一、① $D:\{(x,y)\mid y\le x\le \sqrt[3]{y},\ 0\le y\le 1\}$. 将原积分改变积分次序得

$$\int_0^1 dy\int_y^{\sqrt[3]{y}} \sin x^2 dx = \int_0^1 dx\int_{x^3}^{x} \sin x^2 dy = \int_0^1 (x-x^3)\sin x^2 dx$$

$$= \int_0^1 \frac{1}{2}\sin x^2 dx^2 - \int_0^1 \frac{1}{2}x^2\sin x^2 dx$$

$$= -\frac{1}{2}\cos t\Big|_0^1 - \frac{1}{2}\int_0^1 t\sin t\, dt$$

$$= \frac{1}{2}-\frac{1}{2}\cos 1 - \frac{1}{2}(-\cos 1+\sin 1)$$

$$= \frac{1}{2}-\frac{1}{2}\sin 1.$$

② 变量替换 $\begin{cases} x = ar\cos\theta \\ y = br\sin\theta \end{cases}$ $|J| = abr$.

$$\text{原式} = \int_0^{2\pi} d\theta \int_1^{\infty} e^{-r^2} abr\, dr = \pi ab\cdot\int_1^{\infty} e^{-t} dt$$

$$= \pi ab\left(-e^{-t}\Big|_1^{\infty}\right)$$

$$= \frac{\pi ab}{e}$$

二、由 $\begin{cases} x^2+y^2+z^2=6 \\ x+y+z=0 \end{cases}$ 将 $\begin{cases} F(x,y,z)= x^2+y^2+z^2-6=0 \\ G(x,y,z)= x+y+z=0 \end{cases}$ 分别对 $x,y,z$ 求导得

$$\begin{pmatrix} F_x & F_y & F_z \\ G_x & G_y & G_z \end{pmatrix} = \begin{pmatrix} 2x & 2y & 2z \\ 1 & 1 & 1 \end{pmatrix}$$ 于是切向量 $\vec{\tau} = \left(\frac{\partial(F,G)}{\partial(y,z)}, \frac{\partial(F,G)}{\partial(z,x)}, \frac{\partial(F,G)}{\partial(x,y)}\right)$

在 $P_0=(1,-2,1)$ 处 $\vec{\tau}|_{P_0} = (-6,0,6)$

故切线方程为 $\frac{x-1}{-6} = \frac{y+2}{0} = \frac{z-1}{1}$ 即 $\begin{cases} x+z=2 \\ y=-2. \end{cases}$

法平面方程为 $-6(x-1)+0\cdot(y+2)+6(z-1)=0$ 即 $x=z$.

三、$\begin{cases} x-(u+v)=0 \\ y-(u^2+v^2)=0 \\ z-(u^3+v^3)=0 \end{cases}$ 令 $\begin{cases} F(u,v,z) = x-(u+v) \\ G(u,v,z) = y-(u^2+v^2) \\ H(u,v,z) = z-(u^3+v^3). \end{cases}$

$$|J| = \frac{\partial(F,G,H)}{\partial(u,v,z)} = \begin{vmatrix} -1 & -1 & 0 \\ -2u & -2v & 0 \\ -3u & -3v & 1 \end{vmatrix} = 2(v-u).$$

当 $|J| \neq 0$ 即 $u \neq v$ 即 $(u-v)^2 > 0$ 即 $2y - x^2 > 0$ 时隐函数存在.

再对 $F,G,H$ 对 $x,y$ 求偏导得
$$\begin{cases} 1 - u_x - v_x = 0 \\ -2uu_x - 2vv_x = 0 \\ z_x - 3u^2u_x - 3v^2v_x = 0 \end{cases} \qquad \begin{cases} u_y + v_y = 0 \\ 1 - 2uu_y - 2vv_y = 0 \\ z_y - 3u^2u_y - 3v^2v_y = 0 \end{cases}$$

解得
$$\begin{cases} u_x = \frac{v}{v-u} \\ v_x = \frac{-u}{v-u} \\ z_x = -3uv \end{cases} \qquad \begin{cases} u_y = \frac{1}{2(u-v)} \\ v_y = \frac{-1}{2(u-v)} \\ z_y = \frac{3}{2}(u+v) \end{cases}$$

四. 不妨设大球: $x^2+y^2+z^2=a^2$ 小球 $x^2+y^2+(z-a)^2=b^2$

消去 $z$ 得 $x^2+y^2 = b^2 - \frac{b^4}{4a^2}$ 即为两球相交立体在 $xy$ 平面的投影.

则 $V = \iint\limits_{x^2+y^2 \le b^2 - \frac{b^4}{4a^2}} (z_1 - z_2)\,dxdy = \iint\limits_{x^2+y^2 \le b^2 - \frac{b^4}{4a^2}} \sqrt{a^2-x^2-y^2} - (a - \sqrt{b^2-x^2-y^2})\,dxdy$

再由极坐标变换得 $V = \int_0^{2\pi} d\theta \int_0^{\sqrt{b^2 - \frac{b^4}{4a^2}}} \left[\sqrt{a^2-r^2} - (a - \sqrt{b^2-r^2})\right] r\,dr$

$$= \pi \int_0^{b^2 - \frac{b^4}{4a^2}} \left(\sqrt{a^2-t} + \sqrt{b^2-t} - a\right) dt$$

$$= \pi\left(-a\left(b^2 - \frac{b^4}{4a^2}\right) - \frac{2}{3}(a^2-t)^{\frac{3}{2}}\Big|_0^{b^2-\frac{b^4}{4a^2}} - \frac{2}{3}(b^2-t)^{\frac{3}{2}}\Big|_0^{b^2-\frac{b^4}{4a^2}}\right)$$

$$= \pi\left(\frac{b^4}{4a} - ab^2 - \frac{2}{3}\left(a - \frac{b^2}{2a}\right)^2 + \frac{2}{3}a^3 - \frac{2}{3}\left(\frac{b^2}{2a}\right)^3 + \frac{2}{3}b^3\right)$$

化简得 原式 $= \left(\frac{2}{3}b^3 - \frac{b^4}{4a}\right)\pi$.