Sperner Property of the Boolean Algebra

Haoyang Xu, Nathan Phan

2025 Brown DRP Poster Session

Motivating Problem

Question
What is the maximum number of subsets of $\{1, 2, ..., n\}$ that do not contain each other?

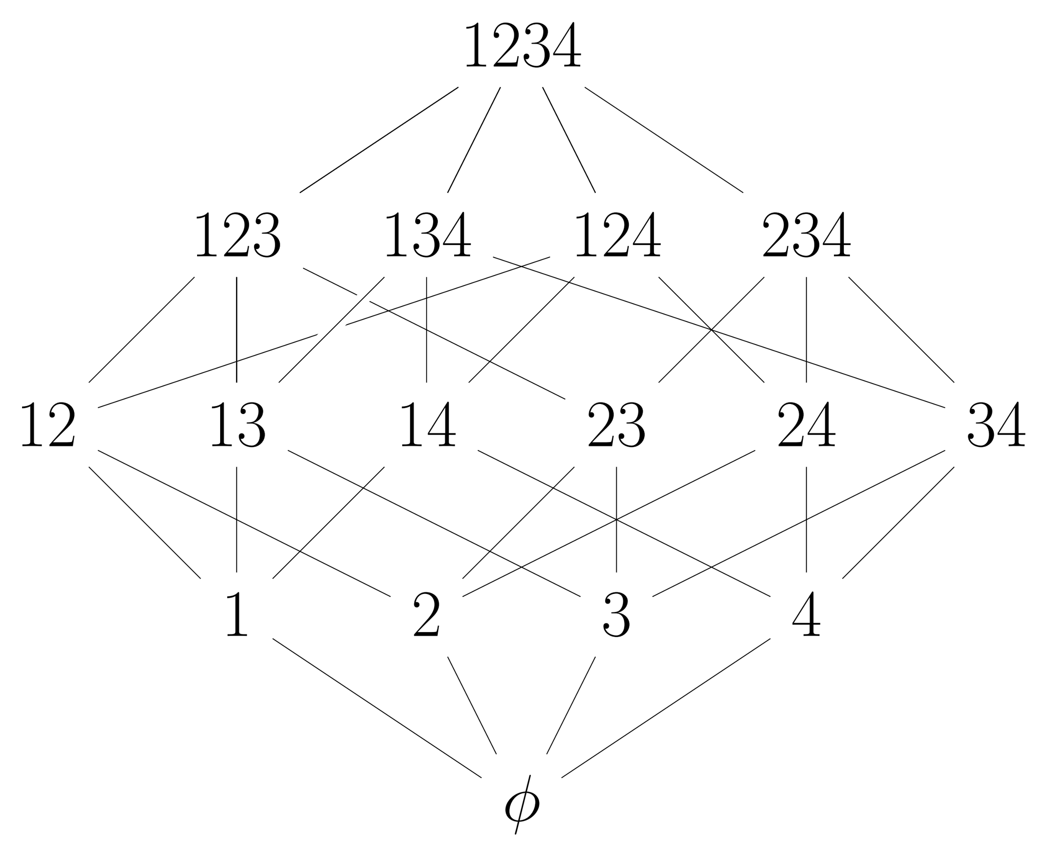

Fig. 1: Hasse diagram of B_4

Possible Thoughts
One may argue that the answer is **the number of subsets in the *middle layer***, but how do we know that this is true?

Sperner Property

Poset and Antichain
A **poset** P is a finite set with a binary operation $\leq$ that is reflexive, antisymmetric, and transitive. An **antichain** in P is a subset $A \subset P$ where no two elements are comparable.

Boolean Algebra
B_n, a **boolean algebra** of rank n, is a set of all subsets of an n-element set. This can be regarded as a poset with $\subseteq$ being the binary relation.

Sperner Property
A graded poset P can be partitioned into *layers* $P = P_1 \sqcup P_2 \sqcup \cdots \sqcup P_n$.
For B_n, $P_i = \{S \subset \{1, \ldots, n\} : |S| = i\}$.
A graded poset P is **Sperner** if

$$\max(|A : \text{antichain}|) = \max(|P_i|).$$

Theorem B_n is **Sperner**!

Answer to our motivating problem

$$\#\text{ of subsets} = \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

A Proof Method

We can find **order-matchings** that partition B_n.

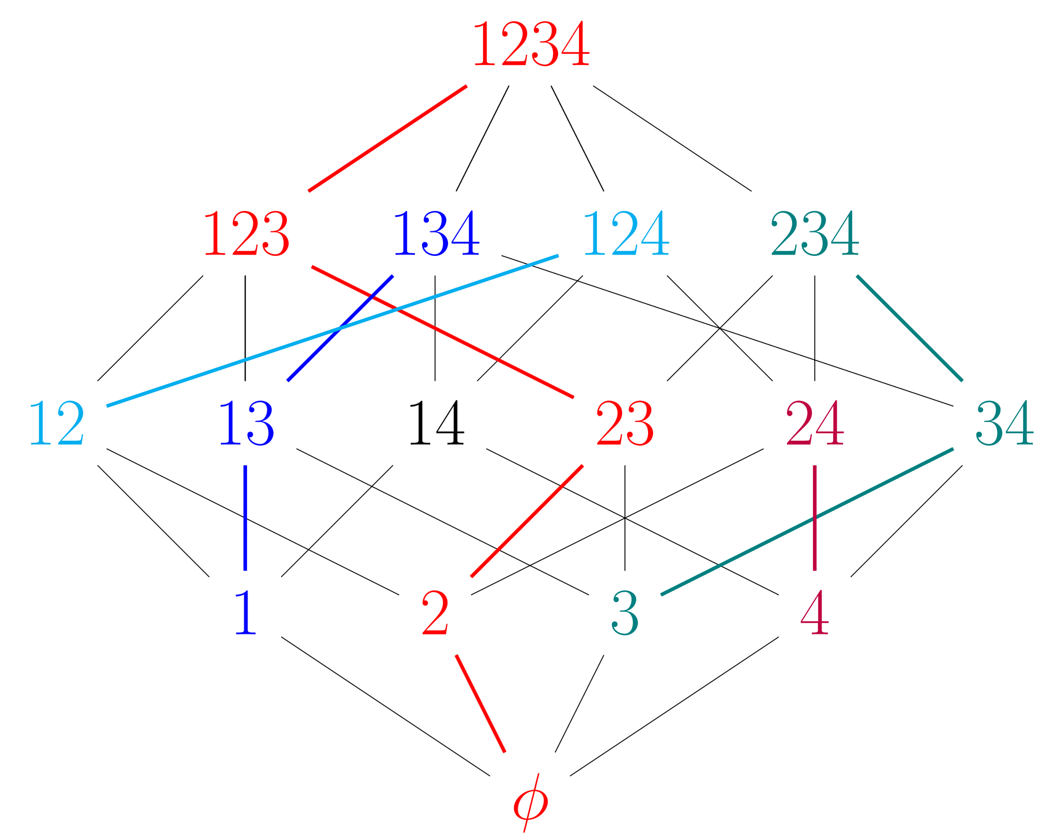

Fig. 2: B_4 with order-matchings

Acknowledgments

We are deeply grateful to Ethan Partida for his exceptional mentorship and huge support. We also wish to acknowledge Prof. Richard P. Stanley and his book *Algebraic Combinatorics* [1] that we enjoyed reading and learned a lot.

References

[1] Richard P. Stanley. *Algebraic Combinatorics*. 2018. DOI: https://doi.org/10.1007/978-3-319-77173-1.