*Lu Sun, and many more.*

# ***A Notebook on Algebra***

*To my family, friends and communities members who have been dedicating to the presentation of this notebook, and to all students, researchers and faculty members who might find this notebook helpful.*

# *Contents*

# *Foreword*

If software or e-books can be made completely open source, why not a notebook?

This brings me back to the summer of 2009 when I started my third year as a high school student in Harbin No. 3 High School. In around the end of August when the results of Gaokao (National College Entrance Examination of China, annually held in July) are released, people from photocopy shops would start selling notebooks photocopies that they claim to be from the top scorers of the exam. Much curious as I was about what these notebooks look like, never have I expected myself to actually learn anything from them, mainly for the following three reasons.

First of all, some (in fact many) of these notebooks were more difficult to understand than the textbooks. I guess we cannot blame the top scorers for being so smart that they sometimes make things extremely brief or overwhelmingly complicated.

Secondly, why would I want to adapt to notebooks of others when I had my own notebooks which in my opinion should be just as good as theirs.

And lastly, as a student in the top-tier high school myself, I knew that the top scorers of the coming year would probably be a schoolmate or a classmate. Why would I want to pay that much money to a complete stranger in a photocopy shop for my friend's notebook, rather than requesting a copy from him or her directly?

However, things had changed after my becoming an undergraduate student in 2010. There were so many modules and materials to learn in a university, and as an unfortunate result, students were often distracted from digging deeply into a module (For those who were still able to do so, you have my highest respect). The situation became even worse as I started pursuing my Ph.D. in 2014. As I had to focus on specific research areas entirely, I could hardly split much time on other irrelevant but still important and interesting contents.

This motivated me to start reading and taking notebooks for selected books and articles, just to force myself to spent time learning new subjects out of my comfort zone. I used to take hand-written notebooks. My very first notebook was on *Numerical Analysis*, an entrance level module for engineering background graduate students. Till today I still have on my hand dozens of these notebooks. Eventually, one day it suddenly came to me: why not digitalize them, and make them accessible online and open source, and let everyone read and edit it?

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As most of the open source software, this notebook (and it applies to the other notebooks in this series as well) does not come with any “warranty” of any kind, meaning that there is no guarantee for the statement and knowledge in this notebook to be absolutely correct as it is not peer reviewed. **Do NOT cite this notebook in your academic research paper or book!** Of course, if you find anything helpful with your research, please trace back to the origin of the citation and double confirm it yourself, then on top of that determine whether or not to use it in your research.

This notebook is suitable as:

- a quick reference guide;
- a brief introduction for beginners of the module;
- a “cheat sheet” for students to prepare for the exam (Don’t bring it to the exam unless it is allowed by your lecturer!) or for lecturers to prepare the teaching materials.

This notebook is NOT suitable as:

- a direct research reference;
- a replacement to the textbook;

because as explained the notebook is NOT peer reviewed and it is meant to be simple and easy to read. It is not necessary brief, but all the tedious explanation and derivation, if any, shall be “fold into appendix” and a reader can easily skip those things without any interruption to the reading experience.

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Although this notebook is open source, the reference materials of this notebook, including textbooks, journal papers, conference proceedings, etc., may not be open source. Very likely many of these reference materials are licensed or copyrighted. Please legitimately access these materials and properly use them.

Some of the figures in this notebook is drawn using Excalidraw, a very interesting tool for machine to emulate hand-writing. The Excalidraw project can be found in GitHub, *excalidraw/excalidraw*.

# *Preface*

This notebook is on *Algebra*. The first part of the notebook is about linear algebra, one of the first few modules a science and engineering undergraduate student would take in his first semester in the collage. It is absolutely the fundamental of almost all the mathematical tools he would use in the future. The second part of the notebook is about abstract algebra, a far more advanced topic and yet still a lot of fun to learn.

The key reference of this notebook is listed below. During the development of the notebook, this list may become longer and longer.

Book *Basic Algebra Second Edition (I and II)* by Nathan Jacobson, published by DOVER [1]

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# Linear Algebra

# Part II

# Abstract algebra

# 1

# *Abstract Algebra Basics*

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What does "abstract" mean in abstract algebra? How is it different from the classic algebra introduced in earlier chapters? In short, classic algebra solves a particular problem using algebra algorithms, whereas abstract algebra studies these algorithms themselves.

As an example, consider the following equation

$$Ax = y$$

where $x$, $y$ are vectors and $A$ a matrix. Given particular $y$ and $A$, solving probable $x$ is a classic algebra problem. It is obvious that $x$ does not necessarily exist or being unique for different $y$ and $A$. Studying the general rules of when $x$ exists and when it is unique for any $y$ and $A$ becomes an abstract algebra problem.

Consider another example where

$$\begin{aligned} a+b &= b+a \\ ab &= ba \end{aligned}$$

which are often used to demonstrate the commutative property of calculations (summation and multiplication, in this example). In classic algebra, they are considered as ground truth and are used to solve practical problems. In abstract algebra, however, the focus shifts to a more formal and generalized understanding of the property. We need to dig deeper into how commutative property is defined, and why it holds true for summation and multiplication, but not for some other operations such as division.

In conclusion, while classic algebra calculates numbers, vectors and matrices, abstract algebra checks the calculations, tools, concepts, and logic derivations and try to explain why they work in the way we desire, and invent new tools that we can use to do powerful calculations and derivations.

## 1.1 A Motivating Example

One of the most famous applications of abstract algebra is to study the analytical solution to the following series of polynomial equations

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0 \tag{1.1}$$

where $n \geq 1$ is the order of the polynomial and $a_1, \ldots, a_n$ are any arbitrary values. The analytical solutions to (1.1) when $n = 1$ and $n = 2$ are obvious. With some effort, the analytical solutions when $n = 3$ and $n = 4$ were found in the 16th century. Since then, people have been struggling to find the analytical solution to the fifth order and beyond $n \geq 5$ polynomial equation.

In the 18th and 19th century, Euler, Lagrange and Gaussian tried to address this problem. Their conclusion was that there is no analytical solution to polynomial equations of fifth order or higher, but they were not be able to fully solve the problem by giving a very solid proof. The methods they used inspired a lot more people that would put hand to this problem.

In the 19th century, Abel was able to prove that there is no solution in radicals to general polynomial equations of degree five or higher with arbitrary coefficients (see Abel–Ruffini theorem). Furthermore, he discusses a set of special cases (with non-arbitrary coefficients in the polynomial equation) that can have analytical solution. These special cases form a set of sufficient condition for a fifth order polynomial equation to have the analytical solution.

The necessary and sufficient condition for a fifth or higher order polynomial to have an analytical solution is finally fully found out and interpreted by the genius Galois at a remarkably young age. Galois was able to create his theorem (known Galois theorem), and use that theorem to find the ultimate answer to this problem that people have been studied for centuries, and his theorem goes far beyond that. Galois theorem will find its usefulness in many areas to come, and eventually it becomes an important building block of a subject known as abstract algebra today.

## 1.2 Algebraic System

An algebraic system is essentially a mathematical system consisting of a (non-empty) set (known as the domain) and a series of operations defined on the domain. There are many algebraic systems, and abstract algebra studies the properties of different algebraic systems. As will be introduced in later parts of the notebook, depending on the properties of the algebraic system, we can categorize them as groups, rings, fields, vector spaces, etc.

### 1.2.1 Set, Mapping, Operation and Relation

Set is one of the most commonly used terms across different mathematical subjects. It is also one of the fundamental concepts in abstract algebra. A set usually refers to a collection of distinct objects. Given a set $U$ and an object $x$, one and only one of the following two statements must be true:

- Object $x$ is a member of set $U$, denoted by $x \in U$;
- Object $x$ is not a member of set $U$, denoted by $x \notin U$.

However, notice that due to the Russell's paradox, it is challenging to give a rigorous mathematical definition to a set that fulfill the above features.

Mapping is used to describe the association of elements in two sets. For example, let $A$ be a set, and $A_0 \subset A$ a subset of $A$. For any element $x \in A_0$, define mapping

$$i \quad : \quad A_0 \to A$$

where

$$i(x) \quad = \quad x$$

In this case, mapping $i$ is called the **embedding mapping** from $A_0$ to $A$.

Let $A$, $B$ be two sets, and $A_0 \subset A$ as subset of $A$. Let $f : A \to B$, and $g : A_0 \to B$. Let $x \in A_0$. If $f(x) = g(x)$, function $f$ is known as an **extension** of function $g$, and function $g$ a **restriction** of function $f$ (on $A_0$). This is denoted by $g = f|_{A_0}$.

# Part III

# Number Theory

# *Bibliography*

[1] Nathan Jacobson. *Basic algebra I and II*. Courier Corporation, 2012.