Notes on clever proofs in "Period 3 Implies Chaos"

This is an explanation on Theorem 1 in Period 3 Implies Chaos [1], a paper that shows a complex and chaotic behaviour can exist even for a map with a simple setting such as period 3. In particular, it shows period 3 implies many periodic orbits and chaos. I really liked how the elementary yet clever proofs deduce such powerful chaotic results.

Theorem 1. *Let J be an interval and let $F : J \to J$ be continuous. If there is a point $a \in J$ where $b = F(a)$, $c = F^2(a)$, $d = F^3(a)$ with $d \leq a < b < c$ or $d \geq a > b > c$, T1 and T2 hold:*
T1. For every $k = 1, 2, \ldots$, there is a periodic point in J having period k.
T2. There is an uncountable set $S \subset J$ (containing no periodic points) which satisfies
(A) For every $p, q \in S$ with $p \neq q$,

$$\limsup_{n\to\infty} |F^n(p) - F^n(q)| > 0 \qquad \textit{and} \qquad \liminf_{n\to\infty} |F^n(p) - F^n(q)| = 0.$$

(B) For every $p \in S$ and every periodic point $q \in J$,

$$\limsup_{n\to\infty} |F^n(p) - F^n(q)| > 0.$$

Intuitively, map F maps a to b, b to c (increasing) but then c to d where d is smaller than a, a property that enables mapped intervals to stretch and fold as in horseshoe. T1 indicates the existence of periodic points for all periods and T2 indicates there are uncountably many pair of points that swing between being indistinguishable and far apart, i.e. chaos. We will prove both by constructing routes of intervals that a point passes through while iterating the map.

T1 Proof Explanation

T1 asserts that for every $k = 1, 2, \ldots$, there is a periodic point in J having period k.
First define two subintervals of J, $K = [a, b]$ and $L = [b, c]$. Since F is continuous, the intermediate value theorem implies $F(K) \supset L$ and $F(L) \supset K \cup L$. This means taht for every $y \in L$ there exists $x \in K$ such that $F(x) = y$ and for every $y \in K \cup L$ there exists $x \in L$ such that $F(x) = y$. In route terms, the allowed transitions are $K \to L$ and $L \to K$ or L.

Lemma 0 Let $G : I \to \mathbb{R}$ be continuous, where I is an interval. If $I_1 \subset G(I)$ is a compact interval, then there exists a compact subinterval $Q \subset I$ such that $G(Q) = I_1$.
This lemma is an extension of intermediate value theorem. Continuity guarantees that you can always find a compact subinterval in domain that gets mapped to the exact interval of interest in codomain.

Lemma 1 Let $F : J \to J$ be continuous. Let $\{I_n\}_{n\geq 0}$ be a sequence of compact intervals such that $I_{n+1} \subset F(I_n)$ for all $n \geq 0$. Then there exist compact intervals $\{Q_n\}_{n\geq 0}$ such that $Q_{n+1} \subset Q_n \subset I_0$ and $F^n(Q_n) = I_n$ for $n \geq 0$. For any $x \in Q = \bigcap Q_n$ we have $F^n(x) \in I_n$ for all n.
This lemma is the result of applying lemma 0 inductively in a nested manner. Continuity guarantees that you can always find a compact subinterval inside a subinterval inside a subinterval ... inside a domain where you can get the exact intervals of interest in its codomain, codomain of codomain ... by iteratively applying the map F to the subintervals. Note that $\bigcap Q_n$ is not empty due to the Cantor intersection property for compact intervals.

Lemma 2 Let $G : J \to R$ be continuous. Let $I \subset J$ be a compact interval such that $I \subset G(I)$. Then there is a point $p \in I$ such that $G(p) = p$.
This lemma is also an extension of intermediate value theorem. Continuity guarantees that if a map $G(I)$ is continuous and $G(I)$ covers I, or equivalently, G_I is surjective onto I, then G has a fixed point in I.
Now let's prove T1.

route Construction Recall from our definition of K and L that $F(K) \supset L$ and $F(L) \supset K \cup L$, which results in a symbolic transition rule $K \to L$, and $L \to K$ or L. Now fix $k \geq 1$ and define a periodic route of intervals of length k over $\{K, L\}$ that is compatible with the transitions. A convenient choice could be as follows.

- For $k = 1$: route $L, L, L, \ldots$ (always L).
- For $k \geq 2$: route
$$I_0 = L,\ I_1 = L,\ \ldots,\ I_{k-2} = L,\ I_{k-1} = K$$
and then repeat with period k.

Exact Intervals in the route Now let Q_n be sets constructed as in Lemma 1. Then $Q_k \subset L = I_0 = Q_0$ and $F^k(Q_k) = L = I_0 = Q_0$. Lemma 2 gives that it is possible to find a point p_k in $Q_0 = I_0 = L$ such that $F^k(p_k) = p_k$, which indicates that the point is of period k or its divisors.

Exact Period By construction, $F^n(p_k) \in I_n$ for all n. The chosen route has a feature that within one block of length k, the set K occurs exactly once (at time $k-1$), while L occurs $k-1$ times. If p_k had a smaller period $m < k$, then its route would repeat every m steps, forcing the pattern of visits to K and L to repeat every m. That cannot match a route that has exactly one K per k steps unless $m = k$. Therefore p_k has least period k.
This proves T1. Setting $d = a$ gives that if a map has period 3, it indeed has periodic points of all possible cycles.

Remark 1. Horseshoe, G-chaos, and D-chaos

- $f : I \to I$ has a horseshoe if there exist an interval $J \subset I$ and two disjoint subintervals $J_0, J_1 \subset J$ such that $f(J_0) \supset J$ and $f(J_1) \supset J$. This stretching and folding operation implies the existence of an invariant Cantor set $\Lambda \subset J$ on which $f|_\Lambda$ is (semi-)conjugate to full shift on two symbols, which is Devaney/$D-$ chaotic, or has sensitivity on initial conditions, dense periodic points, and topological transitivity. In Glendinning's terminology, f is G-chaotic if f^n has a horseshoe for some $n \geq 1$, and symbolic dynamics explains how G-chaos implies D-chaos on an invariant set. For a proper proof, see chapter 7 of `https://www.damtp.cam.ac.uk/user/phh1/dynsys.html`
- Now observe the covering relations $F(K) \supset L$ and $F(L) \supset K \cup L$ can be viewed as symbolic dynamics too. The relation $F(L) \supset K \cup L$ forces L to contain points whose next iterate is in K and points whose next iterate stays in L, where by continuity, L splits into at least two subregions that lead to different next symbols (fold). Meanwhile, $F(K) \supset L$ means that a whole interval is mapped broadly to cover L (stretch). Such folding and stretching creates many distinct routes inducing chaos as will be explained in T2.

Remark 2. Sharkovskii's theorem A natural question to ask after showing that period 3 implies periodic points of all periods would be finding all pairs (m, n) such that period m implies period n.

$$\begin{aligned}
&3 \triangleright 5 \triangleright 7 \triangleright \cdots \triangleright \\
&2 \cdot 3 \triangleright 2 \cdot 5 \triangleright 2 \cdot 7 \triangleright \cdots \triangleright \\
&2^2 \cdot 3 \triangleright 2^2 \cdot 5 \triangleright 2^2 \cdot 7 \triangleright \cdots \triangleright \\
&\vdots \\
&2^k \cdot 3 \triangleright 2^k \cdot 5 \triangleright 2^k \cdot 7 \triangleright \cdots \triangleright \\
&\vdots \\
&2^3 \triangleright 2^2 \triangleright 2^1 \triangleright 1 \triangleright 1
\end{aligned}$$

Then if $f : [0,1] \to [0,1]$ has an n-periodic point and $n \rhd m$, then there is an m-periodic point. See how period 3 implies all periods in the ordering.

T2 Proof Explanation

We will prove T2 with a similar manner to T1, but with more tightly designed routes.

route Construction Let $\mathcal{M}$ be the set of sequences $M = \{M_n\}_{n=1}^{\infty}$ of intervals with

$$M_n = K \quad \text{or} \quad M_n \subset L, \qquad \text{and} \qquad F(M_n) \supset M_{n+1}, \tag{A.1}$$

if $M_n = K$ then

$$n \text{ is the square of an integer and } M_{n+1},\, M_{n+2} \subset L, \tag{A.2}$$

where $K = [a,b]$ and $L = [b,c]$. For $M \in \mathcal{M}$, further denote $P(M,n)$ denote as number of i's in $\{1, \cdots, n\}$ for which $M_i = K$. For each $r \in (3/4, 1)$ choose $M^r = \{M_n^r\}_{n=1}^{\infty}$ to be a sequence in $\mathcal{M}$ such that

$$\lim_{n\to\infty} \frac{P(M^r, n^2)}{n} = r. \tag{A.3}$$

Intuitively, (A.1) constructs symbolic dynamics or restricted routes, (A.2) forces K to appear sparsely (only at the square of an integer) and two Ls to follow when this happens and (A.3) restricts the density at which K appears.

Exact Point in the route Now we will define an uncountable set $\mathcal{M}_0 := \{M^r : \ r \in (3/4, 1)\} \subset \mathcal{M}$ where the uncountability comes from that for distinct rs, the limit condition in (A.3) gives that the route M^rs are also different. Instead of the exact interval as in the proof of T1, we will locate an exact point x_r with $F^n(x_r) \in M_n^r$ for all n, enabled by Lemma 1.

Infinitely Many Distant Maps Let $S = \{x_r : \ r \in (3/4, 1)\}$ which is also uncountable. Then we can define a density function for the points in S in the same manner as in (A.3) and deduce $\rho(x_r) := \lim_{n\to\infty} P(x_r, n^2)/n = r$ Then it follows

$$\text{for } p, q \in S, \text{ with } p \neq q, \text{ there exist infinitely many } n\text{'s such that } F^n(p) \in K \text{ and } F^n(q) \in L \text{ or vice versa.} \tag{A.4}$$

because assuming $\rho(p) > \rho(q)$ gives $P(p,n) - P(q,n) \to \infty$, meaning that the difference between the number of times p, q each has been to K is infinite, deducing that there must be infinitely many ns such that $F^n(p) \in K$ and $F^n(q) \in L$ and vice versa.

Proof for the limsup Condition in (A) Note that the route construction gives that we can never have $F^k(x_r) = b$. By continuity, there exists $\delta > 0$ such that $F^2(x) < (b+d)/2 \in K$ for all $x \in [b-\delta, b] \subset K$. This violates the route construction condition (A.2) which forces a point once mapped to K to be in L in the next two mappings. So we have $F^n(p) < b - \delta$ if $F^n(p) \in K$. We also have $F^n(q) \geq b$ if $F^n(q) \in L$, resulting in $|F^n(p) - F^n(q)| > \delta$. Due to (A.4), we can always find infinitely many n that satisfies this inequality for any $p, q \in S$ with $p \neq q$, the T2 (A) or

$$\limsup_{n\to\infty} |F^n(p) - F^n(q)| \geq \delta > 0.$$

is satisfied.

Proof Sketch for the liminf Condition in (A) and (B)

- (A) is proven by refining the routes so that for two consecutive square times $n^2, (n+1)^2$ assigned to K, the block in between is forced to alternate inside a nested family of subintervals of L shrinking onto a 2-cycle (b^*, c^*). Since this occurs for infinitely many such blocks for any two distinct $x_r, x_{r'}$, their iterates are mapped on to the same small interval at the same time, resulting in $\liminf_{n\to\infty} |F^n(x_r) - F^n(x_{r'})| = 0$
- (B) is proven similarly

References

[1] Li, Tien-Yien, and James A. Yorke. "Period Three Implies Chaos." The American Mathematical Monthly, 1975

* Part II Dynamical Systems Lecture Notes
* GPT 5.2 for the write-up