Chapter 1

Statics

Before reading any of the text of this book, you should read Appendices B and C. The material discussed there (dimensional analysis, checking limiting cases, etc.) is extremely important. It's fairly safe to say that an understanding of these topics is absolutely necessary for an understanding of physics. And they make the subject a lot more fun, too!

For many of you, the material in this chapter will be mainly review. As such, the text here will be relatively short. This is an "extra" chapter. Its main purpose is that it provides me with an excuse to give you some cool statics problems. Try as many as you like, but don't go overboard and spend too much time with them; more important and relevant material will soon be at hand.

1.1 Balancing forces

This chapter deals with "static" situations, that is, ones where all the objects are motionless. If an object is motionless, then $F = ma$ tells us that the total force acting on it must be zero. (The converse is not true, of course. The total force can be zero with a constant non-zero velocity. But we'll deal only with static problems here). The whole goal in a statics problem is to find out what the various forces have to be so that there is zero net force acting on each object (and zero net torque, too; but that's the topic of the next section). Since a force is a vector, this goal involves breaking the force up into its components. You can pick cartesian coordinates, polar coordinates, or perhaps another set. (It is usually clear from the problem which system will make your calculations easiest.) Once this is done, you simply demand that the total force in each direction is zero.

There are many different types of forces in the world, most of which are large-scale effects of complicated things going on at smaller scales. For example, the tension in a rope comes about from the chemical bonds that hold the molecules in the rope together. In doing a mechanics problem, there is of course no need to analyze all the details of the forces taking place in the rope at the molecular scale.

You simply call whatever force there is a "tension" and get on with the problem.

Four types of forces come up repeatedly when doing problems:

Tension

Tension is a general name for a force that a rope, stick, etc., exerts when it is pulled on. Every piece of the rope feels a tension force in both directions, except the end point, which feels a tension on one side and a force on the other side from whatever object is attached to the end.

In some cases, the tension may vary along the rope. (The "Rope wrapped around pole" example at the end of this section is an example of this.) In other cases, the tension must be the same everywhere. For example, in a hanging massless rope, or in a massless rope hanging over a frictionless pulley, the tension must be the same at all points, because otherwise there would be a net force on at least one tiny piece, and then $F = ma$ would give an infinite acceleration for this tiny piece.

Normal force

This is the force perpendicular to a surface that a surface applies to an object. The total force applied by a surface is usually a combination of the normal force and the friction force (see below). But for "frictionless" surfaces such as greasy ones or ice, only the normal force exists. The normal force comes about because the surface actually compresses a tiny bit and acts like a very rigid spring; the surface gets squeezed until the restoring force equals the force the object applies.

REMARK: Technically, the only difference between a "normal force" and a "tension" is the direction of the force. Both situations can be modeled by a spring. In the case of a normal force, the spring (a plane, a stick, or whatever) is compressed, and the force on the given object is directed away from the spring. In the case of a tension, the spring is stretched, and the force on the given object is directed toward the spring. Things like sticks can provide both normal forces and tensions. But a rope, for example, has a hard time providing a normal force. ♣

Friction

Friction is the force parallel to a surface that a surface applies to an object. Some surfaces, such as sandpaper, have a great deal of friction. Some, such as greasy ones, have essentially no friction. There are two types of friction, called "kinetic" friction and "static" friction.

Kinetic friction (which we won't deal with in this chapter) deals with two objects moving relative to each other. It is usually a good approximation to say that the kinetic friction between two objects is proportional to the normal force between them. We call the constant of proportionality μ_k (called the "coefficient of kinetic friction"), where μ_k depends on the two surfaces involved. Thus, $F = \mu_k N$. The direction of the force is opposite to the motion.

Static friction deals with two objects at rest relative to each other. In the static case, all we can say prior to solving a problem is that the static friction force has a

maximum value equal to $F_{\max} = \mu_s N$ (where μ_s is the "coefficient of static friction"). In a given problem, it is most likely less than this. For example, if a block of large mass M sits on a surface with coefficient of friction μ_s, and you give the block a tiny push to the right (tiny enough so that it doesn't move), then the friction force is of course not $\mu_s N = \mu_s Mg$ to the left. Such a force would send the block sailing off to the left. The true friction force is simply equal and opposite to the tiny force you apply. What the coefficient μ_s tells you is that if you apply a force larger than $\mu_s Mg$ (the maximum friction force), then the block will end up moving to the right.

Gravity

Consider two point objects, with masses M and m, separated by a distance R. Newton's law for the gravitational force says that the force between these objects is attractive and has magnitude $F = GMm/R^2$, where $G = 6.67 \cdot 10^{-11}$ m^3/(kg $\cdot$ s^2). As we will show in Chapter 4, the same law applies to spheres. That is, a sphere may be treated like a point mass located at its center. Therefore, an object on the surface of the earth feels a gravitational force equal to

$$F = m\left(\frac{GM}{R^2}\right) \equiv mg, \tag{1.1}$$

where M is the mass of the earth, and R is its radius. This equation defines g. Plugging in the numerical values, we obtain (as you can check) $g \approx 9.8$ m/s^2. Every object on the surface of the earth feels a force of mg downward. If the object is not accelerating, then there must also be other forces present (normal forces, etc.) to make the total force zero.

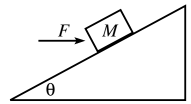

Figure 1.1

Example (Block on plane): A block of mass M rests on a plane inclined at angle θ (see Fig. 1.1). You apply a horizontal force of $F = Mg$ to the block, as shown.

(a) Assume that the friction force between the block and plane is large enough to keep the block still. What are the normal and friction forces (call them N and F_f) that the plane exerts on the block?

(b) Let the coefficient of static friction be μ. For what range of angles θ will the block remain still?

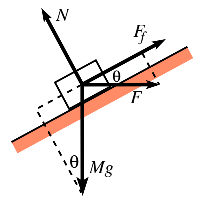

Figure 1.2

Solution:

(a) We will break the forces up into components parallel and perpendicular to the plane ($\hat{x}$ and $\hat{y}$ coordinates would work just as well). The forces are $F = Mg$, F_f, N, and the weight Mg (see Fig. 1.2). Balancing the forces parallel and perpendicular to the plane gives, respectively (with upward along the plane taken to be positive),

$$\begin{aligned} F_f &= Mg\sin\theta - Mg\cos\theta, \qquad \text{and} \\ N &= Mg\cos\theta + Mg\sin\theta. \end{aligned} \tag{1.2}$$

REMARKS: Note that if $\tan\theta < 1$, then F_f is positive; and if $\tan\theta < 1$, then F_f is negative. F_f ranges from $-Mg$ to Mg, as θ ranges from 0 to $\pi/2$ (these limiting cases are fairly obvious). You can show that N is maximum when $\tan\theta = 1$, in which case $F_f = 0$ and $N = \sqrt{2}Mg$. ♣

(b) The coefficient μ tells us that $|F_f| \le \mu N$. So we have, from eqs. (1.2),

$$Mg|\sin\theta - \cos\theta| \le \mu Mg(\cos\theta + \sin\theta). \tag{1.3}$$

The absolute value here signifies that we must consider two cases.

- If $\tan\theta \ge 1$, then eq. (1.3) becomes

$$\begin{aligned} \sin\theta - \cos\theta &\le \mu(\cos\theta + \sin\theta) \\ \implies \quad \tan\theta &\le \frac{1+\mu}{1-\mu}. \end{aligned} \tag{1.4}$$

- If $\tan\theta \le 1$, then eq. (1.3) becomes

$$\begin{aligned} -\sin\theta + \cos\theta &\le \mu(\cos\theta + \sin\theta) \\ \implies \quad \tan\theta &\ge \frac{1-\mu}{1+\mu}. \end{aligned} \tag{1.5}$$

Putting these two ranges for θ together, we have

$$\frac{1-\mu}{1+\mu} \le \tan\theta \le \frac{1+\mu}{1-\mu}. \tag{1.6}$$

Remarks: For very small μ, these bounds both approach 1. (This makes sense. If there is little friction, then the components along the plane of the horizontal and vertical Mg forces must nearly cancel.) A special value for μ is 1. From eq. (1.6), we see that $\mu = 1$ is the cutoff value that allows θ to reach 0 and $\pi/2$. ♣

Let's now do an example involving a rope in which the tension varies with position. We'll need to consider differential pieces of the rope to solve this problem.

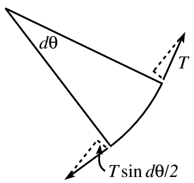

Figure 1.3

Example (Rope wrapped around pole): A rope wraps an angle θ around a pole. You grab one end and pull with a tension T_0. The other end is attached to a large object, say, a boat. If the coefficient of static friction between the rope and the pole is μ, what is the largest force the rope can exert on the boat, if the rope is to not slip around the pole?

Solution: Consider a small piece of the rope. Let this piece subtend and angle $d\theta$, and let the tension in it be T. From Fig. 1.3, we see that the sum of the radially inward components of the tensions at the two ends of this piece is $2T\sin(d\theta/2)$. Therefore, this equals the (radially outward) normal force that the pole exerts on the rope. Note that for small $d\theta$, we may (using $\sin\epsilon \approx \epsilon$) write this normal force as $N_{d\theta} = T\,d\theta$.

The maximum friction force on this little piece of rope is $\mu N_{d\theta} = \mu T\,d\theta$. This friction force is what gives rise to the difference in tension between the two ends of the little piece. In other words, the tension, as a function of θ, satisfies

$$\begin{aligned} T(\theta + d\theta) &\le T(\theta) + \mu N_{d\theta} \\ \implies \quad dT &\le \mu T\,d\theta \end{aligned}$$

$$\begin{aligned} \implies \quad \int \frac{dT}{T} &\le \int \mu\,d\theta \\ \implies \quad \ln T &\le \mu\theta + C \\ \implies \quad T &\le T_0 e^{\mu\theta}, \end{aligned} \tag{1.7}$$

where we have used the fact that $T = T_0$ when $\theta = 0$.

This exponential behavior here is quite strong (as exponential behaviors tend to be). If we let $\mu = 1$, then just a quarter turn around the pole produces a factor of $e^{\pi/2} \approx 5$. One full revolution yields a factor of $e^{2\pi} \approx 530$, and two full revolutions yield a factor of $e^{4\pi} \approx 300,000$. Needless to say, the limiting factor in such a case is not your strength, but rather the structural integrity of the pole around which the rope winds.

1.2 Balancing torques

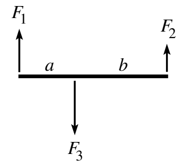

Figure 1.4

In addition to balancing forces in a statics problem, we must also balance torques. We'll have much more to say about torques in Chapters 7 and 8, but we'll need one important fact here.

Consider the situation in Fig. 1.4, where three forces are applied perpendicularly to a stick, which is assumed to remain motionless. F_1 and F_2 are the forces at the ends, and F_3 is the force in the interior. (We have, of course, $F_3 = F_1 + F_2$, because the stick is at rest.)

Claim 1.1 *If the system is motionless, then $F_3 a = F_2(a + b)$. (In other words, the torques around the left end cancel. And you can show that they cancel around any other point, too.)*

We'll prove this claim in another way using angular momentum, in Chapter 7, but let's give a short proof here.

Proof: We'll make one reasonable assumption, which is that the correct relationship between the forces and distances is of the form

$$F_3 f(a) = F_2 f(a + b), \tag{1.8}$$

where $f(x)$ is a function to be determined.[1] Applying this assumption with the roles of "left" and "right" reversed, we have

$$F_3 f(b) = F_1 f(a + b) \tag{1.9}$$

Adding these two equations, and using $F_3 = F_1 + F_2$, gives

$$f(a) + f(b) = f(a + b). \tag{1.10}$$

This implies that $f(x)$ is a linear function, $f(x) = Ax$, as was to be shown.[2] The constant A is irrelevant, since it cancels out in eq. (1.8).) ■

[1] We're simply assuming linearity in F. That is, two forces of F applied at a point should be the same as a force of $2F$ applied at that point. You can't really argue with that.

[2] Another proof of this Claim is given in Problem 11.

The quantities $F_3 a$, $F_2(a+b)$, $F_3 b$, etc., are of course just the torques around various pivot points. Note that dividing eq. (1.8) by eq. (1.9) gives $F_1 f(a) = F_2 f(b)$, and hence $F_1 a = F_2 b$, which says that the torques cancel around the point where F_3 is applied. You can easily show that the torques cancel around any arbitrary pivot point.

When adding up all the torques in a given physical setup, it is of course required that you use the same pivot point when calculating each torque.

In the case where the forces aren't perpendicular to the stick, the claim applies to the components of the forces perpendicular to the stick. (This is fairly obvious. The components parallel to the stick won't have any effect on rotating the stick around the pivot point.) Therefore, referring to the figures shown below, we have

$$F_1 a \sin\theta_a = F_2 b \sin\theta_b. \tag{1.11}$$

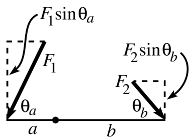

Figure 1.5

This equation can be viewed in two ways:

- $(F_1 \sin\theta_a)a = (F_2 \sin\theta_b)b$. In other words, we effectively have smaller forces acting on the given "lever-arms". (See Fig. 1.5.)

- $F_1(a\sin\theta_a) = F_2(b\sin\theta_b)$ In other words, we effectively have the given forces acting on smaller "lever-arms". (See Fig. 1.6.)

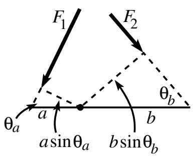

Figure 1.6

Claim 1.1 shows that even if you apply a just a tiny force, you can balance the torque due to a very large force, provided that you make your lever-arm sufficiently long. This fact led a well-know mathematician of long ago to claim that he could move the earth if given a long enough lever-arm.

> One morning while eating my Wheaties,
> I felt the earth move 'neath my feeties.
> The cause for alarm
> Was a long lever-arm,
> At the end of which grinned Archimedes!

One handy fact that is often used is that the torque, due to gravity, on a stick of mass M is the same as the torque due to a point-mass M located at the center of the stick. (The truth of this statement relies on the fact that torque is a linear function of distance to the pivot point.) More generally, the torque on an object due to gravity may be treated simply as the torque due to a force Mg located at the center of mass.

We'll have much more to say about torque in Chapters 7 and 8, but for now we'll simply use the fact that in a statics problem, the torques around any given point must balance.

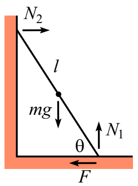

Figure 1.7

Example (Leaning ladder): A ladder leans against a frictionless wall. If the coefficient of friction with the ground is μ, what is the smallest angle the ladder can make with the ground, and not slip?

Solution: As shown in Fig. 1.7, we have three unknown forces: the friction force

F, and the normal forces N_1 and N_2. And we have three equations that will allow us to solve for these three forces: $\Sigma F_{\text{vert}} = 0$, $\Sigma F_{\text{horiz}} = 0$, and $\Sigma\tau = 0$.

Looking at the vertical forces, we see that $N_1 = mg$. And then looking at the horizontal forces, we see that $N_2 = F$. So we have quickly reduced the unknowns from three to one.

We will now use $\Sigma\tau = 0$ to find N_2. But first we must pick the "pivot" point, around which we will calculate the torques. Any stationary point will work fine, but certain choices make the calculations easier than others. The best choice for the pivot is generally the point at which the most forces act, because then the $\Sigma\tau = 0$ equation will have the smallest number of terms in it (because a force provides no torque around the point where it acts, since the lever-arm is zero).

So in this problem, the best choice for pivot is the bottom end of the ladder.[3] Balancing the torques due to gravity and N_2, we have

$$N_2 \ell \sin\theta = mg(\ell/2)\cos\theta \qquad \Longrightarrow \qquad N_2 = \frac{mg}{2\tan\theta}. \tag{1.12}$$

This, then, also equals the friction force F. The condition $F \le \mu N_2 = \mu mg$ therefore becomes

$$\frac{mg}{2\tan\theta} \le \mu mg \qquad \Longrightarrow \qquad \tan\theta \ge \frac{1}{2\mu}. \tag{1.13}$$

Remarks: The factor of 1/2 in the above answer comes from the fact that the ladder behaves like a point mass located half way up itself. You can quickly show that the answer for the analogous problem, but now with a massless ladder and a person standing a fraction f up along it, is $\tan\theta \ge f/\mu$.

Note that the total force exerted on the ladder by the floor points up at an angle given by $\tan\beta = N_1/F = (mg)/(mg/2\tan\theta) = 2\tan\theta$. We see that this force does *not* point along the ladder. There is simply no reason why it should. But there *is* a nice reason why it should point upwards with twice the slop of the ladder. This is the direction which causes the lines of the three forces on the ladder to be collinear, as shown in Fig. 1.8.

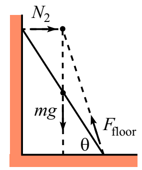

Figure 1.8

This collinearity is a neat little theorem for statics problems involving three forces. The proof is simple. If the three lines weren't collinear, then one force would produce a nonzero torque around the intersection point of the other two lines of force.[4] ♣

That's about all there is to statics problems. All you have to do is balance the forces and torques. To be sure, this sometimes requires a bit of cleverness. There are all sorts of tricks to be picked up by doing problems, so you may as well tackle a few....

[3] But you should verify that other choices of pivot, for example, the middle or top of the ladder, give the same result.

[4] The one exception to this reasoning is when no two of the lines intersect; that is, when all three lines are parallel. Equilibrium is certainly possible in such a scenario. (Of course, you can hang on to our collinearity theorem if you consider the parallel lines to meet at infinity.)

1.3 Problems

Section 1.1: Balancing forces

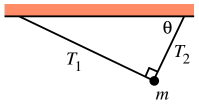

Figure 1.9

1. **Hanging mass**

   A mass m, held up by two strings, hangs from a ceiling (see Fig. 1.9). The strings form a right angle. In terms of the angle θ shown, what is the tension in each string?

2. **Block on a plane**

   A block sits on a plane inclined at an angle θ. Assume that the friction force is large enough to keep the block still. For what θ is the horizontal component of the normal force maximum?

Figure 1.10

3. **Motionless chain** $*$

   A frictionless surface is in the shape of a function which has its endpoints at the same height but is otherwise arbitrary. A chain of uniform mass per unit length rests on this surface (from end to end; see Fig. 1.10). Show that the chain will not move.

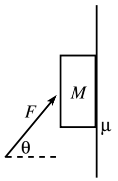

Figure 1.11

4. **Keeping the book up**

   A book of mass M is positioned against a vertical wall. The coefficient of friction between the book and the wall is μ. You wish to keep the book from falling by pushing on it with a force F applied at an angle θ to the horizontal ($-\pi/2 < \theta < \pi/2$). (See Fig. 1.11.) For a given θ, what is the minimum F required? What is the limiting value for θ for which there exists an F which will keep the book up?

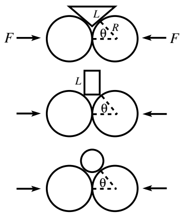

Figure 1.12

5. **Objects between circles** $**$

   Each of the following planar objects is placed, as shown in Fig. 1.12, between two frictionless circles of radius R. The mass density of each object is σ, and the radii to the points of contact make an angle θ with the horizontal. For each case, find the horizontal force that must be applied to the circles to keep them together. For what θ is this force maximum or minimum?

   (a) An isosceles triangle with common side length L.

   (b) A rectangle with height L.

   (c) A circle.

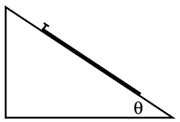

Figure 1.13

6. **Hanging rope** $*$

   (a) A rope with length ℓ and mass density ρ per unit length is suspended from one end. Find the tension along the rope.

   (b) The same rope now lies on a plane inclined at an angle θ (see Fig. 1.13). The top end is nailed to the plane. The coefficient of friction is μ. What is the tension at the top of the rope? (Assume the setup is obtained by initially having the rope lie without any tension on a horizontal plane, and then tilting the plane up to an angle θ.)

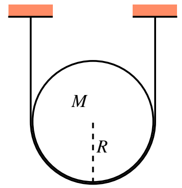

Figure 1.14

7. **Supporting a disc** $***$

   (a) A disc of mass M and radius R is held up by a massless string, as shown in Fig. 1.14. The surface of the disc is frictionless. What is the tension in the string? What is the normal force per unit length the string applies to the disc?

   (b) Let there now be friction between the disc and the string, with coefficient μ. What is the smallest possible tension in the string at its lowest point?

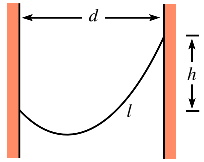

Figure 1.15

8. **Hanging chain** $****$

   (a) A chain of uniform mass density per unit length hangs between two walls. Find the shape of the chain. (Except for an arbitrary additive constant, the function describing the shape should contain one unknown constant.)

   (b) The unknown constant in your answer depends on the horizontal distance d between the walls, the vertical distance h between the support points, and the length ℓ of the chain (see Fig. 1.15). Find an equation involving these given quantities that determines the unknown constant.

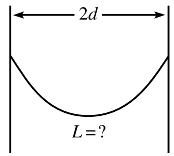

Figure 1.16

9. **Hanging gently** $**$

   A chain hangs between two supports located at the same height, a distance $2d$ apart (see Fig. 1.16). How long should the chain be in order to minimize the magnitude of the force on the supports?

   You may use the fact that the height of the hanging chain is of the form $y(x) = (1/\alpha)\cosh(\alpha x) + a$. You will eventually have to solve an equation numerically in this problem.

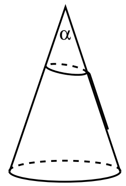

Figure 1.17

10. **Mountain Climber** $****$

    A mountain climber wishes to climb up a frictionless conical mountain. He wants to do this by throwing a lasso (a rope with a loop) over the top and climbing up along the rope. (Assume the mountain climber is of negligible height, so that the rope lies along the mountain; see Fig. 1.17.) At the bottom of the mountain are two stores. One sells "cheap" lassos (made of a segment of rope tied to loop of rope of *fixed* length). The other sells "deluxe" lassos (made of one piece of rope with a loop of *variable* length; the loop's length may change without any friction of the rope with itself). See Fig. 1.18.

    When viewed from the side, this conical mountain has an angle α at its peak. For what angles α can the climber climb up along the mountain if he uses:

    (a) a "cheap" lasso and loops it once around the top of the mountain?

    (b) a "deluxe" lasso and loops it once around the top of the mountain?

    (c) a "cheap" lasso and loops it N times around the top of the mountain? (Assume no friction of the rope with itself.)

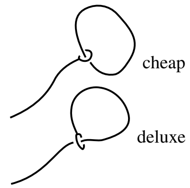

Figure 1.18

(d) a "deluxe" lasso and loops it N times around the top of the mountain? (Assume no friction of the rope with itself.)

Section 1.2: Balancing torques

11. **Equality of torques** $**$

This problem gives another way to demonstrate Claim 1.1, using an inductive argument. We'll get you started, and then you can do the general case.

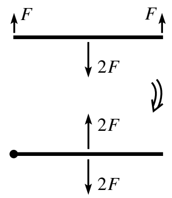

Figure 1.19

Consider the situation where forces F are applied upward at the ends of a stick of length ℓ, and a force $2F$ is applied downward at the midpoint (see Fig. 1.19). The stick will not rotate (by symmetry), and it will not translate (because the net force is zero). We may consider the stick to have a pivot at the left end, if we wish. If we then erase the force F on the right end and replace it with a force $2F$ at the middle, then the two $2F$ forces in the middle will cancel, so the stick will remain still. (There will now be a different force applied at the pivot, namely zero, but the purpose of the pivot is to simply apply whatever force is necessary to keep the end still.) Therefore, we see that a force F applied at a distance ℓ from a pivot is 'equivalent' to a force $2F$ applied at a distance $\ell/2$.

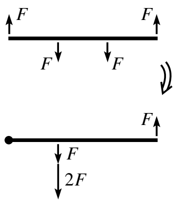

Figure 1.20

Now consider the situation where forces F are applied upward at the ends, and forces F are applied downward at the $\ell/3$ and $2\ell/3$ marks (see Fig. 1.20). The stick will not rotate (by symmetry), and it will not translate (because the net force is zero). Consider the stick to have a pivot at the left end. From the above paragraph, the force F at $2\ell/3$ is equivalent to a force $2F$ at $\ell/3$. Making this replacement, we have the situation shown in Fig. 1.20, with a force $3F$ at the $\ell/3$ mark. Therefore, we see that a force F applied at a distance ℓ is equivalent to a force $3F$ applied at a distance $\ell/3$.

Use induction to show that a force F applied at a distance ℓ is equivalent to a force nF applied at a distance ℓ/n, and then argue why this demonstrates Claim 1.1.

12. **Find the force** $*$

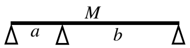

Figure 1.21

A stick of mass M is held up by supports at each end. Each support clearly provides a force of $Mg/2$. Now put another support somewhere in the middle (say, at a distance a from one support, and b from the other; see Fig. 1.21). What forces do the three supports now provide? Can you solve this?

13. **Leaning sticks** $*$

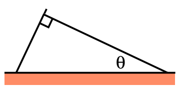

Figure 1.22

One stick leans on another as shown in Fig. 1.22. A right angle is formed where they meet, and the right stick makes an angle θ with the horizontal. The left stick extends infinitesimally beyond the end of the right stick. The coefficient of friction between the two sticks is μ. The sticks have the same mass density per unit length and are both hinged at the ground. What is the minimum angle θ for which the sticks do not fall?

14. **Supporting a ladder** $*$

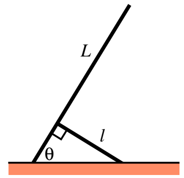

Figure 1.23

A ladder of length L and mass M has its bottom end attached to the ground by a pivot. It makes an angle θ with the horizontal, and is held up by a person of total length ℓ who is attached to the ground by a pivot at his feet (see Fig. 1.23). Assume that the person has zero mass, for simplicity. The person and the ladder are perpendicular to each other. Find the force that the person applies to the ladder.

15. **Stick on a circle** $*$

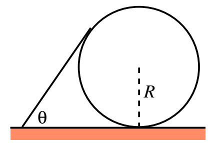

Figure 1.24

A stick of mass per unit length ρ rests on a circle of radius R (see Fig. 1.24). The stick makes an angle θ with the horizontal. The stick is tangent to the circle at its upper end. Friction exists at all points of contact in this problem. Assume that all of these friction forces are large enough to keep the system still. Find the friction force between the ground and the circle.

16. **Leaning sticks and circles** $**$

Figure 1.25

A large number of sticks (of mass per unit length ρ) and circles (of radius R) lean on each other, as shown in Fig. 1.25. Each stick makes an angle θ with the ground. Each stick is tangent to a circle at its upper end. The sticks are hinged to the ground, and every other surface is *frictionless* (unlike in the previous problem). In the limit of a very large number of sticks and circles, what is the normal force between a stick and the circle it rests on, very far to the right? (Assume that the last circle is glued to the floor, to keep it from moving.)

17. **Balancing the stick** $**$

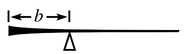

Figure 1.26

Given a semi-infinite stick (that is, one that goes off to infinity in one direction), find how its density should depend on position so that it has the following property: If the stick is cut at an arbitrary location, the remaining semi-infinite piece will balance on a support located a distance b from the end (see Fig. 1.26).

18. **The spool** $**$

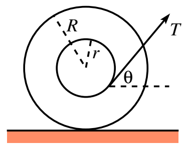

Figure 1.27

A spool consists of an axis of radius r and an outside circle of radius R which rolls on the ground. A thread which is wrapped around the axis is pulled with a tension T (see Fig. 1.27).

(a) Given R and r, what angle, θ, should the thread make with the horizontal so that the spool does not move. Assume there is large enough friction between the spool and ground so that the spool doesn't slip.

(b) Given R, r, and a coefficient of friction μ between the spool and ground, what is the largest T can be (assuming the spool doesn't move)?

(c) Given R and μ, what should r be so that the upper bound on T found in part (b) is as small as possible (assuming the spool doesn't move)? What is the resulting value of T?

1.4 Solutions

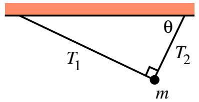

Figure 1.28

1. **Hanging mass**

   Balancing the horizontal and vertical force components on the mass gives (see Fig. 1.28)

$$\begin{aligned} T_1 \sin\theta &= T_2 \cos\theta, \\ T_1 \cos\theta + T_2 \sin\theta &= mg. \end{aligned} \tag{1.14}$$

   The solution to these equations is

$$T_1 = mg\cos\theta, \qquad \text{and} \qquad T_2 = mg\sin\theta. \tag{1.15}$$

   As a double-check, these have the correct limits when $\theta \to 0$ or $\theta \to \pi/2$.

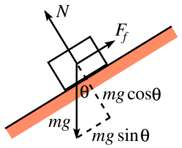

Figure 1.29

2. **Block on a plane**

   The component of the block's weight perpendicular to the plane is $mg\cos\theta$ (see Fig. 1.29). The normal force is therefore $N = mg\cos\theta$. The horizontal component of this is $mg\cos\theta\sin\theta$. To maximize this, we can either take a derivative or we can write it as $(1/2)mg\sin 2\theta$, from which it is clear that the maximum occurs at $\theta = \pi/4$. (The maximum is $mg/2$.)

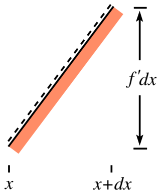

Figure 1.30

3. **Motionless chain**

   Let the curve run from $x = a$ to $x = b$. Consider a little piece of the chain between x and $x + dx$ (see Fig. 1.30). The length of this piece is $\sqrt{1+f'^2}\,dx$. Therefore, its mass is $\rho\sqrt{1+f'^2}\,dx$, where ρ is the mass per unit length. The component of gravity along the curve is $-gf'/\sqrt{1+f'^2}$ (with positive taken to be to the right). So the total force, F, along the curve is

$$\begin{aligned} F &= \int_a^b \left(\frac{-gf'}{\sqrt{1+f'^2}}\right)\left(\rho\sqrt{1+f'^2}\,dx\right) \\ &= -\rho g \int_a^b f'\,dx \\ &= -g\rho\big(f(a) - f(b)\big) \\ &= 0. \end{aligned} \tag{1.16}$$

4. **Keeping the book up**

   The normal force on the wall is $F\cos\theta$. So the friction force holding the book up is at most $\mu F\cos\theta$. The other vertical forces on the book are $-Mg$ and the vertical component of F, which is $F\sin\theta$. If the book is to stay up, we must have

$$\mu F\cos\theta + F\sin\theta - Mg > 0. \tag{1.17}$$

   So F must satisfy

$$F > \frac{Mg}{\mu\cos\theta + \sin\theta}. \tag{1.18}$$

   There is no F that satisfies this if the right-hand-side is infinite. This occurs when

$$\tan\theta = -\mu. \tag{1.19}$$

   So if θ is more negative than this, then it is impossible to keep the book up.

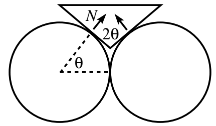

Figure 1.31

5. **Objects between circles**

   (a) Let N be the normal force. The goal in this problem is to find the horizontal component of N, which is $N\cos\theta$.

   The upward force on the triangle from the normal forces is $2N\sin\theta$ (see Fig. 1.31). This must equal the weight of the triangle, which is σ times the area. Since the bottom angle of the isosceles triangle is 2θ, the top side of the triangle has length $2L\sin\theta$, and the altitude to that side is $L\cos\theta$. So the area of the triangle is $L^2\sin\theta\cos\theta$. The mass is therefore $\sigma L^2\sin\theta\cos\theta$. Equating the weight with the upward normal force gives $N = g\sigma L^2\cos\theta/2$, independent of R. The horizontal component is therefore

$$N\cos\theta = \frac{g\sigma L^2\cos^2\theta}{2}. \tag{1.20}$$

   This is 0 at $\theta = \pi/2$, and it grows as θ decreases to $\theta = 0$ (even though the triangle is getting smaller). It has the interesting property of approaching the finite number $g\sigma L^2/2$, as $\theta \to 0$.

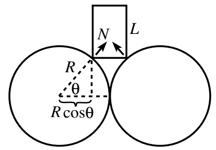

Figure 1.32

   (b) From Fig. 1.32, the base of the rectangle has length $2R(1-\cos\theta)$. The mass is therefore $\sigma 2RL(1-\cos\theta)$. Equating the weight with the upward normal force, $2N\sin\theta$, gives $N = g\sigma LR(1-\cos\theta)/\sin\theta$. The horizontal component is therefore

$$N\cos\theta = \frac{g\sigma LR(1-\cos\theta)\cos\theta}{\sin\theta}. \tag{1.21}$$

   This is 0 at both $\theta = 0$ and $\theta = \pi/2$. Taking the derivative to find where it reaches a maximum, we find (using $\sin^2\theta = 1 - \cos^2\theta$),

$$\cos^3\theta - 2\cos\theta + 1 = 0. \tag{1.22}$$

   An obvious root of this equation is $\cos\theta = 1$ (which we know is not the maximum). Dividing through by the factor $(\cos\theta - 1)$ gives

$$\cos^2\theta + \cos\theta - 1 = 0. \tag{1.23}$$

   The roots of this are

$$\cos\theta = \frac{-1 \pm \sqrt{5}}{2}. \tag{1.24}$$

   We must choose the plus sign, since $|\cos\theta| \le 1$. This root is the golden ratio, $\cos\theta \approx 0.618\ldots$! The angle θ is $\approx 51.8°$.

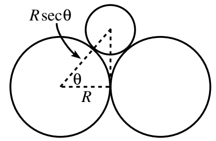

Figure 1.33

   (c) From Fig. 1.33, the length AB is $R\sec\theta$, so the radius of the top circle is $R(\sec\theta - 1)$. The mass is therefore $\sigma\pi R^2(\sec\theta - 1)^2$. Equating the weight with the upward normal force, $2N\sin\theta$, gives $N = g\sigma\pi R^2(\sec\theta - 1)^2/2\sin\theta$. The horizontal component is therefore

$$N\cos\theta = \frac{g\sigma\pi R^2}{2}\frac{\cos\theta}{\sin\theta}\left(\frac{1}{\cos\theta} - 1\right)^2. \tag{1.25}$$

   This is 0 at $\theta = 0$ (from using $\cos\theta \approx 1 - \theta^2/2$ for small θ, so $1/\cos\theta \approx 1 + \theta^2/2$; so there is an extra θ in the numerator in the small θ limit). For $\theta \to \pi/2$, it behaves like $1/\cos\theta$, which goes to infinity. (In this limit, N points almost vertically, but its magnitude is so large that the horizontal component still approaches infinity.)

6. **Hanging rope**

(a) Consider a small piece of the rope between y and $y+dy$ ($0 \le y \le \ell$). The forces on the piece are $T(y+dy)$ upward, $T(y)$ downward, and the weight of the piece (which can be written as $\rho g\, dy$) downward. If the rope is still, then we have $T(y+dy) = T(y) + \rho g\, dy$. Expanding this to first order in dy gives $T'(y) = \rho g$. The tension in the bottom of the rope is 0, so integrating from $y = 0$ up to a position y gives

$$T(y) = \rho g y. \tag{1.26}$$

As a double-check, at the top end we have $T(\ell) = \rho g \ell$, which is just the weight of the whole rope, as it should be.

Of course, one can simply write down the correct answer $T(y) = \rho g y$ by demanding that the tension at a given point accounts for the weight of all the rope below it.

(b) Let z be the coordinate along the plane ($0 \le z \le \ell$). Consider a small piece of the rope between z and $z + dz$. Balancing the forces on the rope along the plane gives $T(z+dz) + F_f(z)dz = T(z) + \rho g \sin\theta\, dz$, where upward is taken to be the positive direction for the friction force $F_f dz$ (where $F_f(z)$ is the friction force per unit length). Expanding this to first order in dz gives

$$T'(z) = \rho g \sin\theta - F_f(z). \tag{1.27}$$

The largest the friction force on a small piece can be is $\mu N dz$, where N is the normal force per unit length (namely $N = \rho g \cos\theta$). But it may not need to be this large, depending on θ and μ. F_f will not be so large that it makes the right-hand-side of eq. (1.27) negative. There are two cases to consider.

- If $\rho g \sin\theta < \mu N$ (i.e., if $\tan\theta < \mu$), then $F_f(z)$ will simply be equal to $\rho g \sin\theta$ (i.e., the friction of each little piece accounts for its weight; so $T'(z) = 0$ everywhere, and so $T(z) = 0$ everywhere).
- If $\rho g \sin\theta > \mu N$ (i.e., if $\tan\theta > \mu$), then $F_f(z) = \mu N = \mu\rho g\cos\theta$. Therefore,

$$T'(z) = \rho g \sin\theta - \mu\rho g \cos\theta. \tag{1.28}$$

Using $T(0) = 0$, this gives

$$T(\ell) = \rho g \ell \sin\theta - \mu\rho g \ell \cos\theta \equiv \rho g (y_0 - \mu x_0), \tag{1.29}$$

where x_0 and y_0 are the width and height of the rope. In the limit $x_0 = 0$ (i.e., a vertical rope), we get the answer from part (a).

The angle $\theta_0 = \arctan(\mu)$ is the minimum angle of inclination for which there is any force on the nail at the top end.

7. **Supporting a disc**

(a) The force down on the disc is Mg, and the force up is $2T$. These forces must balance, so

$$T = \frac{Mg}{2}. \tag{1.30}$$

We can find the normal force per unit length the string applies to the disc in two ways.

First method: Let $N d\theta$ be the normal force on an arc which subtends an angle θ. (So N/R is the desired normal force per unit arclength.) The tension

in the string is uniform, so N is a constant, independent of θ. The upward component of this force is $N d\theta \cos\theta$ (where θ is measured from the vertical, i.e., $-\pi/2 \le \theta \le \pi/2$). The total upward force must be Mg, so we require

$$\int_{-\pi/2}^{\pi/2} N \cos\theta\, d\theta = Mg. \tag{1.31}$$

The integral on the left is $2N$, so $N = Mg/2$. The normal force per unit length, N/R, is $Mg/2R$.

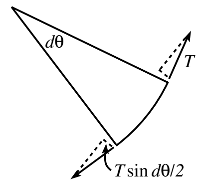

Figure 1.34

Second method: Consider the normal force, $N d\theta$, on a small arc of the circle which subtends and angle $d\theta$. The tension forces on each end of the small piece of string here almost cancel, but they don't exactly, due to the fact that they point in different directions (see Fig. 1.34). Their non-zero sum is what gives the normal force. It's easy to see that the two forces have a sum equal to $2T\sin(d\theta/2)$ (directed radially inward). Since $d\theta$ is small, we may approximate this as $N d\theta = T d\theta$. Hence, $N = T$. The normal force per unit arclength, N/R, is therefore T/R. And since $T = Mg/2$, this equals $Mg/2R$.

(b) Let $T(\theta)$ be the tension, as a function of θ, for $-\pi/2 \le \theta \le \pi/2$. ($T$ will depend on θ now, since there is a tangential force from the friction.) Let $N(\theta)d\theta$ be the normal force, as a function of θ, on an arc which subtends an angle $d\theta$. Then from the second solution above, we have (the existence of friction doesn't affect this equality)

$$T(\theta) = N(\theta). \tag{1.32}$$

Let $F_f(\theta)d\theta$ be the friction force that this little piece of string applies to the disc between θ and $\theta + d\theta$. Balancing the forces on this little piece of (massless) string, we have

$$T(\theta + d\theta) = T(\theta) + F_f(\theta)d\theta. \tag{1.33}$$

(This holds for $\theta > 0$. There would be a minus sign in front of F_f for $\theta < 0$. Since the tension is symmetric around $\theta = 0$, we'll only bother with the $\theta > 0$ case.) Writing $T(\theta + d\theta) \approx T(\theta) + T'(\theta)d\theta$, we find

$$T'(\theta) = F_f(\theta). \tag{1.34}$$

Since the goal is to find the minimum value for $T(0)$, and since we know that $T(\pi/2)$ must be equal to the constant $Mg/2$ (because the tension in the string above the disc is $Mg/2$, from part (a)), we want to look at the case where T' (which equals F_f) is as large as possible. But by the definition of static friction, we have $F_f(\theta)d\theta \le \mu N(\theta)d\theta = \mu T(\theta)d\theta$ (where the second equality comes from eq. (1.32)). Therefore, $F_f \le \mu T$. So eq. (1.34) becomes

$$T'(\theta) \le \mu T(\theta). \tag{1.35}$$

Separating variables and integrating from the bottom of the rope up to an angle θ gives $\ln\big((T(\theta)/T(0)\big) \le \mu\theta$. Exponentiating gives

$$T(\theta) \le T(0)e^{\mu\theta}. \tag{1.36}$$

Letting $\theta = \pi/2$, and noting that T equals $Mg/2$ when $\theta = \pi/2$, yields $Mg/2 \le T(0)e^{\mu\pi/2}$. So we finally have

$$T(0) \ge \frac{Mg}{2}e^{-\mu\pi/2}. \tag{1.37}$$

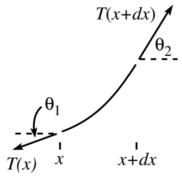

Figure 1.35

This minimum value for $T(0)$ goes to Mg/2 as $\mu \to 0$, as it should. And it goes to zero as $\mu \to \infty$, as it should (imagine a very sticky surface, so that the friction force from the rope near $\theta = \pi/2$ accounts for essentially all the weight).

8. **Hanging chain**

(a) Let the chain be described by the function $Y(x)$. Let the tension in the chain be described by the function $T(x)$. Consider a small piece of the chain, with endpoints having coordinates x and $x + dx$ (see Fig. 1.35). Let the tension at x pull downward at an angle θ_1 with respect to the horizontal. Let the tension at $x + dx$ pull upward at an angle θ_2 with respect to the horizontal. (So $\cos\theta_1 = 1/\sqrt{1+(Y'(x))^2}$, and $\cos\theta_2 = 1/\sqrt{1+(Y'(x+dx))^2} \approx 1/\sqrt{1+(Y'(x)+Y''(x)dx)^2}$.) Balancing the horizontal and vertical forces on the small piece of chain gives

$$\begin{aligned} T(x+dx)\cos\theta_2 &= T(x)\cos\theta_1, \\ T(x+dx)\sin\theta_2 &= T(x)\sin\theta_1 + \frac{g\rho}{\cos\theta_1}\,dx, \end{aligned} \tag{1.38}$$

where ρ is the mass per unit length. The second term on the right above is the weight of the small piece, since $dx/\cos\theta_1$ is its length. (The second of these equations is valid only for x on the right side of the minimum, i.e., where $Y'(x) > 0$. When $Y'(x) < 0$, there should be a minus sign in front of the second term on the right.)

We somehow have to solve these two differential equations for the two unknown functions, $Y(x)$ and $T(x)$. (The angles θ_1 and θ_2 depend on $Y(x)$). There are various ways to do this. Here is one way, broken down into three steps.

- Squaring and adding eqs. (1.38) gives

$$\big(T(x+dx)\big)^2 = \big(T(x)\big)^2 + 2T(x)g\rho\tan\theta_1\,dx + \mathcal{O}(dx^2). \tag{1.39}$$

Writing $T(x+dx) \approx T(x) + T'(x)dx$, and using $\tan\theta_1 = Y'$, we find (neglecting higher order terms in dx)

$$T' = g\rho Y', \tag{1.40}$$

and so

$$T(x) = g\rho Y(x) + C. \tag{1.41}$$

- Now let's see what we can extract from the first equation in (1.38). Expanding things to first order gives (all the functions are evaluated at x, which we won't bother writing, for the sake of neatness)

$$(T + T'dx)\frac{1}{\sqrt{1+(Y'+Y''dx)^2}} = T\frac{1}{\sqrt{1+Y'^2}}. \tag{1.42}$$

Expanding the first square root gives (to first order in dx)

$$(T + T'dx)\frac{1}{\sqrt{1+Y'^2}}\left(1 - \frac{Y'Y''dx}{1+Y'^2}\right) = T\frac{1}{\sqrt{1+Y'^2}}. \tag{1.43}$$

To first order in dx this yields

$$\frac{T'}{T} = \frac{Y'Y''}{1+Y'^2}. \tag{1.44}$$

Integrating both sides yields

$$\ln T + c = \frac{1}{2}\ln(1+Y'^2), \tag{1.45}$$

where c is a constant of integration. Exponentiation then gives

$$b^2T^2 = 1 + Y'^2, \tag{1.46}$$

where $b \equiv e^c$.

- We may now combine eq. (1.46) with eq. (1.40) to solve for T. Eliminating Y' gives $b^2T^2 = 1 + T'^2/(g\rho)^2$. Solving for T' and separating variables yields

$$g\rho\int dx = \int \frac{dT}{\sqrt{b^2T^2-1}}. \tag{1.47}$$

(We took the positive square-root because we are looking at x on the right side, for which $T' > 0$.)

The integral on the left is $g\rho(x-a)$, for some constant a. The integral on the right equals $(1/b)\ln(bT + \sqrt{b^2T^2-1})$. So we find (with $\alpha \equiv bg\rho$)

$$T(x) = \frac{g\rho}{2\alpha}\left(e^{\alpha(x-a)} + e^{-\alpha(x-a)}\right) \equiv \frac{g\rho}{\alpha}\cosh\big(\alpha(x-a)\big). \tag{1.48}$$

Using eq. (1.41) to find Y, we have

$$Y(x) = \frac{1}{\alpha}\cosh\big(\alpha(x-a)\big) + B, \tag{1.49}$$

where B is some constant (which is rather meaningless; it just depends on where you choose the $y = 0$ point). This is valid when $Y'(x) > 0$, that is, when $x > a$. If $Y'(x) < 0$, then there is a minus sign in the second of eqs. (1.38), but the result turns out to be the same.

We may eliminate the need for a if we pick the $x = 0$ point to be at the minimum of the chain. Then $Y'(0) = 0$ implies $a = 0$. So we finally have

$$Y(x) = \frac{1}{\alpha}\cosh(\alpha x) + B. \tag{1.50}$$

This is the shape of the chain.

(b) The constant α may be determined by the locations of the endpoints and the length of the chain. If one hangs a chain between two points, then the given information is (1) the horizontal distance, d, between the two points, (2) the vertical distance, h, between the two points, and (3) the length, ℓ, of the chain (see Fig. 1.36). Note that one does *not* easily know the horizontal distances between the ends and the minimum point (which we have chosen as the $x = 0$ point). If $h = 0$, then these distances are of course $d/2$; but otherwise, they are not obvious.

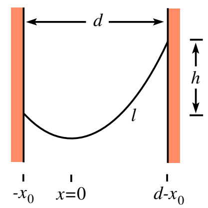

Figure 1.36

If we let the left endpoint be located at $x = -x_0$, then the right endpoint is at $x = d - x_0$. We now have two unknowns, x_0 and α. Our two conditions are

$$Y(d - x_0) - Y(-x_0) = h \tag{1.51}$$

(we take the right end to be higher than the left end, without loss of generality), and the condition that the length equals ℓ, which takes the form (using eq.

(1.50))

$$\ell = \int_{-x_0}^{d-x_0} \sqrt{1+Y'^2}\,dx = \frac{1}{\alpha}\sinh(\alpha x)\Big|_{-x_0}^{d-x_0}. \tag{1.52}$$

If $h = 0$, the limits are simply $\pm d/2$, so we may (numerically) solve for b, using only this equation, $\alpha\ell/2 = \sinh(\alpha d/2)$. If $h \neq 0$, one has to (numerically) solve two equations for two unknowns. Writing out eqs. (1.51) and (1.52) explicitly, we have

$$\begin{aligned}\cosh\big(\alpha(d-x_0)\big) - \cosh(-\alpha x_0) &= \alpha h,\\ \sinh\big(\alpha(d-x_0)\big) - \sinh(-\alpha x_0) &= \alpha\ell.\end{aligned} \tag{1.53}$$

If we take the difference of the squares of these two equations, and use the hyperbolic identities $\cosh^2 x - \sinh^2 x = 1$ and $\cosh x\cosh y - \sinh x \sinh y = \cosh(x-y)$, we obtain

$$2\cosh(\alpha d) - 2 = \alpha^2(\ell^2 - h^2), \tag{1.54}$$

which determines α. (This can be rewritten as $2\sinh(\alpha d/2) = \alpha\sqrt{\ell^2-h^2}$, if desired.)

There are various limits one can check here. If $\ell^2 = d^2 + h^2$ (i.e., the chain forms a straight line), then we have $2\cosh(\alpha d) - 2 = \alpha^2 d^2$; the solution to this is $\alpha = 0$, which does indeed correspond to a straight line. Also, if ℓ is much larger than both d and h, then the solution is a very large α, which corresponds to a 'droopy' chain.

9. **Hanging gently**

We need to calculate the length of the chain, to get its mass. Then we need to find the slope at the support, to break the force there into its components.

The slope as a function of x is

$$y' = \frac{d}{dx}\left(\frac{1}{\alpha}\cosh(\alpha x) + a\right) = \sinh(\alpha x). \tag{1.55}$$

The total length is therefore

$$L = \int_{-d}^{d}\sqrt{1+y'^2}\,dx = \int_{-d}^{d}\cosh(\alpha x) = \frac{2}{\alpha}\sinh(\alpha d). \tag{1.56}$$

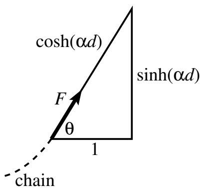

Figure 1.37

The weight of the rope is $W = \rho L g$, where ρ is the mass per unit length. Each support applies a vertical force of $W/2$. This must equal $F\sin\theta$, where F is the support force, and θ is the angle it makes with the horizontal. Since $\tan\theta = y'(d) = \sinh(\alpha d)$, we have $\sin\theta = \tanh(\alpha d)$ (from Fig. 1.37). Therefore,

$$F = \frac{W}{2}\frac{1}{\sin\theta} = \frac{\rho g\sinh(\alpha d)}{\alpha}\frac{1}{\tanh(\alpha d)} = \frac{\rho g}{\alpha}\cosh(\alpha d). \tag{1.57}$$

Taking the derivative of this (as a function of α), and setting it equal to zero gives

$$\tanh(\alpha d) = \frac{1}{\alpha d}. \tag{1.58}$$

This must be solved numerically. The result is

$$\alpha d \approx 1.1997 \equiv \eta. \tag{1.59}$$

The shape of the chain that requires the minimum F is therefore

$$y(x) \approx \frac{d}{\eta}\cosh\left(\frac{\eta x}{d}\right) + a. \tag{1.60}$$

From eqs. (1.56) and (1.59), the length is

$$L = \frac{2d}{\eta}\sinh(\eta) \approx (2.52)d. \tag{1.61}$$

To get an idea of what the chain looks like, we can calculate the ratio of the height, h, to d.

$$\frac{h}{d} = \frac{y(d)-y(0)}{d} = \frac{\cosh(\eta)-1}{\eta} \approx 0.675. \tag{1.62}$$

We can also calculate the angle of the rope at the supports; we find $\theta \approx 56.5°$.

Remark: One can also ask what shape the chain should take in order to minimize the horizontal or vertical component of F.

The vertical component F_y is just the weight, so we clearly want the shortest possible chain, namely a horizontal one (which requires an infinite F.) This corresponds to $\alpha = 0$.

The horizontal component is $F_x = F\cos\theta$. Since $\cos\theta = 1/\cosh(\alpha d)$, eq. (1.57) gives $F_x = \rho g/\alpha$. This goes to zero as $\alpha \to \infty$, which corresponds to a chain of infinite length. ♣

10. **Mountain Climber**

(a) We will take advantage of the fact that a cone is 'flat', in the sense that you can make one out of a piece of paper, without crumpling the paper.

Cut the cone along a straight line emanating from the peak and passing through the knot of the lasso, and roll the cone flat onto a plane. Call the resulting figure, a sector of a circle, S. (See Fig. 1.38.)

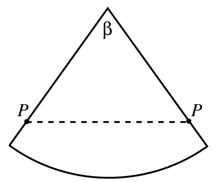

Figure 1.38

If the cone is very sharp, then S will look like a thin 'pie piece'. If the cone is very wide, with a shallow slope, then S will look like a pie with a piece taken out of it. Points on the straight-line boundaries of the sector S are identified with each other. Let P be the location of the lasso's knot. Then P appears on each straight-line boundary, at equal distances from the tip of S. Let β be the angle of the sector S.

The key to the problem is realizing that the path of the lasso's loop must be a straight line on S, as shown in Fig. 1.38. (The rope will take the shortest distance between two points since there is no friction, and rolling the cone onto a plane does not change distances.) Such a straight line between the two identified points P is possible if and only if the sector S is smaller than a semicircle, i.e., $\beta < 180°$.

Let C denote a cross sectional circle, at a distance d (measured along the cone) from the top of the mountain, and let μ equal the ratio of the circumference of C to d. Then a semicircular S implies that $\mu = \pi$. This then implies that the radius of C is equal to $d/2$. Therefore, $\alpha/2 = \sin^{-1}(1/2)$. So we find that if the climber is to be able to climb up along the mountain, then

$$\alpha < 60°. \tag{1.63}$$

Having $\alpha < 60°$ guarantees that there is a loop around the cone of shorter length than the distance straight to the peak and back.

Remark: When viewed from the side, the rope should appear perpendicular to the side of the mountain at the point opposite the lasso's knot. A common mistake is to assume that this implies $\alpha < 90°$. This is not the case, because the loop does not lie in a plane. Lying in a plane, after all, would imply an elliptical loop; but the loop must certainly have a discontinuous change in slope where the knot is. (For planar, triangular mountains, the answer to the problem would be $\alpha < 90°$.) ♣

(b) Use the same strategy. Roll the cone onto a plane. If the mountain very steep, the climber's position can fall by means of the loop growing larger; if the mountain has a shallow slope, the climber's position can fall by means of the loop growing smaller. The only situation in which the climber will not fall is the one where the change in the position of the knot along the mountain is exactly compensated by the change in length of the loop.

In terms of the sector S in a plane, the condition is that if we move P a distance ℓ up (down) along the mountain, the distance between the identified points P decreases (increases) by ℓ. We must therefore have $2\sin(\beta/2) = 1$. So $\beta = 60°$, and hence μ (defined in part (a)) is equal to $\pi/3$. This corresponds to

$$\alpha = 2\sin^{-1}(1/6) \approx 19°. \tag{1.64}$$

We see that there is exactly one angle for which the climber can climb up along the mountain.

Remark: Another way to see that β equals $60°$ is to note that the three directions of rope emanating from the knot all have the same tension, since the deluxe lasso is one continuous piece of rope. Therefore they must have $120°$ angles between themselves. This implies that $\beta = 60°$. ♣

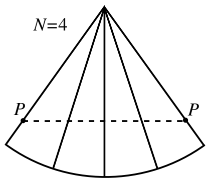

Figure 1.39

(c) Roll the cone N times onto a plane, as shown in Fig. 1.39 for $N = 4$. The resulting figure S_N is a sector of a circle divided into N equal sectors, each representing a copy of the cone. S_N must be smaller than a semicircle, so we must have $\mu < \pi/N$. Therefore,

$$\alpha < 2\sin^{-1}\Big(\frac{1}{2N}\Big). \tag{1.65}$$

(d) Roll the cone N times onto a plane. From the reasoning in part (b), we must have $N\beta = 60°$. Therefore,

$$\alpha = 2\sin^{-1}\Big(\frac{1}{6N}\Big). \tag{1.66}$$

11. **Equality of torques**

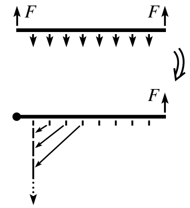

Figure 1.40

The pattern is clear, so let's prove it by induction. Assume that we have shown that a force F applied at a distance d is equivalent to a force kF applied at a distance d/k, for all integers k up to $n-1$. We now want to show that the statement holds for $k = n$.

Consider the situation in Fig. 1.40. Forces F are applied at the ends of a stick, and forces $2F/(n-1)$ are applied at the $j\ell/n$ marks (for $1 \le j \le n-1$). The stick will not rotate (by symmetry), and it will not translate (because the net force is zero). Consider the stick to have a pivot at the left end. Replacing the interior forces by their 'equivalent' ones at the ℓ/n mark (see Fig. 1.40) gives a total force there equal to

$$\frac{2F}{n-1}\Big(1+2+3+\cdots+(n-1)\Big) = \frac{2F}{n-1}\left(\frac{n(n-1)}{2}\right) = nF. \tag{1.67}$$

We therefore see that a force F applied at a distance ℓ is equivalent to a force nF applied at a distance ℓ/n, as was to be shown.

It is now clear that the Claim 1.1 holds. To be explicit, consider a tiny distance ϵ (small compared to a). Then a force F_3 at a distance a is equivalent to a force $F_3(a/\epsilon)$ at a distance ϵ. (Actually, our reasoning above only works if a/ϵ is an integer, but since a/ϵ is very large, we can just pick the closest integer to it, and there will be a negligible error.) But a force $F_3(a/\epsilon)$ at a distance ϵ is equivalent to a force $F_3(a/\epsilon)(\epsilon/(a+b)) = F_3a/(a+b)$ at a distance $(a+b)$. Since this 'equivalent' force at the distance $(a+b)$ cancels the force F_2 there (since the stick is motionless), we have $F_3a/(a+b) = F_2$, which proves the claim.

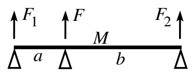

Figure 1.41

12. **Find the force**

In Fig. 1.41, let the supports at the ends exert forces F_1 and F_2, and let the support in the interior exert a force F. Then

$$F_1 + F_2 + F = Mg. \tag{1.68}$$

Balancing torques around the left and right ends gives, respectively,

$$\begin{aligned} Fa + F_2(a+b) &= Mg\frac{a+b}{2}, \\ Fb + F_1(a+b) &= Mg\frac{a+b}{2}. \end{aligned} \tag{1.69}$$

We have used the fact that the mass of the stick can be treated as a point mass at its center. Note that the equation for balancing the torques around the center-of-mass is redundant; it is obtained by taking the difference between the two previous equations. (Balancing torques around the middle pivot also takes the form of a linear combination of these equations.)

It appears as though we have three equations and three unknowns, but we really have only two equations, because the sum of eqs. (1.69) gives eq. (1.68). So, since we have two equations and three unknowns, the system is underdetermined. Solving eqs. (1.69) for F_1 and F_2 in terms of F, we see that any forces of the form

$$(F_1, F, F_2) = \left(\frac{Mg}{2} - \frac{Fb}{a+b}, F, \frac{Mg}{2} - \frac{Fa}{a+b}\right) \tag{1.70}$$

are possible. In retrospect, it is obvious that the forces are not determined. By changing the height of the new support an infinitesimal distance, one can make F be anything from 0 up to $Mg(a+b)/2b$, which is when the stick comes off the left support (assuming $b \geq a$).

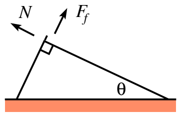

Figure 1.42

13. **Leaning sticks**

Let M_l be the mass of the left stick, and let M_r be the mass of the right stick. Then $M_l/M_r = \tan\theta$ (see Fig. 1.42). Let N be the normal force between the sticks, and let F_f be the friction force between the sticks. (So F_f has a maximum value of μN.) Balancing the torques on the left stick (around the contact point with the ground) gives

$$N = \frac{M_l g}{2}\sin\theta. \tag{1.71}$$

Balancing the torques on the right stick (around the contact point with the ground) gives

$$F_f = \frac{M_r g}{2}\cos\theta. \tag{1.72}$$

The condition $F_f \leq \mu N$ becomes

$$M_r\cos\theta \leq \mu M_l \sin\theta. \tag{1.73}$$

Using $M_l/M_r = \tan\theta$, this becomes

$$\tan^2\theta \geq \frac{1}{\mu}. \tag{1.74}$$

This answer checks in the two extremes: In the limit $\mu = 0$, we see that θ must be very close to $\pi/2$, which makes sense. In the limit $\mu = \infty$ (that is, we have sticky sticks), we see that θ can be anything above a very small lower bound, which also makes sense.

14. **Supporting a ladder**

Let F be the desired force. Note that F must be directed along the person (that is, perpendicular to the ladder), because otherwise there would be a net torque on the person relative to his pivot. This would result in an infinite acceleration of the (massless) person. (If the person has mass m, you can easily show that he must apply an additional force of $mg\sin\theta/2$ down along the ladder.)

Look at torques around the pivot point of the ladder. The gravitational force on the ladder provides a torque of $Mg(L/2)\cos\theta$ (tending to turn it clockwise). The force F provides a torque of $F(\ell/\tan\theta)$ (tending to turn it counterclockwise). Equating these torques gives

$$F = \frac{MgL}{2\ell}\sin\theta. \tag{1.75}$$

Remarks: This F goes to zero as $\theta \to 0$, as it should.[5]

[5] For $\theta \to 0$, we need to lengthen the ladder with a massless extension, because the person will have to be very far to the right if the sticks are to be perpendicular.

F grows to the constant $MgL/2\ell$, as θ increases to $\pi/2$ (which isn't entirely obvious). So if you ever find yourself lifting up a ladder in the (strange) manner where you keep yourself perpendicular to it, you will find that you must apply a larger force, the higher the ladder goes. (However, in the special case where the ladder is exactly vertical, no force is required. You can see that our above calculations are not valid in this case, because we made a division by $\cos\theta$, which is zero when $\theta = \pi/2$.)

The normal force at the pivot of the person (that is, the vertical component of F, if the person is massless) is equal to $MgL\sin\theta\cos\theta/2\ell$. This has a maximum value of $MgL/4\ell$ at $\theta = \pi/4$. ♣

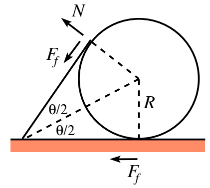

Figure 1.43

15. **Stick on a circle**

Let N be the normal force between the stick and the circle, and let F_f be the friction force between the ground and the circle (see Fig. 1.43). Then we immediately see that the friction force between the stick and the circle is also F_f (since the torques from the two friction forces on the circle must cancel).

Looking at torques on the stick, around the point of contact with the ground, we have $Mg\cos\theta(L/2) = NL$ (since the mass of the stick is effectively all located at its center, as far as torques are concerned), where M is the mass of the stick and L is its length. So $N = (Mg/2)\cos\theta$. Balancing the horizontal forces on the circle gives $N\sin\theta = F_f + F_f\cos\theta$. So we have

$$F_f = \frac{N\sin\theta}{1+\cos\theta} = \frac{Mg}{2}\frac{\sin\theta\cos\theta}{1+\cos\theta}. \tag{1.76}$$

But $M = \rho L$, and from the figure we have $L = R/\tan(\theta/2)$. Using the identity $\tan(\theta/2) = \sin\theta/(1+\cos\theta)$, we finally obtain

$$F_f = \frac{1}{2}\rho g R\cos\theta. \tag{1.77}$$

In the limit $\theta \to \pi/2$, F_f approaches 0, which makes sense. In the limit $\theta \to 0$ (i.e., a very long stick), the friction force approaches the constant $\rho g R/2$, which isn't so obvious.

16. **Leaning sticks and circles**

Let s_i be the ith stick, and let c_i be the ith circle.

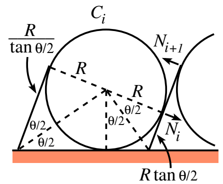

Figure 1.44

The normal forces c_i feels from s_i and from s_{i+1} are equal, because these two forces provide the only horizontal forces on the frictionless circle, so they must cancel. Let N_i be this normal force.

Look at the torques on s_{i+1} (around the hinge on the ground). The torques come from N_i, N_{i+1}, and the weight of s_i. From Fig. 1.44, we see that N_i acts at a point which is a distance $R\tan(\theta/2)$ away from the hinge. Since the stick has a length $R/\tan(\theta/2)$, this point is a fraction $\tan^2(\theta/2)$ up along the stick. Therefore, balancing the torques on s_{i+1} gives

$$\frac{1}{2}Mg\cos\theta + N_i\tan^2\frac{\theta}{2} = N_{i+1}. \tag{1.78}$$

N_0 is by definition 0, so we have $N_1 = (Mg/2)\cos\theta$ (as in the previous problem). Successively using eq. (1.78), we see that N_2 equals $(Mg/2)\cos\theta\big(1+\tan^2(\theta/2)\big)$, and N_3 equals $(Mg/2)\cos\theta\big(1+\tan^2(\theta/2)+\tan^4(\theta/2)\big)$, and so on. In general,

$$N_i = \frac{Mg}{2}\cos\theta\left(1+\tan^2\frac{\theta}{2}+\tan^4\frac{\theta}{2}+\cdots+\tan^{2(i-1)}\frac{\theta}{2}\right). \tag{1.79}$$

In the limit $i \to \infty$, we may write the infinite sum in closed form as

$$\lim_{i\to\infty} N_i \equiv N_\infty = \frac{Mg}{2}\frac{\cos\theta}{1-\tan^2(\theta/2)}. \tag{1.80}$$

(This is the solution to eq. (1.78), with $N_i = N_{i+1}$, so if a limit exists, it must be this.)

Using $M = \rho L = \rho R/\tan(\theta/2)$, we may write N_∞ as

$$N_\infty = \frac{\rho R g}{2}\frac{1}{\tan(\theta/2)}\frac{\cos\theta}{1-\tan^2(\theta/2)}. \tag{1.81}$$

The identity $\cos\theta = \cos^2(\theta/2) - \sin^2(\theta/2)$ may be used to put this in the form

$$N_\infty = \frac{\rho R g}{2}\frac{\cos^3(\theta/2)}{\sin(\theta/2)}. \tag{1.82}$$

This blows up for $\theta \to 0$, which is obvious (N_∞ approaches half the weight of a stick in this limit). And it approaches the constant $\rho Rg/4$ for $\theta \to \pi/2$, which is not at all obvious.

Note that the horizontal force that must be applied to the last circle far to the right is $N_\infty \sin\theta = \rho Rg\cos^4(\theta/2)$. This ranges from ρRg at $\theta = 0$, to $\rho Rg/4$ at $\theta = \pi/2$.

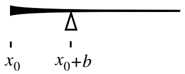

Figure 1.45

17. **Balancing the stick**

Let the stick go off to infinity in the positive x direction. Let it be cut at $x = x_0$, so the pivot point is at $x = x_0 + b$ (see Fig. 1.45). Let the density be $\rho(x)$. Then the condition that the torques around $x_0 + b$ cancel is

$$\int_{x_0}^{x_0+b} \rho(x)\big((x_0+b)-x\big)dx = \int_{x_0+b}^{\infty} \rho(x)\big(x-(x_0+b)\big)dx. \tag{1.83}$$

Combining the two integrals gives

$$I \equiv \int_{x_0}^{\infty} \rho(x)\big((x_0+b)-x\big)dx = 0. \tag{1.84}$$

We want this to equal 0 for all x_0, so the derivative of I with respect to x_0 must be 0. I depends on x_0 through both the limits of integration and the integrand. In taking the derivative, the former dependence requires finding the value of the integrand at the limits, while the latter dependence requires taking the derivative of the integrand w.r.t x_0, and then integrating. We obtain (using the fact that there is zero contribution from the ∞ limit)

$$0 = \frac{dI}{dx_0} = -b\rho(x_0) + \int_{x_0}^{\infty} \rho(x)dx. \tag{1.85}$$

Taking the derivative of this equation with respect to x_0 gives

$$b\rho'(x_0) = -\rho(x_0). \tag{1.86}$$

The solution to this is (rewriting the arbitrary x_0 as x)

$$\rho(x) = Ae^{-x/b}. \tag{1.87}$$

This falls off quickly if b is very small, which makes sense. And it falls off slowly if b is very large. Note that the density at the pivot is $1/e$ times the density at the end. And $1 - 1/e \approx 63\%$ of the mass is contained between the end and the pivot.

18. **The spool**

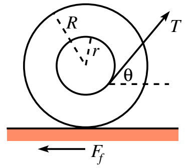

Figure 1.46

(a) Let F_f be the friction force the ground provides. Balancing horizontal forces gives (from Fig. 1.46)

$$T\cos\theta = F_f. \tag{1.88}$$

Balancing torques around the center of the circles gives

$$Tr = F_f R. \tag{1.89}$$

These two equations imply

$$\cos\theta = \frac{r}{R}. \tag{1.90}$$

(b) The normal force from the ground is

$$N = Mg - T\sin\theta. \tag{1.91}$$

The friction force, eq. (1.88), is $F_f = T\cos\theta$. So the statement $F_f \le \mu N$ becomes $T\cos\theta \le \mu(Mg - T\sin\theta)$. Therefore,

$$T \le \frac{\mu Mg}{\cos\theta + \mu\sin\theta}, \tag{1.92}$$

where θ is given by eq. (1.90).

(c) The maximum value of T is given in (1.92). This depends on θ, which in turn depends on r. We want to find the r which minimizes this maximum T.

The θ which maximizes the denominator in eq. (1.92) is easily found to be given by $\tan\theta = \mu$. The value of T for this θ is

$$T = \frac{\mu Mg}{\sqrt{1+\mu^2}} = Mg\sin\theta. \tag{1.93}$$

To find the corresponding r, we can use eq. (1.90) to write $\tan\theta = \sqrt{R^2 - r^2}/r$. The equality $\tan\theta = \mu$ then yields

$$r = \frac{R}{\sqrt{1+\mu^2}}. \tag{1.94}$$

This is the r which yields the smallest upper bound on T.

Remarks: In the limit $\mu = 0$, we have $\theta = 0$, $T = 0$, and $r = R$. In the limit $\mu = \infty$, we have $\theta = \pi/2$, $T = Mg$, and $r = 0$.

We can also ask the question: What should r be so that the upper bound on T found in part (b) is as large as possible? We then want to make the denominator in eq. (1.92) as small as possible. If $\mu < 1$, this is achieved at $\theta = \pi/2$ (with $r = 0$ and $T = Mg$). If $\mu > 1$, this is achieved at $\theta = 0$ (with $r = R$ and $T = \mu Mg$). These answers make sense. ♣