Appendices:

Appendix A: Gravitational Field due to a Uniform Spherical Shell[5]

(Adapted from Lieherherr, Martin. The Physics Teacher. Vol. 42. January 2004. *"Gravitational Field Due to a Sphere: A Geometrical Argument."* p5.)

From the statement of Newton's law of gravitation, we see that it applies only to point masses. This is a good approximation when the sizes of the objects are small compared to the distance between them, e.g. the Moon and the Earth.

Can Newton's law of gravitation be applied to an apple resting on the Earth's surface? From the point of view of the apple, the broad and level Earth, stretching out to the horizon beneath the apple, certainly does not look like a particle.

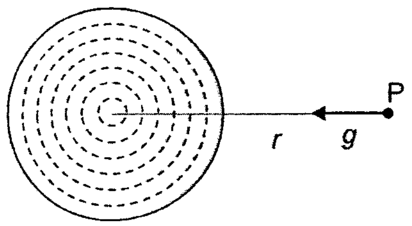
Newton solved this problem by using his law of universal gravitation and calculus to establish two theorems that apply to the gravitational force exerted by a thin spherical shell of uniform density.

Shell Theorem 1:
A uniform spherical shell of matter attracts a particle that is outside the shell as if all the shell's mass were concentrated at its centre.

Shell Theorem 2:
A uniform shell of matter exerts no net gravitational force on a particle located inside it.

Let us now examine each shell theorem in more detail:

- **Shell Theorem 1:**
  From Shell Theorem 1, we can see that any solid spherical object can be treated as many nested spherical thin shells with each of their centre of mass coinciding at the centre of the sphere.



The gravitational field at P due to each spherical shell is as if all the shell's mass were concentrated at the centre.

Since the centre of each spherical shell is common, the gravitational field at P, due to each shell, is directed towards the centre of the solid sphere.

Hence the magnitude of the resultant gravitational field at P can also be written as :

$g = g_1 + g_2 + g_3 + \ldots + g_N$ where g_i : gravitational field strength due to the *ith* shell.

However, $g_i = G\dfrac{m_i}{r^2}$ where m_i : mass of the *ith* shell.

Therefore,
$$g = G\frac{m_1}{r^2} + G\frac{m_2}{r^2} + \ldots + G\frac{m_N}{r^2} = \frac{G}{r^2}(m_1 + m_2 + \ldots + m_N)$$
$$= \frac{G}{r^2}(\text{mass of the sphere})$$

We can also conclude that :

A sphere of uniform composition attracts a particle that is outside the sphere as if the entire sphere's mass were concentrated at its centre.

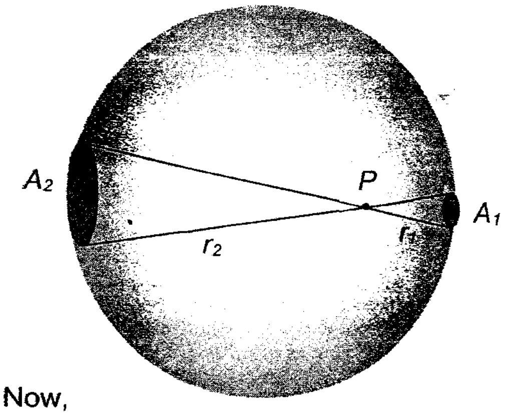
[5] Proof of shell theorems requires techniques of integral calculus. If you are interested, refer to Resnick, Halliday and Krane. Physics -5th Edition. *Proof of Shell Theorems.* pp 306 – 307.

- **<u>Shell Theorem 2:</u>**

Although Newton proved both his theorems using calculus, we will present a simplistic geometrical proof below which makes use of the concepts of similarity.

Consider an arbitrary point P inside a thin spherical shell with homogeneous mass distribution.

Let σ be the mass per unit area of this spherical shell.



Let Point P be at the centre of a double cone which cuts out two circular areas A_1 and A_2 on the spherical shell.

We can therefore see that the gravitational field due to A_1 at P is directed towards centre of A_1 and the gravitational field due to A_2 at P is directed towards centre of A_2, i.e. their directions are opposite to each other.

Now,

$$\frac{g \text{ due to } A_2}{g \text{ due to } A_1} = \frac{G\dfrac{\text{mass of } A_2}{r_2^2}}{G\dfrac{\text{mass of } A_1}{r_1^2}} = \frac{\dfrac{\sigma A_2}{r_2^2}}{\dfrac{\sigma A_1}{r_1^2}} = \frac{A_2}{A_1}\frac{r_1^2}{r_2^2}$$

Since the two circular subtends the same angle, the cones are similar to each other.

Therefore, $\dfrac{A_1}{A_2} = (\dfrac{r_1}{r_2})^2 = \dfrac{r_1^2}{r_2^2}$

Hence, $\dfrac{g \text{ due to } A_2}{g \text{ due to } A_1} = \dfrac{A_2}{A_1}\dfrac{r_1^2}{r_2^2} = \dfrac{r_2^2}{r_1^2}\dfrac{r_1^2}{r_2^2} = 1,$

i.e. the magnitude of the gravitational field due to A_1 = the magnitude of the gravitational field due to A_2.

Since the gravitational field due to A_1 is equal in magnitude and opposite in direction to the gravitational field due to A_2, the resultant gravitational field at P is zero.

Hence, any test mass placed at P will experience no gravitational force acting on it due to the spherical shell.

Furthermore, since P is arbitrarily chosen, we can also conclude that **<u>any mass placed within a uniform spherical shell will not experience any gravitational force due to the shell.</u>**

Appendix B: Variation of the Gravitational Field Strength above and below the Earth's Surface

Consider the Earth to be a non-rotating spherical planet of uniform density ρ. Take the radius of the Earth to be r_E, and the mass of the Earth to be m_E.

To determine the variation of g with respect to the distance from the centre of the Earth r, the following results are useful (see Appendix A above):

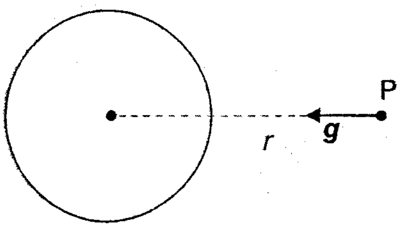
1. The acceleration due to gravity **outside** a spherical body of uniform density is the same as if the entire mass of the body were concentrated at its centre.

2. The acceleration due to gravity at all points **inside** a spherical shell of uniform density is zero.

With these results, we are able to obtain expressions for the acceleration due to gravity both above and below the Earth's surface.

__Gravitational Field Strength above the Earth's surface ($r \geq r_E$)__

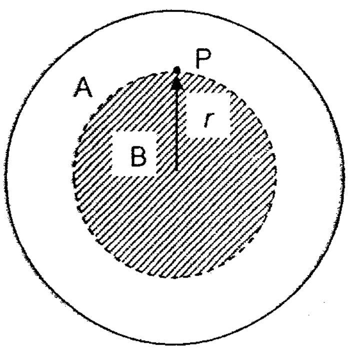
It follows from result 1 that for $r \geq r_E$, $g = G\dfrac{m_E}{r^2}$; and at $r = r_E$, $g = G\dfrac{m_E}{{r_E}^2}$



Since G and m_E are constants, $g \propto \dfrac{1}{r^2}$.

__Gravitational Field Strength below the Earth's surface ($r \leq r_E$)[6]__

Consider a point P below the Earth's surface and at a distance r away from the centre of the Earth.



The gravitational field at P,

g = g at P due to spherical shell A + g at P due to spherical mass B

By result 2,
Gravitation field at P due to shell A = 0, and

By result 1,
Gravitation field at P due to spherical mass B

$= G\dfrac{M_B}{r^2}$ where M_B is the mass of the spherical mass B

[6] The gravitational field strength below the Earth's surface is not explicitly required in the syllabus, it is only introduced here for completeness.

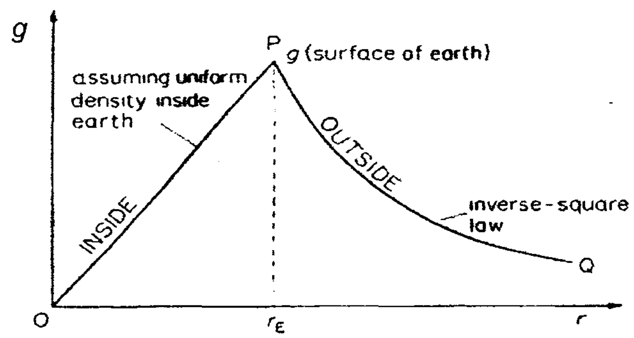
But $\quad M_B = \rho$ (Volume of spherical mass B) = $\rho(\frac{4\pi r^3}{3})$

Hence the gravitational field at P (with r < radius of the Earth) is:

$$g = \frac{G}{r^2}(\frac{4\pi r^3 \rho}{3}) = (\frac{4\pi G\rho}{3})r$$

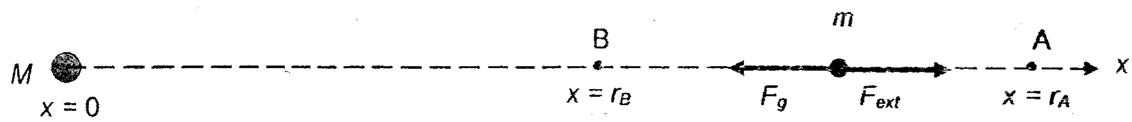
Since $\frac{4\pi G\rho}{3}$ is a constant, we see that for $r < r_E$, $g \propto r$.

A sketch of g against r is shown in below.



Appendix C: Derivation of the Gravitational Potential Energy and Gravitational Potential
Consider a particle of mass m moving between two points A and B in the gravitational field exerted by another particle M. To find an expression for the gravitational potential energy of this system, we need to first find the work done by an external force in moving the mass m between the two points without a change in kinetic energy of m.

Suppose we set up the coordinate system as below.



from B to A

By Newton's Law of Gravitation, the gravitational force[7] F_g **acting on the mass *m* at a distance** x away from M,

$$F_g = -G\frac{Mm}{x^2}$$

The negative sign here indicates that the gravitational force is pointing in the negative x direction.

For the kinetic energy of m to remain constant, we must move m at a **constant velocity** (acceleration = 0).

Therefore, the external force F_{ext} acting on m is opposite in direction and equal in magnitude to F_g.

$$\Rightarrow \qquad F_{ext} = G\frac{Mm}{x^2}$$

Since F_{ext} is not constant but varies with distance x away from M, to find the work done by this external force from moving the test mass m between A and B,

$$W = \int_{r_A}^{r_B} F_{ext}\,dx = \int_{r_A}^{r_B} G\frac{Mm}{x^2}\,dx = GMm\left[-\frac{1}{x}\right]_{r_A}^{r_B} = -GMm\left[\frac{1}{r_B}-\frac{1}{r_A}\right]$$

By conservation of energy, since there is no increase in the kinetic energy of the system,

W = Change in gravitational potential energy of the system, ΔU

Therefore, $$W = \Delta U = U_B - U_A = -GMm\left[\frac{1}{r_B}-\frac{1}{r_A}\right] = \left[-G\frac{Mm}{r_B}\right]-\left[-G\frac{Mm}{r_A}\right]$$

[7] Note that is gravitational force is an internal force to the system of M and m. To consider work done on m, we must always treat M and m as a single system, as m only experiences a force due to the gravitational field exerted by M.

Suppose we choose the reference point A for gravitational potential energy U to be zero at $r_A = \infty$, i.e.

$$U_A = U_\infty = \left[-G\frac{Mm}{r_A}\right] = 0$$

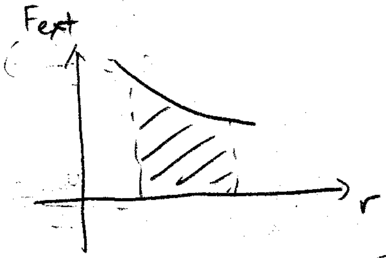
Then,
$$U_B - U_A = U_B = -G\frac{Mm}{r_B}$$

Therefore, the gravitational potential of a mass m placed at a distance of r away from another point mass M, is given by

$$\boxed{U = -G\frac{Mm}{r}}$$

From the definition of the potential energy at a point, the potential at a point at a distance r away from the point mass M,
$$\phi = \frac{W}{m}$$

$$\Rightarrow \quad \boxed{\phi = -\frac{GM}{r}}$$



Work done to move object from 1pt to another point.

$F_{ext} = F_g \Rightarrow \Delta KE = 0$

$\Delta PE = \int F_{ext}\, dr$

$= \int \frac{GMm}{r^2}\, dr$

$$\boxed{\Delta PE = -\frac{GMm}{r} (+c)}$$

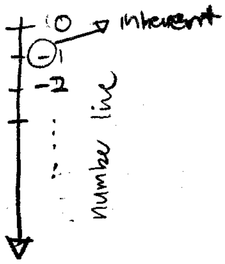
$\left(\text{Potential} = \frac{PE}{M} \right)$

$\boxed{\text{Energy is scalar}}$

B ▭ GPE mgh 0 → ∞ ref as 0 → neutral (Furthest away)

A ▭ 0 −mgh

negative



$U = -\frac{GMm}{r}$

$\varphi = -\frac{GM}{r}$ → source mass

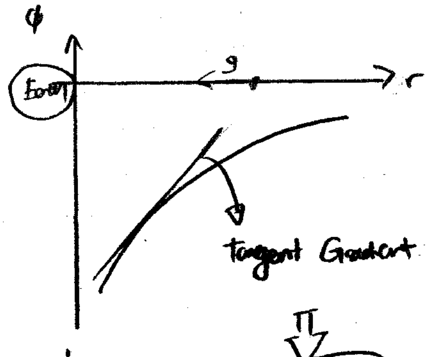
Chapter Summary

$F_g = \frac{GMm}{r^2}$ → Vector ⟷ $\boxed{F_g = -\frac{dF_{Ext}}{dr}}$ $U = -\frac{GMm}{r}$

$\boxed{g = \frac{F_g}{m}}$ $\boxed{\phi = \frac{U}{m}}$

$g = \frac{GM}{r^2}$ $\phi = -\frac{GM}{r}$

$$\boxed{g = -\frac{d\phi}{dr}}$$



tangent Gradient > 0

$g = -\frac{d\phi}{dr}$

Direction of g is in the direction of decreasing potential