Empircal Evaluations of Upper Bounds on Minimum Composition Norms in Finite Product Spaces

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1 Introduction

In this project, we empirically study upper bounds on the minimum composition norm capacity defined on a product space given two marginal capacities. To this end, we developed a Python script to run our experiments on random marginal capacities. Indeed, we developed algorithms to randomly generate additive, superadditive, and supermodular marginal capacities. Due to the computation complexity of finding the minimum composition norm capacity through solving the LP, we limit our study to $\leq 3-$dimensional marginal spaces.

Recall the Mobius transformation on set A is defined as:

$$m^{\xi}(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} \xi(B)$$

where $A \subseteq Z$. The composition norm, which we denote by $\| \cdot \|_c$, of capacity ξ on X is therefore,

$$\|\xi\|_c = \sum_{A \subseteq X} |m^{\xi}(A)|.$$

1.1 Minimum Composition Norm Linear Program

Let $Z := X \times Y$. Given marginal capacities μ and ν on space X and Y respectively, we find the capacity on the product space with minimum composition norm by solving the following LP:

$$\begin{aligned} &\min_{\xi} \|\xi\|_c \\ \text{subject to} \quad & \xi(G \times Y) = \mu(G), && \emptyset \neq G \subset X \\ & \xi(X \times F) = \nu(F), && \emptyset \neq F \subset Y \\ & \xi(A \cup w) - \xi(A) \geq 0, && A \subset Z, w = \{(x,y)\} \notin A \\ & \xi(\emptyset) = 0. \end{aligned}$$

Note that if both μ and ν are normalized, then we implicitly have the constraint $\xi(Z) = 1$ s.t. the product capacity is also normalized.

Project Scope: We are interested in finding solutions to this LP given random marginal capacities, and then empirically evaluating if an upper bound on $\|\xi^*\|_c$ exists where ξ^* is a solution to the LP.

2 Implementation

2.1 Bitmasks

To efficiently characterize subsets computationally, our implementation exploits bit-masks and bit operations. Namely, suppose we have a space with three elements, i.e. $|X| := 3$. Then, every subset can be represented as a number from 0 to 7 ($2^{|X|} - 1$). For example, if our space X was ordered as $\{A, B, C\}$, then the subset $\{A, C\}$ could be characterized by the binary code 101_2 which is equivalent to the integer 5. Suppose we wish to check if $\{A\}$ is a subset of $\{A, C\}$ computationally, then we can check if 100_2 OR $101_2 == 101_2$ which is very efficient computationally. To represent a capacity on set X, we generate a vector of size $2^{|X|}$ where the i^{th} index represents the subset characterized by i in binary.

2.2 Capacity Types

Our code can generate three types of random normalized (and grounded) marginal capacities:

1. **Additive.** Given disjoint sets $A, B \in 2^Z$, the set function satisfies $\xi(A \cup B) = \xi(A) + \xi(B)$.
2. **Superadditive.** Given disjoint sets $A, B \in 2^Z$, the set function satisfies $\xi(A \cup B) \geq \xi(A) + \xi(B)$.
3. **Supermodular.** Given sets $A, B \in 2^Z$, the set function satisfies $\xi(A \cup B) + \xi(A \cap B) \geq \xi(A) + \xi(B)$.

Briefly, we generate these random capacities by first randomly assigning a value from $Unif(0, 1)$ to every singleton subset. In the additive case, it is very easy to build up the capacities of all subsets afterward. In the superadditive and supermodular cases, for a given subset we enumerate all disjoint pair (for superadditive) or all subset pairs A, B (for supermodular) s.t. $A \cup B$ is the mask. Our algorithm uses dynamic programming on all pair enumerations to ensure the inequalities are satisfied in each type by taking a maximum. Then, we add a random value from $Unif(0, 1)$ to further randomize and implement the $\geq$ bound.

2.3 Capacity on the Product Space

We implement LP (1) to solve for vector $M \in \mathbb{R}^{|Z|}$ where $Z := X \times Y$. Suppose $|X| := 3$ and $|Y| := 3$, then $|Z| = 9$ so M is a vector of length 512 and each index of M characterizes a subset of Z.

3 Empirical Results

In Figures (1), (2), and (3), we present the results on additive, superadditive, and supermodular marginal capacities respectively. For each case, we have four experiments: (1) $|X| = 2, |Y| = 2$, (2) $|X| = 2, |Y| = 3$, (3) $|X| = 3, |Y| = 2$, (4) $|X| = 3, |Y| = 3$. For each experiment, we consider $N = 1000$ random marginal capacities and solve the LP for each. Given the optimal solution, we compute the composition norm of the product capacity against the sum of the composition norms of the two random marginals.

For our experiments, the marginal capacities are normalized. In the additive marginal case, the composition norm of each marginal is 1. We see that the product capacity also has composition norm 1 in each of the four varying dimension cases.

In the superadditive and supermodular cases, when both marginal dimensions are 2, the optimal product capacity has composition norm equal to 1. We find that in the higher dimensional cases, an upper bound w.r.t. the sum of comp. norms of the marginals is observed.

Through our small experiments, we see that the upper bound $\|\xi\|_c \leq \|\mu\|_c + \|\nu\|_c - 1$ exists.

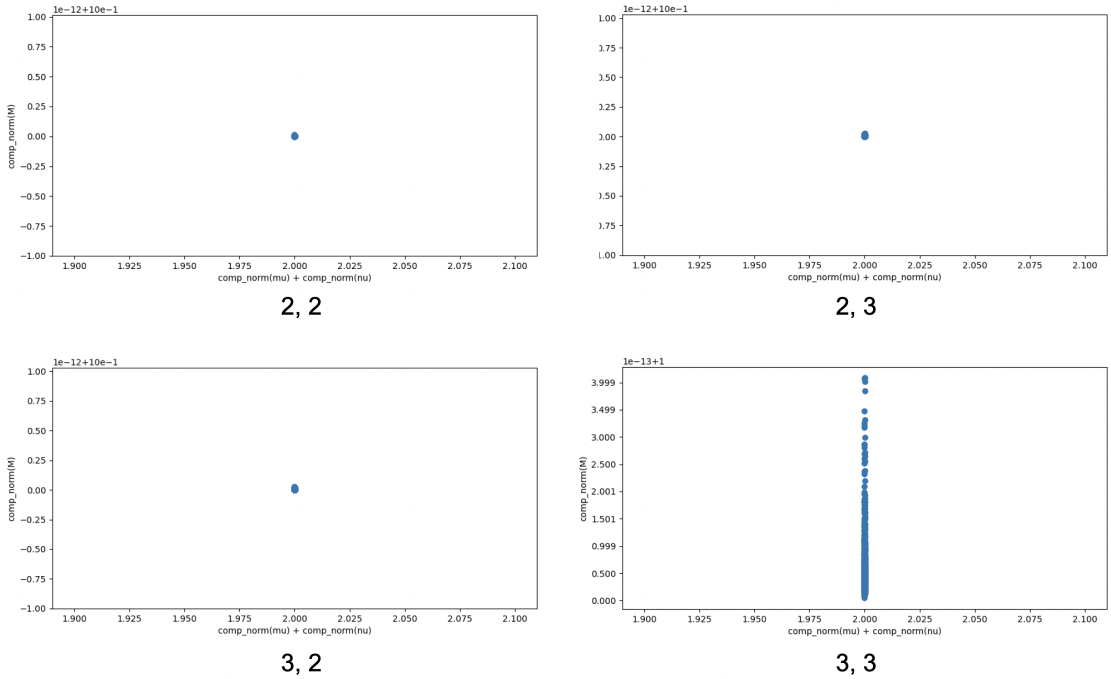

Figure 1: Additive marginal capacities.

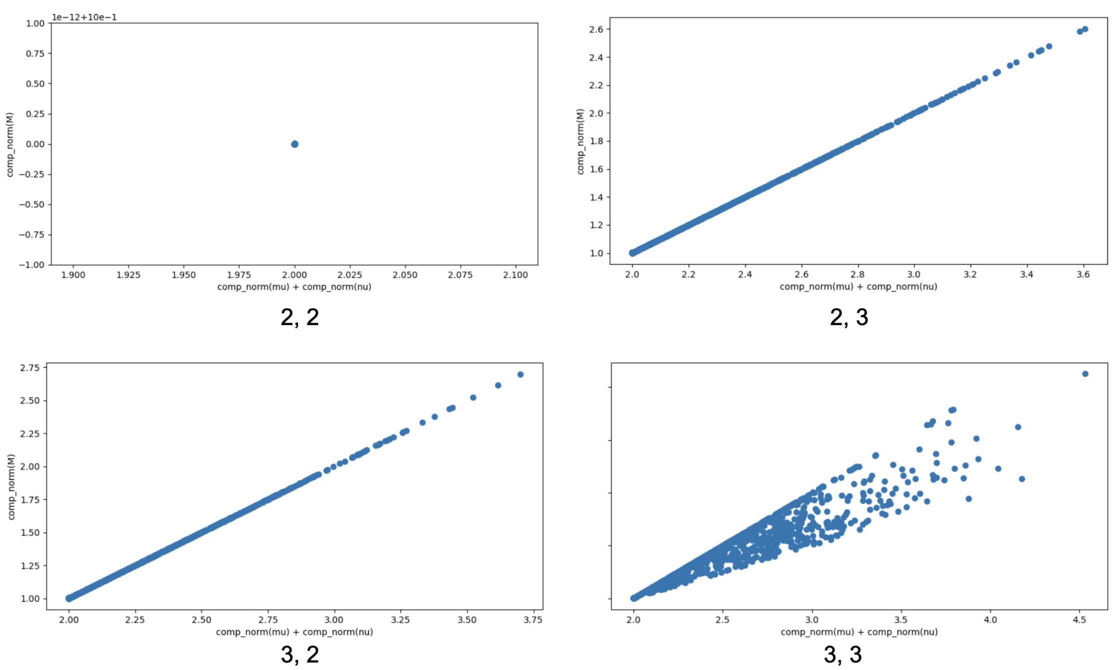

Figure 2: Superadditive marginal capacities.

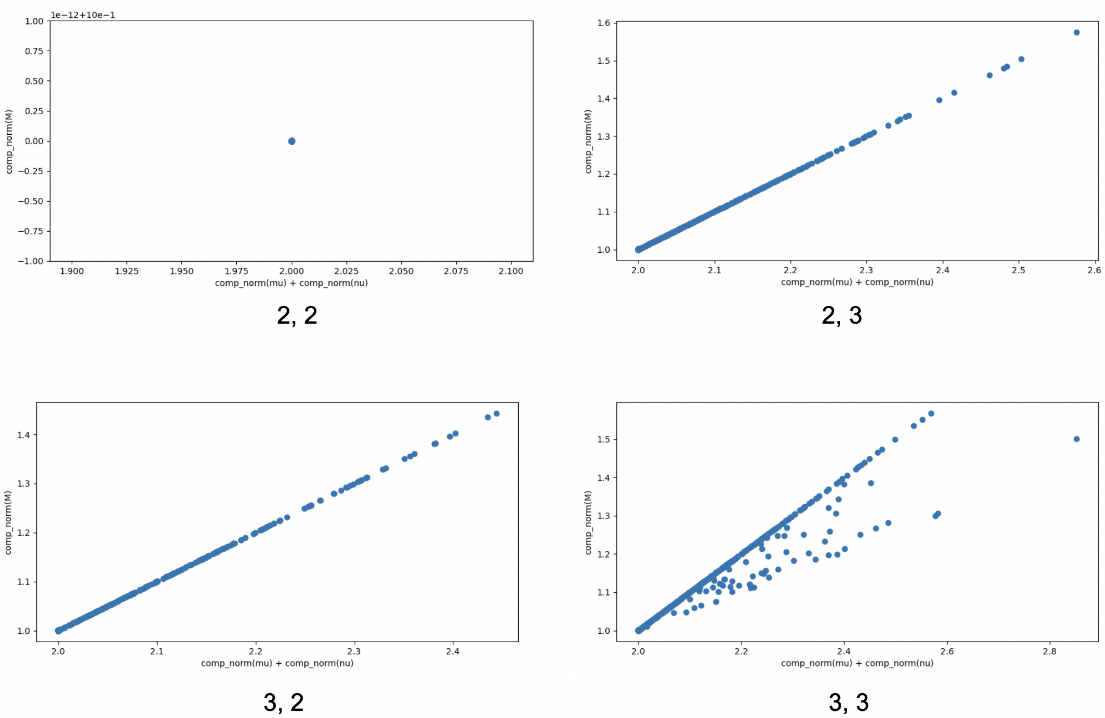

Figure 3: Supermodular marginal capacities.