Lu Sun, and many more.

A Notebook on Calculus

To all family members, friends and communities members who have been dedicating to the presentation of this notebook, and to all students, researchers and faculty members who might find this notebook helpful.

Contents

Foreword

If a piece of software or an e-book can be made completely open source, why not a notebook?

This brings me back to the summer of year 2009, when I just started my third year as a high school student in Harbin No. 3 High School. In around August and September of every year, that is, when the results of Gaokao (National College Entrance Examination of P. R. China, annually held in July) are released, you would find people selling notebooks photocopies claimed to be collected from the top scorers of the exam. Much as I was interested in what these notebooks look like, I myself was not expecting to actually learn anything from them, mainly for the following three reasons.

First of all, some (in fact many) of these notebooks were more difficult to understand than the textbooks. I guess we cannot blame the top scorers for being too smart and make things sometimes extremely brief, or otherwise overwhelmingly complicated.

Secondly, why would I want to adapt to notebooks of others when I have my own? And by the way, I was positive that mine would be as good as theirs, given that I had been putting the same time (three years of high school, only for 6 modules!) and effort learning the courses and preparing the notebooks.

And lastly, as a student in Harbin No. 3 High School, I knew that the top scorers of the coming year would probably be a schoolmate next door, perhaps even a good friend of mine. Why would I want to pay a great amount of penny to a complete stranger in a photocopy shop for his or her notebook, rather than ask from him or her directly?

However, things have changed later on after entering a university as an undergraduate student. I think the main cause of the change is that, since in the university there are so many modules and materials to learn, students are often distracted from digging into one book or module very deeply. (For those who still can concentrate, you have my highest respect.) The situation becomes even worse as I become a Ph.D. student, this time due to that I have to concentrate on one subject entirely, and can hardly split much time on other irrelevant but still important and interesting contents.

This motivates me to start reading and taking notebooks for selected books and articles such as journal papers and magazines. I have a bunch of notebooks with me, most of them are physical. My very first notebook is on *Numerical Analysis*, an entrance level module for engineering background students. Till today I have on my hand dozens of notebooks. One day it suddenly came to me: why not digitalize them, and make them accessible online and open source, and let everyone read and edit it?

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As majority of open source software, this notebook (and it applies to the other notebooks in this series) does not come with any "warranty" of any kind, meaning that there is no guarantee for the statement and knowledge in this notebook to be exactly correct as it is not peer reviewed. **Do NOT cite this notebook in your academic research paper or book!** Of course, if you find anything here useful with your research, please trace back to the origin of the citation, and read it yourself, and on top of that determine whether or not to use it in your research.

This notebook is suitable as:

- a quick reference guide;
- a brief introduction to the subject;
- a "cheat sheet" for students to prepare for the exam (Don't bring it to the exam unless it is allowed by your lecture!) or for lectures to prepare the teaching materials.

This notebook is NOT suitable as:

- a direct research reference;
- a replacement to the textbook;

because as explained the notebook is NOT peer reviewed and it is meant to be simple and easy to read. It is not necessary brief, but all the tedious explanation and derivation, if any, shall be "fold into appendix" and a reader can easily skip those things without any interruption to the reading.

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Although this notebook is open source, the reference materials of this notebook, including many textbooks, journal papers, conference proceedings, etc., may not be open source. Very likely many of these reference materials are licensed or copyrighted. Please legitimately access these materials and properly use them if necessary.

Preface

This notebook is on *Calculas*, a very important mathematical tool that was invented back in Newton's time or even earlier. It has now become entrance level module for mathematics and engineering background students in year one in the university.

Initially, the invention of calculus, including the introduction of differentiation and integration, is of course used to explain things such as the concept of "speed" as a differential of distance over time. You might easily come up with some common use cases of calculus, for example calculating the tangent of a curve, and calculating the volume of an arbitrarily shaped container. Other applications which may not make too much sense for beginners, for example the derivation of cycloid, are also obtained from calculus. Many advanced mathematical tools themselves are built on top of calculus, for example fourier transform, which is widely used in signal processing. Without a solid understanding of calculus, it is hardly possible for one to use these tools confidently and effectively.

The key reference of this notebook is listed below. During the development of the notebook, this list may become longer and longer.

Book *Calculus Metric Version Eighth Edition* by James Stewart, published by Cengage Learning [1].

Book *Calculus* by Gibert Strang (Massachusetts Institute of Technology), published by Wellesley-Cambridge Press [2]. This book is available at MIT Open Courseware (*ocw.mit.edu*). There are countless number of great learning materials there.

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Part I

Limit, Derivative and Integral

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What is Limit

CONTENTS

1.1 The Limit of a Sequence

A motivating example of a convergent sequence is given in Section 1.1.1. The definition of the limit of a sequence is given in Section 1.1.2. Some useful tricks in proving the convergency of a sequence is given in 1.1.3. And finally some additional comments are given in 1.1.4.

1.1.1 A Motivating Example

We use $\{a_n\}$ to denote a sequence. In $\{a_n\}$, the positive integer n is the index of the elements in the sequence, where a_1 represents the first element of $\{a_n\}$, and a_2 the second element, etc.

A sequence has at least one element, and may have in total finite elements (namely, finite sequence) or infinite elements (namely, infinite sequence). In this notebook, we are mostly interested in infinite sequence.

A motivating example is given in Scenario 1 to illustrate the limit of an infinite sequence.

TABLE 1.1
Formulation of a_n as a function of n for small n.

n	$a_n - a_{n-1} = \left(\frac{1}{2}\right)^{n-1}$	a_n
1	—	1
2	0.5	1.5
3	0.25	1.75
4	0.125	1.875
5	0.0625	1.9375
6	0.03125	1.96875
7	0.015625	1.984375
$\vdots$	$\vdots$	$\vdots$

Scenario 1

Consider an infinite sequence $\{a_n\}$ whose elements are recursively derived by

$$a_1 = 1 \tag{1.1}$$

$$a_n = a_{n-1} + \left(\frac{1}{2}\right)^{n-1} \tag{1.2}$$

Q1: Formulate a_n as a function of n.

Q2: When will a_n reach/exceed 1.95? When will a_n reach/exceed 2? When will a_n reach/exceed reach 3?

The old school way of solving Q1 in the above scenario is rather simple: use a table to list down different n and its associated a_n. The value of a_n can be manually calculated for small n, as shown in Table 1.1.

With the results presented in Table 1.1, we can also partially answer Q2. Obviously, for any $n \geq 6$, variable a_n exceeds 1.95.

To find out the answers to the rest of Q2, intuitively we probably need a larger table, say Table 1.2. Table 1.2 pushes the digit display limits of most calculators in the market. From Table 1.2, it can been seen that as n grows larger and larger, the increment $a_n - a_{n-1} = \left(\frac{1}{2}\right)^{n-1}$ becomes smaller and smaller, and the increment is just not enough to top the next a_n to 2.

An alternative way of finding the solution to Q2 is to derive an analytical equation of a_n as a function of n for any arbitrary n. Then we might be able to solve $a_n \geq 2$ and $a_n \geq 3$ for n. Notice that Recursively using (1.2) for $t-1$

TABLE 1.2
Formulation of a_n as a function of n for larger n.

n	$a_n - a_{n-1} = \left(\frac{1}{2}\right)^{n-1}$	a_n
1	1	1
2	0.5	1.5
3	0.25	1.75
4	0.125	1.875
5	0.0625	1.9375
6	0.03125	1.96875
7	0.015625	1.984375
8	0.0078125	1.9921875
9	0.00390625	1.99609375
10	0.001953125	1.998046875
11	0.0009765625	1.9990234375
12	0.00048828125	1.99951171875
13	0.000244140625	1.999755859375
14	0.0001220703125	1.9998779296875
15	0.00006103515625	1.99993896484375
$\vdots$	$\vdots$	$\vdots$

times and substituting (1.1) into (1.2) gives

$$\begin{aligned} a_n &= \sum_{i=1}^{n} \left(\frac{1}{2}\right)^{i-1} & (1.3) \\ &= 1 + \sum_{i=2}^{n} \left(\frac{1}{2}\right)^{i-1} & (1.4) \end{aligned}$$

and multiplying $\frac{1}{2}$ on (1.3) gives

$$\begin{aligned} \frac{1}{2}a_n &= \sum_{i=1}^{n} \left(\frac{1}{2}\right)^{i} \\ &= \sum_{i=2}^{n} \left(\frac{1}{2}\right)^{i-1} + \left(\frac{1}{2}\right)^{n} & (1.5) \end{aligned}$$

Subtracting (1.5) from (1.4) gives

$$\begin{aligned} \frac{1}{2}a_n &= 1 - \left(\frac{1}{2}\right)^{n} \\ a_n &= 2 - \left(\frac{1}{2}\right)^{n-1} & (1.6) \end{aligned}$$

Equation (1.6) can be verified using the results in Table 1.2, and it holds true for any arbitrary n. It suggests that the elements in sequence $\{a_n\}$ will not reach 2 let alone 3 for any n, as $a_n < 2$ for any n, although a_n can be very close to 2 as n increases.

The following features can be observed for sequence $\{a_n\}$:

- The sequence is monotonically increasing, as $a_{n-1} < a_n$ for any n.
- The sequence is bounded, as $a_n < 2$ for any n.
- The sequence can get "as close to 2 as we like", in the sense that for any value smaller than 2, however close to 2 it is (say, 1.9999), a_n will at some point exceeds that value and get even closer to 2, for large enough n.
- Though both 2 and 3 are upper bounds of $\{a_n\}$ in the context, 2 is "special and unique" to $\{a_n\}$ as $\{a_n\}$ can get "as close to 2 as we like". But this does not apply to 3 or any value larger than 2.

This existence of sequence $\{a_n\}$ reveals an important yet not intuitive fact that it is possible to find a monotonically increasing yet bounded sequence. Another way to look at it is that it is possible to add infinite number of positive values together, yet the result is bounded.

1.1.2 Definition of the Limit of a Sequence

Given an infinite sequence $\{a_n\}$. If $\{a_n\}$ is bounded and it gets close to a value L "as close as we like" for large n, then L is called the limit of sequence $\{a_n\}$. To be more precise, the definition of the limit of a sequence is given as follows.

Definition of the limit of sequence:

A sequence $\{a_n\}$ has the limit L if for any $\varepsilon > 0$, there is always a corresponding integer N, such that if $n > N$, $|a_n - L| < \varepsilon$. This is denoted by

$$\lim_{n\to\infty} a_n \quad = \quad L,$$

or

$$a_n \to L \quad \text{as} \quad n \to \infty,$$

and in this case we say "sequence $\{a_n\}$ is a convergent" and "sequence $\{a_n\}$ converges to L as n approaches infinity".

The sequence $\{a_n\}$ in (1.6) given in Section 1.1.1 is an example of a convergent sequence that converges to 2, and we can prove this using the definition of the limit of a sequence as follows. Given any $\varepsilon > 0$, solving

$$|V(N) - 2| < \varepsilon$$

gives

$$\begin{aligned} \left|2 - \left(\frac{1}{2}\right)^{N-1} - 2\right| &< \varepsilon \\ N &> 1 - \log_2 \varepsilon \end{aligned} \tag{1.7}$$

For example, if specify $\varepsilon = 0.05$, using (1.7) gives $N > 5.32$. This implies that from $n \geq 6$ onwards, $|a_n - 2| < 0.05$ can be obtained. This matches the observation given in Table 1.1.

If an infinite sequence $\{a_n\}$ does not have a limit, we say "sequence $\{a_n\}$ is divergent" or "sequence $\{a_n\}$ diverges".

As a special case of divergent sequences, if $\{a_n\}$ becomes unbounded as n approaches infinity, we have

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Definition of sequence diverging to infinity:

For a sequence $\{a_n\}$, if for any arbitrary positive value M, there is always a corresponding integer N, such that if $n > N$, $|a_n| > M$, we say "$\{a_n\}$ diverges to infinity" and denote it by

$$\lim_{n\to\infty} a_n = \infty.$$

We use $\{s_n\} = \sum_{i=1}^{n} a_n$ to denote the sum of the first n elements in the infinite sequence $\{a_n\}$. Apparently, $\{s_n\}$ itself is also an infinite sequence which may or may not converge depending on $\{a_n\}$. As n approaches infinity, $\{s_n\}$ converges to $\sum_{i=1}^{\infty} a_n$, which is called the infinite sum of the series $\{a_n\}$.

1.1.3 Proof of Sequence Convergency

Proving the convergence/divergence of an infinite sequence and obtaining its limit are sometimes not easy, and need to be done case by case. A few examples are given in the following Table 1.3 for the convergency of some commonly seen $\{a_n\}$ and its associated sum $\{s_n\}$.

If an interested sequence falls in one of the above categories, or at least similar to one of the above categories, or can be expressed as a sum/multiplication/division of two sequences from the above categories, then

TABLE 1.3
Convergency of commonly seen $\{a_n\}$ and $\{s_n\}$. Variable c, r are constant real numbers.

Category	a_n	s_n	**Convergency** a_n	**Convergency** s_n
Polynomial	c	nc	c	No(∞)
	n	$\frac{n(n+1)}{2}$	No(∞)	No(∞)
	n^2	$\frac{n(n+1)(2n+1)}{6}$	No(∞)	No(∞)
	n^3	$\frac{n^2(n+1)^2}{4}$	No(∞)	No(∞)
Power Series	$cr^{n-1}, \lvert r\rvert < 1$	$\frac{c(1-r^n)}{1-r}$	0	$\frac{c}{1-r}$
Others	n^{-1}	—	0	No(∞)
	n^{-2}	—	0	$\frac{\pi}{6}$

"No(∞)" stands for "diverges to infinity".

its convergence/divergence might be proved a bit easier. For example, given two sequences $\{a_n\}$ and $\{b_n\}$, if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L \neq 0,$$

then $\{a_n\}$ and $\{b_n\}$ must behave the same in terms of convergency, meaning that both of them must converge or diverge at the same time.

Some other interesting features regarding sequence convergency are given as follows.

For example, the sum $c_n = a_n + b_n$ of two convergent sequence $\{a_n\}$ and $\{b_n\}$ must be convergent to $\lim_{n\to\infty} c_n = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$. This can be proved using the definition of the limitation of the sequence.

It is intuitive and not difficult to prove that for $\{s_n\}$ to converge, i.e. $\lim_{n\to\infty} s_n = s$, it is necessary (but not sufficient) for its associated $\{a_n\}$ to converge to zero, i.e. $\lim_{n\to\infty} a_n = 0$. Do notice that it is possible to have a divergent $\{s_n\}$ even if $\lim_{n\to\infty} a_n = 0$. An example is the harmonic series given in Table 1.3 where $a_n = \frac{1}{n}$.

The famous monotone convergence theorem states that *if a sequence $\{a_n\}$ is monotonically increasing or decreasing, and it is at the same time bounded, i.e. $|a_n| < M$ for all $n \geq 1$, then $\{a_n\}$ must be convergent*. The proof of this theorem is difficult than it appears to be, thus is not included in the notebook.

From the monotone convergence theorem, we know that for a sequence $\{a_n\}$, if $\sum_{i=1}^{\infty} |a_n|$ converges, then $\sum_{i=1}^{\infty} a_n$ must converge. This is can be illustrated simply by splitting $\{a_n\}$ into $\{a_n^+\}$ and $\{a_n^-\}$, where

$$\begin{aligned} a_n^+ &= \begin{cases} a_n & a_n \geq= 0 \\ 0 & a_n < 0 \end{cases}, \\ a_n^- &= \begin{cases} -a_n & a_n < 0 \\ 0 & a_n \geq 0 \end{cases}. \end{aligned}$$

Apparently, $|a_n| = a_n^+ + a_n^-$ and both $\{a_n^+\}$ and $\{a_n^-\}$ are monotonically increasing positive sequences. Since $\sum_{i=1}^{\infty} |a_n|$ is bounded, both $\{a_n^+\}$ and $\{a_n^-\}$ must be bounded, therefore, convergent, according to the monotone convergence theorem. This implies that $a_n = a_n^+ - a_n^-$ must be convergent as the sum of two convergent sequences. If $\sum_{i=1}^{\infty} |a_n|$ converges, then series $\{\sum_{i=1}^{n} a_n\}$ is called "absolutely convergent". Absolutely convergent sequence is convergent, but it might be not true wise versa.

1.1.4 Additional Comments

There are some famous serious widely used in both mathematics and applied mathematics, such as the famous Taylor serious, which will be introduced in later chapters of the book when the definition of derivative is given. The application of Taylor serious, and many more important serious, can be found in varieties of textbooks related to numerical analysis, signal processing, control engineering, etc., and they will not be covered in details in this notebook.

1.2 The Limit of a Function

1.2.1 A Motivating Example

A motivating example is given in Scenario 2 to illustrate the limit of a function.

Scenario 2

Consider function

$$y = f(x) = \begin{cases} (x-1)^2 & x \neq 1 \\ 1 & x = 1 \end{cases} \tag{1.8}$$

Q1: What is the domain (the set of values for the independent variable x) and range (the set of values for the dependent variable y) of this function?

Q2: What is the value of y at $x = 1$?

Q3: What is the value of y when x is somewhere close but not equal to 1 (formally, when x approaches 1)?

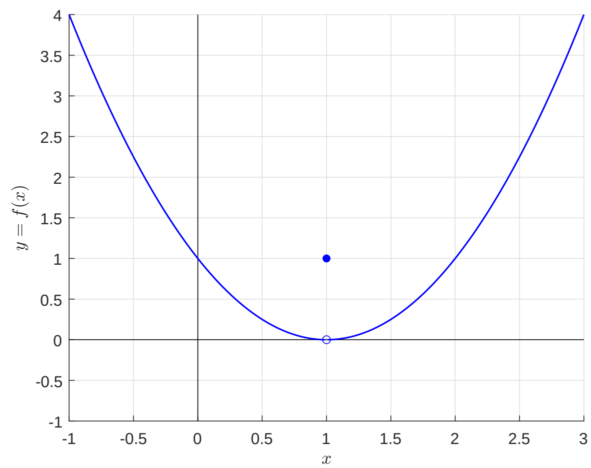

FIGURE 1.1
Plot of y as a function of x in the motivating example.

We can plot y and x in (1.8), which shall give as a straight forward impression on what the function looks like. A plot of y as a function of x is given in Fig 1.1. It is clear from the figure that for Q1 the domain and range of the function are $x \in \mathbb{R}$ and $y \in \mathbb{R}, y > 2$ respectively. For Q2, substituting $x = 1$ into (1.8) gives $y = 1$, as is also illustrated in Fig 1.1.

To answer Q3, we need to quantify "somewhere close but not equal to 1" in a more precise manner. An intuitive way of doing that is to define a small "threshold area" near $x = 1$, say, $-\varepsilon < x - 1 < \varepsilon, x \neq 1$. Notice that $x = 1$ is out of our concern, which suggests that the value of y near $x = 1$ has nothing to do with y at $x = 1$. In practice, ε shall be a rather small positive value as the threshold boundary shall be "close to 1".

Next, we will need to describe y given $-\varepsilon < x - 1 < \varepsilon, x \neq 1$. Clearly, the range of y relates to the choice of ε. Substituting $-\varepsilon < x - 1 < \varepsilon, x \neq 1$ into (1.8) gives $0 < y < \varepsilon^2$, which is the answer to Q3. If ε is chosen extremal small, y will approach 0 (although y cannot be precisely 0).

1.2.2 Definition of the Limit of a Function

The formal definition of the limit of a function follows the same idea given in Section 1.2.1 as follows. Notice that there are a few different but equivalent ways to define the limit of a function. Here the "epsilon-delta definition" is introduced.

Definition of the limit of a function at $x \to a$:

A function $f(x)$ of x has the limit L at $x = a$ if for any $\varepsilon > 0$, there is always a corresponding $\delta > 0$, such that if $|x - a| < \delta$, $|f(x) - L| < \varepsilon$, with prerequisite that $|x - a| < \delta$ is defined for $f(x)$. This is denoted by

$$\lim_{x \to a} f(x) \quad = \quad L,$$

or

$$f(x) \to L \quad as \quad x \to a.$$

Using the definition above, it can be proved easily that for Scenario 2 in Section 1.2.1 the function has a limit of $\lim_{x \to 1} f(x) = 0$. Do notice that $\lim_{x \to a} f(x) = L$ does not necessarily require $f(a) = L$. As a matter of fact, $f(x)$ does not need to be defined at $x = a$, as long as it is defined at the neighbour of $x = a$.

Similar to the definition of the limit of a function, the definition of one-sided limit of a function is given below. The two definitions are similar, but the one-sided limit is weaker in the sense that it only requires one side of the neighbour of $x = a$ to be checked.

Definition of the one-sided limit of a function:

A function $f(x)$ of x has the one-side left limit L_{left} at $x = a$ if for any $\varepsilon > 0$, there is always a corresponding $\delta > 0$, such that if $a - \delta < x < a$, $|f(x) - L_{left}| < \varepsilon$, with prerequisite that $a - \delta < x < a$ is defined for $f(x)$. This is denoted by

$$\lim_{x \to a^-} f(x) \quad = \quad L_{left},$$

or

$$f(x) \to L_{left} \quad as \quad x \to a^-.$$

A function $f(x)$ of x has the one-side right limit L_{right} at $x = a$ if for any $\varepsilon > 0$, there is always a corresponding $\delta > 0$, such that if $a < x < a + \delta$, $|f(x) - L_{right}| < \varepsilon$, with prerequisite that $a < x < a + \delta$ is defined for $f(x)$. This is denoted by

$$\lim_{x \to a^+} f(x) \quad = \quad L_{right},$$

or

$$f(x) \to L_{right} \quad as \quad x \to a^+.$$

Clearly from the definition, a function $f(x)$ has a limit of L at $x = a$ if and only if it has both one-sided left limit L_{left} and one-sided right limit L_{right} at $x = a$ and $L_{left} = L_{right} = L$, i.e.

$$\lim_{x\to a} f(x) = L \quad \Leftrightarrow \quad \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L.$$

Furthermore, if function $f(x)$ has a limit L at $x = a$, and also $f(x) = a$, the function $f(x)$ is called continuous function at $x = a$. The example given in Scenario 2 in Section 1.2.1 is not continuous at $x = 1$ as $\lim_{x\to 1} = 0$ while $f(x)|_{x=1} = 1$, which can be seen from Fig. 1.1.

The definition of the limit of a function $f(x)$ when x approaches infinity is given below. It is quite similar to the definition of the limit of a infinite sequence.

Definition of the limit of a function at $x \to \pm\infty$:

A function $f(x)$ of x has the limit L at $x \to +\infty$ (sometimes denoted as $x \to \infty$ for simplicity) if for any $\varepsilon > 0$, there is always a corresponding δ, such that if $x > \delta$, $|f(x) - L| < \varepsilon$, with prerequisite that $x > \delta$ is defined for $f(x)$. This is denoted by

$$\lim_{x\to+\infty} f(x) \quad = \quad L,$$

or

$$f(x) \to L \quad as \quad x \to +\infty.$$

A function $f(x)$ of x has the limit L at $x \to -\infty$ if for any $\varepsilon > 0$, there is always a corresponding $\delta 0$, such that if $x < \delta$, $|f(x) - L| < \varepsilon$, with prerequisite that $x < \delta$ is defined for $f(x)$. This is denoted by

$$\lim_{x\to-\infty} f(x) \quad = \quad L,$$

or

$$f(x) \to L \quad as \quad x \to -\infty.$$

TABLE 1.4
Limit of commonly seen elementary functions.

Category	a_n	s_n	**Convergency** a_n	s_n
Polynomial	c	nc	c	No(∞)
	n	$\frac{n(n+1)}{2}$	No(∞)	No(∞)
	n^2	$\frac{n(n+1)(2n+1)}{6}$	No(∞)	No(∞)
	n^3	$\frac{n^2(n+1)^2}{4}$	No(∞)	No(∞)
Power Series	$cr^{n-1}, \|r\| < 1$	$\frac{c(1-r^n)}{1-r}$	0	$\frac{c}{1-r}$
Others	n^{-1}	—	0	No(∞)
	n^{-2}	—	0	$\frac{\pi}{6}$

"No(∞)" stands for "diverges to infinity".

1.2.3 Calculation of the Limit of a Function

The calculation of the limit of many commonly seen elementary functions are often obvious and easy. The limit $\lim_{x\to a} f(x)$ can very likely be obtained by substituting $x = a$ into the functions, given that the function is defined at $x = a$. The limit $\lim_{x\to\infty} f(x)$ might be slightly difficult but mostly can be obtained from the definition. Some examples are given below in Table 1.4.

1.2.4 Additional Comments

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What is Derivative

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2.1 Rate of Change of a Function

2.2 Derivative of a Function

Part II

Vector Function, Partial Derivative and Multiple Integral

Part III

Differential Equation

Part IV

Applications and Case Studies

Bibliography

[1] James Stewart. *Calculus Metric Version Eighth Edition*. Cengage Learning, 2015.

[2] Gibert Strang. *Calculus*. Wellesley-Cambridge Press, 1991.