A Glimpse Of Calculus

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Polynomial Equations **2**
Roots of Quadratic Equations 3
Roots of Cubic Equations. 4
Roots of Quartic Equations 7
Roots of Quintic Equations. 9

Complex Numbers **10**
Complex Numbers 11
Complex Functions. 12
Stereographic Projection 16
Matrix Representation 17

Evaluation of π **19**
Liu Hui's Algorithm 20
Integration 24

Series **27**
Power Series 28
Integration 29
Ordinary Differential Equation 30
Term by Term Integration 31
Some Special Series 33

Roots of Quadratic Equations

- Solve the **Quadratic Equation**: $x^2 + bx + c = 0$.
  1. $\left(x + \frac{b}{2}\right)^2 - \frac{b^2}{4} + c = 0.$
  2. $\left(x + \frac{b}{2}\right)^2 = \frac{b^2 - 4c}{4}.$
  3. $x + \frac{b}{2} = \pm\frac{\sqrt{b^2 - 4c}}{2}.$
  4. $\boxed{x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}}$
  - Remove the **Linear Term** by **Substitution**.
    - Let $t = x + \frac{b}{2}$, i.e., $x = t - \frac{b}{2}$.
    - The equation becomes $\boxed{t^2 + d = 0}$

Roots of Cubic Equations

- Solve the **Cubic Equation**: $x^3 + bx^2 + cx + d = 0$.
  1. Use **Cardano's Method** **(1545)**: $\boxed{x = t - b/3}$
     - $t^3 + pt + q = 0$.
  2. Set $t = u + v$:
     - $(u^3 + v^3) + (3uv + p)(u + v) + q = 0.$
  3. Suppose $3uv + p = 0$.
     - $u^3 + v^3 = -q, \quad u^3v^3 = -p^3/27.$
  4. u^3 and v^3 are roots of
     - $z^2 + qz - p^3/27 = 0$.
  5. Solve the equation above to get u and v.
     - $x = t - b/3 = u + v - b/3$.

Roots of Cubic Equations

- Solve the **Cubic Equation**: $x^3 + bx^2 + cx + d = 0$.
  1. Set $x = t - b/3$.
     - $t^3 + pt + q = 0$.
  2. Solve $z^2 + qz - p^3/27 = 0$.
     - $z_1 = u^3$ and $z_2 = v^3$.
  3. $x = u + v - b/3 = \sqrt[3]{z_1} + \sqrt[3]{z_2} - b/3$.
- $t^3 + pt + q = 0$ is called the **Depressed Form**.
  - The **Discriminant** $\boxed{\Delta = 4p^3 + 27q^2}$
    - $\Delta > 0$: 3 distinct real roots.
    - $\Delta = 0$: repeated real roots.
    - $\Delta < 0$: 1 real and 2 nonreal conjugate roots.

Example

- Solve $x^3 - 6x^2 + 9x - 4 = 0$.
  1. $b = -6$. Set $x = t - b/3 = t + 2$.
     - Depressed form: $t^3 - 3t - 2 = 0$.
  2. $p = -3$ and $q = -2$. Solve $z^2 + qz - p^3/27 = 0$:
     - $z^2 - 2z + 1 = 0 \Rightarrow z_1 = z_2 = 1$.
  3. $u = \sqrt[3]{z_1} = 1$ and $v = \sqrt[3]{z_2} = 1$.
     - $x_1 = u + v + 2 = 1 + 1 + 2 = 4$.
  4. Factorize $x^3 - 6x^2 + 9x - 4 = (x-4)(x^2 - 2x + 1)$.
  5. Solve $x^2 - 2x + 1 = 0$: $x_2 = x_3 = 1$.
  6. Therefore, the roots are $x_1 = 4, x_2 = 1, x_3 = 1$.
     - $x^3 - 6x^2 + 9x - 4 = (x-4)(x-1)^2$.

Roots of Quartic Equations

- **Quartic Equation**: $x^4 + bx^3 + cx^2 + dx + e = 0$.
  1. **Ferrari's Method (1522–1565)**: Set $x = t - b/4$.
     - **Depressed Form**: $t^4 + pt^2 + qt + r = 0$.
  2. Solve a **Cubic Equation** in z:
     - $z^3 + \frac{5}{2}pz^2 + (2p^2 - r)z + \left(\frac{p^3}{2} - \frac{pr}{2} - \frac{q^2}{8}\right) = 0$.
  3. $\boxed{x = \frac{\pm\sqrt{p+2z} \pm \sqrt{-\left(3p + 2z \pm \frac{2q}{\sqrt{p+2z}}\right)}}{2} - \frac{b}{4}}$
     - The first and the third $\pm$ are both positive or negative.

Example

- Solve $x^4 - 12x^3 + 43x^2 - 24x - 80 = 0$.
  1. Set $x = t + 3$.
     - $t^4 - 11t^2 + 18t - 8 = 0$.
  2. Solve a cubic equation in z:
     - $z^3 - \frac{55}{2}z^2 + 250z - 750 = 0$.
       - Use Cardano's method to get $z = 15/2$.
  3. $x = \frac{\pm 2 \pm \sqrt{-(-18 \pm 18)}}{2}$.
     - $x = 1, 1, -4, 2$.

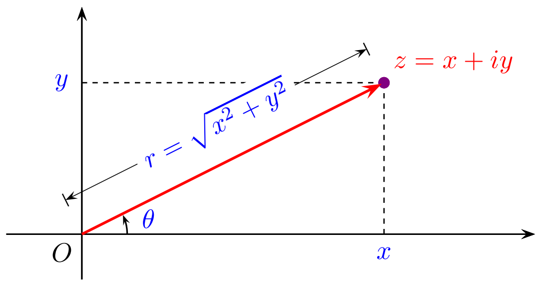
Roots of Quintic Equations

- **Quintic Equation**: $x^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$.
  - ÉVARISTE GALOIS (1811 – 1832) French
  - NIELS HENRIK ABEL (1802 – 1829) Norwegian
- **Abel's Impossibility Theorem**.
  - There is **No** general **Algebraic** solution
    - that is, expression using $+, -, \times, \div, \sqrt[n]{\ }$,

    to **Polynomial Equations** of **Degree** ≥ 5.
- **Remarks**.
  - Approximated solution may be found easily.
    - **Bisection Method**, **Newton-Raphson's Method**.
  - JOHANN CARL FRIEDRICH GAUSS (1777 – 1855)
    - **Fundamental Theorem of Algebra (1799)**: A polynomial equation of degree n has n roots in complex numbers.

Complex Numbers

- **Complex Numbers**: $\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}$, $\boxed{i^2 = -1}$
  - $\mathbb{C}$ is identified with the **Cartesian Plane** $\mathbb{R}^2$:
    - $x + iy \leftrightarrow (x, y)$.
  - **Polar Form**: $z = r(\cos\theta + i\sin\theta)$.
    - **Exponential Form**: $z = re^{i\theta}$, $r = |z|$, $\theta = \arg z$.



Complex Functions

- For any **Real** number x,
  - $e^{ix} = \cos x + i \sin x$,
  - $e^{-ix} = \cos(-x) + i\sin(-x) = \cos x - i \sin x$.

  We solve that

  - $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$ and $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$, $x \in \mathbb{R}$.
- **Definition**. For any **Complex Number** z,define
  - $\boxed{\cos z = \dfrac{e^{iz} + e^{-iz}}{2}}$ and $\boxed{\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}}$

  We can verify that all the **Trigonometric Identities** still hold:

  - $\cos^2 z + \sin^2 z = 1$;
  - $\sin 2z = 2 \sin z \cos z$;
  - $\cos 2z = \cos^2 z - \sin^2 z$; ....

Complex Functions

- Recall that $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$ and $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$.
  - $\cos^2 z + \sin^2 z = \left(\dfrac{e^{iz} + e^{-iz}}{2}\right)^2 + \left(\dfrac{e^{iz} - e^{-iz}}{2i}\right)^2$.
    - $\left(\dfrac{e^{iz} + e^{-iz}}{2}\right)^2 = \dfrac{e^{i2z} + e^{-i2z} + 2}{4}$.
    - $\left(\dfrac{e^{iz} - e^{-iz}}{2i}\right)^2 = \dfrac{e^{i2z} + e^{-i2z} - 2}{-4}$.
  - $\cos^2 z + \sin^2 z = 1$.
- **Definition**. $\boxed{\tan z = \dfrac{\sin z}{\cos z} = \dfrac{i(e^{iz} - e^{-iz})}{e^{iz} + e^{-iz}}}$
  - $\cot z = \dfrac{\cos z}{\sin z}$, $\sec z = \dfrac{1}{\cos z}$, $\csc z = \dfrac{1}{\sin z}$.

Complex Functions

- Recall that $\cos z = \dfrac{e^{iz}+e^{-iz}}{2}$ and $\sin z = \dfrac{e^{iz}-e^{-iz}}{2i}$.
  - $\cos i = \dfrac{e^{i\cdot i}+e^{-i\cdot i}}{2} = \dfrac{e^{-1}+e}{2} = \dfrac{e+e^{-1}}{2} \approx 1.543.$
  - $\sin i = \dfrac{e^{i\cdot i}-e^{-i\cdot i}}{2i} = \dfrac{e^{-1}-e}{2i} = \dfrac{i(e-e^{-1})}{2} \approx 1.175i.$
- The **Logarithmic Function** $\ln x$ is also extendable to $\mathbb{C}\setminus\{0\}$.
  - $\boxed{\log z = \ln|z| + i\arg z}$
    - $\log(-1) = \pi i$; $\log i = \dfrac{\pi i}{2}$; $\log(-i) = -\dfrac{\pi i}{2}$.
- The **Exponential Function** a^x is extendable for any $a, x \in \mathbb{C}$.
  - $\boxed{a^x = \exp(x\log a)}$ if $a \neq 0$.
    - $(-1)^i = e^{-\pi}$; $i^i = e^{-\pi/2}$.

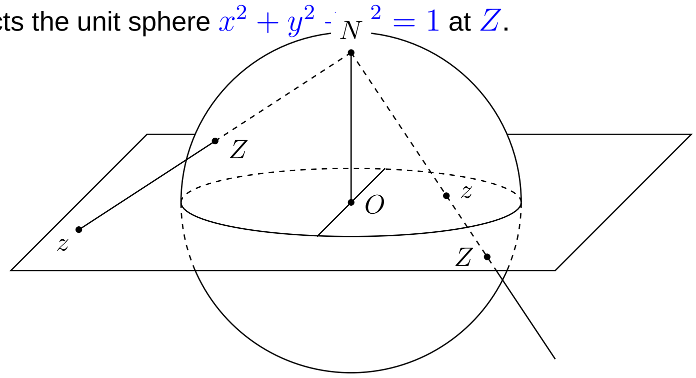
Example

- Let $z = \cos x + i\sin x = e^{ix}$. Let
  - $S = 1 + z + \cdots + z^n = \dfrac{1-z^{n+1}}{1-z}$.

$$\begin{aligned}
S &= \frac{1-[\cos(n+1)x + i\sin(n+1)x]}{1-(\cos x + i\sin x)} \\
&= \frac{1}{2} + \frac{1}{2}\frac{\sin\frac{(n+1)x}{2}}{\sin\frac{x}{2}} + \frac{\sin\frac{(n+1)x}{2}\sin\frac{nx}{2}}{\sin\frac{x}{2}} i \\
\mathrm{Re}(S) &= 1 + \cos x + \cos 2x + \cdots + \cos nx \\
&= \frac{1}{2} + \frac{1}{2}\frac{\sin\frac{(n+1)x}{2}}{\sin\frac{x}{2}} \\
\mathrm{Im}(S) &= \sin x + \sin 2x + \cdots + \sin nx \\
&= \frac{\sin\frac{(n+1)x}{2}\sin\frac{nx}{2}}{\sin\frac{x}{2}}.
\end{aligned}$$

Stereographic Projection

- For any $z = x + iy \leftrightarrow (x, y)$, connect z to $N(0, 0, 1)$.
  - ☐ Line Nz intersects the unit sphere $x^2 + y^2 + z^2 = 1$ at Z.



  - ☐ $S^2 := \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$ the **Unit Sphere**.
    - $\mathbb{C} \leftrightarrow S^2 \setminus \{N\}$ via $z \leftrightarrow Z$.
    - $\mathbb{C} \cup \{\infty\} \leftrightarrow S^2$ via $z \leftrightarrow Z$ and $\infty \leftrightarrow N(0, 0, 1)$.

16 / 36

Matrix Representation

- A **Complex Number** can be identified with a **Real Matrix**:
  - ☐ $z = \boxed{x + iy \leftrightarrow \begin{pmatrix} x & -y \\ y & x \end{pmatrix}} = M_z$.
- All the **Arithmetic Properties** are preserved:
  - ☐ **Addition**: Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.
    - $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$.
    - $M_{z_1} + M_{z_2} = \begin{pmatrix} x_1 + x_2 & -(y_1 + y_2) \\ y_1 + y_2 & x_1 + x_2 \end{pmatrix} = M_{z_1 + z_2}$
  - ☐ **Multiplication**: Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.
    - $z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$.

$$
\begin{aligned}
M_{z_1} M_{z_2} &= \begin{pmatrix} x_1 & -y_1 \\ y_1 & x_1 \end{pmatrix} \begin{pmatrix} x_2 & -y_2 \\ y_2 & x_2 \end{pmatrix} \\
&= \begin{pmatrix} x_1 x_2 - y_1 y_2 & -(x_1 y_2 + y_1 x_2) \\ x_1 y_2 + y_1 x_2 & x_1 x_2 - y_1 y_2 \end{pmatrix} = M_{z_1 z_2}
\end{aligned}
$$

17 / 36

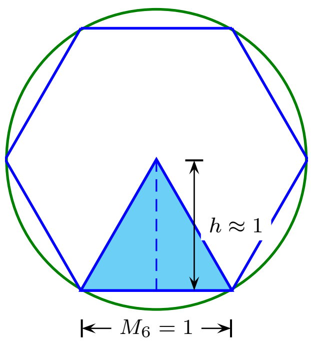
Matrix Representation

- A **Complex Number** can be identified with a **Real Matrix**:
  - $z = \boxed{x + iy \leftrightarrow \begin{pmatrix} x & -y \\ y & x \end{pmatrix}} = M_z.$
- All the **Arithmetic Properties** are preserved:
  - **Modulus**: $|z|^2 = x^2 + y^2 = \det(M_z)$.
  - **Conjugate**: $z^* = x - iy \leftrightarrow \begin{pmatrix} x & y \\ -y & x \end{pmatrix} = (M_z)^T$.
  - **Quotient**: Let $z = x + iy$. Then $1/z = z^{-1} = \dfrac{x - iy}{x^2 + y^2}$.
    - $M_{z^{-1}} = \dfrac{1}{x^2 + y^2} \begin{pmatrix} x & y \\ -y & x \end{pmatrix} = (M_z)^{-1}$.
  - **Power**: $M_{z^n} = M_{\underbrace{z \cdots z}_{n}} = \underbrace{M_z \cdots M_z}_{n} = (M_z)^n$.
  - ....

Evaluation of π

Liu Hui's Algorithm

- Recall that ARCHIMEDES, LIU HUI and ZU CHONGZHI used **regular polygons** to approximation $\boxed{\pi}$ the area of **unit circle**.
- **Liu Hui's Algorithm** for π:
  1. Let $M_n =$ a side of a regular n-gon inscribed in unit circle.
  2. Let $A_n =$ area of the regular n-gon inscribed in the unit circle.



  - $\pi \approx A_n = \frac{1}{2} M_n \cdot h \cdot n \approx \frac{n}{2} \cdot M_n$.

Liu Hui's Algorithm

- Recall that ARCHIMEDES, LIU HUI and ZU CHONGZHI used **regular polygons** to approximation $\boxed{\pi}$ the area of **unit circle**.
- **Liu Hui's Algorithm** for π:
  1. Let M_n = a side of a regular n-gon inscribed in unit circle.
  2. Let A_n = area of the regular n-gon inscribed in the unit circle.
     - $\pi \approx A_n = \frac{1}{2}M_n \cdot h \cdot n \approx \frac{n}{2} \cdot M_n$.
  3. Let $L(n) = 2 - M^2_{3\times 2^n}$. Then $L(1) = 2 - M_6^2 = 1$.
     - LIU HUI found that $\boxed{L(n+1) = \sqrt{2 + L(n)}}$
  4. $\boxed{\pi \approx A_{3\times 2^n} \approx \frac{3\times 2^n}{2} \cdot M_{3\times 2^n} = 3 \times 2^{n-1} \times M_{3\times 2^n}}$
- **Remark.** LIU HUI evaluated up to 96-sided polygon, and used a shortcut to generate the result for 1536-sided polygon.

Liu Hui's Algorithm

- **Liu Hui's Algorithm** for π:

Iteration	Sides	Approximation of π
1	6	3.
2	12	3.1
3	24	3.13
4	48	3.14
5	96	3.141
9	1536	3.14159
12	12288	3.1415926
15	98304	3.141592653
20	1572864	3.141592653589
30	1610612736	3.141592653589793238

- **Remarks**: ZU CHONGZHI found that
  - $\pi \approx 22/7$ and $\pi \approx 355/113$.

Integration

- If $f(x) \le g(x)$ on $[a, b]$, then $\int_a^b f(x)\, dx \le \int_a^b g(x)\, dx$.
- For all $0 \le x \le 1$, $\frac{1}{2} \le \frac{1}{1+x^2} \le 1$.
  - $\frac{1}{2}x^4(1-x)^4 \le \frac{x^4(1-x)^4}{1+x^2} \le x^4(1-x)^4$.
  - $\int_0^1 \frac{x^4(1-x)^4}{2}\, dx \le \int_0^1 \frac{x^4(1-x)^4}{1+x^2}\, dx \le \int_0^1 x^4(1-x)^4\, dx.$
    - $\frac{1}{1260} \le \frac{22}{7} - \pi \le \frac{1}{630}$.

$\therefore$ $\frac{22}{7}$ is a bigger approximation of π with error $\le \frac{1}{630} < 0.0016$.

Integration

- $\int_0^1 \frac{x^8(1-x)^8}{8}\, dx \le \int_0^1 \frac{x^8(1-x)^8}{4(1+x^2)}\, dx \le \int_0^1 \frac{x^8(1-x)^8}{4}\, dx.$
  - $\frac{1}{1750320} \le \pi - \frac{47171}{15015} \le \frac{1}{875160}$.

$\therefore$ $\frac{47171}{15015} \approx 3.14159174\cdots$ is a smaller approximation for π

with error $\le \frac{1}{875160} \approx 10^{-6}$.

- $\int_0^1 \frac{x^{12}(1-x)^{12}}{32}\, dx \le \int_0^1 \frac{x^{12}(1-x)^{12}}{16(1+x^2)}\, dx \le \int_0^1 \frac{x^{12}(1-x)^{12}}{16}\, dx.$
  - $\frac{1}{2163324800} \le \frac{431302721}{137287920} - \pi \le \frac{1}{1081662400}$.

$\therefore$ $\frac{431302721}{137287920} \approx 3.141592654\cdots$ is a bigger approximation for π

with error $\le \frac{1}{1081662400} \approx 10^{-9}$.

Integration

- Approximate π using
  - ☐ $\displaystyle\int_0^1 \frac{x^{4n}(1-x)^{4n}}{4^{n-1}(1+x^2)}\,dx \leq \int_0^1 \frac{x^{4n}(1-x)^{4n}}{4^{n-1}}\,dx < \frac{1}{2^{10n-2}}$

n	Fraction	Decimal
1	$\frac{22}{7}$	3.14
3	$\frac{431302721}{137287920}$	3.141592654
5	$\frac{26856502742629699}{8548690331301120}$	3.141592653589793
10	$\frac{8929347825205334111475899568201677 3}{28422996899365886608045972478361600}$	3.14159265358979323846264338327 9

Series

Power Series

- Recall that a **Power Series** has the form:
  - ☐ $\displaystyle\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots.$

  It plays an important role in **Approximation Theory**.
- **Examples**.
  - ☐ $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots.$
    - $e \approx 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{10!} \approx 2.718281801.$
  - ☐ $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots.$
    - $\sin 2 \approx 2 - \frac{2^3}{3!} + \frac{2^5}{5!} - \frac{2^7}{7!} + \frac{2^9}{9!} \approx 0.9093474427.$

Integration

- Approximate $\int_0^1 \sin(x^2)\, dx$.

1. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots$.
2. $\sin(x^2) \approx x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \frac{x^{18}}{9!} - \frac{x^{22}}{11!}$.
3. Approximate $\int_0^1 \sin(x^2)\, dx$ by
   - $\int_0^1 x^2\, dx - \int_0^1 \frac{x^6}{3!}\, dx + \cdots - \int_0^1 \frac{x^{22}}{11!}\, dx$
4. $\int_0^1 \sin(x^2)\, dx \approx 0.3102683017174579$.

- The exact value is $0.310268301723381101 \cdots$.

Ordinary Differential Equation

- Suppose $f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n + \cdots$.
  - **Term by Term Differentiation**: $f'(x)$:
    - $c_1 + 2c_2 x + 3c_3 x^2 + \cdots + nc_n x^{n-1} + \cdots$.
- **Example**. Suppose $\frac{dy}{dx} = y$ and $y = 1$ at $x = 0$.
  - Let $f(x) = y = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$.
    - $f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + \cdots + nc_n x^{n-1} + \cdots$.
  - Compare coefficients:
    - $c_0 = 1, c_1 = c_0, 2c_2 = c_1, 3c_3 = c_2, 4c_4 = c_3, \ldots$.
    - $c_0 = 1, c_1 = 1, c_2 = 1/2, c_3 = 1/6, c_4 = 1/24, \ldots$
  - $y = f(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$.

Term by Term Integration

- Suppose $f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n + \cdots$.
  - **Term by Term Integration**: $\int f(x)\,dx$:
    - $c_0 x + \frac{c_1}{2}x^2 + \frac{c_2}{3}x^3 + \cdots + \frac{c_n}{n+1}x^{n+1} + \cdots$
- **Examples**.
  - $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots$.
    - $\int \frac{1}{1-x}\,dx = -\ln|1-x| + C$.
    - $x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots + \frac{x^n}{n} + \frac{x^{n+1}}{n+1} + \cdots$.
  - $-\ln|1-x| = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots + \frac{x^n}{n} + \frac{x^{n+1}}{n+1} + \cdots$.

30 / 36

Term by Term Integration

- Suppose $f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n + \cdots$.
  - **Term by Term Integration**: $\int f(x)\,dx$:
    - $c_0 x + \frac{c_1}{2}x^2 + \frac{c_2}{3}x^3 + \cdots + \frac{c_n}{n+1}x^{n+1} + \cdots$
- **Examples**.
  - $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots$.
    - $\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$.
    - $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots$.
  - $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots$

31 / 36

Some Special Series

- $-\ln|1-x| = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^n}{n} + \cdots, -1 < x < 1.$
  - Let $x = 1/2$. Then
    - $\ln 2 = \frac{1}{2} + \frac{1}{2 \cdot 2^2} + \frac{1}{3 \cdot 2^3} + \cdots + \frac{1}{n \cdot 2^n} + \cdots$
    - $\ln 2 \approx \frac{1}{2} + \frac{1}{2 \cdot 2^2} + \cdots + \frac{1}{20 \cdot 2^{20}} = 0.6931471 \cdots$
  - Let $x = -1$. Then
    - $-\ln 2 = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots + \frac{(-1)^n}{n} + \cdots.$
    - $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n-1}}{n} + \cdots.$
  - $\therefore$ The **Alternating Harmonic Series** converges to $\ln 2$.

    Warning: The convergency of alternating Harmonic series is very slow!

Some Special Series

- $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots$
  - Note that $-1 < x < 1$. Take $x = \frac{1}{\sqrt{3}}$:
    - $\frac{\pi}{6} = \frac{1}{\sqrt{3}} - \frac{1}{3(\sqrt{3})^3} + \frac{1}{5(\sqrt{3})^5} - \frac{1}{7(\sqrt{3})^7} + \cdots$
    - $\frac{\pi}{6} \approx \frac{1}{\sqrt{3}} - \frac{1}{3(\sqrt{3})^3} + \frac{1}{5(\sqrt{3})^5} + \cdots + \frac{1}{41(\sqrt{3})^{41}}$
      - $\frac{\pi}{6} \approx 0.52359877559927 \cdots$
      - $\pi \approx 3.14159265359563 \cdots$
  - $\boxed{\pi = 2\sqrt{3} \sum_{n=0}^{\infty} \frac{(-3)^{-n}}{2n+1}}$ is the **Madhava-Leibniz Series**.
    - It is **Efficient** in evaluating π.

Some Special Series

- $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots$
  - Note that $-1 < x < 1$. Take $x = 1$:
    - $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots + \frac{(-1)^n}{2n+1} + \cdots.$

    This is known as the **Leibniz Formula** for π.
  - **Warning**: The convergency is very slow.
- SRINIVASA RAMANUJAN (1887–1920) Indian mathematician.
  - $\boxed{\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!(1103 + 26390n)}{(n!)^4 394^{4n}}.}$
    - The first term gives $\frac{1}{\pi} \approx \frac{2\sqrt{2}}{9801} \cdot 1103.$
    - $\pi \approx \frac{9801\sqrt{2}}{4412} = 3.14159273\cdots.$

Some Special Series

- $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots$
  - Note that $-1 < x < 1$. Take $x = 1$:
    - $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots + \frac{(-1)^n}{2n+1} + \cdots.$

    This is known as the **Leibniz Formula** for π.
  - **Warning**: The convergency is very slow.
- CHUDNOVSKY BROTHERS (1989) American mathematicians.
  - $\boxed{\frac{1}{\pi} = 12 \sum_{n=0}^{\infty} \frac{(-1)^n (6n)!(13591409 + 545140134n)}{(3n)!(n!)^3 640320^{3n+3/2}}}$
    - First term gives $\frac{1}{\pi} \approx 12 \cdot \frac{13591409}{640320^{3/2}}.$
    - $\pi \approx \frac{640320^{3/2}}{12 \cdot 13591409} = 3.141592653589793\cdots.$