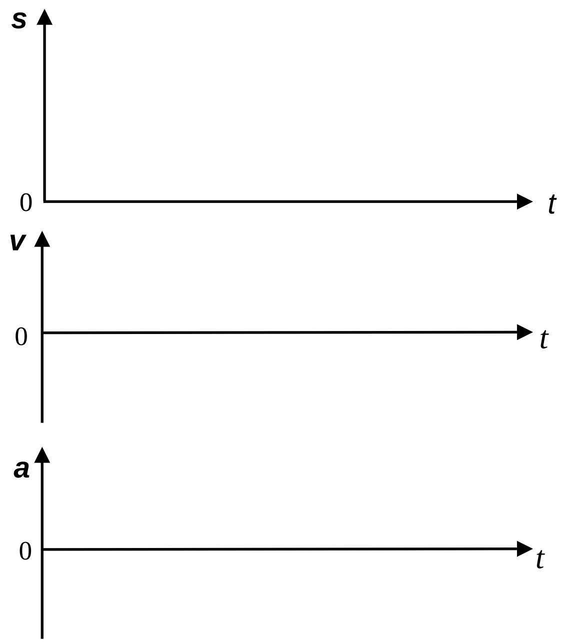
Example 6

A student throws a ball vertically upwards at a speed of 20 m s^{-1} from level A. Neglect air resistance.

i) Sketch the ***s*** - *t*, **v** - *t* and ***a*** - *t* graphs from the moment the ball is thrown to the moment it returns to the student's hand.

ii) Calculate the time it takes for the ball to return to the student's hand.



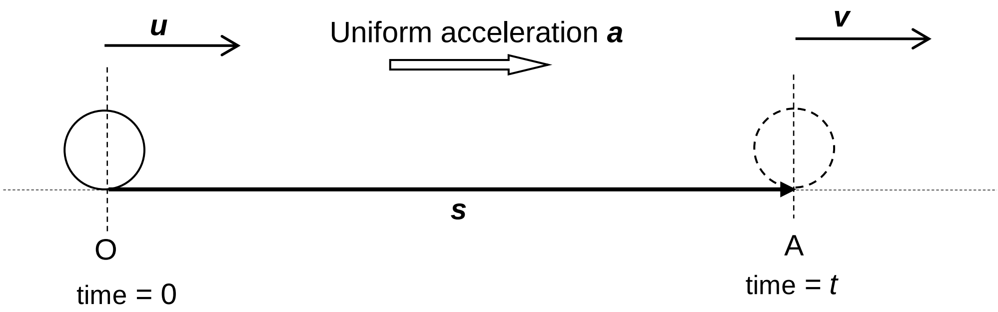


2.4 Describing Motion using Equations (for uniform acceleration)

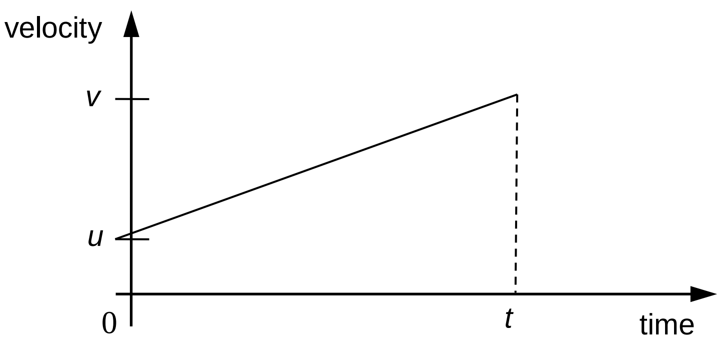
2.4.1 Derivation of the Equations of Motion (for uniform acceleration)

In this section, we shall derive the equations of motion for an object moving in a straight line with **uniform acceleration**.

Suppose at time $= 0$, the object is at point O with an initial velocity of $\boldsymbol{u}$. The object moves at uniform acceleration of $\boldsymbol{a}$ until time $= t$, and has travelled through a displacement of $\boldsymbol{s}$ to arrive at point A with the final velocity of $\boldsymbol{v}$.



Let us look at its velocity-time graph:



Note: acceleration of object, a is given by the gradient of its $\boldsymbol{v}$ - t graph,

$$a = \frac{v - u}{t}$$

$$\boxed{v = u + a\,t} \qquad (1)$$

Note: The displacement of object from point O, s, is given by the area under its $\boldsymbol{v}$ - t graph from time = 0 to time = t,

$$s = \frac{1}{2}(u + v)\,t \qquad (2)$$

Substitute Eqn (1) into (2), we have

$$s = \frac{1}{2}[u + (u + a\,t)]\,t$$

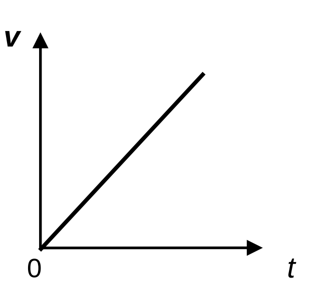
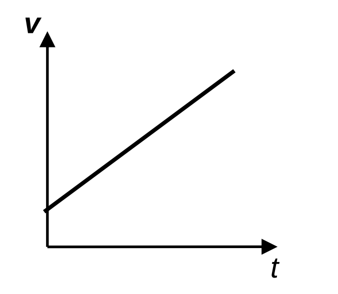
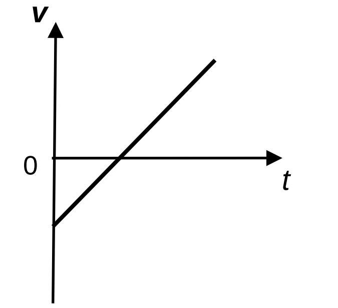
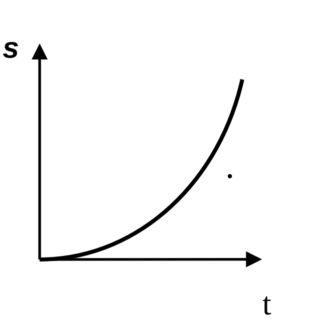
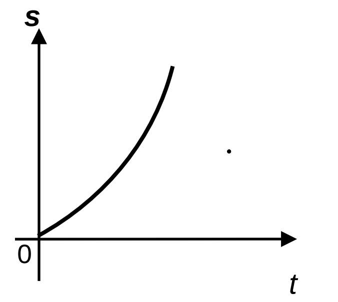
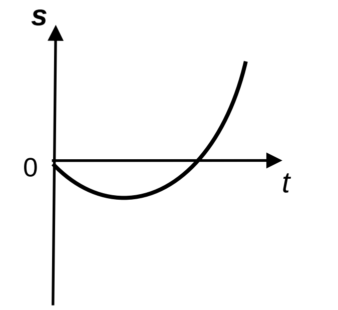
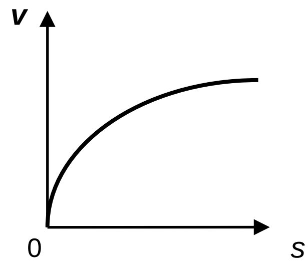
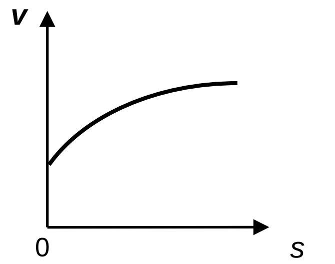
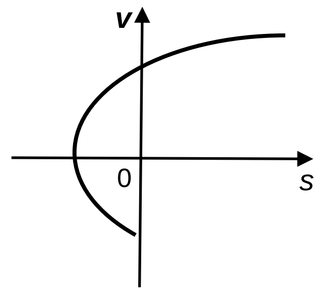
$$\Rightarrow \quad \boxed{s = u\,t + \frac{1}{2}a\,t^2} \qquad (3)$$

From Eqn (1), we have $t = \dfrac{v - u}{a}$ and substitute into Eqn (2), we have

$$s = \frac{1}{2}(u + v)\left(\frac{v - u}{a}\right)$$

$$\Rightarrow \quad \boxed{v^2 = u^2 + 2\,a\,s} \qquad (4)$$

In summary, the equations of motion are tabulated below.

Equations of Motion	Graphical Representation		
	Case 1 (initial velocity is zero, acceleration positive)	**Case 2 (initial velocity is positive, acceleration positive)**	**Case 3 (initial velocity is negative acceleration positive)**
$v = u + at$ Use this equation to relate velocity to time. ***v***-*t* graphs are straight line graphs.			
$s = ut + \frac{1}{2}at^2$ Use this equation to relate displacement to time. ***s***-*t* graphs are *quadratic curves* (The shape is called a *parabola*.)			
$v^2 = u^2 + 2as$ Use this equation to relate velocity to displacement. ***v***-***s*** graphs follow the shape of a **square-root** graph.			

Pause and Think: Why do you think we did not include the graph for $s = \frac{1}{2}(u+v)t$ here?

<u>Note:</u>

- The equation ($s = ut + \frac{1}{2}at^2$) sets the reference point ***s*** = 0 at *t* = 0.
- Equations of motion can only be used if **the acceleration is constant**.
- The equations are **vector equations.** Hence the motion in each dimension is independent of the other direction. Therefore, variables also have to be analysed along the same dimension. For example, if we choose to analyse the motion along the *x*-direction, then we can only use quantities like $\boldsymbol{u_x}$, $\boldsymbol{v_x}$ and $\boldsymbol{a_x}$. Acceleration along the *y*-direction $\boldsymbol{a_y}$ will not affect motion along the *x*-direction.

<u>Example 7:</u> **J82/P1/1**

A motorist traveling at 13 m s^{-1} approaches traffic lights, which turn red when he is 25 m away from the stop line. His reaction time is 0.70 s. If he brakes fully such that the car slows down at a rate of 4.5 m s^{-2},

a) how far from the stop line will he stop?

b) on which side of the stop line will he come to a rest?

2.5 Free Falling Bodies

The simplest example of motion with constant acceleration is a free falling body.

A free falling object is any object moving only under the influence of gravity, *i.e.* the effects of resistive forces (i.e. drag forces) and upthrust are negligible. For free falling objects, whether they are initially thrown upward, downward or released from rest, they will accelerate downwards at a rate of **9.81 m s^{-2}**,[v] regardless of the objects' weight, size and shape. Let us illustrate this below.

Consider a mass *m* under free falling. Since air resistance is negligible, the only force acting on the mass is its own weight. Applying Newton's 2nd Law,

$$\begin{aligned} F_{net} &= ma \\ mg &= ma \\ a &= g \end{aligned}$$

In essence, even though a larger mass experiences a larger gravitational force, its larger inertia also makes it harder to accelerate. These two effects cancel out cleanly. This curious coincidence causes a feather to fall at the same rate as a lead ball.

[v] You will learn in the topic of Gravitation that the value of *g* does vary with location on earth. The average value of 9.81 m s^{-2} is conveniently chosen for general calculations.

<u>Example 8:</u> Are these Objects in Free-Fall?

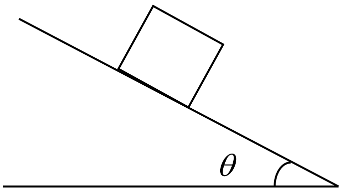
The following are 2 cases of objects in motion.

(i) Determine if each of the objects is in free fall.

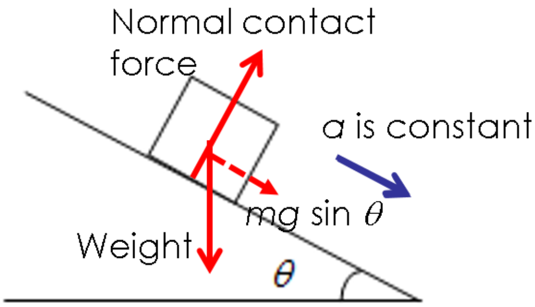
(ii) For each case, if the object is not in free fall, identify the physical quantities that affect the acceleration.

(iii) Hence, explain if the equations of motion can be applied to each scenario.

Case 1: A box sliding down a smooth incline. Assume that air resistance is negligible.



Solution:
Considering the forces on the box,



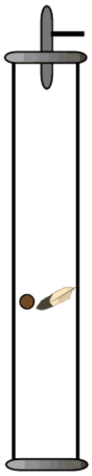
(i) Not in free fall

(ii) Net force: Component of weight of box parallel to the slope
Acceleration = $g \sin \theta$
Factors affecting acceleration : g and θ

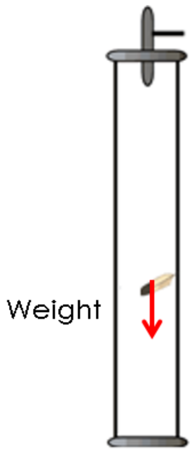
(iii) Since normal contact force and weight are constant, the resultant force acting on the box is constant. Hence, acceleration is constant and equation of motion can be applied in case 1.

Case 2: A feather falling in vacuum.

Watch this:



Solution:
Considering the force on the feather,



(i) In free fall

(ii) Net force: Only weight of feather, W
($W = m_{feather}\, g$)

(iii) $a = g$,
Hence, acceleration is constant and equation of motion can be applied in case 1.

For both cases, can you visualise the acceleration-time and velocity-time graphs?

2.6 Viscous Drag Force

In fluid dynamics, drag (sometimes called air resistance or fluid resistance) refers to forces that oppose the relative motion of an object through a fluid (a liquid or gas). When a body moves through a fluid, it has to push the surrounding fluid molecules away in order to move. Hence it exerts a force on the fluid molecules and by Newton's Third Law, the molecules exert an equal and opposite force back on the body. Hence there is resistance to motion.

It follows that drag force depends on the **velocity** of the body, *i.e*. the faster the motion of the body through the fluid, the larger the resistance. It also means that if the body is at rest relative to the fluid, there is no resistance.

Other factors affecting the magnitude of drag force include the **shape** and **dimension** of the body (a more streamline shape allows a body to experience less drag as it pushes through the fluid) and the type of fluid (**viscosity** of the fluid).

When there is no turbulent flow (*e.g.* small and streamline objects traveling at low speeds) the drag force is proportional to the speed of the object. The magnitude of drag force can be stated empirically as follows:

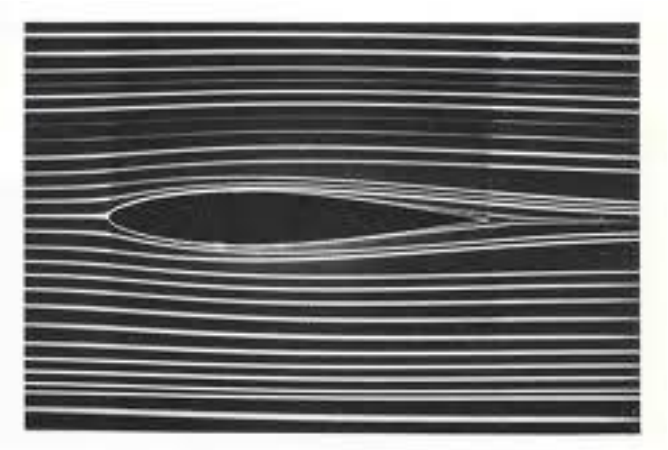
$$F_D = kv$$

F_D : magnitude of drag force
k : a constant that depends on the dimensions of the body and the type of fluid
v : the velocity of the body relative to the fluid

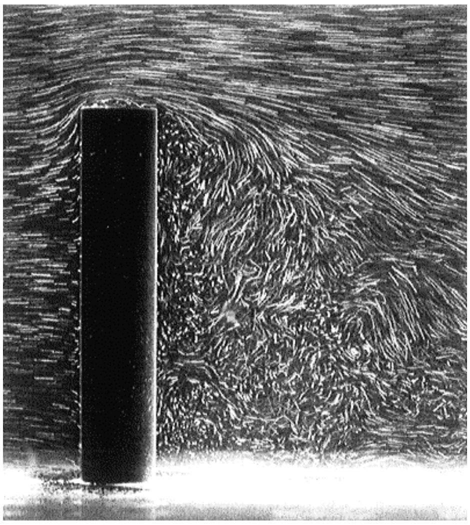
If the fluid flow is turbulent (*e.g.* objects moving at high speeds through air, such as aircraft, skydiver and baseball), the drag force is approximately proportional to the square of the speed. The drag force is then given by:

$$F_D = kv^2$$

F_D : magnitude of drag force
k : a constant that depends on the dimensions of the body and the type of fluid
v : the velocity of the body relative to the fluid



Laminar flow over a streamline body



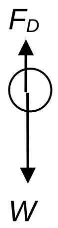
Turbulent flow results in higher drag

2.7 Non Free Falling Bodies

In practice, free fall does not happen because of resistive forces. The effects on a diver or bungee jumper may not be significant, but this is clearly not the case for a parachutist where air resistance is (deliberately) used to limit the maximum downward velocity of the parachutist.

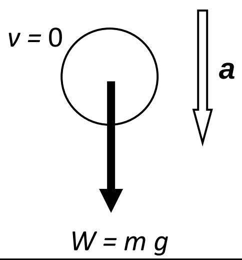
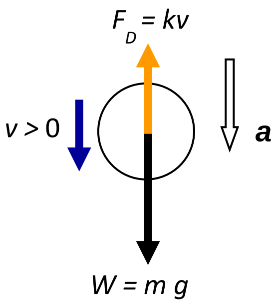
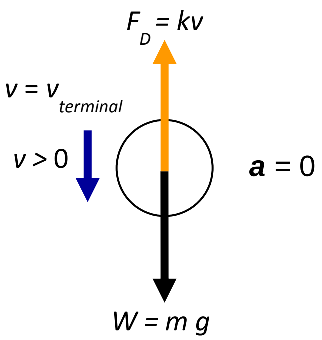
2.7.1 Velocity-time graph of object falling through air

Let us now consider the motion of an object, dropped from rest, falling through air. If air resistance is not negligible, the forces acting on the object are:

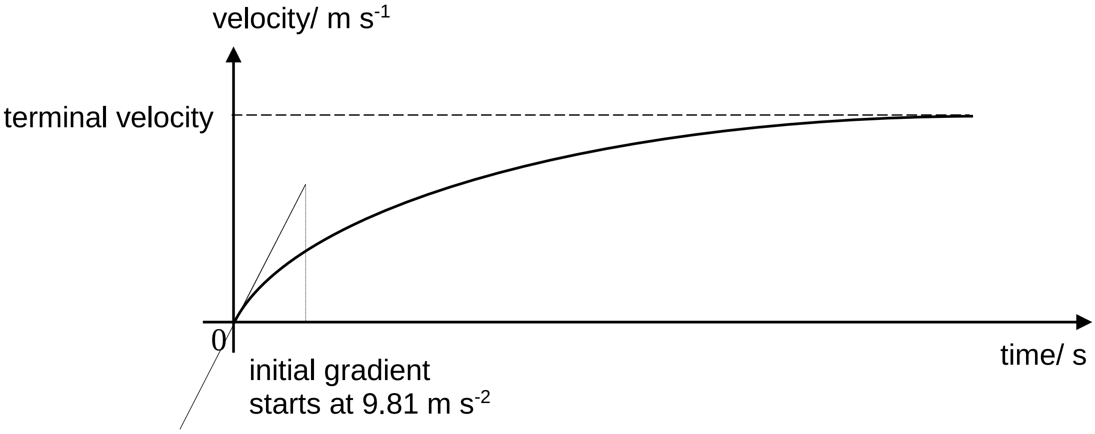


W : Weight of object
F_D : Drag force

For ease of analysis, break the motion up into stages as seen below. Let us assume that drag force is proportional to speed.

Time	Forces acting on the body	Motion of the object
$t = 0$ s		• Object is initially at rest. • The only force acting on the object is the gravitational force (weight, $W = m\,g$). *By Newton's 2nd Law,* $(\downarrow)$ $\Sigma F = ma \Rightarrow mg = ma \Rightarrow a = g$ • The body begins to fall with an acceleration of $\boldsymbol{a = g}$.
$t = t_1$		• As the body gains velocity, air resistance which is dependent on the velocity also increases. *By Newton's 2nd Law* $(\downarrow)$ $\Sigma F = ma \Rightarrow mg - kv = ma \Rightarrow a = g - \frac{kv}{m} < g$ • As F_D increases, the net force decreases and hence the acceleration decreases and falls below $\boldsymbol{g}$. • The object is nevertheless still accelerating and hence the speed still increases (just at a decreasing rate), so the air resistance continues to increase.
$t = t_2$		• Eventually the air resistance increases to a point where it balances the weight of the body. i.e. $W = F_D$ *By Newton's 2nd Law* $(\downarrow)$ $\Sigma F = ma \Rightarrow mg - kv = ma = 0 \Rightarrow a = 0$ • Resultant force acting on the body is zero. Hence acceleration of the body is **zero**. • The object then travels at a constant velocity called the **terminal velocity**.

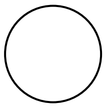
The following graph shows the non-linear velocity change with time for a falling object experiencing air resistance.



2.7.2 Terminal Velocity

We can now explain why heavier objects fall at a faster rate than light ones in the presence of air resistance. For simplicity, let us assume air resistance is proportional to speed *v, i.e.* $F_D = kv$.

Consider the free body diagram of an object falling at terminal velocity,



At terminal velocity, the acceleration is zero and so the net force is zero. Hence,

$$F_D = W$$

Assuming air resistance is proportional to speed, therefore $F_D = kv_t$

$$kv_t = mg$$
$$v_t = \frac{mg}{k}$$

Based on this equation, we can now conclude:

1. Between two objects of the same dimensions but different densities, the denser one reaches a higher terminal velocity because it has a larger *m* but the same *k*.

2. Between two objects of the same mass but different shapes, the more streamline one reaches a higher terminal velocity because it has a smaller *k* but the same *m*.

Example 9

A steel ball bearing is released from a height above a deep pool of oil. Taking the drag force between the ball bearing and oil to be proportional to the relative speed of the ball bearing in oil, sketch the graph to show the variation of the velocity ***v*** of the ball bearing against time *t*. Indicate clearly on your diagram, t_1, the time when the ball bearing hits the oil.

Solution

Assume that upthrust is negligible.

Stage 1: Released from rest ($t = 0$ *to* t_1)

Before the ball bearing enters into the oil,

- Only the weight acts on it and drag force (due to air) is very small compared to weight.

By Newton's 2nd Law,

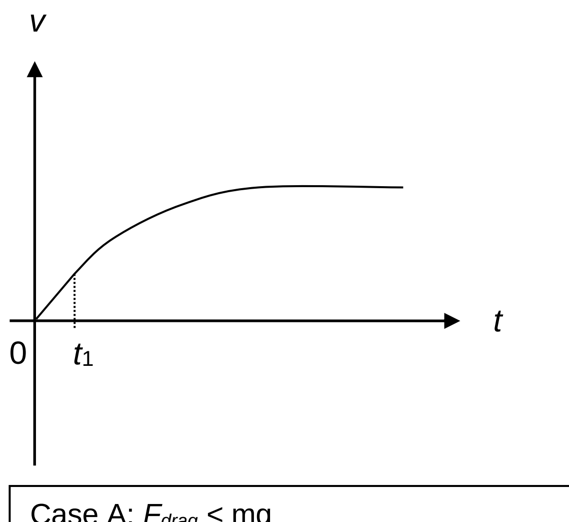
(↓) $\Sigma F = ma$

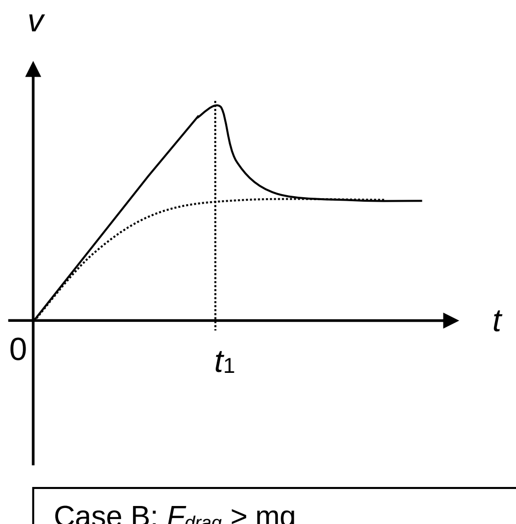
$\Rightarrow$ $mg = ma$

$\Rightarrow$ $a = g$

Stage 2 : Enters the oil

- Enters with some initial velocity
- Drag force due to oil resistance is potentially large compared to the weight, leading to two possible cases:
  Case A: Ball bearing is released at/just above the surface of the oil, entering the oil at low velocity.
  Case B: Ball bearing is released high above the surface of the oil, entering the oil at high velocity.





Case A: $F_{drag} < mg$
- ➢ Resultant force is downwards
- ➢ The ball bearing accelerates but at a value less than g.
- ➢ As v increases, F_{drag} increases until $F_{drag} = mg$
- ➢ Net force = 0
- ➢ Terminal velocity is reached

Case B: $F_{drag} > mg$
- ➢ Resultant force is upwards
- ➢ The ball bearing decelerates (slows down), v decreases
- ➢ As v decreases, F_{drag} decreases until $F_{drag} = mg$
- ➢ Net force = 0
- ➢ Terminal velocity is reached

2.8 Two-Dimensional Kinematics

In this part of the chapter, we deal with the kinematics of a particle moving in a plane, or two-dimensional motion. Some common examples of motion in a plane are the motion of projectiles, motion of satellites and the motion of charged particles in an electric field. For the beginning we shall deal with the special case of motion of **an object with a uniform velocity in one direction and a uniform acceleration in a perpendicular direction.**

2.8.1 Projectile Motion

Anyone who has observed a soccer ball in motion (or, for that matter, any object thrown into air) has observed *projectile motion*.

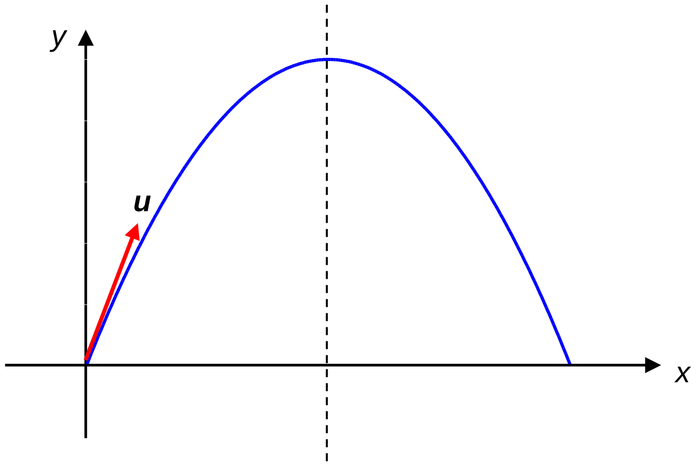
Projectile motion refers to the motion of an object launched in the air in the presence of a gravitational field. In this section, we are interested in the motion of an object projected with an initial velocity at an angle to the horizontal. To analyze this type of motion, we will start with an idealized model. The assumptions made in our idealized model are:

(1) the effect of air resistance is negligible.

(2) the acceleration due to gravity ***g*** *is constant*[vi], *acts downwards* and has a *magnitude equal to* $g = 9.81 \text{ m s}^{-2}$.

With the above assumptions, we find that the path of a projectile, which we call its ***trajectory*****, is always a parabola**[vii]**.**

Key Characteristics of a Parabolic Path:



- The path of the projectile is symmetrical about the highest point.
- The speed of the projectile is symmetrical about the highest point.
- The time it takes for the object to go from $y = 0$ to the highest point

  is equal to

  the time it takes for the object to go from the highest point to $y = 0$.

It was Galileo who first accurately described projectile motion. He showed that it could be understood by analyzing the horizontal and vertical components of the motion separately. He found that:

The horizontal and vertical motions of the projectile are independent of each other.

[vi] We shall restrict ourselves to objects whose distance travelled and maximum height above the Earth are small compared to the Earth's radius (6400 km) so that ***g*** can be considered constant.

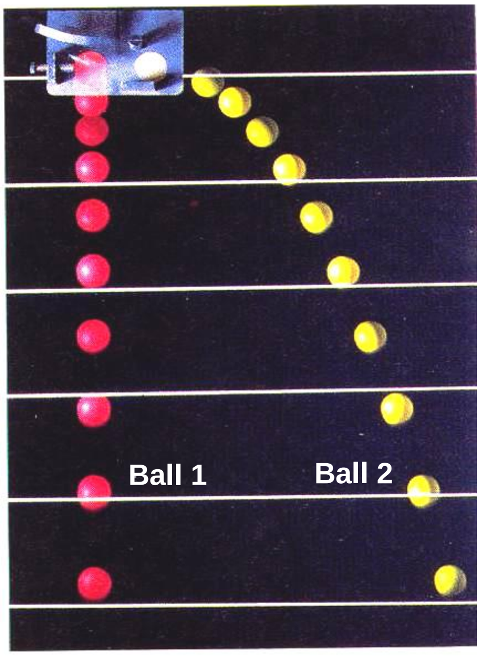
[vii] The proof that trajectory of a projectile is parabolic will be shown in Example 13.

To examine this idea further, let us examine the motion of 2 balls:

(1) Ball 1 is a ball released from rest and allowed to fall vertically through a hole on the table, and

(2) Ball 2 is allowed to roll off the end of a table with an initial velocity $\mathbf{u_x}$ in the horizontal (***x***) direction.

Both the balls leave the table at the same time. The stroboscopic photograph below depicts the motion of both balls. Let us examine the vertical and horizontal motion separately.



- **<u>Vertical Motion:</u>**

  Consider the motion of both balls in the vertical direction.

  Note,

  - The **vertical displacement of Ball 2 matches the vertical displacement of Ball 1**.
  - The **vertical displacement and vertical velocity** of Ball 2 **are only affected by the vertical acceleration *g*** (as in Ball 1).

- **<u>Horizontal Motion:</u>**

  Consider the motion of Ball 2. Note that

  - The **horizontal displacements** made by the ball **at equal intervals of time are approximately the same**.
  - The **horizontal velocity remains constant** with time.
  - The **horizontal acceleration is zero**.

    Consistent with our assumption that only the weight acts on Ball 2 and no other forces act in the horizontal direction. Hence, horizontal acceleration is zero.

  - The **vertical acceleration hence has no effect on the horizontal velocity and horizontal displacements.**

Hence, we see that horizontal and vertical motions are **independent** of each other and projectile motion can be simplified by analysing the vertical and horizontal motions separately. Another result of this analysis, which Galileo himself predicted, is that *an object projected horizontally will reach the ground in the same time as an object dropped vertically.* This is because the vertical motions are the same in both cases.

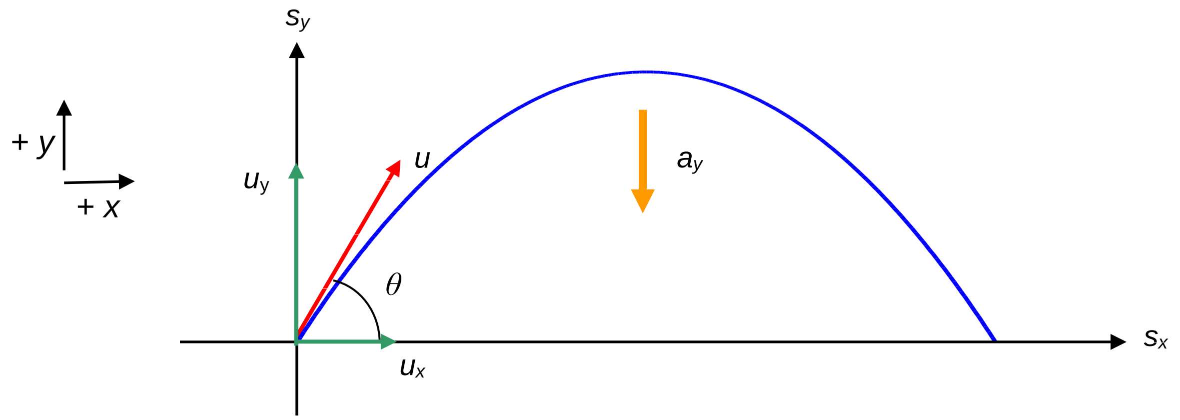
2.8.2 Projectile Motion: General Launch Angle

Let us now use our equations of motion for one-dimensional uniform acceleration motion and extend this to two-dimensional motion.

Recall our three equations of motion for one-dimensional motion:

$$v = u + a\,t$$
$$v^2 = u^2 + 2\,a\,s$$
$$s = u\,t + ½\,a\,t^2$$

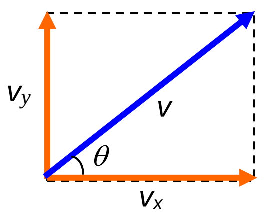
Consider an object projected above the horizontal at an angle of θ with an initial speed of u, as shown in figure below.



Analysing our projectile horizontally and vertically, and choosing our coordinate system as in the diagram, we have:

Horizontally	Vertically
$\boldsymbol{a_x} = 0$	$\boldsymbol{a_y} = -g$
$\boldsymbol{v_x} = \boldsymbol{u_x}$ $\Rightarrow \boldsymbol{v_x} = \boldsymbol{u}\cos\theta$ (1)	$\boldsymbol{v_y} = \boldsymbol{u_y} + \boldsymbol{a_y}\,t$ $\Rightarrow \boldsymbol{v_y} = \boldsymbol{u}\sin\theta - gt$ (3)
$\boldsymbol{s_x} = \boldsymbol{u_x}\,t$ $\Rightarrow \boldsymbol{s_x} = (\boldsymbol{u}\cos\theta)\,t$ (2)	$\boldsymbol{s_y} = \boldsymbol{u_y}\,t + ½\,\boldsymbol{a_y}\,t^2$ $\Rightarrow \boldsymbol{s_y} = (\boldsymbol{u}\sin\theta)\,t - ½\,gt^2$ (4)
	$v_y^2 = u_y^{\,2} + 2\,\boldsymbol{a_y}\,\boldsymbol{s_y}$ $\Rightarrow v_y^2 = u^2\sin^2\theta - 2\,g\,\boldsymbol{s_y}$

In addition, the **resultant velocity *v*** at any instant of time can be found from the horizontal and vertical component of the velocities:



Magnitude of the velocity, $v = \sqrt{v_x^2 + v_y^2}$

Direction of the velocity θ can be determined from:

$$\tan\theta = \frac{v_y}{v_x}$$

IMPORTANT TO KNOW

In exam questions, it is essential to give both the magnitude and direction for any vector quantity.

From the above equations, we can easily use them to obtain other information of the projectile, like:

- **Maximum Height:** Maximum vertical displacement with respect to the initial starting point (vertical velocity, $\boldsymbol{v_y}$ = 0 m s^{-1}).
- **Horizontal Range:** Maximum horizontal displacement with respect to the initial starting point.
- **Angle of Projection for Maximum Horizontal Range.**
- **Equation of the trajectory of the projectile.**

TIPS Solving Projectile Motion Problems:

In solving problems involving projectile motion, follow the following procedures:

1. *Read* the question carefully.
2. *Draw* a careful diagram summarising the given information.
3. Choose an *x*-*y* coordinate system.
4. Identify the accelerations in the *x* and *y* directions.
5. If you are given the initial velocity, you may want to resolve it into its *x* and *y* components.
6. Analyse the horizontal and vertical motion separately.
7. Apply the relevant equations. Remember to take sign conventions into consideration.

We shall now work through several examples of projectile motion to familiarize ourselves with the strategy and the application of the equations.

Example 10

A long-jumper leaves the ground at an angle of 20^{o} to the horizontal and at a speed of 11 m s^{-1}.
(a) How far does he jump?
(b) What is his maximum height reached?

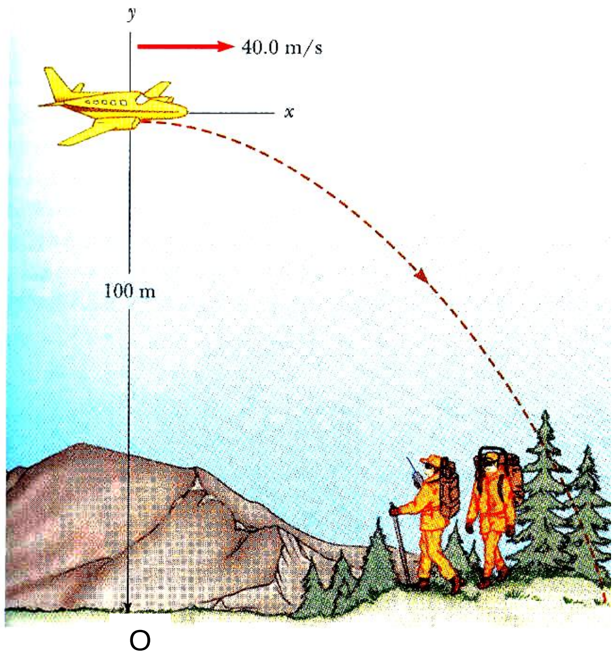
Example 11

A stone is thrown from the top of a building upward at an angle of 30° to the horizontal with an initial speed of 20 m s^{-1}. If the height of the building is 45 m,
(i) how long is the stone in flight?
(ii) what is the speed of the stone just before it strikes the ground?

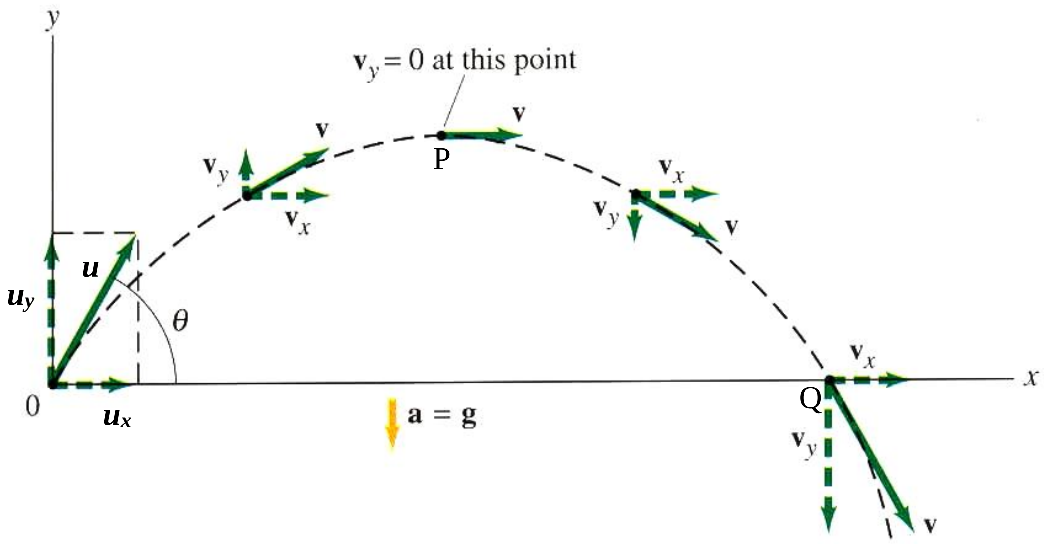
Exercise Where does the stone strike the ground?
Answer 73.0 m from the base of the building.

Example 12

An Alaskan rescue plane drops a package of emergency rations to a stranded party of explorers, as shown in the figure below. If the plane is travelling horizontally at 40.0 m s^{-1} and is 100 m above a point O on the ground, how far from the point O does the package strike the ground?



Example 13: More about Projectile Motion



Consider a projectile projected with an initial velocity ***u*** at an angle above the horizontal. (Diagram above) Assuming that air resistance is negligible and the acceleration of free fall ***g*** remains constant, we can derive using the equations of motion expressions of:

(i) maximum height *H* reached by the projectile,

(ii) duration of time of flight t_{flight},

(iii) horizontal range *R,*

(iv) optimal angle of projection θ_{max} (for maximum range)

(v) equation of the trajectory of the projectile.

Adopt the sign conventions as given in the diagram above.

Solution:

(i) Maximum Height, *H*

The maximum height reached is the maximum vertical displacement reached by the projectile (Point P on the diagram above). At the point of maximum height, the vertical component of the velocity is zero, hence we have

Taking upward direction as positive, consider vertical motion

$$s_y = H$$
$$a_y = -g$$
$$u_y = u\sin\theta$$
$$v_y = 0$$

Using $\quad v_y{}^2 = u_y{}^2 + 2a_y s_y$

$$\Rightarrow \quad 0 = (u\sin\theta)^2 + 2(-g)(H)$$

$$\Rightarrow \quad H = \frac{u^2\sin^2\theta}{2g}$$

(ii) Duration of Flight, t_{flight} :

The time of flight of the projectile is the time the projectile spends in the air, i.e. the time interval between the point the projectile leaves the point of projection O to the point where it lands on the ground Q.

At point Q, the vertical displacement of the projectile from point O is zero. Hence we have
Taking upward direction as positive, consider vertical motion,

$$s_y = 0$$
$$a_y = -g$$
$$u_y = u\sin\theta$$

Using $s_y = u_y\, t + \frac{1}{2} a_y\, t^2$

$\Rightarrow \quad 0 = (u\sin\theta)\, t_{flight} + \frac{1}{2}(-g)\, t_{flight}{}^2$

$\Rightarrow \quad t_{flight} = \dfrac{2u\sin\theta}{g}$ or $t_{flight} = 0$ (at the time of beginning of throw when $s_y = 0$)

(iii) Horizontal Range, *R*:

Since there are no horizontal forces acting on the projectile, the horizontal velocity remains constant during the time of flight and the horizontal acceleration is zero. Using, the result for the time of flight that we derived in (ii), hence, we have

Consider horizontal motion,

$$v_x = u_x = (u\cos\theta)$$
$$a_x = 0$$
$$t_{flight} = \frac{2u\sin\theta}{g}$$
$$s_x = R$$

Using $s_x = u_x\, t + \frac{1}{2} a_x\, t^2$

$\Rightarrow \quad R = (u\cos\theta)\left(\dfrac{2u\sin\theta}{g}\right) + 0$

But $2\cos\theta\sin\theta = \sin 2\theta$

Hence, $R = \dfrac{u^2\sin 2\theta}{g}$

(iv) Optimal Angle of Projection θ_{max} (for maximum range):

Now, $R = \dfrac{u^2\sin 2\theta}{g}$

But $-1 \leq \sin 2\theta \leq 1 \quad \Rightarrow$ maximum $\sin 2\theta = 1$

Therefore, the maximum possible range, $R_{max} = \dfrac{u^2}{g}$

This occurs when $\sin 2\theta = 1 \Rightarrow 2\theta = 90^o \Rightarrow \theta = 45^o$

(v) The Trajectory Equation:

To show that the path followed by any projectile is a parabola, we need to find the *y* as a function of *x* by eliminating *t* between the vertical motion and the horizontal motion.

We can write following equations for both the horizontal and vertical displacements:

$$x = u_x t \qquad (1)$$
$$y = u_y t - \tfrac{1}{2} g t^2 \qquad (2)$$

From (1), we have $t = x/u_x$, and we substitute it into (2) to obtain,

$$y = \left(\frac{u_y}{u_x}\right)x - \left(\frac{g}{2{u_x}^2}\right)x^2$$

If we write $u_x = u \cos\theta$ and $u_y = u \sin\theta$, we can therefore write

$$y = (\tan\theta)x - \left(\frac{g}{2u\cos^2\theta}\right)x^2$$

We can see that *y* as a function of *x* has the form

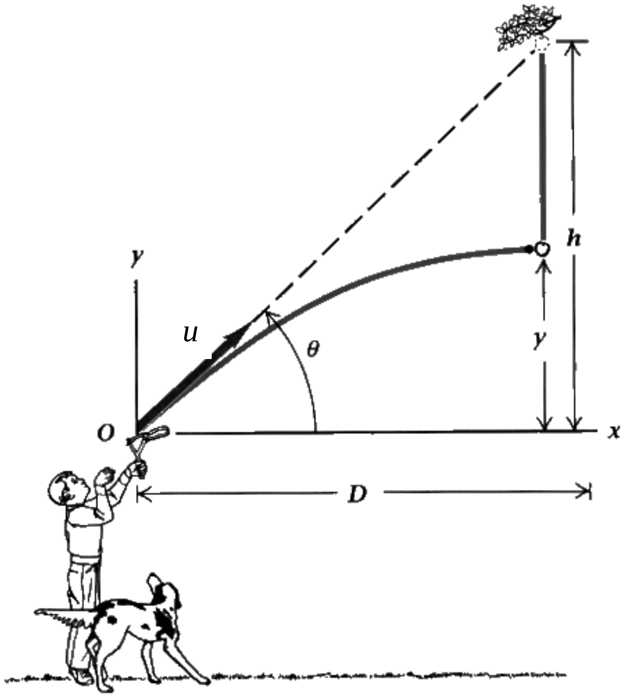
$$y = ax - bx^2$$

This is an equation of a parabola.

We do NOT recommend you to memorise the formulae derived in this example. During examinations, you will NOT be given any credit for simply quoting these results directly without showing derivation.

Challenging Problem: Example 14

A boy aims his slingshot directly at an apple hanging in a tree. At the moment he shoots a pebble, the apple drops. If pebble hits target, both pebble and target (i.e. the apple) are at SAME location at the SAME time. Horizontal distance, D, travelled by stone is known. Thus time t, time for the pebble to travel the distance D can be determined. Show that at time t, apple is at the same height as pebble.



Solution

For apple, vertical displacement from slingshot,

$(\downarrow) s_{apple} = h - (u_y t + \frac{1}{2} a_y t^2)$

$s_{apple} = h - (0 + \frac{1}{2} g t^2)$

$s_{apple} = h - \frac{1}{2} g t^2$ -----(1)

For pebble,

horizontal displacement from slingshot, $(\rightarrow) s_{pebble(x)} = u \cos\theta \times t = D$ ----- (2)

vertical displacement from the slingshot, $(\uparrow) s_{pebble(y)} = u_y t + \frac{1}{2} a_y t^2$

$(\uparrow) s_{pebble(y)} = u \sin\theta \times t - \frac{1}{2} g t^2$ ----(3)

(3) / (2):

$s_{pebble(y)} = (\tan\theta - \frac{1}{2} g t^2 / u \cos\theta \times t) D$

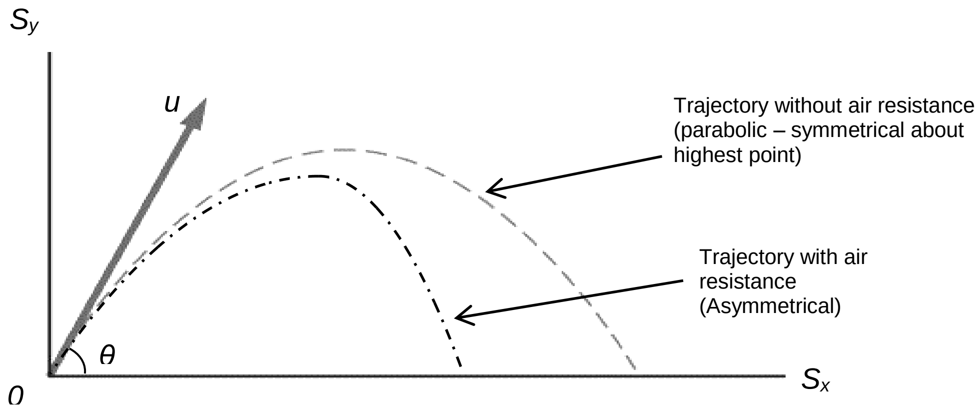
Since $h = \tan\theta \times D$

$s_{pebble(y)} = h - \frac{1}{2} g t^2$ -----(4)

Equations (1) and (4) are identical, thus both apple and pebble is at the same height at the same time so the pebble will hit the apple.

2.8.3 Projectile Motion with Air Resistance

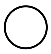
Consider an object projected above the horizontal with an initial speed of u and at angle of θ with respect to the horizontal.



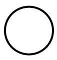
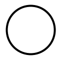
Suppose that the air resistance experienced by the projectile is not negligible. Then **throughout the motion, the projectile will experience a drag force in opposite direction to its motion** (velocity).

Let us further analyse the horizontal motion and vertical motion independently,

Horizontally,

	The drag force acts in ______________ direction to the horizontal velocity. ⇒ The horizontal velocity ___________ with time. ⇒ The **horizontal range is reduced** (as compared with the case if no air resistance present).

Vertically,

On its flight up,	On its flight down,
	
The drag force acts in ______________ direction as the weight mg. The magnitude of the net acceleration is _____________ than g. ⇒ **Maximum height reached** is lower than if there were no air resistance present.	The drag force acts in ______________direction to the weight mg. The magnitude of the net acceleration is _____________than g. ⇒ The final (vertical) speed v_y will be ________________ than the initial (vertical) speed u_y (at the point of projection). ⇒ The average (vertical) speed upwards is __________ than the average (vertical) speed downwards. Since, distance travelled up = distance to come down and Average speed = Total distance / Time taken ⇒ **The time taken to travel up is shorter than the time taken to travel down.**

Furthermore, comparing the vertical motion of the projectile on its flight up and on its flight down, we see that:

- The (vertical) acceleration experienced by the projectile when it is going up is different from the (vertical) acceleration experienced by the projectile when it is coming down. ($a_{y,up} \neq a_{y,down}$)
- For a particular height, the speed of the projectile when it is going up is also different for the speed of the projectile when it is coming down.

Hence, we can see that the **path of the projectile will not be symmetrical about the highest point**, *i.e.* the path is asymmetrical about the highest point.

In summary,

The key characteristics of a projectile motion with air resistance (compared to if air resistance is negligible) are:

1. Horizontal range is shorter.
2. Maximum height is lower.
3. The time taken to reach maximum height must be shorter than the time taken to return to the starting point.
4. Path is asymmetrical about a vertical line through its highest point.

Further Readings and References

Appendix I : Understanding the Calculus Notations

Although rigorous treatment with calculus is not required in the A-level syllabus, it is however useful for us to understand the physical significance of some of the calculus notations as they are commonly used to define physical quantities.

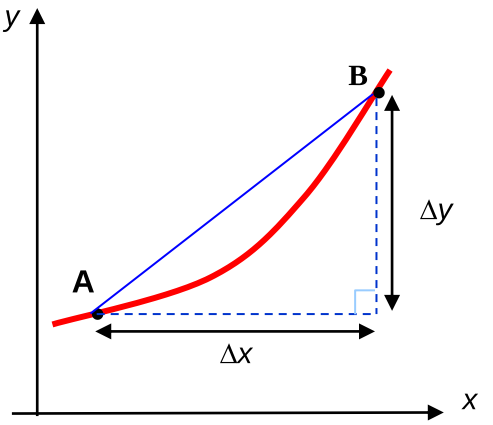


Figure 1. Graph of *y* vs. *x*

The Derivative : $\dfrac{dy}{dx}$

Consider a graph of *y* vs. *x* as indicated by the curve in Figure 1. $\dfrac{\Delta y}{\Delta x}$ indicates the gradient of the line joining the 2 points A and B on the line.

Figure 2 shows that if we move point B closer towards point A, eventually the 2 points will meet. At this point gradient of the line AB will be exactly equal to the gradient of the tangent of the curve at A. (as represented by the bold straight line). Mathematically, we say:

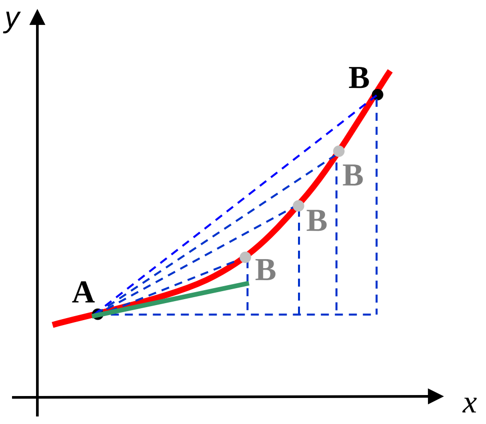


Figure 2. Tangent of the curve at **A**.

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = \text{gradient of the tangent of the } y - x \text{ graph}$$

This physical interpretation is very important in physics as many physical quantities are expressed in this derivative form.

For example, in our study of kinematics, we have :

Velocity, *v* = Rate of change of displacement with respect to time

$= \dfrac{ds}{dt}$

= Gradient of tangent of the *s* - *t* graph

and

Acceleration, *a* = Rate of change of velocity with respect to time

$= \dfrac{dv}{dt}$

= Gradient of the tangent of the *v* - *t* graph

The Integral: $\int y\,dx$

The reverse process to taking the derivative is referred to either as *integration* or finding the *anti-derivative*. Graphically, it is equivalent to finding the area under the curve. Let us see why.

Consider a y - x graph as shown in Figure 3. Suppose we wish to find the area under the graph from some initial x value, x_{intial}, to some final x value, x_{final}. One way to do so is break the area up into n number of small little rectangles each of some area, ΔA.

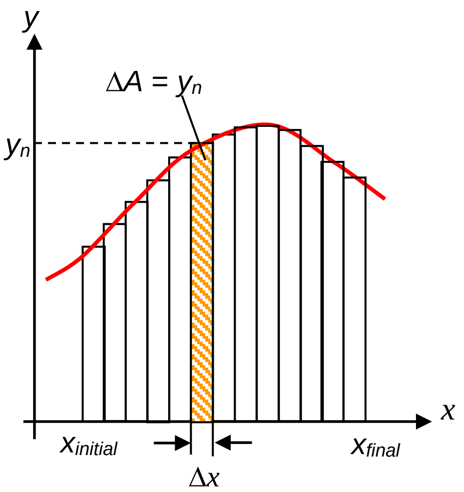


Figure 3. Finding the area under a curve. The area of the shaded rectangle is equal to the product of the width of the rectangle Δx and the y value of the curve at the particular x, y_n.

Hence, the area of each small rectangle can be written as : $\Delta A = y_n\,\Delta x$

and the total area under the curve can be estimated by summing the area of all these little rectangles, i.e.

$$\text{Total area } A \approx \sum_n y_n \Delta x$$

where the symbol Σ (upper case Greek sigma) signifies a sum over all terms, i.e. over all values of n.

For the approximation of the area to be even better, we can try to divide the area into smaller and smaller rectangles, by decrease the interval Δx, i.e. we say that we let $\Delta x \to 0$.

Therefore, $$A = \lim_{\Delta x \to 0} \sum_n y_n \Delta x$$

This is also written as $\int_{x_{initial}}^{x_{final}} y\,dx$, therefore $\int_{x_{initial}}^{x_{final}} y\,dx = \text{Area under the } y - x \text{ graph}$

We now use the defining equations for acceleration and velocity to illustrate the use of the above concept.

The defining equation for acceleration of an object is $a = \dfrac{dv}{dt}$

Therefore, the change in velocity of an object is $v_{final} - v_{initial} = \int_{t_{initial}}^{t_{final}} a\,dt$ = Area under a - t graph

Similarly, velocity of an object is $v = \dfrac{ds}{dt}$

Rearranging the equation, we also have the change in displacement of an object given as

$$s_{final} - s_{initial} = \int_{t_{initial}}^{t_{final}} v\,dt \text{ = Area under } v - t \text{ graph}$$

Tutorial 2A: Kinematics - 1D Motion

Self Review Questions

Use these questions to test your familiarity with the concepts. These questions should be sufficiently easy such that you can solve them on your own, with a little bit of thinking, without help from the tutors. The solutions are made available on Moodle for self-check.

S1 An athlete swims the length of a 50 m pool in 20 s and makes the return trip to the starting point in 22 s. Determine his average velocity in

(a) the first half of the swim,

(b) the second half of the swim, and

(c) the round trip.

S2 A motorist drives north for 35 minutes at 85 km h^{-1} and then stops for 15 minutes. He then continues northeast travelling 130 km in 2 h. (Hint: Draw a vector diagram to illustrate the motion)

(a) What is his total displacement?

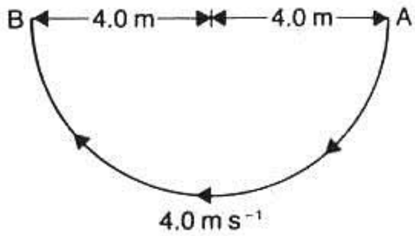
(b) What is his average velocity?

S3 An object moves along a semi-circular path AB of radius 4.0 m as shown in the figure on the right, at a constant speed of 4.0 m s^{-1}. Calculate

(a) the time taken,

(b) the average velocity, and

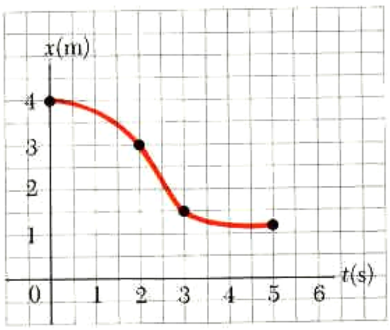
(c) the change in velocity.



S4 Using the figure on the right, determine

(a) the average velocity between t = 0.0 s and t = 3.0 s,

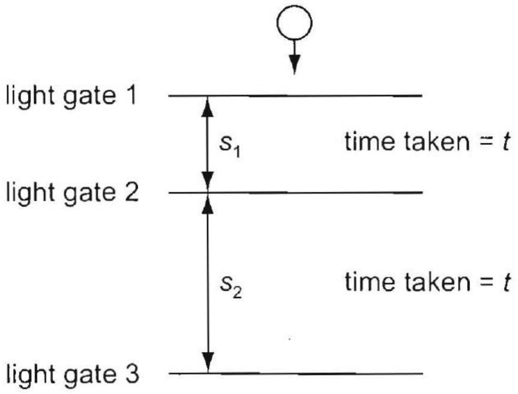
(b) the instantaneous velocity at t = 3.0 s.



S5 An aeroplane lands on the runway with a velocity of 50 m s^{-1} and decelerates at 10 m s^{-2} to a velocity of 20 m s^{-1}. What is the distance travelled on the runway?

A 90 m **B** 105 m **C** 125 m **D** 160 m

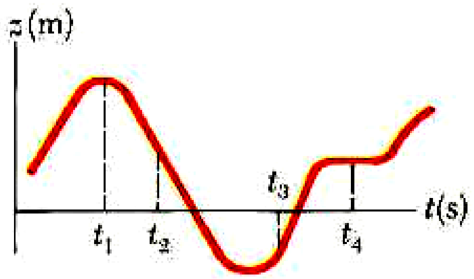
S6 **[N10/I/5]** An object falls freely with a constant acceleration *a* from above three light gates. It is found that it takes a time *t* to fall between the first two light gates a distance of s_1 apart. It then takes an additional time, also *t*, to fall between the second and third light gates a distance s_2 apart.



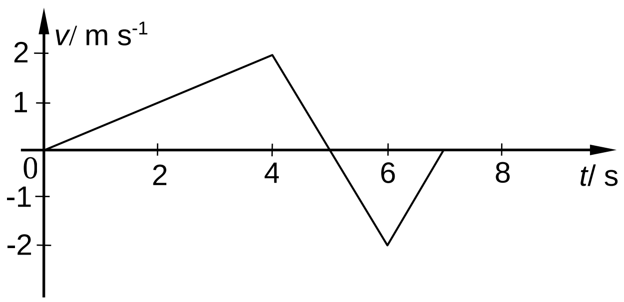
What is the acceleration in terms of s_1, s_2 and t ?

A $(s_2 - s_1)/t^2$ **B** $(s_2 - s_1)/2t^2$ **C** $2(s_2 - s_1)/3t^2$ **D** $2(s_2 - s_1)/t^2$

S7 The position-time graph for a particle moving along the *z* axis is as shown in Figure below. Determine whether the velocity is positive, negative, or zero at times (a) t_1, (b) t_2, (c) t_3, (d) t_4.



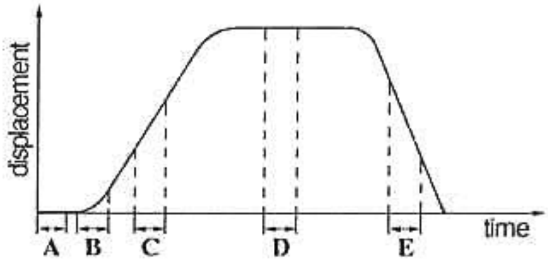
S8 The velocity-time graph of a particle along a straight line is shown below.



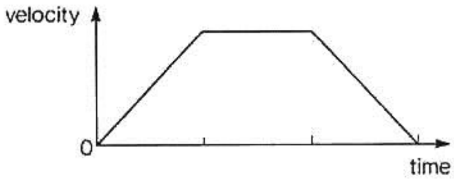
The displacement of the particle after 7.0 s is

A 1.0 m **B** 3.0 m **C** 5.0 m **D** 7.0 m

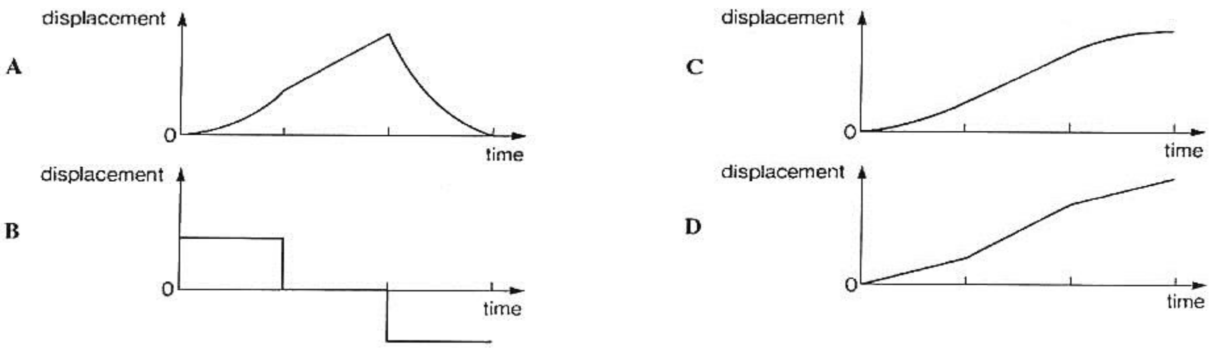
S9 **[J92/I/3]** The graph represents how displacement varies with time for a vehicle moving along a straight line. During which time interval does the acceleration of the vehicle have its greatest numerical value?



S10 **[N95/I/5]** The graph of velocity against time for a moving object is shown.



Which of the following is the corresponding graph of displacement against time?



S11 A hot air balloon is travelling vertically upward at a constant speed of 5.0 m s^{-1}. When it is 21.0 m above the ground, a package is released from the balloon.

a) How long after being released is the package in the air?

b) What is the velocity of the package just before it hits the ground?

Discussion Questions

D1 Give examples to answer the following questions.

a) Is it possible for a body to be accelerating while travelling at constant speed?

b) Can a body have a constant velocity and a varying speed?

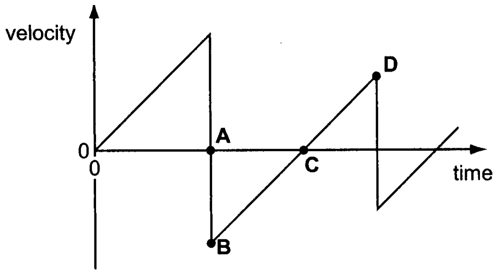
c) Can the direction of a body's velocity change when it has a constant acceleration?

d) If a body has zero velocity at an instant, can it be accelerating?

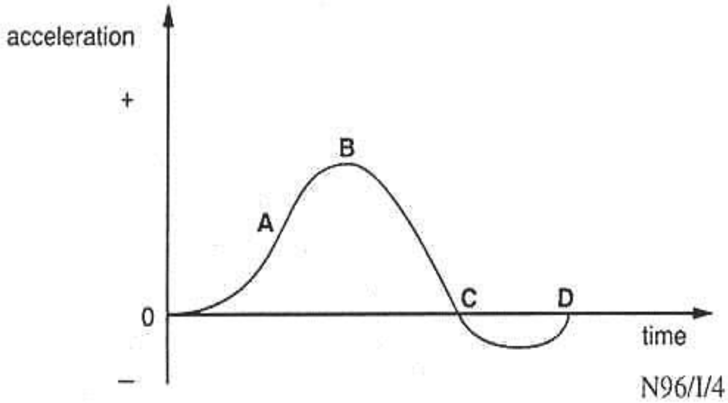
e) Can a body be accelerating in a direction opposite to that of its velocity?

f) If the average velocity of an object is non-zero for some interval, does this mean that the instantaneous velocity is never zero during this interval?

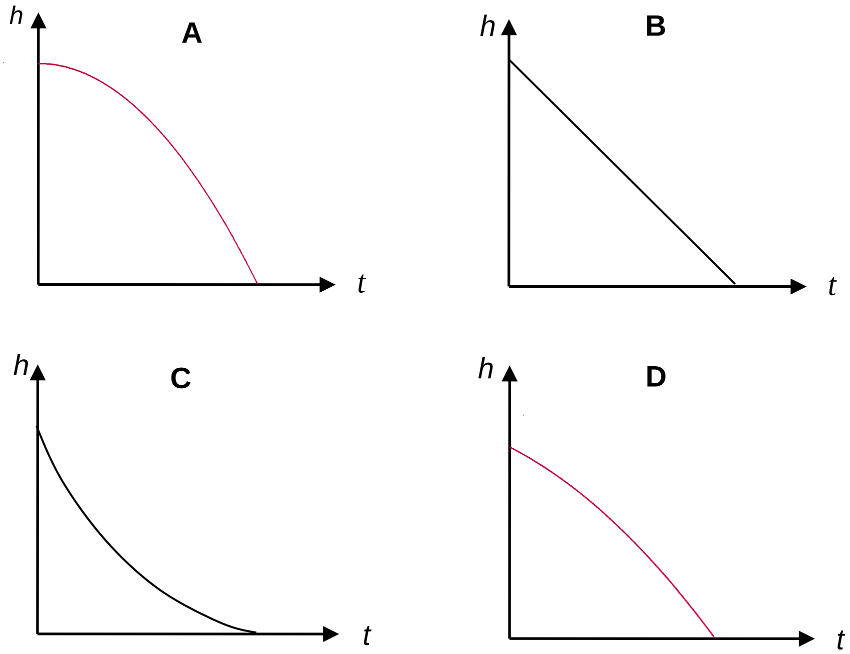
D2 A ball is released from rest above a hard, horizontal surface. The graph shows how the velocity of the bouncing ball varies with time. At which point on the graph does the ball reach its maximum height after the first bounce?



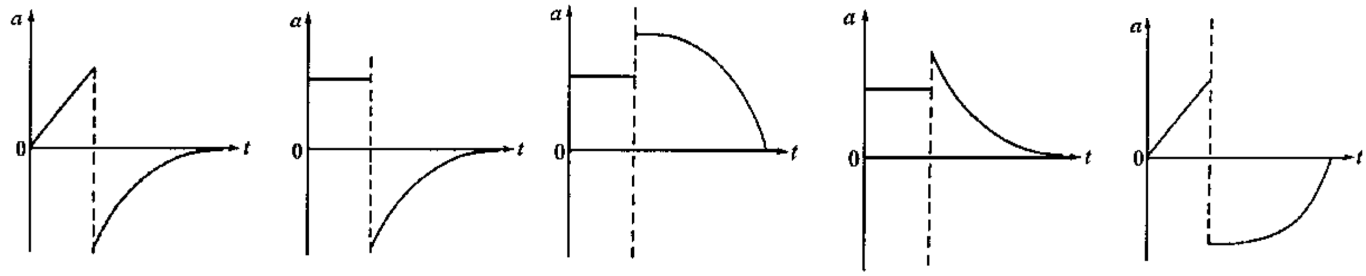
D3 [N96/I/4] A car is travelling along a straight road. The graph shows the variation with time of its acceleration during part of the journey. At which point on the graph does the car have its greatest velocity?



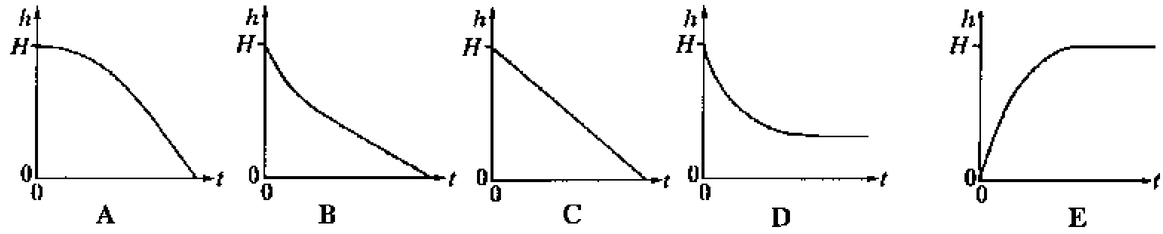
D4 A small ball falls freely under gravity after being released from rest. Which graph best represents the variation of the height of the ball with time *t*?



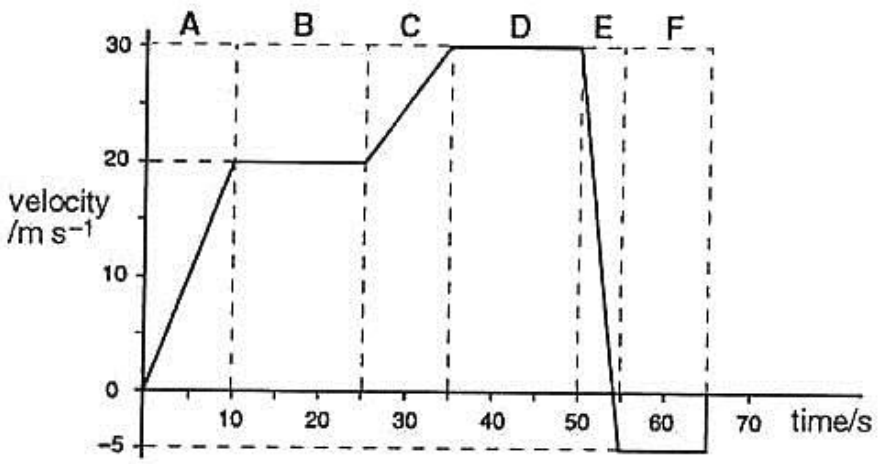
D5 [J83/II/5] A small metal sphere is released from rest at a height of a few centimetres above the surface of a viscous liquid. On entering the liquid, the sphere experiences a viscous drag proportional to its velocity. Which one of the following graphs most closely represents the variation of acceleration a with time t of the sphere?



D6 [N83/II/4] A steel ball bearing is released at the surface of a viscous liquid contained in a tall, wide jar. In falling through the liquid, the ball bearing experiences a retarding force proportional to its velocity. If the depth of the liquid in the jar is H, which one of the following graphs best represents the variation of the height h of the ball bearing above the base with t?



D7 [N97/II/1] Figure below shows a velocity-time graph for a journey lasting 65 s. It has been divided up into six sections for ease of reference.



(a) Using the information from the graph, obtain:

(i) the velocity 10 s after the start,

(ii) the acceleration in section A,

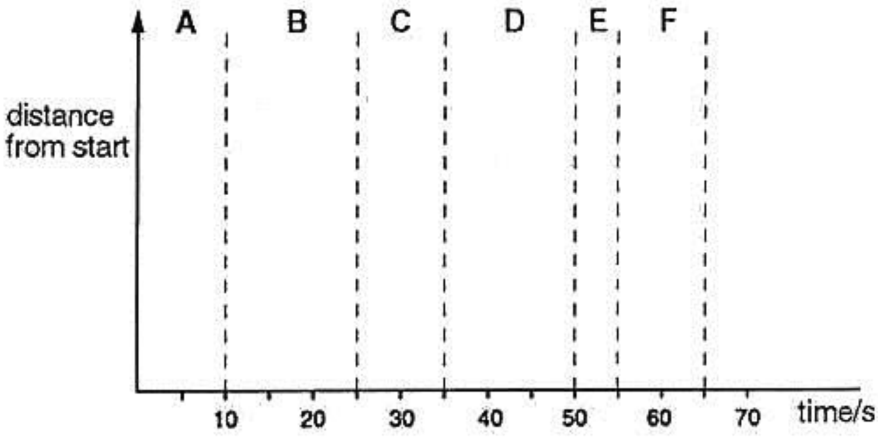
(iii) the acceleration in section E,

(iv) the distance travelled in section B,

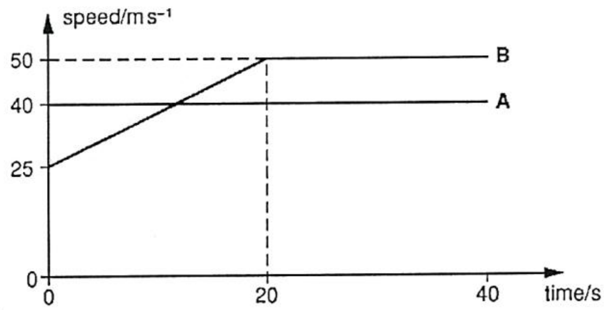
(v) the distance travelled in section C. [5]

(b) Describe qualitatively in words what happens in sections E and F of the journey. [4]

(c) Copy the figure below into your answer script and sketch the shape of the corresponding distance-time graph. You are not expected to make detailed calculations of the distance travelled. [3]



D8 [N93/II/2] The graph below shows the speeds of two cars **A** and **B** which are travelling in the same direction over a period of time of 40 s. Car **A**, travelling at a constant speed of 40 m s^{-1}, overtakes car **B** at time t = 0. In order to catch up with car **A**, car **B** immediately accelerates uniformly for 20 s to reach a constant speed of 50 m s^{-1}.



(a) How far does car **A** travel during the first 20 s? [1]

(b) Calculate the acceleration of car **B** in the first 20 s. [1]

(c) How far does car **B** travel in this time? [2]

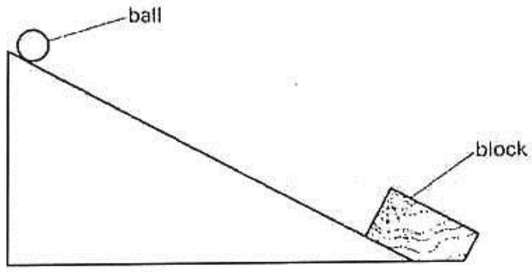
(d) What additional time will it take for car **B** to catch up with car **A**? [2]

(e) How far will each car have then travelled since t = 0? [1]

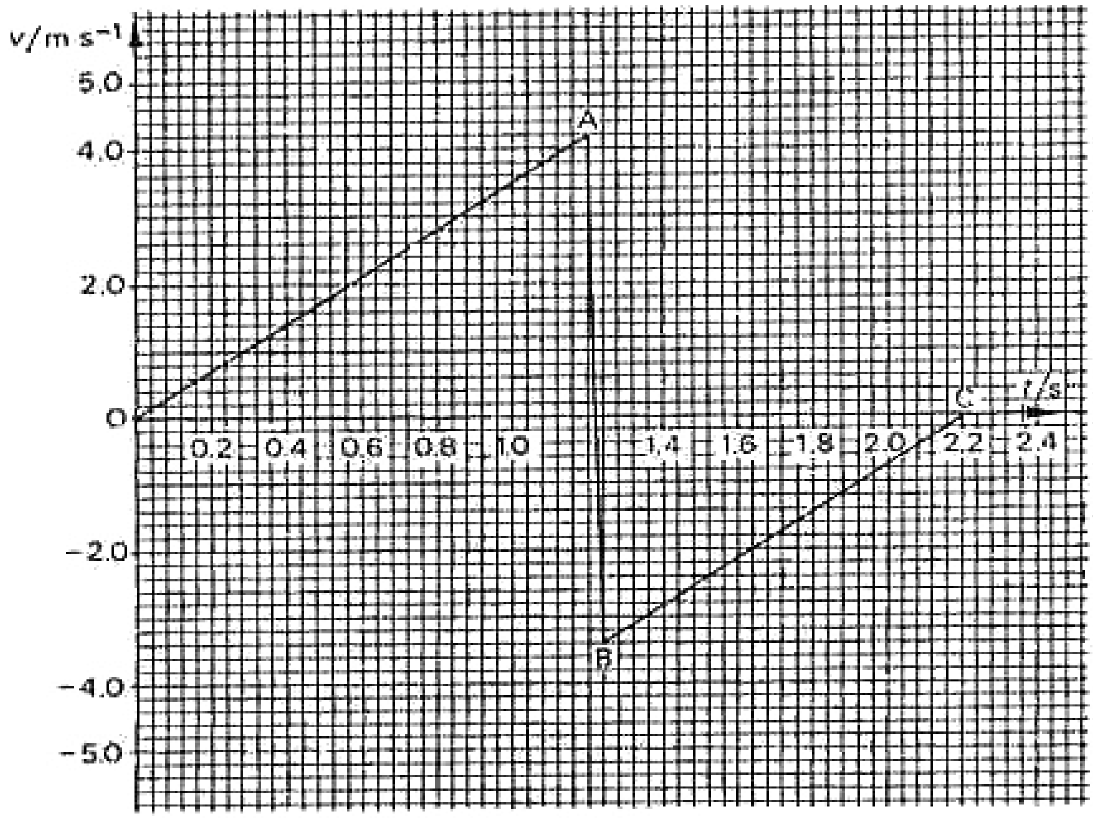
(f) What is the maximum distance between the cars before car **B** catches up with car **A**? [3]

D9 [J92/III/1(part)]

A ball is placed at the top of a slope as shown in the figure below.



A block is fixed rigidly to the lower end of the slope. The ball of mass 0.70 kg is released at time t = 0 from the top of the incline and v, the velocity of the ball down the slope is found to vary with t as shown in the graph.



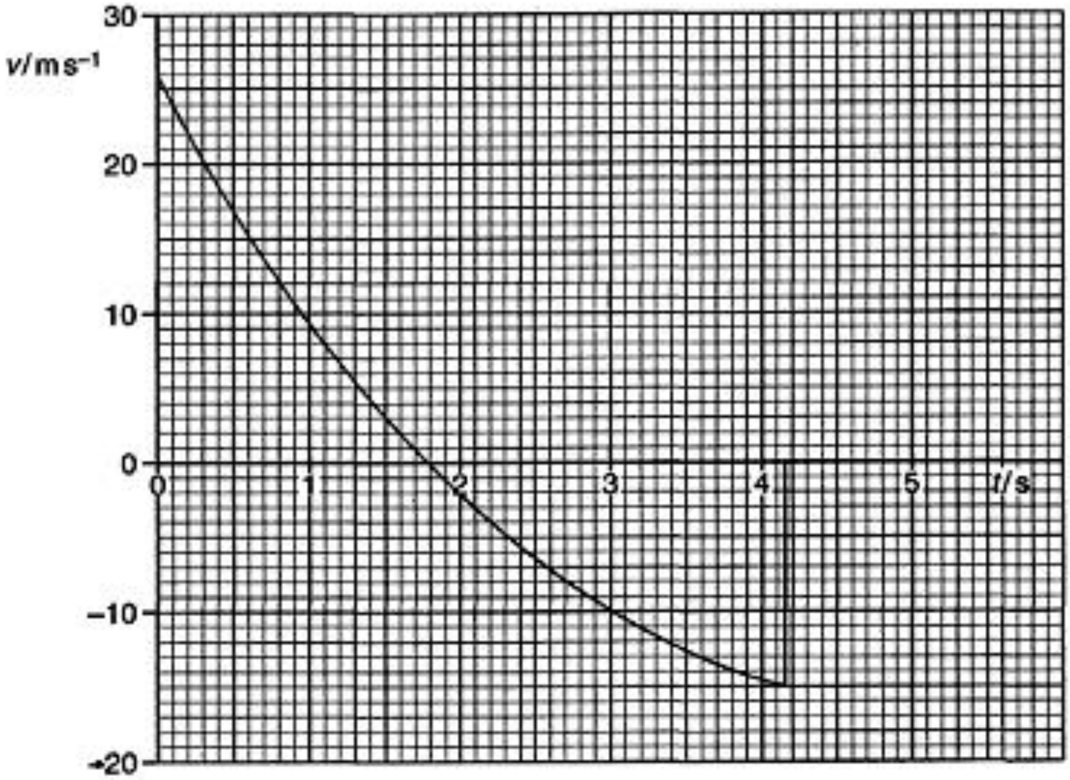
(i) Describe qualitatively the motion of the ball during the periods OA, AB and BC.

(ii) Calculate

1. the acceleration of the ball down the incline,
2. the length of the incline,
3. the mean force experienced by the ball during impact with the block.

D10 [N03/III/1(c), (d) and (e)]

(c) The graph below shows the variation with time *t* of the velocity *v* of a ball from the moment it is thrown with a velocity of 26 m s^{-1} vertically upwards.



(i) State the time at which the ball reaches its maximum height.

(ii) State the feature of a velocity-time graph that enables the acceleration to be determined.

(iii) Just after the ball leaves the thrower's hand, it has a downward acceleration of approximately 20 m s^{-2}. Explain how this is possible.

(iv) State the time at which acceleration is *g*. Explain why the acceleration has this value only at this particular time.

(v) Sketch an acceleration-time graph for the motion (in the air). Show the value of *g* on the acceleration axis.

(d) Explain why, for all real vertical throws, the time taken to reach maximum height must be shorter than the time taken to return to the starting point.

(e) The ball in (c) starts with kinetic energy of 54 J.

(i) Calculate the mass of the ball.

(ii) Describe qualitatively how the amount of kinetic energy changes during the motion.

Answers to Tutorial 2A: Kinematics - 1D Motion

S1 (a) 2.5 m s^{-1}, away from starting point (b) 2.3 m s^{-1}, towards starting point (c) 0

S2 (a) 169 km, bearing of 33.0° (b) 60 km h^{-1}, bearing of 33.0°

S3 (a) 3.14 s (b) 2.55 m s^{-1}, A to B (c) 8.0 m s^{-1}, vertically upwards

S4 (a) -0.83 m s^{-1}, -0.67 m s^{-1}, (allow -0.61 m s^{-1} to -0.88 m s^{-1})

S5 B

S6 A

S7 (a) zero (b) negative (c) positive (d) zero

S8 B

S9 B

S10 C

S11 (a) 2.64 s (b) -20.9 m s^{-1}

D7 (a)(i) 20 m s^{-1} (ii) 2 m s^{-2} (iii) -7 m s^{-2} (iv) 300 m (v) 250 m

D8 (a) 800 m (b) 1.25 m s^{-2} (c) 750 m (d) 5 s (e) 1 km (f) 90 m

D9 (ii) (1) 3.50 m s^{-1} (2) 2.52 m (3) 133 N upwards along the slope

D10 (c)(i) 1.80 s (iv) 1.80 s (e) (i) 0.160 kg

Tutorial 2B: Kinematics - 2D Motion

Self Review Questions

S1 As a projectile moves through its parabolic trajectory, which of these quantities, if any, remain constant: (a) speed, (b) acceleration, (c) horizontal component velocity, (d) vertical component velocity?

S2 A transport plane travelling horizontally at a steady speed of 50 m s^{-1} at an altitude of 300 m releases a parcel when directly above a point X on level ground. Calculate

a) the time of flight of the parcel,

b) the velocity of impact of the parcel,

c) the horizontal distance from X to the point of impact.

S3 The angle of elevation of an anti-aircraft gun is 70^{o} and the muzzle velocity is 900 m s^{-1}. How long after firing will the shell be at an altitude of 1700 m?

A 2.0 s **B** 4.0 s **C** 6.0 s **D** 8.0 s

S4 A ball is projected horizontally from the top of a cliff on the surface of the Earth with a speed of 40 m s^{-1}. Assuming that there is no air resistance, what will its speed be after 3 s?

A 30 m s^{-1} **B** 40 m s^{-1} **C** 50 m s^{-1} **D** 60 m s^{-1}

S5 A stone thrown horizontally at a speed of 24 m s^{-1} from the top of a cliff takes 4.0 s to hit the sea. What is the height *h* of the cliff-top above the sea and the distance *d* from the base of the cliff to the point of impact?

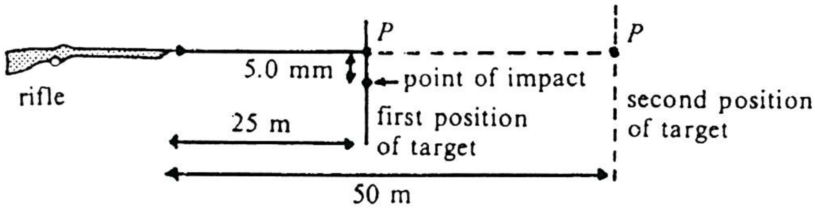
	Height *h*/m	Distance *d*/m
A	78	96
B	78	116
C	90	96
D	90	116

S6 Water shoots out from a horizontal pipe which is at a height of 52 cm from the floor. If the horizontal distance travelled by the water before it hits the floor is 100 cm, what is the velocity of the water when it leaves the pipe?

S7 A particle can be projected at 40 m s^{-1}. Find the angle(s) of projection required for it to achieve a horizontal range of 150 m.

Discussion Questions

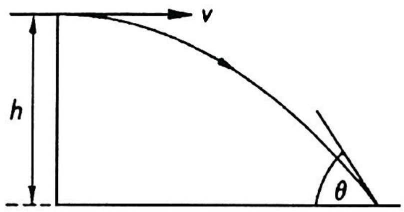
D1 [N82/II/4] When a rifle is fired horizontally at a target *P* on a screen at a range of 25 m, the bullet strikes the screen at a point 5.0 mm below *P*. The screen is now moved to a distance of 50 m and the rifle again fire horizontally at *P* in its new position. See figure below.



Assuming that the air resistance may be neglected, what is the new distance below *P* at which the screen would now be struck?

A $5\sqrt{2}$ mm **B** 10 mm **C** 15 mm **D** 20 mm **E** 25 mm

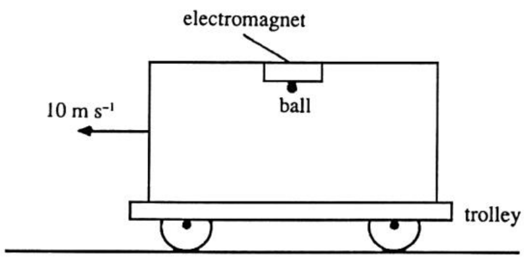
D2 [J88/I/3] The diagram shows the path of a projectile fired with a horizontal velocity *v* from the top of a cliff of height *h*.



Which of the following values for *v* and *h* will give the greatest value of the angle θ?

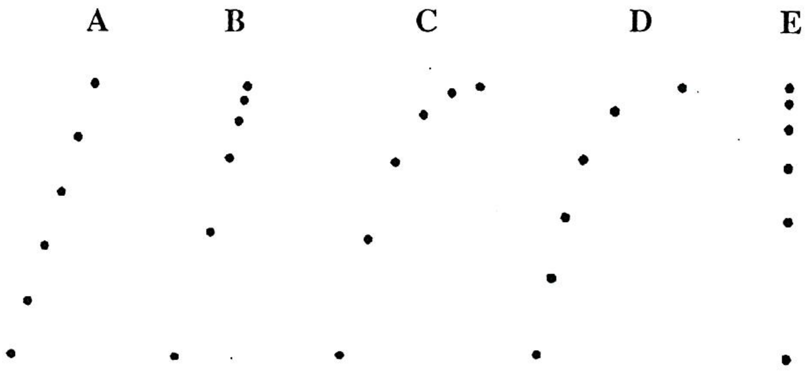
	v/m s^{-1}	h/m
A	10	30
B	10	50
C	30	30
D	30	50
E	50	10

D3 [N91/I/14] A ball is suspended from an electromagnet attached to a trolley which is travelling at a constant speed of 10 m s^{-1} to the left. The trolley is illuminated by a stroboscope flashing at a constant rate. The diagram represents the viewpoint of a stationary camera.



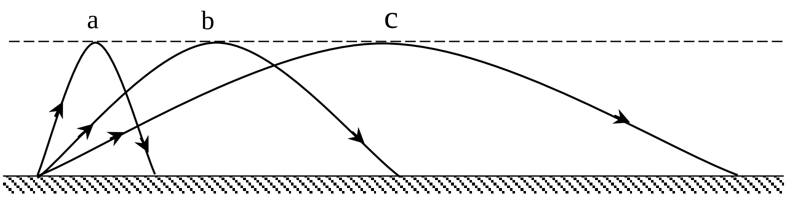
The ball is released and a series of stroboscopic images of the ball are recorded on a single photographic plate.

Which diagram best represents what is seen on the photographic plate?



N91/I/4

D4 Trajectories are shown in the figure below for three kicked footballs.



Ignoring the effects for air resistance on the footballs, order the trajectories from smallest to largest, indicating any ties, according to:

a) initial vertical velocity component,

b) time of flight,

c) initial horizontal velocity component and

d) initial speed.

D5 A Hollywood daredevil plans to jump the canyon shown in the figure. If he desires a 3.0 s flight time,

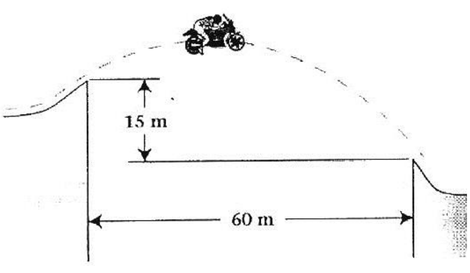
Calculate

(a) the correct angle for his launch ramp,

(b) his correct launch speed,

(c) the correct angle for his landing ramp, and

(d) his predicted landing speed.



D6 A cannon ball is projected at an angle of 30° above horizontal and at a speed of 15.0 m s^{-1} from the top of a cliff. The top of the cliff is at a height of 100.0 m above the sea level. A hot-air balloon is at a horizontal distance of 50.0 m away from the bottom of the cliff and rises vertically at a constant speed. When the hot-air balloon is at a height of 30.0 m from sea level, the cannon ball is launched. What is the speed of the balloon if the cannon ball hit it? (Hint: Refer to lecture example 14)

D7 A ball is projected horizontally from the top of a building. One second later, another ball is projected horizontally from the same point with the same velocity.

(a) At what point in the motion will the balls be closest to each other?

(b) Will the first ball always be travelling faster than the second ball?

(c) How much time passes between the moments the first ball and the second one hit the ground?

(d) Can the horizontal projection velocity of the second ball be changed so that the balls arrive at the ground at the same time?

D8 [2010 C1 BT/II/2] A person stands on top of a hill at point **O** and throws a stone at an initial angle of 60° with respect to the horizontal with a speed of 10 m s^{-1}. The stone landed at point A on the slope *t* seconds later. Point **A** is *d* metres from point **O**, in the direction 45° down slope as shown in Figure 9. You may neglect air resistance.

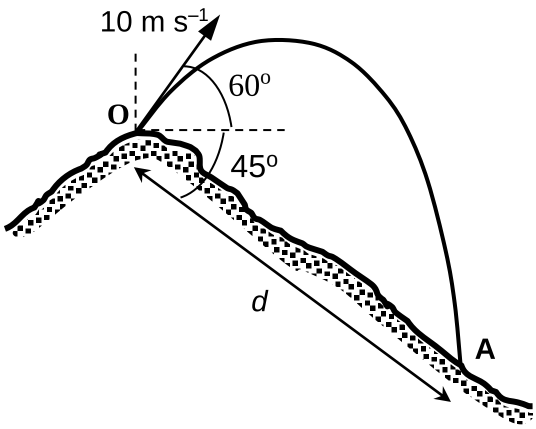


Figure 9

(i) Write down the expressions, in terms of *t*, for
1. the horizontal displacement and [1]
2. the vertical displacement of the stone at point **A** with respect to point **O**. [1]

(ii) Calculate *t*. [3]

(iii) Find *d*. [1]

D9 From a point O at a height h above the ground, two particles P and Q are simultaneously projected with initial speed u at an angle of θ with the horizontal as shown below. The acceleration due to gravity is represented by g.

(i) Copy the diagram above into your foolscap and sketch the trajectories of P and Q in your diagram

(ii) Obtain an expression in terms of u, g and θ, the distance between the points of impact of P and Q.

<u>Answers to Tutorial 2B: Kinematics - 2D Motion</u>

S1 (a) Not constant (b) Constant (c) Constant (d) Not constant

S2 (a) 7.82 s (b) 91.6 m s^{-1}, 57.0° below the horizontal (c) 391 m

S3 A

S4 C

S5 A

S6 3.07 m s^{-1};

S7 33.4° and 56.6°;

D5 (a) 26° (b) 22 m s^{-1} (c) 45° (d) 28 m s^{-1}

D7 (a) At the start of projection of the 2nd ball (b) Yes (c) 1 s (d) No

D8 (ii) 2.78 s (iii) 19.7 m

D9 (ii) $\dfrac{u^2 \sin 2\theta}{g}$