On teaching mathematics

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Mathematics is a part of physics. Physics is an experimental science, a part of natural science. Mathematics is the part of physics where experiments are cheap.

The Jacobi identity (which forces the heights of a triangle to cross at one point) is an experimental fact in the same way as that the Earth is round (that is, homeomorphic to a ball). But it can be discovered with less expense.

In the middle of the twentieth century it was attempted to divide physics and mathematics. The consequences turned out to be catastrophic. Whole generations of mathematicians grew up without knowing half of their science and, of course, in total ignorance of any other sciences. They first began teaching their ugly scholastic pseudo-mathematics to their students, then to schoolchildren (forgetting Hardy's warning that ugly mathematics has no permanent place under the Sun).

Since scholastic mathematics that is cut off from physics is fit neither for teaching nor for application in any other science, the result was the universal hate towards mathematicians – both on the part of the poor schoolchildren (some of whom in the meantime became ministers) and of the users.

The ugly building, built by undereducated mathematicians who were exhausted by their inferiority complex and who were unable to make themselves familiar with physics, reminds one of the rigorous axiomatic theory of odd numbers. Obviously, it is possible to create such a theory and make pupils admire the perfection and internal consistency of the resulting structure (in which, for example, the sum of an odd number of terms and the product of any number of factors are defined). From this sectarian point of view, even numbers could either be declared a heresy or, with passage of time, be introduced into the theory supplemented with a few "ideal" objects (in order to comply with the needs of physics and the real world).

Unfortunately, it was an ugly twisted construction of mathematics like the one above which predominated in the teaching of mathematics for decades. Having originated in France, this pervertedness quickly spread to teaching of foundations of mathematics, first to university students, then to school pupils of all lines (first in France, then in other countries, including Russia).

To the question "what is 2 + 3" a French primary school pupil replied: "3 + 2, since addition is commutative". He did not know what the sum was equal to and could not even understand what he was asked about!

Another French pupil (quite rational, in my opinion) defined mathematics as follows: "there is a square, but that still has to be proved".

[25]This is an extended text of the address at the discussion on teaching of mathematics in Palais de Découverte in Paris on 7 March 1997

Judging by my teaching experience in France, the university students' idea of mathematics (even of those taught mathematics at the École Normale Supérieure – I feel sorry most of all for these obviously intelligent but deformed kids) is as poor as that of this pupil.

For example, these students have never seen a paraboloid and a question on the form of the surface given by the equation $xy = z^2$ puts the mathematicians studying at ENS into a stupor. Drawing a curve given by parametric equations (like $x = t^3 - 3t$, $y = t^4 - 2t^2$) on a plane is a totally impossible problem for students (and, probably, even for most French professors of mathematics).

Beginning with l'Hospital's first textbook on calculus ("calculus for understanding of curved lines") and roughly until Goursat's textbook, the ability to solve such problems was considered to be (along with the knowledge of the times table) a necessary part of the craft of every mathematician.

Mentally challenged zealots of "abstract mathematics" threw all the geometry (through which connection with physics and reality most often takes place in mathematics) out of teaching. Calculus textbooks by Goursat, Hermite, Picard were recently dumped by the student library of the Universities Paris 6 and 7 (Jussieu) as obsolete and, therefore, harmful (they were only rescued by my intervention).

ENS students who have sat through courses on differential and algebraic geometry (read by respected mathematicians) turned out be acquainted neither with the Riemann surface of an elliptic curve $y^2 = x^3 + ax + b$ nor, in fact, with the topological classification of surfaces (not even mentioning elliptic integrals of first kind and the group property of an elliptic curve, that is, the Euler-Abel addition theorem). They were only taught Hodge structures and Jacobi varieties!

How could this happen in France, which gave the world Lagrange and Laplace, Cauchy and Poincaré, Leray and Thom? It seems to me that a reasonable explanation was given by I. G. Petrovskii, who taught me in 1966: genuine mathematicians do not gang up, but the weak need gangs in order to survive. They can unite on various grounds (it could be super-abstractness, anti-Semitism or "applied and industrial" problems), but the essence is always a solution of the social problem – survival in conditions of more literate surroundings.

By the way, I shall remind you of a warning of L. Pasteur: there never have been and never will be any "applied sciences", there are only *applications of sciences* (quite useful ones!).

In those times I was treating Petrovskii's words with some doubt, but now I am being more and more convinced of how right he was. A considerable part of the super-abstract activity comes down simply to industrialising shameless grabbing of discoveries from discoverers and then systematically assigning them to epigons-generalizers. Similarly to the fact that America does not carry Columbus's name, mathematical results are almost never called by the names of their discoverers.

In order to avoid being misquoted, I have to note that my own achievements were for some unknown reason never expropriated in this way, although it always happened to both my teachers (Kolmogorov, Petrovskii, Pontryagin, Rokhlin) and my pupils. Prof. M. Berry once formulated the following two principles:

The Arnold Principle. If a notion bears a personal name, then this name is not the name of the discoverer.

The Berry Principle. The Arnold Principle is applicable to itself.

Let's return, however,to teaching of mathematics in France.

When I was a first-year student at the Faculty of Mechanics and Mathematics of the Moscow State University, the lectures on calculus were read by the set-theoretic topologist L. A. Tumarkin, who conscientiously retold the old classical calculus course of French type in the Goursat version. He told us that integrals of rational functions along an algebraic curve can be taken if the corresponding Riemann surface is a sphere and, generally speaking, cannot be taken if its genus is higher, and that for the sphericity it is enough to have a sufficiently large number of double points on the curve of a given degree (which forces the curve to be unicursal: it is possible to draw its real points on the projective plane with one stroke of a pen).

These facts capture the imagination so much that (even given without any proofs) they give a better and more correct idea of modern mathematics than whole volumes of the Bourbaki treatise. Indeed, here we find out about the existence of a wonderful connection between things which seem to be completely different: on the one hand, the existence of an explicit expression for the integrals and the topology of the corresponding Riemann surface and, on the other hand, between the number of double points and genus of the corresponding Riemann surface, which also exhibits itself in the real domain as the unicursality.

Jacobi noted, as mathematics' most fascinating property, that in it one and the same function controls both the presentations of a whole number as a sum of four squares and the real movement of a pendulum.

These discoveries of connections between heterogeneous mathematical objects can be compared with the discovery of the connection between electricity and magnetism in physics or with the discovery of the similarity between the east coast of America and the west coast of Africa in geology.

The emotional significance of such discoveries for teaching is difficult to overestimate. It is they who teach us to search and find such wonderful phenomena of harmony of the Universe.

The de-geometrisation of mathematical education and the divorce from physics sever these ties. For example, not only students but also modern algebro-geometers on the whole do not know about the Jacobi fact mentioned here: an elliptic integral of first kind expresses the time of motion along an elliptic phase curve in the corresponding Hamiltonian system.

Rephrasing the famous words on the electron and atom, it can be said that a hypocycloid is as inexhaustible as an ideal in a polynomial ring. But teaching ideals to students who have never seen a hypocycloid is as ridiculous as teaching addition of fractions to children who have never cut (at least mentally) a cake or an apple into equal parts. No wonder that the children will prefer to add a numerator to a numerator and a denominator to a denominator.

From my French friends I heard that the tendency towards super-abstract generalizations is their traditional national trait. I do not entirely disagree that this might be a question of a hereditary disease, but I would like to underline the fact that I borrowed the cake-and-apple example from Poincaré.

The scheme of construction of a mathematical theory is exactly the same as that in any other natural science. First we consider some objects and make some observations in special cases. Then we try and find the limits of application of our observations, look for counter-examples which would prevent unjustified extension of our observations onto a too wide range of events (example: the number of partitions of consecutive odd numbers 1, 3, 5, 7, 9 into an odd number of natural summands gives the sequence 1, 2, 4, 8, 16, but then comes 29).

As a result we formulate the empirical discovery that we made (for example, the Fermat conjecture or Poincaré conjecture) as clearly as possible. After this there comes the difficult period of checking as to how reliable are the conclusions .

At this point a special technique has been developed in mathematics. This technique, when applied to the real world, is sometimes useful, but can sometimes also lead to self-deception. This technique is called modelling. When constructing a model, the following idealisation is made: certain facts which are only known with a certain degree of probability or with a certain degree of accuracy, are considered to be "absolutely" correct and are accepted as "axioms". The sense of this "absoluteness" lies precisely in the fact that we allow ourselves to use these "facts" according to the rules of formal logic, in the process declaring as "theorems" all that we can derive from them.

It is obvious that in any real-life activity it is impossible to wholly rely on such deductions. The reason is at least that the parameters of the studied phenomena are never known absolutely exactly and a small change in parameters (for example, the initial conditions of a process) can totally change the result. Say, for this reason a reliable long-term weather forecast is impossible and will remain impossible, no matter how much we develop computers and devices which record initial conditions.

In exactly the same way a small change in axioms (of which we cannot be completely sure) is capable, generally speaking, of leading to completely different conclusions than those that are obtained from theorems which have been deduced from the accepted axioms. The longer and fancier is the chain of deductions ("proofs"), the less reliable is the final result.

Complex models are rarely useful (unless for those writing their dissertations).

The mathematical technique of modelling consists of ignoring this trouble and speaking about your deductive model in such a way as if it coincided with reality. The fact that this path, which is obviously incorrect from the point of view of natural science, often leads to useful results in physics is called “the inconceivable effectiveness of mathematics in natural sciences” (or “the Wigner principle”).

Here we can add a remark by I. M. Gel’fand: there exists yet another phenomenon which is comparable in its inconceivability with the inconceivable effectiveness of mathematics in physics noted by Wigner – this is the equally inconceivable ineffectiveness of mathematics in biology.

“The subtle poison of mathematical education” (in F. Klein’s words) for a physicist consists precisely in that the absolutised model separates from the reality and is no longer compared with it. Here is a simple example: mathematics teaches us that the solution of the Malthus equation $dx/dt = x$ is uniquely defined by the initial conditions (that is that the corresponding integral curves in the (t, x)-plane do not intersect each other). This conclusion of the mathematical model bears little relevance to the reality. A computer experiment shows that all these integral curves have common points on the negative t-semi-axis. Indeed, say, curves with the initial conditions $x(0) = 0$ and $x(0) = 1$ practically intersect at $t = -10$ and at $t = -100$ you cannot fit in an atom between them. Properties of the space at such small distances are not described at all by Euclidean geometry. Application of the uniqueness theorem in this situation obviously exceeds the accuracy of the model. This has to be respected in practical application of the model, otherwise one might find oneself faced with serious troubles.

I would like to note, however, that the same uniqueness theorem explains why the closing stage of mooring of a ship to the quay is carried out manually: on steering, if the velocity of approach would have been defined as a smooth (linear) function of the distance, the process of mooring would have required an infinitely long period of time. An alternative is an impact with the quay (which is damped by suitable non-ideally elastic bodies). By the way, this problem had to be seriously confronted on landing the first descending apparata on the Moon and Mars and also on docking with space stations – here the uniqueness theorem is working against us.

Unfortunately, neither such examples, nor discussing the danger of fetishising theorems are to be met in modern mathematical textbooks, even in the better ones. I even got the impression that scholastic mathematicians (who have little knowledge of physics) believe in the principal difference of the axiomatic mathematics from modelling which is common in natural science and which always requires the subsequent control of deductions by an experiment.

Not even mentioning the relative character of initial axioms, one cannot forget about the inevitability of logical mistakes in long arguments (say, in the form of a computer breakdown caused by cosmic rays or quantum oscillations). Every working mathematician knows that if one does not control oneself (best of all by examples), then after some ten pages half of all the signs in formulae will be wrong and twos will find their way from denominators into numerators.

The technology of combatting such errors is the same external control by experiments or observations as in any experimental science and it should be taught from the very beginning to all juniors in schools.

Attempts to create “pure” deductive-axiomatic mathematics have led to the rejection of the scheme used in physics (observation – model – investigation of the model – conclusions – testing by observations) and its substitution by the scheme: definition – theorem – proof. It is impossible to understand an unmotivated definition but this does not stop the criminal algebraists-axiomatisators. For example, they would readily define the product of natural numbers by means of the long multiplication rule. With this the commutativity of multiplication becomes difficult to prove but it is still possible to deduce it as a theorem from the axioms. It is then possible to force poor students to learn this theorem and its proof (with the aim of raising the standing of both the science and the persons teaching it). It is obvious that such definitions and such proofs can only harm the teaching and practical work.

It is only possible to understand the commutativity of multiplication by counting and re-counting soldiers by ranks and files or by calculating the area of a rectangle in the two ways. Any attempt to do without this interference by physics and reality into mathematics is sectarianism and isolationism which destroy the image of mathematics as a useful human activity in the eyes of all sensible people.

I shall open a few more such secrets (in the interest of poor students).

The *determinant* of a matrix is an (oriented) volume of the parallelepiped whose edges are its columns. If the students are told this secret (which is carefully hidden in the purified algebraic education), then the whole theory of determinants becomes a clear chapter of the theory of poly-linear forms. If determinants are defined otherwise, then any sensible person will forever hate all the determinants, Jacobians and the implicit function theorem.

What is a *group*? Algebraists teach that this is supposedly a set with two operations that satisfy a load of easily-forgettable axioms. This definition provokes a natural protest: why would any sensible person need such pairs of operations? "Oh, curse this maths" – concludes the student (who, possibly, becomes the Minister for Science in the future).

We get a totally different situation if we start off not with the group but with the concept of a transformation (a one-to-one mapping of a set onto itself) as it was historically. A collection of transformations of a set is called a group if along with any two transformations it contains the result of their consecutive application and an inverse transformation along with every transformation.

This is all the definition there is. The so-called "axioms" are in fact just (obvious) *properties* of groups of transformations. What axiomatisators call "abstract groups" are just groups of transformations of various sets considered up to isomorphisms (which are one-to-one mappings preserving the operations). As Cayley proved, there are no "more abstract" groups in the world. So why do the algebraists keep on tormenting students with the abstract definition?

By the way, in the 1960s I taught group theory to Moscow *schoolchildren*. Avoiding all the axiomatics and staying as close as possible to physics, in half a year I got to the Abel theorem on the unsolvability of a general equation of degree five in radicals (having on the way taught the pupils complex numbers, Riemann surfaces, fundamental groups and monodromy groups of algebraic functions). This course was later published by one of the audience, V. Alekseev, as the book *The Abel theorem in problems*.

What is a *smooth manifold*? In a recent American book I read that Poincaré was not acquainted with this (introduced by himself) notion and that the "modern" definition was only given by Veblen in the late 1920s: a manifold is a topological space which satisfies a long series of axioms.

For what sins must students try and find their way through all these twists and turns? Actually, in Poincaré's *Analysis Situs* there is an absolutely clear definition of a smooth manifold which is much more useful than the "abstract" one.

A smooth k-dimensional submanifold of the Euclidean space R^N is its subset which in a neighbourhood of its every point is a graph of a smooth mapping of $\mathbf{R}^k$ into $\mathbf{R}^{N-k}$ (where $\mathbf{R}^k$ and $\mathbf{R}^{N-k}$ are coordinate subspaces). This is a straightforward generalization of most common smooth curves on the plane (say, of the circle $x^2 + y^2 = 1$) or curves and surfaces in the three-dimensional space.

Between smooth manifolds smooth mappings are naturally defined. Diffeomorphisms are mappings which are smooth, together with their inverses.

An "abstract" smooth manifold is a smooth submanifold of a Euclidean space considered up to a diffeomorphism. There are no "more abstract" finite-dimensional smooth manifolds in the world (Whitney's theorem). Why do we keep on tormenting students with the abstract definition? Would it not be better to prove them the theorem about the explicit classification of closed two-dimensional manifolds (surfaces)?

It is this wonderful theorem (which states, for example, that any compact connected oriented surface is a sphere with a number of handles) that gives a correct impression of what modern mathematics is and not the super-abstract generalizations of naive submanifolds of a Euclidean space which in fact do not give anything new and are presented as achievements by the axiomatisators.

The theorem of classification of surfaces is a top-class mathematical achievement, comparable with the discovery of America or X-rays. This is a genuine discovery of mathematical natural science and it is even difficult to say whether the fact itself is more attributable to physics or to mathematics. In its significance for both the applications and the development of correct Weltanschauung it by far surpasses such "achievements" of mathematics as the proof of Fermat's last theorem or the proof of the fact that any sufficiently large whole number can be represented as a sum of three prime numbers.

For the sake of publicity modern mathematicians sometimes present such sporting achievements as the last word in their science. Understandably this not only does not contribute to the society's appreciation of mathematics but, on the contrary, causes a healthy distrust of the necessity of wasting energy on (rock-climbing-type) exercises with these exotic questions needed and wanted by no one.

The theorem of classification of surfaces should have been included in high school mathematics courses (probably, without the proof) but for some reason is not included even in university mathematics courses (from which in France, by the way, all the geometry has been banished over the last few decades).

The return of mathematical teaching at all levels from the scholastic chatter to presenting the important domain of natural science is an espessially hot problem for France. I was astonished that all the best and most important in methodical approach mathematical books are almost unknown to students here (and, seems to me, have not been translated into French). Among these are *Numbers and figures* by Rademacher and Töplitz, *Geometry and the imagination* by Hilbert and Cohn-Vossen, *What is mathematics?* by Courant and Robbins, *How to solve it* and *Mathematics and plausible reasoning* by Polya, *Development of mathematics in the 19th century* by F. Klein.

I remember well what a strong impression the calculus course by Hermite (which does exist in a Russian translation!) made on me in my school years.

Riemann surfaces appeared in it, I think, in one of the first lectures (all the analysis was, of course, complex, as it should be). Asymptotics of integrals were investigated by means of path deformations on Riemann surfaces under the motion of branching points (nowadays, we would have called this the Picard-Lefschetz theory; Picard, by the way, was Hermite's son-in-law – mathematical abilities are often transferred by sons-in-law: the dynasty Hadamard – P. Levy – L. Schwarz – U. Frisch is yet another famous example in the Paris Academy of Sciences).

The "obsolete" course by Hermite of one hundred years ago (probably, now thrown away from student libraries of French universities) was much more modern than those most boring calculus textbooks with which students are nowadays tormented.

If mathematicians do not come to their senses, then the consumers who preserved a need in a modern, in the best meaning of the word, mathematical theory as well as the immunity (characteristic of any sensible person) to the useless axiomatic chatter will in the end turn down the services of the undereducated scholastics in both the schools and the universities.

A teacher of mathematics, who has not got to grips with at least some of the volumes of the course by Landau and Lifshitz, will then become a relict like the one nowadays who does not know the difference between an open and a closed set.

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