Investigating the Local-Global Conjecture in Polyhedral Circle Packings

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Abstract

Integral circle packings are fractal arrangements of mutually tangent circles with integer curvatures. The longstanding **Local-Global Conjecture** [1] predicts which sufficiently large integers will appear in a given packing. Haag, Kertzer, Rickards and Stange disprove this conjecture for certain Apollonian, or tetrahedral circle packings [2]. We disprove the Local-Global Conjecture for certain octahedral, cubic, square, triangular, and hexagonal packings, following the approach of [2]. In each packing type, we find obstructions to the conjecture for certain packings, but not others. We give computational evidence that the obstructions we find may be the only ones.

Polyhedral Packings

Polyhedral circle packings have tangency restrictions based on polyhedra [3]. Our research focuses on two cases, the cubic and octahedral packings. These are constructed from configurations of 8 and 6 mutually tangent base circles, with 6 and 8 dual circles, respectively. Reflecting through these dual circles generates the rest of the packing.

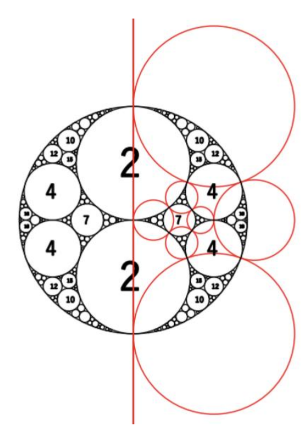

Figure 1. Octahedral packing, dual circles in red.

Packings from Tilings

Other circle packings have tangency restrictions based on tilings of the plane [4]. These are constructed from configurations of infinitely many base and dual circles. We focus on three cases, the triangular, square, and hexagonal packings.

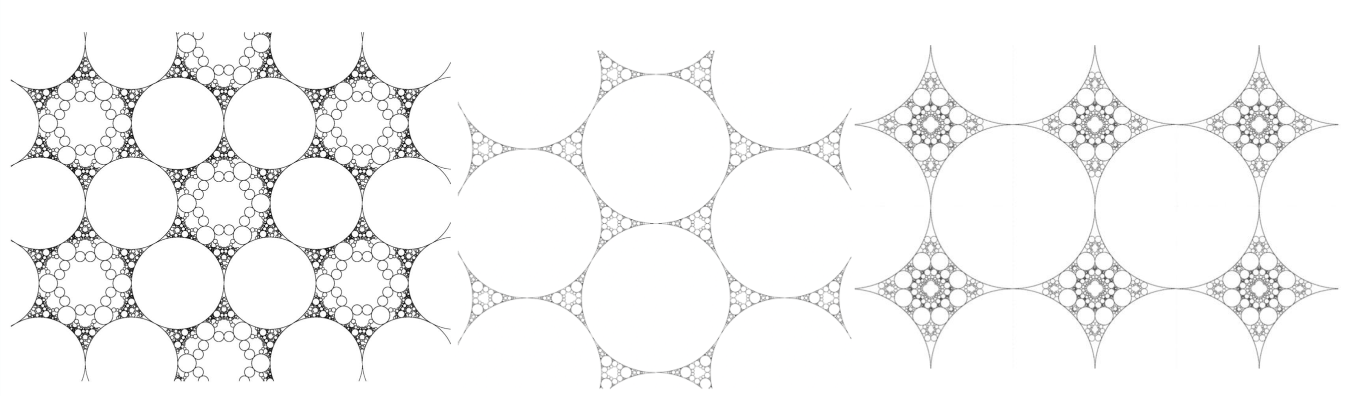

Figure 2. From left to right: hexagonal, triangular, and square tilings [4]

Modular Restrictions

The algebraic relations among curvatures in a packing lead to certain congruence restrictions. We give a complete description of congruence restrictions in all the packing types we study. For example, in any integral octahedral packing, all curvatures are restricted to either {0,1,2}, {0, 3, 6}, {4,5,6}, or {2,4,7} mod 8.

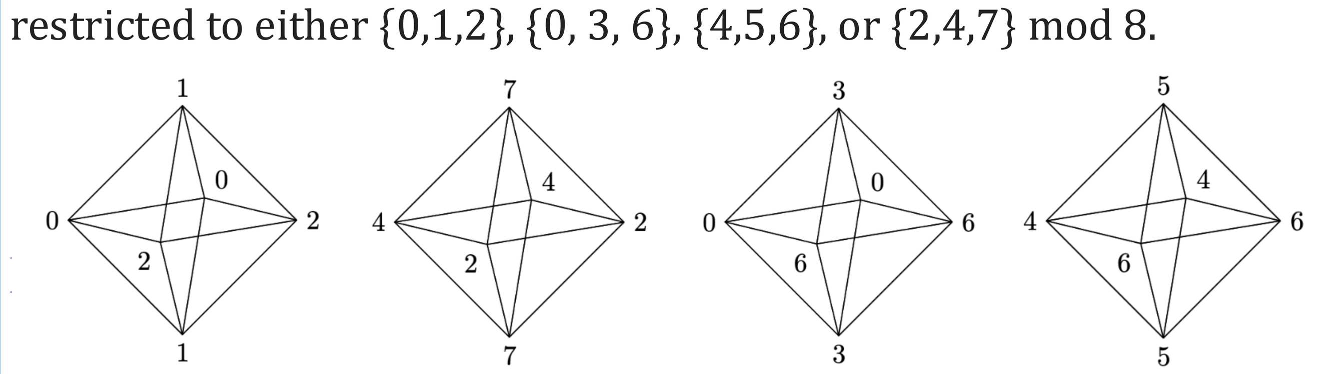

Figure 3. Possible octahedral circle configuration modulo 8

What Integers are Allowed to Appear?

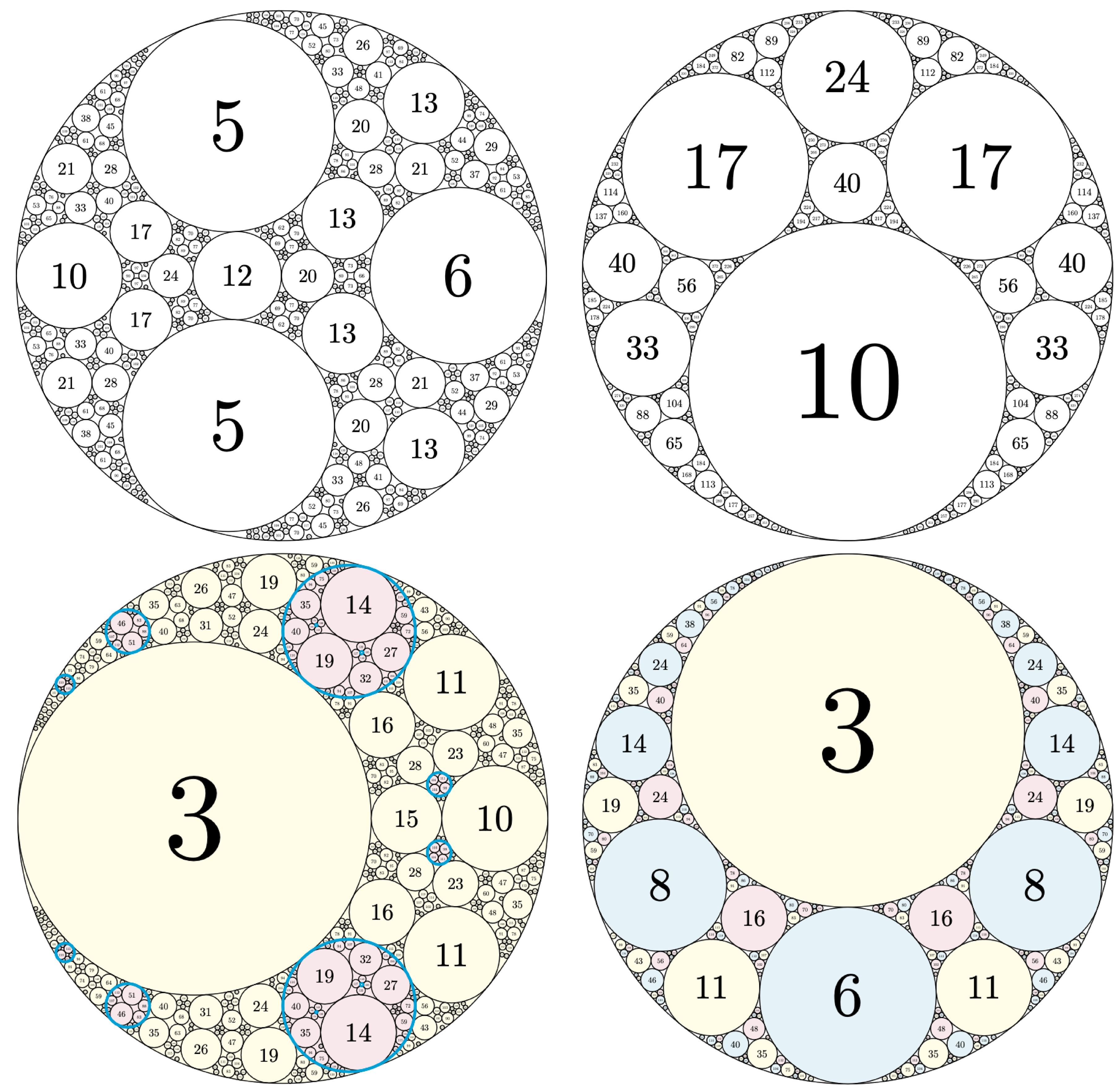

Figure 4: Top row: cubic and octahedral packings with full quadratic obstructions, not satisfying the Local-Global Property. Bottom row: cubic and octahedral packings with partial quadratic obstructions.

Our Theorem:

The Local-Global Conjecture is false for Octahedral, Cubic, Triangular, Square, and Hexagonal Packings.

What is the Local-Global Conjecture?

In an integral Apollonian circle packing, the curvatures that appear must fall into one of six or eight residue classes modulo 24. The Local-Global Conjecture states that every sufficiently large integer in one of these residue classes will appear as a curvature in the packing. For more general integral packings, the Local-Global Conjecture states that every sufficiently large integer satisfying some finite list of congruence restrictions appears as a curvature.

Disproving the Local-Global Conjecture

Our strategy to disprove the Local-Global Conjecture involves a **quadratic invariant** χ_2 based on the Kronecker symbol. The Kronecker symbol is a function $\left(\frac{a}{n}\right)$ that depends on whether there exists a square number congruent to a modulo n. Our χ_2 is the Kronecker symbol for two adjacent curvatures in the packing, or a modified version. In some cases, the quadratic invariant leads to a **quadratic obstruction**: certain integers, of the form a constant times a perfect square, cannot appear as curvatures.

Full and Partial Quadratic Invariants

The most interesting case is when χ_2 is constant across the whole packing. For example, the packings in the top row of Figure 4 have $\chi_2 = -1$. That is, no circle has a curvature which is a square modulo any of the curvatures of its neighbors. Thus, there is a quadratic obstruction: no curvature in the packing is a square number. **This disproves the Local-Global Conjecture.**

For χ_2 to be constant across the packing, two criteria must be met:

1) Any tangent circles share a χ_2 value.
2) If a circle is fixed, its χ_2 is the same no matter which tangent circle is used to compute it.

The proof of (1) uses quadratic reciprocity, while (2) uses an explicit parametrization of circles.

However, sometimes one or both of the conditions fail. In those cases, we can still define a **partial quadratic invariant** χ_2. In the bottom left packing in Figure 4, the first condition fails and χ_2 changes sign at the tangencies which cross blue circles. In the bottom right packing, the second condition fails as χ_2 is not well defined for a yellow circle: the red and blue circles necessarily yield different χ_2 values.

In these cases, we find **partial quadratic obstructions** which do not disprove the Local-Global conjecture. For example, in the bottom left packing, no curvature within an even number of blue circles is a square number. In the bottom right packing, no curvature of a blue circle is a square number.

Future Work

More work is needed to analyze some of the most complicated congruence cases. These include certain examples of triangular and hexagonal tilings, as well as the cubeoctahedral packing shown below. In these cases, both (1) and (2) fail and the behavior of the Kronecker symbol is more difficult to predict. We hope our work points the way to a more complete understanding of quadratic obstructions in group orbits.

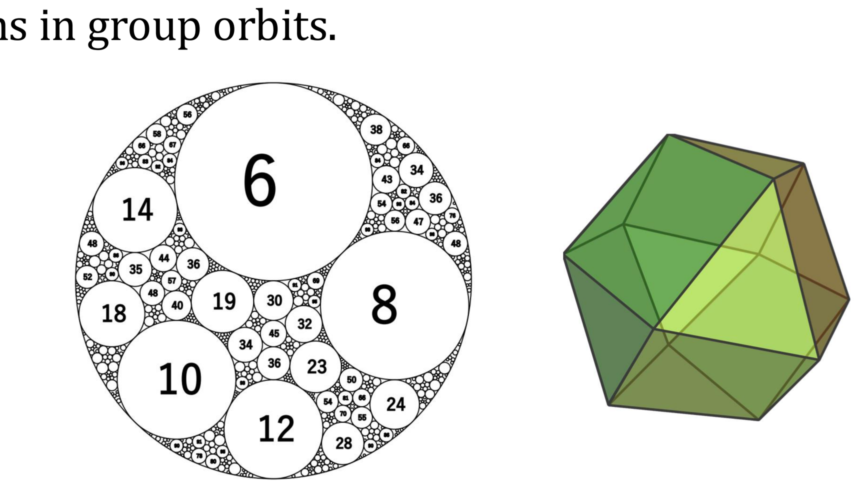

Figure 5. Cuboctahedron packing, its geometry is on the right

References

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[2] S. Haag, C. Kertzer, J. Rickards and K. Stange, The local-global conjecture for Apollonian circle packings is false, *Ann. of Math*. (2) (2024), no. 2, 749-770
[3] A.V. Kontorovich and K. Nakamura, Geometry and arithmetic of crystallographic sphere packings, *Proc. Natl. Acad. Sci. USA* (2019), no. 2, 436-441
[4] P. Rehwinkel and I. Whitehead and D. Yang and M. Yang, Circle packings from tilings of the plane, *J. Geom*. (2024), no. 1, Paper No. 17, 28 pp