Real Analysis, Fourth Edition

Solutions to Exercises

David Malison

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1 The Real Numbers: Sets, Sequences, and Functions

1.1 The Field, Positivity, and Completeness Axioms

1. For any $a \neq 0$ and $b \neq 0$, we have

$$\begin{aligned}(ab)(a^{-1}b^{-1}) &= (ba)(a^{-1}b^{-1}) \\ &= b(aa^{-1})b^{-1} \\ &= bb^{-1} \\ &= 1\end{aligned}$$

Since multiplicative inverses are unique, we conclude that $(ab)^{-1} = a^{-1}b^{-1}$.

2. (i) I first prove some elementary results. Observe that if $a + c = b + c$, then

$$\begin{aligned}a &= a + 0 \\ &= a + (c - c) \\ &= (a + c) - c \\ &= (b + c) - c \\ &= b + (c - c) \\ &= b + 0 \\ &= b\end{aligned}$$

so that $a = b$. Since $0 + 0a = 0a = (0 + 0)a = 0a + 0a$, the above result implies $0 = 0a = a0$ for all a. But this means

$$\begin{aligned}ab + (-a)b &= (a - a)b \\ &= 0b \\ &= 0\end{aligned}$$

and

$$
\begin{aligned}
ab + a(-b) &= a(b - b) \\
&= a0 \\
&= 0
\end{aligned}
$$

so that $(-a)b = -ab = a(-b)$. Also, observe that

$$-a + a = a - a = 0$$

so that $a = -(-a)$. Together, the above results imply

$$
\begin{aligned}
(-a)(-b) &= -(a(-b)) \\
&= -(-ab) \\
&= ab
\end{aligned}
$$

In particular $(-a)^2 = a^2$ for all a.

Now if a is positive, $a^2 = a \cdot a > 0$. If a is not positive and $a \neq 0$, then $-a$ is positive. But this implies $(-a)^2 = a^2$ is positive.

(ii) Suppose a is positive but a^{-1} is not positive. Since $aa^{-1} = 1$, a^{-1} cannot be zero. Therefore $-a^{-1}$ must be positive, which implies $(-a^{-1})a = -1$ is positive. But -1 cannot be positive since 1 is positive by part (i).

(iii) If $a - b > 0$ and $c > 0$, then $(a - b)c > 0$. This implies $ac - bc > 0$, so that $ac > bc$. If $c < 0$, then $(-c)(a - b) > 0$. But this implies $(-c)a + (-b)(-c) = bc - ac > 0$, so $bc > ac$.

3. Suppose $E = \{x\}$ for some $x \in \mathbb{R}$. Since $x \leq x$, x is an upper bound of E. If $y < x$, then y is not an upper bound of E. Therefore x is the least upper bound. A similar argument can be used to show that x is also the greatest lower bound.

   To prove the converse, first observe that $a \leq b$ and $b \leq a$ if and only if $a = b$. Now suppose $\inf E = \sup E$ and let x be a point in E. Then $\sup E = \inf E \leq x \leq \sup E$, which implies $\inf E = \sup E = x$.

4. (i) Suppose $a \neq 0$ and $b \neq 0$. Then $1 = a\frac{1}{a}b\frac{1}{b} = ab\frac{1}{a}\frac{1}{b}$. If $ab = 0$, this expression would imply $1 = 0$. But this contradicts the non-triviality assumption.

   (ii) Observe that

$$
\begin{aligned}
(a - b)(a + b) &= a(a + b) - b(a + b) \\
&= a^2 + ab - ba - b^2 \\
&= a^2 - b^2
\end{aligned}
$$

   Thus if $a^2 - b^2 = 0$, either $a - b = 0$ or $a + b = 0$ by part (i).

   (iii) Let $x = \frac{c}{1+c}$. Then $x < c$ and $0 < x < 1$, so $x^2 < c$. But this means $x \in E$, so E is non-empty.

   Now suppose $x > c + 1$. Then $x^2 > c^2$ so $x \notin E$. Thus $c + 1$ is an upper bound of E.

The completeness axiom therefore implies that there exists a least upper bound $x_0 > 0$ of E. Suppose $x_0^2 < c$. Choose h such that $0 < h < 1$ and

$$h < \frac{c - x_0^2}{2(x_0 + 1)}$$

Then

$$(x_0 + h)^2 - x_0^2 = 2x_0 h + h^2 < 2h(x_0 + h) < 2h(x_0 + 1) < c - x_0^2$$

This implies $(x_0 + h)^2 < c$ so that $x_0 + h \in E$. But $x_0 + h > x_0$, contradicting the assumption that x_0 is an upper bound of E.

Now suppose $x_0^2 > c$. Pick $k = \frac{x_0^2 - c}{2x_0}$. Then $0 < k < x_0$. If $x \geq x_0 - k$, then

$$x_0^2 - x^2 \leq x_0^2 - (x_0 - k)^2 = 2x_0 k - k^2 < 2x_0 k = x_0^2 - c$$

This implies $c < x^2$ so $x \notin E$. But this means $x_0 - k$ is an upper bound of E smaller than x_0, contradicting the assumption that x_0 is the least upper bound.

The above results imply $x_0^2 = c$. Now suppose $x^2 = c$ for some $x > 0$. Then $x^2 - x_0^2 = 0$, so either $x - x_0 = 0$ or $x + x_0 = 0$ by part (ii). Since x and x_0 are positive, we must have $x - x_0 = 0$. This implies $x = x_0$.

5. (i) By using the field axioms and the fact that $1 + 1 = 2$, we can complete the square to obtain

$$\begin{aligned}
ax^2 + bx + c &= ax^2 + bx + \frac{1}{a}\left(\frac{b}{2}\right)^2 - \frac{1}{a}\left(\frac{b}{2}\right)^2 + c \\
&= ax^2 + \frac{bx}{2} + \frac{bx}{2} + \frac{1}{a}\left(\frac{b}{2}\right)^2 - \frac{1}{a}\left(\frac{b}{2}\right)^2 + c \\
&= ax\left(x + \frac{b}{2a}\right) + \frac{b}{2}\left(x + \frac{b}{2a}\right) - \frac{1}{a}\left(\frac{b}{2}\right)^2 + c \\
&= \left(ax + \frac{b}{2}\right)\left(x + \frac{b}{2a}\right) - \frac{1}{a}\left(\frac{b}{2}\right)^2 + c \\
&= a\left(x + \frac{b}{2a}\right)^2 - \frac{1}{a}\left(\frac{b}{2}\right)^2 + c
\end{aligned}$$

Setting the right-hand equal to zero and rearranging, we obtain

$$\begin{aligned}
& ax^2 + bx + c = 0 && (1) \\
\iff \quad & \left(x + \frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a} \\
\iff \quad & (2ax + b)^2 = b^2 - 4ac && (2)
\end{aligned}$$

If $b^2 - 4ac > 0$, there exists a unique positive square root of the right-hand side by Problem 4(iii).

This means

$$(2ax+b)^2 = \left(\sqrt{b^2-4ac}\right)^2$$

But then either

$$2ax+b+\sqrt{b^2-4ac}=0$$

or

$$2ax+b-\sqrt{b^2-4ac}=0$$

by Problem 4(ii). Solving these two equations, we obtain

$$x = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

or

$$x = \frac{-b+\sqrt{b^2-4ac}}{2a}$$

(ii) If $b^2-4ac<0$, then there are no real solutions to equation (2) because $z^2 \geq 0$ for all $z \in \mathbf{R}$ (see Problem 2(i)). Since the solutions to (2) are the same as the solutions to (1), the quadratic equation has no real solutions.

6. Let E denote a non-empty set that is bounded below by b. Define the set $-E = \{-x \,|\, x \in E\}$. Since $x \geq b$ for all $x \in E$, $-x \leq -b$ for all $x \in E$. Therefore $-b$ is an upper bound of $-E$ and the completeness axiom implies there exists a least upper bound x_0 of $-E$.

   Now pick any $x \in E$. Then $-x \in -E$, so that $-x \leq x_0 \implies x \geq -x_0$. Thus $-x_0$ is a lower bound of E. Now suppose there exists another lower bound x_0' of E which satisfies $x_0' > -x_0$. Then for any $x \in E$, $x \geq x_0' \implies -x \leq -x_0'$. This means $-x_0'$ is an upper bound of $-E$. But $x_0' > -x_0 \implies -x_0' < x_0$, contradicting the assertion that x_0 is the least upper bound of E.

   We conclude that $-x_0$ is the greatest lower bound of E, which by construction means

$$\inf E = -x_0 = -\sup(-E) = -\sup\{-x \,|\, x \in E\}$$

7. (i) First, observe that $|0| = 0$ and $|a| > 0$ if $a \neq 0$.

   Now if $a \geq 0$ then $|a| \geq 0$ and $|a|^2 = a^2$. If $-a > 0$, then $|a|^2 = (-a)^2 = a^2$. Therefore $|a|^2 = a^2$ for all a. This implies

$$|ab|^2 = (ab)^2 = a^2b^2 = |a|^2|b|^2 = (|a||b|)^2$$

   If either a or b is zero, then $|ab| = |0| = 0 = |a||b|$. If both a and b are non-zero, then $|a||b|$ and $|ab|$ are positive. Since positive square roots are unique, the above expression implies $|ab| = |a||b|$.

(ii) If either a or b is zero, the result is trivial. Assume both a and b are nonzero. Observe that

$$\begin{aligned}
|a+b|^2 &= (a+b)^2 \\
&= a^2 + 2ab + b^2 \\
&\leq |a|^2 + 2|ab| + |b|^2 \\
&= |a|^2 + 2|a||b| + |b|^2 \\
&= (|a|+|b|)^2
\end{aligned}$$

where the inequality follows from the fact that $x \leq |x|$ for all real x. If $|a+b| > |a|+|b|$, then $|a+b|^2 > (|a|+|b|)^2$. But this would contradict the above equation. We therefore must have $|a+b| \leq |a|+|b|$.

(iii) Suppose $|x-a| < \epsilon$. Since $x - a \leq |x-a|$ and $-(x-a) \leq |x-a|$, we have

$$\begin{aligned}
x - a < \epsilon \quad &\Longrightarrow \quad x < \epsilon + a \\
-x + a < \epsilon \quad &\Longrightarrow \quad a - \epsilon < x
\end{aligned}$$

Conversely, suppose $a - \epsilon < x$ and $x < a + \epsilon$. If $x - a \geq 0$, then $|x-a| = x - a < \epsilon$. If $x - a < 0$, then $|x-a| = -(x-a) = a - x < \epsilon$.

1.2 The Natural and Rational Numbers

8. $S(n)$: There does not exist a natural number between n and $n+1$.

Suppose there exists $x \in \mathbf{N}$ satisfying $1 < x < 2$. Consider the set

$$E = \{1\} \cup \{k \in \mathbf{N} \,|\, k \geq 2\}$$

If $k = 1$, then $k + 1 = 2 \in \mathbf{N}$ so that $k+1 \in E$. If $k \in \mathbf{N}$ and $k \geq 2$, then $k+1 \geq 2$ and $k+1 \in \mathbf{N}$ so that $k+1 \in E$. This means E is inductive, which implies $\mathbf{N} \subseteq E$. But $x \notin E$, a contradiction. Therefore $(1,2) \cap \mathbf{N} = \varnothing$, so $S(1)$ is true.

Now suppose $S(k)$ is true for some $k \in \mathbf{N}$ and suppose there exists $x \in \mathbf{N}$ satisfying $k+1 < x < k+2$. Since $S(k)$ is true and $k < x-1 < k+1$, $x - 1 \notin \mathbf{N}$. Define $E = \mathbf{N} \sim \{x\}$. Since $x > 1$ and $1 \in \mathbf{N}$, $1 \in E$. If $n \in E$, $n \in \mathbf{N}$ and therefore $n \neq x - 1$. But this means $n + 1 \in \mathbf{N} \sim \{x\} = E$, so that E is an inductive set. This implies $\mathbf{N} \subseteq E$. But $x \notin E$, a contradiction. Therefore $S(k+1)$ is true.

9. $S(n)$: If $n > 1$ is a natural number, then $n - 1$ is a natural number.

$S(1)$ is true because $1 > 1$ is a false statement.

Suppose $S(k)$ is true for some $k \in \mathbf{N}$. Since $(k+1) - 1 = k \in \mathbf{N}$, $S(k+1)$ is true.

$S(n)$: If m and n are natural numbers with $m > n$, then $m - n$ is a natural number.

$S(1)$ was proven above.

Suppose $S(k)$ is true for some $k \in \mathbf{N}$. Let m be a natural number such that $m > k+1$. Then $m-1$ is a natural number by $S(1)$. Since $m-1 > k$, $(m-1)-k = m-(k+1)$ is a natural number by $S(k)$. Thus $S(k+1)$ is true.

10. The only integer contained in the interval $[0, 1)$ is 0 because the positive integers bounded below by 1. Thus the statement is true if $n = 0$.

    Suppose n is a natural number. By Problem 8, we know that $(n, n+1)$ does not contain any natural number. Therefore n is the only integer contained in $[n, n+1)$. Now suppose z is an integer in $[-n, -n+1)$. Then $-z$ is a natural number in $(n-1, n]$, so we must have $-z = n$ by Problem 8. But this means $-n$ is the only integer in $[-n, -n+1)$.

11. Let E be a non-empty set of integers that is bounded above. By the completeness axiom, there exists a least upper bound c of E. Then there must exist $m \in E$ such that $m+1 > c$. Otherwise $m+1 \leq c$ for all $m \in E$, which would imply $c-1$ is an upper bound of E that is smaller than c.

    I claim that $m \geq n$ for all $n \in E$. If not, there exists $n \in E$ such that

    $$m < n \leq c < m+1$$

    But this would imply n is an integer in $(m, m+1)$, contradicting the result from Problem 10.

12. Let a and b be two real numbers with $a < b$. Suppose $a \geq 0$. By Theorem 2, there exists a rational number r such that

    $$\frac{a}{\sqrt{2}} < r < \frac{b}{\sqrt{2}}$$

    The previous expression implies

    $$a < r\sqrt{2} < b$$

    The number $i = r\sqrt{2}$ is an irrational number lying between a and b. For if i were rational, i could be written as $\frac{m}{n}$ for some pair of natural numbers m and n. There also exist natural numbers p and q such that $r = \frac{p}{q}$. But this would imply $\sqrt{2} = \frac{mq}{np}$, which would mean that $\sqrt{2}$ is rational. However it was shown in the text that $\sqrt{2}$ is irrational, a contradiction.

    Now suppose $a < 0$. By the Archimedean property, there exists a natural number n such that $n > -a$. We can infer from the previous case that there exists an irrational number i satisfying $n+a < i < n+b$. The number $i' = i-n$ is an irrational number lying between a and b. For if i' were rational, there would exist a pair of natural numbers r and s satisfying $i' = \frac{r}{s}$. But this would mean $i = \frac{ns+r}{s}$, contradicting the assumption that i is irrational.

13. Let a be a real number and define the set

    $$E = \{x \mid x \text{ is rational and } x \leq a\}$$

    By construction, a is an upper bound of E. Now suppose there exists another upper bound a' of E satisfying $a' < a$. By Theorem 2, there exists a rational number x satisfying $a' < x < a$. But then $x \in E$, contradicting the assumption that a' is an upper bound of E. We can therefore conclude that a is the least upper bound of E.

Now suppose we define the set

$$E' = \{x \,|\, x \text{ is irrational and } x \leq a\}$$

Using the result from Problem 12, we can apply the same argument to conclude that a is the least upper bound of E'.

14. $S(n)$: If $r > 0$, $(1+r)^n \geq 1 + n \cdot r$

    $S(1)$ is true because $1 + r \geq 1 + r$.

    Suppose $S(k)$ is true for some $k \in \mathbf{N}$ and fix $r > 0$. Then

$$\begin{aligned}(1+r)^{k+1} &= (1+r)(1+r)^k \\ &\geq (1+r)(1+k\cdot r) \\ &= 1 + (k+1)r + k \cdot r^2 \\ &\geq 1 + (k+1)r\end{aligned}$$

    where the first inequality follows from the induction hypothesis and the second inequality follows because k and r^2 are positive. Therefore $S(k+1)$ is true.

15. (i) $S(n)$:

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

    $S(1)$ is true because

$$1^2 = \frac{1(1+1)(2(1)+1)}{6}$$

    Suppose $S(k)$ is true for some $k \in \mathbf{N}$. Then

$$\begin{aligned}\sum_{j=1}^{k+1} j^2 &= \sum_{j=1}^{k} j^2 + (k+1)^2 \\ &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= (k+1)\left(\frac{k(2k+1)+6(k+1)}{6}\right) \\ &= (k+1)\left(\frac{2k^2+7k+6}{6}\right) \\ &= \frac{(k+1)(k+2)(2k+3)}{6} \\ &= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}\end{aligned}$$

    Thus $S(k+1)$ is true.

    (ii) As a preliminary step, I first use induction to prove that $\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$ for all $n \in \mathbf{N}$.

$S(n)$:

$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$$

$S(1)$ is true because

$$1 = \frac{1(1+1)}{2}$$

Suppose $S(k)$ is true for some $k \in \mathbf{N}$. Then

$$\begin{aligned}
\sum_{j=1}^{k+1} j &= \sum_{j=1}^{k} j + (k+1) \\
&= \frac{k(k+1)}{2} + (k+1) \\
&= (k+1)\left(\frac{k}{2}+1\right) \\
&= \frac{(k+1)(k+2)}{2} \\
&= \frac{(k+1)((k+1)+1)}{2}
\end{aligned}$$

Thus $S(k+1)$ is true.

$S(n)$:

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$$

$S(1)$ is true because

$$1^3 = 1^2$$

Suppose $S(k)$ is true for some $k \in \mathbf{N}$. Then

$$\begin{aligned}
1^3 + 2^3 + \cdots + k^3 + (k+1)^3 &= (1 + 2 + \cdots + k)^2 + (k+1)^3 \\
&= (1 + 2 + \cdots + k + (k+1) - (k+1))^2 + (k+1)^3 \\
&= (1 + 2 + \cdots + k + (k+1))^2 - 2(1 + 2 + \cdots + k + (k+1))(k+1) \\
&\quad + (k+1)^2 + (k+1)^3 \\
&= (1 + 2 + \cdots + k + (k+1))^2 - 2\left(\frac{(k+1)(k+2)}{2}\right)(k+1) \\
&\quad + (k+1)^2 + (k+1)^3 \\
&= (1 + 2 + \cdots + k + (k+1))^2 - (k+1)^2((k+2)-1) \\
&\quad + (k+1)^3 \\
&= (1 + 2 + \cdots + k + (k+1))^2
\end{aligned}$$

where the fourth equality follows from the preliminary result. Thus $S(k+1)$ is true.

(iii) $S(n)$: If $r \neq 1$, then

$$1 + r + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}$$

$S(1)$ is true because

$$\begin{aligned} 1 + r &= \frac{1 - r^2}{1 - r} \\ &= \frac{(1-r)(1+r)}{1-r} \\ &= 1 + r \end{aligned}$$

Suppose $S(k)$ is true for some $k \in \mathbf{N}$. Then

$$\begin{aligned} 1 + r + \cdots + r^k + r^{k+1} &= \frac{1 - r^{k+1}}{1 - r} + r^{k+1} \\ &= \frac{1 - r^{k+1} + r^{k+1} - r^{k+2}}{1 - r} \\ &= \frac{1 - r^{k+2}}{1 - r} \end{aligned}$$

Thus $S(k+1)$ is true.

1.3 Countable and Uncountable Sets

16. Consider the mapping from $\mathbf{N}$ to $\mathbf{Z}$ defined by

$$f(n) = \begin{cases} 0 & \text{if } n = 1 \\ \dfrac{n}{2} & \text{if } n \text{ is even} \\ -\dfrac{n+1}{2} & \text{if } n \text{ is odd and } n > 1 \end{cases}$$

If n is a natural number, then $f(2n) = n$ and $f(2n-1) = -n$. We also have $f(1) = 0$. Therefore f is onto.

Now suppose $f(n) = f(n')$. If $f(n)$ equals 0, then $n = n' = 1$. If $f(n)$ is positive, then $\frac{n}{2} = \frac{n'}{2} \implies n = n'$. If $f(n)$ is negative, then $-\frac{n+1}{2} = -\frac{n'+1}{2} \implies n = n'$. Therefore f is one-to-one.

17. Let A be a non-empty set. Suppose A is countable. Then there exists a one-to-one correspondence f between a subset of $\mathbf{N}$ and A. The inverse of f defines a one-to-one mapping of A to $\mathbf{N}$.

Conversely, suppose there exists a one-to-one mapping f of A to $\mathbf{N}$. Then f provides a one-to-one correspondence between A and $f(A)$, so the inverse of f provides a one-to-one mapping of $f(A)$ onto A. Since $f(A)$ is countable by Theorem 3, we can conclude that A is countable by Theorem 5.

18. As a preliminary result, I first show that every finite set of numbers contains a maximal element.

$S(n)$: Let $S \subset \mathbf{R}$ be a non-empty set. If there exists a one-to-one correspondence between $\{1, \cdots, n\}$ and S, then S contains a maximal element.

Suppose there exists a one-to-one correspondence f between $\{1\}$ and S. Then $S = \{f(1)\}$, so $s \leq f(1)$ for all $s \in S$. Thus $S(1)$ is true.

Now assume $S(k)$ is true and suppose there exists a one-to-one correspondence between $\{1, \cdots, k+1\}$ and S. Then $S = \{f(i) \,|\, 1 \leq i \leq k\} \cup \{f(k+1)\}$. By the induction hypothesis, $\{f(i) \,|\, 1 \leq i \leq k\}$ has a maximal element $\hat{s}$. If $\hat{s} \geq f(k+1)$, then $\hat{s}$ is a maximal element of S. If $\hat{s} < f(k+1)$, then $f(k+1)$ is a maximal element of S. We conclude that $S(k+1)$ must be true.

$S(n)$: The Cartesian product $\overbrace{\mathbf{N} \times \cdots \times \mathbf{N}}^{n \text{ times}}$ is countably infinite.

The identity function establishes a one-to-one correspondence between $\mathbf{N}$ and $\mathbf{N}$, so $\mathbf{N}$ is countable. Now suppose $\mathbf{N}$ were finite. Then by the preliminary result, there would exist a maximal element m of $\mathbf{N}$. But $m+1$ would then be a natural number larger than m, a contradiction. We conclude that $\mathbf{N}$ is countably infinite, so $S(1)$ is true.

Suppose $S(k)$ is true. Then there exists a one-to-one mapping f of $\mathbf{N}$ onto $\overbrace{\mathbf{N} \times \cdots \times \mathbf{N}}^{k \text{ times}}$. Consider the mapping from $\overbrace{\mathbf{N} \times \cdots \times \mathbf{N}}^{k+1 \text{ times}}$ to $\mathbf{N}$ defined by

$$g(n_1, \cdots, n_k, n_{k+1}) = (f^{-1}(n_1, \cdots, n_k) + n_{k+1})^2 + n_{k+1}$$

It is straight-forward to check that g is one-to-one using the argument in the text. Thus $\overbrace{\mathbf{N} \times \cdots \times \mathbf{N}}^{k+1 \text{ times}}$ is equipotent to $g(\overbrace{\mathbf{N} \times \cdots \times \mathbf{N}}^{k+1 \text{ times}})$, a subset of the countable set $\mathbf{N}$. We infer from Theorem 3 that $\overbrace{\mathbf{N} \times \cdots \times \mathbf{N}}^{k+1 \text{ times}}$ is countable.

Now suppose $\overbrace{\mathbf{N} \times \cdots \times \mathbf{N}}^{k+1 \text{ times}}$ is finite. Then there exists a one-to-one mapping f from $\{1, \cdots, n\}$ onto $\overbrace{\mathbf{N} \times \cdots \times \mathbf{N}}^{k+1 \text{ times}}$ for some $n \in \mathbf{N}$. Consider the mapping from $\overbrace{\mathbf{N} \times \cdots \times \mathbf{N}}^{k \text{ times}}$ to $\{1, \cdots, n\}$ defined by

$$g(n_1, \cdots, n_k) = f^{-1}(n_1, \cdots, n_k, 1)$$

This establishes a one-to-one correspondence between $\overbrace{\mathbf{N} \times \cdots \times \mathbf{N}}^{k \text{ times}}$ and a subset of $\{1, \cdots, n\}$, implying that $\overbrace{\mathbf{N} \times \cdots \times \mathbf{N}}^{k \text{ times}}$ is finite. This contradicts the assumption that $\overbrace{\mathbf{N} \times \cdots \times \mathbf{N}}^{k \text{ times}}$ is countably infinite. We conclude that $\overbrace{\mathbf{N} \times \cdots \times \mathbf{N}}^{k+1 \text{ times}}$ is countably infinite, so $S(k+1)$ is true.

19. Let Λ be a non-empty finite set and for each $\lambda \in \Lambda$, let E_λ be a countable set. Then for some $m \in \mathbf{N}$, there exists a one-to-one mapping h of $\{1, \cdots, m\}$ onto Λ. For $n \in \{1, \cdots, m\}$, define $\lambda_n = h(n)$ and

keep E, $N(n)$ and f_n defined as in the text. Define

$$E' = \{(n,k) \in \mathbf{N} \times \mathbf{N} \,|\, 1 \le n \le m,\ E_{\lambda_n} \text{ is nonempty and } 1 \le k \le N(n) \text{ if } E_{\lambda_n} \text{ is also finite}\}$$

and define the mapping f of E' onto E by $f(n,k) = f_n(k)$. The proof then proceeds as in the text.

20. Suppose $g(f(a)) = g(f(a'))$. Since g is one-to-one, we must have $f(a) = f(a')$. Since f is one-to-one, we must also have $a = a'$. But this means $g \circ f$ is one-to-one. Now fix $c \in C$. Since g is onto, there exists $b \in B$ such that $g(b) = c$. Since f is onto, there also exists $a \in A$ such that $f(a) = b$. But this means $g(f(a)) = c$, so $g \circ f$ is onto.

    Suppose $f^{-1}(b) = f^{-1}(b')$. Then $b = f(f^{-1}(b)) = f(f^{-1}(b')) = b'$, so f^{-1} must be one-to-one. Now suppose $a \in A$. Then $a = f^{-1}(f(a))$, so f^{-1} is onto.

21. $S(n)$: There does not exist a one-to-one mapping from $\{1, \cdots, m+n\}$ to $\{1, \cdots, n\}$ for any natural number m.

    Fix $m \in \mathbf{N}$ and let f be a mapping from $\{1, \cdots, m+1\}$ to $\{1\}$. Then $f(1) = f(2) = 1$, so f cannot be one-to-one. Thus $S(1)$ is true.

    Suppose $S(k)$ is true. Fix $m \in \mathbf{N}$ and let f be a mapping from $\{1, \cdots, m+(k+1)\}$ to $\{1, \cdots, k+1\}$. If there does not exist $n \in \{1, \cdots, m+(k+1)\}$ such that $f(n) = k+1$, then f defines a mapping from $\{1, \cdots, (m+1)+k\}$ to $\{1, \cdots, k\}$. The induction hypothesis then implies f is not one-to-one. Now suppose $f(n) = k+1$ for some $n \in \{1, \cdots, m+(k+1)\}$. Define the mapping

$$g(i) = \begin{cases} f(i) & \text{if } 1 \le i < n \\ f(i+1) & \text{if } n \le i \le m+k \end{cases}$$

    If f is one-to-one, then g defines a one-to-one mapping from $\{1, \cdots, m+k\}$ to $\{1, \cdots, k\}$. But this contradicts the induction hypothesis, so f cannot be one-to-one. Therefore $S(k+1)$ must be true.

22. Suppose $2^{\mathbf{N}}$ is countable. Let $\{X_n \,|\, n \in \mathbf{N}\}$ denote an enumeration of $2^{\mathbf{N}}$ and define

$$D = \{n \in \mathbf{N} \,|\, n \text{ is not in } X_n\}$$

    Then $D \in 2^{\mathbf{N}}$, so $D = X_d$ for some $d \in \mathbf{N}$. If d is not in D, then we would have a contradiction because d would have to be in D by construction. Likewise if d is in D, then we have a contradiction because d could not be in D by construction. We can conclude that no enumeration can exist, so $2^{\mathbf{N}}$ is uncountable.

23. Define Λ, m, λ_n, $N(n)$ and f_n as in Problem 19 and define $E = E_{\lambda_1} \times \cdots \times E_{\lambda_m}$. If E_{λ_n} is empty for any $n \in \{1, \cdots, m\}$, then E is empty and therefore countable. So suppose that E_{λ_n} is non-empty for all $n \in \{1, \cdots, m\}$. Define

$$E' = \{(k_1, \cdots, k_m) \in \overbrace{\mathbf{N} \times \cdots \times \mathbf{N}}^{m \text{ times}} \,|\, 1 \le k_n \le N(n) \text{ if } E_{\lambda_n} \text{ is finite}\}$$

    Define the mapping f of E' onto E by $f(k_1, \cdots, k_m) = (f_1(k_1), \cdots, f_m(k_m))$. E' is a subset of a countable set by Corollary 4, so E' is countable by Theorem 3. Theorem 5 then implies that E is also countable.

Define the set $\{1,2\}^{\mathbf{N}}$ as the set of functions from $\mathbf{N}$ to $\{1,2\}$. Consider the mapping f from $2^{\mathbf{N}}$ to $\{1,2\}^{\mathbf{N}}$ defined by

$$[f(S)](n) = \begin{cases} 1 & \text{if } n \in S \\ 2 & \text{otherwise} \end{cases}$$

Now suppose $f(S) = f(S')$. Since $n \in S \iff [f(S)](n) = 1 \iff [f(S)'](n) = 1 \iff n \in S'$, we must have $S = S'$. Thus f is one-to-one. Now suppose g is a mapping from $\mathbf{N}$ to $\{1,2\}$. Define $S = g^{-1}(\{1\})$. Then $f(S) = g$, so f is onto. Thus $|\{1,2\}^{\mathbf{N}}| = |2^{\mathbf{N}}|$ and therefore $\{1,2\}^{\mathbf{N}}$ is uncountable by Problem 22. Since $\{1,2\}^{\mathbf{N}}$ is a subset of $\mathbf{N}^{\mathbf{N}}$, $\mathbf{N}^{\mathbf{N}}$ must also be uncountable.

24. Let I be a non-degenerate interval of real numbers. In Problem 25, we show that $|I| = |\mathbf{R}|$. But $\mathbf{R}$ is not finite. For if it was, it would contain a maximal element by Problem 18. However, no maximal element of $\mathbf{R}$ exists.

25. I proceed by defining a series of one-to-one correspondences between various intervals. I do not explicitly show that each mapping is one-to-one and onto, although it is easy to do so.

    - $|(a,b)| = |(0,1)|$, where $a < b$: Define $f : (a,b) \to (0,1)$ as
    $$f(x) = \frac{1}{b-a}(x-a)$$
    - $|(0,1)| = |(1,\infty)|$: Define $f : (0,1) \to (1,\infty)$ as
    $$f(x) = \frac{1}{x}$$
    - $|(1,\infty)| = |(c,\infty)|$: Define $f : (1,\infty) \to (c,\infty)$ as
    $$f(x) = x + c - 1$$
    - $|(-\infty,c)| = |(c,\infty)|$: Define $f : (-\infty,c) \to (c,\infty)$ as
    $$f(x) = 2c - x$$
    - $|(-\infty,\infty)| = |(-1,1)|$: Define $f : (-\infty,\infty) \to (-1,1)$ as
    $$f(x) = \begin{cases} \dfrac{x}{1+x} & \text{if } x \geq 0 \\ \dfrac{x}{1-x} & \text{if } x < 0 \end{cases}$$
    - The preceding mappings establish the equipotence of all non-degenerate open intervals. We can

also use these mappings to prove the following relations:

$$\begin{aligned}
|[a,b]| &= |[0,1]| \\
|[a,b)| &= |[0,1)| \\
|(a,b]| &= |(0,1]| \\
|(0,1]| &= |[c,\infty)| \\
|(-\infty,c]| &= |[c,\infty)|
\end{aligned}$$

- $|[-1,1]| \to |(-\infty,\infty)|$: Define $f : [-1,1] \to (-\infty,\infty)$ as

$$f(x) = \begin{cases} 1 & \text{if } x = 1 \\ \dfrac{1}{1-x} & \text{if } x \geq 0 \text{ and } \dfrac{x}{1-x} \in \mathbf{N} \\ \dfrac{x}{1-x} & \text{if } x \geq 0 \text{ and } \dfrac{x}{1-x} \notin \mathbf{N} \\ -1 & \text{if } x = -1 \\ -\dfrac{1}{1+x} & \text{if } x < 0 \text{ and } -\dfrac{x}{1+x} \in \mathbf{N} \\ \dfrac{x}{1+x} & \text{if } x < 0 \text{ and } -\dfrac{x}{1+x} \notin \mathbf{N} \end{cases}$$

  In essence, this function makes room for the interval's endpoints by displacing $\mathbf{Z} \sim \{0\}$.

- $|(-1,1]| = |(-\infty,\infty)|$: Define $f : (-1,1] \to (-\infty,\infty)$ as

$$f(x) = \begin{cases} 1 & \text{if } x = 1 \\ \dfrac{1}{1-x} & \text{if } x \geq 0 \text{ and } \dfrac{x}{1-x} \in \mathbf{N} \\ \dfrac{x}{1-x} & \text{if } x \geq 0 \text{ and } \dfrac{x}{1-x} \notin \mathbf{N} \\ \dfrac{x}{1+x} & \text{if } x < 0 \end{cases}$$

- $|[-1,1)| = |(-\infty,\infty)|$: Define $f : [-1,1) \to (-\infty,\infty)$

$$f(x) = \begin{cases} \dfrac{x}{1-x} & \text{if } x \geq 0 \\ -1 & \text{if } x = -1 \\ -\dfrac{1}{1+x} & \text{if } x < 0 \text{ and } -\dfrac{x}{1+x} \in \mathbf{N} \\ \dfrac{x}{1+x} & \text{if } x < 0 \text{ and } -\dfrac{x}{1+x} \notin \mathbf{N} \end{cases}$$

26. Let G denote the set of irrational numbers in $(0,1)$ and let $\{q_n \mid n \in \mathbf{N}\}$ denote an enumeration of the rationals in $(0,1)$. Define

$$i_n = \frac{\sqrt{2}}{2^n}$$

and construct the mapping $f : (0,1) \to G$ as

$$f(x) = \begin{cases} i_{2n} & \text{if } x = q_n \\ i_{2n-1} & \text{if } x = i_n \\ x & \text{otherwise} \end{cases}$$

f defines a one-to-one correspondence between $(0,1)$ and G, so $|(0,1)| = |G|$.

In Problem 25 we showed that $|\mathbf{R}| = |(0,1)|$, so the above result implies $|\mathbf{R}| = |G|$. This means we can find a one-to-one mapping g from $\mathbf{R}$ onto G. Now consider the mapping $h : \mathbf{R} \times \mathbf{R} \to G \times G$ defined by

$$h(x,y) = (g(x), g(y))$$

h defines a one-to-one mapping from $\mathbf{R} \times \mathbf{R}$ onto $G \times G$, so $|\mathbf{R} \times \mathbf{R}| = |G \times G|$.

Recall that if x is an irrational number in $(0,1)$, it can be uniquely written as

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}} =: [a_1, a_2, a_3, \cdots]$$

where $a_1, a_2, a_3, \cdots$ is an infinite sequence of natural numbers. (This representation is called the *continued fraction expansion* of x.) Let $x = [a_1, a_2, \cdots]$ and $y = [b_1, b_2, \cdots]$ denote two elements of G and consider the mapping $m : G \times G \to G$ defined by

$$m(x,y) = [a_1, b_1, a_2, b_2, \cdots]$$

Then m defines a one-to-one correspondence between $G \times G$ and G, so $|G \times G| = |G|$. Combining the above results, we have $|\mathbf{R} \times \mathbf{R}| = |G \times G| = |G| = |\mathbf{R}|$.

1.4 Open Sets, Closed Sets, and Borel Sets of Real Numbers

27. The set of rational numbers is not open because every interval around a rational number contains an irrational number (see Problem 12). The set of irrational numbers is also not open because every interval around an irrational number contains a rational number (see Theorem 2). By Proposition 11, this implies that the set of rational numbers is also not closed.

28. From Propositions 8 and 11, we know that $\mathbf{R}$ and $\varnothing$ are both open and closed.

    Suppose A is a non-empty, proper subset of $\mathbf{R}$ that is both open and closed. Then there exists $x \in A$ and $y \in \mathbf{R} \sim A$. Suppose without loss of generality that $x < y$ and define

$$E = \{x \in A : x < y\}$$

Then E is non-empty ($x \in E$) and bounded above (by y). The completeness axiom implies that there exists a least upper bound of E. Let $x^* = \sup E$ and suppose $x^* \in A$. Since $y \notin A$ and y is an upper bound of E, we must have $x^* < y$. Therefore there exists $r > 0$ such that $x^* + r < y$. But since A is open, we can also find $r^* \in (0, r)$ such that $(x^* - r^*, x^* + r^*) \subset A$. But this implies $x^* + \frac{r}{2} \in E$, so x^* is not an upper bound for E. This contradicts the definition of x^*. Now suppose $x^* \in \mathbf{R} \sim A$. Since A is closed, $\mathbf{R} \sim A$ is open. Therefore there exists $r > 0$ such that $(x^* - r, x^* + r) \subset \mathbf{R} \sim A$. Thus if $x \in A$, $x \leq x^* - r$. But this means $x^* - r$ is an upper bound of E, contradicting the assumption that x^* is the least upper bound.

The above argument shows that E cannot have a least upper bound, a contradiction of the completeness axiom. We conclude that no non-empty, proper subset of $\mathbf{R}$ that is both open and closed can exist.

29. Let $A = (0, 1)$ and $B = \{1\}$. Then $\bar{A} = [0, 1]$ and $\bar{B} = B$, so $\bar{A} \cap \bar{B} = \{1\} \neq \varnothing$.

30. (i) Let $x \in \bar{E'}$. Suppose there exists an open interval I_x containing x such that $I_x \cap E' = \{x\}$. Then $x \in E'$. Now suppose for any open interval I_x containing x, $I_x \cap E' \neq \{x\}$. Fix an interval I_x. Since x is a limit point of E', we know $I_x \cap E' \neq \varnothing$. Therefore there must exist $x' \in I_x \cap E'$ satisfying $x' \neq x$. Since I_x is open, there exists $r \in (0, |x - x'|)$ such that $(x' - r, x' + r) \subseteq I_x$. Now $(x' - r, x' + r)$ is an open interval containing x' and $x' \in E'$, so there must exist $x'' \in (x' - r, x' + r) \cap (E \sim \{x'\})$. But since

$$0 < |x - x'| - |x'' - x'| \leq |x'' - x|$$

we know that $x'' \neq x$. Therefore $x'' \in I_x \cap (E \sim \{x\})$. Since I_x was arbitrary, we can conclude that x is a point of closure of $E \sim \{x\}$. This means $x \in E'$. Since every point of closure of E' is in E', E' is closed.

(ii) Suppose $x \in \bar{E}$. If $x \in E$, then $x \in E \cup E'$. Suppose $x \in \mathbf{R} \sim E$ and let I_x be an open interval containing x. Since x is a point of closure of E, there exists $x' \in E \cap I_x$. But since $x \in \mathbf{R} \sim E$, $x' \neq x$. Therefore $x' \in I_x \cap (E \sim \{x\})$. Since I_x was arbitrary, x is a point of closure of $E \sim \{x\}$. Thus $x \in E'$ and therefore $x \in E \cup E'$. We conclude that $\bar{E} \subseteq E \cup E'$.

Now suppose $x \in E \cup E'$. If $x \in E$, then $x \in \bar{E}$ because $E \subseteq \bar{E}$. If $x \in E'$ and I_x is an interval containing x, then there exists $x' \in I_x \cap (E \sim \{x\})$. But then $x' \in I_x \cap E$. Since I_x was arbitrary, x is a point of closure of E. Therefore $x \in \bar{E}$ and we conclude $E \cup E' \subseteq \bar{E}$.

31. Suppose E is a set containing only isolated points. For each $x \in E$, define $f(x) = (p, q)$ where p and q are rational numbers such that $p < x < q$ and $(p, q) \cap E = \{x\}$. f defines a one-to-one mapping from E to $\mathbf{Q} \times \mathbf{Q}$. By Corollary 4 and Problem 23, $\mathbf{Q} \times \mathbf{Q}$ is a countable set. This means there exists a one-to-one mapping g from $\mathbf{Q} \times \mathbf{Q}$ onto $\mathbf{N}$. The composition $g \circ f$ defines a one-to-one mapping from E to $\mathbf{N}$ (see Problem 20), which implies E is countable (see Problem 17).

32. (i) Suppose E is open and $x \in E$. Then there exists an $r > 0$ such that the interval $(x - r, x + r)$ is contained in E. But this means $x \in \operatorname{int} E$, so $E \subseteq \operatorname{int} E$. Since $\operatorname{int} E \subseteq E$ by definition, $E = \operatorname{int} E$.

Conversely, suppose $E = \operatorname{int} E$. If x is a point in E, then $x \in \operatorname{int} E$. But this means there exists an $r > 0$ such that the interval $(x - r, x + r)$ is contained in E, so E is open.

(ii) Let E be dense in $\mathbf{R}$ and suppose $x \in \operatorname{int}(\mathbf{R} \sim E)$. Then there exists $r > 0$ such that $(x - r, x + r) \subseteq$

$\mathbf{R} \sim E$. But this means there does not exist an element of E between any two numbers in $(x-r, x+r)$, contradicting the assumption that E is a dense set. We conclude that no such x can be found, so $\text{int}(\mathbf{R} \sim E) = \varnothing$.

Conversely, suppose $\text{int}(\mathbf{R} \sim E) = \varnothing$. Let x and y be two real numbers satisfying $x < y$ and suppose $(x, y) \subset \mathbf{R} \sim E$. Let $z \in (x, y)$ and choose $r \in (0, \min(z-x, y-z))$. Then $(z-r, z+r) \subset (x, y)$, so $(z-r, z+r) \subset \mathbf{R} \sim E$. But this means $z \in \text{int}(\mathbf{R} \sim E)$, contradicting the assumption that $\text{int}(\mathbf{R} \sim E) = \varnothing$. Therefore $(x, y) \not\subset \mathbf{R} \sim E$, which means there must be an element of E between x and y. But since x and y were arbitrary, this means E is dense in $\mathbf{R}$.

33. Let $F_n = [n, \infty)$. Then $\{F_n\}_{n=1}^{\infty}$ is a descending, countable collection of non-empty closed sets. The intersection $\bigcap_{n=1}^{\infty} F_n$ is empty because of the Archimedean property.

34. The text establishes the following relationships:

$$\begin{aligned} \textbf{Completeness Axiom} \quad &\Longrightarrow \quad \textbf{Heine-Borel Theorem + Archimedean Property} \\ \textbf{Heine-Borel Theorem} \quad &\Longrightarrow \quad \textbf{Nested Set Theorem} \end{aligned}$$

I show below that

$$\textbf{Nested Set Theorem + Archimedean Property} \quad \Longrightarrow \quad \textbf{Completeness Axiom}$$

This implies that the following equivalences hold:

$$\begin{aligned} &\textbf{Completeness Axiom} \quad \Longleftrightarrow \\ &\textbf{Heine-Borel Theorem + Archimedean Property} \quad \Longleftrightarrow \\ &\textbf{Nested Set Theorem + Archimedean Property} \end{aligned}$$

Suppose the assertions of the Nested Set Theorem and the Archimedean Property are true. Let E be a non-empty set bounded above by b and let a be an element of E. Define the following sequence of sets:

$$[a_1, b_1] = [a, b]$$

$$[a_{n+1}, b_{n+1}] = \begin{cases} \left[a_n, \dfrac{a_n + b_n}{2}\right] & \text{if } \dfrac{a_n + b_n}{2} \text{ is an upper bound of } E \\ \left[\dfrac{a_n + b_n}{2}, b_n\right] & \text{otherwise} \end{cases}$$

By construction, each b_n is an upper bound of E and there exists at least one element of E in $[a_n, b_n]$ for all $n \in \mathbf{N}$. $[a_1, b_1]$ is bounded and $\{[a_n, b_n]\}_{n=1}^{\infty}$ is a descending, countable collection of nonempty closed sets of real numbers, so the Nested Set Theorem implies that the intersection $\bigcap_{n=1}^{\infty}[a_n, b_n]$ is non-empty. Let x^* denote a point in the intersection.

It is easy to establish via induction that

$$|b_n - a_n| = \frac{|b_1 - a_1|}{2^{n-1}}$$

for all $n \in \mathbf{N}$. We can also establish via induction that $2^{n-1} \geq n$ for all $n \in \mathbf{N}$. Now fix any $x > x^*$. By the Archimedean property, there exists $n \in \mathbf{N}$ such that

$$\frac{|x - x^*|}{|b_1 - a_1|}n > 1 \qquad \Longrightarrow \qquad |x - x^*| > \frac{|b_1 - a_1|}{n} \geq \frac{|b_1 - a_1|}{2^{n-1}} = |b_n - a_n| \qquad \Longrightarrow \qquad x > b_n$$

Since b_n is an upper bound of E, we must have $x \notin E$. But since every number greater than x^* is not in E, x^* is an upper bound of E.

Now suppose $x < x^*$. Arguing as before, we can find an $n \in \mathbf{N}$ such that $x < a_n$. But since there is an element of E greater than a_n, x cannot be an upper bound of E. Thus if x is an upper bound of E, we must have $x \geq x^*$. We conclude that x^* is the least upper bound of E.

35. For any collection of sets $\mathcal{F}$, let $\sigma(\mathcal{F})$ denote the smallest σ-algebra containing $\mathcal{F}$.

    Let $\mathcal{F}$ denote the collection of closed subsets of $\mathbf{R}$ and fix $F \in \mathcal{F}$. Since $\mathbf{R} \sim F$ is an open set, $\mathbf{R} \sim F \in \mathcal{B}$. We conclude that $F = \mathbf{R} \sim (\mathbf{R} \sim F) \in \mathcal{B}$, which means $\mathcal{F} \subset \mathcal{B}$. This implies $\sigma(\mathcal{F}) \subseteq \mathcal{B}$. Now let $\mathcal{G}$ denote the collection of open subsets of $\mathbf{R}$ and pick $G \in \mathcal{G}$. Then $\mathbf{R} \sim G \in \mathcal{F}$, so $G = \mathbf{R} \sim (\mathbf{R} \sim G) \in \sigma(\mathcal{F})$. But this implies $\mathcal{G} \subset \sigma(\mathcal{F})$, so $\mathcal{B} = \sigma(\mathcal{G}) \subseteq \sigma(\mathcal{F})$.

36. Let $\mathcal{F}$ denote the collection of intervals of the form $[a, b)$, where $a < b$. Fix $[a, b) \in \mathcal{F}$. Then

    $$[a, b) = \bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, b\right)$$

    Since $\left(a - \frac{1}{n}, b\right) \in \mathcal{B}$ for all $n \in \mathbf{N}$ and $\mathcal{B}$ is closed under countable intersections, we have $[a, b) \in \mathcal{B}$. This implies $\mathcal{F} \subseteq \mathcal{B}$, so $\sigma(\mathcal{F}) \subseteq \mathcal{B}$.

    Now suppose (a, b) is an open interval, where $a < b$. Pick $m \in \mathbf{N}$ such that $\frac{1}{m} < b - a$. Then

    $$(a, b) = \bigcup_{n=m}^{\infty} \left[a + \frac{1}{n}, b\right)$$

    Since $\left[a + \frac{1}{n}, b\right) \in \mathcal{F}$ for all $n \geq m$ and $\sigma(\mathcal{F})$ is closed under countable unions, we have $(a, b) \in \sigma(\mathcal{F})$. We can likewise show $(-\infty, b) \in \sigma(\mathcal{F})$ and $(a, \infty) \in \sigma(\mathcal{F})$ by using the expressions

    $$(-\infty, b) = \bigcup_{n=1}^{\infty} \Big[b - n, b\Big)$$
    $$(a, \infty) = \bigcup_{n=1}^{\infty} \left[a + \frac{1}{n}, a + n + 1\right)$$

    Thus all open intervals are in $\sigma(\mathcal{F})$. By Proposition 9, every non-empty open set can be written as the countable union of open intervals. Since every open interval is in $\sigma(\mathcal{F})$ and $\sigma(\mathcal{F})$ is closed under countable unions, every non-empty open set is in $\sigma(\mathcal{F})$. $\sigma(\mathcal{F})$ also contains $\varnothing$, so all open sets are in $\sigma(\mathcal{F})$. This mean $\mathcal{B} \subseteq \sigma(\mathcal{F})$.

37. Let (a, b) be an open interval, where $a < b$. There exists $m \in \mathbf{N}$ such that $\frac{1}{m} < \frac{b-a}{2}$. We can then

write

$$(a, b) = \bigcup_{n=m}^{\infty} \left[a + \frac{1}{n}, b - \frac{1}{n}\right]$$

so (a, b) is a countable union of closed sets. We can also write

$$(-\infty, b) = \bigcup_{n=1}^{\infty} \left[b - n, b - \frac{1}{n}\right]$$

$$(a, \infty) = \bigcup_{n=1}^{\infty} \left[a + \frac{1}{n}, a + n\right]$$

Therefore any open interval can be written as a countable union of closed sets. By Proposition 9, every non-empty open set can be written as a countable union of open intervals. Since the empty set is closed and a countable collection of countable sets is countable, we can conclude that any open set is an F_σ set.

1.5 Sequences of Real Numbers

As a preliminary result, I prove that the conclusion of Theorem 18 holds for sequences that converge to extended real numbers.

Theorem 18 (extended reals): Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers. Suppose $\{a_n\} \to a$ and $\{b_n\} \to b$, where a and b are extended real numbers. Then for each pair of non-zero real numbers α and β,

$$\{\alpha \cdot a_n + \beta \cdot b_n\} \to \alpha \cdot a + \beta \cdot b \tag{1}$$

provided that $\alpha \cdot a + \beta \cdot b$ is not of the form $\infty - \infty$. Moreover,

$$\text{if } a_n \leq b_n \text{ for all } n, \text{ then } a \leq b \tag{2}$$

Proof: All that remains to be shown is that (1) and (2) hold when a or b is infinite. Suppose $a = \infty$, $b \in \mathbf{R}$, and $\alpha > 0$. For any real number c, choose N such that

$$a_n \geq \frac{c - \beta \cdot b + 1}{\alpha}, \quad b_n \geq \frac{\beta \cdot b - 1}{\beta}$$

for all $n \geq N$. Then

$$\alpha \cdot a_n + \beta \cdot b_n \geq \alpha \cdot a_n + \beta \cdot b - 1 \geq c$$

for all $n \geq N$. This implies $\{\alpha \cdot a_n + \beta \cdot b_n\} \to \infty$. The cases where $a = -\infty$ or $\alpha < 0$ can be handled similarly. If $a = b = \infty$ and $\alpha, \beta > 0$, choose N such that

$$a_n \geq \frac{c}{2 \cdot \alpha}, \qquad b_n \geq \frac{c}{2 \cdot \beta}$$

for all $n \geq N$. Then

$$\alpha \cdot a_n + \beta \cdot b_n \geq c$$

for all $n \geq N$, so $\{\alpha \cdot a_n + \beta \cdot b_n\} \to \infty$. Other cases in which a and b are both infinite can be dealt with analogously.

Now suppose $a_n \leq b_n$ and $a = \infty$. For any c, there exists N such that $a_n \geq c$ for all $n \geq N$. But this means $b_n \geq c$ for all $n \geq N$, so $b = \infty$. The case when $b = -\infty$ can be handled with a similar argument.

38. Define $b_n = \sup\{a_k \mid k \geq n\}$ and let $b = \limsup \{a_n\} = \lim_{n\to\infty} b_n$.

Suppose $b = \infty$. Then by Proposition 19(ii), $\{a_n\}$ is not bounded above. This means it is possible to define the sequence

$$n_1 = \min\{n \geq 1 : a_n > 1\}, \qquad n_{k+1} = \min\{n \geq n_k + 1 : a_n > k+1\}$$

For any c, we have $a_{n_k} \geq c$ for all $k \geq c$. This implies $\lim_{k\to\infty} a_{n_k} = \infty$.

Now suppose that $\{a_k\}$ is bounded above. Consider the sequence

$$n_0 = 0, \qquad n_1 = \min\{n \geq 1 : b_1 < a_n + 1\}, \qquad n_{k+1} = \min\left\{n \geq n_k + 1 : b_{n_k+1} < a_n + \frac{1}{k+1}\right\}$$

Suppose $b > -\infty$ and fix $\epsilon > 0$. Because b_k converges to b and $n_{k-1} + 1 \geq k$, by Theorem 17 there exists an N_1 such that $|b_{n_{k-1}+1} - b_k| < \epsilon/3$ for all $k \geq N_1$. We can also find N_2 such that $|b - b_k| < \epsilon/3$ for $k \geq N_2$ and N_3 such that $1/k < \epsilon/3$ for $k \geq N_3$. Define $N = \max\{N_1, N_2, N_3\}$. Then

$$|a_{n_k} - b| \leq |a_{n_k} - b_{n_{k-1}+1}| + |b_{n_{k-1}+1} - b_k| + |b - b_k| < \epsilon$$

for all $k \geq N$. But since ϵ was arbitrary, we have $\lim_{k\to\infty} a_{n_k} = b$.

Now suppose $b = -\infty$. Fix c and pick N such that $b_{n_{k-1}+1} \leq c - 1$ for all $k \geq N$. By construction,

$$\left|a_{n_k} - b_{n_{k-1}+1}\right| < \frac{1}{k} \leq 1$$

which implies

$$a_{n_k} \leq |a_{n_k} - b_{n_{k-1}+1}| + b_{n_{k-1}+1} < c$$

for $k \geq N$. Thus $\lim_{k\to\infty} a_{n_k} = -\infty$.

We conclude that $\limsup\{a_n\}$ is a cluster point. To see that $\limsup\{a_n\}$ is the largest cluster point, let a denote any cluster point of $\{a_n\}$. Then there exists a subsequence $\{a_{n_k}\}$ such that $\lim_{k\to\infty} a_{n_k} = a$. If $a = \infty$, then $\{a_n\}$ is not bounded above so $\limsup\{a_n\} = \infty$ by Proposition 19(ii). If $a < \infty$, then for any $\epsilon > 0$ there exists N such that $a_{n_k} > a - \epsilon$ for all $k \geq N$. But this implies $b_k \geq a - \epsilon$ for all k, so $\limsup\{a_n\} = \lim_{k\to\infty} b_k \geq a - \epsilon$ by Proposition 14. Since ϵ was arbitrary, we conclude that $\limsup\{a_n\} \geq a$.

To see that $\liminf\{a_k\}$ is a cluster point, observe that $\liminf\{a_n\} = -\limsup\{-a_n\}$ by Proposition 19(iii). We know there exists a subsequence $\{-a_{n_k}\}$ which satisfies $\lim_{k\to\infty} -a_{n_k} = \limsup\{-a_n\}$. This implies $\lim_{k\to\infty} a_{n_k} = \lim_{k\to\infty} -(-a_{n_k}) = -\limsup\{-a_n\} = \liminf\{a_n\}$ by the extension to Theorem 18 proven above. To see that $\liminf\{a_k\}$ is the smallest cluster point, suppose a is a cluster point of

$\{a_n\}$. Then there exists a subsequence $\{a_{n_k}\}$ such that $\lim_{k\to\infty} a_{n_k} = a$. This means $\lim_{k\to\infty} -a_{n_k} = -a \le \limsup\{-a_n\}$, so we must have $a \ge -\limsup\{-a_n\} = \liminf\{a_n\}$.

39. (i) Suppose $\limsup\{a_n\} = \ell \in \mathbf{R}$ and fix $\epsilon > 0$. Then there exists N such that $\sup\{a_k \mid k \ge n\} > \ell - \epsilon$ for all $n \ge N$. But this means for any k, there exists $n \ge k$ such that $a_n > \ell - \epsilon$. Thus there must be infinitely many indices n for which $a_n > \ell - \epsilon$. Likewise, there exists N such that $\sup\{a_k \mid k \ge n\} < \ell + \epsilon$ for all $n \ge N$. But this means $a_n < \ell + \epsilon$ for all $n \ge N$. Thus $a_n > \ell + \epsilon$ for at most N indices.

    Conversely, suppose that for any $\epsilon > 0$, there are infinitely many indices n for which $a_n > \ell - \epsilon$. Then $\sup\{a_k \mid k \ge n\} \ge \ell - \epsilon$ for all n, so $\limsup\{a_n\} \ge \ell - \epsilon$. Since ϵ was arbitrary, we must have $\limsup\{a_n\} \ge \ell$. Now suppose there exists an N such that $a_n \le \ell + \epsilon$ for all $n \ge N$. Then $\sup\{a_k \mid k \ge n\} \le \ell + \epsilon$ for all $n \ge N$, so $\limsup\{a_n\} \le \ell + \epsilon$. Since ϵ was arbitrary, we must have $\limsup\{a_n\} \le \ell$.

    (ii) Suppose $\{a_n\}$ is unbounded above. Fix $n \in \mathbf{N}$. For any c, there must be $k \ge n$ such that $a_k \ge c$. Otherwise, $\max\{a_1, \cdots, a_n, c\}$ would be an upper bound of $\{a_n\}$. Thus $\{a_k \mid k \ge n\}$ is unbounded for all n, so $\sup\{a_k \mid k \ge n\} = \infty$ for all n. This means $\limsup\{a_n\} = \infty$.

    Conversely, observe that if $a_n \le c$ for all n, then $\sup\{a_k \mid k \ge n\} \le c$ for all n. But this means $\limsup\{a_n\} \le c$. Thus if $\limsup_n\{a_n\} = \infty$, $\{a_n\}$ cannot be bounded.

    (iii) From Problem 6, we know that $\sup\{a_k \mid k \ge n\} = -\inf\{-a_k \mid k \ge n\}$. Therefore $\limsup\{a_n\} = -\liminf\{-a_n\}$ by the extension to Theorem 18 proven above.

    (iv) Combining parts (i) and (iii), we see that $\liminf\{a_n\} = \ell \in \mathbf{R}$ if and only if for each $\epsilon > 0$, $a_n < \ell + \epsilon$ for infinitely many indices and $a_n < \ell - \epsilon$ for finitely many indices.

    If $\liminf\{a_n\} = \limsup\{a_n\} = a \in \mathbf{R}$, then for any $\epsilon > 0$ there exists N such that $a_n \ge a - \epsilon$ and $a_n \le a + \epsilon$ for all $n \ge N$. But this means $\{a_n\} \to a$. If $\liminf\{a_n\} = \limsup\{a_n\} = \infty$, then $\{a_n\} \to \infty$ by part (ii). The case when $\liminf\{a_n\} = \limsup\{a_n\} = -\infty$ follows analogously.

    To check the converse, suppose that $\{a_n\} \to a \in \mathbf{R}$. Then for any $\epsilon > 0$ there are only finitely many indices for which $a_n \ge a + \epsilon$ and $a_n \le a - \epsilon$. But this means there are infinitely many indices for which $a_n < a + \epsilon$ and $a_n > a - \epsilon$, so $\liminf\{a_n\} = \limsup\{a_n\} = a$ by part (i). Now suppose that $\{a_n\} \to \infty$. Then for any c there exists N such that $a_n \ge c$ for all $n \ge N$. But this implies $\inf\{a_k \mid k \ge n\} \ge c$ for $n \ge N$. Thus $\liminf\{a_n\} = \limsup\{a_n\} = \infty$. The case when $\{a_n\} \to -\infty$ can be handled similarly.

    (v) If $a_n \le b_n$ for all n, then $\sup\{a_k \mid k \ge n\} \le \sup\{b_k \mid k \ge n\}$ for all n. But this implies $\limsup\{a_n\} \le \limsup\{b_n\}$ by the extension to Theorem 18 proven above.

40. By Proposition 19 (iv), we know that $\{a_n\} \to a$ if and only if $\liminf\{a_n\} = \limsup\{a_n\} = a$. But by Problem 38, $\liminf\{a_n\} = \limsup\{a_n\} = a$ if and only if all the cluster points are the same.

41. This follows immediately from the result in Problem 38.

42. I first prove the following preliminary result:

    Claim: If $\{x_n\} \to x$ and $\{y_n\} \to y$, where $x \cdot y$ is not of the form $0 \cdot \infty$, then $\{x_n \cdot y_n\} \to x \cdot y$.

Proof: Suppose x and y are both in $\mathbf{R}$. Observe that

$$x_n \cdot y_n - x \cdot y = (x_n - x) \cdot (y_n - y) + y \cdot (x_n - x) + x \cdot (y_n - y)$$

For any $\epsilon > 0$, we can pick N such that $|x_n - x|$ and $|y_n - y|$ are smaller than $\sqrt{\epsilon}$ for all $n \geq N$. This implies

$$|x_n - x| \cdot |y_n - y| < \epsilon$$

for all $n \geq N$, so $\{(x_n - x) \cdot (y_n - y)\} \to 0$. Since $\{y \cdot (x_n - x)\} \to 0$ and $\{x \cdot (y_n - x)\} \to 0$, we must have $\{x_n \cdot y_n - x \cdot y\} \to 0$.

Now suppose $x = \infty$ and $0 < y < \infty$. Fix c and choose N large enough that $|y_n - y| < \dfrac{y}{2}$ and $x_n \geq \dfrac{2 \cdot c}{y}$ for all $n \geq N$. Then

$$x_n \cdot y_n > \frac{x_n \cdot y}{2} \geq c$$

for all $n \geq N$, so $\{x_n \cdot y_n\} \to \infty$. The case when $x = \infty$ and $0 > y > -\infty$ can be handled similarly.

Suppose $x = \infty$ and $y = \infty$. Fix c and choose N such that $x_n \geq \sqrt{c}$ and $y_n \geq \sqrt{c}$ for all $n \geq N$. Then

$$x_n \cdot y_n > c$$

for all $n \geq N$, so $\{x_n \cdot y_n\} \to \infty$. The case when $x = \infty$ and $y = -\infty$ can be handled similarly.

Observe that for any n, $a_k \leq \sup\{a_k \,|\, k \geq n\}$ and $b_k \leq \sup\{b_k \,|\, k \geq n\}$ for all $k \geq n$. Since a_k and b_k are positive, we have

$$a_k \cdot b_k \leq \sup\{a_k \,|\, k \geq n\} \cdot \sup\{b_k \,|\, k \geq n\}$$

for all $k \geq n$. But this implies

$$\sup\{a_k \cdot b_k \,|\, k \geq n\} \leq \sup\{a_k \,|\, k \geq n\} \cdot \sup\{b_k \,|\, k \geq n\}$$

The preliminary result then implies

$$\limsup\{a_n\} \leq \limsup\{a_n\} \cdot \limsup\{b_n\}$$

provided the product on the right is not of the form $0 \cdot \infty$.

43. Let $\{a_n\}$ be a sequence of real numbers. Define the set

$$E = \{n \in \mathbf{N} : a_n > a_k \text{ for all } k > n\}$$

Suppose E is infinite. Since E is a subset of $\mathbf{N}$, E is countable. Let $\{n_k \,|\, k \in \mathbf{N}\}$ denote an enumeration of E such that $n_1 < n_2 < \cdots$. By construction $a_{n_1} > a_{n_2} > \cdots$, so $\{a_{n_k}\}$ is a monotonically decreasing subsequence. Now suppose E is finite. Then there exists N such that for all $n \geq N$, we can find $k_n > n$ such that $a_n \leq a_{k_n}$. Define the sequence

$$n_1 = N, \qquad n_{j+1} = k_{n_j}$$

By construction $a_{n_1} \leq a_{n_2} \leq \cdots$, so $\{a_{n_k}\}$ is a monotonically increasing subsequence.

Let $\{a_n\}$ be a bounded sequence. By the above result, we can find a monotone subsequence of $\{a_n\}$. Since this subsequence is bounded, it must converge by Theorem 15.

44. Define the floor function $\lfloor\cdot\rfloor : \mathbf{R} \to \mathbf{Z}$ as

$$\lfloor x \rfloor = \max\{m \in \mathbf{Z} : m \leq x\}$$

This function is well-defined by Problem 11. It is easy to check that $0 \leq x - \lfloor x \rfloor < 1$ for all $x \in \mathbf{R}$. Now fix $x \in (0,1)$ and consider the sequence

$$a_1 = \lfloor xp \rfloor, \qquad a_{n+1} = \left\lfloor \left(x - \sum_{i=1}^{n} \frac{a_i}{p^i} \right) p^{n+1} \right\rfloor$$

Since $0 \leq xp < p$, we must have $0 \leq a_1 < p$. The expression

$$\begin{aligned} a_{n+1} &= \left\lfloor \left(x - \sum_{i=1}^{n-1} \frac{a_i}{p^i} - \frac{a_n}{p^n} \right) p^{n+1} \right\rfloor \\ &= \left\lfloor \left(\left(x - \sum_{i=1}^{n-1} \frac{a_i}{p^i} \right) p^n - a_n \right) p \right\rfloor \end{aligned}$$

implies $0 \leq a_{n+1} < p$ for $n \geq 1$, since by construction $0 \leq \left(x - \sum_{i=1}^{n-1} \frac{a_i}{p^i} \right) p^n - a_n < 1$. Thus all the terms in the sequence $\{a_n\}$ are integers satisfying $0 \leq a_n < p$.

Next, observe that

$$\begin{aligned} \left| x - \sum_{i=1}^{n} \frac{a_i}{p^i} \right| &= \left| x - \sum_{i=1}^{n-1} \frac{a_i}{p^i} - \frac{a_n}{p^n} \right| \\ &= \frac{1}{p^n} \left| \left(x - \sum_{i=1}^{n-1} \frac{a_i}{p^i} \right) p^n - a_n \right| \\ &< \frac{1}{p^n} \end{aligned}$$

This implies $\left\{ \sum_{i=1}^{n} \frac{a_i}{p^i} \right\} \to x$ by the Archimedean property, so we can write $x = \sum_{n=1}^{\infty} \frac{a_n}{p^n}$.

Now suppose there exists a sequence of integers $\{b_n\}$ with $0 \leq b_n < p$ such that $\sum_{n=1}^{\infty} \frac{b_n}{p^n}$. Define

$$E = \{n \in \mathbf{N} \mid a_n \neq b_n\}$$

If E is non-empty, we can define $k = \min E$ by Theorem 1. Suppose without loss that $a_k < b_k$. Then

$$\begin{aligned} 0 &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{a_i - b_i}{p^i} \\ &= \sum_{i=1}^{k} \frac{a_i - b_i}{p^i} + \lim_{n\to\infty} \sum_{i=k+1}^{n} \frac{a_i - b_i}{p^i} \\ &= \frac{a_k - b_k}{p^k} + \lim_{n\to\infty} \sum_{i=k+1}^{n} \frac{a_i - b_i}{p^i} \end{aligned}$$

Since $a_k - b_k \geq 1$ and $b_n - a_n \leq p - 1$, this expression implies

$$\begin{aligned} \frac{1}{p^k} \leq \frac{a_k - b_k}{p^k} &= \sum_{n=k+1}^{\infty} \frac{b_n - a_n}{p^n} \\ &\leq \frac{p-1}{p^{k+1}} \sum_{n=0}^{\infty} \frac{1}{p^n} \\ &= \frac{p-1}{p^{k+1}} \frac{1}{1-p^{-1}} \\ &= \frac{1}{p^k} \end{aligned}$$

We can therefore conclude

$$a_k = b_k + 1 \text{ and } \sum_{n=1}^{\infty} \frac{b_{n+k} - a_{n+k}}{p^n} = 1$$

But the latter equality holds if and only if $b_{n+k} = p - 1$ and $a_{n+k} = 0$ for all n. This means we can write x as

$$x = \sum_{i=1}^{k} \frac{a_i}{p^i} = \frac{\sum_{i=1}^{k} a_i p^{k-i}}{p^k}$$

and the sequence $\{b_n\}$ must be defined as

$$b_n = \begin{cases} a_n & \text{if } n < k \\ a_k - 1 & \text{if } n = k \\ p - 1 & \text{if } n \geq k \end{cases}$$

To check the converse, suppose $\{a_n\}$ is a sequence of integers with $0 \leq a_n < p$. Then

$$0 \leq \sum_{k=n}^{n+m} \frac{a_k}{p^k} \leq \sum_{k=n}^{n+m} \frac{p-1}{p^k} = \frac{p-1}{p^n} \sum_{k=0}^{m} \frac{1}{p^k} = \frac{p^{m+1} - 1}{p^{m+n}}$$

Fix $\epsilon > 0$ and pick N such that $\frac{1}{p^{n-1}} < \frac{\epsilon}{2}$ for all $n \geq N$. Then

$$\left| \sum_{k=n}^{n+m} \frac{a_k}{p^k} \right| \leq \frac{1}{p^{n-1}} + \frac{1}{p^{m+n}} < \epsilon$$

for all $n, m \geq N$. But this means the series converges by Proposition 20. We also have

$$0 \leq \sum_{k=1}^{n} \frac{a_k}{p^k} \leq \sum_{k=1}^{n} \frac{p-1}{p^k} = 1 - \frac{1}{p^n} < 1$$

which implies $0 \leq \displaystyle\sum_{k=1}^{\infty} \frac{a_k}{p^k} \leq 1$ by Theorem 18.

45. (i) Let $\{s_n\}$ denote the sequence of partial sums. Suppose $\{s_n\}$ converges and fix $\epsilon > 0$. By Theorem 17, there exists an N such that

$$\text{if } n, n' \geq N, \text{ then } |s_{n'} - s_n| < \epsilon$$

Define $N' = N + 1$. Then for any $n \geq N'$, we have

$$\left| \sum_{k=n}^{n+m} a_k \right| = |s_{n+m} - s_{n-1}| < \epsilon$$

for any natural number m.

To prove the converse, first fix $\epsilon > 0$. Let N be an index such that

$$\left| \sum_{k=n}^{n+m} a_k \right| < \frac{\epsilon}{2} \text{ for } n \geq N \text{ and any natural number } m$$

Choose natural numbers n and n' such that $n, n' \geq N$. Without loss of generality, assume $n' > n$. If $n' > n + 1$, define $m = n' - n - 1$. Then

$$|s_{n'} - s_n| = \left| \sum_{k=n+1}^{n+1+m} a_k \right| < \frac{\epsilon}{2} < \epsilon$$

If $n' = n + 1$, then

$$\begin{aligned}
|s_{n'} - s_n| &= |a_{n+1}| \\
&= |a_{n+1} + a_{n+2} + a_{n+3} - (a_{n+2} + a_{n+3})| \\
&\leq |a_{n+1} + a_{n+2} + a_{n+3}| + |a_{n+2} + a_{n+3}| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&< \epsilon
\end{aligned}$$

Therefore the sequence $\{s_n\}$ is Cauchy, which implies that $\{s_n\}$ converges by Theorem 17.

(ii) Suppose $\sum_{k=1}^{\infty} |a_k|$ is summable and fix $\epsilon > 0$. By part (i), there exists N such that

$$\left| \sum_{k=n}^{n+m} a_k \right| \leq \sum_{k=n}^{n+m} |a_k| = \left| \sum_{i=n}^{n+m} |a_k| \right| < \epsilon$$

for $n \geq N$ and any natural number m. But this implies $\sum_{k=1}^{\infty} a_k$ is summable by part (i).

(iii) If each term a_n is non-negative, the sequence of partial sums forms an increasing sequence. Thus by Theorem 15, the sequence of partial sums converges if and only if it is bounded.

46. The text establishes the following relationships:

$$\textbf{Completeness Axiom} \quad \Longrightarrow \quad \textbf{Monotone Convergence Theorem} + \textbf{Archimedean Property}$$

We proved in Problem 43 that

$$\textbf{Monotone Convergence Theorem} \quad \Longrightarrow \quad \textbf{Bolzano-Weierstrass Theorem}$$

I show below that

$$\textbf{Bolzano-Weierstrass Theorem} \quad \Longrightarrow \quad \textbf{Nested Set Theorem}$$

From Problem 34, we know

$$\textbf{Nested Set Theorem} + \textbf{Archimedean Property} \quad \Longrightarrow \quad \textbf{Completeness Axiom}$$

Thus we will have shown the following equivalences:

$$\begin{aligned}
&\textbf{Completeness Axiom} \iff \\
&\textbf{Monotone Convergence Theorem} + \textbf{Archimedean Property} \iff \\
&\textbf{Bolzano-Weierstrass Theorem} + \textbf{Archimedean Property} \iff \\
&\textbf{Nested Set Theorem} + \textbf{Archimedean Property}
\end{aligned}$$

Suppose the assertion of the Bolzano-Weierstrass Theorem holds. Let $\{F_n\}_{n=1}^{\infty}$ denote a descending, countable collection of nonempty closed sets of real numbers for which F_1 is bounded. Let $a_n = \sup F_n$ for all $n \in \mathbf{N}$. Since F_n is closed, $a_n \in F_n$. Since $\{F_n\}_{n=1}^{\infty}$ is descending, $a_n \in F_1$. Since F_1 is bounded, the sequence $\{a_n\}$ is bounded. By the Bolzano-Weierstrass Theorem, there exists a subsequence $\{a_{n_k}\}$ that converges. Let $x = \lim_{k\to\infty} a_{n_k}$. Suppose $x \notin F_n$ for some n. Since F_n is closed, $\mathbf{R} \sim F_n$ is open. Therefore there exists $r > 0$ such that $(x - r, x + r) \subseteq \mathbf{R} \sim F_n$. But since $\{a_{n_k}\} \to x$, we can find an index N such that $|a_{n_k} - x| < r$ for all $k \geq N$. This means there exists $n_k \geq n$ such that $a_{n_k} \notin F_n$. But $a_{n_k} \in F_{n_k} \subseteq F_n$, a contradiction. Therefore $x \in F_n$ for all n, so $\bigcap_{n=1}^{\infty} F_n \neq \varnothing$.

1.6 Continuous Real-Valued Functions of a Real Variable

47. Since E is closed, $\mathbf{R} \sim E$ is open. By Proposition 9, we can write $\mathbf{R} \sim E = \bigcup_{n=1}^{\infty} (a_n, b_n)$ where $\{(a_n, b_n)\}$ is a collection of disjoint open intervals. If $x \in E$, define $g(x) = f(x)$. If $x \in (a_n, b_n)$, define

$g(x)$ as

$$g(x) = \begin{cases} f(a_n) + \dfrac{f(b_n) - f(a_n)}{b_n - a_n}(x - a_n) & \text{if } a_n, b_n \in \mathbf{R} \\ f(a_n) & \text{if } a_n \in \mathbf{R} \text{ and } b_n = \infty \\ f(b_n) & \text{if } b_n \in \mathbf{R} \text{ and } a_n = -\infty \\ 0 & \text{if } a_n = -\infty \text{ and } b_n = \infty \end{cases}$$

48. I first show that f is continuous at 0. By construction, $f(0) = 0$. Fix any $\epsilon > 0$. Since $\lim_{n\to\infty} n \sin \frac{1}{n} = 1$, there exists an index N such that

$$\left| n \sin \frac{1}{n} \right| < 1 + \epsilon$$

for all $n \geq N$. Define $\delta = \min\left\{\frac{1}{N}, \frac{\epsilon}{1+\epsilon}\right\}$. For any non-zero rational number $x = \frac{p}{q}$, where $\frac{p}{q}$ is in lowest terms and q is chosen to be positive for convenience, we have

$$\begin{aligned} |x| = \left|\frac{p}{q}\right| < \delta \qquad &\implies \qquad \frac{1}{\delta} \leq \frac{|p|}{\delta} < q \\ &\implies \qquad q \geq N, \qquad |p| \leq \delta \cdot q \\ &\implies \qquad \left| q \sin \frac{1}{q} \right| < 1 + \epsilon, \qquad |p| \leq \delta \cdot q \\ &\implies \qquad |f(x)| = \left| p \cdot \sin \frac{1}{q} \right| \leq \delta \cdot \left| q \cdot \sin \frac{1}{q} \right| < \delta \cdot (1 + \epsilon) \leq \epsilon \end{aligned}$$

If x is irrational, then

$$|x| < \delta \qquad \implies \qquad |f(x)| = |x| < \epsilon$$

since $\delta < \epsilon$. Thus $|x - 0| < \delta$ implies $|f(x) - 0| < \epsilon$, so $f(x)$ is continuous at zero.

Next, I show that f is discontinuous at all non-zero rational numbers. Let $x = \frac{p}{q}$ be a non-zero rational number. Define $z = |p \cdot q| + 1$ and

$$p_n = p \cdot (z^n + q), \qquad q_n = q \cdot z^n$$

Since p, q and z are pairwise coprime, p_n and q_n are relatively prime for all n. Define $x_n = \frac{p_n}{q_n}$ and observe that

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \frac{p_n}{q_n} \cdot q_n \cdot \sin \frac{1}{q_n} = \left(\lim_{n\to\infty} \frac{p_n}{q_n}\right) \cdot \left(\lim_{n\to\infty} q_n \cdot \sin \frac{1}{q_n}\right) = \frac{p}{q}$$

However

$$f(x) = p \sin \frac{1}{q} \neq \frac{p}{q}$$

because the left-hand side is an irrational number. Since $\lim_{n\to\infty} x_n = x$ but $\lim_{n\to\infty} f(x_n) \neq f(x)$, f is not continuous at x.

Lastly, we can also show that f is continuous at all irrational numbers. Let x be an irrational number and fix $\epsilon > 0$. Let $x = [a_0; a_1, a_2, \cdots]$ denote the continued fraction expansion of x and define the sequences

$$
\begin{aligned}
h_n &= a_n h_{n-1} + h_{n-2} \qquad h_{-1} = 1 \qquad h_{-2} = 0 \\
k_n &= a_n k_{n-1} + k_{n-2} \qquad k_{-1} = 0 \qquad k_{-2} = 1
\end{aligned}
$$

Since $a_n \geq 1$, $k_n \geq n$ for all n. It is possible to show $\left\{\dfrac{h_n}{k_n}\right\} \to x$ and

$$\left|x - \frac{h_n}{k_n}\right| < \left|x - \frac{p}{q}\right|$$

for any rational number $\frac{p}{q}$ satisfying $0 < q < k_n$. This means we can find an index N such that

$$\left|x - \frac{h_n}{k_n}\right| < \frac{\epsilon \cdot |x|}{\epsilon + 2 \cdot |x|}$$

and

$$\left|1 - n \cdot \sin \frac{1}{n}\right| < \frac{\epsilon}{2 \cdot |x|}$$

for all $n \geq N$. Let $\delta = \left|x - \frac{h_N}{k_N}\right|$. Then for any rational $\frac{p}{q}$ satisfying $\left|x - \frac{p}{q}\right| < \delta$, we must have $q \geq k_N \geq N$. But this implies

$$
\begin{aligned}
\left|x - p \cdot \sin \frac{1}{q}\right| &\leq \left|x - \frac{p}{q}\right| + \left|\frac{p}{q} - p \cdot \sin \frac{1}{q}\right| \\
&= \left|x - \frac{p}{q}\right| + \left|\frac{p}{q}\right| \left|1 - q \cdot \sin \frac{1}{q}\right| \\
&\leq \left|x - \frac{p}{q}\right| + \left|x - \frac{p}{q}\right| \left|1 - q \cdot \sin \frac{1}{q}\right| + |x| \left|1 - q \cdot \sin \frac{1}{q}\right| \\
&= \left|x - \frac{p}{q}\right| \left(1 + \left|1 - q \cdot \sin \frac{1}{q}\right|\right) + |x| \left|1 - q \cdot \sin \frac{1}{q}\right| \\
&< \delta \left(1 + \frac{\epsilon}{2 \cdot |x|}\right) + \frac{\epsilon}{2} \\
&< \epsilon
\end{aligned}
$$

For any irrational x' satisfying $|x' - x| < \delta$, we have $|x' - x| < \epsilon$ because $\delta < \epsilon$. Therefore $|x' - x| < \delta$ implies $|f(x') - f(x)| < \epsilon$, so f is continuous at x.

49. (i) • $f + g$: Fix $\epsilon > 0$ and pick $x \in E$. There exists $\delta_1, \delta_2 > 0$ such that

$$|x' - x| < \delta_1 \implies |f(x') - f(x)| < \frac{\epsilon}{2}$$

$$|x' - x| < \delta_2 \implies |g(x') - g(x)| < \frac{\epsilon}{2}$$

for all $x' \in E$. Define $\delta = \min\{\delta_1, \delta_2\}$. Then

$$|x' - x| < \delta \implies |f(x') + g(x') - (f(x) + g(x))| \leq |f(x') - f(x)| + |g(x') - g(x)| < \epsilon$$

for all $x' \in E$. Thus $f+g$ is continuous at x.

- fg: Fix $\epsilon > 0$ and pick $x \in E$. There exists $\delta_1, \delta_2 > 0$ such that

$$|x'-x| < \delta_1 \implies |f(x') - f(x)| < \min\left\{\sqrt{\frac{\epsilon}{3}}, \frac{\epsilon}{3\max\{2|g(x)|, 1\}}\right\}$$

$$|x'-x| < \delta_2 \implies |g(x') - g(x)| < \min\left\{\sqrt{\frac{\epsilon}{3}}, \frac{\epsilon}{3\max\{2|f(x)|, 1\}}\right\}$$

for all $x' \in E$. Define $\delta = \min\{\delta_1, \delta_2\}$. Then $|x'-x| < \delta$ implies

$$\begin{aligned} |f(x') - f(x)| \cdot |g(x') - g(x)| &< \frac{\epsilon}{3} \\ |g(x)| \cdot |f(x') - f(x)| &\le \frac{\epsilon}{3} \cdot \frac{|g(x)|}{\max\{2|g(x)|, 1\}} < \frac{\epsilon}{3} \\ |f(x)| \cdot |g(x') - g(x)| &\le \frac{\epsilon}{3} \cdot \frac{|f(x)|}{\max\{2|f(x)|, 1\}} < \frac{\epsilon}{3} \end{aligned}$$

for all $x' \in E$. But this implies

$$\begin{aligned} |f(x') \cdot g(x') - f(x) \cdot g(x)| &= \begin{array}{l} |(f(x') - f(x)) \cdot (g(x') - g(x)) \\ \quad + f(x) \cdot (g(x') - g(x)) + g(x) \cdot (f(x') - f(x))| \end{array} \\ &\le \begin{array}{l} |f(x') - f(x)| \cdot |g(x') - g(x))| \\ \quad + |f(x)| \cdot |g(x') - g(x)| + |g(x)| \cdot |f(x') - f(x)| \end{array} \\ &< \epsilon \end{aligned}$$

for all $x' \in E$ satisfying $|x'-x| < \delta$. Thus fg is continuous at x.

(ii) Let E' denote the domain of h. Fix $\epsilon > 0$ and choose $x \in E'$. Define $y = h(x)$. There exists $\delta > 0$ such that for any $y' \in E$,

$$|y' - y| < \delta \implies |f(y') - f(y)| < \epsilon$$

There also exists $\eta > 0$ such that for any $x' \in E'$,

$$|x' - x| < \eta \implies |h(x') - h(x)| < \delta$$

But this means

$$|x' - x| < \eta \implies |f(h(x')) - f(h(x))| < \epsilon$$

for any $x' \in E'$, so $f \circ h$ is continuous at x.

(iii) Observe that

$$\max\{f(x), g(x)\} = \frac{1}{2}\left(f(x) + g(x) + |f(x) - g(x)|\right)$$

The result then follows from parts (i) and (iv).

(iv) For any $x, x' \in \mathbf{R}$ we have

$$|x| - |x'| \leq |x - x'| \text{ and } |x'| - |x| \leq |x - x'|$$

which means

$$\big||x| - |x'|\big| \leq |x - x'|$$

But then for any $\epsilon > 0$, we have

$$|x - x'| < \epsilon \implies \big||x| - |x'|\big| < \epsilon$$

so $|\cdot|$ is continuous.

50. Suppose f is Lipschitz on E. Then there exists a $c \geq 0$ such that

$$|f(x') - f(x)| \leq c|x' - x| \text{ for all } x', x \in E$$

Fix $\epsilon > 0$ and define $\delta = \dfrac{\epsilon}{\max\{2c, 1\}}$. Then for all $x, x' \in E$, we have

$$|x' - x| < \delta \implies |f(x') - f(x)| \leq \frac{c}{\max\{2c, 1\}}\epsilon < \epsilon$$

Thus f is uniformly continuous on E.

Consider the function $f(x) = \sqrt{x}$ on the set $E = [0, 1]$. Fix $\epsilon > 0$ and $x \in E$. If $x = 0$, define $\delta = \epsilon^2$. Then

$$|x'| < \delta \implies \left|\sqrt{x'}\right| = \sqrt{|x'|} < \epsilon$$

If $x > 0$, define $\delta = \epsilon\sqrt{x}$. Then

$$\left|\sqrt{x'} - \sqrt{x}\right| = \left|\sqrt{x'} - \sqrt{x}\right| \cdot \frac{\sqrt{x'} + \sqrt{x}}{\sqrt{x'} + \sqrt{x}} = \frac{|x' - x|}{\sqrt{x'} + \sqrt{x}} \tag{1}$$

so that

$$|x' - x| < \delta \implies \left|\sqrt{x'} - \sqrt{x}\right| \leq \frac{|x' - x|}{\sqrt{x}} < \epsilon$$

Thus f is continuous on E. Since E is closed and bounded, f is uniformly continuous on E by Theorem 23. To see that f is not Lipschitz, fix any $c \geq 0$. From equation (1), we have

$$\left|\frac{\sqrt{x'} - \sqrt{x}}{x' - x}\right| = \frac{1}{\sqrt{x'} + \sqrt{x}}$$

as long as x and x' do not both equal 0. If $c = 0$, then $|f(x') - f(x)| > c|x' - x|$ for all $x', x \in E$ satisfying $x \neq x'$. If $c > 0$, choose $x, x' \in \left(0, \max\left\{\frac{1}{4c^2}, 1\right\}\right)$ with $x \neq x'$. Then $\sqrt{x'} + \sqrt{x} < \frac{1}{c}$, so

$$\left|\frac{\sqrt{x'} - \sqrt{x}}{x' - x}\right| > c$$

Thus for any $c \geq 0$, we can find $x', x \in E$ violating the Lipschitz condition.

51. Let $A_k = \{a_{k,1}, a_{k,2}, \cdots, a_{k,2^k+1}\}$ denote the sequence of sets defined recursively by

$$a_{1,1} = a, \quad a_{1,2} = \frac{a+b}{2}, \quad a_{1,3} = b$$

$$a_{k+1,i} = \begin{cases} a_{k,\frac{i+1}{2}} & \text{if } i \in \{1, \cdots, 2^{k+1}+1\} \text{ is odd} \\ \dfrac{a_{k,\frac{i}{2}+1} - a_{k,\frac{i}{2}}}{2} & \text{if } i \text{ is even} \end{cases}$$

The elements of A_k partition the interval $[a,b]$ into subintervals of length $\frac{b-a}{2^k}$. Define the function $\varphi_k : [a,b] \to \mathbf{R}$ as

$$\varphi_k(x) = f(a_{k,i}) + \frac{f(a_{k,i+1}) - f(a_{k,i})}{a_{k,i+1} - a_{k,i}}(x - a_{k,i}) \quad \text{if } a_{k,i} \le x \le a_{k,i+1}$$

By construction, φ_k is piecewise linear for each k. If $a_{k,i} \le x \le a_{k,i+1}$, then

$$\begin{aligned} |f(x) - \varphi_k(x)| &= \left| \frac{a_{k,i+1} - x}{a_{k,i+1} - a_{k,i}}(f(x) - f(a_{k,i})) + \frac{x - a_{k,i}}{a_{k,i+1} - a_{k,i}}(f(x) - f(a_{k,i+1})) \right| \\ &\le |f(x) - f(a_{k,i})| + |f(x) - f(a_{k,i+1})| \end{aligned}$$

By Theorem 23, there exists a positive number δ such that for all $x, x' \in [a,b]$,

$$|x - x'| < \delta \implies |f(x) - f(x')| < \frac{\epsilon}{2}$$

Choose K such that $\frac{b-a}{2^K} < \delta$. Then

$$\begin{aligned} &a_{K,i} \le x \le a_{K,i+1} \implies |x - a_{K,i}| < \delta \text{ and } |x - a_{K,i+1}| < \delta \\ &\implies |f(x) - f(a_{K,i})| < \frac{\epsilon}{2} \text{ and } |f(x) - f(a_{K,i+1})| < \frac{\epsilon}{2} \implies |f(x) - \varphi_K(x)| < \epsilon \end{aligned}$$

Since any $x \in [a,b]$ must belong to $[a_{K,i}, a_{K,i+1}]$ for some $i \in \{1, \cdots, 2^K\}$, $|f(x) - \varphi_K(x)| < \epsilon$ for all $x \in [a,b]$.

52. Suppose E is not bounded above. Then for any c, there exists $x \in E$ satisfying $x > c$. But this means the function $f(x) = x$ does not obtain its maximum on E. Likewise, if E is not bounded below the function $f(x) = -x$ does not obtain its maximum on E. Now suppose E is open. Then for any $x \in E$, there exists an $x' \in E$ such that $x' > x$. But this means the function $f(x) = x$ does not obtain its maximum on E. We conclude that if every continuous, real-valued function on E takes its maximum on E, then E must be closed and bounded.

    The converse is given by the Extreme Value Theorem.

53. Suppose E is unbounded and consider the open cover $\{(x-1, x+1)\}_{x \in E}$. Let $\{(x_i - 1, x_i + 1)\}_{i=1}^n$ denote a finite collection of sets from this open cover. There exists $x \in E$ satisfying $x > \max\{x_1, \cdots, x_n\} + 1$, so $\{(x_i - 1, x_i + 1)\}_{i=1}^n$ cannot cover E. Now suppose E is a set which is not closed. Then there must

exist a point of closure x_* of E which is not in E. For each $x \in E$, define I_x as

$$I_x = \begin{cases} \left(-\infty, \dfrac{x+x_*}{2}\right) & \text{if } x < x_* \\ \left(\dfrac{x+x_*}{2}, \infty\right) & \text{if } x > x_* \end{cases}$$

Then $\bigcup_{x\in E} I_x$ is an open cover of E. Let $\bigcup_{i=1}^n I_{x_i}$ denote a finite collection of sets from this open cover. Since x_* is a point of closure of E, there exists a point $x \in E$ satisfying $|x - x_*| < \frac{1}{2}\min\{|x_1 - x_*|, \cdots, |x_n - x_*|\}$. But by construction, $x' \notin \bigcup_{i=1}^n I_{x_i}$. Therefore $\bigcup_{i=1}^n I_{x_i}$ cannot cover E.

The converse is given by the Heine-Borel Theorem.

54. Let E be an interval and suppose y and y' are two points in $f(E)$. Then there exists $x, x' \in E$ such that $f(x) = y$ and $f(x') = y'$. Without loss of generality, assume $y < y'$ and pick any $z \in (y, y')$. Since E is an interval, $[\min\{x, x'\}, \max\{x, x'\}] \subseteq E$. By the Intermediate Value Theorem, there is a point $x_0 \in (\min\{x, x'\}, \max\{x, x'\})$ at which $f(x_0) = z$. But this implies $z \in f(E)$.

    Conversely, suppose every continuous real-valued function on E has an interval as its image. Take f to be the identity function. Then $f(E) = E$, so E is an interval.

55. Let E be an open interval and let f be a monotone function. If f is continuous, then $f(E)$ is an interval by Problem 54.

    To prove the converse, fix $x_0 \in E$ and suppose $\{x_n\}$ is a sequence in E converging upward to x_0. By the Monotone Convergence Theorem for Real Sequences, $f(x_n)$ must converge to a limit $f(x_0^+)$. Because f is monotone, we have

    $$x < x_0 \implies f(x) \le f(x_0^+), \qquad x_0 \le x \implies f(x_0) \le f(x)$$

    for any $x \in E$. Therefore if $f(x_0^+) < f(x_0)$, there does not exist any $x \in E$ such that $f(x) \in (f(x_0^+), f(x_0))$. An analogous argument applies to sequences converging downward to x_0. We can conclude that if $f(E)$ is an interval, any monotone sequence $\{x_n\}$ in E converging to x_0 must satisfy $\{f(x_n)\} \to f(x_0)$.

    Now suppose f is not continuous at x_0. Then we can find a positive ϵ such that for any n there exists an $x_n \in E$ that satisfies $|x_n - x_0| < \frac{1}{n}$ and $|f(x_n) - f(x_0)| > \epsilon$. But this means we can find a monotone subsequence $\{x_{n_k}\}$ such that $\{x_{n_k}\}$ converges to x but $\{f(x_{n_k})\}$ does not converge to $f(x)$. Therefore if any monotone sequence $\{x_n\}$ in E converging to x_0 satisfies $\{f(x_n)\} \to f(x_0)$, f must be continuous at x_0. Thus if $f(E)$ is an interval, f is continuous on E.

56. For any $x \in \mathbf{R}$ and $n \in \mathbf{N}$, say x satisfies property C_n if there exists a $\delta_x > 0$ such that for all $x', x'' \in \mathbf{R}$,

    $$|x' - x| < \delta_x \text{ and } |x'' - x| < \delta_x \implies |f(x') - f(x'')| < \frac{1}{n} \tag{2}$$

    Let A_n denote the collection of points in E which satisfy property C_n. Suppose $x \in A_n$ and pick y such that $|x - y| < \frac{\delta_x}{2}$. Then for all $x', x'' \in \mathbf{R}$,

    $$|x' - y| < \frac{\delta_x}{2} \text{ and } |x'' - y| < \frac{\delta_x}{2} \implies |x' - x| < \delta_x \text{ and } |x'' - x| < \delta_x \implies |f(x') - f(x'')| < \frac{1}{n}$$

Therefore y satisfies property C_n for $\delta_y = \frac{\delta_x}{2}$, so $y \in A_n$. But this means $\left(x - \frac{\delta_x}{2}, x + \frac{\delta_x}{2}\right) \subseteq A_n$, so x is an interior point of A_n. Since x was an arbitrary point of A_n, A_n is an open set. Thus $\bigcap_{n=1}^{\infty} A_n$ is a G_δ set. I now show that this set is equal to the set of all points at which f is continuous.

Now suppose $x \in \bigcap_{n=1}^{\infty} A_n$. Fix $\epsilon > 0$ and pick n such that $\epsilon > \frac{1}{n}$. Since $x \in A_n$, there exists a $\delta_x > 0$ such that (2) is satisfied. By choosing $x'' = x$ in (2), we see that

$$|x' - x| < \delta \implies |f(x') - f(x)| < \frac{1}{n} < \epsilon$$

for any $x' \in \mathbf{R}$. Therefore f is continuous at x.

Now suppose f is continuous at x. Then for any $n \in N$, there exists a $\delta > 0$ such that

$$|x' - x| < \delta \implies |f(x') - f(x)| < \frac{1}{2n}$$

for all $x' \in \mathbf{R}$. But this implies

$$|x' - x| < \delta \text{ and } |x'' - x| < \delta \implies |f(x') - f(x'')| \leq |f(x') - f(x)| + |f(x'') - f(x)| < \frac{1}{n}$$

so x satisfies property C_n for all n. But this means $x \in \bigcap_{n=1}^{\infty} A_n$.

57. For any $k, n, N \in \mathbf{N}$, define the set

$$A_{k,n}(N) = \left\{ x \in \mathbf{R} : |f_k(x) - f_n(x)| \leq \frac{1}{N} \right\}$$

Since the function $|f_k(x) - f_n(x)|$ is continuous and $\left(\infty, \frac{1}{N}\right]$ is closed, $A_{k,n}(N)$ is closed for all k, n, N. This implies that the set

$$B_n(N) = \bigcap_{k=n}^{\infty} A_{k,n}(N) = \left\{ x \in \mathbb{R} : \sup_{k \geq n} |f_k(x) - f_n(x)| \leq \frac{1}{N} \right\}$$

is closed for all n, N. Define the set

$$B = \bigcap_{N=1}^{\infty} \bigcup_{n=1}^{\infty} B_n(N)$$

This set is the intersection of a countable collection of F_σ sets. I now show that B is the set of points for which $\{f_n(x)\}$ converges.

Suppose $x \in B$. Fix $\epsilon > 0$ and pick N such that $\epsilon > \frac{2}{N}$. There exists n such that $x \in B_n(N)$. For any $n', m' \geq n$, we have

$$|f_{n'}(x) - f_{m'}(x)| \leq |f_{n'}(x) - f_n(x)| + |f_{m'}(x) - f_n(x)| \leq 2 \sup_{k \geq n} |f_k(x) - f_n(x)| \leq \frac{2}{N} < \epsilon$$

But this means $\{f_n(x)\}$ is Cauchy, so $\{f_n(x)\}$ converges to a real number by Theorem 17.

Conversely, suppose $\{f_n(x)\}$ converges to a real number. By Theorem 17, $\{f_n(x)\}$ is Cauchy. Now fix any N. There exists n such that for all $n', m' \geq n$

$$|f_{n'}(x) - f_{m'}(x)| < \frac{1}{N}$$

But this means

$$\sup_{n' \geq n} |f_{n'}(x) - f_n(x)| \leq \frac{1}{N}$$

so $x \in B_n(N)$. Since N was arbitrary, $x \in B$.

58. Suppose $A \subseteq \mathbf{R}$ is an open set. Then $f^{-1}(A)$ is open by Proposition 22.

Suppose $A \subseteq \mathbf{R}$ is a closed set. Then $\mathbf{R} \sim A$ is open, so $f^{-1}(\mathbf{R} \sim A)$ is open. Since $f^{-1}(\mathbf{R} \sim A) = \{x \in \mathbf{R} : f(x) \notin A\} = \mathbf{R} \sim f^{-1}(A)$, $f^{-1}(A)$ must be closed.

Define $\mathcal{C}$ as

$$\mathcal{C} := \left\{A \in \mathcal{B} : f^{-1}(A) \in \mathcal{B}\right\}$$

$\mathcal{C}$ is a σ-algebra:

- $f^{-1}(\varnothing) = \varnothing \in \mathcal{B}$
- Suppose $A \in \mathcal{C}$. Then $f^{-1}(\mathbf{R} \sim A) = \mathbf{R} \sim f^{-1}(A) \in \mathcal{B}$ because $f^{-1}(A) \in \mathcal{B}$ and $\mathcal{B}$ is closed under complements.
- Suppose $A_\lambda \in \mathcal{C}$ for $\lambda \in \Lambda$, where Λ is a countable set. Then $f^{-1}\left(\bigcup_{\lambda \in \Lambda} A_\lambda\right) = \{x \in \mathbf{R} : f(x) \in \bigcup_{\lambda \in \Lambda} A_\lambda\} = \bigcup_{\lambda \in \Lambda} \{x \in \mathbf{R} : f(x) \in A_\lambda\} = \bigcup_{\lambda \in \Lambda} f^{-1}(A_\lambda) \in \mathcal{B}$ because $f^{-1}(A_\lambda) \in \mathcal{B}$ for all $\lambda \in \Lambda$ and $\mathcal{B}$ is closed under countable intersections.

We already showed that all open subsets of $\mathbf{R}$ are in $\mathcal{C}$, so $\mathcal{B} \subseteq \mathcal{C}$. But $\mathcal{C} \subseteq \mathcal{B}$ by construction, so $\mathcal{B} = \mathcal{C}$.

59. Fix $\epsilon > 0$ and $x_* \in E$. Choose N such that for all $x \in E$ and $n \geq N$, $|f_n(x) - f(x)| < \frac{\epsilon}{3}$. Since f_N is continuous, it is also possible to find a $\delta > 0$ such that for all $x \in E$,

$$|x - x_*| < \delta \implies |f_N(x) - f_N(x_*)| < \frac{\epsilon}{3}$$

Thus for all $x \in E$, we have

$$|x - x_*| < \delta \implies |f(x) - f(x_*)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_*)| + |f(x_*) - f_N(x_*)| < \epsilon$$

Therefore f is continuous at x_*. Since x_* was an arbitrary point in E, f is continuous on E.

60. Suppose f is a continuous, real-valued function defined on a set E. Fix $x_* \in E$ and let $\{x_n\}$ be a sequence in E converging to x_*. Pick $\epsilon > 0$ and choose $\delta > 0$ such that for all $x \in E$,

$$|x_* - x| < \delta \implies |f(x_*) - f(x)| < \epsilon$$

Since $\{x_n\} \to x_*$, we can find an N such that $|x - x_n| < \delta$ for all $n \geq N$. But this means $|f(x_n) - f(x_*)| < \epsilon$ for all $n \geq N$. Therefore $\{f(x_n)\}$ converges to $f(x_*)$.

Now fix any $x_* \in E$ and suppose $f(x_*)$ is not continuous at x_*. Then there exists $\epsilon > 0$ such that for any $n \in \mathbf{N}$, we can find $x_n \in E$ satisfying

$$|x_n - x_*| < \frac{1}{n} \text{ and } |f(x_n) - f(x_*)| \geq \epsilon$$

The sequence $\{x_n\}$ is in E and converges to x_* but its image sequence $\{f(x_n)\}$ fails to converge to $f(x_*)$.

Let f be a continuous real-valued function on the non-empty, closed and bounded set E. The argument in the text can be used to establish that f is bounded. Define $y = \sup\{f(x) : x \in E\}$. For each n, we can find $x_n \in E$ such that $|y - f(x_n)| < \frac{1}{n}$. Thus the sequence $\{f(x_n)\}$ converges to y. By the Bolzano-Weierstrass Theorem, there exists a subsequence of $\{x_n\}$, call it $\{x_{n_k}\}$, that converges to some x_*. Since $\{x_{n_k}\} \in E$ for all k and E is closed, x_* must also be in E. But Proposition 21 then implies $\{f(x_{n_k})\}$ converges to $f(x_*)$. We therefore have

$$f(x_*) = \lim_{k\to\infty} f(x_{n_k}) = \lim_{n\to\infty} f(x_n) = y$$

2 Lebesgue Measure

2.1 Introduction

1. Since m is countably additive and $B = A \cup (B \sim A)$, we have

$$m(B) = m(A) + m(B \sim A) \geq m(A)$$

where the inequality follows because $m(B \sim A) \geq 0$.

2. Since $A = A \cup \varnothing$, we have

$$m(A) = m(A) + m(\varnothing)$$

by the countable additivity property of m. If $m(A) < \infty$, then this expression implies $m(\varnothing) = 0$.

3. Define the sequence of sets

$$E_1' = E_1, \qquad E_{n+1}' = E_{n+1} \sim \bigcup_{i=1}^{n} E_i$$

Then $E_n' \subseteq E_n$, the collection of sets $\{E_n'\}_{n=1}^{\infty}$ is pairwise disjoint, and

$$\bigcup_{n=1}^{\infty} E_n' = \bigcup_{n=1}^{\infty} E_n$$

By the countable additivity and monotonicity of m, we have

$$m\left(\bigcup_{n=1}^{\infty} E_n\right) = m\left(\bigcup_{n=1}^{\infty} E_n'\right) = \sum_{n=1}^{\infty} m(E_n') \leq \sum_{n=1}^{\infty} m(E_n)$$

4. Let E be a subset of $\mathbf{R}$ and fix $y \in \mathbf{R}$. The sets E and $E + y$ have the same number of elements, so

$c(E) = c(E + y)$.

Let $\{E_n\}_{n=1}^{\infty}$ denote a countable, disjoint collection of sets. If E_N is infinite for some N, then

$$\infty \geq \sum_{n=1}^{\infty} c(E_n) \geq \sum_{n=1}^{N} c(E_n) = \infty$$

But $\bigcup_{n=1}^{\infty} E_n \supseteq E_N$, so $\bigcup_{n=1}^{\infty} E_n$ is infinite and thus $c\left(\bigcup_{n=1}^{\infty} E_n\right) = \infty$. Therefore $c\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} c(E_n) = \infty$.

Now suppose E_n is finite for all n. Define

$$I = \{n : c(E_n) > 0\}$$

If I is infinite, then

$$\infty \geq \sum_{n=1}^{\infty} c(E_n) = \sum_{n \in I} c(E_n) \geq \sum_{n \in I} 1 = \infty$$

so that $\sum_{n=1}^{\infty} c(E_n) = \infty$. $\bigcup_{n=1}^{\infty} E_n$ is infinite because the sets in the collection are disjoint. Therefore $c\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} c(E_n) = \infty$. If I is finite, there must exist an index N such that $E_n = \varnothing$ for all $n \geq N$. But this implies

$$c\left(\bigcup_{n=1}^{\infty} E_n\right) = c\left(\bigcup_{n=1}^{N} E_n\right) = \sum_{n=1}^{N} c\left(E_n\right) = \sum_{n=1}^{\infty} c\left(E_n\right)$$

where the second equality follows because the number of elements in a finite union of finite disjoint sets equals the sum of the number of elements in each set.

2.2 Lebesgue Outer Measure

5. By Proposition 1, the interval $[0, 1]$ has outer measure equal to 1. The example on page 31 shows that any countable set has outer measure 0. Therefore $[0, 1]$ cannot be countable.

6. Because the outer measure is monotone, $m^*(A) \leq m^*([0, 1]) = 1$. Since the outer measure of a countable set is 0 and the set of rational numbers in $[0, 1]$ is countable, $m^*([0, 1] \sim A) = 0$. By the subadditivity of m^*, we have $1 = m^*([0, 1]) \leq m^*([0, 1] \sim A) + m^*(A) = m^*(A)$. We can therefore conclude $m^*(A) = 1$.

7. Since E is bounded $m^*(E)$ is finite. Because $m^*(E)$ is finite, for any k there exists a countable collection of nonempty, open, bounded intervals $\{I_{k,n}\}_{n=1}^{\infty}$ such that $E \subseteq \bigcup_{n=1}^{\infty} I_{k,n}$ and

$$m^*(E) + \frac{1}{k} > \sum_{n=1}^{\infty} \ell(I_{k,n})$$

Define $G_k = \bigcup_{n=1}^{\infty} I_{k,n}$. G_k is open for all k, so $G := \bigcap_{n=1}^{\infty} G_k$ is a G_δ set. By the definition of m^*, we must have $m^*(G_k) \le \sum_{n=1}^{\infty} \ell(I_{k,n})$. Since $E \subseteq G \subseteq G_k$, we can conclude

$$m^*(E) \le m^*(G) \le m^*(G_k) \le \sum_{n=1}^{\infty} \ell(I_{k,n}) < m^*(E) + \frac{1}{k}$$

by the monotonicity of m^*. Since this expression holds for all k, $m^*(E) = m^*(G)$.

8. I first prove a preliminary result:

   Let A_i, $i = 1, \cdots, n$, be a finite collection of sets of real numbers and define $B = \bigcup_{i=1}^n A_i$. Then $\bar{B} = \bigcup_{i=1}^n \bar{A}_i$.

   Proof: Suppose $x \in \bigcup_{i=1}^n \bar{A}_i$ and let I be an open interval containing x. Because x is in $\bar{A}_i$ for some i, there must exist a point y in I that is also in A_i. But y is also in B, so we can conclude $x \in \bar{B}$. Thus $\bigcup_{i=1}^n \bar{A}_i \subseteq \bar{B}$.

   To see the reverse inclusion, suppose that $x \in \bigcap_{i=1}^n \bar{A}_i^C$. Then for each i, there exists $\epsilon_i > 0$ such that $(x - \epsilon_i, x + \epsilon_i) \subseteq A_i^C$. Define $\epsilon = \min_i \epsilon_i$. Then $(x - \epsilon, x + \epsilon) \subseteq A_i^C$ for all i, so $(x - \epsilon, x + \epsilon) \subseteq \bigcap_{i=1}^n A_i^C = B^C$. But this means $x \in \bar{B}^C$. Thus $\bigcap_{i=1}^n \bar{A}_i^C \subseteq \bar{B}^C$, which implies $\bar{B} \subseteq \bigcup_{i=1}^n \bar{A}_i$.

   Now since the rationals are dense in $\mathbf{R}$, $\bar{B} = [0, 1]$. We therefore have

   $$[0,1] = \bar{B} \subseteq \overline{\bigcup_{k=1}^n I_k} = \bigcup_{k=1}^n \bar{I}_k$$

   where the last equality follows from the preliminary result. But Proposition 1 and the monotonicity and subadditivity of outer measure then imply

   $$1 = \mu^*([0,1]) \le \mu^* \left(\bigcup_{k=1}^n \bar{I}_k \right) \le \sum_{k=1}^n \mu^*(\bar{I}_k) = \sum_{k=1}^n \mu^*(I_k)$$

9. By the monotonicity of m^*, $m^*(B) \le m^*(A \cup B)$. Since m^* is countably subadditive, we have $m^*(A \cup B) \le m^*(A) + m^*(B) = m^*(B)$. Therefore $m^*(A \cup B) = m^*(B)$.

10. By the subadditivity property of m^*, we know that $m^*(A \cup B) \le m^*(A) + m^*(B)$. Therefore we only need to show that the reverse inequality holds. Fix $\epsilon > 0$. Since A and B are bounded, $A \cup B$ is bounded and $m^*(A \cup B)$ is finite. We can therefore find a countable collection of non-empty, open, bounded intervals $\{I_k\}_{k=1}^{\infty}$ which covers $A \cup B$ and satisfies

    $$m^*(A \cup B) > \sum_{k=1}^{\infty} \ell(I_k) - \epsilon$$

    Without loss, assume the length of each interval in the collection is less than $\frac{\alpha}{2}$ (the intervals can be subdivided until this condition holds). Then by construction, each interval only intersects either A or B. Define

    $$\mathcal{A} = \{k : I_k \cap A \neq \varnothing\}, \qquad \mathcal{B} = \{k : I_k \cap B \neq \varnothing\}$$

Since $\{I_k\}_{k\in\mathcal{A}}$ and $\{I_k\}_{k\in\mathcal{B}}$ form open covers of A and B, respectively, we can conclude

$$m^*(A\cup B) > \sum_{k\in\mathcal{A}} \ell(I_k) + \sum_{k\in\mathcal{B}} \ell(I_k) - \epsilon \geq m^*(A) + m^*(B) - \epsilon$$

This expression holds for all $\epsilon > 0$, so we must have $m^*(A\cup B) \geq m^*(A) + m^*(B)$. Therefore $m^*(A\cup B) = m^*(A) + m^*(B)$.

2.3 The σ-Algebra of Lebesgue Measurable Sets

11. Since σ-algebras are closed under countable unions, intersections and complements, the result follows from the following identities:

$$\begin{aligned}
[a,\infty) &= \bigcap_{k=1}^{\infty}\left(a-\frac{1}{k},\infty\right) \\
(-\infty,a] &= (a,\infty)^C \\
(-\infty,a) &= [a,\infty)^C \\
(a,b) &= (-\infty,b)\cap(a,\infty) \\
[a,b) &= (-\infty,b)\cap[a,\infty) \\
(a,b] &= (-\infty,b]\cap(a,\infty) \\
[a,b] &= (-\infty,b]\cap[a,\infty)
\end{aligned}$$

12. Since all intervals of the form (a,∞) are open sets and thus in the Borel σ-algebra, the result follows from Problem 11.

13.

(i) and (ii) I first prove two preliminary results:

Claim 1: The translate of an open set is open.

Proof: Let A be an open set and suppose $x\in A+y$. Then $x-y\in A$. Since A is open, there exists $r>0$ such that $(x-y-r, x-y+r)\subseteq A$. Now pick $z\in(x-r,x+r)$. Then $z-y\in A$, so $z=(z-y)+y$ is in $A+y$. Therefore $(x-r,x+r)\subseteq A+y$, so $A+y$ is open.

Claim 2: For any set of real numbers A, $A^C+y=(A+y)^C$ for all $y\in\mathbf{R}$.

Proof: Since $(A^C+y)\cap(A+y)=\varnothing$, we must have $A^C+y\subseteq(A+y)^C$. To see the reverse inclusion, suppose $x\in(A+y)^C$. Then for all $z\in A$, we must have $x\neq z+y$. But this implies $x-y\in A^C$, so $x=(x-y)+y$ is in A^C+y.

Now suppose G is a G_δ set. Then there exists a countable collection of open sets $\{G_k\}_{k=1}^{\infty}$ such that $G=\bigcap_{k=1}^{\infty}G_k$. Fix $y\in\mathbf{R}$ and suppose $x\in G+y$. Then $x=z+y$ for some $z\in G$. But since $z\in G_k$ for all k, $x\in\bigcap_{k=1}^{\infty}(G_k+y)$. Therefore $G+y\subseteq\bigcap_{k=1}^{\infty}(G_k+y)$. Now suppose $x\in\bigcap_{k=1}^{\infty}(G_k+y)$. Then for every k, there exists $x_k\in G_k$ such that $x=x_k+y$. But this implies $x-y\in G_k$ for all k, so $x-y\in G$. Thus $x=(x-y)+y$ is in $G+y$, so $\bigcap_{k=1}^{\infty}(G_k+y)\subseteq G+y$. Therefore $G+y=\bigcap_{k=1}^{\infty}(G_k+y)$. By Claim 1, G_k+y is open for all k. Thus $G+y$ is a G_δ set.

Now suppose F is an F_σ set. Then there exists a countable collection of closed sets $\{F_k\}_{k=1}^\infty$ such that $F = \bigcup_{k=1}^\infty F_k$. Since $F^C = \bigcap_{k=1}^\infty F_k^C$ is a G_δ set, we have

$$(F+y)^C = F^C + y = \bigcap_{k=1}^\infty (F_k^C + y) = \bigcap_{k=1}^\infty (F_k + y)^C$$

where the first and last equality follow from Claim 2 and the middle equality follows from the result for G_δ sets. Taking complements, we have $F + y = \bigcup_{k=1}^\infty (F_k + y)$. Since $F_k^C + y$ is open and $F_k + y = (F_k^C + y)^C$ by Claim 2, $F_k + y$ is closed for all k. Thus $F + y$ is an F_σ set.

(iii) This follows directly from Propositions 2 and 4.

14. Suppose every bounded subset of E has outer measure equal to zero. Then for all n, we have

$$\sum_{i=1}^n m^*((-n,n) \cap E) = 0$$

so that $\sum_{n=1}^\infty m^*((-n,n)\cap E) = 0$. By the countable subadditivity of m^*, we must have

$$m^*(E) = m^*\left(\bigcup_{n=1}^\infty (-n,n) \cap E\right) \leq \sum_{n=1}^\infty m^*((-n,n)\cap E) = 0$$

Thus if $m^*(E) > 0$, there must exist an n such that the $m^*((-n,n)\cap E) > 0$.

15. Define I_n as

$$I_n = \left[-n \cdot \frac{\epsilon}{2}, (-n+1)\cdot\frac{\epsilon}{2}\right) \cup \left[(n-1)\cdot\frac{\epsilon}{2}, n\cdot\frac{\epsilon}{2}\right)$$

Then $\{I_n\}_{n=1}^\infty$ is a collection of disjoint and measurable sets that cover $\mathbf{R}$, so $E = \bigcup_{n=1}^\infty (E \cap I_n)$. We also have $m^*(E \cap I_n) \leq m^*(I_n) = \epsilon$ for all n. To obtain a finite collection of sets, observe that

$$m^*(E) = m^*\left(\bigcup_{n=1}^\infty E \cap I_n\right) = \sum_{n=1}^\infty m^*(E\cap I_n)$$

by countable additivity. Since $m^*(E) < \infty$, we can find an index m such that

$$m^*\left(\bigcup_{n=m+1}^\infty (E\cap I_n)\right) = \sum_{n=m+1}^\infty m^*(E\cap I_n) < \epsilon$$

Thus we can write E as the union of the finite collection of sets

$$\{E \cap I_n\}_{n=1}^m \cup \left\{E \cap \bigcup_{n=m+1}^\infty I_n\right\}$$

where the sets are disjoint and measurable and each set has measure at most ϵ.

2.4 Outer and Inner Approximation of Lebesgue Measurable Sets

16. • Measurability + (i) $\implies$ (iii): Let E be a measurable set and fix $\epsilon > 0$. Since E is measurable, E^C is measurable. By part (i), there exists an open set $\mathcal{O}$ containing E^C for which $m^*(\mathcal{O} \sim E^C) < \epsilon$.

Define $F = \mathcal{O}^C$. Then F is closed, contained in E and

$$m^*(E \sim F) = m^*(E \cap F^C) = m^*(\mathcal{O} \cap E) = m^*(\mathcal{O} \sim E^C) < \epsilon$$

- (iii) $\implies$ (iv): By part (iii), for any k there exists a closed set F_k contained in E that satisfies $m^*(E \sim F_k) < \frac{1}{k}$. Define $F = \bigcup_{k=1}^{\infty} F_k$. F is an F_σ set contained in E, and by the monotonicity of m^* we have
$$m^*(E \sim F) = m^*\left(\bigcap_{k=1}^{\infty}(E \sim F_k)\right) \leq m^*(E \sim F_k) < \frac{1}{k}$$
for all k. Thus $m^*(E \sim F) = 0$.

- (iv) $\implies$ Measurability: Since a set of measure zero is measurable, as is an F_σ set, and the measurable sets are an algebra, the set
$$E = (E \sim F) \cup F$$
is measurable.

17. Suppose E is a measurable set and fix $\epsilon > 0$. By Theorem 11, there is an open set $\mathcal{O}$ containing E and a closed set F contained in E for which $m^*(\mathcal{O} \sim E) < \frac{\epsilon}{2}$ and $m^*(E \sim F) < \frac{\epsilon}{2}$. By the measurability of E, we have

$$\begin{aligned} m^*(\mathcal{O} \sim F) &= m^*(\mathcal{O} \cap F^C \cap E) + m^*(\mathcal{O} \cap F^C \cap E^C) \\ &= m^*(E \sim F) + m^*(\mathcal{O} \sim E) \\ &< \epsilon \end{aligned}$$

Now fix $\epsilon > 0$ and suppose that there is a closed set F and an open set $\mathcal{O}$ for which $F \subseteq E \subseteq \mathcal{O}$ and $m^*(\mathcal{O} \sim F) < \epsilon$. Then

$$m^*(\mathcal{O} \sim E) \leq m^*(\mathcal{O} \sim F) < \epsilon$$

If such sets can be found for all ϵ, E is measurable by Theorem 11(i).

18. Since E has finite outer measure, we can use the argument in Problem 7 to construct a G_δ set G satisfying $E \subseteq G$ and $m^*(E) = m^*(G)$.

According to Theorem 11(iv) and the excise property, E is measurable if and only if we can find an F_σ set F that satisfies $F \subseteq E$ and $m^*(F) = m^*(E)$.

19. By Theorem 11(i), we know that if E is not measurable there exists an $\epsilon_0 > 0$ such that $m^*(\mathcal{O} \sim E) \geq \epsilon_0$ for any open set $\mathcal{O}$ containing E. Since E has finite outer measure, we can find a collection of open intervals $\{I_k\}_{k=1}^{\infty}$ that contains E and satisfies

$$m^*(E) > \sum_{k=1}^{\infty} \ell(I_k) - \epsilon_0$$

Let $\mathcal{O} = \bigcup_{k=1}^{\infty} I_k$. Then

$$m^*(\mathcal{O}) - m^*(E) \leq \sum_{k=1}^{\infty} \ell(I_k) - m^*(E) < \epsilon_0 \leq m^*(\mathcal{O} \sim E)$$

20. Suppose E is measurable. Then

$$b - a = m^*((a,b)) = m^*((a,b) \cap E) + m^*((a,b) \sim E) \tag{1}$$

where the first equality follows from Proposition 1 and the second equality from the definition of measurability.

Conversely, suppose (1) holds for all open, bounded intervals. Fix a set A with finite outer measure. For any $\epsilon > 0$, we can choose a countable collection of nonempty, open, bounded intervals $\{(a_k, b_k)\}_{k=1}^{\infty}$ that covers A and satisfies

$$m^*(A) > \sum_{k=1}^{\infty} (b_k - a_k) - \epsilon \tag{2}$$

We then have

$$\begin{aligned} m^*(A) &> \sum_{k=1}^{\infty} \left(m^*((a_k, b_k) \cap E) + m^*((a_k, b_k) \sim E)\right) - \epsilon \\ &\geq m^* \left(\bigcup_{k=1}^{\infty} (a_k, b_k) \cap E \right) + m^* \left(\bigcup_{k=1}^{\infty} (a_k, b_k) \sim E \right) - \epsilon \\ &\geq m^*(A \cap E) + m^*(A \sim E) - \epsilon \end{aligned}$$

where the first inequality follows from applying (1) to (2), the second inequality follows from the countable subadditivity of m^*, and the third inequality follows the monotonicity of m^*. Since this expression must hold for all ϵ, we conclude that $m^*(A) \geq m^*(A \cap E) + m^*(A \sim E)$. Thus E is measurable.

21. Let E_1 and E_2 be two sets of real numbers. Suppose we can find G_δ sets G_1 and G_2 that satisfy

$$E_1 \subseteq G_1 \text{ and } m^*(G_1 \sim E_1) = 0$$
$$E_2 \subseteq G_2 \text{ and } m^*(G_2 \sim E_2) = 0$$

Write $G_1 = \bigcap_{k=1}^{\infty} G_{1,k}$ and $G_2 = \bigcap_{k=1}^{\infty} G_{2,k}$, where $\{G_{1,k}\}_{k=1}^{\infty}$ and$\{G_{2,k}\}_{k=1}^{\infty}$ are countable collections of open sets. Without loss, assume that $\{G_{1,k}\}_{k=1}^{\infty}$ and $\{G_{2,k}\}_{k=1}^{\infty}$ are decreasing sequences (otherwise, use the sequences defined by $\tilde{G}_{1,k} = \bigcap_{n=1}^{k} G_{1,n}$ and $\tilde{G}_{2,k} = \bigcap_{n=1}^{k} G_{2,n}$). Define $G = \bigcap_{k=1}^{\infty} (G_{1,k} \cup G_{2,k})$. Then $G_1 \cup G_2 \subseteq G$, so G is a G_δ set containing $E_1 \cup E_2$. We also have

$$G \subseteq \limsup G_{1,n} \cup \limsup G_{2,n} = G_1 \cup G_2$$

where the equality follows because the sequences are decreasing. Therefore $G = G_1 \cup G_2$, so

$$\begin{aligned} m^*(G \sim (E_1 \cup E_2)) &= m^*((G_1 \cup G_2) \sim (E_1 \cup E_2)) \\ &\leq m^*(G_1 \sim (E_1 \cup E_2)) + m^*(G_2 \sim (E_1 \cup E_2)) \\ &\leq m^*(G_1 \sim E_1) + m^*(G_2 \sim E_2) \\ &= 0 \end{aligned}$$

Now suppose we can find F_σ sets F_1 and F_2 that satisfy

$$\begin{aligned} F_1 \subseteq E_1 &\text{ and } m^*(E_1 \sim F_1) = 0 \\ F_2 \subseteq E_2 &\text{ and } m^*(E_2 \sim F_2) = 0 \end{aligned}$$

Write $F_1 = \bigcup_{k=1}^\infty F_{1,k}$ and $F_2 = \bigcup_{k=1}^\infty F_{2,k}$, where $\{F_{1,k}\}_{k=1}^\infty$ and$\{F_{2,k}\}_{k=1}^\infty$ are countable collections of closed sets. Define $F = \bigcup_{k=1}^\infty (F_{1,k} \cup F_{2,k}) = F_1 \cup F_2$. Then F is an F_σ set contained in $E_1 \cup E_2$ and

$$\begin{aligned} m^*(E_1 \cup E_2 \sim F) &= m^*(E_1 \cup E_2 \sim (F_1 \cup F_2)) \\ &\leq m^*(E_1 \sim (F_1 \cup F_2)) + m^*(E_2 \sim (F_1 \cup F_2)) \\ &\leq m^*(E_1 \sim F_1) + m^*(E_2 \sim F_2) \\ &= 0 \end{aligned}$$

22. Let A denote a set of real numbers. Suppose $\{I_k\}_{k=1}^\infty$ is a countable collection of bounded, open intervals that contains A. Then

$$\sum_{k=1}^\infty \ell(I_k) \geq m^*\left(\bigcup_{k=1}^\infty I_k\right) \geq m^{**}(A)$$

so $m^*(A) \geq m^{**}(A)$.

If $m^{**}(A) = \infty$, then $m^{**}(A) \geq m^*(A)$ trivially. Suppose $m^{**}(A) < \infty$. Then for any ϵ, we can find an open set $\mathcal{O}$ containing A such that

$$m^{**}(A) > m^*(\mathcal{O}) - \epsilon \geq m^*(A) - \epsilon$$

Since this inequality holds for all ϵ, $m^{**}(A) \geq m^*(A)$. We therefore conclude that the two set functions are equal.

23. Claim 1: Suppose A is a bounded set of real numbers. Then $m^{***}(A) = m^*(A)$ if and only if A is measurable.

Proof: Suppose $m^{***}(A) = m^*(A)$. For any $\epsilon > 0$, we can find a closed set $F \subseteq A$ such that $m^{***}(A) < m^*(F) + \epsilon$. But the excision property then implies

$$m^*(A \sim F) = m^*(A) - m^*(F) = m^{***}(A) - m^*(F) < \epsilon$$

Since ϵ was arbitrary, we can apply Theorem 11(iii) to conclude that A is measurable.

Conversely, suppose A is measurable. Fix $\epsilon > 0$. By Theorem 11(iii), there exists a closed set F contained in A for which $m^*(A \sim F) < \epsilon$. Applying the excision property, we have

$$m^*(A) < m^*(F) + \epsilon \leq m^{***}(A) + \epsilon$$

Since this expression holds for all ϵ, $m^*(A) \leq m^{***}(A)$. Since $m^*(F) \leq m^*(A)$ for all $F \subseteq A$, we also have $m^{***}(A) \leq m^*(A)$. We therefore conclude that $m^*(A) = m^{***}(A)$.

Claim 2: Suppose A is an unbounded set of real numbers. Then $m^{***}(A \cap I) = m^*(A \cap I)$ for every bounded interval I if and only if A is measurable.

Proof: Suppose $m^{***}(A \cap I) = m^*(A \cap I)$ for every bounded interval I. Let $I_n = (-n, n)$. By Claim 1, $A \cap I_n$ is measurable for all n. Therefore $A = \bigcup_{n=1}^{\infty}(A \cap I_n)$ is measurable by Proposition 7.

Conversely, suppose A is measurable. Pick any bounded interval I. By Proposition 8, I is measurable. Because the set of measurable sets is closed under finite intersections, $A \cap I$ is measurable. Claim 1 then implies $m^{***}(A \cap I) = m^*(A \cap I)$.

2.5 Countable Additivity, Continuity, and the Borel-Cantelli Lemma

24. Using the identities

$$\begin{aligned} E_1 &= (E_1 \sim E_2) \cup (E_1 \cap E_2) \\ E_2 &= (E_2 \sim E_1) \cup (E_1 \cap E_2) \\ E_1 \cup E_2 &= (E_1 \sim E_2) \cup (E_2 \sim E_1) \cup (E_1 \cap E_2) \end{aligned}$$

and the finite additivity of Lebesgue measure, we obtain

$$\begin{aligned} m(E_1 \cup E_2) + m(E_1 \cap E_2) &= m(E_1 \sim E_2) + m(E_1 \cap E_2) + m(E_2 \sim E_1) + m(E_1 \cap E_2) \\ &= m(E_1) + m(E_2) \end{aligned}$$

25. Consider the sequence of sets $\{E_k\}_{k=1}^{\infty}$ defined by $E_k = (k, \infty)$. Then $m(E_k) = \infty$ for all k and $\bigcap_{k=1}^{\infty} E_k = \varnothing$, so

$$0 = m\left(\bigcap_{k=1}^{\infty} E_k\right) \neq \lim_{k \to \infty} m(E_k) = \infty$$

26. By countable subadditivity of m^*, we know that

$$m^*\left(A \cap \bigcup_{k=1}^{\infty} E_k\right) = m^*\left(\bigcup_{k=1}^{\infty}(A \cap E_k)\right) \leq \sum_{k=1}^{\infty} m^*(A \cap E_k)$$

If $m^*\left(A \cap \bigcup_{k=1}^{\infty} E_k\right) = \infty$, the reserve inequality holds trivially. If $m^*\left(A \cap \bigcup_{k=1}^{\infty} E_k\right) < \infty$, we can use the result from Problem 7 to find a G_δ set G that contains $A \cap \bigcup_{k=1}^\infty E_k$ and satisfies

$$m^*(G) = m^*\left(A \cap \bigcup_{k=1}^\infty E_k\right)$$

Since G is measurable, $\{G \cap E_k\}_{k=1}^\infty$ is a countable collection of measurable, disjoint sets. We therefore have

$$\begin{aligned}
m^*\left(A \cap \bigcup_{k=1}^\infty E_k\right) &= m^*(G) \\
&\geq m^*\left(G \cap \bigcup_{k=1}^\infty E_k\right) \\
&= m^*\left(\bigcup_{k=1}^\infty (G \cap E_k)\right) \\
&= \sum_{k=1}^\infty m^*(G \cap E_k) \\
&\geq \sum_{k=1}^\infty m^*(A \cap E_k)
\end{aligned}$$

where the inequalities follow from the monotonicity of m^*.

27. (i) Let $\{M_k\}_{k=1}^n$ be a finite collection of disjoint sets in $\mathcal{M}'$. Define $M_k = \varnothing$ for $k = n+1, n+2, \cdots$ and consider the countable collection $\{M_k\}_{k=1}^\infty$. Then $\bigcup_{k=1}^\infty M_k \in \mathcal{M}'$ and

$$m'\left(\bigcup_{k=1}^n M_k\right) = m'\left(\bigcup_{k=1}^\infty M_k\right) = \sum_{k=1}^\infty m'\left(M_k\right) = \sum_{k=1}^n m'\left(M_k\right) + \sum_{k=n+1}^\infty m'\left(M_k\right) = \sum_{k=1}^n m'\left(M_k\right)$$

where the second equality follows by countable additivity and the final equality follows because $m'(\varnothing) = 0$. Thus m' is finitely additive.

Let A and B be sets in $\mathcal{M}'$ satisfying $A \subseteq B$. Then $B \sim A \in \mathcal{M}'$ and

$$m'(B) = m'(A \cup (B \sim A)) = m'(A) + m'(B \sim A) \geq m'(A)$$

where the second equality follows from finite additivity and the inequality follows because $m'(B \sim A) \geq 0$. Thus m' is monotone.

Fix $M \in \mathcal{M}'$ and let $\{M_k\}_{k=1}^\infty$ be a countable collection of sets in $\mathcal{M}'$ that covers M. Define $\tilde{M}_0 = \varnothing$ and $\tilde{M}_k = M_k \sim \bigcup_{n=0}^{k-1} M_n$. Then $\tilde{M}_k \subseteq M_k$ for all k, $\bigcup_{k=1}^\infty M_k = \bigcup_{k=1}^\infty \tilde{M}_k$, and the collection of sets $\{\tilde{M}_k\}_{k=1}^\infty$ is pairwise disjoint. We therefore have

$$m'(M) \leq m'\left(\bigcup_{k=1}^\infty M_k\right) = m'\left(\bigcup_{k=1}^\infty \tilde{M}_k\right) = \sum_{k=1}^\infty m'\left(\tilde{M}_k\right) \leq \sum_{k=1}^\infty m'(M_k)$$

where the first inequality follows from monotonicity, the second equality follows from countable additivity, and the final inequality follows monotonicity.

Let A and B be sets in $\mathcal{M}'$ satisfying $A \subseteq B$. Suppose $m'(B) < \infty$. By finite additivity,

$$m'(B) = m'(A) + m'(B \sim A)$$

so $m'(B \sim A) = m'(B) - m'(A)$.

(ii) We can follow the proof of Theorem 15 in the text with m' replacing m.

28. Let $\mathcal{M}'$ be a σ-algebra of subsets of $\mathbf{R}$ and let m' be a set function on $\mathcal{M}'$ which takes values in $[0, \infty]$. Assume m' is finitely additive and that if $\{M_k\}_{k=1}^\infty$ is an ascending collection of sets in $\mathcal{M}'$, then

$$m'\left(\bigcup_{k=1}^\infty M_k\right) = \lim_{k\to\infty} m'(M_k)$$

Let $\{M_k\}_{k=1}^\infty$ be a collection of disjoint sets in $\mathcal{M}'$. Define $\tilde{M}_k = \bigcup_{n=1}^k M_n$. Then $\{\tilde{M}_k\}_{k=1}^\infty$ is an ascending collection of sets in $\mathcal{M}'$ and $\bigcup_{k=1}^\infty M_k = \bigcup_{k=1}^\infty \tilde{M}_k$. We therefore have

$$m'\left(\bigcup_{k=1}^\infty M_k\right) = m'\left(\bigcup_{k=1}^\infty \tilde{M}_k\right) = \lim_{k\to\infty} m'(\tilde{M}_k) = \lim_{k\to\infty} \sum_{n=1}^k m'(M_n) = \sum_{k=1}^\infty m'(M_k)$$

where the second equality follows from the continuity property of m' and the third equality follows from the finite additivity of m'.

2.6 Nonmeasurable Sets

29. Let $\sim$ denote the rational equivalence relation.

(i) Let x be a real number. Then $x - x = 0 \in \mathbf{Q}$, so $x \sim x$. Therefore $\sim$ is reflexive.

Suppose $x \sim y$. Then $x-y$ is a rational number, so $y-x = (-1)(x-y)$ is a rational number because -1 is a rational number and the rational numbers are closed under multiplication. Therefore $\sim$ is symmetric.

Suppose $x \sim y$ and $y \sim z$. Then $x - y$ and $y - z$ are rational numbers, so $x - z = x - y + (y - z)$ is a rational number because the rational numbers are closed under addition. Therefore $\sim$ is transitive.

(ii) Since the difference between any two rational numbers is rational, all the elements of $\mathbf{Q}$ belong to a single equivalence class. Thus the singleton set $\{q\}$ for any $q \in \mathbf{Q}$ is a valid choice set for $\sim$ on $\mathbf{Q}$.

(iii) Since x is not irrationally equivalent to itself, irrational equivalence is not reflexive. Therefore irrational equivalence is not an equivalence relation on $\mathbf{R}$ or $\mathbf{Q}$.

30. Let E be a nonempty set of real numbers and let $\mathcal{C}_E$ denote a choice set for the rational equivalence relation on E. For any $x \in \mathcal{C}_E$, let $E_x \subseteq E$ denote the equivalence class to which x belongs. Then

$$E_x \subseteq \{x + q : q \in \mathbf{Q}\}$$

for all $x \in \mathcal{C}_E$. Since $\mathbf{Q}$ is countable, E_x is countable by Theorems 3 and 5 of Chapter 1. If $\mathcal{C}_E$ is countable, then

$$E = \bigcup_{x \in \mathcal{C}_E} E_x$$

is countable by Corollary 6 of Chapter 1. A countable set has outer measure equal to zero (see page 31). Therefore if E has positive outer measure, $\mathcal{C}_E$ must be uncountably infinite.

31. It suffices to only consider bounded sets because if a set has positive outer measure, it contains a bounded subset with positive outer measure (Problem 14).

32. Define $E = (0, 1)$ and $\Lambda = \{1, 2\}$. Then E is a bounded, measurable set of real numbers, Λ is a finite set of real numbers, and $\{\lambda + E\}_{\lambda \in \Lambda}$ is disjoint. If we take $\Lambda = \mathbf{N}$, then Λ is a countably infinite, unbounded set of real numbers, and $\{\lambda + E\}_{\lambda \in \Lambda}$ is disjoint. However, in both cases $m(E) = 1 \neq 0$. Therefore the lemma does not remain true if Λ is allowed to be finite or unbounded.

    If Λ is uncountably infinite and bounded, we can find a countable subset of Λ that satisfies the conditions of the lemma. Thus the conclusion of the lemma still holds.

33. By Problem 7, we can find a G_δ set G that contains E and satisfies $m^*(G) = m^*(E)$. If $m^*(G \sim E) = 0$, then $G \sim E$ is measurable. But if $G \sim E$ is measurable, then $E = G \cap (G \sim E)^C$ is measurable. Therefore if E is not measurable, $m^*(G \sim E) > 0$.

2.7 The Cantor Set and the Cantor-Lebesgue Function

34. Consider the function $f(x) = \frac{\psi(x)}{2}$, where $\psi : [0, 1] \to [0, 2]$ is defined in Proposition 21. f is continuous and strictly increasing because it is the composition of two continuous, strictly increasing functions. The range of f is $[0, 1]$ by the Intermediate Value Theorem. Define $\mathcal{O} = [0, 1] \sim \mathbf{C}$. Through a minor adaptation of the argument in the proof of Proposition 21, we see that $f(\mathcal{O})$ and $f(\mathbf{C})$ are disjoint and $m(f(\mathcal{O})) = \frac{1}{2}$. But since $[0, 1] = f([0, 1]) = f(\mathcal{O}) \cup f(\mathbf{C})$, we must have $m(f(\mathbf{C})) = \frac{1}{2}$ by the additivity of m. Therefore the inverse of f is a continuous, strictly increasing function defined on $[0, 1]$ that maps $f(\mathbf{C})$ (a set of positive measure) onto the Cantor set (a set of measure zero).

35. Suppose f is continuous at $x_0 \in I$. Since I is open, there exists $r > 0$ such that $(x_0 - r, x_0 + r) \subseteq I$. Pick sequences $b_n \in (x_0, x_0 + r)$ and $a_n \in (x_0 - r, x_0)$ that converge to x_0. Then $a_n < x_0 < b_n$, the sequences $\{a_n\}$ and $\{b_n\}$ are in I, and $\lim_{n\to\infty}[f(b_n) - f(a_n)] = 0$ by Proposition 21 and Theorem 18 of Chapter 1.

    Conversely, suppose there are sequences $\{a_n\}$ and $\{b_n\}$ in I such that for each n, $a_n < x_0 < b_n$ and $\lim_{n\to\infty}[f(b_n) - f(a_n)] = 0$. Fix $\epsilon > 0$ and choose n such that $|f(b_n) - f(a_n)| < \epsilon$. Define $\delta = \min\{x_0 - a_n, b_n - x_0\}$. If $|x - x_0| < \delta$, then

    $$a_n < x_0 - \delta < x < x_0 + \delta < b_n$$

    which implies

    $$f(a_n) \leq f(x) \leq f(b_n)$$

because f is increasing. But this means

$$f(a_n) - f(b_n) \leq f(x) - f(x_0) \leq f(b_n) - f(a_n)$$

which implies

$$|f(x) - f(x_0)| \leq |f(b_n) - f(a_n)| < \epsilon$$

36. Define $\mathcal{O} = [0,1] \sim \mathbf{C}$ and fix any point $x \in (0,1) \cap \mathbf{C}$. Then $\varphi(t) = f(t) \leq f(x)$ for any $t \in \mathcal{O} \cap [0,x)$. Therefore $\varphi(x) \leq f(x)$. To prove the reverse inequality, fix $\epsilon > 0$. Since φ is continuous, there exists a $\delta > 0$ such that $\varphi(t) < \varphi(x) + \epsilon$ for all $t \in [0,1]$ satisfying $|x - t| < \delta$. Since $m(\mathbf{C}) = 0$, $\mathbf{C}$ cannot contain the open interval $(x, x+\delta)$. Therefore there exists $t \in \mathcal{O} \cap (x, x+\delta)$ such that

$$f(x) \leq f(t) = \varphi(t) < \varphi(x) + \epsilon$$

But since $f(x) \leq \varphi(x) + \epsilon$ for arbitrary positive ϵ, $f(x) \leq \varphi(x)$.

Note that under the given assumptions, f may not agree with φ at 0 or 1.

37. The strictly increasing, continuous function function ψ defined in Proposition 21 maps a measurable set onto a non-measurable set. Since ψ is one-to-one, the inverse of ψ is a continuous function that maps a non-measurable set onto a measurable set. Thus the pre-image of a measurable set need not be measurable.

38. Let $E \subseteq [a,b]$ be a non-empty set of measure zero. Fix $\epsilon > 0$. By Theorem 11(a), there exists an open set $\mathcal{O}$ containing E for which $m(\mathcal{O}) = m(\mathcal{O} \sim E) < \frac{\epsilon}{c}$. By Proposition 9 of Chapter 1, we can write $\mathcal{O}$ as the union of a countable collection of pairwise disjoint open intervals $\{I_k\}_{k=1}^{\infty}$. For any interval I_k, define $I'_k = I_k \cap [a,b]$, $a_k = \inf_{u \in I'_k} f(u)$ and $b_k = \sup_{u \in I'_k} f(u)$. We then have

$$m(f(I'_k)) \leq b_k - ak_k = \sup_{u,v \in I'_k} |f(u) - f(v)| \leq c \sup_{u,v \in I'_k} |u - v| \leq c\,\ell(I_k)$$

where the first inequality follows because $f(I'_k) \subseteq (a_k, b_k)$ and the second inequality follows from the Lipschitz condition. We therefore have

$$\begin{aligned}
m(f(E)) &\leq m(f(\mathcal{O} \cap [a,b])) \\
&= m\left(f\left(\bigcup_{k=1}^{\infty} I'_k\right)\right) \\
&= m\left(\bigcup_{k=1}^{\infty} f\left(I'_k\right)\right) \\
&\leq \sum_{k=1}^{\infty} m(f(I'_k)) \\
&\leq c \sum_{k=1}^{\infty} \ell(I_k) \\
&= c\, m(\mathcal{O}) \\
&< \epsilon
\end{aligned}$$

Since this expression holds for any positive ϵ, we must have $m(f(E)) = 0$.

Before proceeding to the next step, I prove a preliminary lemma.

Claim: A continuous image of a closed and bounded set of real numbers is also closed and bounded.

Proof: Let A be a closed and bounded set of real numbers and let f be a continuous function on A. Let $\{E_k\}_{k=1}^{\infty}$ be an open cover of $f(A)$. By Proposition 22 of Chapter 1, $f^{-1}(E_k) = A \cap \mathcal{U}_k$ where $\mathcal{U}_k$ is an open set. Then

$$\bigcup_{k=1}^{\infty} f^{-1}(E_k) = f^{-1}\left(\bigcup_{k=1}^{\infty} E_k\right) \supseteq f^{-1}(f(A)) \supseteq A$$

Therefore $\{\mathcal{U}_k\}_{k=1}^{\infty}$ is an open cover of A. By the Heine-Borel Theorem, $\{\mathcal{U}_k\}_{k=1}^{\infty}$ contains a finite subcover $\{\mathcal{U}_k\}_{k=1}^{N}$. But then

$$f(A) = f\left(\bigcup_{k=1}^{N} (A \cap \mathcal{U}_k)\right) = f\left(f^{-1}\left(\bigcup_{k=1}^{N} E_k\right)\right) \subseteq \bigcup_{k=1}^{N} E_k$$

Therefore $\{E_k\}_{k=1}^{N}$ is a finite cover of $f(A)$. Since every open cover of $f(A)$ contains a finite subcover, $f(A)$ is closed and bounded by Problem 53 of Chapter 1.

Now suppose $E \subseteq [a, b]$ is an F_σ set. Then there exist a countable collection of closed sets $\{F_k\}_{k=1}^{\infty}$ such that $E = \bigcup_{k=1}^{\infty} F_k$. By the preliminary lemma and the fact that Lipschitz functions are continuous (see page 25), $f(F_k)$ is closed and bounded for all k. Thus $f(E) = \bigcup_{k=1}^{\infty} f(F_k)$ is an F_σ set.

Now suppose $E \subseteq [a, b]$ is measurable. By Theorem 11(iv), we can find an F_σ set F contained in E for which $m(E \sim F) = 0$. But since $f(E) = f(F) \cup f(E \sim F)$ and both $f(F)$ and $f(E \sim F)$ are measurable, $f(E)$ is measurable.

39. Let F_k denote the set of points that remain after k removal operations. F_k is the union of 2^k disjoint closed intervals, each of length $l_k := 2^{-k}\left(1 - \alpha + \alpha\left(\frac{2}{3}\right)^k\right)$. Since a finite union of closed sets is closed, each F_k is closed. Since an intersection of closed sets is closed and $F = \bigcap_{k=1}^{\infty} F_k$, F is closed. Define $\mathcal{O} = [0, 1] \sim F$ and pick $x, y \in [0, 1]$. Since $\mathcal{O}$ is open, if either x or y belongs to $\mathcal{O}$ we can find a point between x and y which also belongs to $\mathcal{O}$. So suppose both x and y belong to F. Choose $k \in \mathbf{N}$ such that $l_k < |x - y|$. Write $F_k = \bigcup_{n=1}^{2^k} I_n$, where each I_n is a closed interval of length l_k. Since x and y both belong to F_k, they must belong to one of the intervals in $\{I_n\}_{n=1}^{2^k}$. However x and y cannot belong to the same interval since $l_k < |x - y|$. Since the intervals are disjoint, there must exist a point between x and y which is not in F_k and therefore not in F. Thus $\mathcal{O}$ is dense in $[0, 1]$. Finally, observe that $\mathcal{O}$ is the countable union of the disjoint collection of open intervals which are removed during the construction of F. At the k-th deletion stage, 2^{k-1} intervals of length $\alpha 3^{-k}$ are removed. Therefore the length of $\mathcal{O}$ is given by

$$m(\mathcal{O}) = \frac{\alpha}{2} \sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^k = \alpha$$

which implies $m(F) = 1 - \alpha$ by the excision property.

40. Let F denote a generalized Cantor set (see Problem 39) of measure $1 - \alpha$. Define $\mathcal{O} = [0, 1] \sim F$. Since $\mathcal{O}$ is open, it does not contain any of its boundary points. But since $\mathcal{O}$ is dense in $[0, 1]$, every point in F is a boundary point of $\mathcal{O}$. Thus the boundary of $\mathcal{O}$ equals F and therefore has measure $1 - \alpha$.

41. Suppose $x \in \mathbf{C}$ and fix $\epsilon > 0$. Choose k such that $3^{-k} < \epsilon$ and let C_k denote the points that remain after k removal operations. C_k is the union of a pairwise disjoint collection of 2^k closed intervals, each of which has length 3^{-k}. Since $x \in C_k$, there exists an interval $I = [a, b]$ in this collection that contains x. The endpoints of the intervals in the collection are never removed in the construction of the Cantor set, so both a and b belong to $\mathbf{C}$. Furthermore, both a and b belong to $(x - \epsilon, x + \epsilon)$ since $I \subseteq (x - \epsilon, x + \epsilon)$. But x cannot equal both a and b, so one of the endpoints is in the set $\mathbf{C} \cap (x - \epsilon, x + \epsilon) \sim \{x\}$. This process can be repeated for all $k' > k$ to generate an infinite collection of points in $\mathbf{C} \cap (x - \epsilon, x + \epsilon)$. Since ϵ was arbitrary and $\mathbf{C}$ is closed, $\mathbf{C}$ is perfect.

42. Suppose X is countable. Let $\{x_k\}_{k=1}^{\infty}$ denote an enumeration of X. Define $n_1 = 1$ and $U_1 = (x_{n_1} - 1, x_{n_1} + 1)$. Since X is a perfect set, $U_1 \cap X$ is infinite. Therefore we can define $n_2 = \min\{k : x_k \in U_1 \sim \{x_{n_1}\}\}$. Let U_2 be an open interval satisfying

$$\begin{aligned} x_{n_2} &\in U_2 \\ x_{n_1} &\notin \bar{U}_2 \\ \bar{U}_2 &\subset U_1 \end{aligned}$$

By continuing in this fashion, we obtain a sequence of sets $\{U_n\}_{n=1}^{\infty}$ that satisfies

$$\begin{gathered} x_k \notin \bar{U}_{n+1} \text{ for } k = 1, \cdots, n \qquad (1) \\ U_n \cap X \neq \varnothing \text{ for } n \in \mathbf{N} \\ \bar{U}_{n+1} \subset U_n \text{ for } n \in \mathbf{N} \end{gathered}$$

Let $E_n = \bar{U}_n \cap X$. Then $\{E_n\}_{n=1}^{\infty}$ is a decreasing sequence of non-empty closed sets, and E_1 is bounded. Therefore by the Nested Set Theorem, $\bigcap_{n=1}^{\infty} E_n$ is non-empty. Since $\bigcap_{n=1}^{\infty} E_n \subseteq X$, $x_k \in \bigcap_{n=1}^{\infty} E_n$ for some k. But then $x_k \in \bar{U}_{k+1}$, a contradiction with the condition in (1).

43. This conclusion follows immediately from Problems 41 and 42.

44. Observe that $\mathbf{C}$ cannot contain an open interval. For if we could find an open interval $I \subseteq \mathbf{C}$, then $m(\mathbf{C}) \geq m(I) > 0$ by the monotonicity of measure. However this expression contradicts the result from Proposition 19 that $m(\mathbf{C}) = 0$.

    Now fix any interval I. By the previous argument, there must exist $x \in I \sim \mathbf{C}$. Since $\mathbf{C}$ is closed, $I \sim \mathbf{C}$ is open. Therefore, we can find an interval I_x that contains x and is contained in $I \sim \mathbf{C}$. But then I_x is an open interval contained in I that is disjoint from $\mathbf{C}$. Since any open set contains an open interval, the proof is complete.

45. Suppose A is an interval and f is a strictly increasing function defined on A. Define $B = f(A)$. Since f is strictly monotone, it defines a one-to-one correspondence between the sets A and B. Therefore the inverse function $f^{-1} : B \to A$ is well-defined.

    Now fix $y \in B$ and define the function

$$f_y(\alpha) = f\left(f^{-1}(y) + \alpha\right)$$

    for $\alpha \in A - f^{-1}(y)$. Note that $f_y(0) = y$, f_y is strictly increasing in α, and $f_y^{-1}(y') = f^{-1}(y') - f^{-1}(y)$

for $y' \in B$.

Fix $\epsilon > 0$. If $f^{-1}(y)$ is on the interior of A, we can find $\epsilon' \in (0, \epsilon)$ such $-\epsilon', \epsilon' \in A - f^{-1}(y)$. Define

$$\delta = \min\{y - f_y(-\epsilon'), f_y(\epsilon') - y\}$$

If $f^{-1}(y)$ is a lower (upper) bound of A, choose $\epsilon' \in (0, \epsilon)$ such that $\epsilon' \in A - f^{-1}(y)$ ($-\epsilon' \in A - f^{-1}(y)$) and define $\delta = f_y(\epsilon') - y$ ($\delta = y - f_y(-\epsilon')$). Now pick $y' \in B$ satisfying $|y - y'| < \delta$. If $y' \geq y$, then

$$f_y(f_y^{-1}(y')) - y = |y - y'| < \delta \leq f_y(\epsilon') - y \quad \Longrightarrow \quad f_y^{-1}(y') < \epsilon$$

If $y' < y$, then

$$y - f_y(f_y^{-1}(y')) = |y - y'| < \delta \leq y - f_y(-\epsilon') \quad \Longrightarrow \quad f_y^{-1}(y') > -\epsilon$$

Therefore $|f_y^{-1}(y')| = |f^{-1}(y') - f^{-1}(y)| < \epsilon$.

46. Define

$$\mathcal{F} = \{E \subseteq \mathbf{R} : f^{-1}(E) \in \mathcal{B}\}$$

Suppose $\mathcal{O}$ is an open set. Then $f^{-1}(\mathcal{O})$ is open (and thus Borel) by Proposition 22 of Chapter 1. Therefore $\mathcal{O} \in \mathcal{F}$.

Next observe that $\mathcal{F}$ is a σ-algebra:

- $f^{-1}(\varnothing) = \varnothing \in \mathcal{B}$, so $\varnothing \in \mathcal{F}$.
- Suppose $E \in \mathcal{F}$. Then $f^{-1}(E^C) = f^{-1}(E)^C \in \mathcal{B}$, since $f^{-1}(E)$ is in $\mathcal{B}$ and $\mathcal{B}$ is closed under complements. Therefore $E^C \in \mathcal{F}$.
- Suppose $E_k \in \mathcal{F}$ for $k \in \mathbf{N}$. Then $f^{-1}\left(\bigcup_{k=1}^{\infty} E_k\right) = \bigcup_{k=1}^{\infty} f^{-1}(E_k) \in \mathcal{B}$, since $f^{-1}(E_k)$ is in $\mathcal{B}$ for all k and $\mathcal{B}$ is closed under countable unions. Therefore $\bigcup_{k=1}^{\infty} E_k \in \mathcal{B}$.

Since $\mathcal{B}$ is the smallest σ-algebra containing the open sets, we must have $\mathcal{B} \subseteq \mathcal{F}$. But this implies $f^{-1}(B) \in \mathcal{B}$ for all $B \in \mathcal{B}$.

47. Let f be a continuous, strictly increasing function defined on an interval A. Let g denote the inverse of f. By Problem 45, g is a continuous function. Therefore if B is a Borel set contained in the range of g, $g^{-1}(B) \in \mathcal{B}$ by Problem 46. But $g^{-1}(B) = \{y : g(y) \in B\} = \{f(x) : x \in B\} = f(B)$, so $f(B)$ is a Borel set.

3 Lebesgue Measurable Functions

3.1 Sums, Products, and Compositions

1. Suppose $f(x) \neq g(x)$ for some $x \in [a, b]$. Then the set

$$E := \{x \in [a, b] : f(x) \neq g(x)\} = (f - g)^{-1}((-\infty, 0) \cup (0, \infty))$$

is non-empty. Since f and g are continuous functions, $f - g$ is continuous. Therefore E must be open by Proposition 22 of Chapter 1. But then E must contain an open interval and therefore have non-zero measure. Since $\{x \in [a, b] : f(x) \neq g(x)\}$ cannot be contained in a set of measure zero, f cannot equal g a.e. on $[a, b]$.

To see that the statement does not hold for an arbitrary measurable set, suppose f is a continuous function defined over a set E of measure zero. Let $g(x) = f(x) + 1$. Then $f(x) \neq g(x)$ for all $x \in E$, but $f = g$ a.e. on E.

2. If f is continuous on $D \cup E$, then f is continuous on its restrictions to D and E. However, the converse is false. For example, suppose $D = [0, 1)$, $E = [1, 2]$, define f as

$$f(x) = \begin{cases} 1 & \text{if } 0 \leq x < 1 \\ 2 & \text{if } 1 \leq x \leq 2 \end{cases}$$

3. Let E denote the measurable domain of f and define

$$E_0 = \{x \in E : f \text{ is not continuous at } x\}$$

Since E_0 is finite, $m(E_0) = 0$ and f is measurable on E_0. By Proposition 3, f is measurable on $E \sim E_0$. We conclude from Proposition 5(ii) that f is measurable on E.

4. Let E denote a non-measurable subset of $(0, 1)$. We know such a set exists from Theorem 17 of Chapter 2. Consider the function f defined as

$$f(x) = e^x \cdot (2\chi_E - 1)$$

where χ_E is the characteristic function of the set E. Then $\{x \in \mathbf{R} : f(x) > 0\} = E$ is not a measurable set. However f is one-to-one, so $f^{-1}(c)$ is either empty or a singleton set and therefore measurable.

5. Fix $c \in \mathbf{R}$. Then

$$\{x \in E : f(x) > c\} = \bigcup_{q \in \mathbf{Q} \,:\, q > c}^{\infty} \{x \in E : f(x) > q\}$$

Since $\{q \in \mathbf{Q} : q > c\}$ is a countable set and each $\{x \in E : f(x) > q\}$ is measurable, the above expression implies $\{x \in E : f(x) > c\}$ is a measurable set.

6. From Proposition 5(ii), we know that g is measurable on $\mathbf{R}$ if and only if the restrictions of g to D and $\mathbf{R} \sim D$ are measurable. But since the restriction of g to D is f and the restriction of g to $\mathbf{R} \sim D$ is always measurable, the result follows immediately.

7. By Proposition 2, f is measurable if and only if for each open set $\mathcal{O}$, $f^{-1}(\mathcal{O})$ is measurable. If $f^{-1}(A)$ is measurable for each Borel set A, then f is measurable because all open sets are Borel sets. Conversely, suppose f is measurable. Define $\mathcal{F}$ as

$$\mathcal{F} = \{A \subseteq \mathbf{R} : f^{-1}(A) \text{ is measurable}\}$$

Then $\mathcal{F}$ is a σ-algebra:

- Since $f^{-1}(\mathbf{R}) = E$ is measurable, $\mathbf{R} \in \mathcal{F}$.
- Suppose $A \in \mathcal{F}$. Then $f^{-1}(A^C) = (f^{-1}(A))^C$ is measurable because $f^{-1}(A)$ is measurable and the collection of measurable sets is closed under complementation.
- Suppose $\{A_n\}_{n=1}^{\infty}$ is a countable collection of sets satisfying $A_n \in \mathcal{F}$ for all n. Then $f^{-1}(\bigcup_{n=1}^{\infty} A_n) = \bigcup_{n=1}^{\infty} f^{-1}(A_n)$ is measurable because $f^{-1}(A_n)$ is measurable for all n and the collection of measurable sets is closed under countable unions.

Since all open sets are in $\mathcal{F}$ and $\mathcal{B}$ is the smallest σ-algebra containing the Borel sets, we must have $\mathcal{B} \subseteq \mathcal{F}$. But this implies $f^{-1}(A)$ is measurable for each Borel set A.

8. The proof of Proposition 1 only relies on the fact that the collection of Lebesgue measurable sets is a σ-algebra. Since the collection of Borel sets is also a σ-algebra, the statement remains valid. The proof of Theorem 6 also remains valid if we define a Borel measurable function f on a Borel set E to be finite almost everywhere if there exists a *Borel* set $B_0 \subseteq E$ of measure zero that satisfies

$$\{x \in E : f \text{ is finite}\} \supseteq E \sim B_0$$

(This slight modification is necessary because not all sets of measure zero are Borel sets.)

(i) This statement follows immediately from the fact that all Borel sets are Lebesgue measurable.

(ii) This statement follows from the fact that the collection of sets $\{A \subseteq \mathbf{R} : f^{-1}(A) \text{ is a Borel set}\}$ is a σ-algebra containing the open sets.

(iii) Observe that for any real number c, we have

$$(f \circ g)^{-1}((c, \infty)) = g^{-1}(f^{-1}((c, \infty)))$$

Since (c, ∞) is a Borel set, $g^{-1}(f^{-1}((c, \infty)))$ is a Borel set by part (ii). Therefore $f \circ g$ is a Borel measurable function.

(iv) Since f is Borel measurable, $f^{-1}((c, \infty))$ is a Borel set for any real number c. By Problem 7, $(f \circ g)^{-1}(-\infty, c) = g^{-1}(f^{-1}((-\infty, c))) = g^{-1}(f^{-1}((c, \infty)))$ is a Lebesgue measurable set. Therefore $f \circ g$ is a Lebesgue measurable function.

9. Since $|f_n(x) - f_m(x)|$ is a measurable function by Theorem 6 and Proposition 7, each set of the form $\left\{x \in E : |f_n(x) - f_m(x)| < \frac{1}{k}\right\}$ is measurable. Because a sequence of points converges if and only if it is Cauchy (Theorem 17, Chapter 1), we have

$$E_0 = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n,m \geq N} \left\{x \in E : |f_n(x) - f_m(x)| < \frac{1}{k}\right\}$$

Since the collection of measurable sets is a σ-algebra, E_0 is measurable.

10. See the counter-example on page 57.

11. Let h denote the inverse of g. For any open set $\mathcal{O}$, we have

$$(f \circ g)^{-1}(\mathcal{O}) = g^{-1}\left(f^{-1}(\mathcal{O})\right) = \{x : g(x) \in f^{-1}(\mathcal{O})\} = \{h(y) : y \in f^{-1}(\mathcal{O})\} = h\left(f^{-1}(\mathcal{O})\right)$$

Since f is measurable and h is Lipschitz, $f^{-1}(\mathcal{O})$ and $h(f^{-1}(\mathcal{O}))$ are measurable sets by Proposition 3 and Problem 38 of Chapter 2. Thus $(f \circ g)^{-1}(\mathcal{O})$ is measurable, so $f \circ g$ is a measurable function.

3.2 Sequential Pointwise Limits and Simple Approximation

12. For any natural number n, we can use the Simple Approximation Lemma to construct simple functions φ_n and ψ_n such that

$$\varphi_n \leq f \leq \psi_n \text{ and } 0 \leq \psi_n - \varphi_n < \frac{1}{n} \text{ on } E$$

Fix $\epsilon > 0$ and pick $N > \frac{1}{\epsilon}$. Then

$$|f - \varphi_n| = f - \varphi_n \leq \psi_n - \varphi_n < \frac{1}{n} \leq \frac{1}{N} < \epsilon \text{ on } E \text{ for all } n \geq N$$

and

$$|f - \psi_n| = \psi_n - f \leq \psi_n - \varphi_n < \frac{1}{n} \leq \frac{1}{N} < \epsilon \text{ on } E \text{ for all } n \geq N$$

Thus the sequences $\{\varphi_n\}$ and $\{\psi_n\}$ converge to f uniformly on E. By replacing φ_n with $\max\{\varphi_1, \cdots, \varphi_n\}$ and ψ_n with $\min\{\psi_1, \cdots, \psi_n\}$, we have $\{\varphi_n\}$ increasing and $\{\psi_n\}$ decreasing.

13. For any natural number n and integer k, define

$$E_{n,k} = \left\{ x \in E : f(x) \in \left(\frac{k}{n}, \frac{k+1}{n} \right] \right\}$$

Since the function f is bounded on $E_{n,k}$, we can use the Simple Approximation Lemma to find a simple function $\varphi_{n,k}$ such that

$$|f - \varphi_{n,k}| < \frac{1}{n} \text{ on } E_{n,k}$$

Extend each $\varphi_{n,k}$ to E by setting $\varphi_{n,k}(x) = 0$ if $x \notin E_{n,k}$. Let $\{k_i\}$ denote an enumeration of the integers and define

$$\varphi_n = \sum_{i=1}^{\infty} \varphi_{n,k_i}$$

The range of φ_n is a countable union of finite sets of real numbers and is therefore countable (see Corollary 6 of Chapter 1). Since $\varphi_n(x) = \lim_{N\to\infty} \sum_{i=1}^{N} \varphi_{n,k_i}(x)$, φ_n is the pointwise limit of a sequence of measurable functions and therefore measurable by Proposition 9. We can conclude that φ_n is a semisimple function on E.

Now fix $\epsilon > 0$. Choose $x \in E$ and $N > \frac{1}{\epsilon}$. For any $n \geq N$, there exists k such that $\frac{k}{n} < f(x) \leq \frac{k+1}{n}$. But this means $x \in E_{k,n}$ and $x \notin E_{k',n}$ for all $k' \neq k$, so

$$|f(x) - \varphi_n(x)| = |f(x) - \varphi_{n,k}(x)| < \frac{1}{n} \leq \frac{1}{N} < \epsilon$$

Therefore φ_n converges uniformly to f on E.

14. Define

$$E_n = \{x \in E : |f(x)| < n\}$$

f is bounded on each E_n and each E_n is a measurable subset of E. Since f is finite a.e. on E, we know that

$$m\left(E \sim \bigcup_{n=1}^{\infty} E_n\right) = m\left(\{x \in E : f(x) \text{ is not finite}\}\right) = 0$$

Since $m(E \sim E_1) \leq m(E) < \infty$ and $E \sim E_1 \supseteq E \sim E_2 \supseteq \cdots$, Theorem 15(ii) of Chapter 2 implies

$$0 = m\left(E \sim \bigcup_{n=1}^{\infty} E_n\right) = m\left(\bigcap_{n=1}^{\infty} (E \sim E_n)\right) = \lim_{n \to \infty} m(E \sim E_n)$$

Thus for any $\epsilon > 0$, we can find an n such that $m(E \sim E_n) < \epsilon$.

15. By Problem 14, for any $\epsilon > 0$ we can find a measurable set F contained in E such that f is bounded on F and $m(E \sim F) < \epsilon$. By Problem 12, we can find a sequence of measurable functions $\{\varphi_n\}$ on F that converges uniformly to f on F. Each simple function can be extended to E by defining $\varphi_n(x) = 0$ if $x \in E \sim F$.

16. By Theorem 12 of Chapter 2, there exists a finite disjoint collection of open intervals $\{I_k\}$ such that $m\left(E \Delta \bigcup_{k=1}^{n} I_k\right) < \epsilon$. Define $\mathcal{O} = \bigcup_{k=1}^{n} I_k$, $h = \chi_{\mathcal{O}} = \sum_{k=1}^{n} \chi_{I_k}$ and

$$F = I \sim (E \Delta \mathcal{O}) = I \cap ((E \cap \mathcal{O}) \cup (E^c \cap \mathcal{O}^c))$$

If $x \in I \cap \mathcal{O} \cap E$, then $h(x) = 1 = \chi_E(x)$; if $x \in I \cap \mathcal{O}^c \cap E^c$, then $h(x) = 0 = \chi_E(x)$. Thus h is a step function on I for which $h = \chi_E$ on F and $m(I \sim F) = m(E \Delta \mathcal{O}) < \epsilon$.

17. Let $\sum_{k=1}^{n} c_k \chi_{E_k}$ denote the canonical representation of ψ. By Problem 16, we know that for each E_k we can find a step function h_k defined on I and a measurable subset F_k of I for which $h_k = \chi_{E_k}$ on F_k and $m(I \sim F_k) < \frac{\epsilon}{n}$. Let $h = \sum_{k=1}^{n} c_k h_k$ and $F = \bigcap_{k=1}^{n} F_k$. If $x \in F$, then $h_k(x) = \chi_{E_k}(x)$ for $k = 1, \cdots, n$. Thus $h(x) = \sum_{k=1}^{n} c_k h_k(x) = \sum_{k=1}^{n} c_k \chi_{E_k}(x) = \psi(x)$, so $h = \psi$ on F. We also know that $m(I \sim F) = m\left(\bigcup_{k=1}^{n} (I \sim F_k)\right) \leq \sum_{k=1}^{n} m(I \sim F_k) < \epsilon$ by the subadditivity of Lebesgue measure.

18. By the Simple Approximation Lemma, we can find a simple function ψ defined on E such that $|\psi - f| < \epsilon$ on E. By Problem 17, we can find a step function h on I and a measurable subset F of I for which $h = \psi$ on F and $m(I \sim F) < \epsilon$.

19. Let φ and ψ be two simple functions and let n and m denote the finite number of distinct values each function takes, respectively. Then nm is the maximum number of distinct values $\varphi + \psi$, $\varphi\psi$, $\min\{\varphi, \psi\}$ or $\max\{\varphi, \psi\}$ can take. Since these compositions are measurable by Theorem 6 and Proposition 8, they are simple functions.

20. If $x \in A \cap B$, then $\chi_{A \cap B}(x) = \chi_A(x) = \chi_B(x) = 1$. If $x \notin A \cap B$, then $\chi_{A \cap B}(x) = 0$ and either $\chi_A(x) = 0$ or $\chi_B(x) = 0$. In either case, $\chi_{A \cap B}(x) = \chi_A(x)\chi_B(x)$.

    Suppose A and B are disjoint. If $x \in A$, then $\chi_{A \cup B}(x) = 1 = \chi_A(x)$ and $\chi_B(x) = 0$. If $x \in B$, then $\chi_{A \cup B}(x) = 1 = \chi_B(x)$ and $\chi_A(x) = 0$. If $x \notin A \cup B$, then $\chi_{A \cup B}(x) = 0 = \chi_A(x) = \chi_B(x)$. In each case, $\chi_{A \cup B}(x) = \chi_A(x) + \chi_B(x)$. If A and B are not disjoint, decompose $A \cup B$ into the disjoint sets

$A \cap B^C$, $A^C \cap B$, and $A \cap B$. Then using our result for disjoint sets, we have

$$\chi_{A \cup B} = \chi_{A \cap B^C} + \chi_{B \cap A^C} + \chi_{A \cap B} =$$
$$(\chi_{A \cap B^C} + \chi_{A \cap B}) + (\chi_{A^C \cap B} + \chi_{A \cap B}) - \chi_{A \cap B} = \chi_A + \chi_B - \chi_{A \cap B}$$

where the final equality follows because $A = (A \cap B) \cup (A \cap B^C)$ and $B = (A^C \cap B) \cup (A \cap B)$.

Since A and A^C are disjoint and $\chi_{A \cup A^C} = 1$, we have $\chi_A + \chi_{A^C} = \chi_{A \cup A^C} = 1$. Thus $\chi_{A^C} = 1 - \chi_A$.

21. Fix $c \in \mathbf{R}$ and suppose $\inf_n f_n(x) < c$. Then there exists n such that $f_n(x) < c$, for otherwise c would be a lower bound of $\{f_n(x)\}$ that exceeds $\inf_n f_n(x)$. Thus $x \in \bigcup_{n=1}^{\infty} \{x \in E : f_n(x) < c\}$. Now suppose $f_n(x) < c$ for some n. Then $\inf_n f_n(x) < c$ because $\inf_n f_n(x)$ is a lower bound of $\{f_n(x)\}$. Therefore $x \in \{x \in E : \inf_n f_n(x) < c\}$. We conclude that

$$\left\{x \in E : \inf_n f_n(x) < c\right\} = \bigcup_{n=1}^{\infty} \{x \in E : f_n(x) < c\}$$

Each set $\{x \in E : f_n(x) < c\}$ is measurable because each function f_n is measurable. Since the collection of measurable sets is closed under countable unions, the above expression implies $\{x \in E : \inf_n f_n(x) < c\}$ is measurable.

Now $\sup_n f_n = -\inf_n(-f_n)$, so $\sup_n f_n$ is measurable by Theorem 6. Since $\liminf_n f_n = \lim_n \inf_{k \geq n} f_k$ and $\limsup_n f_n = \lim_n \sup_{k \geq n} f_k$, $\liminf_n f_n$ and $\limsup_n f_n$ are measurable by Proposition 9.

22. Fix $\epsilon > 0$. For each natural number n, define

$$E_n = (-\infty, a) \cup \{x \in [a, b] \mid f(x) - f_n(x) < \epsilon\} \cup (b, \infty)$$

Suppose $x \in E_n$. If $x \in (-\infty, a) \cup (b, \infty)$, then we can find a $\delta > 0$ such that $(x - \delta, x + \delta) \subseteq (-\infty, a) \cup (b, \infty)$. If $x \in \{x \in [a, b] \mid f(x) - f_n(x) < \epsilon\}$, pick $\tilde{\epsilon}$ such that $f(x) - f_n(x) < \tilde{\epsilon} < \epsilon$. Since f and f_n are continuous on $[a, b]$, there exist $\tilde{\delta}, \tilde{\delta}_n > 0$ such that

$$\text{If } x' \in [a, b] \text{ and } |x - x'| < \tilde{\delta}, \text{ then } |f(x) - f(x')| < \frac{\epsilon - \tilde{\epsilon}}{2}$$
$$\text{If } x' \in [a, b] \text{ and } |x - x'| < \tilde{\delta}_n, \text{ then } |f_n(x) - f_n(x')| < \frac{\epsilon - \tilde{\epsilon}}{2}$$

Define $\delta = \min\left\{\tilde{\delta}, \tilde{\delta}_n\right\}$ and suppose $|x - x'| < \delta$. If $x' \notin [a, b]$, then $x' \in (-\infty, a) \cup (b, \infty) \subseteq E_n$. If $x' \in [a, b]$, then

$$f(x') - f_n(x') = f(x') - f(x) + f(x) - f_n(x) + f_n(x) - f_n(x') \leq$$
$$|f(x') - f(x)| + f(x) - f_n(x) + |f_n(x) - f_n(x')| < \epsilon$$

so that $x' \in \{x \in [a, b] \mid f(x) - f_n(x) < \epsilon\} \subseteq E_n$. Thus $(x - \delta, x + \delta) \subseteq E_n$. We conclude that each E_n is open. Since $\lim_n f_n(x) = f(x)$, for any $x \in [a, b]$ there exists an n such that $f(x) - f_n(x) < \epsilon$. Thus $\{E_n\}$ is an open cover of $[a, b]$. By the Heine-Borel Theorem, we can find a finite subcover $\{E_{n_k}\}_{k=1}^{K}$. Define $N = \max\{n_k\}_{k=1}^{K}$. Since $\{E_{n_k}\}_{k=1}^{K}$ covers $[a, b]$, we know that for any $x \in [a, b]$ there exists $n' \leq N$ such that $f(x) - f_{n'}(x) < \epsilon$. But since the sequence $\{f_n\}$ is increasing, we must have $f(x) - f_n(x) \leq f(x) - f_{n'}(x) < \epsilon$ for all $n \geq N$. Thus $\{f_n\}$ converges to f uniformly on $[a, b]$.

23. Let f be an extended real-value function on a measurable set E. Observe that $f = f^+ - f^-$, where $f^+ = \max\{f, 0\}$ and $f^-(x) = \max\{-f, 0\}$. Since $f^+ \geq 0$ and $f^- \geq 0$, the proof for the non-negative case implies the existence of sequences of non-negative simple functions $\{\varphi_n\}$, $\{\psi_n\}$ on E that satisfy

$$\{\varphi_n\} \uparrow f^+ \text{ pointwise on } E$$
$$\{\psi_n\} \uparrow f^- \text{ pointwise on } E$$

Let $\phi_n = \varphi_n - \psi_n$. By Problem 19, ϕ_n is a simple function on E for all n. If $f(x) \geq 0$, then $f^-(x) = 0$. Thus $\psi_n(x) = 0$ for all n and $\phi_n(x) = \varphi_n(x) \to f^+(x) = f(x)$. Likewise, if $f(x) < 0$ then $\phi_n(x) = -\psi_n(x) \to -f^-(x) = f(x)$. Therefore ϕ_n converges pointwise to f on E. We also have

$$|\phi_n| \leq \varphi_n + \psi_n \leq f^+ + f^- = |f|$$

on E for all n.

24. For each natural number n, define $g_n(x) = f(x) + x/n$. Fix a real number c and define

$$A = \{x \in I : g_n(x) > c\}$$

Suppose $x \in A$ and $x' \in A$ and without loss of generality assume that $x < x'$. Pick any x'' lying between x and x'. Since g_n is strictly increasing, we must have $c < g_n(x) < g_n(x'') < g_n(x')$. But this implies $x'' \in A$. Thus the set A is an interval and therefore measurable by Proposition 8 of Chapter 2. Since g_n is measurable for all n and f is the pointwise limit of $\{g_n\}$, f is measurable by Proposition 9.

3.3 Littlewood's Three Principles, Egoroff's Theorem, and Lusin's Theorem

25. See Problem 47 of Chapter 1.

26. Pick $x \in F$ and fix $\epsilon > 0$. Since g is continuous, there exists a $\delta > 0$ such that

$$\text{if } |x' - x| < \delta, \text{ then } |g(x') - g(x)| < \epsilon$$

Since $f = g$ on F, we have

$$\text{if } x' \in F \text{ and } |x' - x| < \delta, \text{ then } |f(x') - f(x)| < \epsilon$$

Therefore f is continuous on F.

Let $E = \mathbf{R}$, $A = \{x \in \mathbf{R} : x \text{ is rational}\}$ and $f = \chi_A$. Then f is real-valued and measurable but not continuous on E at any point.

27. Let $E = \mathbf{R}$ and define $f_n = \chi_{(n,\infty)}$ and $f = 0$. Let $\epsilon = 1$ and choose any closed set $F \subseteq E$. If F is bounded, then $m(F)$ is finite and $m(E \sim F) = m(E) - m(F) = \infty > 1 = \epsilon$. Therefore any closed set satisfying $m(E \sim F) < \epsilon$ cannot be bounded. So assume F is unbounded and pick any natural number

N. Since F is unbounded, there must exist $x \in F$ satisfying $x > N$. But for this x, we have

$$|f_N(x) - f(x)| = 1 \geq \epsilon$$

Thus there is no index N such that

$$|f_n - f| < \epsilon \text{ on } F \text{ for all } n \geq N$$

Therefore even though $\{f_n\}$ converges pointwise to f on E, $\{f_n\}$ cannot converge uniformly to f on any closed set $F \subseteq E$ satisfying $m(E \sim F) < \epsilon$.

28. Assume E has finite measure and suppose $\{f_n\}$ is a sequence of measurable functions on E that converges pointwise a.e. to a function f that is finite a.e. Define $E_0 = \{x \in E : \lim_{n\to\infty} f_n(x) = f(x)\}$ and $E_1 = \{x \in E : f(x) \text{ is finite}\}$. By assumption, $m(E \sim E_0) = m(E \sim E_1) = 0$. Let $\tilde{E} = E_0 \cap E_1$ and observe that

$$0 \leq m\left(E \sim \tilde{E}\right) \leq m(E \sim E_1) + m(E \sim E_2) = 0$$

Therefore $m(E \sim \tilde{E}) = 0$.

Fix $\epsilon > 0$. By Egoroff's Theorem, we can find a closed set F contained in $\tilde{E}$ for which

$$\{f_n\} \to f \text{ uniformly on } F \text{ and } m\left(\tilde{E} \sim F\right) < \epsilon$$

But then F is also a closed set contained in E that satisfies

$$m(E \sim F) \leq m\left(\tilde{E} \sim F\right) + m\left(E \sim \tilde{E} \sim F\right) < \epsilon + m\left(E \sim \tilde{E}\right) = \epsilon$$

29. Let f be a real-valued measurable function on a set of real numbers E and fix $\epsilon > 0$. Let $\{k_n : n \in \mathbf{N}\}$ denote an enumeration of the integers and define

$$E_n = E \cap [k_n, k_n + 1)$$

Since $m(E_n) \leq 1$ for all n, we know from the finite case of Lusin's Theorem that we can find a collection of continuous functions $\{g_n\}$ on $\mathbf{R}$ and a collection of closed sets $\{F_n\}$ such that

$$f = g_n \text{ on } F_n,\ F_n \subseteq E_n, \text{ and } m(E_n \sim F_n) < \epsilon/2^n$$

for all n. Let $F = \bigcup_{n=1}^{\infty} F_n$ and let

$$g = \sum_{n=1}^{\infty} \chi_{E_n} g_n$$

Pick $x \in F$ and $\epsilon > 0$. Since the sets in $\{F_n\}$ are closed and disjoint, there must exist an index n and a $\tilde{\delta} \in (0, 1/2)$ such that the interval $I = (x - \tilde{\delta}, x + \tilde{\delta})$ satisfies $I \cap F_n \neq \varnothing$ and $I \cap \bigcup_{n' \neq n} F_{n'} = \varnothing$. Otherwise, x would be a limit point (and thus an element of) two sets in $\{F_n\}$. Since g_n is continuous, there exists a $\delta' > 0$ such that

$$\text{if } |x' - x| < \delta', \text{ then } |g_n(x) - g_n(x')| < \epsilon$$

Let $\delta = \min\{\tilde{\delta}, \delta'\}$. Notice that if $x' \in F$ and $|x - x'| < \delta$, then $x' \in F_n$ and $g(x') = g_n(x')$. We can therefore conclude

$$\text{if } x' \in F \text{ and } |x' - x| < \delta \text{ then } |g(x) - g(x')| < \epsilon$$

Thus g is continuous on F.

Now suppose $\{x_k\}$ is a sequence in F that converges to x. Then there exists an index n such that $x_k \in F_n$ for all k large enough. But since F_n is closed, we must have $x \in F_n \subseteq F$. Thus F contains all its limit points and is therefore a closed set.

Since g is continuous on F, we can find a continuous extension of g to all of $\mathbf{R}$ by Problem 25. To check that $m(E \sim F) < \epsilon$, observe that

$$m(E \sim F) = m\left(\bigcup_{n=1}^{\infty}(E_n \sim F_n)\right) \leq \sum_{n=1}^{\infty} m\left(E_n \sim F_n\right) = \epsilon$$

30. Suppose f is an extended real-valued measurable function on E that is finite a.e. Fix $\epsilon > 0$ and let $E_0 = \{x \in E : f \text{ is finite}\}$. We know from Lusin's Theorem that there exists a continuous function g on $\mathbf{R}$ and a closed set F contained in E_0 for which

$$f = g \text{ on } F \text{ and } m(E_0 \sim F) < \epsilon$$

But since $E = (E \sim E_0) \cup E_0$ and $m(E \sim E_0) = 0$, we also have

$$m(E \sim F) \leq m(E \sim E_0 \sim F) + m(E_0 \sim F) = m(E_0 \sim F) < \epsilon$$

31. Let $\{k_n : n \in \mathbf{N}\}$ denote an enumeration of the integers and define

$$\tilde{E}_n = E \cap [k_n, k_n + 1)$$

Since $m(\tilde{E}_n) \leq 1$, we know from Egoroff's Theorem that for any natural number m we can find a closed set $F_{n,m}$ contained in $\tilde{E}_n$ for which

$$\{f_n\} \to f \text{ uniformly on } F_{n,m} \text{ and } m(\tilde{E}_n \sim F_{n,m}) < \frac{1}{m2^n}$$

Let $E_0 = E \sim \bigcup_{m=1}^{\infty}\bigcup_{n=1}^{\infty} F_{n,m}$. Observe that

$$m\left(E \sim \bigcup_{n=1}^{\infty} F_{n,m}\right) = m\left(\bigcup_{n=1}^{\infty}\left(\tilde{E}_n \sim F_{n,m}\right)\right) \leq \sum_{n=1}^{\infty} m(\tilde{E}_n \sim F_{n,m}) < \frac{1}{m}$$

By the continuity of measure, we have

$$\begin{aligned}
m(E_0) &= m\left(E \sim \bigcup_{m=1}^{\infty}\bigcup_{n=1}^{\infty} F_{n,m}\right) \\
&= m\left(\bigcap_{m=1}^{\infty}\left(E \sim \bigcup_{n=1}^{\infty} F_{n,m}\right)\right) \\
&= \lim_{N\to\infty} m\left(\bigcap_{m=1}^{N}\left(E \sim \bigcup_{n=1}^{\infty} F_{n,m}\right)\right) \\
&\le \lim_{N\to\infty} m\left(E \sim \bigcup_{n=1}^{\infty} F_{n,N}\right) \\
&\le \lim_{N\to\infty} \frac{1}{N} \\
&= 0
\end{aligned}$$

Let $g(n) = (g_1(n), g_2(n))$ denote a 1-to-1 correspondence from $\mathbf{N}$ onto $\mathbf{N}\times\mathbf{N}$ and define $E_n = F_{g_1(n),g_2(n)}$. Then $E = \bigcup_{n=0}^{\infty} E_n$, each E_n is measurable, $\{f_n\} \to f$ uniformly on E_n if $n \ge 1$, and $m(E_0) = 0$.

4 Lebesgue Integration

4.1 The Riemann Integral

1. Let $\epsilon = 1$ and let N denote a natural number. Then

$$|f(q_n) - f_n(q_n)| = |1 - 0| = 1 \ge \epsilon$$

for all $n \ge N$. Thus there is no index N such that $|f - f_n| < \epsilon$ on $[0,1]$ for all $n \ge N$.

2. Let $P = \{x_0, x_1, \cdots, x_n\}$ denote a partition of $[a,b]$ and let P' denote a refinement of P. Let n_i denote the number of points in P' that lie between x_i and x_{i+1} for $i = 0, \cdots, n-1$. Label the points of P' as

$$P' = \{x_{0,0}, x_{0,1}, \cdots, x_{0,n_0}, x_{1,0}, x_{1,1}, \cdots, x_{1,n_1}, \cdots, x_{n-1,0}, x_{n-1,1}, \cdots, x_{n-1,n_{n-1}}, x_{n,0}\}$$

where

$$x_i = x_{i,0} < x_{i,1} < \cdots < x_{i,n_i} < x_{i+1,0} = x_{i+1}, \quad i = 0, \cdots, n-1$$

Define

$$\begin{aligned}
m_i &= \inf\{f(x) \mid x_i < x < x_{i+1}\} \\
M_i &= \sup\{f(x) \mid x_i < x < x_{i+1}\} \\
m_{i,n_i} &= \inf\{f(x) \mid x_{i,n_i} < x < x_{i+1,0}\} \\
M_{i,n_i} &= \sup\{f(x) \mid x_{i,n_i} < x < x_{i+1,0}\}
\end{aligned}$$

and

$$
\begin{aligned}
m_{i,j} &= \inf\{f(x) \mid x_{i,j} < x < x_{i,j+1}\}, \qquad j = 0, \cdots, n_i - 1 \\
M_{i,j} &= \sup\{f(x) \mid x_{i,j} < x < x_{i,j+1}\}, \qquad j = 0, \cdots, n_i - 1
\end{aligned}
$$

if $n_i \geq 1$. Since $(x_{i,n_i}, x_{i+1,0}) \subseteq (x_i, x_{i+1})$, we know that

$$m_{i,n_i} \geq m_i \text{ and } M_{i,n_i} \leq M_i$$

Likewise,

$$m_{i,j} \geq m_i \text{ and } M_{i,j} \leq M_i \text{ for } j = 0, \cdots, n_i - 1$$

if $n_i \geq 1$. Also observe that $x_{i+1} - x_i = x_{i+1,0} - x_{i,n_i}$ if $n_i = 0$ and

$$x_{i+1} - x_i = x_{i+1,0} - x_{i,n_i} + \sum_{j=0}^{n_i-1} (x_{i,j+1} - x_{i,j}) \tag{1}$$

if $n_i \geq 1$. We therefore have

$$
\begin{aligned}
L(f, P) &= \sum_{i=0}^{n-1} m_i (x_{i+1} - x_i) \\
&= \sum_{i=0}^{n-1} m_i \left(x_{i+1,0} - x_{i,n_i}\right) + \sum_{i:\, n_i \geq 1} \sum_{j=0}^{n_i-1} m_i \left(x_{i,j+1} - x_{i,j}\right) \\
&\leq \sum_{i=0}^{n-1} m_{i,n_i} \left(x_{i+1,0} - x_{i,n_i}\right) + \sum_{i:\, n_i \geq 1} \sum_{j=0}^{n_i-1} m_{i,j} \left(x_{i,j+1} - x_{i,j}\right) \\
&= L(f, P')
\end{aligned}
$$

and

$$
\begin{aligned}
U(f, P) &= \sum_{i=0}^{n-1} M_i (x_{i+1} - x_i) \\
&= \sum_{i=0}^{n-1} M_i \left(x_{i+1,0} - x_{i,n_i}\right) + \sum_{i:\, n_i \geq 1} \sum_{j=0}^{n_i-1} M_i \left(x_{i,j+1} - x_{i,j}\right) \\
&\geq \sum_{i=0}^{n-1} M_{i,n_i} \left(x_{i+1,0} - x_{i,n_i}\right) + \sum_{i:\, n_i \geq 1} \sum_{j=0}^{n_i-1} M_{i,j} \left(x_{i,j+1} - x_{i,j}\right) \\
&= U(f, P')
\end{aligned}
$$

3. Let P and P' be two partitions of $[a, b]$. Define P'' to be a refinement of both P and P'. It is apparent from the definitions of Darboux sums that $L(f, P'') \leq U(f, P'')$. Combining this observation with the result from Problem 2, we can conclude

$$L(f, P) \leq L(f, P'') \leq U(f, P'') \leq U(f, P')$$

Thus $U(f,P')$ is an upper bound of the set $\{L(f,P) \mid P \text{ a partition of } [a,b]\}$, so $U(f,P')$ must exceed $(R)\underline{\int_a^b} f = \sup\{L(f,P) \mid P \text{ a partition of } [a,b]\}$. But then $(R)\underline{\int_a^b} f$ is a lower bound of the set $\{U(f,P) \mid P \text{ a partition of } [a,b]\}$, which means $(R)\underline{\int_a^b} f$ is no greater than $(R)\overline{\int_a^b} f\{U(f,P) \mid P \text{ a partition of } [a,b]\}$.

4. Let $\{P_n'\}$ and $\{P_n''\}$ denote sequences of partitions of $[a,b]$ that satisfy

$$\lim_{n\to\infty} L(f,P_n') = (R)\underline{\int_a^b} f \quad \text{and} \quad \lim_{n\to\infty} U(f,P_n'') = (R)\overline{\int_a^b} f$$

For each n, define P_n to be a refinement of both P_n' and P_n''. Then

$$0 \le U(f,P_n) - L(f,P_n) \le U(f,P_n'') - L(f,P_n')$$

where the latter equality follows from Problem 2. Taking limits of this expression, we conclude that

$$\lim_{n\to\infty} [U(f,P_n) - L(f,P_n)] = 0$$

5. By definition of upper and lower Riemann integrals, we know that

$$(R)\overline{\int_a^b} f \le U(f,P_n) \quad \text{and} \quad (R)\underline{\int_a^b} f \ge L(f,P_n)$$

for all n. Therefore

$$0 \le (R)\overline{\int_a^b} f - (R)\underline{\int_a^b} f \le U(f,P_n) - L(f,P_n)$$

where the first inequality follows from Problem 3. Taking limits, we can conclude that

$$0 = (R)\overline{\int_a^b} f - (R)\underline{\int_a^b} f \quad \Longrightarrow \quad (R)\overline{\int_a^b} f = (R)\underline{\int_a^b} f$$

Therefore f is Riemann integrable.

6. Since f is a continuous function on the closed, bounded interval $[a,b]$, we know by the Extreme Value Theorem that f is bounded. For each natural number n, define $P_n = \{x_{0,n}, x_{1,n}, \cdots, x_{n,n}\}$ to be a partition of $[a,b]$ into n subintervals of length $(b-a)/n$. Fix $\epsilon > 0$. By Theorem 23 of Chapter 1, there exists a $\delta > 0$ such that for all $x, x' \in [a,b]$,

$$\text{if } |x - x'| < \delta, \text{ then } |f(x) - f(x')| < \frac{\epsilon}{2(b-a)}$$

Choose an index N that satisfies $N > (b-a)/\delta$ and let n denote a natural number greater than N. Let

$$M_{i,n} = \sup\{f(x) \mid x_{i-1,n} < x < x_{i,n}\} \quad \text{and} \quad m_{i,n} = \inf\{f(x) \mid x_{i-1,n} < x < x_{i,n}\}$$

Then

$$\begin{aligned}
M_{i,n} - m_{i,n} &= \sup\left\{ f(x) - f\left(\frac{x_{i,n} - x_{i-1,n}}{2}\right) \;\middle|\; x_{i-1,n} < x < x_{i,n} \right\} + \\
&\qquad \sup\left\{ f\left(\frac{x_{i,n} - x_{i-1,n}}{2}\right) - f(x) \;\middle|\; x_{i-1,n} < x < x_{i,n} \right\} \\
&\le \frac{\epsilon}{2(b-a)} + \frac{\epsilon}{2(b-a)} \\
&= \frac{\epsilon}{b-a}
\end{aligned}$$

where the second line follows because $|x - (x_{i,n} - x_{i-1,n})/2| < \delta$ for all x satisfying $x_{i-1,n} < x < x_{i,n}$. We therefore have

$$U(f, P_n) - L(f, P_n) = \sum_{i=1}^{n} (M_{i,n} - m_{i,n}) \frac{b-a}{n} \le \epsilon$$

for all $n \ge N$. We can conclude that $\lim_{n\to\infty}[U(f, P_n) - L(f, P_n)] = 0$, so f is Riemann integrable over $[a, b]$ by Problem 5.

7. Since f is a real-valued increasing function on the interval $[a, b]$, we know that f is bounded below by $f(a)$ and bounded above by $f(b)$. We also know that

$$\sup\left\{ f(x) \;\middle|\; \frac{i-1}{n} < x < \frac{i}{n} \right\} \le f\left(\frac{i}{n}\right) \quad \text{and} \quad \inf\left\{ f(x) \;\middle|\; \frac{i-1}{n} < x < \frac{i}{n} \right\} \ge f\left(\frac{i-1}{n}\right)$$

Therefore

$$0 \le U(f, P_n) - L(f, P_n) \le \sum_{i=1}^{n} \left(f\left(\frac{i}{n}\right) - f\left(\frac{i-1}{n}\right) \right) \frac{1}{n} = \frac{f(1) - f(0)}{n}$$

Taking limits of the above expression, we conclude that $\lim_{n\to\infty}[U(f, P_n) - L(f, P_n)] = 0$. Thus f is Riemann integrable over $[a, b]$ by Problem 5.

8. We first check that f is bounded on $[a, b]$. Since $\{f_n\}$ converges uniformly to f on $[a, b]$, there exists an index N such that

$$|f - f_n| < 1 \text{ on } [a, b] \text{ for all } n \ge N \tag{2}$$

Pick a natural number n that exceeds N. Since f_n is bounded, there exists a real number M such that $|f_n| \le M$ on $[a, b]$. If f is unbounded on $[a, b]$, there exists $x \in [a, b]$ such that $|f(x)| \ge M + 1$. But then

$$|f(x) - f_n(x)| \ge |f(x)| - |f_n(x)| \ge M - (M+1) = 1$$

which is a contradiction to (2).

Since f_n is Riemann integrable, we know from Problem 4 that we can find a partition $P_n = \{x_{0,n}, x_{1,n}, \cdots, x_{N_n,n}\}$ of $[a, b]$ such that $U(f_n, P_n) - L(f_n, P_n) < 1/n$. Define

$$\begin{aligned}
M_{i,n} &= \sup\{f_n(x) \mid x_{i-1,n} < x < x_{i,n}\} \\
m_{i,n} &= \inf\{f_n(x) \mid x_{i-1,n} < x < x_{i,n}\} \\
M'_{i,n} &= \sup\{f(x) \mid x_{i-1,n} < x < x_{i,n}\} \\
m'_{i,n} &= \inf\{f(x) \mid x_{i-1,n} < x < x_{i,n}\}
\end{aligned}$$

for $i \in \{1, \cdots, N_n\}$. Because $\{f_n\}$ converges to f uniformly on $[a,b]$, for any $\epsilon > 0$ there exists an index N such that

$$f(x) < f_n(x) + \frac{\epsilon}{b-a} \text{ and } f_n(x) < f(x) + \frac{\epsilon}{b-a} \text{ on } [a,b]$$

for all $n \geq N$. Taking sups and infs of these inequalities over the interval $(x_{i-1,n}, x_{i,n})$, we see that

$$|M'_{i,n} - M_{i,n}| \leq \frac{\epsilon}{b-a} \text{ and } |m'_{i,n} - m_{i,n}| \leq \frac{\epsilon}{b-a}$$

if $n \geq N$. We therefore have

$$\begin{aligned} |U(f,P_n) - U(f_n,P_n)| &\leq \sum_{i=1}^{N_n} \left|M_{i,n} - M'_{i,n}\cdot\right| (x_{i,n} - x_{i-1,n}) \leq \epsilon \\ |L(f,P_n) - L(f_n,P_n)| &\leq \sum_{i=1}^{N_n} \left|m_{i,n} - m'_{i,n}\cdot\right| (x_{i,n} - x_{i-1,n}) \leq \epsilon \end{aligned}$$

for all $n \geq N$. We conclude that $\lim_{n\to\infty} |U(f,P_n) - U(f_n,P_n)| = \lim_{n\to\infty} |L(f,P_n) - L(f_n,P_n)| = 0$. We also know that

$$\begin{aligned} U(f,P_n) - L(f,P_n) &\leq |U(f,P_n) - U(f_n,P_n)| + U(f_n,P_n) - L(f_n,P_n) + |L(f_n,P_n) - L(f,P_n)| \\ &\leq |U(f,P_n) - U(f_n,P_n)| + \frac{1}{n} + |L(f_n,P_n) - L(f,P_n)| \end{aligned}$$

Taking limits of this expression, we see that $\lim_{n\to\infty}[U(f,P_n) - L(f,P_n)] = 0$. Therefore, f is Riemann integrable by Problem 5.

Since $\left|U(f,P_n) - (R)\int_a^b f\right| \leq |U(f,P_n) - L(f,P_n)| \to 0$ and $\left|U(f_n,P_n) - (R)\int_a^b f_n\right| \leq |U(f_n,P_n) - L(f_n,P_n)| \leq 1/n \to 0$, we also have

$$\begin{aligned} \left|(R)\int_a^b f - (R)\int_a^b f_n\right| &\leq \left|(R)\int_a^b f - U(f,P_n)\right| + |U(f,P_n) - U(f_n,P_n)| + \left|U(f_n,P_n) - (R)\int_a^b f_n\right| \\ &\to 0 \end{aligned}$$

Therefore,

$$(R)\int_a^b f = \lim_{n\to\infty} (R)\int_a^b f_n$$

4.2 The Lebesgue Integral of a Bounded Measurable Function over a Set of Finite Measure

9. Since any set contained in a set of measure zero is measurable, $\{x \in E : f(x) \leq c\} \subseteq E$ is measurable for any real number c. Therefore f is measurable.

   Since f is bounded, there exists a real number M such that $|f| \leq M$ on E. We then have

$$\left|\int_E f\right| \leq \int_E |f| \leq \int_E M \cdot \chi_E = M \cdot m(E) = 0$$

where the first inequality follows from Corollary 7 and the second inequality from Theorem 5. We can therefore conclude $\int_E f = 0$.

10. Let $\{\varphi_n\}$ and $\{\psi_n\}$ be two sequences of simple functions that satisfy

$$\varphi_n \leq f \cdot \chi_A \leq \psi_n \text{ on } E \text{ and } \lim_{n\to\infty} \int_E \varphi_n = \int_E f \cdot \chi_A = \lim_{n\to\infty} \int_E \psi_n$$

Then $\varphi_n \leq f \leq \psi_n$ on A, so we must have

$$\int_A \varphi_n \leq \int_A f \leq \int_A \psi_n$$

for all n. Taking limits of this expression, we see that $\int_A f = \int_E f \cdot \chi_A$.

11. See Dirichlet's Function from Section 4.1 for a counter-example.

12. There exists a set $E_0 \subseteq E$ such that $m(E_0) = 0$ and $f = g$ on $E \sim E_0$. We therefore have

$$\int_E f = \int_{E\sim E_0} f + \int_{E_0} f = \int_{E\sim E_0} f = \int_{E\sim E_0} g = \int_{E\sim E_0} g + \int_{E_0} g = \int_E g$$

where the first and last equality follow from Corollary 6, the second and fourth equalities follow because $\int_{E_0} f = \int_{E_0} g = 0$ by Problem 9, and the third equality follows because $f = g$ on $E \sim E_0$.

13. See the counter-example on page 78.

14. If $\{f_n\}$ converges uniformly to f on E, then f is bounded (see the solution to Problem 8). Thus there exists a real number M such that $|f| \leq M$ on E. We also know there exists an index N such that

$$|f_n - f| < 1 \text{ on } E$$

for all $n \geq N$. But then $|f_n| - |f| \leq |f_n - f| < 1$, so $|f_n| \leq M + 1$ for all $n \geq N$. Since each f_n is bounded, for each natural number n there exists a real number M_n such that $|f_n| \leq M_n$ on E. Define $M' = \max\{M_1, M_2, \cdots, M_N, M+1\}$. Then $|f_n| \leq M'$ for all n, so $\{f_n\}$ is a uniformly pointwise bounded sequence on E that converges pointwise to f. The Bounded Convergence Theorem then implies $\lim_{n\to\infty} \int_E f_n = \int_E f$.

15. The assertions can be verified by thoroughly reviewing the proofs in the text.

16. For any natural number n, define

$$E_n = \left\{ x \in E \;\middle|\; f(x) > \frac{1}{n} \right\}$$

Since f is measurable, E_n is a measurable set. Because f is non-negative, $f \geq f \cdot \chi_{E_n} \geq \frac{1}{n} \cdot \chi_{E_n}$ for all n. By Theorem 5, we have

$$0 = \int_E f \geq \int_E f \cdot \chi_{E_n} \geq \int_{E_n} \frac{1}{n} \cdot \chi_{E_n} = \frac{1}{n} \cdot m(E_n) \geq 0$$

We can therefore conclude $m(E_n) = 0$ for all n. Now define

$$E_0 = \{x \in E : f(x) > 0\}$$

and observe that $\{E_n\}$ converges upwards to E_0. By continuity of measure, we have

$$m(E_0) = \lim_{n\to\infty} m(E_n) = 0$$

Since f is non-negative, $f = 0$ on $E \sim E_0$, Therefore $f = 0$ a.e.

4.3 The Lebesgue Integral of a Measurable Nonnegative Function

17. Let h be a bounded, measurable, non-negative function on E. Then $0 \le h \le f$ on E and $\int_E h = 0$ by Problem 9. Therefore $\int_E f = 0$.

18. Let f be a bounded, measurable function on a set E. Suppose E_0 and E_1 are measurable subsets of E with finite measure such that $f \equiv 0$ on $E \sim E_0$ and $E \sim E_1$. Then $f \equiv 0$ on $E_0 \sim E_1$ and $E_1 \sim E_0$. Since $E_0 \sim E_1$ and $E_1 \sim E_0$ have finite measure, we know that $\int_{E_0\sim E_1} f = \int_{E_1\sim E_0} f = 0$ from the definition of the Lebesgue integral for bounded, measurable functions over sets of finite measure. We therefore have
$$\int_{E_0} f = \int_{E_0\sim E_1} f + \int_{E_0\cap E_1} f = \int_{E_1\sim E_0} f + \int_{E_0\cap E_1} f = \int_{E_1} f$$
where the first and last equality follow by Corollary 6. Since the value of the integral does not depend on the choice of set of finite measure outside of which f vanishes, the integral of a bounded measurable function of finite support is well-defined.

19. Let $E = [0, 1]$ and $D = (0, 1]$. Since f is a monotonic function defined on D, f is measurable on E by Propositions 4 and 5 of Chapter 3.

    Let $\{f_n\}$ denote the sequence of step functions defined by

    $$f_n = \sum_{k=0}^{n2^n-1} 2^{-\alpha k 2^{-n}} \chi_{\left(2^{-(k+1)2^{-n}},2^{-k2^{-n}}\right]}$$

    If $\alpha \ge 0$, then $\{f_n\}$ is a uniformly bounded sequence of measurable functions on E converging pointwise to f. If $\alpha < 0$, then $\{f_n\}$ is an increasing sequence of non-negative measurable functions on E converaging pointwise to f. Therefore by the Bounded and Monotone Convergence Theorems, we know that $\int_E f_n \to \int_E f$. Since

    $$m\left(2^{-(k+1)2^{-n}}, 2^{-k2^{-n}}\right] = \left(1 - 2^{-2^{-n}}\right) 2^{-k2^{-n}}$$

    we have

    $$\int_E f_n = \left(1 - 2^{-2^{-n}}\right) \sum_{k=0}^{n2^n-1} \left(2^{-(1+\alpha)2^{-n}}\right)^k$$

If $\alpha \neq -1$, we can use the geometric summation formula to obtain

$$\int_E f_n = \left(1 - 2^{-2^{-n}}\right) \cdot \frac{1 - 2^{-(1+\alpha)n}}{1 - 2^{-(1+\alpha)2^{-n}}}$$

(See Problem 15(iii) of Chapter 1.) Using L'Hôpital's rule, it is straight-forward to show that

$$\lim_{n\to\infty} \int_E f_n = \begin{cases} \frac{1}{1+\alpha} & \text{if } \alpha > -1 \\ \infty & \text{if } \alpha < -1 \end{cases}$$

If $\alpha = -1$, we have

$$\int_E f_n = \left(1 - 2^{-2^{-n}}\right) n2^n \to \infty$$

We can therefore conclude

$$\int_E f = \begin{cases} \infty & \text{if } \alpha \leq -1 \\ \frac{1}{1+\alpha} & \text{if } \alpha > -1 \end{cases}$$

20. It is necessary and sufficient to show that $\int_E h \leq M$ for all bounded, measurable functions of finite support h that satisfy $0 \leq h \leq f$ on E. Let h denote such a function. For each natural number n, define

$$h_n = \min\{h, f_n\} \text{ on } E$$

Since $h_n \leq f_n$ on E, we know that $\int_E h_n \leq \int_E f_n \leq M$ for all n. Therefore $\int_E h_n = \lim_{n\to\infty} \int_E h_n \leq M$, where the first equality is proven in the text.

To see that this result implies Fatou's Lemma, suppose $\liminf \int_E f_n < \infty$. Let $\{f_{n_k}\}$ denote a subsequence of $\{f_n\}$ that satisfies

$$\lim_{k\to\infty} \int_E f_{n_k} = \liminf \int_E f_n$$

For any $\epsilon > 0$, there exists an index K such that $\int_E f_{n_k} \leq \liminf \int_E f_n + \epsilon$ for all $k \geq K$. The above result then implies $\int_E f \leq \liminf \int_E f_n + \epsilon$. Since ϵ can be made arbitrarily small, we can conclude that $\int_E f \leq \liminf \int_E f_n$.

To see that Fatou's Lemma implies this result, suppose there exists a real number M such that $\int f_n \leq M$ for all n. Then $\liminf \int f_n \leq M$, so by Fatou's Lemma we know that $\int f \leq \liminf \int f_n \leq M$.

21. By the definition of the Lebesgue integral, we can find a bounded, measurable function of finite support g that satisfies $0 \leq g \leq f$ on E and

$$\int_E g > \int_E f - \frac{\epsilon}{2}$$

Let $E_0 = \{x \in E : g(x) \neq 0\}$ and note that $m(E_0) < \infty$. By the definition of the upper Lebesgue integral for bounded functions on a set of finite measure, we can find a simple function η that satisfies $\eta \leq g$ on E_0 and

$$\int_{E_0} \eta > \int_{E_0} g - \frac{\epsilon}{2}$$

Since it is always possible to replace η with $\max\{0, \eta\}$, we can assume without loss of generality that

$\eta \geq 0$. Extend η to E by defining $\eta = 0$ on $E \sim E_0$. Then $0 \leq \eta \leq f$ on E, η has finite support, and

$$\int_E |f - \eta| = \int_E f - \int_E \eta < \int_E g - \int_E \eta + \frac{\epsilon}{2} = \int_{E_0} g - \int_{E_0} \eta + \frac{\epsilon}{2} < \epsilon$$

Now suppose E is a closed and bounded interval. Since g is bounded, there exists a positive real number M such that $|g| \leq M$ on E. By Lusin's Theorem, we can find a continuous function $\tilde{g}$ defined on E and a closed set $F \subseteq E$ that satisfy

$$\tilde{g} = g \text{ on } F \text{ and } m(E \sim F) < \frac{\epsilon}{8M}$$

The function $\tilde{g}$ may be chosen such that $|\tilde{g}| \leq M$ (see the construction in Problem 47 of Chapter 1). A continuous function on a closed and bounded interval is Riemann integrable (see Problem 6), so by the definition of the lower Riemann integral we can find a step function $h \leq \tilde{g}$ on E that satisfies

$$\int_E h > \int_E \tilde{g} - \frac{\epsilon}{4}$$

Define $h(x) \equiv 0$ for $x \notin E$. Since E has finite measure, the step function h has finite support and

$$\begin{aligned}
\int_E |f - h| &\leq \int_E |f - g| + \int_E |g - \tilde{g}| + \int_E |\tilde{g} - h| \\
&< \frac{\epsilon}{2} + \int_{E\sim F} |g - \tilde{g}| + \frac{\epsilon}{4} \\
&\leq \frac{3\epsilon}{4} + \int_{E\sim F} |g| + \int_{E\sim F} |\tilde{g}| \\
&\leq \frac{3\epsilon}{4} + 2 \cdot m(E \sim F) \cdot M \\
&< \frac{3\epsilon}{4} + \frac{\epsilon}{4} \\
&= \epsilon
\end{aligned}$$

22. Let E be a measurable set. Because of Fatou's Lemma, it suffices to show that $\limsup \int_E f_n \leq \int_E f$. Since $\limsup \int_{\mathbf{R}} f_n \leq \int_{\mathbf{R}} f < \infty$, we can assume without loss that $\int_{\mathbf{R}} f_n$ is finite for all n. Since f_n is non-negative, we know that $f_n \cdot \chi_E \leq f_n$ and $f_n \cdot \chi_{\mathbf{R}\sim E} \leq f_n$. Therefore $\int_E f_n$ and $\int_{\mathbf{R}\sim E} f_n$ are finite and $\int_E f_n = \int_{\mathbf{R}} f_n - \int_{\mathbf{R}\sim E} f_n$. We can thus conclude

$$\begin{aligned}
\limsup \int_E f_n &= \limsup \left(\int_{\mathbf{R}} f_n - \int_{\mathbf{R}\sim E} f_n \right) \\
&\leq \limsup \int_{\mathbf{R}} f_n - \liminf \int_{\mathbf{R}\sim E} f_n \\
&\leq \int_{\mathbf{R}} f - \int_{\mathbf{R}\sim E} f \\
&= \int_E f
\end{aligned}$$

where the third line follows from Fatou's Lemma.

23. Let $f_n = \sum_{k=1}^{n} a_k \chi_{[k,k+1)}$. Then $\{f_n\}$ is an increasing sequence of measurable simple functions on E

that converges upward to f. By Lemma 1 and the Monotone Convergence Theorem, we have

$$\int_E f = \lim_{n\to\infty} \int_E f_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \sum_{k=1}^{\infty} a_k$$

24. (i) By the Simple Approximation Theorem, we can find an increasing sequence of simple functions $\{\psi_n\}$ on E that converges pointwise on E to f. Since we can always replace ψ_n with $\max\{0, \psi_n\}$, we can assume without loss that each ψ_n is non-negative. Define $\varphi_n = \psi_n \cdot \chi_{[-n,n]}$. Then each φ_n is a non-negative simple function on E of finite support and $\{\varphi_n\}$ converges pointwise on E to f.

    (ii) Let

$$A = \{\varphi \mid \varphi \text{ simple, of finite support and } 0 \le \varphi \le f \text{ on } E\}$$

and let

$$x = \sup \left\{ \int_E \varphi \;\middle|\; \varphi \in A \right\}$$

Suppose $\varphi \in A$. Since simple functions are bounded and measurable, we know that $\int_E f \ge \int_E \varphi$ by the definition of the Lebesgue integral for nonnegative measurable functions. Therefore $\int_E f \ge x$. Applying Part (i) of this problem, we can construct an increasing sequence of simple functions $\{\varphi_n\}$ in A that converges pointwise on E to f. By the Monotone Convergence Theorem, we know that $\int_E f = \lim_{n\to\infty} \int_E \varphi_n$. Thus for any $\epsilon > 0$, we can find an natural number n such that

$$\int_E f \le \int \varphi_n + \epsilon \le x + \epsilon$$

Since this expression holds for all ϵ, we must have $\int_E f \le x$.

25. Since $f_n \le f$ on E for all n, we know that $\int_E f_n \le \int_E f$. Thus $\limsup \int_E f_n \le \int_E f$, which along with Fatou's Lemma implies $\lim_{n\to\infty} \int_E f_n = \int_E f$.

26. Let $f_n = \chi_{(-\infty,-n]}$. Then f_n converges downward to $f \equiv 0$ but $\lim_{n\to\infty} \int f_n = \infty \ne 0 = \int f$.

27. Let $g_n = \inf_{k\ge n} f_k$. Then $\{g_n\}$ is an increasing sequence of non-negative, measurable functions on E that converges pointwise on E to $\liminf f_n$. By the Monotone Convergence Theorem, we know that $\int_E \liminf f_n = \lim_{n\to\infty} \int_E g_n$. But since $g_n \le f_n$, we also know that $\int_E g_n \le \int_E f_n$ for all n. Therefore $\lim_{n\to\infty} \int_E g_n \le \liminf \int_E f_n$. Combining these results, we can conclude

$$\int \liminf f_n \le \liminf \int f_n$$

4.4 The General Lebesgue Integral

28. Suppose f is non-negative. By Problem 24(i), we can construct an increasing sequence of non-negative simple functions $\{\varphi_n\}$ on E, each of finite support, which converges pointwise on E to f. Then $\varphi_n \cdot \chi_C$ is an increasing sequence of non-negative measurable functions converging upwards to $f \cdot \chi_C$. We

therefore have

$$\int_E f \cdot \chi_C = \lim_{n\to\infty} \int_E \varphi_n \cdot \chi_C = \lim_{n\to\infty} \int_C \varphi_n = \int_C f$$

where the first and last equalities follow from the Monotone Convergence Theorem and the middle equality follows from Problem 10.

Now suppose f is integrable. By the previous result and the monotonicity of integration, we have

$$\int_C f^+ = \int_E f^+ \cdot \chi_C < \int_E f^+ < \infty$$

and

$$\int_C f^- = \int_E f^- \cdot \chi_C < \int_E f^- < \infty$$

Since $f^+ \cdot \chi_C = [f \cdot \chi_C]^+$ and $f^- \cdot \chi_C = [f \cdot \chi_C]^-$, we have

$$\int_C f = \int_C f^+ - \int_C f^- = \int_E f^+ \cdot \chi_C - \int_E f^- \cdot \chi_C = \int_E [f \cdot \chi_C]^+ - \int_E [f \cdot \chi_C]^- = \int_E f \cdot \chi_C$$

29. Let $E = [1, \infty)$ and define

$$E' = \bigcup_{n=1}^{\infty} \left(n - \frac{1}{2}, n\right]$$

Consider the function

$$f = \chi_{E'} - \chi_{E \sim E'}$$

Then $a_n = 0$ for all n, so the series $\sum_{n=1}^{\infty} a_n$ converges absolutely and therefore also converges. However, f is not integrable. Thus convergence and absolute convergence of $\sum_{n=1}^{\infty} a_n$ does not imply integrability of f.

Now suppose f is integrable. Then

$$\sum_{i=1}^{n} |a_i| \leq \sum_{i=1}^{n} \int_i^{i+1} |f| = \int_1^{n+1} |f| = \int_E |f| \cdot \chi_{[1,n+1)}$$

But $\left\{|f| \cdot \chi_{[1,n+1)}\right\}$ is an increasing sequence of non-negative, measurable functions converging pointwise to $|f|$ on E. We can therefore conclude

$$\sum_{n=1}^{\infty} |a_n| \leq \lim_{n\to\infty} \int_E |f| \cdot \chi_{[1,n+1)} = \int_E |f| < \infty$$

by the Monotone Convergence Theorem. Thus integrability of f implies that $\sum_{n=1}^{\infty} a_n$ converges and converges absolutely.

30. Let $g_n = \inf_{k\geq n} f_k$. Then g_n converges pointwise on E to $\liminf f_n$, $|g_n| \leq g$ on E, and $g_n \leq f_n$ for all

n. By the Lebesgue Dominated Convergence Theorem and monotonicity of integration, we have

$$\begin{aligned}\int_E \liminf f_n &= \lim_{n\to\infty} \int_E g_n \\ &\leq \liminf \int_E f_n\end{aligned}$$

Applying the above inequality, we can also conclude

$$\int_E \limsup f_n = -\int_E \liminf(-f_n) \geq -\liminf\left(-\int_E f_n\right) = \limsup \int_E f_n$$

31. Let g' be a finite, integrable function and h' be a nonnegative function that together satisfy $g + h = g' + h'$. Then

$$g^+ - g^- + h = [g']^+ - [g']^- + h'$$

which implies

$$g^+ + [g']^- + h = [g']^+ + g^- + h'$$

By the linearity of integration for non-negative functions, we have

$$\int_E g^+ + \int_E [g']^- + \int_E h = \int_E [g']^+ + \int_E g^- + \int_E h'$$

which implies

$$\int_E g^+ - \int_E g^- + \int_E h = \int_E [g']^+ - \int_E [g']^- + \int_E h'$$

We can conclude from this expression that

$$\int_E g + \int_E h = \int_E g' + \int_E h'$$

32. I first prove the following preliminary result:

Claim: Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers. Suppose $\{a_n\}$ converges to $a \in \mathbf{R}$. Then $\liminf(a_n + b_n) = a + \liminf b_n$ and $\limsup(a_n + b_n) = a + \limsup b_n$.

Proof: For any n, we have $\inf_{k\geq n}(a_k + b_k) \geq \inf_{k\geq n} a_k + \inf_{k\geq n} b_k$. Therefore

$$\liminf(a_n + b_n) \geq \liminf a_n + \liminf b_n = a + \liminf b_n$$

Now let $\{b_{n_k}\}$ be a subsequence of $\{b_n\}$ that converges to $\liminf b_n$. Then

$$\liminf(a_n + b_n) \leq \lim_{k\to\infty}(a_{n_k} + b_{n_k}) = a + \liminf b_n$$

where the inequality follows because $\liminf(a_n + b_n)$ is the smallest subsequential limit of the sequence $\{b_n\}$. We can therefore conclude $\liminf(a_n + b_n) = a + \liminf b_n$. This result then implies

$$\limsup(a_n + b_n) = -\liminf(-(a_n + b_n)) = -(-a + \liminf(-b_n)) = a + \limsup b_n$$

Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise a.e. on E to f. Let $\{g_n\}$ be a sequence of nonnegative measurable functions on E that converges pointwise a.e. on E to g and that satisfies $|f_n| \leq g_n$ on E for all n. Suppose $\lim_{n\to\infty} \int_E g_n = \int_E g < \infty$. We may assume without loss that $\int_E g_n < \infty$ for all n. By the monotonicity of integration, we know that $\int_E |f_n| \leq \int_E g_n < \infty$. We also have

$$\int_E |f| \leq \liminf \int_E |f_n| \leq \liminf \int_E g_n = \int_E g < \infty$$

where the first inequality follows from Fatou's Lemma. By possibly excising from E a countable collection of sets of measure zero and applying Proposition 15, we may assume that f_n, g_n, f and g are finite on E. The functions $g - f$ and $g_n - f_n$ for each n are then properly defined, non-negative and measurable. Moreover, the sequence $\{g_n - f_n\}$ converges a.e. on E to $g - f$. We therefore have

$$\begin{aligned}
\int_E g - \int_E f &= \int_E (g - f) \\
&\leq \liminf \int_E (g_n - f_n) \\
&= \lim_{n\to\infty} \int_E g_n - \limsup \int_E f_n \\
&= \int_E g - \limsup \int_E f_n
\end{aligned}$$

where the second line follows from Fatou's Lemma and the third line follows from linearity of integration and the preliminary result. We can thus conclude

$$\limsup \int_E f_n \leq \int_E f$$

An analogous argument applied to the sequence $\{f_n - g_n\}$ can be used to show

$$\int_E f \leq \liminf \int_E f_n$$

33. Suppose $\int_E |f - f_n| \to 0$. Since $\big||f_n| - |f|\big| \leq |f - f_n|$ on E for all n and $|f_n| - |f| \to 0$ a.e. on E, we have

$$\lim_{n\to\infty} \int_E (|f_n| - |f|) = 0$$

by the General Lebesgue Dominated Convergence Theorem. Since f is integrable, this expression implies

$$\lim_{n\to\infty} \int_E |f_n| = \int_E |f|$$

by linearity of integration.

Conversely, suppose $\lim_{n\to\infty} \int_E |f_n| = \int_E |f|$. Then $\lim_{n\to\infty} \int_E (|f_n| + |f|) = 2\int_E |f| < \infty$. Since $|f_n - f| \leq |f_n| + |f|$ on E for all n and $|f - f_n| \to 0$ a.e. on E, we have

$$\lim_{n\to\infty} \int_E |f - f_n| = 0$$

by the General Lebesgue Dominated Convergence Theorem.

34. Let $f_n = f \cdot \chi_{[-n,n]}$. Then $\{f_n\}$ is an increasing sequence of non-negative measurable functions on E converging pointwise to f on $\mathbf{R}$. By the Monotone Convergence Theorem and Problem 28, we have

$$\int_{\mathbf{R}} f = \lim_{n\to\infty} \int_{\mathbf{R}} f_n = \lim_{n\to\infty} \int_{\mathbf{R}} f \cdot \chi_{[-n,n]} = \lim_{n\to\infty} \int_{-n}^{n} f$$

35. Let $\{y_n\}$ be a sequence in $[0,1]$ that converges to 0. Define $f_n(x) = f(x, y_n)$. Then $|f_n| \le g$ on $[0,1]$ and f_n converges pointwise to f on $[0,1]$. By the Lebesgue Dominated Convergence Theorem, we have

$$\int_0^1 f(x)dx = \lim_{n\to\infty} \int_0^1 f_n(x)dx = \lim_{n\to\infty} \int_0^1 f(x, y_n)dx$$

Since this expression holds for any sequence $\{y_n\}$ in $[0,1]$ converging to 0, we can conclude

$$\int_0^1 f(x)dx = \lim_{y\to 0} \int_0^1 f(x,y)dx$$

Now suppose $f(x,y)$ is continuous in y for each x. Fix $y \in [0,1]$ and let $\{y_n\}$ denote a sequence in $[0,1]$ converges to y. Let $f_n(x) = f(x, y_n)$ and let $f(x) = f(x,y)$. Then $|f_n| \le g$ on $[0,1]$ and f_n converges pointwise to f on $[0,1]$. By the Lebesgue Dominated Convergence Theorem, we have

$$\begin{aligned} h(y) &= \int_0^1 f(x,y)dx \\ &= \int_0^1 f(x)dx \\ &= \lim_{n\to\infty} \int_0^1 f_n(x)dx \\ &= \lim_{n\to\infty} \int_0^1 f(x,y_n)dx \\ &= \lim_{n\to\infty} h(y_n) \end{aligned}$$

Since this expression holds for any sequence $\{y_n\}$ in $[0,1]$ converging to y, we can conclude

$$h(y) = \lim_{y'\to y} h(y')$$

Thus h is continuous at y.

36. Notice that the left-hand side of the expression we are asked to verify may not be well-defined. For example, suppose we define $f(x,y)$ on Q as

$$f(x,y) = \begin{cases} \dfrac{1}{x} + y & \text{if } 0 < x \le 1 \\ 0 & \text{if } x = 0 \end{cases}$$

Although this function satisfies the conditions given in the problem, the derivative of $\int_0^1 f(x,y)dx = \infty$ with respect to y is not defined.

To avoid this problem, assume there exists $y_0 \in [0,1]$ such that $f(x, y_0)$ is an integrable function of x on $[0,1]$. For any $(x,y) \in [0,1]^2$, we have

$$\begin{aligned} |f(x,y)| &= \left| f(x,y_0) + \int_{y_0}^{y} \frac{\partial f}{\partial y}(x,t)dt \right| \\ &\leq |f(x,y_0)| + \int_{y_0}^{y} \left| \frac{\partial f}{\partial y}(x,t) \right| dt \\ &\leq |f(x,y_0)| + g(x) \end{aligned}$$

By monotonicity and linearity of integration, we can conclude from this expression that $f(x,y)$ is an integrable function of x on $[0,1]$ for any $y \in [0,1]$.

Now fix $y \in [0,1]$. Let $\{h_n\}$ denote a sequence that converges to zero and assume $y + h_n \in [0,1]$ for all n. Let $f(x) = \frac{\partial f}{\partial y}(x,y)$ and define

$$f_n(x) = \frac{f(x, y+h_n) - f(x,y)}{h_n}$$

Then $f_n(x)$ converges pointwise to $f(x)$ on $[0,1]$ and $|f_n(x)| \leq g(x)$ by the Mean Value Theorem. We can therefore conclude

$$\begin{aligned} \frac{d}{dy}\left[\int_0^1 f(x,y)dx\right] &= \lim_{n\to\infty} \frac{\int_0^1 f(x,y+h_n)dx - \int_0^1 f(x,y)dx}{h_n} \\ &= \lim_{n\to\infty} \int_0^1 f_n(x)dx \\ &= \int_0^1 f(x)dx \\ &= \int_0^1 \frac{\partial f}{\partial y}(x,y)dx \end{aligned}$$

where the second equality follows from linearity of integration and the third equality follows from the Lebesgue Dominated Convergence Theorem.

4.5 Countable Additivity and Continuity of Integration

37. Since $\{E_n\}$ is a descending countable collection of measurable subsets of E and $m\left(\cap_{n=1}^{\infty} E_n\right) = m(\emptyset) = 0$, we have

$$0 = \int_{\cap_{n=1}^{\infty} E_n} f = \lim_{n\to\infty} \int_{E_n} f$$

by continuity of integration. The desired result then follows from the formal definition of a limit.

38. (i) For any natural number $n \geq 2$, we have

$$\int_1^n f = \int_1^n \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \cdot \chi_{[k,k+1)} = \sum_{k=1}^{n-1} \frac{(-1)^k}{k}$$

This series converges by the alternating series test, so $\lim_{n\to\infty} \int_1^n f$ is well-defined. However, we also have

$$\int_1^\infty |f| = \lim_{n\to\infty}\int_1^n |f| = \lim_{n\to\infty}\int_1^n \sum_{k=1}^\infty \frac{1}{k}\cdot\chi_{[k,k+1)} = \lim_{n\to\infty}\sum_{k=1}^{n-1}\frac{1}{k} = \infty$$

where the first equality follows from the Monotone Convergence Theorem and the last equality follows from the divergence of the harmonic series. Therefore f is not integrable over $[1,\infty)$.

(ii) Extend the function to $[0,\infty)$ by defining

$$f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

For any natural number n, we have

$$\begin{aligned}
\int_{2\pi(n-1)}^{2\pi n} f &= \int_{(2n-2)\pi}^{(2n-1)\pi} \frac{\sin x}{x}dx + \int_{(2n-1)\pi}^{2n\pi} \frac{\sin x}{x}dx \\
&= \int_0^\pi \frac{\sin(x+(2n-2)\pi)}{x+(2n-2)\pi}dx + \int_0^\pi \frac{\sin(x+(2n-1)\pi)}{x+(2n-1)\pi}dx \\
&= \int_0^\pi \frac{\sin x}{x+(2n-2)\pi}dx - \int_0^\pi \frac{\sin x}{x+(2n-1)\pi}dx
\end{aligned}$$

Thus

$$\begin{aligned}
\int_0^{2\pi n} f &= \sum_{k=1}^n \int_{2\pi(k-1)}^{2\pi k} f \\
&= \sum_{i=0}^{2n-1} (-1)^i \int_0^\pi \frac{\sin x}{x+i\pi}dx
\end{aligned}$$

But $\int_0^\pi \frac{\sin x}{x+i\pi}$ is positive and decreasing in i, so the above series converges by the alternating series test. Therefore $\lim_{n\to\infty}\int_0^{2\pi n} f$ is well-defined. Since f is continuous, $\int_0^1 f$ is finite and $\lim_{n\to\infty}\int_1^n f = \lim_{n\to\infty}\int_0^{2\pi n} f - \int_0^1 f$ is well-defined.

We also see that

$$\begin{aligned}
\int_{2\pi(n-1)}^{2\pi n} |f| &= \int_0^\pi \left|\frac{\sin(x+(2n-2)\pi)}{x+(2n-2)\pi}\right| dx + \int_0^\pi \left|\frac{\sin(x+(2n-1)\pi)}{x+(2n-1)\pi}\right| dx \\
&= \int_0^\pi \frac{\sin x}{x+(2n-2)\pi}dx + \int_0^\pi \frac{\sin x}{x+(2n-1)\pi}dx \\
&\geq \frac{1}{(2n-1)\pi}\int_0^\pi \sin x\,dx + \frac{1}{2n\pi}\int_0^\pi \sin x\,dx \\
&= \frac{2}{(2n-1)\pi} + \frac{2}{2n\pi}
\end{aligned}$$

which means

$$\int_0^\infty |f| = \lim_{n\to\infty} \int_0^{2\pi n} |f|$$
$$\geq \lim_{n\to\infty} \frac{2}{\pi} \sum_{i=1}^{2n} \frac{1}{i}$$
$$= \infty$$

by the Monotone Convergence Theorem and the divergence of the harmonic series. Since $\int_0^1 |f|$ is finite, this result implies $\int_1^\infty |f|$ is infinite. Thus f is not integrable over $[1,\infty)$.

The above results do not contradict continuity of integration because continuity only applies to integrable functions.

39. Let f be integrable over E and suppose $\{E_n\}_{n=1}^\infty$ is an ascending countable collection of measurable subsets of E. Define $E_0 = \emptyset$ and $C_k = E_k \sim E_{k-1}$. Then

$$\{C_k\}_{k=1}^\infty \text{ is disjoint, } \bigcup_{k=1}^n C_k = E_n \text{ for all } n, \text{ and } \bigcup_{k=1}^\infty E_k = \bigcup_{k=1}^\infty C_k$$

We therefore have

$$\int_{\cup_{n=1}^\infty E_n} f = \int_{\cup_{n=1}^\infty C_n} f = \lim_{n\to\infty} \sum_{k=1}^n \int_{C_k} f = \lim_{n\to\infty} \int_{\cup_{i=1}^n C_k} f = \lim_{n\to\infty} \int_{E_n} f$$

where the second equality follows from countable additivity of integration and the third equality from the additivity over domains property.

Now suppose $\{E_n\}_{n=1}^\infty$ is a descending countable collection of measurable subsets of E. We then have

$$\int_{\cap_{n=1}^\infty E_n} f = \int_E f - \int_{E\sim\cap_{n=1}^\infty E_n} f$$
$$= \int_E f - \int_{\cup_{n=1}^\infty (E\sim E_n)} f$$
$$= \int_E f - \lim_{n\to\infty} \int_{E\sim E_n} f$$
$$= \lim_{n\to\infty} \left(\int_E f - \int_{E\sim E_n} f \right)$$
$$= \lim_{n\to\infty} \int_{E_n} f$$

where the first and last lines follow from the additivity over domains property and the third line follows from continuity of integration for an ascending countable collection of measurable sets.

4.6 Uniform Integrability: The Vitali Convergence Theorem

40. For any $x \in \mathbf{R}$, we have

$$\int_{-\infty}^{x} |f| \leq \int_{\mathbf{R}} |f| < \infty$$

by monotonicity of integration. Therefore $F(x) = \int_{-\infty}^{x} f$ is well-defined.

Now fix $\epsilon > 0$. By Proposition 23, there exists $\delta > 0$ such that if $A \subseteq E$ is measurable and $m(A) < \delta$, then $\int_A |f| < \epsilon$. Suppose $|x' - x| < \delta$ and without loss of generality assume $x' > x$. Then

$$|F(x') - F(x)| = \left| \int_{x}^{x'} f \right| \leq \int_{x}^{x'} |f| < \epsilon$$

Therefore F is continuous.

Now suppose

$$f(x) = \begin{cases} \dfrac{1}{2\sqrt{x}} & \text{if } 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Then

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \sqrt{x} & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

But F is not Lipschitz (see the solution to Problem 50 of Chapter 1).

41. Let $f_n = \chi_{[0,n)}$. Fix $\epsilon > 0$ and let $\delta = \epsilon$. Then for any measurable set $A \subseteq \mathbf{R}$ with $m(A) < \delta$, we have

$$\int_A |f_n| \leq \int_A 1 = m(A) < \epsilon$$

Therefore the sequence of functions $\{f_n\}$ is uniformly integrable. But $\{f_n\} \to \chi_{[0,\infty)}$ on $\mathbf{R}$, but $\chi_{[0,\infty)}$ is not integrable.

42. Let $E = [0, 1]$ and define $h_n = n\chi_{[0,1/n)} - n\chi_{[1-1/n,1]}$. Then E has finite measure, each h_n is integrable and $\{h_n\}$ converges pointwise a.e. to $h \equiv 0$ on E. We also have $\int_E h_n = 0$ for all n, so

$$\lim_{n\to\infty} \int_E h_n = 0$$

Let $\epsilon = 1/2$. Fix $\delta > 0$ and choose $N > 1/\delta$. Let $A = [0, 1/N)$. Then $A \subseteq E$ is measurable and $m(A) < \delta$, but

$$\int_A h_n = 1 \geq \epsilon$$

for all $n \geq N$. Therefore $\{h_n\}$ is not uniformly integrable.

43. Fix $\epsilon > 0$ and choose $\alpha \in \mathbf{R}$. Suppose $\alpha = 0$. Then for any measurable set $A \subseteq E$, $\int_A |\alpha f_n| = 0 < \epsilon$ for all n. Thus $\{\alpha f_n\}$ is uniformly integrable. If $\alpha \neq 0$, choose δ to satisfy the uniform integrability

challenge for $\epsilon/|\alpha|$. If $A \subseteq E$ is measurable and $m(A) < \delta$, then

$$\int_A |\alpha f_n| = |\alpha| \int_A |f_n| < |\alpha| \cdot \frac{\epsilon}{|\alpha|} = \epsilon$$

for all n. Therefore $\{\alpha f_n\}$ is uniformly integrable for all α.

Now choose δ_f and δ_g to satisfy the $\epsilon/2$ uniform integrability challenge for for $\{f_n\}$ and $\{g_n\}$, respectively. Let $\delta = \min\{\delta_f, \delta_g\}$. If $A \subseteq E$ is measurable and $m(A) < \delta$, then

$$\int_A |f_n + g_n| \leq \int_A |f_n| + \int_A |g_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all n. Therefore $\{f_n + g_n\}$ is uniformly integrable.

Combining the above results, we conclude that $\{\alpha f_n + \beta g_n\}$ is uniformly integrable for all α and β.

44. (i) Suppose f is non-negative. By Problem 24, we can find an increasing sequence of simple functions $\{\varphi_n\}$, each of finite support, that converges pointwise to f. By the Monotone Convergence Theorem, there exists an index N such that

$$\int_{\mathbf{R}} |f - \varphi_n| = \int_{\mathbf{R}} f - \int \varphi_n < \epsilon$$

for all $n \geq N$.

In general, we can write $f = f^+ - f^-$ where f^+ and f^- denote the positive and negative parts of f. By the previous result, we can find simple functions of finite support η^+ and η^- that satisfy

$$\int_{\mathbf{R}} |f^+ - \eta^+| < \frac{\epsilon}{2} \qquad \int_{\mathbf{R}} |f^- - \eta^-| < \frac{\epsilon}{2}$$

Let $\eta = \eta^+ - \eta^-$. Then η is a simple function of finite support that satisfies

$$\int_{\mathbf{R}} |f - \eta| = \int_{\mathbf{R}} \left|f^+ - \eta^+ - (f^- - \eta^-)\right| \leq \int_{\mathbf{R}} \left|f^+ - \eta^+\right| + \int_{\mathbf{R}} \left|f^- - \eta^-\right| < \epsilon$$

(ii) Suppose f is non-negative. Let $f_n = \min\{f, n\} \cdot \chi_{[-n,n]}$. Then $\{f_n\}$ converges upward to f, so by the Monotone Convergence Theorem

$$\lim_{n\to\infty} \int_{\mathbf{R}} f_n = \int_{\mathbf{R}} f$$

Pick n such that

$$\int_{\mathbf{R}} f - \int_{\mathbf{R}} f_n < \frac{\epsilon}{2}$$

f_n is a bounded, measurable function on the closed and bounded interval $I = [-n, n]$. By Problem 18 of Chapter 3, there exists a step function s on I and a measurable subset F of I for which

$$f_n = s \text{ on } F \text{ and } m(I \sim F) < \frac{\epsilon}{4n}$$

Without loss, we can assume $0 \leq s \leq n$ (if not, replace s with $\max\{0, \min\{s, n\}\}$). By the additivity over domains and monotonicity properties of integration, we have

$$\int_I |f_n - s| = \int_F |f_n - s| + \int_{I\sim F} |f_n - s| \leq \int_{I\sim F} |f_n| + \int_{I\sim F} |s| \leq 2n \cdot m(I \sim F) < \frac{\epsilon}{2}$$

Extend s to $\mathbf{R}$ by defining $s = 0$ on $\mathbf{R} \sim I$. Then

$$\int_{\mathbf{R}} |f - s| \leq \int_{\mathbf{R}} |f - f_n| + \int_{\mathbf{R}} |f_n - s| = \int_{\mathbf{R}} |f - f_n| + \int_I |f_n - s| < \epsilon$$

The result for general f can be proven using the same argument employed in part (i).

(iii) We can proceed as in part (ii), replacing s with a continuous function g and the result from Problem 18 of Chapter 3 with Lusin's Theorem.

45. By the additive domains property, we have

$$\int_{\mathbf{R}} |\hat{f}| = \int_E |\hat{f}| + \int_{\mathbf{R}\sim E} |\hat{f}| = \int_E |f| < \infty$$

Therefore $\hat{f}$ is integrable. The additive domains property likewise implies $\int_{\mathbf{R}} \hat{f} = \int_E f$.

By Problem 45, we can find a simple function $\hat{\eta}$ on $\mathbf{R}$ that satisfies

$$\int_{\mathbf{R}} |\hat{f} - \hat{\eta}| < \epsilon$$

Let η denote the restriction of $\hat{\eta}$ to E. Then

$$\int_E |f - \eta| = \int_E |\hat{f} - \hat{\eta}| \leq \int_{\mathbf{R}} |\hat{f} - \hat{\eta}| < \epsilon$$

An analogous argument can be used to find a continuous function satisfying the desired property.

46. Since $|f(x)\cos nx| < |f(x)|$, $f(x)\cos nx$ is integrable over $\mathbf{R}$. By Problem 44(ii), we can find a step function s that vanishes outside a closed, bounded interval and satisfies $\int_{-\infty}^{\infty} |f - s| < \epsilon/2$. Since s is a step function, there exists a partition $P = \{x_0, x_1, \cdots, x_N\}$ and numbers $c_1, \cdots, c_N$ such that

$$s(x) = \begin{cases} c_i & \text{if } x_{i-1} < x < x_i,\ 1 \leq i \leq N \\ 0 & \text{otherwise} \end{cases}$$

We therefore have

$$\begin{aligned} \left| \int_{-\infty}^{\infty} s(x)\cos(nx)dx \right| &= \left| \sum_{i=1}^N c_i \int_{x_{i-1}}^{x_i} \cos(nx)dx \right| \\ &= \frac{1}{n} \left| \sum_{i=1}^N c_i \left(\sin(nx_i) - \sin(nx_{i-1})\right) \right| \\ &\leq \frac{2 \cdot N \cdot \max_i |c_i|}{n} \end{aligned}$$

For all $n > \frac{4 \cdot N \cdot \max_i |c_i|}{\epsilon}$, we have

$$\begin{aligned}\left|\int_{-\infty}^{\infty} f(x)\cos(nx)dx\right| &\leq \left|\int_{-\infty}^{\infty} \big(f(x)-s(x)\big)\cos(nx)dx\right| + \left|\int_{-\infty}^{\infty} s(x)\cos(nx)dx\right| \\ &\leq \int_{-\infty}^{\infty} |f(x)-s(x)|\, dx + \left|\int_{-\infty}^{\infty} s(x)\cos(nx)dx\right| \\ &< \epsilon\end{aligned}$$

Since ϵ can be made arbitrarily small, the result follows.

47. (i) Suppose $f = \chi_E$ for a measurable set E. For any real number t, we have

$$\chi_E(x+t) = \chi_{E-t}(x) \text{ for all } x \in \mathbf{R}$$

Translation invariance of Lebesgue measure then implies

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \chi_E(x)dx = m(E) = m(E-t) =$$
$$\int_{-\infty}^{\infty} \chi_{E-t}(x)dx = \int_{-\infty}^{\infty} \chi_E(x+t)dx = \int_{-\infty}^{\infty} f(x+t)dx$$

The result extends to simple functions by linearity of integration. If f is a measurable non-negative function, then by the Simple Approximation Theorem there exists an increasing sequence of simple functions $\{\varphi_n\}$ that converges pointwise to f. For each x, $\{\varphi_n(x+t)\}$ is an increasing sequence converging to $f(x+t)$. The Monotone Convergence Theorem therefore implies

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{n\to\infty}\int_{-\infty}^{\infty} \varphi_n(x)dx = \lim_{n\to\infty}\int_{-\infty}^{\infty} \varphi_n(x+t)dx = \int_{-\infty}^{\infty} f(x+t)dx$$

The result can be extended to a general integrable function f by applying the result for non-negative functions to the positive and negative parts of f.

(ii) Let $\{t_n\}$ denotes a sequence of real numbers converging to 0. Since g is bounded, there exists a number M such that $|g| < M$ on $\mathbf{R}$. We can use the result from Problem 44(iii) to find a continuous function $\hat{f}$ on $\mathbf{R}$ that vanishes outside a closed and bounded set E and that satisfies $\int_{\mathbf{R}} |f - \hat{f}| < \frac{\epsilon}{2M}$. We therefore have

$$\begin{aligned}&\left|\int_{-\infty}^{\infty} g(x)(f(x)-f(x+t))dx\right| \\ &\leq M\int_{-\infty}^{\infty}\left|\hat{f}(x)-\hat{f}(x+t_n)\right|dx + M\int_{-\infty}^{\infty}\left|f(x)-\hat{f}(x)\right|dx + M\int_{-\infty}^{\infty}\left|f(x+t_n)-\hat{f}(x+t_n)\right|dx \\ &< M\int_{-\infty}^{\infty}\left|\hat{f}(x)-\hat{f}(x+t_n)\right|dx + \epsilon\end{aligned} \tag{1}$$

where the last line follows from part (i). By The Extreme Value Theorem, $\hat{f}$ is bounded above by some constant $\hat{M}$ on E. Thus $\left|\hat{f}(x+t_n)-\hat{f}(x)\right|$ converges pointwise to 0 on a bounded set $\hat{E} \supseteq E$, vanishes outside $\hat{E}$, and is bounded above by $2\hat{M}$. The Dominated Convergence Theorem then implies

$$0 = \lim_{n\to\infty} \int_{\hat{E}} \left|\hat{f}(x+t_n) - \hat{f}(x)\right| = \lim_{n\to\infty} \int_{-\infty}^{\infty} \left|\hat{f}(x+t_n) - \hat{f}(x)\right|$$

Taking the lim sup of both sides of (1), we obtain

$$\limsup_{n} \left| \int_{-\infty}^{\infty} g(x)(f(x) - f(x+t_n))dx \right| \leq \epsilon$$

Since ϵ can be made arbitrarily small, we have

$$\lim_{n\to\infty} \left| \int_{-\infty}^{\infty} g(x)(f(x) - f(x+t_n))dx \right| = 0$$

Since this expression holds for all sequences $\{t_n\}$ converging to zero, the desired result follows.

48. Since g is bounded, there exists a real number M such that $|g| \leq M$ on E. Monotonicity of integration then implies

$$\int_E |f \cdot g| = \int_E |f| \cdot |g| \leq M \int_E |f| < \infty$$

49.

(i) $\to$ (ii): If f is 0 a.e. on $\mathbf{R}$ and g is a bounded function, then $fg = 0$ a.e. on $\mathbf{R}$. But this implies $\int_{\mathbf{R}} fg = 0$ by Proposition 9.

(ii) $\to$ (iii): Fix a measurable set A. If (ii) is true, then $\int_A f = \int_{\mathbf{R}} f \cdot \chi_A = 0$ by Problem 28.

(iii) $\to$ (iv): (iv) follows from (iii) because every open set is measurable.

(iv) $\to$ (i): Let $f_n = |f| \cdot \chi_{(-n,n)}$. Then $\{f_n\}$ converges upward to $|f|$. If (iv) is true, then

$$\int_{\mathbf{R}} |f| = \lim_{n\to\infty} \int_{\mathbf{R}} f_n = \lim_{n\to\infty} \int_{-n}^{n} |f| = 0$$

where the first equality follows from the Monotone Convergence Theorem. (i) then follows from Proposition 9.

50. Suppose $\mathcal{F}$ is uniformly integrable. Fix $\epsilon > 0$ and let $\delta > 0$ correspond to the ϵ challenge in the uniform integrability criteria. For any measurable set $A \subseteq E$ satisfying $m(A) < \delta$, we have

$$\left| \int_A f \right| \leq \int_A |f| < \epsilon$$

for all $f \in \mathcal{F}$.

To show the converse, choose $\delta > 0$ such that if $A \subseteq E$ is measurable and $m(A) < \delta$, then $\left|\int_A f\right| < \frac{\epsilon}{2}$. Fix an $A \subseteq E$ that satisfies these conditions. Let $B = \{x \in E : f(x) \geq 0\}$. Then $f = f^+$ on B and $f = -f^-$ on $E \sim B$, where f^+ and f^- denote the positive and negative parts of f. Moreover, B is a measurable set and both $m(A \cap B)$ and $m(A \cap B^C)$ are less than δ. We therefore have

$$\begin{aligned} \int_A |f| &= \int_{A\cap B} |f| + \int_{A\cap B^C} |f| \\ &= \int_{A\cap B} f^+ + \int_{A\cap B^C} f^- \\ &= \left|\int_{A\cap B} f\right| + \left|\int_{A\cap B^C} f\right| \\ &< \epsilon \end{aligned}$$

for all $f \in \mathcal{F}$. Thus $\mathcal{F}$ is uniformly integrable.

51. If $\mathcal{U}$ is an open set, then $E \cap \mathcal{U}$ is a measurable set contained in E. Uniform integrability then immediately implies the condition given in the problem.

    To show the converse, fix $\epsilon > 0$ and let δ correspond to the ϵ challenge given in the problem. Let $A \subseteq E$ be a measurable set that satisfies $m(A) < \delta$. By Theorem 11 of Chapter 2, we can find an open set $\mathcal{U}$ containing A for which $m(\mathcal{U} \sim A) < \delta - m(A)$. Then

    $$m(E \cap \mathcal{U}) = m(A) + m((E \cap \mathcal{U}) \sim A) \leq m(A) + m(\mathcal{U} \sim A) < \delta$$

    Therefore

    $$\int_A |f| \leq \int_{E\cap\mathcal{U}} |f| < \epsilon$$

    for all $f \in \mathcal{F}$. Thus $\mathcal{F}$ is uniformly integrable.

52. (a) Define $f_n = n \cdot \chi_{[0,1/n]}$ and $\mathcal{F} = \{f_n\}$. Then $\int_0^1 |f_n| = 1$ for all n. Let $\epsilon = 1/2$ and fix any $\delta > 0$. Choose $n > 1/\delta$ and define $A = [0, 1/n]$. Then $m(A) = 1/n < \delta$ but $\int_A |f_n| = 1 > \epsilon$. Therefore $\mathcal{F}$ is not uniformly integrable.

    (b) Let $\mathcal{F}$ be a family of functions on $[0,1]$ that satisfies the given properties. Fix $\epsilon > 0$ and let $\delta = \epsilon/2$. Choose $f \in \mathcal{F}$. Since f is bounded and measurable, it is integrable over $[0,1]$ by Theorem 4. If $A \subseteq [0,1]$ is a measurable set satisfying $m(A) < \delta$, then

    $$\int_A |f| \leq \int_A 1 = m(A) < \delta < \epsilon$$

    by monotonicity of integration. Therefore $\mathcal{F}$ is uniformly integrable.

    (c) Let $\mathcal{F}$ be a family of functions on $[0,1]$ that satisfies the given properties. Extend each $f \in \mathcal{F}$ to the real line by defining $f \equiv 0$ on $\mathbf{R} \sim [0,1]$. Then $\int_a^b |f| \leq b - a$ for all $(a,b) \subseteq \mathbf{R}$. Fix $\epsilon > 0$ and choose a measurable set $A \subseteq [0,1]$ such that $m(A) < \epsilon$. By Theorem 11 of Chapter 2, we can find an open set $\mathcal{O} \supseteq A$ such that $m(\mathcal{O} \sim A) < \epsilon - m(A)$. By Proposition 9 of Chapter 1, $\mathcal{O}$ can be written as

    $$\mathcal{O} = \bigcup_{i=1}^{\infty} (a_i, b_i)$$

where $a_1 < b_1 < a_2 < \cdots$. Since $m(\mathcal{O}) = m(A) + m(\mathcal{O} \sim A) < \epsilon$, we have

$$\int_A |f| \le \int_{\mathcal{O}} |f| = \sum_{n=1}^{\infty} \int_{a_n}^{b_n} |f| \le \sum_{n=1}^{\infty} (b_n - a_n) = m(\mathcal{O}) < \epsilon$$

by monotonicity and countable additivity of integration. Therefore $\mathcal{F}$ is uniformly integrable.

5 Lebesgue Integration: Further Topics

5.1 Uniform Integrability and Tightness: A General Vitali Convergence Theorem

1. If $\{h_n\}$ is uniformly integrable and tight over E, then $\lim_{n\to\infty} \int_E h_n = 0$ by the Vitali Convergence Theorem.

   Conversely, suppose $\lim_{n\to\infty} \int_E h_n = 0$. Pick N such that $\int_E h_n < \epsilon$ for all $n \ge N$. By Problem 2, we know that the finite collection of functions $\{h_n\}_{n=1}^N$ is tight. We can therefore find a set of finite measure $E_0 \subseteq E$ such that
   $$\int_{E\sim E_0} |h_n| < \epsilon$$
   for all $n < N$. Since
   $$\int_{E\sim E_0} |h_n| \le \int_E h_n < \epsilon$$
   for all $n \ge N$, we can conclude that $\{h_n\}$ is tight over E. The proof that $\{h_n\}$ is uniformly integrable remains unchanged from the proof of uniform integrability in Theorem 26 of Chapter 4.

2. For each f_k, we can apply Proposition 1 to find a set of finite measure $E_k \subseteq E$ that satisfies
   $$\int_{E\sim E_k} |f_k| < \epsilon$$
   Let $E_0 = \bigcup_{k=1}^n E_k$. Then $m(E_0) \le \sum_{k=1}^n m(E_k) < \infty$ and
   $$\int_{E\sim E_0} |f_k| \le \int_{E\sim E_k} |f_k| < \epsilon$$
   for $k = 1, \cdots, n$. Therefore $\{f_k\}_{k=1}^n$ is tight.

   $\{f_k\}_{k=1}^n$ is uniformly integrable by Proposition 24 of Chapter 4.

3. Fix $\epsilon > 0$ and choose $\alpha \in \mathbf{R}$. Suppose $\alpha = 0$. Then $\int_{E\sim\emptyset} |\alpha f_n| = 0 < \epsilon$, so $\{\alpha f_n\}$ is tight. If $\alpha \ne 0$, let $E_0 \subseteq E$ denote a set finite measure that satisfies
   $$\int_{E\sim E_0} |f_n| < \frac{\epsilon}{|\alpha|}$$
   for all n. Then
   $$\int_{E\sim E_0} |\alpha f_n| = |\alpha| \int_{E\sim E_0} |f_n| < \epsilon$$

for all n. Therefore $\{\alpha f_n\}$ is tight.

Now choose E_f and E_g to satisfy the $\epsilon/2$ tightness challenge for $\{f_n\}$ and $\{g_n\}$, respectively. Let $E_0 = E_f \cup E_g$. Then $m(E_0) \leq m(E_f) + m(E_g) < \infty$ and

$$\int_{E \sim E_0} |f_n + g_n| \leq \int_{E \sim E_0} |f_n| + \int_{E \sim E_0} |g_n| \leq \int_{E \sim E_f} |f_n| + \int_{E \sim E_g} |g_n| < \epsilon$$

for all n. Therefore $\{f_n + g_n\}$ is tight.

Combining the above results, we conclude that $\{\alpha f_n + \beta g_n\}$ is tight for all α and β.

$\{\alpha f_n + \beta g_n\}$ is uniformly integrable from Problem 43 of Chapter 4.

4. Suppose $\{f_n\}$ is uniformly integrable and tight over E. Let $E_0 \subseteq E$ denote a set of finite measure that satisfies

$$\int_{E \sim E_0} |f_n| < \frac{\epsilon}{2}$$

for all n. Let δ correspond to the $\epsilon/2$ uniform integrability challenge. Choose a measurable set $A \subseteq E$ that satisfies $m(A \cap E_0) < \delta$. Then $\int_{A \cap E_0} |f_n| < \epsilon/2$ for all n, so

$$\int_A |f_n| = \int_{A \cap E_0} |f_n| + \int_{A \sim E_0} |f_n| \leq \int_{A \cap E_0} |f_n| + \int_{E \sim E_0} |f_n| < \epsilon$$

for all n.

Conversely, suppose the given statement is true. Let δ and E_0 correspond to the given ϵ challenge. Pick a measurable set $A \subseteq E$ that satisfies $m(A) < \delta$. Then $m(A \cap E_0) < \delta$, so $\int_A |f_n| < \epsilon$ for all n. Therefore $\{f_n\}$ is uniformly integrable. Now define $A = E \sim E_0$. Then $m(A \cap E_0) = m(\emptyset) = 0 < \delta$, so $\int_{E \sim E_0} |f_n| < \epsilon$ for all n. Therefore $\{f_n\}$ is tight.

5. Suppose $\{f_n\}$ is uniformly integrable and tight. Fix $\epsilon > 0$. Let δ correspond to the $\epsilon/3$ uniform integrability challenge. Since $\{f_n\}$ is tight, we can find a set of finite measure E_0 that satisfies

$$\int_{\mathbf{R} \sim E_0} |f_n| < \frac{\epsilon}{3}$$

for all n. Let $E_r = (-r, r) \cap E_0$. Then $E_r \uparrow E_0$, so $\lim_{r \to \infty} m(E_r) = m(E_0)$ by continuity of measure. Choose r such that $m(E_0) < m(E_r) + \delta$. Then $m(E_0 \sim E_r) = m(E_0 \sim (-r, r)) < \delta$, so $\int_{E_0 \sim (-r,r)} |f_n| < \epsilon/3$ for all n by uniform integrability. Let $\mathcal{O}$ denote an open subset of $\mathbf{R}$ that satisfies $m(\mathcal{O} \cap (-r, r)) < \delta$. Then $\int_{\mathcal{O} \cap (-r,r)} |f_n| < \epsilon/3$ for all n by uniform integrability, so we have

$$\begin{aligned}
\int_{\mathcal{O}} |f_n| &= \int_{\mathcal{O} \cap (-r,r)} |f_n| + \int_{\mathcal{O} \sim (-r,r)} |f_n| \\
&\leq \int_{\mathcal{O} \cap (-r,r)} |f_n| + \int_{\mathbf{R} \sim (-r,r)} |f_n| \\
&\leq \int_{\mathcal{O} \cap (-r,r)} |f_n| + \int_{E_0 \sim (-r,r)} |f_n| + \int_{\mathbf{R} \sim E_0} |f_n| \\
&< \epsilon
\end{aligned}$$

for all n.

Conversely, suppose the statement is true. Let δ and r correspond to the given ϵ challenge. Let A denote a measurable subset of $\mathbf{R}$ that satisfies $m\left(A \cap (-r, r)\right) < \delta$. By Theorem 11 of Chapter 2, we can find an open subset $\mathcal{O}$ of $\mathbf{R}$ that contains A and that satisfies $m(\mathcal{O} \sim A) < \delta - m(A \cap (-r, r))$. Then

$$m(\mathcal{O} \cap (-r, r)) \leq m(A \cap (-r, r)) + m(\mathcal{O} \sim A) < \delta$$

so that

$$\int_A |f_n| \leq \int_{\mathcal{O}} |f_n| < \epsilon$$

for all n. But then $\{f_n\}$ is uniformly integrable and tight by Problem 4.

5.2 Convergence in Measure

6. Suppose $\{f_n\} \to g$ in measure on E. Let

$$E_k = \left\{ x \in E : |f(x) - g(x)| > \frac{1}{k} \right\}$$

Then

$$m(E_k) \leq m\left(|f_n - f| > \frac{1}{2k}\right) + m\left(|f_n - g| > \frac{1}{2k}\right) \to 0 \text{ as } n \to \infty$$

Therefore $m(E_k) = 0$ for all k, so $m\left(\bigcup_{k=1}^{\infty} E_k\right) = 0$. Since $f = g$ on $E \sim \bigcup_{k=1}^{\infty} E_k$, $f = g$ a.e. on E.

Conversely, suppose $f = g$ a.e. on E. Then there exists a set E_0 with $m(E_0) = 0$ such that $f = g$ on $E \sim E_0$. We then have

$$\begin{aligned} m\left(|f_n - g| > \eta\right) &= m\left(x \in E \sim E_0 : |f_n(x) - g(x)| > \eta\right) \\ &= m\left(x \in E \sim E_0 : |f_n(x) - f(x)| > \eta\right) \\ &= m\left(|f_n - f| > \eta\right) \\ &\to 0 \end{aligned}$$

Thus $\{f_n\} \to g$ in measure on E.

7. Fix $\eta > 0$. First suppose that g is a simple function. Then g can be written as $\sum_{k=1}^{K} c_k \chi_{A_k}$, where each c_k is non-zero scalar, each $A_k \subseteq E$ is a measurable set, and $\{A_k\}_{k=1}^{K}$ are disjoint. We have

$$\begin{aligned} m\left(|f_n \cdot g - f \cdot g| > \eta\right) &= \sum_{k=1}^{K} m\left(\left\{ x \in A_k : |f_n(x) - f(x)| > \frac{\eta}{|c_k|} \right\}\right) \\ &\leq \sum_{k=1}^{K} m\left(|f_n - f| > \frac{\eta}{|c_k|}\right) \\ &\to 0 \text{ as } n \to \infty \end{aligned}$$

where the last line follows because $\{f_n\} \to f$ in measure on E. Thus $\{f_n \cdot g\} \to f \cdot g$ in measure on E.

Now let g be a measurable function on E that is finite a.e. on E. By the Simple Approximation

Theorem, we can construct a sequence $\{\phi_k\}$ of simple functions on E that converges pointwise on E to g. By Egoroff's Theorem, for any $\epsilon > 0$ we can find a closed set F contained in E for which

$$\{\phi_k\} \to g \text{ uniformly on } F \text{ and } m(E \sim F) < \epsilon$$

Choose K such that $|g - \phi_k| < 1$ on F for all $k \geq K$. Then for any $k \geq K$, we have

$$\begin{aligned}
&m\left(\{|f_n \cdot g - f \cdot g| > \eta\}\right) \leq \\
&\qquad m\left(\left\{x \in F : |f_n(x) - f(x)| \cdot |g(x) - \phi_k(x)| > \frac{\eta}{2}\right\}\right) + \\
&\qquad m\left(\left\{x \in E \sim F : |f_n(x) - f(x)| \cdot |g(x) - \phi_k(x)| > \frac{\eta}{2}\right\}\right) + m\left(|f_n \cdot \phi_k - f \cdot \phi_k| > \frac{\eta}{2}\right) \\
&\qquad < m\left(|f_n - f| > \frac{\eta}{2}\right) + \epsilon + m\left(|f_n \cdot \phi_k - f \cdot \phi_k| > \frac{\eta}{2}\right) \\
&\qquad \to \epsilon
\end{aligned}$$

where the last follows because $\{f_n\} \to f$ in measure on E and from the result for simple functions. This expression implies $\limsup_n m\left(\{|f_n \cdot g - f \cdot g| > \eta\}\right) \leq \epsilon$ for all $\epsilon > 0$, so we can conclude that

$$\lim_{n\to\infty} m\left(\{|f_n \cdot g - f \cdot g| > \eta\}\right) = 0$$

Thus $\{f_n \cdot g\} \to f \cdot g$ in measure on E.

Now suppose $\{g_n\} \to g$ in measure on E. Applying our previous result, we obtain

$$\begin{aligned}
m\left(\{|f_n \cdot g_n - f \cdot g| > \eta\}\right) &\leq m\left(\left\{|f_n - f| > \sqrt{\frac{\eta}{3}}\right\}\right) + m\left(\left\{|g_n - g| > \sqrt{\frac{\eta}{3}}\right\}\right) + \\
&\qquad m\left(\left\{|f_n \cdot g - f \cdot g| > \frac{\eta}{3}\right\}\right) + m\left(\left\{|g_n \cdot f - g \cdot f| > \frac{\eta}{3}\right\}\right) \\
&\to 0
\end{aligned}$$

Therefore $\{f_n \cdot g_n\} \to f \cdot g$ in measure on E. From this result, we can immediately infer that $\{f_n^2\} \to f^2$ in measure on E.

8. Let $\{f_n\}$ denote a sequence of nonnegative measurable functions on E. Suppose $\{f_n\} \to f$ in measure on E. Let $\{n_k\}$ denote an increasing sequence of natural numbers that satisfies

$$\int_E f_{n_k} \to \liminf_{n\to\infty} \int_E f_n$$

Since $\{f_{n_k}\} \to f$ in measure on E, we can use Theorem 4 to find a further subsequence $\{f_{n_{k_j}}\}$ that converges a.e. on E to f. By applying Fatou's Lemma for pointwise convergence to this subsequence, we can conclude

$$\int_E f \leq \liminf_{j\to\infty} \int_E f_{n_{k_j}} = \liminf_{n\to\infty} \int_E f_n$$

Fatou's Lemma therefore holds if "pointwise convergence a.e." is replaced with "convergence in measure". Since the Monotone and Dominated Convergence Theorems follow directly from Fatou's Lemma, "pointwise convergence a.e." can be replaced with "convergence in measure" for these results as well.

Now suppose $\{f_n\}$ is uniformly integrable and tight over E. By Theorem 4, there exists a subsequence

$\{f_{n_j}\}$ that converges a.e. on E to f. We can infer that f is integrable by applying the Vitali Convergence Theorem to this subsequence. Now fix $\epsilon > 0$. By uniform integrability of $\{f_n\}$ and Proposition 23 of Chapter 4, $|f_n - f|$ is uniformly integrable. Furthermore, $|f_n - f|$ is tight by tightness of $\{f_n\}$ and Proposition 1. Since $|f_n - f| \to 0$ in measure on E, Corollary 5 implies

$$\lim_{n\to\infty} \int_E |f_n - f| = 0$$

Since

$$-\int_E |f_n - f| \le \int_E f_n - \int_E f \le \int_E |f_n - f|$$

we can conclude that

$$\lim_{n\to\infty} \int_E f_n = \int_E f$$

Thus the Vitali Convergence Theorem also remains valid if "pointwise convergence a.e." is replaced with "convergence in measure".

9. Let $E = \mathbf{R}$, $f_n = \chi_{[n,\infty)}$ and $f = 0$. Then $\{f_n\} \to f$ pointwise on E. However, for any $\epsilon \in (0,1)$ we have

$$m(\{|f_n - f| > \epsilon) = m([n,\infty)) = \infty \not\to 0$$

Therefore $\{f_n\}$ does not converge in measure to f on E.

10. Suppose $\{f_n\} \to f$ in measure on E. Fix a scalar $\alpha \in \mathbf{R}$ and choose any $\eta > 0$. If $\alpha = 0$, then

$$\lim_{n\to\infty} m(|\alpha f_n| > \eta) = \lim_{n\to\infty} m(\emptyset) = 0$$

so that $\{\alpha f_n\} \to 0$ in measure on E. If $\alpha \neq 0$, then

$$\lim_{n\to\infty} m\left(|\alpha f_n - \alpha f| > \eta\right) = \lim_{n\to\infty} m\left(|f_n - f| > \frac{\eta}{|\alpha|}\right) = 0$$

so that $\{\alpha f_n\} \to \alpha f$ in measure on E.

Now suppose $\{g_n\} \to g$ in measure on E. Then

$$m\left(|f_n + g_n - f - g| > \eta\right) \le m\left(|f_n - f| > \frac{\eta}{2}\right) + m\left(|g_n - g| > \frac{\eta}{2}\right) \to 0$$

so that $\{f_n + g_n\} \to f + g$ converges in measure on E.

Combining the above results, we can conclude that $\{\alpha f_n + \beta g_n\} \to \alpha f + \beta g$ in measure on E for all α and β.

11. Suppose $\{f_n\} \to f$ in measure on E. Let $\{f_{n_j}\}$ denote a subsequence of $\{f_n\}$. Then $\{f_{n_j}\} \to f$ in measure on E, so there exists a further subsequence that converges pointwise a.e. on E to f by Theorem 4.

Conversely, suppose $\{f_n\}$ does not converge in measure on E to f. Then there exists some $\eta > 0$ and $\epsilon > 0$ such that

$$m(|f_n - f| > \eta) > \epsilon \text{ infinitely often}$$

We can therefore find a subsequence $\{f_{n_j}\}$ that satisfies

$$m(|f_{n_j} - f| > \eta) > \epsilon$$

for all j. Since $m(E) < \infty$, for any further subsequence $\left\{f_{n_{j_k}}\right\}$, we have

$$\begin{aligned} m\left(\left\{x \in E : \left|f_{n_{j_k}}(x) - f(x)\right| > \eta \text{ infinitely often}\right\}\right) &= m\left(\bigcap_{l=1}^{\infty}\bigcup_{k=l}^{\infty}\{|f_{n_{j_k}} - f| > \eta\}\right) \\ &= \lim_{l\to\infty} m\left(\bigcup_{k=l}^{\infty}\{|f_{n_{j_k}} - f| > \eta\}\right) \\ &\geq \epsilon \end{aligned}$$

by continuity of measure. But since

$$\left\{x \in E : \left|f_{n_{j_k}}(x) - f(x)\right| > \eta \text{ infinitely often}\right\} \subseteq \left\{x \in E : \ \{f_{n_{j_k}}(x)\} \text{ does not converge to } f_n(x)\right\}$$

we cannot have $\left\{f_{n_{j_k}}\right\}$ converging pointwise a.e. to f on E.

12. Let $\{a_j\}$ denote a sequence of real numbers that satisfies $|a_{j+1} - a_j| \leq 2^{-j}$ for all j. Fix $\epsilon > 0$ and let N denote an index satisfying $2^{-N+1} < \epsilon$. Pick $m, n \geq N$. Without loss of generality, assume $m > n$. We then have

$$\begin{aligned} |a_m - a_n| &\leq \sum_{j=n}^{m-1} |a_{j+1} - a_j| \\ &\leq 2^{-n} \sum_{j=0}^{m-n-1} 2^{-j} \\ &\leq 2^{-n+1} \\ &< \epsilon \end{aligned}$$

Therefore $\{a_j\}$ is Cauchy and converges by Theorem 17 of Chapter 1.

13. Suppose $\{f_n\}$ is Cauchy in measure. Then for each natural number j, there exists an index N_j such that

$$m\left(|f_m - f_n| > 2^{-j}\right) < 2^{-j}$$

for all $m, n \geq N_j$. Let $\{n_j\}$ denote a strictly increasing sequence of natural numbers defined recursively by

$$n_1 = N_1, \qquad n_{j+1} = \max\{n_j + 1, N_{j+1}\}$$

and let

$$E_j = \left\{x \in E : \left|f_{n_{j+1}}(x) - f_{n_j}(x)\right| > 2^{-j}\right\}, \qquad E_0 = \bigcup_{j=1}^{\infty} E_j$$

Since $n_j, n_{j+1} \geq N_j$, we must have

$$m(E_j) < 2^{-j}$$

for all j. Countable subadditivity of measure then implies

$$m(E_0) \leq \sum_{j=1}^{\infty} m(E_j) < 1$$

If $x \in E \sim E_0$, then the sequence $\{f_{n_j}(x)\}$ converges by Problem 12. We can therefore define the function

$$f(x) = \begin{cases} \lim\limits_{j\to\infty} f_{n_j}(x) & \text{if } x \in E \sim E_0 \\ 0 & \text{if } x \in E_0 \end{cases}$$

Since $\left|f_{n_{j+1}}(x) - f_{n_j}(x)\right|$ is a measurable function of x for all j, each E_j is a measurable set. Therefore E_0 is also measurable, so $\chi_{E\sim E_0}$ is a measurable function. Thus $g_j := f_{n_j} \cdot \chi_{E\sim E_0}$ is a measurable function for all j. Since $\{g_j\} \to f$ pointwise, f is measurable.

Now fix $\eta > 0$ and $\epsilon > 0$. For any natural number j, we have

$$m\left(|f_n - f| > \eta\right) \leq m\left(|f_n - f_{n_j}| > \frac{\eta}{2}\right) + m\left(|f_{n_j} - f| > \frac{\eta}{2}\right)$$

Since $\{f_n\}$ is Cauchy in measure, there exists an index N such that

$$m\left(|f_n - f| > \eta\right) < \epsilon + m\left(|f_{n_j} - f| > \frac{\eta}{2}\right) \tag{1}$$

for all $n, n_j \geq N$. Pick J that satisfies $2^{-J} < \eta/4$. If $x \in E \sim E_0$, then for any $j \geq J$ we have

$$\begin{aligned} \left|f_{n_j}(x) - f(x)\right| &\leq \sum_{i=j}^{j+m-1} \left|f_{n_{i+1}}(x) - f_{n_i}(x)\right| + \left|f_{n_{j+m}}(x) - f(x)\right| \\ &\leq \sum_{i=j}^{\infty} 2^{-i} + \left|f_{n_{j+m}}(x) - f(x)\right| \\ &= 2^{-j+1} + \left|f_{n_{j+m}}(x) - f(x)\right| \\ &\leq \frac{\eta}{2} + \left|f_{n_{j+m}}(x) - f(x)\right| \end{aligned}$$

for any natural number m. Since $\{f_{n_j}\} \to f$ on $E \sim E_0$, by letting m go to infinity we can conclude that

$$\left|f_{n_j}(x) - f(x)\right| \leq \frac{\eta}{2}$$

for all $j \geq J$. Therefore

$$E \sim E_0 \subseteq \left\{x \in E : \left|f_{n_j}(x) - f(x)\right| \leq \frac{\eta}{2}\right\}$$

which implies

$$E_j' := \left\{x \in E : \left|f_{n_j}(x) - f(x)\right| > \frac{\eta}{2}\right\} \subseteq E_0$$

for all $j \geq J$. Since $\chi_{E_j'} \leq \chi_{E_0}$ for $j \geq J$, Fatou's Lemma implies

$$\int_E \liminf_{j\to\infty} \left(\chi_{E_0} - \chi_{E_j'}\right) \leq \liminf_{j\to\infty} \int_E \left(\chi_{E_0} - \chi_{E_j'}\right)$$

Upon rearrangement, this expression can be written as

$$m(E_0) - \int_E \limsup_{j\to\infty} \chi_{E'_j} \leq m(E_0) - \limsup_{j\to\infty} m\left(E'_j\right) \tag{2}$$

Since

$$\limsup_{j\to\infty} \chi_{E'_j}(x) = \begin{cases} 1 & \text{if } x \in E'_j \text{ infinitely often} \\ 0 & \text{otherwise} \end{cases}$$

and $m(E_0) < \infty$, (2) implies

$$\limsup_{j\to\infty} m\left(|f_{n_j} - f| > \frac{\eta}{2}\right) \leq m\left(x \in E'_j \text{ infinitely often}\right)$$

But since $E'_j \subseteq E_0$ for all $j \geq J$, we must have

$$m\left(\bigcup_{j=J}^{\infty} E'_j\right) \leq m(E_0) < \infty$$

The Borel-Cantelli Lemma then implies $m\left(x \in E'_j \text{ infinitely often}\right) = 0$. Therefore

$$\limsup_{j\to\infty} m\left(|f_{n_j} - f| > \frac{\eta}{2}\right) = 0$$

We can therefore conclude

$$m(|f_n - f| > \eta) \leq \epsilon$$

for all $n \geq N$ by taking the $\limsup$ of (1) over j. Since ϵ can be chosen to be arbitrarily small, $\{f_n\} \to f$ in measure on E.

14. Suppose $\{f_n\} \to f$ in measure on E. Fix $\epsilon > 0$ and let $\eta = \epsilon/m(E)$. Let

$$E' := \{x \in E : |f_n(x) - f(x)| > \eta\}$$

Then

$$\begin{aligned} \rho(f_n, f) &= \int_E \frac{|f_n - f|}{1 + |f_n - f|} \\ &= \int_{E'} \frac{|f_n - f|}{1 + |f_n - f|} + \int_{E\sim E'} \frac{|f_n - f|}{1 + |f_n - f|} \\ &\leq m(E') + \eta \cdot m(E) \\ &< m(\{x \in E : |f_n(x) - f(x)| > \eta) + \epsilon \end{aligned}$$

where the third line follows because $\frac{|f_n - f|}{1+|f_n-f|} \leq 1$ on E' and $\frac{|f_n - f|}{1+|f_n-f|} \leq \eta$ on $E \sim E'$. But by convergence in measure, this expression implies

$$\limsup_{n\to\infty} \rho(f_n, f) \leq \epsilon$$

Since ϵ can be chosen to be arbitrarily small and $\rho(f_n, f) \geq 0$ for all n, we must have $\lim_{n\to\infty} \rho(f_n, f) = 0$.

Conversely, suppose $\lim_{n\to\infty} \rho(f_n, f) = 0$. Fix $\eta > 0$. Then

$$\begin{aligned}
m\left(|f_n - f| > \eta\right) &= m\left(\frac{|f_n - f|}{1 + |f_n - f|} > \frac{\eta}{1+\eta}\right) \\
&\leq \frac{1+\eta}{\eta} \int_E \frac{|f_n - f|}{1 + |f_n - f|} \\
&= \frac{1+\eta}{\eta} \cdot \rho(f_n, f) \\
&\to 0 \text{ as } n \to \infty
\end{aligned}$$

where the first line follows because the function $\frac{x}{1+x}$ is strictly increasing over $[0, \infty)$ and the second line follows from Markov's inequality. Therefore $\{f_n\} \to f$ in measure on E.

5.3 Characterizations of Riemann and Lebesgue Integrability

15. Let E_f, E_g, and $E_{f\cdot g}$ denote the sets of points in $[a, b]$ at which f, g and $f \cdot g$ fail to be continuous, respectively. By Theorem 8, we know that $m(E_f)$ and $m(E_g)$ have measure 0.

    Now suppose $x \in [a, b] \sim (E_f \cup E_g)$. Fix $\epsilon > 0$. There exists a $\delta > 0$ such that for all $x' \in [a, b]$ satisfying $|x' - x| < \delta$, we have

    $$\begin{aligned}
    |f(x') - f(x)| &< \min\left\{\frac{\epsilon}{3(|g(x)| + 1)}, \sqrt{\frac{\epsilon}{3}}\right\} \\
    |g(x') - g(x)| &< \min\left\{\frac{\epsilon}{3(|f(x)| + 1)}, \sqrt{\frac{\epsilon}{3}}\right\}
    \end{aligned}$$

    But then for all $x' \in [a, b]$ satisfying $|x' - x| < \delta$, we have

    $$|f(x')g(x') - f(x)g(x)| \leq |f(x') - f(x)||g(x') - g(x)| + |f(x') - f(x)||g(x)| + |g(x') - g(x)||f(x)| < \epsilon$$

    Thus $f \cdot g$ is continuous at x, so $x \in [a, b] \sim E_{f\cdot g}$. We can conclude that

    $$E_{f\cdot g} \subseteq E_f \cup E_g$$

    But this implies $m(E_{f\cdot g}) \leq m(E_f) + m(E_g) = 0$. Since $f \cdot g$ is bounded, $f \cdot g$ is continuous by Theorem 8.

16. If f is a bounded function, then f is Riemann integrable over $[a, b]$ by Theorem 8. But then f is Lebesgue integrable over $[a, b]$ by Theorem 3 of Chapter 4. Theorem 7 then implies f is measurable.

    Let E_0 denote the set of discontinuity points of f. Since f is continuous on $E \sim E_0$, the restriction of f to $E \sim E_0$ is measurable by Proposition 3 of Chapter 3. But since $m(E_0) = 0$, the restriction of f to E_0 is also measurable. Therefore f is measurable by Proposition 5 of Chapter 3.

17. For each natural number n, define

$$\begin{aligned} f\left(\frac{4n+1}{(4n+4)n}\right) &= 2n(n+1) \\ f\left(\frac{4n+2}{(4n+4)n}\right) &= 0 \\ f\left(\frac{4n+3}{(4n+4)n}\right) &= -2n(n+1) \\ f\left(\frac{4n+4}{(4n+4)n}\right) &= 0 \end{aligned}$$

Extend f to all of $(0,1]$ by linear interpolation. Set $f(0) = 0$. By construction, f is continuous on $(0,1]$ and

$$\int_{[1/n,1]} f = 0$$

for all n. Therefore the sequence $\left\{\int_{[1/n,1]} f\right\}$ converges. However,

$$\int_{[0,1]} |f| = \lim_{n\to\infty} \int_{[1/n,1]} |f| = \lim_{n\to\infty} n = \infty$$

Therefore f is not Lebesgue integrable over $[0,1]$.

If f is non-negative, then $\lim_{n\to\infty} \int_{[1/n,1]} f = \int_{[0,1]} f$ by the Monotone Convergence Theorem. In this case, $\int_{[0,1]} f$ must be integrable if $\lim_{n\to\infty} \int_{[1/n,1]} f$ converges.

6 Differentiation and Integration

6.1 Continuity of Monotone Functions

1. By Proposition 2, we can find an increasing function f on (a,b) that is continuous only at $(a,b) \sim C$ and that satisfies $0 \leq f \leq 1$. If $b \in C$, define $f(b) = 2$. Then f is not continuous at b because $|f(b) - f(x)| \geq 1$ for all $x \in (a,b)$. If $b \notin C$, define $f(b) = \sup\{f(x) : x \in (a,b)\}$. Pick a sequence $\{x_n\}$ in (a,b) that converges upward to b. Since f is bounded and increasing on (a,b), the sequence $\{f(x_n)\}$ converges to some real number y. Since $f(x_n) \leq f(b)$ for all n, we must have $y \leq f(b)$. For any $x \in (a,b)$, there is an index n such that $x < x_n < b$. But then $f(x) \leq f(x_n) \leq y$ because f is increasing. Therefore y must also be an upper bound of $\{f(x) : x \in (a,b)\}$, which implies $y \geq f(b)$. We conclude that $\lim_{n\to\infty} f(x_n) = f(b)$, so f is continuous at b.

   If we also define $f(a) = -1$ if $a \in C$ and $f(a) = \inf\{f(x) : x \in (a,b)\}$ if $a \notin C$, then f satisfies the desired property.

2. Since the rational numbers are countable, we can use Problem 1 to construct an increasing function $\tilde{f}$ on $[0,1]$ that is continuous only at the irrational numbers in $[0,1]$. Define $f(x) = \tilde{f}(x) + x$. Then f is strictly increasing, continuous at the irrational numbers in $[0,1]$, and discontinuous at each rational number in $[0,1]$.

3. Extend f to $\mathbf{R}$ by defining

$$g(x) = \begin{cases} \inf\{f(y) : y \in E\} & \text{if } x \leq \inf E \\ f(x) & \text{if } x > \inf E \text{ and } x \in E \\ \sup\{f(y) : y \in E \text{ and } y < x\} & \text{if } x > \inf E \text{ and } x \notin E \end{cases}$$

Now suppose f is not continuous at some $x_0 \in E$. Then there exists $\epsilon > 0$ such that for all $\delta > 0$, there exists $x \in E$ satisfying $|x - x_0| < \delta$ and $|f(x) - f(x_0)| > \epsilon$. But this implies that for any $\delta > 0$, there exists $x \in \mathbf{R}$ satisfying $|x - x_0| < \delta$ and $|g(x) - g(x_0)| > \epsilon$. Therefore g is not continuous at x_0, implying that the set of points at which f fails to be continuous must be contained in the set of points at which g fails to be continuous. But since g is monotone, g is continuous except possibly at a countable number of points by Theorem 1. We conclude that the set of points at which f fails to be continuous must be countable, since it is contained in a countable set.

4. Let $E = \{0, 1\}$ and $C = \{0\}$. Any function defined on E is continuous on E, so no function on E can only be continuous at $E \sim C = \{1\}$.

6.2 Differentiability of Monotone Functions: Lebesgue's Theorem

5. Let $E = [0, 1]$ and let $\mathcal{F}$ denote the collection of all degenerate intervals of the form $[x, x]$ for $x \in E$. Then for each $x \in E$ and $\epsilon > 0$, there is an interval $I \in \mathcal{F}$ that contains x and has $\ell(I) < \epsilon$. However, for any finite disjoint subcollection $\{I_k\}_{k=1}^n$ of $\mathcal{F}$ we have

$$m^*\left(E \sim \bigcup_{k=1}^n I_k\right) = 1$$

6. Let E denote a set of finite outer measure and let $\mathcal{F}$ denote a collection of general, non-degenerate intervals. Suppose for each point $x \in E$ and $\epsilon > 0$, there is an interval $I \in \mathcal{F}$ that contains x and satisfies $\ell(I) < \epsilon$. Without loss of generality, assume all the intervals in $\mathcal{F}$ are bounded. Define $\bar{\mathcal{F}}$ as

$$\bar{\mathcal{F}} = \{\bar{I} : I \in \mathcal{F}\}$$

Then $\bar{\mathcal{F}}$ is a collection of closed, bounded, nondegenerate intervals that covers E in the sense of Vitali. By the Vitali Covering Lemma, for any $\epsilon > 0$ there is a finite disjoint subcollection $\{\bar{I}_k\}_{k=1}^n$ of intervals in $\bar{\mathcal{F}}$ for which

$$m^*\left(E \sim \bigcup_{k=1}^n \bar{I}_k\right) < \epsilon$$

For each k, there exists $I_k \in \mathcal{F}$ such that $\bar{I}_k \supseteq I_k$ and $m^*(\bar{I}_k \sim I_k) = 0$. By the monotonicity and subadditivity of outer measure, we have

$$
\begin{aligned}
m^*\left(E \sim \bigcup_{k=1}^n I_k\right) &\leq m^*\left(E \sim \bigcup_{k=1}^n \bar{I}_k\right) + m^*\left(\bigcup_{k=1}^n \bar{I}_k \sim \bigcup_{k=1}^n I_k\right) \\
&< \epsilon + m^*\left(\bigcup_{k=1}^n \bar{I}_k \sim I_k\right) \\
&\leq \epsilon + \sum_{k=1}^n m^*\left(\bar{I}_k \sim I_k\right) \\
&= \epsilon
\end{aligned}
$$

7. Consider the Weierstrass function

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} \sin(2^n x)$$

It can be shown that f is continuous everywhere but nowhere differentiable. But if f were monotone on an open interval (a, b), then f would be differentiable almost everywhere on (a, b) by Lebesgue's Theorem. Therefore f cannot be monotone on any open interval.

8. Let z denote the midpoint of I. Fix $x \in I \cap J$ and pick any $y \in J$. Then

$$|y - x| \leq \ell(J) \leq \frac{\ell(I)}{\gamma}, \qquad |x - z| \leq \frac{\ell(I)}{2}$$

By the Triangle Inequality, we have

$$|y - z| \leq |y - x| + |x - z| \leq \left(\frac{1}{2} + \frac{1}{\gamma}\right) \cdot \ell(I)$$

If $\gamma \geq 1/2$, then $|y - z| \leq 5/2 \cdot \ell(I)$. Thus $y \in 5 * I$.

The result does not hold if $0 < \gamma < 1/2$. For example, if $J = [0, 10]$ and $I = [-1, 1]$ then $5 * I = [-5, 5] \not\supseteq J$.

9. Suppose E has measure zero. Then for any index n, there exists a countable collection of nonempty open, bounded intervals $\mathcal{I}_n = \{I_{n,k}\}_{k=1}^{\infty}$ that covers E and that satisfies

$$\sum_{k=1}^{\infty} \ell(I_{n,k}) < 2^{-n} \tag{1}$$

The collection $\mathcal{I} = \bigcup_{n=1}^{\infty} \mathcal{I}_n$ is a union of countable sets and is therefore countable. Now fix $x \in E$. Since $\mathcal{I}_1$ covers E, there exists some $I_1 \in \mathcal{I}_1$ that contains x. Now suppose we have found intervals $I_1, \cdots, I_J$ in $\mathcal{I}$ that satisfy

$$x \in I_j \text{ for } j = 1, \cdots, J \qquad \ell(I_1) > \cdots > \ell(I_J) \tag{2}$$

Choose n such that $2^{-n} < \ell(I_J)$. Since $\mathcal{I}_n$ covers E, there exists some $I_{J+1} \in \mathcal{I}_n$ that contains x. Moreover, $\ell(I_{J+1}) < 2^{-n}$ by (1). Continuing in this way, we can recursively construct an infinite series of distinct intervals in $\mathcal{I}$ that contain x. Moreover, (1) implies

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\ell(I_{n,k}) < 1 < \infty$$

Conversely, suppose $\{I_k\}_{k=1}^{\infty}$ is a countable collection of open intervals for which each point in E belongs to infinitely many I_k's and

$$\sum_{k=1}^{\infty}\ell(I_k) < \infty$$

By the Borel-Cantelli Lemma, there exists a set E_0 of measure zero such that all points not in E_0 belong to finitely many of the I_k's. This implies $E \subseteq E_0$, so E must have measure zero by monotonicity of measure.

10. Fix $x, x' \in E$ and assume $x < x'$. Then

$$\ell((c_k, d_k) \cap (-\infty, x)) \leq \ell((c_k, d_k) \cap (-\infty, x'))$$

for all k. Therefore f is increasing.

Since x is in infinitely many of the intervals, we can assume without loss that $c_k < x < d_k$ for all k. For each index k, pick $\epsilon_k > 0$ such that

$$(x, x+\epsilon_k) \subseteq (c_k, d_k)$$

For each index n, define $\tilde{\epsilon}_n = \min\{\epsilon_1, \epsilon_2, \cdots, \epsilon_n\}$. Then for all $0 < h < \tilde{\epsilon}_n$, we have

$$\begin{aligned}
\frac{f(x+h)-f(x)}{h} &\geq \frac{1}{h}\sum_{k=1}^{n}\Big(\ell((c_k, d_k)\cap(-\infty, x+h)) - \ell((c_k, d_k)\cap(-\infty, x))\Big) \\
&= \frac{1}{h}\sum_{k=1}^{n}\Big(\ell(c_k, x+h) - \ell(c_k, x)\Big) \\
&= n
\end{aligned}$$

Thus for any $M > 0$, we can find an $\epsilon > 0$ such that

$$\frac{f(x+h)-f(x)}{h} > M$$

for all $0 < h < \epsilon$. Therefore $\bar{D}(x) = \infty$. Since $\bar{D}(x)$ is not finite, f is not differentiable at x.

11. Applying Problem 47(i) of Chapter 4, we have

$$\begin{aligned}\int_a^b g(t+\gamma)dt &= \int_{\mathbf{R}} g(t+\gamma)\chi_{(a,b)}(t)dt \\ &= \int_{\mathbf{R}} g(t)\chi_{(a,b)}(t-\gamma)dt \\ &= \int_{\mathbf{R}} g(t)\chi_{(a+\gamma,b+\gamma)}(t)dt \\ &= \int_{a+\gamma}^{b+\gamma} g(t)dt\end{aligned}$$

Suppose f is integrable over the closed, bounded interval $[a,b]$. Extend f to take the value $f(b)$ on $(b,b+1]$. Then for any $a \leq u < v \leq b$ and $0 < h \leq 1$, we have

$$\begin{aligned}\int_u^v \frac{f(x+h)-f(x)}{h}dx &= \int_u^v \frac{f(x+h)}{h}dx - \int_u^v \frac{f(x)}{h}dx \\ &= \int_{u+h}^{v+h} \frac{f(x)}{h}dx - \int_u^v \frac{f(x)}{h}dx \\ &= \left(-\int_u^{u+h} \frac{f(x)}{h}dx + \int_u^v \frac{f(x)}{h}dx + \int_v^{v+h} \frac{f(x)}{h}dx\right) - \int_u^v \frac{f(x)}{h}dx \\ &= \int_v^{v+h} \frac{f(x)}{h}dx - \int_u^{u+h} \frac{f(x)}{h}dx\end{aligned}$$

12. Fix $x \in \mathbf{Q}$ and $h > 0$. Choose $q \in \mathbf{Q}$ satisfying $x < q < x+h$ and define $t = q - x$. Then $0 < |t| \leq h$ and $\text{Diff}_t\chi_{\mathbf{Q}}(x) = 0$. Since $\text{Diff}_t\chi_{\mathbf{Q}}(x) \leq 0$ for any $t \neq 0$, we must have $\sup_{0<|t|\leq h} \text{Diff}_t\chi_{\mathbf{Q}}(x) = 0$ for all h. Therefore $\bar{D}\chi_{\mathbf{Q}}(x) = 0$.

   Now suppose $x \notin \mathbf{Q}$. Fix $M > 0$ and $h > 0$ and define $\epsilon = \min\left\{h, \frac{1}{M}\right\}$. Pick $q \in \mathbf{Q}$ satisfying $x < q < x + \epsilon$ and define $t = q - x$. Then $0 < |t| \leq h$ and

$$\text{Diff}_t\chi_{\mathbf{Q}}(x) = \frac{1}{t} > \frac{1}{\epsilon} \geq M$$

   Since M was arbitrary, $\sup_{0<|t|\leq h} \text{Diff}_t\chi_{\mathbf{Q}}(x) = \infty$ for all h. Therefore $\bar{D}\chi_{\mathbf{Q}}(x) = \infty$.

   Putting these results together, we conclude

$$\bar{D}\chi_{\mathbf{Q}}(x) = \begin{cases} 0 & \text{if } x \in \mathbf{Q} \\ \infty & \text{if } x \notin \mathbf{Q} \end{cases}$$

   By instead choosing $q \notin \mathbf{Q}$ in the above proof, it is straight-forward to show

$$\underline{D}\chi_{\mathbf{Q}}(x) = \begin{cases} -\infty & \text{if } x \in \mathbf{Q} \\ 0 & \text{if } x \notin \mathbf{Q} \end{cases}$$

13. Consider the sequence of disjoint intervals $\{I_n\}_{k=1}^{\infty}$ constructed recursively in the proof of the Vitali Covering Lemma. From the proof, we know that for any $\epsilon > 0$ there exists an n such that

$$0 \leq m^* \left[E \sim \bigcup_{k=1}^{\infty} I_k \right] \leq m^* \left[E \sim \bigcup_{k=1}^{n} I_k \right] < \epsilon$$

Since ϵ can be made arbitrarily small, we have

$$m^* \left[E \sim \bigcup_{k=1}^{\infty} I_k \right] = 0$$

14. Suppose $\mathcal{I} := \{I_\alpha\}$ is a collection of closed, bounded, nondegenerate intervals indexed by α. Define $\mathcal{F}$ as

$$\mathcal{F} = \{[a, b] : -\infty < a < b < \infty \text{ and } [a, b] \subseteq I_\alpha \text{ for some } I_\alpha \in \mathcal{I}\}$$

Then $\mathcal{F}$ is a collection of closed, bounded, nondegenerate intervals that covers $\bigcup_\alpha I_\alpha$ in the sense of Vitali. By the Vitali Covering Lemma, for each $\epsilon > 0$ there exists a finite disjoint subcollection $\{\tilde{I}_k\}_{k=1}^{n}$ of $\mathcal{F}$ for which

$$m^* \left[E \sim \bigcup_{k=1}^{n} \tilde{I}_k \right] < \epsilon$$

Then $\bigcup_{k=1}^{n} \tilde{I}_k$ is a closed set contained in $\bigcup_\alpha I_\alpha$. We conclude $\bigcup_\alpha I_\alpha$ is measurable by Theorem 11(iii) of Chapter 2.

15. For any $t \neq 0$, we have

$$\text{Diff}_t f(0) = \sin\left(\frac{1}{t}\right)$$

Choose $h > 0$. Pick a natural number k such that

$$\frac{\pi}{2} + 2\pi k \geq \frac{1}{h}$$

Set $t = 1/(\frac{\pi}{2} + 2\pi k)$. Then $0 < |t| \leq h$ and $\text{Diff}_t f(0) = 1$. Since h was arbitrary, we must have

$$\bar{D} f(0) = \lim_{h \to 0} \sup_{0 < |t| \leq h} \text{Diff}_t f(0) \geq 1$$

Since $\sin(\cdot)$ is bounded above by 1, we must also have $\bar{D} f(0) \leq 1$. Thus $\bar{D} f(0) = 1$.

An analogous argument can be used to show $\underline{D} f(0) = -1$.

16. Suppose g is non-negative. Then f is increasing on $[a, b]$, so f is differentiable almost everywhere on (a, b).

In the general case, write $g = g^+ - g^-$ where g^+ and g^- denote the positive and negative parts of g. By the above result, the functions $h_1(x) = \int_a^x g^+$ and $h_2(x) = \int_a^x g^-$ are differentiable almost everywhere on (a, b). Thus there exists sets E_1 and E_2 of measure zero such that h_1 and h_2 are differentiable at all points in $(a, b) \sim E_1$ and $(a, b) \sim E_2$, respectively. By Problem 20 and the linearity of differentiation,

we have

$$h_1' - h_2' = \underline{D}(h_1) + \underline{D}(-h_2) \leq \underline{D}(h_1 - h_2) \leq \overline{D}(h_1 - h_2) \leq \overline{D}(h_1) + \overline{D}(-h_2) = h_1' - h_2'$$

on $(a, b) \sim (E_1 \cup E_2)$. Since $f = h_1 - h_2$, this expression implies that f is differentiable on $(a, b) \sim (E_1 \cup E_2)$. But $E_1 \cup E_2$ has measure zero, so f is differentiable almost everywhere on (a, b).

17. Extend f to the closed interval $[a, b]$ by setting $f(a) = \inf_{x \in (a,b)} f(x)$ and $f(b) = \sup_{x \in (a,b)} f(x)$. Since f is bounded, $-\infty < f(a) \leq f(b) < \infty$. The extended function is increasing on $[a, b]$, so we can apply Corollary 4 to conclude

$$\int_a^b f' \leq \sup_{x \in (a,b)} f(x) - \inf_{x \in (a,b)} f(x)$$

18. Suppose $c \in (a, b)$ is a local minimizer for f. Then there exists some $h^* > 0$ such that $f(c) \leq f(x)$ for all $x \in (c - h^*, c + h^*)$. For any $h \in (0, h^*)$, we have

$$\frac{f(c+h) - f(c)}{h} \geq 0$$

which implies

$$\overline{D}f(c) = \lim_{h \to 0} \sup_{0 < |t| \leq h} \frac{f(c+t) - f(c)}{t} \geq 0$$

We also have

$$\frac{f(c+h) - f(c)}{h} \leq 0$$

for any $h \in (-h^*, 0)$. Thus

$$\underline{D}f(c) = \lim_{h \to 0} \inf_{0 < |t| \leq h} \frac{f(c+t) - f(c)}{t} \leq 0$$

19. **Corrected exercise: let f be continuous on $[a, b]$ with $\underline{D}f \geq 0$. Show that f is increasing on $[a, b]$.**

Choose u, v, and ϵ such that $a < u < v < b$ and $\epsilon > 0$. Let

$$S = \{x \in [u, v] : f(x) + \epsilon x \geq f(u) + \epsilon u\}.$$

S is not empty as $u \in S$ and S is bounded above by v. Therefore we can define $s^* = \sup S$. S is also closed because f is continuous. Therefore $s^* \in S$.

We will show $s^* = v$ by way of contradiction. Suppose $s^* < v$. For each $h > 0$, define

$$g(h) = \inf_{0 < |t| \leq h} \frac{f(s^* + t) - f(s^*)}{t}$$

Since $\lim_{h \to 0} g(h) = \underline{D}f(s^*) \geq 0$, there exists $h_0 \in (0, v - s^*)$ such that $g(h_0) > -\epsilon$. But this means

$$\frac{f(s^* + t) - f(s^*)}{t} > -\epsilon \qquad \text{for all } 0 < |t| \leq h_0 \tag{3}$$

Pick $t_0 \in (0, h_0)$ and define $s_0 = s^* + t_0$. By (3), we have

$$\frac{f(s_0) - f(s^*)}{s_0 - s^*} > -\epsilon$$

Rearranging, we obtain

$$f(s_0) + \epsilon s_0 > f(s^*) + \epsilon s^* \geq f(u) + \epsilon u$$

where the second inequality follows because $s^* \in S$. But this implies $s_0 \in S$ and $s_0 > s^*$, a contradiction.

We can conclude that

$$f(v) + \epsilon v \geq f(u) + \epsilon u$$

Since this expression holds for all $\epsilon > 0$, we have $f(v) \geq f(u)$. Thus f is increasing on (a, b). Since f is continuous, this result can be extended to all of $[a, b]$ by keeping v fixed and letting $u \to a$ and keeping u fixed and letting $v \to b$.

20. Fix $h > 0$. Then

$$\begin{aligned}
\inf_{0<|t|\leq h} \frac{f(t+h) - f(t)}{h} + \inf_{0<|t|\leq h} \frac{g(t+h) - g(t)}{h} &\leq \inf_{0<|t|\leq h} \frac{f(t+h) + g(t+h) - (f(t) + g(t))}{h} \\
&\leq \sup_{0<|t|\leq h} \frac{f(t+h) + g(t+h) - (f(t) + g(t))}{h} \\
&\leq \sup_{0<|t|\leq h} \frac{f(t+h) - f(t)}{h} + \sup_{0<|t|\leq h} \frac{g(t+h) - g(t)}{h}
\end{aligned}$$

Taking limits as $h \to 0$ yields the desired result.

21. (i) We first show that $\overline{D}(f \circ g)(\gamma) \leq \overline{D}f(c) \cdot g'(\gamma)$. If $\overline{D}f(c) = \infty$, this inequality holds trivially. So suppose $\overline{D}f(c) < \infty$. Then for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\frac{f(y) - f(c)}{y - c} \leq \overline{D}f(c) + \epsilon \tag{4}$$

for all y satisfying $0 < |y - c| \leq \delta$. Since $g'(\gamma) > 0$, there also exists an open interval I containing γ such that

$$\frac{g(x) - g(\gamma)}{x - \gamma} > 0 \tag{5}$$

for all $x \in I \sim \{\gamma\}$. Let $\{x_n\}$ denote a sequence in $I \sim \{\gamma\}$ that converges to γ. By continuity of g, $|g(x_n) - g(\gamma)| < \delta$ for all n sufficiently large. From (4), we therefore have

$$\frac{(f \circ g)(x_n) - (f \circ g)(\gamma)}{g(x_n) - g(\gamma)} \leq \overline{D}f(c) + \epsilon$$

for all n sufficiently large. But this implies

$$\frac{(f \circ g)(x_n) - (f \circ g)(\gamma)}{x_n - \gamma} = \frac{(f \circ g)(x_n) - (f \circ g)(\gamma)}{g(x_n) - g(\gamma)} \cdot \frac{g(x_n) - g(\gamma)}{x_n - \gamma} \leq \left(\overline{D}f(c) + \epsilon\right) \frac{g(x_n) - g(\gamma)}{x_n - \gamma}$$

for all n sufficiently large. Taking the lim sup of both sides, we obtain

$$\limsup_{n\to\infty} \frac{(f\circ g)(x_n) - (f\circ g)(\gamma)}{x_n - \gamma} \leq \left(\overline{D}f(c) + \epsilon\right) g'(\gamma)$$

Since this expression holds for arbitrarily small ϵ, we can conclude

$$\limsup_{n\to\infty} \frac{(f\circ g)(x_n) - (f\circ g)(\gamma)}{x_n - \gamma} \leq \overline{D}f(c) \cdot g'(\gamma)$$

But since $\{x_n\}$ was an arbitrary sequence converging to γ, we must have $\overline{D}(f\circ g)(\gamma) \leq \overline{D}f(c)\cdot g'(\gamma)$.

We now prove $\overline{D}(f\circ g)(\gamma) \geq \overline{D}f(c)\cdot g'(\gamma)$. If $\overline{D}(f\circ g)(\gamma) = \infty$, then the inequality holds trivially. So suppose $\overline{D}(f\circ g)(\gamma) < \infty$. For any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\frac{(f\circ g)(x) - (f\circ g)(\gamma)}{x - \gamma} \leq \overline{D}(f\circ g)(\gamma) + \epsilon$$

for all x satisfying $0 < |x - \gamma| \leq \delta$. Now g is a strictly increasing on I by (5). By Problem 45 of Chapter 2, g has a continuous inverse on $g(I)$. Thus for any sequence $\{y_n\}$ in $g(I)$ converging to c, there exists a sequence $\{x_n\}$ in I converging to γ such that $y_n = g(x_n)$ for all n. We therefore have

$$\frac{f(y_n) - f(c)}{y_n - c} \cdot \frac{g(x_n) - g(\gamma)}{x_n - \gamma} = \frac{(f\circ g)(x_n) - (f\circ g)(\gamma)}{x_n - \gamma} \leq \overline{D}(f\circ g)(c) + \epsilon$$

for n large enough. Rearranging, we obtain

$$\frac{f(y_n) - f(c)}{y_n - c} \leq (\overline{D}(f\circ g)(c) + \epsilon) \cdot \left(\frac{g(x_n) - g(\gamma)}{x_n - \gamma}\right)^{-1}$$

for n large enough. Taking the lim sup of both sides, we obtain

$$\limsup_{n\to\infty} \frac{f(y_n) - f(c)}{y_n - c} \cdot \leq \frac{\overline{D}(f\circ g)(c) + \epsilon}{g'(\gamma)}$$

Since this expression holds for arbitrarily small ϵ, we can conclude

$$\limsup_{n\to\infty} \frac{f(y_n) - f(c)}{y_n - c} \cdot \leq \frac{\overline{D}(f\circ g)(c)}{g'(\gamma)}$$

But since this result holds for any sequence $\{y_n\}$ converging to c, we have $\overline{D}f(c) \cdot g'(\gamma) \leq \overline{D}(f\circ g)(\gamma)$.

(ii) Since $\overline{D}f(c)$ and $\underline{D}f(c)$ are finite, there exists $\delta_0 > 0$ and finite positive number M such that

$$\left|\frac{f(y) - f(c)}{y - c}\right| < M$$

for all y satisfying $|y - c| < \delta_0$. Since $g'(\gamma) = 0$, for any $\epsilon > 0$ there exists $\delta_1 > 0$ such that

$$\left|\frac{g(x) - g(\gamma)}{x - \gamma}\right| < \frac{\epsilon}{M}$$

if $|x-\gamma| < \delta_1$. Now let $\{x_n\}$ denote a sequence converging to γ. Then there exists N_1 such that $|x_n - \gamma| < \delta_1$ for all $n \geq N_1$. By continuity of g, there also exists N_0 such that $|g(x_n) - g(\gamma)| < \delta_0$ for $n \geq N_0$. We therefore have

$$\left|\frac{(f \circ g)(x_n) - (f \circ g)(\gamma)}{x_n - \gamma}\right| \leq \left|\frac{(f \circ g)(x_n) - (f \circ g)(\gamma)}{g(x_n) - g(\gamma)}\right| \cdot \left|\frac{g(x_n) - g(\gamma)}{x_n - \gamma}\right| < M \cdot \frac{\epsilon}{M} = \epsilon$$

for all $n \geq \max\{N_0, N_1\}$. But since ϵ can be made arbitrarily small, this expression implies

$$\lim_{n\to\infty} \frac{(f \circ g)(x_n) - (f \circ g)(\gamma)}{x_n - \gamma} = 0$$

for any sequence $\{x_n\}$ converging to γ. We can therefore conclude $\overline{D}(f \circ g)(\gamma) = 0$.

22. Suppose f is an increasing function on an interval I. Fix a real number y and consider the set

$$A = \{x \in I : f(x) \leq y\}$$

If A is empty, it is trivially measurable. Now suppose A is non-empty. Let x_0 and x_1 denote two points in A that satisfy $x_0 \leq x_1$. Fix $\lambda \in [0, 1]$ and define

$$x_\lambda = \lambda x_0 + (1 - \lambda)x_1$$

Since f is increasing and $x_\lambda \leq x_1$, we must have

$$f(x_\lambda) \leq f(x_1) \leq y$$

Therefore $x_\lambda \in A$. We conclude that A is an interval and therefore measurable by Proposition 8 of Chapter 2. Since A is measurable for any choice of y, f is a measurable function.

If f is decreasing, then $-f$ is increasing and therefore measurable. f is then measurable by Theorem 6 of Chapter 3.

23. Let $M \geq 0$ be a real number that satisfies

$$-M \leq \underline{D}f(x) \leq \overline{D}f(x) \leq M$$

for all $x \in (a, b)$. Fix $x_0, x_1 \in (a, b)$ and assume without loss that $x_0 < x_1$. Define the function

$$g(x) = f(x) - \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) \tag{6}$$

Since g is a continuous function defined on the closed and bounded interval $[x_0, x_1]$, the Extreme Value Theorem implies the existence of some $c \in [x_0, x_1]$ such that $g(c) \leq g(x)$ for all $x \in [x_0, x_1]$. Suppose $c \in (x_0, x_1)$. Then $\overline{D}g(c) \geq 0$ by Problem 18. Using the result from Problem 20, we have

$$0 \leq \overline{D}g(c) \leq \overline{D}f(c) - \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Upon rearrangement, this expression yields

$$f(x_1) - f(x_0) \leq \overline{D}f(c) \cdot (x_1 - x_0) \leq M \cdot (x_1 - x_0)$$

Now suppose $c \in \{x_0, x_1\}$. Then $g(c) = f(x_0)$, so we have

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} \leq \frac{f(x) - f(x_0)}{x - x_0}$$

for all $x \in (x_0, x_1)$. Thus for all $t \in (0, x_1 - x_0)$, we have

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} \leq \frac{f(x_0 + t) - f(x_0)}{t}$$

which implies

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} \leq \lim_{h \to 0} \sup_{0 < |t| \leq h} \frac{f(x_0 + t) - f(x_0)}{t} = \overline{D}f(x_0) \leq M$$

We conclude that

$$f(x_1) - f(x_0) \leq M \cdot (x_1 - x_0)$$

An analogous argument using lower derivatives in place of upper derivatives can be used to show

$$f(x_0) - f(x_1) \geq M \cdot (x_1 - x_0)$$

Therefore $|f(x_1) - f(x_0)| \leq M|x_1 - x_0|$, so f is Lipschitz on (a, b).

It remains to show that the Lipschitz condition is also satisfied when $x_0 = a$ or $x_1 = b$. Suppose $x_0 = a$ but continue to assume $x_1 \in (a, b)$. Fix $\epsilon > 0$. Since f is continuous at a, there exists $0 < \delta < b - a$ such that

$$|f(x) - f(a)| < \epsilon$$

for all $x \in [a, a + \delta)$. Let $\{a_n\}$ denote a sequence in $(a, a + \delta)$ that converges to a. Then

$$|f(x_1) - f(a)| \leq |f(x_1) - f(a_n)| + |f(a_n) - f(a)| \leq M \cdot |x_1 - a_n| + \epsilon$$

for all n. Taking the limit as $n \to \infty$, we obtain

$$|f(x_1) - f(a)| \leq M \cdot |x_1 - a| + \epsilon$$

Since ϵ can be made arbitrarily small, we have

$$|f(x_1) - f(a)| \leq M \cdot |x_1 - a|$$

An analogous argument can be applied to the other end point to show that f is Lipschitz on $[a, b]$.

24. The derivative of f on (0,1] is given by

$$f'(x) = -\frac{2}{x} \cdot \cos\left(\frac{1}{x^2}\right) + 2x \cdot \sin\left(\frac{1}{x^2}\right)$$

For each natural number k, the function f' satisfies

$$f'(x) \geq \frac{1}{x}$$

on the interval

$$E_k = \left(\frac{1}{\sqrt{\pi + 2\pi \cdot k}}, \frac{1}{\sqrt{\frac{3\pi}{4} + 2\pi \cdot k}} \right)$$

The integral of $|f'|$ over E_k therefore satisfies

$$\int_{E_k} |f'| \geq \int_{E_k} \frac{1}{x} = \frac{1}{2} \cdot \log \left(\frac{1 + 2 \cdot k}{\frac{3}{4} + 2 \cdot k} \right) \geq \frac{M}{k}$$

where M is any constant larger than $\frac{1}{2} \cdot \log \left(\frac{12}{11} \right) \approx \frac{1}{23}$. Since $\{E_k\}_{k=1}^{\infty}$ is a sequence of disjoint sets in $(0, 1]$, we must have

$$\int_0^1 |f'| \geq \sum_{k=1}^{\infty} \int_{E_k} |f'| \geq M \sum_{k=1}^{\infty} \frac{1}{k} = \infty$$

Thus $|f'|$ is not integrable.

6.3 Functions of Bounded Variation: Jordan's Theorem

25. The Weierstrass function is continuous everywhere on $[0, 1]$ but is nowhere differentiable. Therefore by the contrapositive to Corollary 6, f must not be of bounded variation on any subinterval $[a, b]$ of $[0, 1]$.

26. Fix a natural number n. Define $c_n = \frac{1}{n \cdot \sqrt{2}}$ and consider the partition

$$P_n = \left\{ 0, c_n, \frac{1}{n}, \frac{1}{n} + c_n, \frac{2}{n}, \frac{2}{n} + c_n, \cdots, \frac{n-1}{n}, \frac{n-1}{n} + c_n, 1 \right\}$$

Then $V(f, P_n) = 2 \cdot n$. Since $2 \cdot n$ diverges, f is not of bounded variation.

27. The total variation of f on $[0, x]$ is given by

$$TV(f_{[0,x]}) = \begin{cases} \sin x & 0 \leq x < \frac{\pi}{2} \\ 2 - \sin x & \frac{\pi}{2} < x \leq \frac{3 \cdot \pi}{2} \\ 4 + \sin x & \frac{3 \cdot \pi}{2} < x \leq 2 \cdot \pi \end{cases}$$

We can write $f(x) = h(x) - g(x)$ where

$$h(x) = f(x) + TV(f_{[0,x]}) = \begin{cases} 2 \cdot \sin x & 0 \leq x < \frac{\pi}{2} \\ 2 & \frac{\pi}{2} < x \leq \frac{3 \cdot \pi}{2} \\ 4 + 2 \cdot \sin x & \frac{3 \cdot \pi}{2} < x \leq 2 \cdot \pi \end{cases}$$

$$g(x) = TV(f_{[0,x]})$$

By Lemma 5, h and g are increasing.

28. Since f is a step function on $[a,b]$, there exists a partition $P = \{x_0, x_1, \cdots, x_n\}$ of $[a,b]$ and numbers $c_1, \cdots, c_n$ such that

$$f(x) = c_i \text{ if } x_{i-1} < x < x_i$$

Now consider the partition

$$P' = \left\{x_0, \frac{x_0 + x_1}{2}, x_1, \cdots, x_{n-1}, \frac{x_{n-1} + x_n}{2}, x_n\right\}$$

Any refinement of P' does not change the variation of f, so we have

$$TV(f) = V(f, P') = \sum_{k=1}^{n} \left(|f(x_{k-1}) - c_k| + |f(x_k) - c_k|\right)$$

29. (a) Fix a natural number n and consider the partition

$$P_n = \left\{0, \sqrt{\frac{2}{2 \cdot n \cdot \pi}}, \sqrt{\frac{2}{(2 \cdot n - 1) \cdot \pi}}, \cdots, \sqrt{\frac{2}{2 \cdot \pi}}, \sqrt{\frac{2}{\pi}}\right\}$$

We then have

$$V\left(f_{\left[0, \sqrt{\frac{2}{\pi}}\right]}, P_n\right) = \frac{2}{\pi} \cdot \left(1 + \frac{1}{2} + \cdots \frac{1}{n-1} + \frac{1}{n}\right)$$

Since the harmonic series diverges and $V\left(f_{\left[0, \sqrt{\frac{2}{\pi}}\right]}, P_n\right) \leq V\left(f_{[-1,1]}, P_n\right)$, f is not of bounded variation on $[-1, 1]$.

(b) Observe that

$$g'(x) = -\sin(1/x) + 2 \cdot x \cdot \cos(1/x)$$

for $x \neq 0$. We also have

$$\text{Diff}_h\, g(0) = h \cdot \cos(1/h)$$

Taking the limit as h goes to zero, we can conclude

$$g'(0) = 0$$

Therefore $g(\cdot)$ is differentiable on $[-1, 1]$ and its derivative satisfies $|g'(x)| \leq 3$. This means $g(\cdot)$ is Lipschitz on $[-1, 1]$, which implies $g(\cdot)$ is of bounded variation on $[-1, 1]$.

30. That linear combinations of functions of bounded variation are also of bounded variation follows immediately from Problem 34.

Suppose f and g are real-valued functions of bounded variation on the closed, bounded interval $[a,b]$. Then both f and g must be bounded on $[a,b]$. Thus there exists a real number $M > 0$ such that $|f(x)| < M$ and $|g(x)| < M$ for all $x \in [a,b]$. Now let $P = \{x_0, \cdots, x_k\}$ denote a partition of $[a,b]$.

Then

$$\begin{aligned}
V(f \cdot g, P) &= \sum_{i=1}^{k} |f(x_i) \cdot g(x_i) - f(x_{i-1}) \cdot g(x_{i-1})| \\
&\leq \sum_{i=1}^{k} (|f(x_i) - f(x_{i-1})| \cdot |g(x_i)| + |g(x_i) - g(x_{i-1})| \cdot |f(x_{i-1}|) \\
&\leq (V(f, P) + V(g, P)) \cdot M \\
&\leq (TV(f, P) + TV(g, P)) \cdot M
\end{aligned}$$

Since $V(f \cdot g, P)$ is bounded above, $f \cdot g$ is of bounded variation on $[a, b]$.

31. Let $P' = \{x_0, \cdots, x_k\}$ denote a partition of $[a, b]$. Suppose $x \in (a, b)$ is not in P'. Let P denote the partition obtained from adjoining x to P'. Then there exists $n \in \{1, \cdots, k\}$ such that $x_{n-1} < x < x_n$. By the Triangle Inequality, we have

$$\begin{aligned}
V(f, P) &= \sum_{i \neq n} |f(x_i) - f(x_{i-1})| + |f(x) - f(x_{n-1})| + |f(x_n) - f(x)| \\
&\geq \sum_{i \neq n} |f(x_i) - f(x_{i-1})| + |f(x_n) - f(x_{n-1})| \\
&= V(f, P')
\end{aligned}$$

But this proves the desired result, since any refinement of P' can be formed by sequentially adjoining a finite number of additional partition points.

32. From the definition of total variation, we have

$$TV(f) = \sup\{V(f, P) \mid P \text{ a partition of } [a, b]\}$$

Now fix a partition P' of $[a, b]$ and define

$$s = \sup\{V(f, P) \mid P \text{ a refinement of } P'\}$$

For any partition P of $[a, b]$, we can form a refinement P'' of P' by combining the partition points of P and P'. By Problem 31, we have $V(f, P) \leq V(f, P'') \leq s$. Since P was arbitrary, $s \geq TV(f)$. Furthermore, we must have $TV(f) \geq s$ since all refinements of P' are partitions of $[a, b]$. We can therefore conclude $TV(f) = s$.

Now set $P_0 = \{a, b\}$. Since $TV(f) = \sup\{V(f, P) \mid P \text{ a refinement of } P_0\}$, we can find a refinement P_1 of P_0 that satisfies

$$|TV(f) - V(f, P_1)| < 1$$

Moreover, by Problem 31 we have $V(f, P_0) \leq V(f, P_1)$. Continuing in this manner, we can construct a sequence of partitions $\{P_n\}$ that satisfies

$$|TV(f) - V(f, P_n)| < \frac{1}{n}$$

and $V(f, P_{n-1}) \leq V(f, P_n)$ for all natural numbers n.

33. Suppose $TV(f) < \infty$. Fix $\epsilon > 0$ and choose a partition $P = \{x_0, \cdots, x_k\}$ of $[a, b]$ such that

$$V(f, P) \geq TV(f) - \frac{\epsilon}{2}$$

By pointwise convergence, there exists an index N such that

$$|f_n(x_i) - f(x_i)| < \frac{\epsilon}{4 \cdot k}, \qquad i = 0, 1, \cdots, k$$

for all $n \geq N$. We therefore have

$$\begin{aligned}
TV(f) &\leq V(f, P) + \frac{\epsilon}{2} \\
&= \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| + \frac{\epsilon}{2} \\
&\leq \sum_{i=1}^{k} |f(x_i) - f_n(x_i)| + \sum_{i=1}^{k} |f_n(x_i) - f_n(x_{i-1})| + \sum_{i=1}^{k} |f(x_{i-1}) - f_n(x_{i-1})| + \frac{\epsilon}{2} \\
&< V(f_n, P) + \epsilon \\
&\leq TV(f_n) + \epsilon
\end{aligned}$$

for all $n \geq N$. But this implies

$$TV(f) \leq \liminf TV(f_n)$$

Now suppose $TV(f) = \infty$. Fix a real number $M > 0$ and choose a partition $P = \{x_0, \cdots, x_k\}$ of $[a, b]$ such that

$$V(f, P) \geq M$$

By pointwise convergence, for any $\epsilon > 0$ there exists an index N such that

$$|f_n(x_i) - f(x_i)| < \frac{\epsilon}{2 \cdot k}, \qquad i = 0, 1, \cdots, k$$

for all $n \geq N$. Using the same steps as before, we can conclude

$$M \leq V(f, P) \leq TV(f_n)$$

for all $n \geq N$. Since M can be arbitrarily large, we must have $\liminf TV(f_n) = \infty$.

34. By the triangle inequality, we have

$$\begin{aligned}
V(f + g, P) &= \sum_{i=1}^{k} |f(x_i) - f(x_{i-1}) + g(x_i) - g(x_{i-1})| \\
&\leq \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| + \sum_{i=1}^{k} |g(x_i) - g(x_{i-1})| \\
&= V(f, P) + V(g, P)
\end{aligned}$$

for any partition P of $[a,b]$. But this implies

$$\begin{aligned} TV(f+g) &\le \sup\{V(f,P)+V(g,P) \mid P \text{ a partition of } [a,b]\} \\ &\le \sup\{V(f,P) \mid P \text{ a partition of } [a,b]\} + \sup\{V(g,P) \mid P \text{ a partition of } [a,b]\} \\ &= TV(f)+TV(g) \end{aligned}$$

We also have

$$\begin{aligned} V(\alpha f, P) &= \sum_{i=1}^{k} |\alpha f(x_i) - \alpha f(x_{i-1})| \\ &= |\alpha| \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| \\ &= |\alpha| \cdot V(f,P) \end{aligned}$$

which implies

$$\begin{aligned} TV(\alpha f) &= \sup\{|\alpha| \cdot V(f,P) \mid P \text{ a partition of } [a,b]\} \\ &= |\alpha| \cdot \sup\{V(f,P) \mid P \text{ a partition of } [a,b]\} \\ &= |\alpha| \cdot TV(f) \end{aligned}$$

35. The derivative of f on $(0,1]$ is given by

$$f'(x) = -\beta \cdot x^{\alpha-\beta-1} \cdot \cos(1/x^{\beta}) + \alpha \cdot x^{\alpha-1} \cdot \sin(1/x^{\beta})$$

Suppose $\alpha > \beta > 0$. Then f' is integrable:

$$\int_0^1 |f'| \le \int_0^1 (\beta \cdot x^{\alpha-\beta-1} + \alpha \cdot x^{\alpha-1}) dx = \frac{\beta}{\alpha-\beta} + 1 < \infty$$

By the Fundamental Theorem of Calculus, we have

$$\int_y^x f' = f(x) - f(y)$$

for any $x > y > 0$. Taking the limit as $y \to 0$, we obtain

$$f(x) = \int_0^x f'$$

by continuity of integration. Now let $P = \{x_0, \cdots, x_k\}$ be a partition of $[0,1]$. We then have

$$\begin{aligned}
V(f,P) &= \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| \\
&= \sum_{i=1}^{k} \left| \int_0^{x_i} f' - \int_0^{x_{i-1}} f' \right| \\
&= \sum_{i=1}^{k} \left| \int_{x_i}^{x_{i-1}} f' \right| \\
&\le \sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} |f'| \\
&= \int_0^1 |f'| \\
&< \infty
\end{aligned}$$

Since P was arbitrary, f is of bounded variation.

Now suppose $\beta \ge \alpha > 0$. Fix an index n and consider the following partition:

$$P_n = \left\{0, \left(\frac{2}{2n\pi}\right)^{1/\beta}, \left(\frac{2}{(2n-1)\pi}\right)^{1/\beta}, \cdots, \left(\frac{2}{\pi}\right)^{1/\beta}, 1\right\}$$

It can be shown

$$\begin{aligned}
V(f,P_n) &= c\sum_{k=0}^{n-1} \frac{1}{(1+2k)^{\alpha/\beta}} + \sin(1) - c \\
&\ge \frac{c}{2^{\alpha/\beta}} \sum_{k=1}^{n} \frac{1}{k^{\alpha/\beta}} + \sin(1) - c
\end{aligned}$$

where

$$c = 2\left(\frac{2}{\pi}\right)^{\alpha/\beta}$$

Since $0 < \alpha/\beta \le 1$, the series on the right-hand side diverges. Therefore f is not of bounded variation.

36. **Corrected exercise: Let f fail to be of bounded variation on $[0,1]$. Show that there is a point x_0 in $[0,1]$ such that f fails to be of bounded variation on each subinterval of $[0,1]$ that contains x_0 in its interior.**

Suppose f is not of bounded variation on an interval $[a_n, b_n] \subseteq [0,1]$. Define

$$a_{n+1} = \begin{cases} a_n & \text{if } TV(f_{[a_n,(a_n+b_n)/2]}) = \infty \\ (a_n+b_n)/2 & \text{otherwise} \end{cases}$$

$$b_{n+1} = \begin{cases} (a_n+b_n)/2 & \text{if } TV(f_{[a_n,(a_n+b_n)/2]}) = \infty \\ b_n & \text{otherwise} \end{cases}$$

Since $TV(f_{[a_n,b_n]}) = TV(f_{[a_n,(a_n+b_n)/2]}) + TV(f_{[(a_n+b_n)/2,b_n]}) = \infty$, f must not be of bounded variation on $[a_{n+1}, b_{n+1}]$. Starting at $a_1 = 0$ and $b_1 = 1$, we can use this recursion to define a descending, countable collection of nonempty closed intervals $\{[a_n, b_n]\}_{n=1}^{\infty}$ that satisfy $TV(f_{[a_n,b_n]}) = \infty$ for all n. By the Nested Set Theorem, there exists an $x_0 \in \bigcap_{n=1}^{\infty}[a_n, b_n]$. Choose a subinterval $[a, b]$ of $[0, 1]$ that contains x_0 in its interior. Then there exists n large enough such that $[a_n, b_n] \subseteq [a, b]$. But this implies $TV(f_{[a,b]}) \geq TV(f_{[a_n,b_n]}) = \infty$, so f is not of bounded variation on $[a, b]$.

6.4 Absolutely Continuous Functions

37. (i) Let

$$f(x) = \begin{cases} x\cos(\pi/2x) & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0 \end{cases}$$

It was shown in Section 6.3 that f is not of bounded variation on $[0, 1]$. Therefore by Theorem 8, f is not absolutely continuous on $[0, 1]$. But for each $\epsilon > 0$, f is continuous with bounded first derivative on $[\epsilon, 1]$. Therefore f is Lipschitz on $[\epsilon, 1]$ by the Mean Value Theorem, implying that f is absolute continuous on $[\epsilon, 1]$ by Proposition 7.

(ii) Fix $\epsilon > 0$. Let δ_1 correspond to the $\epsilon/2$ challenge for continuity at 0. Let δ_2 correspond to the $\epsilon/2$ challenge of absolute continuity on $[\delta_1, 1]$. Define $\delta = \min(\delta_1, \delta_2)$. Choose a finite, disjoint collection of open intervals $\{(a_k, b_k)\}_{k=1}^{n}$ contained in $[0, 1]$ that satisfy $\sum_{k=1}^{n}(b_k - a_k) < \delta$. Assume the intervals are ordered so that

$$a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n$$

Also assume there exists an index m such that $\{(a_k, b_k)\}_{k=1}^{m}$ is contained in $[0, \delta_1)$ and $\{(a_k, b_k)\}_{k=m+1}^{n}$ is contained in $[\delta_1, 1]$ (if $\delta_1 \in (a_k, b_k)$ for some k, replace (a_k, b_k) with (a_k, δ_1) and (δ_1, b_k)). Since $b_m < \delta_1$, we have

$$\begin{aligned} \sum_{k=1}^{m} |f(b_k) - f(a_k)| &= f(b_m) - f(a_1) - \sum_{k=2}^{m}(f(a_k) - f(b_{k-1})) \\ &\leq f(b_m) - f(0) \\ &< \epsilon/2 \end{aligned}$$

where the first two lines follow because f is increasing and the third line follows from continuity at 0. Since $\sum_{m+1}^{n}(b_k - a_k) < \delta_2$, we also have

$$\sum_{k=m+1}^{n} |f(b_k) - f(a_k)| < \epsilon/2$$

by absolute continuity of f on $[\delta_1, 1]$. Combining these results, we can conclude

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \epsilon$$

which implies that f is absolutely continuous on $[0,1]$.

(iii) Fix $\epsilon \in (0,1)$. Then $|f'(x)| \le \frac{1}{2\sqrt{\epsilon}}$ for all $x \in [\epsilon, 1]$, implying that f is Lipschitz and thus absolutely continuous on $[\epsilon, 1]$. Since f is also increasing on $[0,1]$, we can conclude that f is absolutely continuous on $[0,1]$ by part (ii).

To see that f is not Lipschitz on $[0,1]$, fix a constant $c \ge 0$. If $c = 0$, then

$$|f(x_1) - f(x_0)| > 0 = c \cdot |x_1 - x_0|$$

for $x_0, x_1 \in [0,1]$ satisfying $x_0 \ne x_1$. If $c > 0$, let $x_0 = 0$ and choose $x_1 \in (0, \min(1/c^2, 1))$. We then have

$$|f(x_1) - f(x_0)| = \sqrt{x_1} > c \cdot x_1 = c \cdot |x_1 - x_0|$$

Therefore there can exist no Lipschitz constant for f on $[0,1]$.

38. Suppose f is absolutely continuous on $[a,b]$. Fix $\epsilon > 0$ and let $\delta > 0$ correspond to the $\epsilon/2$-challenge for absolute continuity. Let $\{(a_k, b_k)\}_{k=1}^{\infty}$ be a countable disjoint collection of open intervals in $[a,b]$ satisfying $\sum_{k=1}^{\infty}(b_k - a_k) < \delta$. Then for each index n, we have

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| < \epsilon/2$$

by absolute continuity. But since this expression holds for all n, we have

$$\sum_{k=1}^{\infty} |f(b_k) - f(a_k)| \le \epsilon/2 < \epsilon$$

as desired.

Since every finite collection is countable, the converse follows immediately from the definition of absolute continuity.

39. Suppose f is absolutely continuous on $[a,b]$. Fix $\epsilon > 0$ and let δ correspond to the ϵ-challenge in Problem 38. Pick a measurable subset E of $[a,b]$ that satisfies $m(E) < \delta$. By Theorem 11 of Chapter 2, there exists a countable collection of disjoint open intervals $\{(a_k, b_k)\}_{k=1}^{\infty}$ such that $E \subseteq \bigcup_{k=1}^{\infty}(a_k, b_k)$ and $m\left(\bigcup_{k=1}^{\infty}(a_k, b_k) \sim E\right) < \delta - m(E)$. We then have

$$\sum_{k=1}^{\infty}(b_k - a_k) = m\left(\bigcup_{k=1}^{\infty}(a_k, b_k)\right) = m\left(\bigcup_{k=1}^{\infty}(a_k, b_k) \sim E\right) + m(E) < \delta$$

But this implies

$$\begin{aligned}
m^*(f(E)) &\le m^*\left(f\left(\bigcup_{k=1}^{\infty}(a_k,b_k)\right)\right) \\
&= m^*\left(\bigcup_{k=1}^{\infty} f((a_k,b_k))\right) \\
&\le m\left(\bigcup_{k=1}^{\infty}[f(a_k),f(b_k)]\right) \\
&= \sum_{k=1}^{\infty}(f(b_k)-f(a_k)) \\
&< \epsilon
\end{aligned}$$

where the first line follows from monotonicity of outer measure, the second line follows because the image of a union equals the union of images, the third line follows from the Intermediate Value Theorem and monotoncity of f, the fourth line follows from countable subadditivity, and the final line follows from Problem 38.

Correction: The converse requires f to be continuous. Suppose the condition in the problem is true. Fix $\epsilon > 0$ and let δ correspond to the ϵ-challenge in the problem. Let $\{(a_k,b_k)\}_{k=1}^{n}$ denote a collection of disjoint open intervals contained in $[a,b]$ that satisfy $\sum_{k=1}^{n}(b_k-a_k) < \delta$. Since f is increasing, $\{(f(a_k),f(b_k))\}_{k=1}^{n}$ is also a collection of disjoint open intervals. We therefore have

$$\begin{aligned}
\sum_{k=1}^{n}|f(b_k)-f(a_k)| &= \sum_{k=1}^{n} m\left((f(a_k),f(b_k))\right) \\
&= m\left(\bigcup_{k=1}^{n}(f(a_k),f(b_k))\right) \\
&\le m\left(\bigcup_{k=1}^{n} f((a_k,b_k))\right) \\
&= m\left(f\left(\bigcup_{k=1}^{n}(b_k-a_k)\right)\right) \\
&< \epsilon
\end{aligned}$$

where the first line follows by definition of the measure of an interval, the second line follows by additivity of measure, the third line follows from the Intermediate Value Theorem and monotonicity of f, the fourth line follows because the image of a union equals the union of images, and the last line follows from the condition in the problem.

40. Suppose f is an increasing, absolutely continuous function on $[a,b]$. Let $E \subseteq [a,b]$ denote a set of measure zero. By Problem 39, for any $\epsilon > 0$ we have $m^*(f(E)) < \epsilon$. Since ϵ can be chosen to be arbitrarily small, we must have $m^*(f(E)) = 0$.

    The above result implies $\psi(x)$ is not absolutely continuous. But since $\psi(x) = \varphi(x) - x$ and x is absolutely continuous, φ must not be absolutely continuous by Problem 42.

41. Suppose E is measurable. By Theorem 11 of Chapter 2, there exists an F_σ set F such that $F \subseteq E$ and $m^*(E \sim F) = 0$. By (i), $f(F)$ is an F_σ set and is also therefore measurable. By (ii), $f(E \sim F)$ has measure 0 and is therefore measureable. Since $f(E) = f(F \cup (E \sim F)) = f(F) \cup f(E \sim F)$, we can infer that $f(E)$ is measurable as well.

   (i) Let E denote a closed subset of $[a, b]$. Since E is closed and bounded, E is compact. The continuous image of a compact set is compact, so $f(E)$ is also closed and bounded.

   Now suppose F is an F_σ set. Then there exists a countable collection of closed sets $\{E_n\}_{n=1}^\infty$ such that $F = \bigcup_{n=1}^\infty E_n$. But this implies $f(F) = \bigcup_{n=1}^\infty f(E_n)$. By the above result, we know that $f(E_n)$ is closed for all n; therefore $f(F)$ is also an F_σ set.

   (ii) Follows from Problem 40.

42. Sum: Suppose f and g are absolutely continuous on $[a, b]$. Fix $\epsilon > 0$. Let δ_1 and δ_2 correspond to the $\epsilon/2$-challenge of absolute continuity for f and g on $[a, b]$, respectively. Define $\delta = \min(\delta_1, \delta_2)$ and let $\{(a_k, b_k)\}_{k=1}^n$ denote a finite collection of disjoint open intervals contained in $[a, b]$ that satisfy $\sum_{k=1}^n (b_k - a_k) < \delta$. Then

$$\begin{aligned}\sum_{k=1}^n |(f(b_k) + g(b_k)) - (f(a_k) + g(a_k))| &\leq \sum_{k=1}^n |f(b_k) - f(a_k)| + \sum_{k=1}^n |g(b_k) - g(a_k)| \\ &< \epsilon\end{aligned}$$

   where the first line follows from the Triangle Inequality and the second line follows from the definition of absolute continuity. We can therefore conclude that $f + g$ is absolutely continuous on $[a, b]$.

   Product: Suppose f and g are absolutely continuous on $[a, b]$. Fix $\epsilon > 0$. Since f and g are continuous on $[a, b]$, they must be uniformly bounded on $[a, b]$ by the Extreme Value Theorem. Thus there exists $M \geq 0$ such that $|f| \leq M$ and $|g| \leq M$ on $[a, b]$. Let δ_1 and δ_2 correspond to the $\frac{\epsilon}{2 \cdot M}$-challenge of absolutely continuity for f and g on $[a, b]$, respectively. Let $\delta = \min(\delta_1, \delta_2)$ and let $\{(a_k, b_k)\}_{k=1}^n$ denote a finite collection of disjoint open intervals contained in $[a, b]$ that satisfy $\sum_{k=1}^n (b_k - a_k) < \delta$. Then

$$\begin{aligned}\sum_{k=1}^n |f(b_k) \cdot g(b_k) - f(a_k) \cdot g(a_k)| &= \sum_{k=1}^n |(f(b_k) - f(a_k)) \cdot g(b_k) + (g(b_k) - g(a_k)) \cdot f(a_k)| \\ &\leq \sum_{k=1}^n |f(b_k) - f(a_k)| \cdot |g(b_k)| + \sum_{k=1}^n |g(b_k) - g(a_k)| \cdot |f(a_k)| \\ &\leq \sum_{k=1}^n |f(b_k) - f(a_k)| \cdot M + \sum_{k=1}^n |g(b_k) - g(a_k)| \cdot M \\ &< \epsilon\end{aligned}$$

   Therefore $f \cdot g$ is absolutely continuous on $[a, b]$.

43. (i) By Problem 37, we know that f is continuous on $[0, 1]$. An analogous argument can be used to show that f is absolutely continuous on $[-1, 0]$. But if f is absolutely continuous on $[a, b]$ and on $[b, c]$, then f is absolutely continuous on $[a, c]$. To show this, fix $\epsilon > 0$. Let δ_1 and δ_2 correspond to the $\epsilon/2$ challenge of absolutely continuity for f on $[a, b]$ and $[b, c]$ respectively. Define $\delta = \min(\delta_1, \delta_2)$ and choose a finite disjoint collection $\{(a_k, c_k)\}_{k=1}^n$ of open intervals in

(a, c) that satisfy $\sum_{k=1}^n (c_k - a_k) < \delta$. Without loss, assume there exists an index m such that $(a_k, c_k) \subseteq [a, b]$ for $k = 1, \cdots, m$ and $(a_k, c_k) \subseteq [b, c]$ for $k = m+1, \cdots, n$. Then

$$\sum_{k=1}^{n} |f(c_k) - f(a_k)| = \sum_{k=1}^{m} |f(c_k) - f(a_k)| + \sum_{k=m+1}^{n} |f(c_k) - f(a_k)| < \epsilon$$

The first derivative of g is given by

$$g'(x) = \begin{cases} \sin\left(\frac{\pi}{2x}\right) \cdot \frac{\pi}{2} + 2 \cdot x \cdot \cos\left(\frac{\pi}{2x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Therefore $|g'(x)| \leq 4$ for all $x \in [-1, 1]$, which implies that g is Lipschitz and thus absolutely continuous on $[-1, 1]$.

(ii)

$$\begin{aligned} V(f \circ g, P_n) &= \frac{2}{(2n)^{2/3}} + \frac{2}{(2n-2)^{2/3}} + \cdots \frac{2}{2^{2/3}} \\ &= 2^{1/3}\left(\frac{1}{n^{2/3}} + \frac{1}{(n-1)^{2/3}} + \cdots + 1\right) \\ &= 2^{1/3} \sum_{k=1}^{n} \frac{1}{k^{2/3}} \end{aligned}$$

(iii) Since $\sum_{k=1}^n \frac{1}{k^p}$ diverges for $0 < p \leq 1$, $V(f \circ g, P_n)$ is unbounded above. Therefore $f \circ g$ is not of bounded variation and thus not absolutely continuous on $[-1, 1]$.

44. Fix $\epsilon > 0$ and let $c \geq 0$ denote a Lipschitz constant for f. If $c = 0$, then $f \circ g$ is constant and the result is trivial. If $c > 0$, let δ correspond to the ϵ/c-challenge for absolutely continuity of g on $[a, b]$. Let $\{(a_k, b_k)\}_{k=1}^n$ be a finite disjoint collection of open intervals in (a, b) that satisfy $\sum_{k=1}^n (b_k - a_k) < \delta$. Then

$$\sum_{k=1}^{n} |(f \circ g)(b_k) - (f \circ g)(a_k)| \leq c \sum_{k=1}^{n} |g(b_k) - g(a_k)| < \epsilon$$

Therefore $f \circ g$ is absolutely continuous on $[a, b]$.

45. Assume without loss that g is strictly increasing on $[a, b]$. Fix $\epsilon > 0$ and let δ' correspond to the ϵ-challenge for absolutely continuity of f on $\mathbf{R}$. Let δ correspond to the δ'-challenge for absolutely continuity of g on $[a, b]$. Let $\{(a_k, b_k)\}_{k=1}^n$ be a finite disjoint collection of open intervals in (a, b) that satisfy $\sum_{k=1}^n [b_k - a_k] < \delta$. Because g is strictly increasing, $\{(g(a_k), g(b_k))\}_{k=1}^n$ is a finite disjoint collection of open intervals in $\mathbf{R}$. Moreover, $\sum_{k=1}^n [g(b_k) - g(a_k)] < \delta'$ by absolute continuity of g. Therefore $\sum_{k=1}^n |(f \circ g)(b_k) - (f \circ g)(a_k)| < \epsilon$ by absolute continuity of f.

46.
- Assertion 1: $\mathcal{F}_{Lip}$, $\mathcal{F}_{AC}$ and $\mathcal{F}_{BV}$ are closed under linear combinations:

  That $\mathcal{F}_{BV}$ and $\mathcal{F}_{AC}$ are closed under linear combinations follows directly from Problems 34 and 42, respectively. Suppose f and g are Lipschitz on E. Let c_1 and c_2 denote their respective

Lipschitz constants. Then for any x, $x' \in E$, we have

$$|(f(x)+g(x))-(f(x')+g(x'))| \leq |f(x)-f(x')|+|g(x)-g(x')| \leq (c_1+c_2)\cdot|x-x'|$$

Therefore $f+g$ is also Lipschitz, so $\mathcal{F}_{Lip}$ is closed under addition. Now let $\alpha \in \mathbf{R}$ denote an arbitrary scalar. Then

$$|\alpha \cdot f(x) - \alpha \cdot f(x')| = |\alpha| \cdot |f(x)-f(x')| \leq |\alpha| \cdot c \cdot |x-x'|$$

Therefore $\alpha \cdot f$ is also Lipschitz and $\mathcal{F}_{Lip}$ is closed under scalar multiplication. We conclude that $\mathcal{F}_{Lip}$ is closed under linear combinations.

- Assertion 2: A function in one of these collections has its total variation function in the same collection:

  This claim was shown for $\mathcal{F}_{AC}$ in the proof of Theorem 8.

  Suppose f is bounded variation on $[a,b]$. Since $g(x) = TV(f_{[a,x]})$ is increasing and $g(a)=0$, $TV(g_{[a,b]}) = TV(f_{[a,b]}) < \infty$. Thus $TV(f_{[a,x]})$ is also of bounded variation.

  Now suppose f is Lipschitz on $[a,b]$ with Lipschitz constant c. Pick $x, x' \in [a,b]$ and assume without loss that $x \leq x'$. Then

$$\begin{aligned}|TV(f_{[a,x]}) - TV(f_{[a,x']})| &= |TV(f_{x,x'})| \\ &\leq c \cdot |x-x'|\end{aligned}$$

  where the first line follows from (20) on page 117 in the textbook, and the second line follows from the example on the top of page 117. Therefore $TV(f_{[a,x]})$ is also Lipschitz.

47. If f is absolutely continuous, then the condition in the problem follows immediately from the triangle inequality.

    Conversely, suppose the condition in the problem holds. Fix $\epsilon > 0$ and let δ correspond to the $\epsilon/2$-challenge in the prompt. Let $\{(a_k,b_k)\}_{k=1}^n$ be a finite disjoint collection of open intervals in (a,b) that satisfy $\sum_{k=1}^n (b_k - a_k) < \delta$. Then

$$\sum_{k:f(b_k)\geq f(a_k)} [b_k - a_k] < \delta$$

    which implies

$$\sum_{k:f(b_k)\geq f(a_k)} |f(b_k)-f(a_k)| = \left| \sum_{k:f(b_k)\geq f(a_k)} [f(b_k)-f(a_k)] \right| < \epsilon/2$$

    Likewise,

$$\sum_{k:f(b_k)< f(a_k)} [b_k - a_k] < \delta$$

which implies

$$\sum_{k:f(b_k)<f(a_k)} |f(b_k) - f(a_k)| = \left| - \sum_{k:f(b_k)<f(a_k)} [f(b_k) - f(a_k)] \right|$$
$$= \left| \sum_{k:f(b_k)<f(a_k)} [f(b_k) - f(a_k)] \right| < \epsilon/2$$

Therefore

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| = \sum_{k:f(b_k)\geq f(a_k)} |f(b_k) - f(a_k)| + \sum_{k:f(b_k)<f(a_k)} |f(b_k) - f(a_k)| < \epsilon$$

Thus δ satisfies the ϵ-challenge for absolute continuity, so f is absolutely continuous on $[a, b]$.

6.5 Integrating Derivatives: Differentiating Indefinite Integrals

48. φ is singular, but $\varphi(0) = 0$ and $\varphi(1) = 1$. We therefore have

$$\int_0^1 \varphi' = 0 \neq 1 = \varphi(1) - \varphi(0)$$

Theorem 10 then implies that φ is not absolutely continuous.

In contrast with the reasoning in Problem 40, the above argument exploits a property of the derivative of φ.

49. By Problem 11, we know that

$$\int_a^b \mathrm{Diff}_{1/n} f = \big(\mathrm{Av}_{1/n} f(b) - \mathrm{Av}_{1/n} f(a)\big) \tag{1}$$

for all n. Since f is continuous at a, for any $\epsilon > 0$, there exists an index N such that

$$\text{if } |x - a| < \frac{1}{N} \text{ then } |f(x) - f(a)| < \epsilon$$

Thus for all $n \geq N$, we have

$$\begin{aligned}
\big|\mathrm{Av}_{1/n} f(a) - f(a)\big| &= \left| n \int_a^{a+1/n} f - f(a) \right| \\
&= \left| n \int_a^{a+1/n} (f - f(a)) \right| \\
&\leq \frac{1}{n} \int_a^{a+1/n} |f(x) - f(a)| \\
&\leq \frac{1}{n} \int_a^{a+1/n} \epsilon \\
&= \epsilon
\end{aligned}$$

Since $\epsilon > 0$ can be made arbitrarily small, we conclude that $\lim_{n\to\infty} \mathrm{Av}_{1/n} f(a) = f(a)$. Likewise, $\lim_{n\to\infty} \mathrm{Av}_{1/n} f(b) = f(b)$. Taking limits of (1), we obtain

$$\lim_{n\to\infty} \int_a^b \mathrm{Diff}_{1/n} f = f(b) - f(a)$$

Since $f' = \lim_{n\to\infty} \mathrm{Diff}_{1/n} f$ a.e. on $[a, b]$, we also know that

$$\int_a^b f' = \int_a^b \lim_{n\to\infty} \mathrm{Diff}_{1/n} f$$

Therefore if $\lim_{n\to\infty} \int_a^b \mathrm{Diff}_{1/n} f = \int_a^b \lim_{n\to\infty} \mathrm{Diff}_{1/n} f$, then

$$\int_a^b f' = \lim_{n\to\infty} \int_a^b \mathrm{Diff}_{1/n} f = \int_a^b \lim_{n\to\infty} \mathrm{Diff}_{1/n} f = f(b) - f(a)$$

Conversely, if $\int_a^b f' = f(b) - f(a)$ then,

$$\lim_{n\to\infty} \int_a^b \mathrm{Diff}_{1/n} f = \int_a^b f' = f(b) - f(a) = \lim_{n\to\infty} \mathrm{Diff}_{1/n} f$$

50. By the Vitali Convergence Theorem, we have

$$\int_a^b f' = \int_a^b \lim_{n\to\infty} \mathrm{Diff}_{1/n} f = \lim_{n\to\infty} \int_a^b \mathrm{Diff}_{1/n} f$$

Therefore $\int_a^b f' = f(b) - f(a)$ by Problem 49.

51. By the Dominated Convergence Theorem, we know that

$$\int_a^b f' = \int_a^b \lim_{n\to\infty} \mathrm{Diff}_{1/n} f = \lim_{n\to\infty} \int_a^b \mathrm{Diff}_{1/n} f$$

Therefore $\int_a^b f' = f(b) - f(a)$ by Problem 49.

52. $f \cdot g$ is absolutely continuous by Problem 42. Theorem 10 then implies

$$\int_b^a (f \cdot g)' = f(b) \cdot g(b) - f(a) \cdot g(a)$$

By the Product Rule, $(f \cdot g)' = f' \cdot g + g' \cdot f$ a.e. $[a, b]$. Therefore

$$\int_b^a f' \cdot g + \int_b^a g' \cdot f = f(b) \cdot g(b) - f(a) \cdot g(a)$$

by linearity of integration. Since g is continuous on $[a, b]$, there exists $M > 0$ such that $|g| \leq M$ on $[a, b]$. We also know that f' is integrable by Theorem 10. Therefore

$$\int_b^a |f' \cdot g| = \int_b^a |f'| \cdot |g| \leq M \int_b^a |f'| < \infty$$

by monotonicity of integration. We can thus conclude

$$\int_b^a f' \cdot g = f(b) \cdot g(b) - f(a) \cdot g(a) - \int_b^a f' \cdot g$$

53. Suppose f is Lipschitz. Then there exists $c \geq 0$ such that

$$\left| \frac{f(x+t)}{t} \right| \leq c$$

for any x and t that satisfy $x + t \in [a, b]$. Thus if f is differentiable at x, then

$$|f'(x)| = \lim_{t \to 0} \left| \frac{f(x+t)}{t} \right| \leq c$$

Since f is differentiable a.e. by Theorem 10, we must have $|f'| \leq c$ a.e. on $[a, b]$.

Conversely, suppose there exists $c \geq 0$ such that $|f'| \leq c$ a.e. on $[a, b]$. If x, x' are in $[a, b]$ and $x \leq x'$, then f is absolutely continuous on $[x, x']$. We therefore have

$$|f(x) - f(x')| = \left| \int_x^{x'} f' \right| \leq \int_x^{x'} |f'| \leq c|x - x'|$$

by Theorem 10 and monotonicity of integration. Therefore f is Lipschitz on $[a, b]$.

54. (i) Fix $\epsilon > 0$ and $\delta > 0$. Define the collection of closed, bounded, non-degenerate intervals

$$\mathcal{F} = \left\{ [c, d] \subseteq (a, b) : c < d \text{ and } f(d) - f(c) < \epsilon \cdot \frac{d - c}{b - a} \right\}$$

Since f is singular, there exists a set E_0 with $m(E_0) = 0$ such that $f'(x) = 0$ for all $x \in (a, b) \sim E_0$. Let $E = (a, b) \sim E_0$. Because f is increasing and $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = 0$ for all $x \in E$, $\mathcal{F}$ covers E in the sense of Vitali. By the Vitali Covering Lemma, there exists a finite, disjoint subcollection $\{(c_k, d_k)\}_{k=1}^N$ of $\mathcal{F}$ for which

$$m\left(E \sim \bigcup_{k=1}^N [c_k, d_k] \right) < \delta$$

Without loss, assume the subcollection is ordered so that

$$c_1 < d_1 < c_2 < d_2 < \cdots < c_N < d_N$$

Then

$$(a, b) \sim \bigcup_{k=1}^N [c_k, d_k] = \bigcup_{k=1}^n (a_k, b_k)$$

where

$$n = N + 1$$

$$a_k = \begin{cases} a & \text{if } k = 1 \\ d_{k-1} & \text{if } k = 2, \cdots, n \end{cases}$$

$$b_k = \begin{cases} c_k & \text{if } k = 1, \cdots, n-1 \\ b & \text{if } k = n \end{cases}$$

Then

$$\begin{aligned} m\left(\bigcup_{k=1}^{n}(a_k, b_k)\right) &= m\left(E \sim \bigcup_{k=1}^{N}(c_k, d_k) \sim E_0\right) + m\left(E \sim \bigcup_{k=1}^{N}(c_k, d_k) \cap E_0\right) \\ &\leq m\left(E \sim \bigcup_{k=1}^{N}(c_k, d_k)\right) + m\left(E_0\right) \\ &< \delta \end{aligned}$$

Since $[c_k, d_k] \in \mathcal{F}$ for $k = 1, \cdots, N$, we also have

$$\begin{aligned} \sum_{k=1}^{n}[f(b_k) - f(a_k)] &= f(b) - f(a) - \sum_{k=1}^{N}[f(d_k) - f(c_k)] \\ &> f(b) - f(a) - \epsilon \cdot \frac{\sum_{k=1}^{N}(d_k - c_k)}{b-a} \\ &\geq f(b) - f(a) - \epsilon \end{aligned}$$

(ii) Fix $\epsilon > 0$. By Corollary 4, f' is integrable over $[a, b]$. By Proposition 23 of Chapter 4, there exists $\delta > 0$ such that

$$\text{if } A \subseteq [a, b] \text{ is measurable and } m(A) < \delta, \text{ then } \int_A f' < \frac{\epsilon}{2} \tag{2}$$

Choose a finite disjoint collection $\{(a_k, b_k)\}_{k=1}^{n}$ of open intervals in (a, b) for which

$$\sum_{k=1}^{n}[b_k - a_k] < \delta \text{ and } f(b) - f(a) - \sum_{k=1}^{n}[f(b_k) - f(a_k)] < \frac{\epsilon}{2}$$

Then

$$\int_{\bigcup_{k=1}^{n}(a_k, b_k)} f' < \frac{\epsilon}{2}$$

by (2). Assume without loss that the intervals are ordered so that

$$a = b_0 \leq a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n \leq a_{n+1} = b$$

We then have

$$f(b) - f(a) - \sum_{k=1}^{n}[f(b_k) - f(a_k)] = \sum_{k=1}^{n+1}[f(a_k) - f(b_{k-1})] \geq \sum_{k=1}^{n+1}\int_{b_{k-1}}^{a_k} f' = \int_{[a,b]\sim\bigcup_{k=1}^{n}(a_k,b_k)} f'$$

where the inequality follows from Corollary 4. Thus

$$\int_a^b f' = \int_{\bigcup_{k=1}^{n}(a_k,b_k)} f' + \int_{[a,b]\sim\bigcup_{k=1}^{n}(a_k,b_k)} f' < \epsilon$$

Since ϵ can be made arbitrarily small, we must have $\int_a^b f' = 0$. But $f' \geq 0$ because f is increasing, so $f' = 0$ a.e. by Proposition 9 of Chapter 4.

(iii) Fix $\epsilon > 0$ and $\delta > 0$. Since $\lim_{N\to\infty}\sum_{n=1}^{N}(f_n(b) - f_n(a)) = f(b) - f(a)$, there exists an index N for which

$$f(b) - f(a) - \frac{\epsilon}{2} < \sum_{n=1}^{N}[f_n(b) - f_n(a)] \tag{3}$$

The function $\sum_{n=1}^{N}[f_n(x) - f_n(x)]$ is increasing and singular on $[a, b]$, so by part (i) there exists a finite disjoint collection $\{(a_k, b_k)\}_{k=1}^{K}$ of open intervals in (a, b) for which

$$\sum_{k=1}^{K}[b_k - a_k] < \delta$$

and

$$\sum_{k=1}^{K}\sum_{n=1}^{N}[f_n(b_k) - f_n(a_k)] > \sum_{n=1}^{N}[f_n(b) - f_n(a)] - \frac{\epsilon}{2} \tag{4}$$

for $k = 1, \cdots, K$. Combining (3) and (4), we obtain

$$f(b) - f(a) - \epsilon < \sum_{k=1}^{K}\sum_{n=1}^{N}[f_n(b_k) - f_n(a_k)] \tag{5}$$

Since each f_n is increasing, we also know that

$$\sum_{n=1}^{N}[f_n(b_k) - f_n(a_k)] \leq \sum_{n=1}^{\infty}[f_n(b_k) - f_n(a_k)] = f(b_k) - f(a_k)$$

for $k = 1, \cdots, K$. Substituting into (5), we obtain

$$f(b) - f(a) - \epsilon < \sum_{k=1}^{K}[f(b_k) - f(a_k)]$$

But then f satisfies the property in part (i), so we can conclude that f is singular by part (ii).

55. (i) Since f is of bounded variation, v is of bounded variation by the remark at the end of Section 6.4. Therefore both f and v are differentiable almost everywhere on $[a, b]$ by Corollary 6. But then there exists a set $E \subseteq [a, b]$ of measure zero such that f and v are differentiable on $[a, b] \sim E$.

Now for any interval $[c,d] \subseteq [a,b]$, we have

$$f(d) - f(c) \le |f(d) - f(c)| \le TV(f_{[c,d]}) = TV(f_{[a,d]}) - TV(f_{[a,c]}) = v(d) - v(c)$$

Therefore

$$f(d) - v(d) \le f(c) - v(c) \text{ for all } a \le c < d \le b$$

Thus $f(x) - v(x)$ is decreasing on $[a,b]$. We also know that $f(x) + v(x)$ is increasing on $[a,b]$ by Lemma 5. Therefore for all $x \in [a,b] \sim E$, we have

$$f'(x) - v'(x) \le 0 \text{ and } f'(x) + v'(x) \ge 0$$

which implies $|f'| \le v'$ almost everywhere on $[a,b]$. By monotonicity of integration and Corollary 4, we have

$$\int_a^b |f'| \le \int_a^b v' \le v(b) - v(a) = TV(f_{[a,b]})$$

(ii) Suppose f is absolutely continuous. Fix $\epsilon > 0$ and choose a partition $P = \{x_0, x_1, \cdots, x_k\}$ that satisfies

$$\sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| > TV(f_{[a,b]}) - \epsilon$$

Since f is absolutely continuous, we have

$$\sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| = \sum_{i=1}^{k} \left| \int_{x_{i-1}}^{x_i} f' \right| \le \sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} |f'| \le \int_a^b |f'|$$

by Theorem 10. We can therefore conclude

$$TV(f_{[a,b]}) - \epsilon \le \int_a^b |f'|$$

for any $\epsilon > 0$, which implies $TV(f_{[a,b]}) \le \int_a^b |f'|$. As the reverse inequality was proven in part (i), we must have $TV(f_{[a,b]}) = \int_a^b |f'|$.

Conversely, suppose $TV(f_{[a,b]}) = \int_a^b |f'|$. Then

$$TV(f_{[a,b]}) = \int_a^b |f'| \le \int_a^b v' \le TV(f_{[a,b]})$$

by part (i). But this expression implies

$$\int_a^b (v' - |f'|) = 0$$

Since $v' - |f'| \ge 0$, $v' = |f'|$ a.e. on $[a,b]$ by Proposition 9 of Chapter 4. Therefore for any $a \le c < d \le b$, we have

$$\int_c^d |f'| = \int_c^d v' = TV(f_{[c,d]}) \ge |f(d) - f(c)|$$

Now fix $\epsilon > 0$. Since f' is integrable, there exists a $\delta > 0$ such that for any finite, disjoint collection $\{(a_k, b_k)\}_{k=1}^n$ of open intervals contained in $[a, b]$,

$$\text{if} \quad \sum_{k=1}^{n} [b_k - a_k] < \delta \quad \text{then} \quad \int_{\cup_{k=1}^n (a_k, b_k)} |f'| < \epsilon$$

But since $|f(b_k) - f(a_k)| \leq \int_{a_k}^{b_k} |f'|$ for $k = 1, \cdots, n$, we have

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| \leq \sum_{k=1}^{n} \int_{a_k}^{b_k} |f'| = \int_{\cup_{k=1}^n (a_k, b_k)} |f'| < \epsilon$$

for any finite, disjoint collection $\{(a_k, b_k)\}_{k=1}^n$ of open intervals contained in $[a, b]$ satisfying $\sum_{k=1}^n [b_k - a_k] < \delta$. Therefore f is absolutely continuous on $[a, b]$.

(iii) Corollaries 4 and 12 can be viewed as an application of the results in parts (i) and (ii) to increasing functions.

56. (i) By Proposition 9 of Chapter 1, $\mathcal{O} = \bigcup_{k=1}^\infty (a_k, b_k)$ for some countable collection of disjoint open intervals $\{(a_k, b_k)\}_{k=1}^\infty$. The Intermediate Value Theorem and Theorem 10 then imply

$$m(g((a_k, b_k))) = g(b_k) - g(a_k) = \int_{a_k}^{b_k} g'$$

for $k = 1, 2, \cdots$. By countable additivity of measure and integration, we have

$$\begin{aligned} m(g(\mathcal{O})) &= m\left(g\left(\bigcup_{k=1}^\infty (a_k, b_k)\right)\right) \\ &= m\left(\bigcup_{k=1}^\infty g((a_k, b_k))\right) \\ &= \sum_{k=1}^\infty m\left(g((a_k, b_k))\right) \\ &= \sum_{k=1}^\infty \int_{a_k}^{b_k} g' \\ &= \int_{\mathcal{O}} g' \end{aligned}$$

(ii) Since E is a G_δ-set, there exists a countable collection of open sets $\{G_k\}_{k=1}^\infty$ such that $E = \bigcap_{k=1}^\infty G_k$. Because g is injective, we have

$$g(E) = g\left(\bigcap_{k=1}^\infty G_k\right) = \bigcap_{k=1}^\infty g(G_k)$$

Since g is continuous and strictly increasing, g maps open intervals to open intervals. Therefore g maps open sets to open sets, which implies that $g(E)$ is a G_δ-set. Therefore $g(E)$ is measurable.

Now assume without loss that $G_1 \subseteq (a, b)$. Define $\mathcal{O}_n = \bigcap_{k=1}^n G_k$ and let $E_n = \mathcal{O}_n \sim E$. By

Problem 39, for any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\text{if } m(E_n) < \delta, \text{ then } m(g(E_n)) < \epsilon$$

Since $E_n \downarrow \varnothing$ and $m(E_1) \leq b - a$, $\lim_{n\to\infty} m(E_n) = 0$ by continuity of measure. Therefore there exists an index N such that $m(E_n) < \delta$ for all $n \geq N$. The above expression then implies that $m(g(E_n)) < \epsilon$ for all $n \geq N$. Since ϵ can be made arbitrarily small, we must have $\lim_{n\to\infty} m(g(E_n)) = 0$.

Now monotonicity and subadditivity of outer measure imply

$$\begin{aligned} m(g(E)) &\leq m(g(\mathcal{O}_n)) \\ &= m(g(E \cup E_n)) \\ &= m(g(E) \cup g(E_n)) \\ &\leq m(g(E)) + m(g(E_n)) \end{aligned}$$

By part (i), we also know that

$$\int_{\mathcal{O}_n} g' = m(g(\mathcal{O}_n))$$

for $n = 1, 2, \cdots$. Combining these expressions, we obtain

$$m(g(E)) \leq \int_{\mathcal{O}_n} g' \leq m(g(E)) + m(g(E_n))$$

Taking limits of both sides and applying continuity of integration, we obtain $m(g(E)) = \int_E g'$.

(iii) Suppose E has measure zero. Then for any $\epsilon > 0$, $m^*(g(E)) < \epsilon$ by Problem 39. Therefore $g(E)$ has measure zero. We also know that $\int_E g' = 0$ by Problem 9 of Chapter 4. Therefore $m(g(E)) = 0 = \int_E g'$, as desired.

(iv) By Theorem 11 of Chapter 2, there exists a G_δ set E containing A for which $m(E \sim A) = 0$. Corollary 18 of Chapter 4 and parts (ii) and (iii) then imply

$$\begin{aligned} \int_A g' &= \int_E g' + \int_{E\sim A} g' \\ &= \int_E g' \\ &= m(g(E)) \end{aligned}$$

But monotonicity and subadditivity of measure imply

$$m(g(A)) \leq m(g(E)) \leq m(g(A)) + m(g(E \sim A)) = m(g(A))$$

Therefore $m(g(A)) = m(g(E))$. Combining these results, we have $\int_A g' = m(g(A))$.

(v) Since φ is simple, it can be written as $\varphi = \sum_{k=1}^n c_k \cdot \chi_{E_k}$ where $c_1, \cdots, c_k$ are distinct values and $E_k = \{y \in [c,d] : \varphi(y) = c_k\}$. Since $g : [a,b] \to [c,d]$ is surjective by the Intermediate Value Theorem, $g\left(g^{-1}(E_k)\right) = E_k$. We can therefore apply linearity of integration and part (iv) to conclude

$$\begin{aligned}
\int_a^b \varphi(g(x))g'(x)dx &= \sum_{k=1}^n c_k \int_a^b \chi_{E_k}(g(x))g'(x)dx \\
&= \sum_{k=1}^n c_k \cdot \int_{g^{-1}(E_k)} g' \\
&= \sum_{k=1}^n c_k \cdot m\left(g\left(g^{-1}(E_k)\right)\right) \\
&= \sum_{k=1}^n c_k \cdot m(E_k) \\
&= \sum_{k=1}^n c_k \cdot \int_c^d \chi_{E_k} \\
&= \int_c^d \varphi(y)dy
\end{aligned}$$

(vi) By the Simple Approximation Theorem, there exists an increasing sequence $\{\varphi_n\}$ of simple functions on $[c, d]$ that converges pointwise to f on $[c, d]$. The Monotone Convergence Theorem and part (v) then imply

$$\begin{aligned}
\int_c^d f(y)dy &= \lim_{n\to\infty} \int_c^d \varphi_n(y)dy \\
&= \lim_{n\to\infty} \int_a^b \varphi_n(g(x))g'(x)dx \\
&= \int_a^b f(g(x))g'(x)dx
\end{aligned}$$

(vii) Since g is injective, $\mathcal{O} = g^{-1}(g(\mathcal{O}))$. Therefore $\chi_{g(\mathcal{O})}(g(x)) = \chi_{\mathcal{O}}(x)$. Applying part (vi), we conclude

$$m(g(\mathcal{O})) = \int_c^d \chi_{g(\mathcal{O})}(y)d(y) = \int_a^b \chi_{g(\mathcal{O})}(g(x))g'(x)dx = \int_a^b \chi_{\mathcal{O}}(x)g'(x)dx = \int_{\mathcal{O}} g'(x)dx$$

57. Let $a = 0$ and $b = 1$. Let g denote the Cantor-Lebesgue function and let $f(y) = 1$ on $[0, 1]$. Then $g' = 0$ a.e., so

$$1 = \int_c^d f(y)dy \neq \int_a^b f(g(x))g'(x)dx = 0$$

58. Let F denote a generalized Cantor set of measure $1 - \alpha$ for some $\alpha \in (0, 1)$. Let $E = [0, 1] \sim F$ and let

$$f(x) = \int_0^x \chi_E$$

Then f is absolutely continuous by Theorem 11. Now pick any $x, x' \in [0, 1]$ satisfying $x < x'$. By Problem 39 of Chapter 2, E is an open set that is dense in $[0, 1]$. Therefore there exists $y \in (x, x')$ and

$r > 0$ such that $(y - r/2, y + r/2) \subseteq E \cap (x, x')$. But then

$$f(x') - f(x) = \int_x^{x'} \chi_E = m(E \cap [x, x']) \geq r > 0$$

by monotonicity of measure. Therefore $f(x') > f(x)$, so f is strictly increasing on $[0,1]$. By Theorem 14, there exists a set E_0 of measure zero such that $f' = \chi_E$ on $[0,1] \sim E_0$. But then $f' = 0$ on $F \sim E_0$, a set of positive measure.

59. Define $h(x) = \int_{g(a)}^{g(x)} f(s)ds - \int_a^x f(g(t))g'(t)dt$. The claim is that $h(b) - h(a) = 0$. However showing that $h' = 0$ a.e. $[a, b]$ does not prove this claim (the Cantor-Lebesgue function is a counter-example).

60. By Theorems 10 and 11, $f(x) = f(a) + \int_a^x f' = f(a)$ for all $x \in [a, b]$.

    Suppose $g_0 + h_0 = g_1 + h_1$ on $[a, b]$, where g_0 and g_1 are absolutely continuous and h_0 and h_1 are singular. Furthermore, assume $h_0(a) = h_1(a) = 0$. Then $g_0' = g_1'$ on $[a, b]$ and $g_0(a) = g_1(a)$, which implies

$$g_0(x) = g_0(a) + \int_a^x g_0' = g_1(a) + \int_a^x g_1' = g_1(x)$$

    for $x \in [a, b]$. Since $h_0 - h_1 = g_1 - g_0$, the above expression also implies $h_0 = h_1$ on $[a, b]$.

6.6 Convex Functions

61. Suppose φ is convex. The statement in the prompt holds trivially for $n = 1$. Now suppose the statement is true for some index n. Let $x_1, \cdots, x_n, x_{n+1}$ denote points in (a, b) and let $\lambda_1, \cdots, \lambda_n, \lambda_{n+1}$ denote nonnegative numbers that satisfy $\sum_{k=1}^{n+1} \lambda_k = 1$. Define $\lambda = \sum_{k=1}^{n} \lambda_k$ and $x = \sum_{k=1}^{n} \lambda_k x_k / \lambda$. By the induction hypothesis, we know that

$$\varphi(x) \leq \sum_{k=1}^{n} \lambda_k \varphi(x_k)/\lambda$$

    We therefore have

$$\begin{aligned}
\varphi\left(\sum_{k=1}^{n+1} \lambda_k x_k\right) &= \varphi\left(\lambda x + (1-\lambda)x_{n+1}\right) \\
&\leq \lambda\varphi(x) + (1-\lambda)\varphi(x_{n+1}) \\
&\leq \sum_{k=1}^{n} \lambda_k \varphi(x_k) + \lambda_{n+1}\varphi(x_{n+1}) \\
&= \sum_{k=1}^{n+1} \lambda_k \varphi(x_k)
\end{aligned}$$

    where the second line follows from convexity of φ. Therefore the statement is true for $n + 1$, so we conclude by induction that the statement holds for all natural numbers n.

    The converse follows immediately because the definition of convexity is a special case of the statement in the prompt.

Now suppose f is a simple function over $[0,1]$. Let $\sum_{k=1}^{n} c_k \chi_{E_k}$ denote the canonical representation of f. Then $m(E_k) \geq 0$ for all k and $\sum_{k=1}^{n} m(E_k) = 1$, so

$$\begin{aligned}
\varphi\left(\int_0^1 f(x)dx\right) &= \varphi\left(\sum_{k=1}^{n} c_k m(E_k)\right) \\
&\leq \sum_{k=1}^{n} m(E_k)\varphi\left(c_k\right) \\
&= \int_0^1 (\varphi \circ f)(x)dx
\end{aligned}$$

where the second line applies the statement in the prompt and the first and third lines follow from the definition of the integral of a simple function.

62. If φ is convex, then the prompt statement follows immediately from the definition of convexity.

Conversely, suppose the prompt statement is true. Fix $x_1, x_2 \in (a,b)$ and $\lambda \in [0,1]$. Write the binary representation of λ as

$$\lambda = \sum_{k=1}^{\infty} c_k 2^{-k}$$

where $c_k \in \{0,1\}$ for all k. Since $1 = \sum_{k=1}^{\infty} 2^{-k}$, we also have

$$1 - \lambda = \sum_{k=1}^{\infty} (1-c_k)2^{-k}$$

We can now derive the expression

$$\begin{aligned}
\varphi\left(\lambda x_0 + (1-\lambda)x_1\right) &= \varphi\left(\sum_{k=1}^{\infty} c_k 2^{-k} x_0 + \sum_{k=1}^{\infty}(1-c_k)2^{-k}x_1\right) \\
&= \varphi\left(\frac{c_1 x_1 + (1-c_1)x_2}{2} + \frac{\sum_{k=1}^{\infty} c_{k+1}2^{-k}x_0 + \sum_{k=1}^{\infty}(1-c_{k+1})2^{-k}x_1}{2}\right) \\
&\leq \frac{\varphi(c_1 x_1 + (1-c_1)x_2)}{2} + \frac{\varphi\left(\sum_{k=1}^{\infty} c_{k+1}2^{-k}x_0 + \sum_{k=1}^{\infty}(1-c_{k+1})2^{-k}x_1\right)}{2} \\
&= \frac{c_1\varphi(x_1) + (1-c_1)\varphi(x_2)}{2} + \frac{\varphi\left(\sum_{k=1}^{\infty} c_{k+1}2^{-k}x_0 + \sum_{k=1}^{\infty}(1-c_{k+1})2^{-k}x_1\right)}{2}
\end{aligned}$$

where the inequality follows from the prompt statement and the last line follows because $c_1 \in \{0,1\}$. If we continue recursively, we obtain the inequality

$$\varphi\left(\lambda x_0 + (1-\lambda)x_1\right) \leq \sum_{k=1}^{n} c_k 2^{-k}\varphi(x_1) + \sum_{k=1}^{n}(1-c_k)2^{-k}\varphi(x_2) + 2^{-n}\varphi\left(\sum_{k=1}^{\infty} c_{k+n}2^{-k}x_0 + \sum_{k=1}^{\infty}(1-c_{k+n})2^{-k}x_1\right)$$

for any index n. Since φ is continuous, it is uniformly bounded on the interval $[x_1, x_2]$. Therefore the last term can be made arbitrarily small by taking n large enough. We can therefore take limit to obtain

$$\varphi\left(\lambda x_0 + (1-\lambda)x_1\right) \leq \sum_{k=1}^{\infty} c_k 2^{-k}\varphi(x_1) + \sum_{k=1}^{\infty}(1-c_k)2^{-k}\varphi(x_2) = \lambda\varphi(x_1) + (1-\lambda)\varphi(x_2)$$

63. Consider the function $\varphi(x) = -\sqrt{1-x^2}$ over the interval $[0, 1]$. Then

$$\varphi'(x) = \frac{x}{\sqrt{1-x^2}} \qquad \varphi''(x) = \frac{1}{(1-x^2)^{3/2}}$$

on $(0, 1)$. Since $\varphi'' \geq 0$ on $(0, 1)$, φ is convex on $(0, 1)$ by Proposition 15. Since φ is continuous on $[0, 1]$, φ must also be convex on $[0, 1]$. However φ' is unbounded on $(0, 1)$ and therefore φ cannot be Lipschitz on $[0, 1]$.

64. If $\varphi'' \geq 0$ on (a, b), then φ is convex on (a, b) by Proposition 15.

Conversely, suppose φ is convex. Fix $x \in (a, b)$. By Lemma 16, we know that

$$\frac{\varphi'(x+h) - \varphi'(x)}{h} \geq 0$$

for all $h \neq 0$ satisfying $x + h \in (a, b)$. Taking the limit as $h \to 0$, we conclude that $\varphi''(x) \geq 0$.

65. Taking derivatives, we obtain the expressions

$$\varphi'(t) = bp(a+bt)^{p-1} \qquad \varphi''(t) = b^2 p(p-1)(a+bt)^{p-2}$$

for $t \in (0, \infty)$. If $p \geq 1$, $\varphi'' \geq 0$ on $(0, \infty)$. Therefore φ is convex on $(0, \infty)$ by Proposition 15. Since φ is continuous at 0, φ is convex on $[0, \infty)$.

66. Claim: Suppose φ is a convex function on $(-\infty, \infty)$ that satisfies

$$\varphi\left(\int_0^1 f(x)dx\right) = \int_0^1 (\varphi \circ f)(x)dx$$

for all integrable functions f over $[0, 1]$. Then there exist constants c_0 and c_1 such that $\varphi(x) = c_0 + c_1 x$.

Proof: Let $c_0 = \varphi(0)$ and $c_1 = \varphi(1) - \varphi(0)$. Then for $x \in [0, 1]$, we have

$$\begin{aligned}
\varphi(x) &= \varphi\left(\int_0^1 \chi_{[0,x]}(t)dt\right) \\
&= \int_0^1 (\varphi \circ \chi_{[0,x]})(t)dt \\
&= x\varphi(1) + (1-x)\varphi(0) \\
&= c_0 + c_1 x
\end{aligned}$$

For $x \in (1, \infty)$, we have

$$\begin{aligned}
\varphi(1) &= \varphi\left(\int_0^1 x \cdot \chi_{[0,1/x]}(t)dt\right) \\
&= \int_0^1 \varphi(x \cdot \chi_{[0,1/x]}(t))dt \\
&= \frac{1}{x}\varphi(x) + \left(1 - \frac{1}{x}\right)\varphi(0) \\
\Rightarrow \qquad \varphi(x) &= c_0 + c_1 x
\end{aligned}$$

For $x \in (-\infty, 0)$, we have

$$
\begin{aligned}
\varphi(0) &= \varphi\left(\int_0^1 \left(x \cdot \chi_{[0,1/(1-x)]}(t) + \cdot\chi_{(1/(1-x),1]}(t)\right) dt\right) \\
&= \int_0^1 \varphi\left(x \cdot \chi_{[0,1/(1-x)]}(t) + \cdot\chi_{(1/(1-x),1]}(t)\right) dt \\
&= \frac{1}{1-x}\varphi(x) + \frac{-x}{1-x}\varphi(1) \\
\Rightarrow \qquad \varphi(x) &= c_0 + c_1 x
\end{aligned}
$$

67. Claim: Let φ be a convex function on $(-\infty, \infty)$ and let f be an integrable function over the closed, bounded, non-degenerate interval $[a, b]$ such that $\varphi \circ f$ is also integrable over $[a, b]$. Then

$$
\varphi\left(\frac{1}{b-a}\int_a^b f(x)dx\right) \leq \frac{1}{b-a}\int_a^b (\varphi \circ f)(x)dx
$$

Proof: Define

$$
\alpha = \frac{1}{b-a}\int_a^b f(x)dx
$$

Choose m to lie between $\varphi(\alpha^-)$ and $\varphi(\alpha^+)$. Since φ is convex, we know that

$$
\varphi\left(f(x)\right) \geq m\left(f(x) - \alpha\right) + \tilde{\varphi}(\alpha)
$$

Thus by monotonicity and linearity of integration, we have

$$
\int_a^b (\varphi \circ f)(x)dx \geq m \cdot \left(\int_a^b f(x)dx - (b-a)\cdot\alpha\right) + \varphi(\alpha)\cdot(b-a)
$$

Dividing both sides by $b - a$ yields the desired result.

68. Since $\frac{d^2}{dx^2}\exp(x) = \exp(x) \geq 0$ for all $x \in \mathbf{R}$, $\exp(\cdot)$ is convex by Proposition 15. The desired result then follows immediately from Jensen's Inequality.

69. **Correction:** $\lim_{k\to\infty}\prod_{n=1}^k \zeta_n^{\alpha_n}$ **might not exist. For example, if** $\alpha_n = 2^{-n}$ **and** $\zeta_n = \exp\left((-2)^n\right)$**, then** $\prod_{n=1}^k \zeta_n^{\alpha_n} = \exp((1+(-1)^{k+1})/2)$**. We therefore prove** $\limsup_{k\to\infty}\prod_{n=1}^k \zeta_n^{\alpha_n} \leq \sum_{n=1}^\infty \alpha_n\zeta_n$**.** Let $\alpha_0 = 0$ and define $E_n = [\alpha_{n-1}, \alpha_{n-1} + \alpha_n)$. Then $m(E_n) = \alpha_n$ for all n. For each k, define $f_k = \sum_{n=1}^k \zeta_n \cdot \chi_{E_n}$. Then f_k is a positive measurable function over $[0, 1]$, so by Problem 70 we have

$$
\log\left(\sum_{n=1}^k \alpha_n \cdot \zeta_n\right) = \log\left(\int_0^1 f_k(x)dx\right) \geq \int_0^1 \log\left(f_k(x)\right)dx = \sum_{n=1}^k \alpha_n \cdot \log(\zeta_n) = \log\left(\prod_{n=1}^k \zeta_n^{\alpha_n}\right)
$$

Taking exponentials, we obtain the expression

$$
\sum_{n=1}^k \alpha_n \cdot \zeta_n \geq \prod_{n=1}^k \zeta_n^{\alpha_n}
$$

for all k. Since $\{\alpha_n \cdot \zeta_n\}$ is a sequence of positive numbers, the series $\sum_{n=1}^k \alpha_n \cdot \zeta_n$ converges. We therefore have

$$\sum_{n=1}^{\infty} \alpha_n \cdot \zeta_n \geq \limsup_{k\to\infty} \prod_{n=1}^{k} \zeta_n^{\alpha_n}$$

70. The function $-\log(\cdot)$ is convex on $(0,\infty)$ by Proposition 15. Since g is positive on $[0,1]$, we can use the Jensen's Inequality proof in the text to show

$$-\log\left(\int_0^1 g(x)dx\right) \leq -\int_0^1 \log(g(x))dx$$

The desired result then follows by multiplying both sides of the inequality by -1.

71. Suppose there exist constants c_1 and c_2 for which

$$|\varphi(t)| \leq c_1 + c_2|t|$$

for all $t \in \mathbf{R}$. If f is integrable over $[0,1]$, then f is finite a.e. on $[0,1]$. Therefore

$$|(\varphi \circ f)(x)| \leq c_1 + c_2|f(x)|$$

for almost $x \in [0,1]$. Monotonicity of measure then implies

$$\int_0^1 |(\varphi \circ f)(x)|dx \leq c_1 + c_2 \int_0^1 |f(x)|dx < \infty$$

Thus $\varphi \circ f$ is integrable whenever f is.

The following preliminary result will be used in the proof of the converse.

Claim: Let $\{b_n\}$ be an increasing sequence of real numbers and suppose $\lim_{n\to\infty} b_n = \infty$. There exists a non-negative sequence $\{a_n\}$ of real numbers such that

$$\sum_{n=1}^{\infty} a_n < \infty \text{ and } \sum_{n=1}^{\infty} a_n \cdot b_n = \infty$$

Proof: Assume without loss of generality that $b_1 > 0$ (otherwise just set $a_n = 0$ until $b_n > 0$). Define

$$k_n = \lfloor \log_2 b_n \rfloor$$

For any fixed index n, let m_n denote the total number of indicies n_0 satisfying $k_{n_0} = k_n$. Now define

$$a_n = \frac{1}{m_n 2^{k_n}}$$

Then $a_n > 0$ for all n, and

$$\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty$$

We also have

$$2^{k_n} \leq b_n$$

which implies

$$\sum_{n=1}^{\infty} a_n \cdot b_n \geq \sum_{n=1}^{\infty} 1 = \infty$$

as desired.

Now suppose $\varphi \circ f$ is integrable over $[0,1]$ whenever f is. Let g denote the function

$$g(x) = \left| \frac{\varphi(x)}{x} \right|$$

We first show there must exist $x_0 \geq 0$ and $c_1 \geq 0$ such that

$$|g(x)| \leq c_1 \text{ if } |x| \geq x_0$$

To prove this claim, suppose g is unbounded on every set of the form $(\infty, -x_0] \cup [x_0, \infty)$. Since φ is continuous, we can define the sequence

$$y_n = \max_{1 \leq |x| \leq n} g(x)$$

As g is unbounded on $(\infty, -1] \cup [1, \infty)$, we must have $y_n \uparrow \infty$. We can therefore apply the preliminary lemma to construct a non-negative sequence $\{a_n\}$ such that

$$0 < \sum_{n=1}^{\infty} a_n < \infty \text{ and } \sum_{n=1}^{\infty} a_n \cdot y_n = \infty$$

Let $\{x_n\}$ denote a sequence satisfying $g(x_n) = y_n$ and $1 \leq |x_n| \leq n$. Let $a = \sum_{n=1}^{\infty} a_n$ and define

$$d_n = \frac{a_n}{|x_n| \cdot a}$$

Define

$$E_1 = [0,\ d_1)\,, \quad E_n = \left[\sum_{k=1}^{n-1} d_k,\ \sum_{k=1}^{n} d_k\right) \quad n = 2, 3, ...$$

(If $d_n = 0$, set $E_n = \varnothing$.) Since

$$\sum_{n=1}^{\infty} d_n \leq \sum_{n=1}^{\infty} \frac{a_n}{a} = 1$$

we must have $E_n \subseteq [0,1]$ for all n. Let f denote the function

$$f = \sum_{n=1}^{\infty} |x_n| \cdot \chi_{E_n}$$

Then

$$\int_0^1 |f(x)| dx = \sum_{n=1}^{\infty} |x_n| \cdot d_n = 1$$

but

$$\int_0^1 |(\varphi \circ f)(x)| dx = \sum_{n=1}^{\infty} |\varphi(x_n)| \cdot d_n = \sum_{n=1}^{\infty} y_n \cdot |x_n| \cdot d_n = \frac{1}{a} \sum_{n=1}^{\infty} a_n \cdot y_n = \infty$$

Thus $\varphi \circ f$ is not integrable over $[0, 1]$ even though f is.

We conclude that there must exist $x_0 \geq 0$ and $c_1 \geq 0$ such that

$$|\varphi(x)| \leq c_1 \cdot |x| \text{ if } |x| \geq x_0$$

Let $c_0 = \max_{|x| \leq x_0} |\varphi(x)|$. Since $c_0 \geq 0$ and $c_1 \cdot |x| \geq 0$, we must have

$$|\varphi(x)| \leq c_0 + c_1 \cdot |x|$$

for all $x \in \mathbf{R}$, as desired.

7 The L^p spaces: Completeness and Approximation

7.1 Normed Linear Spaces

1. Let f and g belong to $C[a, b]$. Then by monotonicity and linearity of integration, we have

$$||f + g||_1 = \int_a^b |f + g| \leq \int_a^b (|f| + |g|) = \int_a^b |f| + \int_a^b |g| = ||f||_1 + ||g||_1$$

Therefore $|| \cdot ||_1$ satisfies the triangle inequality. For any real number α, we have

$$||\alpha f||_1 = \int_a^b |\alpha f| = \int_a^b |\alpha| \cdot |f| = |\alpha| \int_a^b |f| = |\alpha| \cdot ||f||_1$$

by linearity of integration. Therefore $|| \cdot ||_1$ satisfies positive homogeneity. Since $|f| \geq 0$, $||f||_1 \geq 0$ by monotonicity of integration. Moreover, $||f||_1 = 0$ if and only if $f = 0$ by Proposition 9 of Chapter 4. Thus $|| \cdot ||_1$ satisfies nonnegativity, so $|| \cdot ||_1$ is a norm on $C[a, b]$.

Now fix $c \geq 0$. Define $x_n = \frac{1}{n}b + \frac{n-1}{n}a$ and consider the following sequence of functions $\{f_n\}$:

$$f_n(x) = \begin{cases} \dfrac{2n}{b-a}\left(1 - \dfrac{x-a}{x_n - a}\right) & \text{for } x \in [a, x_n] \\ 0 & \text{for } x \in (x_n, b] \end{cases}$$

Then $f_n \in C[a, b]$ and

$$||f_n||_{\max} = \frac{2n}{b-a}, \qquad ||f_n||_1 = 1$$

for all n. Therefore $||f_n||_{\max} > c||f_n||_1$ for all $n > \frac{c(b-a)}{2}$. Thus there is no $c \geq 0$ for which $||f||_{\max} \leq c||f||_1$ for all $f \in C[a, b]$.

Now let $c = b - a$. Since $|f| \leq ||f||_{\max}$, we must have

$$||f||_1 = \int_a^b |f| \leq \int_a^b ||f||_{\max} = (b-a) \cdot ||f||_{\max} = c||f||_{\max}$$

for all $f \in C[a, b]$.

2. Let $f(x) = \sum_{i=0}^{n} a_i x^k$ and $g(x) = \sum_{i=0}^{m} b_i x^i$, where $\{a_i\}_{i=0}^{n}$ and $\{b_i\}_{i=0}^{m}$ are sets of real numbers and n and m are non-negative integers. Assume without loss of generality that $n \geq m$ and define $b_i = 0$ for $i = m+1, m+2, \cdots, n$. Then for any scalars α and β, we have

$$\alpha f(x) + \beta g(x) = \sum_{i=0}^{n} (\alpha a_i + \beta b_i) x^i$$

Thus $\alpha f + \beta g$ is in X, so X is a linear space. The above expression also implies

$$||f+g|| = \sum_{i=0}^{n} |a_i + b_i| \leq \sum_{i=0}^{n} |a_i| + \sum_{i=1}^{m} |b_i| = ||f|| + ||g||$$

so that the triangle inequality holds. We also have

$$||\alpha f|| = \sum_{i=0}^{n} |\alpha a_i| = |\alpha| \sum_{i=0}^{n} |a_i| = |\alpha| \cdot ||f||$$

so that $||\cdot||$ satisfies positive homogeneity. Non-negativity of $||\cdot||$ follows directly from non-negativity of the absolute value function. Therefore $||\cdot||$ is a norm on X.

3. Let f and g denote functions in $L^1[a,b]$. We then have

$$||f+g|| = \int_a^b x^2 |f(x)+g(x)| dx \leq \int_a^b x^2 (|f(x)|+|g(x)|) dx = \int_a^b x^2 |f(x)| dx + \int_a^b x^2 |g(x)| dx = ||f|| + ||g||$$

so the triangle inequality holds. For any real number α, we have

$$||\alpha f|| = \int_a^b x^2 |\alpha f(x)| dx = |\alpha| \int_a^b x^2 |f(x)| dx = |\alpha| \cdot ||f||$$

Therefore positive homogeneity also holds. Since $x^2|f(x)| \geq 0$ on $[a,b]$, $||f|| \geq 0$. Moreover, $||f|| = 0$ if and only if $x^2|f| = 0$ a.e. $[a,b]$ by Proposition 9 of Chapter 4. But since $x^2 \neq 0$ a.e. $[a,b]$, $x^2|f| = 0$ if and only if $f = 0$ a.e. $[a,b]$. Thus $||\cdot||$ satisfies non-negativity.

4. We first show that the set

$$U = \Big\{ M : m\{x \in [a,b] \;\big|\; |f(x)| > M\} = 0 \Big\}$$

is equal to the set of essential upper bounds of f. Suppose M is an essential upper bound of f. Then there exists a set E_0 of measure zero such that $|f| \leq M$ on $[a,b] \sim E_0$. But then

$$\{x \in [a,b] \;\big|\; |f(x)| > M\} \subseteq E_0$$

which implies

$$m\left(\Big\{x \in [a,b] \;\big|\; |f(x)| > M\Big\}\right) = 0$$

by monotonicity and non-negativity of measure. That each element of M is an essential upper bound of f follows immediately from the definition of an essential upper bound.

We therefore conclude that $||f||_\infty = \inf U$. To show that $||f||_\infty \in U$, let M_n denote a sequence in U

converging downward to $||f||_\infty$. Then

$$\bigcap_{n=1}^{\infty} \{x \in [a,b] \mid |f(x)| \le M_n\} = \{x \in [a,b] \mid |f(x)| \le ||f||_\infty\}$$

Since

$$\begin{aligned} b - a &= m\left(\{x \in [a,b] \mid |f(x)| \le M_n\}\right) + m\left(\{x \in [a,b] \mid |f(x)| > M_n\}\right) \\ &= m\left(\{x \in [a,b] \mid |f(x)| \le M_n\}\right) \end{aligned}$$

for all n, continuity of measure implies

$$b - a = m\left(\{x \in [a,b] \mid |f(x)| \le ||f||_\infty\}\right)$$

The excision property of measure then implies

$$\begin{aligned} m\left(\{x \in [a,b] \mid |f(x)| > ||f||_\infty\}\right) &= m([a,b]) - m\left(\{x \in [a,b] \mid |f(x)| \le ||f||_\infty\}\right) \\ &= (b-a) - (b-a) \\ &= 0 \end{aligned}$$

Therefore $||f||_\infty \in U$, so $||f||_\infty = \min U$.

Now suppose f is continuous. Since $f \le ||f||_{\max}$ on $[a,b]$, $||f||_{\max}$ is an essential upper bound of f. Therefore $||f||_{\max} \ge ||f||_\infty$. To prove the converse, let $M \ge 0$ denote an essential upper bound of f. Then there exists a set E of measure zero such that $f \le M$ on $[a,b] \sim E$. Now suppose there exists $x \in [a,b]$ such that $f(x) > M$. Because f is continuous at x, there must exist a non-degenerate interval $I \subseteq [a,b]$ that contains x and that satisfies $f(x') > M$ for all $x' \in I$. Then $I \subseteq E$, so $0 < m(I) \le m(E)$. However $m(E) = 0$ by assumption. Therefore $f(x) \le M$ for all x, so $||f||_{\max} \le M$. We conclude that $||f||_{\max}$ is a lower bound of the set of essential upper bounds of f, so $||f||_{\max} \le ||f||_\infty$.

5. Let $\{a_k\}$ and $\{b_k\}$ denote two sequences in ℓ^∞. Then $|a_k + b_k| \le |a_k| + |b_k|$ for all k, so

$$\begin{aligned} ||\{a_k\} + \{b_k\}||_\infty &= \sup_{1 \le k < \infty} |a_k + b_k| \\ &\le \sup_{1 \le k < \infty} (|a_k| + |b_k|) \\ &\le \sup_{1 \le k < \infty} |a_k| + \sup_{1 \le k < \infty} |b_k| \\ &= ||\{a_k\}||_\infty + ||\{b_k\}||_\infty \end{aligned}$$

For any scalar α, we have

$$||\alpha \cdot \{a_k\}||_\infty = \sup_{1 \le k < \infty} |\alpha \cdot a_k| = \sup_{1 \le k < \infty} |\alpha| \cdot |a_k| = |\alpha| \sup_{1 \le k < \infty} |a_k| = |\alpha| \cdot ||\{a_k\}||_\infty$$

Since $|a_k| \ge 0$ for all k, we also must have

$$||\{a_k\}||_\infty = \sup_{1 \le k < \infty} |a_k| \ge 0$$

Since $||\{a_k\}||_\infty \geq |a_k|$ for any k, $|a_k| = 0$ for all k if and only if $||\{a_k\}||_\infty = 0$.

Now let $\{a_k\}$ and $\{b_k\}$ denote two sequences in ℓ^1. We then have

$$\begin{aligned}
||\{a_k\} + \{b_k\}||_1 &= \sum_{k=1}^{\infty} |a_k + b_k| \\
&\leq \sum_{k=1}^{\infty} (|a_k| + |b_k|) \\
&= \sum_{k=1}^{\infty} |a_k| + \sum_{k=1}^{\infty} |b_k| \\
&= ||\{a_k\}||_1 + ||\{b_k\}||_1
\end{aligned}$$

For any scalar α, we have

$$||\alpha \cdot \{a_k\}||_1 = \sum_{k=1}^{\infty} |\alpha \cdot a_k| = \sum_{k=1}^{\infty} |\alpha| \cdot |a_k| = |\alpha| \sum_{k=1}^{\infty} |a_k| = |\alpha| \cdot ||\{a_k\}||_1$$

Since $|a_k| \geq 0$ for all k, we also must have

$$||\{a_k\}||_1 = \sum_{k=1}^{\infty} |a_k| \geq 0$$

Since $||\{a_k\}||_1 \geq |a_k|$ for any k, $|a_k| = 0$ for all k if and only if $||\{a_k\}||_1 = 0$.

7.2 The Inequalities of Young, Hölder, and Minkowski

6. Let E be a measurable set, $1 \leq p < \infty$, and q the conjugate of p. Choose $f \in L^p(E)$ and $g \in L^q(E)$ and assume $f \neq 0$ and $g \neq 0$. By linearity of integration, we have

$$\begin{aligned}
\left(\int_E \left|\frac{f}{||f||_p}\right|^p\right)^{1/p} &= \frac{1}{||f||_p}\left(\int_E |f|^p\right)^{1/p} = 1 \\
\left(\int_E \left|\frac{g}{||g||_q}\right|^q\right)^{1/q} &= \frac{1}{||g||_q}\left(\int_E |g|^q\right)^{1/q} = 1
\end{aligned}$$

If Hölder's Inequality is true for normalized functions, we have

$$\int_E \left|\frac{f}{||f||_p} \cdot \frac{g}{||g||_q}\right| \leq 1$$

which implies $\int_E |f \cdot g| \leq ||f||_p \cdot ||g||_q$ by linearity of integration.

7. **Example 1**: By Problem 19 of Chapter 4, $\int_0^1 x^{\alpha \cdot p} < \infty$ if and only if $\alpha \cdot p > -1$. Fix $-\frac{1}{p_1} < \alpha \leq -\frac{1}{p_2}$. Then $\int_0^1 |x^\alpha|^{p_1} < \infty$ but $\int_0^1 |x^\alpha|^{p_2} = \infty$, so $x^\alpha \in L^{p_1}(E) \sim L^{p_2}(E)$.

   Example 2: (The example given in the text is incorrect. Instead define $f(x) = \frac{x^{-1/2}}{1+|\ln x|}$ and $E =$

$(0, \infty))$. Suppose $p = 2$. For any $M \geq 1$, we have

$$\int_1^M |f|^p = \int_1^M \frac{x^{-1}}{(1+\ln x)^2} dx = \int_1^M \frac{d}{dx}\left(-\frac{1}{1+\ln x}\right) dx = 1 - \frac{1}{1+\ln M}$$

$$\int_{1/M}^1 |f|^p = \int_{1/M}^1 \frac{x^{-1}}{(1-\ln x)^2} dx = \int_{1/M}^1 \frac{d}{dx}\left(\frac{1}{1-\ln x}\right) dx = 1 - \frac{1}{1+\ln M}$$

Applying the Monotone Convergence Theorem, we obtain

$$\int_0^\infty |f|^p = \lim_{M\to\infty} \int_{1/M}^M |f|^p = \lim_{M\to\infty} \left(\int_{1/M}^1 |f|^p + \int_1^M |f|^p\right) = 2 < \infty$$

Now suppose $p < 2$. Applying L'Hôpital's Rule, it is possible to show that

$$\lim_{x\to\infty} \frac{x^{1-p/2}}{(1+\ln x)^p} = \infty$$

Therefore there exists an $M \geq 1$ such that

$$\frac{x^{1-p/2}}{(1+\ln x)^p} \geq 1$$

for all $x \geq M$. But then for all $x \geq M$, we have

$$|f(x)|^p = \frac{x^{-p/2}}{(1+\ln x)^p} \geq x^{-1}$$

Since $\int_M^\infty x^{-1} dx = \infty$, we must have $\int_M^\infty |f|^p = \infty$ by monotonicity of integration. But this implies $\int_0^\infty |f|^p = \infty$.

Now suppose $p > 2$. Again applying L'Hôpital's Rule, it is possible to show that

$$\lim_{x\to 0} \frac{x^{1-p/2}}{(1-\ln x)^p} = \infty$$

Therefore there exists an $M \geq 1$ such that

$$\frac{x^{1-p/2}}{(1-\ln x)^p} \geq 1$$

for all $x \in (0, 1/M)$. But then for all $x \in (0, 1/M)$, we have

$$|f(x)|^p = \frac{x^{-p/2}}{(1-\ln x)^p} \geq x^{-1}$$

Since $\int_0^{1/M} x^{-1} dx = \infty$, we must have $\int_0^{1/M} |f|^p dx = \infty$ by monotonicity of integration. But this again implies $\int_0^\infty |f|^p = \infty$.

8. For any $\lambda \in \mathbf{R}$, we have

$$0 \le \int_E (\lambda|f| + |g|)^2 = \int_E (\lambda^2 f^2 + 2\lambda \cdot |f| \cdot |g| + g^2) = \lambda^2 \int_E f^2 + 2\lambda \int_E |f \cdot g| + \int_E g^2 \tag{1}$$

by monotonicity and linearity of integration. Since $L^2(E)$ is a linear space, we also know that $\int_E (f+g)^2 < \infty$. Applying the above expression for $\lambda = -1$, we obtain

$$\int_E |f \cdot g| \le \frac{1}{2}\left(\int_E f^2 + \int_E g^2\right) < \infty$$

Thus $\lambda^2 \int_E f^2 + 2\lambda \int_E |f \cdot g| + \int_E g^2$ is a well-defined quadratic equation whose minimum occurs at $\lambda^* = -\int_E |f \cdot g| / \int_E f^2$. Plugging λ^* into (1), we can infer

$$0 \le -\frac{\left(\int_E |f \cdot g|\right)^2}{\int_E f^2} + \int_E g^2$$

Rearranging this expression, we obtain

$$\left(\int_E |f \cdot g|\right)^2 \le \int_E f^2 \int_E g^2$$

Taking square roots of both sides yields the Cauchy-Schwartz Inequality.

9. (Correction: Show that in Young's Inequality there is equality if and only if $a^p = b^q$.)

   Suppose $a^p = b^q$. Then
$$\frac{a^p}{p} + \frac{b^q}{q} = a^p = a \cdot a^{p/q} = ab$$

   Now suppose $a^p > b^q$. Then
$$\frac{a^p}{p} + \frac{b^q}{q} > a^p = a \cdot a^{p/q} > ab$$

   If $a^p < b^q$, then
$$\frac{a^p}{p} + \frac{b^q}{q} > b^q = b \cdot b^{q/p} > ab$$

   Therefore $a^p \ne b^q$ implies $\frac{a^p}{p} + \frac{b^q}{q} \ne ab$.

10. Suppose there exists α and β, both not equal to zero, such that $\alpha|f|^p = \beta|g|^q$. Without loss of

generality, assume $\beta \neq 0$. Then

$$\begin{aligned}
\int_E |f \cdot g| &= \int_E |f| \cdot \left(\frac{\alpha |f|^p}{\beta}\right)^{1/q} \\
&= \int_E |f|^p \left(\frac{\alpha}{\beta}\right)^{1/q} \\
&= \left(\int_E |f|^p\right)^{1/p} \left(\frac{\alpha}{\beta}\int_E |f|^p\right)^{1/q} \\
&= \left(\int_E |f|^p\right)^{1/p} \left(\int_E |g|^q\right)^{1/q} \\
&= ||f||_p \cdot ||g||_q
\end{aligned}$$

If f or g is 0 the converse holds by setting β or α equal to 0, respectively. So suppose f and g are both nonzero and assume

$$\int_E |f \cdot g| = ||f||_p \cdot ||g||_q$$

We can rearrange this expression to obtain

$$\int_E \left(\left(\frac{|f|}{||f||_p}\right)^p \cdot \frac{1}{p} + \left(\frac{|g|}{||g||_q}\right)^q \cdot \frac{1}{q} - \frac{|f|}{||f||_p} \cdot \frac{|g|}{||g||_q}\right) = 0$$

By Young's Inequality, the integrand is non-negative. Therefore by Proposition 9 of Chapter 4,

$$\left(\frac{|f|}{||f||_p}\right)^p \cdot \frac{1}{p} + \left(\frac{|g|}{||g||_q}\right)^q \cdot \frac{1}{q} = \frac{|f|}{||f||_p} \cdot \frac{|g|}{||g||_q}$$

a.e. on E. Problem 9 then implies

$$\left(\frac{|f|}{||f||_p}\right)^p = \left(\frac{|g|}{||g||_q}\right)^q$$

a.e. on E. We conclude that $\alpha|f|^p = \beta|g|^q$ a.e. on E, where $\alpha = 1/||f||_p^p$ and $\beta = 1/||g||_q^q$.

11. For any $x \in \mathbf{R}^n$, we have

$$||x||_p = \begin{cases} \left(\sum_{k=1}^n |x_k|^p\right)^{1/p} & \text{if } 1 \leq p < \infty \\ \max_k |x_k| & \text{if } p = \infty \end{cases}$$

It is obvious that $||x||_p \geq 0$ and that $||x||_p = 0$ if $x = 0$. $||x||_p = 0$ implies $x = 0$ because $||x||_p \geq |x_k|$ for $k = 1, \cdots, n$. Therefore $||\cdot||_p$ satisfies non-negativity. Since T is linear, for any $\alpha \in \mathbf{R}$ we have

$$||\alpha \cdot x||_p = ||T_{\alpha \cdot x}||_p = ||\alpha \cdot T_x||_p = |\alpha| \cdot ||T_x||_p = |\alpha| \cdot ||x||_p$$

Therefore $||\cdot||_p$ satisfies positive homogeneity. Now fix $x, y \in \mathbf{R}^n$. Then

$$||x + y||_p = ||T_{x+y}||_p = ||T_x + T_y||_p \leq ||T_x||_p + ||T_y||_p = ||x||_p + ||y||_p$$

Therefore $||\cdot||_p$ satisfies the Triangle (Minkowski) Inequality. We conclude $||\cdot||_p$ is a norm on $\mathbf{R}^n$.

Hölder's Inequality in $\mathbf{R}^n$ is given by

$$\sum_{k=1}^{n} |x_k \cdot y_k| \leq ||x||_p \cdot ||y||_q \quad \text{ for all } x, y \in \mathbf{R}^n$$

This inequality follows from applying Hölder's Inequality to T_x and T_y:

$$\sum_{k=1}^{n} |x_k \cdot y_k| = \int_1^{n+1} |T_x \cdot T_y| \leq ||T_x||_p \cdot ||T_y||_q = ||x||_p \cdot ||y||_q$$

12. If $1 \leq p < \infty$, we have

$$||T_a||_p = \left(\int_1^{\infty} a_k \chi_{[k,k+1)} \right)^{1/p} = \left(\sum_{k=1}^{\infty} |a_k|^p \right)^{1/p} = ||a||_p$$

If $p = \infty$, then

$$||T_a||_p = \sup_{1 \leq k < \infty} |a_k| = ||a||_p$$

Therefore if $a \in \ell^p$, then $T_a \in L^p[1, \infty)$. For any two sequences $a, b \in \ell^p$, we have

$$||a + b||_p = ||T_{a+b}||_p = ||T_a + T_b||_p \leq ||T_a||_p + ||T_b||_p = ||a||_p + ||b||_p$$

Thus the Minkowski Inequality holds. Hölder's Inequality in ℓ^p can be written as

$$\sum_{k=1}^{\infty} |a_k \cdot b_k| \leq ||a||_p \cdot ||b||_q \quad \text{ for all } a \in \ell^p, b \in \ell^q$$

This inequality follows from applying Hölder's Inequality to T_a and T_b:

$$\sum_{k=1}^{\infty} |a_k \cdot b_k| = \int_1^{\infty} |T_a \cdot T_b| \leq ||T_a||_p \cdot ||T_b||_q = ||a||_p \cdot ||b||_q$$

Moreover, if $a \neq 0$, the sequence a^* defined by

$$a_k^* = ||a||_p^{1-p} \cdot \operatorname{sgn}(a_k) \cdot |a_k|^{p-1}$$

satisfies

$$\sum_{k=1}^{\infty} a_k \cdot a_k^* = ||a||_p^{1-p} = \sum_{k=1}^{\infty} |a_k|^p = ||a||_p$$

and

$$||a^*||_q = ||a||_p^{1-p} \cdot \left(\sum_{k=1}^{\infty} |a_k|^{q(p-1)} \right)^{1/q} = ||a||_p^{1-p} \cdot ||a||_p^{p/q} = 1$$

13. Fix $f \in L^{p_1}(E)$ and assume there exists $M \geq 0$ such that $|f| \leq M$ on E. Define

$$E_0 = \{x \in E : |f(x)| \leq 1\}$$

Since $|f|^{p_1} > 1$ on $E \sim E_0$, we must have

$$m(E \sim E_0) \leq \int_{E \sim E_0} |f|^{p_1} \leq \int_E |f|^{p_1} < \infty$$

Since $|f|^{p_2} \leq |f|^{p_1}$ on E_0, we also have

$$\int_{E_0} |f|^{p_2} \leq \int_{E_0} |f|^{p_1} \leq \int_E |f|^{p_1} < \infty$$

Therefore

$$\int_E |f|^{p_2} = \int_{E \sim E_0} |f|^{p_2} + \int_{E_0} |f|^{p_2} \leq M \cdot m(E \sim E_0) + \int_{E_0} |f|^{p_2} < \infty$$

14. Since $|\ln(1/x)| = \frac{d}{dx}(x \ln(1/x) + x)$ on $(0,1]$, we have

$$\int_c^1 |\ln(1/x)| dx = 1 - (c \ln(1/c) + c)$$

for any $c \in (0,1]$. By the Monotone Convergence Theorem, $\int_0^1 |\ln(1/x)| dx = \lim_{c \to 0} \int_c^1 |\ln(1/x)| dx = 1$. Therefore $f \in L^1(0,1]$.

Now fix any $p \in (1, \infty)$. Using L'Hôpital's Rule, it is straight-forward to show

$$\lim_{x \to 0} \frac{|\ln(1/x)|^p}{1/x} = 0$$

Therefore there exists $c \in (0,1]$ such that

$$|\ln(1/x)|^p \leq \frac{1}{x}$$

for all $x \in (0, c]$. But this implies

$$\begin{aligned}
\int_0^1 |\ln(1/x)|^p dx &= \int_0^c |\ln(1/x)|^p dx + \int_c^1 |\ln(1/x)|^p dx \\
&\leq \int_0^c \frac{1}{x^{1/p}} dx + \int_c^1 |\ln(1/x)|^p dx \\
&\leq \frac{p}{p-1} + (1-c) \cdot |\ln(1/c)|^p \\
&< \infty
\end{aligned}$$

Therefore $f \in L^p(0,1]$.

For any $M \geq 0$, $\ln(1/x) > M$ for $x \in (0, e^{-M})$. Therefore f is not essentially bounded on $(0,1]$, so $f \notin L^\infty(0,1]$.

15. Let E denote a set of real numbers. Suppose $f \in L^p(E)$, $g \in^q (E)$, and $h \in L^r(E)$, where $r, p, q \geq 1$ and

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$$

Then

$$\int_E |f \cdot g \cdot h| \leq ||f||_p \cdot ||g||_q \cdot ||h||_r$$

Proof: Let $t = \frac{p \cdot q}{p+q}$ denote the conjugate of r. Then

$$\int_E |f|^{t \cdot (p+q)/q} = \int_E |f|^p < \infty$$
$$\int_E |g|^{t \cdot (p+q)/p} = \int_E |g|^q < \infty$$

Therefore $|f|^t \in L^{(p+q)/q}$ and $|g|^t \in L^{(p+q)/p}$. Since $(p+q)/q$ and $(p+q)/p$ are conjugates, we have

$$\int_E |f \cdot g|^t \leq \left(\int_E |f|^p\right)^{q/(p+q)} \left(\int_E |g|^q\right)^{p/(p+q)} = ||f||_p^t \cdot ||g||_q^t < \infty$$

by Hölder's Inequality. But this implies $|f \cdot g| \in L^t(E)$ and $||f \cdot g||_t \leq ||f||_p \cdot ||g||_q$. Since t and r are conjugates, we have

$$\int_E |f \cdot g \cdot h| = \int_E |f \cdot g| \cdot |h| \leq ||f \cdot g||_t \cdot ||h||_r \leq ||f||_p \cdot ||g||_q \cdot ||h||_r$$

16. Let $f_1(x) = 1$ and let

$$f_n(x) = \begin{cases} \dfrac{n}{2} & \text{if } 0 \leq x < \dfrac{1}{n} \\ \dfrac{n}{2(n-1)} & \text{if } \dfrac{1}{n} \leq x \leq 1 \end{cases}$$

for $n = 2, 3, \cdots$. Then

$$\int_0^1 |f_n| = 1$$

for all n, so $\{f_n\}$ is bounded in $L^1[0,1]$.

Now fix $\epsilon \in (0, 1/2)$. For any $\delta > 0$, we can find n such that $\frac{1}{n} < \delta$. But then $m([0, 1/n)) < \delta$ and

$$\int_0^{1/n} f_n = \frac{1}{2} > \epsilon$$

Therefore $\{f_n\}$ is not uniformly integrable.

17. Fix $p \geq 1$ and let

$$f_n = \chi_{[n,n+1)}$$

Then $\int_{\mathbf{R}} |f_n|^p = 1$ for all n, so $\{f_n\}$ is bounded in $L^p(\mathbf{R})$. Fix $\epsilon \in (0,1)$ and let E_0 denote a set of finite measure. Let $E_n = E_0 \sim (-n, n)$. Since $m(E_0) < \infty$ and $\bigcap_{n=1}^{\infty} E_n = \varnothing$, $\lim_{n \to \infty} m(E_n) = 0$ by continuity of measure. Pick n such that $m(E_n) < 1 - \epsilon$. Then

$$\int_{E_0} |f_n| = \int_{E_0 \cap (-n,n)} |f_n| + \int_{E_n} |f_n| \leq m(E_n) < 1 - \epsilon$$

Therefore

$$\int_{\mathbf{R}\sim E_0} |f_n| = \int_{\mathbf{R}} |f_n| - \int_{E_0} |f_n| > 1 - (1-\epsilon) = \epsilon$$

We conclude that $\{f_n\}$ is not tight over $\mathbf{R}$.

18. If $m(E) = 0$, $||f||_p = 0$ for all $p \geq 1$ and the result holds trivially. So assume $m(E) > 0$. By Corollary 3, we know that

$$||f||_p \leq [m(E)]^{1/p} \cdot ||f||_\infty$$

for all $p \in (1, \infty)$. Since $\lim_{p\to\infty}[m(E)]^{1/p} = 1$, we have

$$\limsup_{p\to\infty} ||f||_p \leq ||f||_\infty \tag{2}$$

Now define

$$E_n = \left\{ x \in E : |f(x)| > ||f||_\infty - \frac{1}{n} \right\}$$

If $m(E_n) = 0$, then $||f||_\infty - \frac{1}{n}$ would be an essential upper bound for f that is less than $||f||_\infty$. But that would contradict the definition of $||f||_\infty$. Therefore $m(E_n) > 0$ for all n. From Hölder's Inequality, we know that

$$\int_{E_n} |f| = \int_E |f \cdot \chi_{E_n}| \leq ||f||_p \cdot (m(E_n))^{(p-1)/p}$$

for all $p \in (1, \infty)$. Since $||f||_\infty - \frac{1}{n} < |f|$ on E_n, we must have

$$\left(||f||_\infty - \frac{1}{n} \right) \cdot m(E_n) \leq \int_{E_n} |f|$$

Combining expressions and simplifying, we can conclude

$$||f||_\infty - \frac{1}{n} \leq ||f||_p \cdot m(E_n)^{-1/p}$$

Since $m(E_n) > 0$, $\lim_{p\to\infty} m(E_n)^{-1/p} = 1$. Therefore

$$||f||_\infty - \frac{1}{n} \leq \liminf_{p\to\infty} ||f||_p$$

for all n. Letting $n \to \infty$, we obtain

$$||f||_\infty \leq \liminf_{p\to\infty} ||f||_p \tag{3}$$

Combining (2) and (3), we conclude

$$\lim_{p\to\infty} ||f||_p = ||f||_\infty$$

19. If $f = 0$, then $\int_E f \cdot g = 0$ for all g and $||f||_p = 0$ for $1 \leq p < \infty$. Thus the result holds.

Now assume $f \neq 0$. For any $g \in L^q(E)$ with $||g||_q \leq 1$, we have

$$\int_E f \cdot g \leq \int_E |f \cdot g| \leq ||f||_p \cdot ||g||_q = ||f||_p$$

by Hölder's Inequality. Therefore

$$\sup_{g\in L^q(E),\,||g||_q\le 1} \int_E f\cdot g \le ||f||_p$$

We also know that

$$\int_E f\cdot f^* = ||f||_p$$

where f^* denotes the conjugate function of f. Since $||f^*||_q = 1$, we must have

$$\sup_{g\in L^q(E),\,||g||_q\le 1} \int_E f\cdot g \ge ||f||_p$$

We can conclude that

$$\sup_{g\in L^q(E),\,||g||_q\le 1} \int_E f\cdot g = ||f||_p$$

Since the supremum is obtained at $g = f^*$, we may write

$$\max_{g\in L^q(E),\,||g||_q\le 1} \int_E f\cdot g = ||f||_p$$

20. Suppose $f = 0$. If $g \in L^q(E)$, then g is finite a.e. E. Therefore $f\cdot g = 0$ a.e. E, so $\int_E f\cdot g = 0$.

    Conversely, suppose $\int_E f\cdot g = 0$ for all $g \in L^q(E)$. By Problem 20, $||f||_p = 0$. By non-negativity of $||\cdot||_p$, $f = 0$.

21. Claim: The expression

$$\lim_{\epsilon\to 0^+} \frac{\int_0^1 f}{\epsilon^\lambda} = 0 \quad \text{ for all } f \in L^p[0,1] \tag{4}$$

    holds if and only if $\lambda \le \frac{1}{q}$, where q denotes the conjugate of p. (Define $\frac{1}{q} = 0$ if $p = 1$.)

    Proof: Let q denote the conjugate of p and suppose $\lambda < \frac{1}{q}$. Then

$$\left|\frac{\int_0^\epsilon f}{\epsilon^\lambda}\right| \le ||f||_p \cdot \epsilon^{1/q-\lambda}$$

    by Hölder's Inequality. Since $\lim_{\epsilon\to 0^+} \epsilon^{1/q-\lambda} = 0$, (4) holds. Now suppose $\lambda = \frac{1}{q}$. Then

$$\left|\frac{\int_0^\epsilon f}{\epsilon^\lambda}\right| \le \left(\int_0^\epsilon |f|^p\right)^{1/p}$$

    by Hölder's Inequality. By continuity of integration, $\lim_{\epsilon\to 0^+} \int_0^\epsilon |f|^p = 0$. Therefore (4) again holds.

    Now suppose $\lambda > 1/q$. Let $f(x) = x^{-1+\lambda}$. Then

$$|f(x)|^p = x^{-1+p\cdot(\lambda-1/q)}$$

Since $p \cdot (\lambda - 1/q) > 0$, $f \in L^p[0,1]$. However

$$\lim_{\epsilon \to 0^+} \frac{\int_0^\epsilon f}{\epsilon^\lambda} = \frac{1}{\lambda} \neq 0$$

Therefore (4) does not hold.

22. Let $\{x_0, \cdots, x_n\}$ denote an arbitrary partition of $[a,b]$, Since F is indefinite integral of an $L^p[a,b]$ function, there exists $f \in L^p[a,b]$ such that

$$F(x_k) - F(x_{k-1}) = \int_{x_{k-1}}^{x_k} f \quad k = 1, \cdots, n$$

By Hölder's Inequality,

$$\int_{x_{k-1}}^{x_k} |f| \leq ||f||_p \cdot |x_k - x_{k-1}|^{1/q} \quad k = 1, \cdots, n$$

Combining expressions and raising both sides to the p, we obtain

$$|F(x_k) - F(x_{k-1})|^p \leq |x_k - x_{k-1}|^{p-1} \int_{x_{k-1}}^{x_k} |f|^p \quad k = 1, \cdots, n$$

Rearranging and summing over k, we can infer

$$\sum_{k=1}^{n} \frac{|F(x_k) - F(x_{k-1})|^p}{|x_k - x_{k-1}|^{p-1}} \leq \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} |f|^p = \int_a^b |f|^p < \infty$$

Therefore the result holds by setting $M = \int_a^b |f|^p$.

7.3 L^p is Complete: The Riesz-Fischer Theorem

23. Define

$$a_n = \begin{cases} 0 & \text{if } n = 1 \\ \displaystyle\sum_{i=1}^{n-1} \frac{1}{i^2} & \text{if } n = 2, 3, \cdots \end{cases}$$

Then

$$\lim_{n \to \infty} a_n = \sum_{i=1}^{\infty} \frac{1}{i^2} < \infty$$

By Theorem 17 of Chapter 1, $\{a_n\}$ is Cauchy.

We also have

$$|a_{n+1} - a_n| = \frac{1}{n^2} \text{ for all } n$$

Thus if $\{\epsilon_n\}$ is a sequence of positive numbers that satisfies

$$|a_{n+1} - a_n| \leq \epsilon_n^2 \text{ for all } n$$

then

$$\epsilon_n \geq \frac{1}{n} \text{ for all } n$$

But this implies

$$\sum_{n=1}^{\infty} \epsilon_n \geq \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

Therefore $\{a_n\}$ is not rapidly Cauchy.

24. That each $\alpha f_n + \beta g_n$ and $\alpha f + \beta g$ belong to X follows because X is a linear space. Using properties of a norm, we can also infer

$$\begin{aligned} 0 \leq ||\alpha f_n + \beta g_n - (\alpha f + \beta g)|| &\leq ||\alpha(f_n - f)|| + ||\beta(g_n - g)|| \\ &= |\alpha| \cdot ||f_n - f|| + |\beta| \cdot ||g_n - g|| \end{aligned}$$

But since $\lim_{n\to\infty} ||f_n - f|| = \lim_{n\to\infty} ||g_n - g|| = 0$ by assumption, the above expression implies

$$\lim_{n\to\infty} ||\alpha f_n + \beta g_n - (\alpha f + \beta g)|| = 0$$

Therefore $\{\alpha f_n + \beta g_n\} \to \alpha f + \beta g$ in X.

25. By Corollary 3, each f_n and f are in $L^{p_1}(E)$ and

$$0 \leq ||f_n - f||_{p_1} \leq c \cdot ||f_n - f||_{p_2}$$

for some $c \in [0, \infty)$. Since $\lim_{n\to\infty} ||f_n - f||_{p_2} = 0$ by assumption, the above expression implies $||f_n - f||_{p_1} = 0$. Therefore $\{f_n\} \to f$ in $L^{p_1}(E)$.

26. Since $|f_n| \leq g$ for all n a.e. on E, we must have $|f| \leq g$ a.e. on E. We can infer that $|f_n|^p \leq g^p$ and $|f|^p \leq g^p$ a.e. on E, which implies

$$\int_E |f_n|^p \leq \int_E g^p < \infty \quad \text{and} \quad \int_E |f|^p \leq \int_E g^p < \infty$$

Thus f_n and f are both in $L^p(E)$. By the Lebesgue Dominated Convergence Theorem, we have

$$\lim_{n\to\infty} \int_E |f_n|^p = \int_E |f|^p$$

which implies that $\{f_n\} \to f$ in $L^p(E)$ by Theorem 7.

27. By the Riesz-Fischer Theorem, $\{f_n\}$ contains a subsequence that converges pointwise to f a.e. on E. By Propositions 4 and 5, this subsequence itself contains a rapidly Cauchy subsequence. To save on notation, assume that $\{f_n\}$ itself satisfies these properties. Define

$$g = \sup_{1 \leq n < \infty} |f_n|$$

If we show $g \in L^p(E)$, the proof is complete since $g \geq |f_n|$ for all n. Since

$$g \leq \sup_{1 \leq n < \infty} |f_n - f| + |f|$$

we have

$$|g|^p \le 2^p \left\{ \left| \sup_{1\le n<\infty} |f_n - f| \right|^p + |f|^p \right\}$$

Since $f \in L^p(E)$, it suffices to show $\sup_{1\le n<\infty} |f_n - f| \in L^p(E)$.

To do so, we first prove the statement

$$\max_{1\le k\le n} |f_k - f| \le \sum_{k=1}^{n} |f_{k+1} - f_k| + |f_{n+1} - f| \tag{1}$$

for all n. (1) follows immediately from the Triangle Inequality if $n = 1$. Now suppose (1) is true for some n. Then

$$\begin{aligned}
\max_{1\le k\le n+1} |f_k - f| &= \max\left\{ \max_{1\le k\le n} |f_k - f|, |f_{n+1} - f| \right\} \\
&\le \max\left\{ \sum_{k=1}^{n} |f_{k+1} - f_k| + |f_{n+1} - f|, |f_{n+1} - f| \right\} \\
&= \sum_{k=1}^{n} |f_{k+1} - f_k| + |f_{n+1} - f| \\
&= \sum_{k=1}^{n+1} |f_{k+1} - f_k| + |f_{n+1} - f| - |f_{n+2} - f_{n+1}| \\
&\le \sum_{k=1}^{n+1} |f_{k+1} - f_k| + |f_{n+2} - f|
\end{aligned}$$

Therefore (1) holds for all n by induction.

Now (1) implies

$$\sup_{1\le n<\infty} |f_n - f| \le \sum_{n=1}^{\infty} |f_{n+1} - f_n| \tag{2}$$

So it suffices to show that $\sum_{n=1}^{\infty} |f_{n+1} - f_n| \in L_p(E)$. Because $\{f_n\}$ is rapidly Cauchy, there exists a convergent series $\sum_{n=1}^{\infty} \epsilon_n$ such that

$$||f_{n+1} - f_n||_p \le \epsilon_n^2$$

for all n. By Minkowski's Inequality, we know that

$$\left| \sum_{k=1}^{n} |f_{k+1} - f_k| \right|^p \le \sum_{k=1}^{n} |f_{k+1} - f_k|^p \le \left(\sum_{k=1}^{n} \epsilon_k^2 \right)^p \tag{3}$$

for all n. By the Monotone Convergence Theorem, we can take limits of (3) to conclude

$$\int_E \left(\sum_{n=1}^{\infty} |f_{n+1} - f_n| \right)^p \le \left(\sum_{n=1}^{\infty} ||f_{n+1} - f_n||_p \right)^p \le \left(\sum_{n=1}^{\infty} \epsilon_n^2 \right)^p < \infty$$

as desired.

28. By assumption, there exists a real number M such that $||f_n||_{p+\theta} \le M$ for all n. By Fatou's Lemma,

we have

$$\int_E |f|^{p+\theta} \leq \liminf_{n\to\infty} \int_E |f_n|^{p+\theta} \leq M^{p+\theta} < \infty$$

Therefore $f \in L^{p+\theta}(E)$. By Corollary 3, f and $\{f_n\}$ belong to $L^p(E)$. It remains to show that $||f_n - f||_p \to 0$. By Hölder's Inequality, we have

$$\int_A |f_n - f|^p \leq ||f_n - f||_{p+\theta}^p \cdot m(A)^{\frac{\theta}{p+\theta}}$$

for any measurable set $A \subseteq E$. By Minkowski's Inequality,

$$||f_n - f||_{p+\theta} \leq ||f_n||_{p+\theta} + ||f||_{p+\theta} \leq 2 \cdot M$$

Therefore

$$\int_A |f_n - f|^p \leq 2^p \cdot M^p \cdot m(A)^{\frac{\theta}{p+\theta}}$$

Thus for any $\epsilon > 0$, we have

$$\int_A |f_n - f|^p < \epsilon$$

as long as $m(A) < \left(\dfrac{\epsilon}{2^p \cdot M^p}\right)^{\frac{p+\theta}{\theta}}$. Therefore $\{|f_n - f|^p\}$ is uniformly integrable over E, which implies

$$\lim_{n\to\infty} \int_E |f_n - f|^p = 0$$

by the Vitali Convergence Theorem.

29. Consider the sequence of polynomials

$$p_n(x) = \sum_{k=0}^{n} \frac{x^n}{n!}$$

on $[0,1]$. By Taylor's Theorem, we know that

$$p_n(x) = e^x - e^{u(x)} \frac{x^{n+1}}{(n+1)!}$$

where $u(x) \in [0,1]$. Therefore

$$||p_n - p_m||_{\max} \leq \frac{e}{(n+1)!} + \frac{e}{(m+1)!}$$

for all n and m. Thus $\{p_n\}$ is Cauchy in $||\cdot||_{\max}$. However, $\{p_n\}$ converges to e^x in $||\cdot||_{\max}$, which is not a polynomial. Therefore the space of polynomials on $[0, \log 2]$ is not complete.

30. By construction,

$$|f_{n+k}(x) - f_n(x)| \leq ||f_{n+k} - f_n||_{\max} \tag{4}$$

for all $x \in [a,b]$. Since $||\cdot||_{\max}$ is a norm, we also have

$$||f_{n+k} - f_n||_{\max} \leq \sum_{j=n}^{n+k-1} ||f_{j+1} - f_j||_{\max} \leq \sum_{j=n}^{n+k-1} a_j \leq \sum_{j=n}^{\infty} a_j \tag{5}$$

by the Triangle Inequality.

Now fix $\epsilon > 0$. Since $\sum_{j=1}^{\infty} a_j$ is a convergent series of positive numbers, there exists an index N such that $\sum_{j=n}^{\infty} a_j < \epsilon$ for all $n \geq N$. Now fix $n, m \geq N$ and assume without loss that $m > n$. Substituting $k = m - n$ into (4) and (5), we have

$$|f_m(x) - f_n(x)| \leq ||f_m - f_n||_{\max} \leq \sum_{j=n}^{\infty} a_j < \epsilon \tag{6}$$

Thus $\{f_n(x)\}$ is a Cauchy sequence of real numbers. Since the reals are complete, $\{f_n(x)\}$ converges to a real number. We can therefore define a real-valued function $f(\cdot)$ on $[a, b]$ by

$$\lim_{n\to\infty} f_n(x) = f(x) \tag{7}$$

To see that $\{f_n\} \to f$ uniformly on $[a, b]$, observe that

$$|f_n(x) - f(x)| \leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)| \leq ||f_m - f_n||_{\max} + |f_m(x) - f(x)|$$

for all $x \in [a, b]$ and all indicies n and m. By (6), there exists an index N such that

$$|f_n(x) - f(x)| < \epsilon + |f_m(x) - f(x)|$$

for $m, n \geq N$. Letting $m \to \infty$, we have

$$|f_n(x) - f(x)| \leq \epsilon$$

for all $n \geq N$. Therefore $\{f_n\} \to f$ uniformly on $[a, b]$.

It remains to show that f is continuous. For any $x, x' \in [a, b]$, we have

$$|f(x) - f(x')| \leq |f_n(x) - f(x)| + |f_n(x) - f_n(x')| + |f_n(x') - f(x')|$$

Since $\{f_n\} \to f$ uniformly on $[a, b]$, there exists an index N such that $|f_n - f| < \epsilon/3$ on $[a, b]$. Therefore if we fix $n \geq N$, we have

$$|f(x) - f(x')| < \frac{2 \cdot \epsilon}{3} + |f_n(x) - f_n(x')|$$

Since f_n is continuous, there exists $\delta > 0$ such that $|f_n(x) - f_n(x')| < \epsilon/3$ if $|x - x'| < \delta$. But then

$$|x - x'| < \delta \text{ implies } |f(x) - f(x')| < \epsilon$$

Therefore $f \in C[a, b]$.

31. Suppose $\{f_n\}$ is Cauchy in $C[a, b]$. Choose a sequence $1 \leq n_1 < n_2 < \cdots$ such that $||f_m - f_n||_{\max} < 1/2^k$ for all $m, n \geq n_k$. Then

$$||f_{n_{k+1}} - f_{n_k}||_{\max} < \frac{1}{2^k}$$

for all k. Since $\sum_{k=1}^{\infty} 2^{-k}$ is a convergent sequence of positive numbers, by Problem 30 there exists a function $f \in C[a, b]$ such that $\{f_{n_k}\} \to f$ uniformly on $[a, b]$. But this implies $||f_{n_k} - f||_{\max} \to 0$. By

Proposition 4, $\{f_n\} \to f$ in $C[a,b]$. Thus $C[a,b]$ is complete.

32. For any n and k, there exists a set $E_{n,k} \subseteq E$ with $m(E_{n,k}) = 0$ such that $|f_{n+k} - f_n| \leq ||f_{n+k} - f_n||_\infty$ on $E \sim E_{n,k}$. Let $E_0 = \bigcup_{n=1}^\infty \bigcup_{k=1}^\infty E_{n,k}$. Then $m(E_0) = 0$ and

$$|f_{n+k}(x) - f_n(x)| \leq ||f_{n+k} - f_n||_\infty \leq \sum_{j=n}^{n+k-1} a_j \leq \sum_{j=n}^{\infty} a_j$$

for all k, n and all $x \in E \sim E_0$. Now following the argument in Problem 30, we can show there exists a function f on E such that $\{f_n\} \to f$ uniformly on $E \sim E_0$. It remains to show that $f \in L^\infty(E)$. Observe that

$$|f(x)| \leq |f_n(x) - f(x)| + |f_n(x)| \leq ||f_n - f||_\infty + |f_n(x)|$$

on $E \sim E_0$ for all n. Since $||f_n - f||_\infty \to 0$, there must exist an index N such that $||f_N - f||_\infty < \infty$. Since $f_N \in L^\infty(E)$, there must exist a set E_N with $m(E_N) = 0$ such that $|f_N| < \infty$ on $E \sim E_N$. Since $|f|$ is bounded on $E \sim (E_0 \cup E_N)$ and $m(E_0 \cup E_N) = 0$, we conclude that $f \in L^\infty(E)$.

33. Suppose $\{f_n\}$ is Cauchy in $L^\infty(E)$. Using the same argument as in Problem 31, we can show there exists $f \in L^\infty(E)$, a subsequence $\{f_{n_k}\}$, and $E_0 \subseteq E$ with $m(E_0) = 0$ such that $\{f_{n_k}\} \to f$ uniformly on $E \sim E_0$. But this implies $||f_{n_k} - f||_\infty \to 0$. By Proposition 4, $\{f_n\} \to f$ in $L^\infty(E)$. Thus $L^\infty(E)$ is complete.

34. Fix $1 \leq p \leq \infty$. Let $\{a_n\}_{n=1}^\infty$ denote a Cauchy sequence in ℓ^p, where $a_n = (a_{n,1}, a_{n,2}, \cdots)$. Following Problem 12, we can construct a linear operator $T(\cdot)$ mapping a sequence $a \in \ell^p$ to a function $T(a) \in L^p[1,\infty)$ that satisfies $||T(a)||_p = ||a||_p$. Let $f_n = T(a_n)$. By linearity of T, we have

$$||f_n - f_m||_p = ||T(a_n) - T(a_m)||_p = ||T(a_n - a_m)||_p = ||a_n - a_m||_p$$

Therefore the sequence $\{f_n\}_{n=1}^\infty$ is Cauchy in $L^p[1,\infty)$. By the Riesz-Fischer Theorem, there exists a function f such that $f_n \to f$ in $L^p[1,\infty)$. Moreover, there exists a subsequence $\{f_{n_i}\}_{i=1}^\infty$ that converges pointwise to f a.e. on $[1,\infty)$. For each k, we have

$$f = \lim_{i\to\infty} f_{n_i} = \lim_{i\to\infty} a_{n_i,k} \quad \text{a.e. on } [k, k+1) \tag{8}$$

Let $b_k = \lim_{i\to\infty} a_{n_i,k}$ and consider the sequence $b = (b_1, b_2, \cdots)$. Equation (8) implies $f = T(b)$ a.e. on $[1,\infty)$. We can therefore infer

$$||b||_p = ||T(b)||_p = ||f||_p < \infty$$

and

$$||a_{n_i} - b||_p = ||T(a_{n_i} - b)||_p = ||T(a_{n_i}) - T(b)||_p = ||f_{n_i} - f||_p \to 0$$

Thus $\{a_{n_i}\} \to b$ in ℓ^p. By Proposition 4, $\{a_n\} \to b$ in ℓ^p. We conclude that ℓ^p is complete and therefore a Banach space.

35. We first most show that c and c_0 are linear spaces. Let $a = (a_1, a_2, \cdots)$ and $b = (b_1, b_2, \cdots)$ denote two sequences in c. Let $a_\infty = \lim_{k\to\infty} a_k$ and $b_\infty = \lim_{k\to\infty} b_k$. For any two scalars α and β, we have

$$\lim_{k\to\infty} (\alpha \cdot a_k + \beta \cdot b_k) = \alpha \cdot a_\infty + \beta \cdot b_\infty$$

Therefore the sequence $\alpha \cdot a + \beta \cdot b$ converges. Thus c is a linear space. Moreover if $a_\infty = b_\infty = 0$, then $\alpha \cdot a + \beta \cdot b$ converges to 0. Thus c_0 is also a linear space.

It remains to show that c and c_0 are complete. Let $\{a_n\}$ denote a Cauchy sequence in c, where $a_n = (a_{n,1}, a_{n,2}, \cdots)$. By Problem 34, there exists a sequence $b = (b_1, b_2, \cdots)$ such that $\{a_n\} \to b$ in ℓ^∞. For any set of indicies k, l, and n, we have

$$\begin{aligned} |b_k - b_l| &\leq |b_k - a_{n,k}| + |a_{n,k} - a_{n,l}| + |a_{n,l} - b_l| \\ &\leq 2 \cdot ||a_n - b||_\infty + |a_{n,k} - a_{n,l}| \end{aligned}$$

Since $\{a_n\} \to b$ in ℓ^∞, there exists an index n such that $||a_n - b||_\infty < \epsilon/3$. Since a_n converges, there exists an index N such that $|a_{n,k} - a_{n,l}| < \epsilon/3$ for all $k, l \geq N$. We can thus conclude

$$|b_k - b_l| < \epsilon$$

for all $k, l \geq N$. But then b is a Cauchy sequence, and therefore must converge. Thus b is in c, so c is complete. Now if $\{a_n\}$ is in c_0, then

$$|b_k| \leq |b_k - a_{n,k}| + |a_{n,k}| \leq ||a_n - b||_\infty + |a_{n,k}|$$

Since a_n converges to zero, there exists an index N such that $|a_{n,k}| < 2 \cdot \epsilon/3$ for all $k \geq N$. Therefore

$$|b_k| < \epsilon$$

for all $k \geq N$, implying that b converges to zero. Thus b is in c_0, so c_0 is also complete.

7.4 Approximation and Separability

36. Suppose $\mathcal{S}$ is dense in X. Fix $g \in X$. For each index n, there exists $f_n \in \mathcal{S}$ such that $||f_n - g|| < \frac{1}{n}$. But then $\lim_{n\to\infty} ||f_n - g|| = 0$, so $\{f_n\} \to g$ in X. Conversely, suppose for any $g \in X$ there exists a sequence $\{f_n\}$ in $\mathcal{S}$ such that $\{f_n\} \to g$ in X. Fix $\epsilon > 0$ and choose n such that $||f_n - g|| < \epsilon$. Since $f_n \in \mathcal{S}$, it follows that $\mathcal{S}$ is dense in X.

37. Fix $h \in \mathcal{H}$ and $\epsilon > 0$. Since $\mathcal{G}$ is dense in $\mathcal{H}$, there exists $g \in \mathcal{G}$ such that $||h - g|| < \epsilon/2$. Likewise, since $\mathcal{F}$ is dense in $\mathcal{G}$, there exists $f \in \mathcal{F}$ such that $||f - g|| < \epsilon/2$. By the Triangle Inequality, we have

$$||h - f|| \leq ||h - g|| + ||g - f|| < \epsilon$$

Therefore $\mathcal{F}$ is dense in $\mathcal{H}$.

38. Let P_n denote the set of polynomials of degree n with rational coefficients. There is a 1-to-1 mapping between P_n and $\mathbf{Q}^n$. By Problem 23 of Chapter 1, $\mathbf{Q}^n$ is countable. Therefore each P_n is countable. By Corollary 6 of Chapter 1, $\bigcup_{n=1}^{\infty} P_n$ is countable.

39. Fix $g \in L^p(E)$. Suppose $g \neq 0$ so that $||g||_p > 0$. By Problem 20, there exists $h \in L^q(E)$ such that

$$\left|\int_E g \cdot h\right| > 0$$

Fix $\epsilon \in \left(0, \left|\int_E g \cdot h\right|\right)$ and choose $f \in \mathcal{S}$ such that

$$||h - f||_q < \frac{\epsilon}{||g||_p}$$

Then

$$\begin{aligned}\left|\int_E f \cdot g\right| &\geq \left|\int_E g \cdot h\right| - \left|\int_E g \cdot (h - f)\right| \\ &> \epsilon - ||g||_p \cdot ||h - f||_q \\ &> 0\end{aligned}$$

40. To complete the proof, we must show $\mathcal{S}'[a, b]$ is a countable set. Let $\mathcal{S}'_n[a, b]$ denote the step functions ψ on $[a, b]$ that take rational values and for which there is a partition $P = \{x_0, \cdots, x_n\}$ of $[a, b]$ with ψ constant on (x_{k-1}, x_k) and x_k rational for $1 \leq k \leq n - 1$. Each step function in this set is fully characterized by $n-1$ rational partition points and n rational values. Therefore there is 1-to-1 mapping between $\mathcal{S}'_n[a, b]$ and $\mathbf{Q}^{2n-1}$. By Problem 23 of Chapter 1, $\mathbf{Q}^{2n-1}$ is countable. Therefore $\mathcal{S}'_n[a, b]$ is countable for each n. By Corollary 6 of Chapter 1, $\mathcal{S}'[a, b] = \bigcup_{n=1}^{\infty} \mathcal{S}'_n[a, b]$ is countable.

41. Let $E = (0, 1]$ and let

$$\alpha_n = -\frac{n-1}{n} \cdot \frac{1}{p_2}$$

Define $f_n(x) = x^{\alpha_n}$ and $f(x) = x^{-1/p_2}$. Each $f_n \in L^{p_2}(E)$ and

$$\int_E |f_n|^{p_1} = \int_0^1 x_n^{-\frac{p_1}{p_2} \cdot \frac{n-1}{n}} = \frac{1}{1 - \frac{p_1}{p_2} \cdot \frac{n-1}{n}}$$

But this implies

$$\lim_{n \to \infty} \int_E |f_n|^{p_1} = \frac{1}{1 - \frac{p_1}{p_2}} = \int_E |f|^{p_1}$$

Since $\{f_n\} \to f$ pointwise a.e. on E, $\{f_n\} \to f$ in L^{p_1} by Theorem 7. By Proposition 4, $\{f_n\}$ is Cauchy in $|| \cdot ||_{p_1}$. But $f \notin L^{p_2}$, so L^{p_2} normed by $|| \cdot ||_{p_1}$ is not complete.

42. Suppose E is countable and consider the singleton class of functions $\mathcal{S} = \{0\}$. For any $f \in L^{\infty}(E)$ and $\epsilon > 0$, $||f - 0||_{\infty} = 0 < \epsilon$. Thus $\mathcal{S}$ is dense in $L^{\infty}(E)$, so $L^{\infty}(E)$ is separable.

    Now suppose E contains a non-degenerate interval. Then E must contain a closed, bounded, non-degenerate interval $[a, b]$. Let $\mathcal{S}$ be dense in $L^{\infty}(E)$. Then the functions in $\mathcal{S}$ restricted to $[a, b]$ must be dense in $L^{\infty}([a, b])$. But it was proven in the text that $L^{\infty}([a, b])$ is not separable, so the restriction of functions in $\mathcal{S}$ to $[a, b]$ cannot be countable. But this implies $\mathcal{S}$ is not countable. Therefore $L^{\infty}(E)$ is not separable.

43. Fix $f \in X$ and $\epsilon > 0$. For each n, there exists $g_n \in X_0$ such that $||f - g_n|| < 1/n$. For all $n, m > 2/\epsilon$,

we have

$$\begin{aligned}||g_n - g_m|| &\le ||f - g_n|| + ||f - g_m|| \\ &< \frac{1}{n} + \frac{1}{m} \\ &< \epsilon\end{aligned}$$

Therefore $\{g_n\}$ is a Cauchy sequence in X_0. Since X_0 is complete, there exists $g \in X_0$ such that $\{g_n\} \to g$ in X_0. But we also have

$$0 \le ||f - g|| \le ||f - g_n|| + ||g - g_n||$$

by the Triangle Inequality. Taking limits, we conclude that $||f - g|| = 0$, which implies $f = g$. Thus $f \in X_0$, so $X_0 \supseteq X$. Since $X_0 \subseteq X$ by construction, we must have $X_0 = X$.

44. Fix $1 \le p < \infty$. For each natural number n, define

$$\mathcal{F}_n = \{q = (q_1, q_2, \cdots) \in \ell^p : q_k \in \mathbf{Q} \text{ for all } k \text{ and } q_k = 0 \text{ for } k \ge n\}$$

There exists a one-to-one correspondence between $\mathcal{F}_n$ and $\mathbf{Q}^n$. Since $\mathbf{Q}^n$ is countable by Problem 23 of Chapter 1, $\mathcal{F}_n$ is countable. By Corollary 6 of Chapter 1, $\mathcal{F} := \cup_{n=1}^{\infty} \mathcal{F}_n$ is also countable. We claim that $\mathcal{F}$ is dense in ℓ^p. To show this, fix $\epsilon > 0$ and $a \in \ell^p$. Since $\sum_{k=1}^{\infty} |a_k|^p < \infty$, there exists an index n such that

$$\sum_{k=n+1}^{\infty} |a_k|^p < \frac{\epsilon^p}{2}$$

For $k = 1, \cdots, n$, choose $q_k \in \mathbf{Q}$ so that

$$|a_k - q_k| < \frac{\epsilon}{2^{(k+1)/p}}$$

Then

$$\begin{aligned}||a - q||_p &= \left(\sum_{k=1}^{n} |a_k - q_k|^p + \sum_{k=n+1}^{\infty} |a_k|^p \right)^{1/p} \\ &< \left(\sum_{k=1}^{n} 2^{-k} \cdot \frac{\epsilon^p}{2} + \frac{\epsilon^p}{2} \right)^{1/p} \\ &< \left(\frac{\epsilon^p}{2} + \frac{\epsilon^p}{2} \right)^{1/p} \\ &= \epsilon\end{aligned}$$

We show ℓ^∞ is not separable by contradiction. Suppose $\{q_n\}_{n=1}^{\infty} \subseteq \ell^\infty$ is dense in ℓ^∞. For any set $A \in 2^{\mathbf{N}}$, define

$$a_n(A) = \begin{cases} 1 & \text{if } n \in A \\ 0 & \text{otherwise} \end{cases}$$

and let $a(A) = (a_1(A), a_2(A), \cdots) \in \ell^\infty$. Since $\{q_n\}_{n=1}^{\infty}$ is dense in ℓ^∞, there exists an index $\eta(A)$ such that

$$||a(A) - q_{\eta(A)}||_\infty < \frac{1}{2}$$

Now if A, A' are in $2^{\mathbf{N}}$ with $A \neq A'$, we have

$$\begin{aligned}
1 &= ||a(A) - a(A')||_\infty \\
&\leq ||a(A) - q_{\eta(A')}||_\infty + ||q_{\eta(A)} - q_{\eta(A')}||_\infty + ||a(A') - q_{\eta(A')}||_\infty \\
&< ||q_{\eta(A)} - q_{\eta(A')}||_\infty + 1
\end{aligned}$$

Therefore $||q_{\eta(A)} - q_{\eta(A')}||_\infty > 0$, which implies $\eta(A) \neq \eta(A')$. Thus η is a one-to-one mapping from $2^{\mathbf{N}}$ to $\mathbf{N}$. But this implies $2^{\mathbf{N}}$ is countable by Problem 17 of Chapter 1. However, Problem 22 of Chapter shows that $2^{\mathbf{N}}$ is uncountable, a contradiction.

45. Suppose E is bounded. Then there exists $M \geq 0$ such that $E \subseteq (-M, M)$. Fix $f \in L^p(E)$ and $\epsilon > 0$. By Proposition 23 of Chapter 4, there exists $\delta > 0$ such that

$$\text{if } A \subseteq E \text{ is measurable and } m(A) < \delta, \text{ then } \int_A |f|^p < \frac{\epsilon^p}{2^{p+1}}$$

By Lusin's Theorem, there exists a continuous function g on $\mathbf{R}$ and a closed set $F \subseteq E$ such that

$$g = f \text{ on } F \text{ and } m(E \sim F) < \delta$$

Since F is closed, the set $(-M, M) \sim F$ is open. Therefore by Proposition 9 of Chapter 1, $(-M, M) \sim F = \bigcup_{k=1}^{\infty} (a_k, b_k)$ where $\{(a_k, b_k)\}_{k=1}^{\infty}$ is a collection of disjoint open intervals. Now define the sequence of functions

$$g_n(x) = \begin{cases} g(a_k) - \dfrac{n \cdot g(a_k)}{c_k} \cdot (x - a_k) & \text{if} \quad a_k < x < a_k + \dfrac{c_k}{n} \\ 0 & \text{if} \quad a_k + \dfrac{c_k}{n} \leq x \leq b_k - \dfrac{d_k}{n} \\ g(b_k) + \dfrac{n \cdot g(b_k)}{d_k} \cdot (x - b_k) & \text{if} \quad b_k - \dfrac{d_k}{n} < x < b_k \\ g(x) & \text{if} \quad x \in F \end{cases}$$

where

$$c_k = 2^{-k} \cdot \min\left\{\frac{1}{\max\{|g(a_k)|^p, 1\}}, b_k - a_k\right\} \qquad d_k = 2^{-k} \cdot \min\left\{\frac{1}{\max\{|g(b_k)|^p, 1\}}, b_k - a_k\right\}$$

We then have

$$\begin{aligned}
\int_{a_k}^{b_k} |g_n|^p &\leq \frac{|g(a_k)|^p \cdot c_k}{2 \cdot n} + \frac{|g(b_k)|^p \cdot d_k}{2 \cdot n} \\
&\leq \frac{1}{n \cdot 2^k}
\end{aligned}$$

which implies

$$\int_{E\sim F}|g_n|^p \leq \int_{(-M,M)\sim F}|g_n|^p = \sum_{k=1}^{\infty}\int_{a_k}^{b_k}|g_n|^p \leq \frac{1}{2\cdot n}$$

Now pick $n > 2^p/\epsilon^p$. Then

$$\int_E |f-g_n|^p = \int_{E\sim F}|f-g_n|^p \leq 2^p\cdot\int_{E\sim F}(|f|^p+|g_n|^p) < \epsilon^p$$

By construction, g_n is continuous on $(-M,M)$. We extend g_n to $\mathbf{R}$ by defining

$$g_n(x) = \begin{cases} 0 & \text{if} \quad x \leq -M-1 \\ g(-M)+g(-M)\cdot(x+M) & \text{if} \quad -M-1 < x \leq -M \\ g(M)-g(M)\cdot(x-M) & \text{if} \quad M < x < M+1 \\ 0 & \text{if} \quad x \geq M+1 \end{cases}$$

Then g_n vanishes outside the bounded set $[-M-1, M+1]$, so $g_n \in C_c(E)$. Since $||f-g_n||_p < \epsilon$, $C_c(E)$ is dense in $L^p(E)$.

Now suppose E is unbounded. By the Dominated Convergence Theorem, there exists $n > 0$ such that

$$\int_{E\sim(-n,n)}|f|^p < \frac{\epsilon^p}{2^{p+1}}$$

We previously showed how to construct a continuous function g that satisfies

$$\int_{E\cap(-n,n)}|f-g|^p < \frac{\epsilon^p}{2}$$

and for which there exists an $M \geq 0$ such that $g(x) = 0$ if $|x| \geq M$. Assume without loss that $n \geq M$. We then have

$$\begin{aligned}\int_E|f-g|^p &= \int_{E\sim(-n,n)}|f-g|^p + \int_{E\cap(-n,n)}|f-g|^p \\ &< 2^p\cdot\int_{E\sim(-M,M)}(|f|^p+|g|^p)+\frac{\epsilon^p}{2} \\ &< \epsilon^p\end{aligned}$$

Since $g \in C_c(E)$ and $||f-g||_p < \epsilon$, $C_c(E)$ is dense in $L^p(E)$.

46. If $a = 0$ or $b = 0$, the expression reduces to

$$\max\{|a|,|b|\} \leq 2^p\cdot\max\{|a|,|b|\}$$

which is obviously true.

Now assume a and b are both non-zero. We first consider the case when $\operatorname{sgn}(a) = \operatorname{sgn}(b)$. Define $f(x) = x^{1/p}$. Then

$$\begin{aligned}\left|\operatorname{sgn}(a)\cdot|a|^{1/p} - \operatorname{sgn}(b)\cdot|b|^{1/p}\right| &= |f(|a|) - f(|b|)| \\ &\leq \left|f\left(\big||a|-|b|\big|\right) - f(0)\right| \\ &= \big||a|-|b|\big|^{1/p} \\ &\leq \left|a-b\right|^{1/p} \\ &\leq 2\cdot\left|a-b\right|^{1/p}\end{aligned}$$

where the first inequality follows because f is concave over non-negative numbers.

It remains to show the inequality holds when $\operatorname{sgn}(a) = -\operatorname{sgn}(b)$. In this case,

$$\max\{|a|, |b|\} \leq |a-b|$$

which implies

$$\max\{|a|^{1/p}, |b|^{1/p}\} \leq |a-b|^{1/p}$$

We therefore have

$$\begin{aligned}\left|\operatorname{sgn}(a)\cdot|a|^{1/p} - \operatorname{sgn}(b)\cdot|b|^{1/p}\right| &= |a|^{1/p} + |b|^{1/p} \\ &\leq 2\cdot\max\{|a|^{1/p}, |b|^{1/p}\} \\ &\leq 2\cdot\left|a-b\right|^{1/p}\end{aligned}$$

47. If $a = 0$ or $b = 0$, the expression reduces to

$$\max\{|a|^p, |b|^p\} \leq p\cdot\max\{|a|^p, |b|^p\}$$

which is obviously true. The inequality also holds trivially whenever $|a| = |b|$.

Now assume a and b are both non-zero with $|a| \neq |b|$. We first consider the case when $\operatorname{sgn}(a) = \operatorname{sgn}(b)$. Define $f(x) = x^p$. Then

$$\begin{aligned}\left|\frac{\operatorname{sgn}(a)\cdot|a|^p - \operatorname{sgn}(b)\cdot|b|^p}{|a|-|b|}\right| &= \left|\frac{f(|a|) - f(|b|)}{|a|-|b|}\right| \\ &\leq f'\left(|a|+|b|\right) \\ &= p\cdot(|a|+|b|)^{p-1}\end{aligned}$$

where the inequality follows from the convexity of f. Rearranging this expression, we obtain

$$\begin{aligned}|\operatorname{sgn}(a)\cdot|a|^p - \operatorname{sgn}(b)\cdot|b|^p| &\leq p\cdot\big||a|-|b|\big|\cdot(|a|+|b|)^{p-1} \\ &\leq p\cdot|a-b|\cdot(|a|+|b|)^{p-1}\end{aligned}$$

Now assume $\text{sgn}(a) = -\text{sgn}(b)$. Then

$$\Big|\text{sgn}(a) \cdot |a|^p - \text{sgn}(b) \cdot |b|^p\Big| = |a|^p + |b|^p$$

and

$$|a-b| \cdot (|a| + |b|)^{p-1} = (|a| + |b|)^p$$

Thus the inequality reduces to

$$|a|^p + |b|^p \leq p \cdot (|a| + |b|)^p$$

If $p \geq 2$, then

$$|a|^p + |b|^p \leq 2 \cdot \max\{|a|^p, |b|^p\} \leq 2 \cdot (|a| + |b|)^p \leq p \cdot (|a| + |b|)^p$$

If $1 < p < 2$, define $g(x) = x^{p-1}$. We then have

$$\begin{aligned}
\frac{|a|^p + |b|^p}{|a| + |b|} &= \frac{|a|}{|a| + |b|} \cdot g(|a|) + \frac{|b|}{|a| + |b|} \cdot g(|b|) \\
&\leq g\left(\frac{|a|^2 + |b|^2}{|a| + |b|}\right) \\
&\leq (|a| + |b|)^{p-1}
\end{aligned}$$

where the first inequality follows from concavity of g and the second inequality follows because $|a|^2 + |b|^2 \leq (|a| + |b|)^2$. But this expression implies

$$|a|^p + |b|^p \leq (|a| + |b|)^p \leq p \cdot (|a| + |b|)^p$$

48. Since $f \in L^1(E)$, we know that

$$\int_E |\Phi(f)|^p = \int_E |f| < \infty$$

Therefore $\Phi(f) \in L^p(E)$. For any f, g in $L^1(E)$, we also have

$$\begin{aligned}
||\Phi(f) - \Phi(g)||_p^p &= \int_E \left|\text{sgn}(f) \cdot |f|^{1/p} - \text{sgn}(g) \cdot |g|^{1/p}\right|^p \\
&\leq \int_E 2^p \cdot |f - g| \\
&= 2^p \cdot ||f - g||_1
\end{aligned}$$

where the inequality follows from Problem 46. Thus if $\{f_n\} \to f$ in $L^1(E)$, then

$$0 \leq \lim_{n\to\infty} ||\Phi(f_n) - \Phi(f)||_p \leq 2 \cdot \lim_{n\to\infty} ||f_n - f||_1^{1/p} = 0$$

We conclude that $\Phi(\cdot)$ is a continuous map from $L^1(E)$ to $L^p(E)$.

Now suppose $\Phi(f) = \Phi(g)$. Then there exists a set E_0 with $m(E_0) = 0$ such that

$$\text{sgn}(f) \cdot |f|^{1/p} = \text{sgn}(g) \cdot |g|^{1/p} \tag{1}$$

on $E \sim E_0$. Fix $x \in E \sim E_0$. If $\operatorname{sgn}(f(x)) = \operatorname{sgn}(g(x))$, then (1) implies $|f(x)| = |g(x)| = 0$ so that $f(x) = g(x)$. If $\operatorname{sgn}(f(x)) = -\operatorname{sgn}(g(x))$, then (1) implies $|f(x)| + |g(x)| = 0$ so that $f(x) = 0 = g(x)$. Therefore $f = g$ on $E \sim E_0$. We conclude that $\Phi(\cdot)$ is one-to-one.

For $g \in L^p(E)$, define the function $\Phi^{-1}(g)$ on E as

$$\Phi^{-1}(g)(x) = \operatorname{sgn}(g(x)) \cdot |g(x)|^p$$

Then since $g \in L^p(E)$, we have

$$\int_E |\Phi^{-1}(g)| = \int_E |g|^p < \infty$$

Thus $\Phi^{-1}(g) \in L^1(E)$ and

$$\Phi(\Phi^{-1}(g)) = \operatorname{sgn}(\operatorname{sgn}(g) \cdot |g|^p) \cdot |\operatorname{sgn}(g) \cdot |g|^p|^{1/p} = \operatorname{sgn}(g) \cdot |g| = g$$

Therefore the image of Φ on $L^1(E)$ is $L^p(E)$ and its inverse is given by Φ^{-1}. To show that the inverse mapping is continuous, observe that for all f, g in $L^p(E)$ we have

$$\begin{aligned}
||\Phi^{-1}(f) - \Phi^{-1}(g)||_1 &= \int_E \big|\operatorname{sgn}(f) \cdot |f|^p - \operatorname{sgn}(g) \cdot |g|^p\big| \\
&\leq p \cdot \int_E |f - g| \cdot (|f| + |g|)^{p-1} \\
&\leq p \cdot ||f - g||_p \cdot \left(\int_E (|f| + |g|)^p\right)^{(p-1)/p} \\
&\leq p \cdot ||f - g||_p \cdot (||f||_p + ||g||_p)^{p-1} \\
&\leq p \cdot ||f - g||_p \cdot (||f - g||_p + 2 \cdot ||g||_p)^{p-1}
\end{aligned}$$

where the first inequality follows from Problem 47, the second inequality from Hölder's Inequality, and the third inequality by Minkowski's Inequality. Thus if $\{g_n\} \to g$ in $L^p(E)$, then

$$0 \leq \lim_{n\to\infty} ||\Phi^{-1}(g_n) - \Phi^{-1}(g)||_1 \leq p \cdot \lim_{n\to\infty} ||g_n - g||_p \cdot (||g_n - g||_p + 2 \cdot ||g||_p)^{p-1} = 0$$

49. Suppose $L^1(E)$ is separable. Then there exists a countable subset $\mathcal{S} \subseteq L^1(E)$ that is dense in $L^1(E)$. We claim that $\Phi(\mathcal{S}) := \{\Phi(f) : f \in \mathcal{S}\}$ is dense in $L^p(E)$, where $\Phi(\cdot)$ was defined in Problem 48. To show this, pick $g \in L^p(E)$ and define $f = \Phi^{-1}(g)$. By Problem 36, there exists a sequence $\{f_n\}$ in $\mathcal{S}$ that converges to f in $L^1(E)$. By Problem 48, $\{\Phi(f_n)\}$ is a sequence in $\Phi(\mathcal{S})$ that converges to $\Phi(f) = \Phi(\Phi^{-1}(g)) = g$. We conclude by Problem 36 that $L^p(E)$ is dense in $\Phi(\mathcal{S})$. Since Φ is one-to-one, $\Phi(\mathcal{S})$ is countable. Thus $L^p(E)$ is separable.
50. Suppose there exists a continuous mapping Φ from $L^1([a,b])$ onto $L^\infty([a,b])$. Then we could use the argument in Problem 49 to conclude that $L^\infty([a,b])$ is separable. However $L^\infty([a,b])$ was shown not to be separable in the text, a contradiction.
51. See Problem 45.

8 The L^p spaces: Duality and Weak Convergence

8.1 The Riesz Representation for the Dual of L^p, $1 \leq p < \infty$

1. Let $||T||_{**} = \sup\{T(f) \mid f \in X, ||f|| \leq 1\}$. Suppose $M \geq 0$ satisfies

$$|T(f)| \leq M \cdot ||f|| \text{ for all } f \in X$$

Then for any $f \in X$ with $||f|| \leq 1$, we have $T(f) \leq |T(f)| \leq M \cdot ||f|| \leq M$. Therefore $||T||_{**} \leq M$, which implies $||T||_{**} \leq ||T||_*$

Now fix $f \in X$ and suppose $f \neq 0$. By nonnegativity positive homogeneity of $|| \cdot ||$ and linearity of T, we have

$$\frac{1}{||f||} \cdot T(f) = T\left(\frac{f}{||f||}\right) \leq ||T||_{**}$$

Thus $T(f) \leq ||T||_{**} \cdot ||f||$. If $f = 0$, then $T(f) = 0 = ||T||_{**} \cdot 0 = ||T||_{**} \cdot ||f||$. We conclude that $T(f) \leq ||T||_{**} \cdot ||f||$ for all $f \in X$, so $||T||_{**} \geq ||T||_*$.

2. Let T_1 and T_2 denote two bounded linear functionals on X. For any real numbers α, β, γ, and δ and any f and g in X, we have

$$\begin{aligned}(\alpha \cdot T_1 + \beta \cdot T_2)(\gamma \cdot f + \delta \cdot g) &= (\alpha \cdot T_1 + \beta \cdot T_2)(\gamma \cdot f + \delta \cdot g) \\ &= \gamma \cdot (\alpha \cdot T_1(f) + \beta \cdot T_2(f)) + \delta \cdot (\alpha \cdot T_1(g) + \beta \cdot T_2(g)) \\ &= \gamma \cdot (\alpha \cdot T_1 + \beta \cdot T_2)(f) + \delta \cdot (\alpha \cdot T_1 + \beta \cdot T_2)(g)\end{aligned}$$

$$|\alpha \cdot T_1(f) + \beta \cdot T_2(f)| \leq |\alpha| \cdot |T_1(f)| + |\beta| \cdot |T_2(f)| \leq (|\alpha| \cdot ||T_1||_* + |\beta| \cdot ||T_2||_*) \cdot ||f||$$

Thus $\alpha \cdot T_1 + \beta \cdot T_2$ is also a bounded linear functional. We conclude that the bounded linear functionals on X form a linear space.

To see that $|| \cdot ||_*$ is a norm on this space, first observe that $T(0) = 0$ by linearity. Since $||0|| \leq 1$, $||T||_* \geq 0$. If $T = 0$ for all $f \in X$, $||T||_* = 0$. Conversely, if $||T||_* > 0$ there must exist $f \in X$ with $||f|| \leq 1$ such that $T(f) > 0$. But this implies $T \neq 0$. We conclude that $|| \cdot ||_*$ satisfies non-negativity.

Next, observe that

$$\begin{aligned}||T_1 + T_2||_* &= \sup\{T_1(f) + T_2(f) \mid f \in X, ||f|| \leq 1\} \\ &\leq \sup\{T_1(f) \mid f \in X, ||f|| \leq 1\} + \sup\{T_2(f) \mid f \in X, ||f|| \leq 1\} \\ &= ||T_1||_* + ||T_2||_*\end{aligned}$$

Therefore $|| \cdot ||_*$ satisfies the triangle inequality.

Lastly, fix a real number $\alpha \neq 0$. Since

$$|(\alpha \cdot T)(f)| = |\alpha| \cdot |T(f)| \leq |\alpha| \cdot ||T||_* \cdot ||f|| \text{ for all } f \in X$$

we must have $||\alpha \cdot T||_* \leq |\alpha| \cdot ||T||_*$. We also have

$$|\alpha| \cdot |T(f)| = |(\alpha \cdot T)(f)| \leq ||\alpha \cdot T||_* \cdot ||f|| \text{ for all } f \in X$$

which implies $\frac{||\alpha \cdot T||_*}{|\alpha|} \geq ||T||_*$. We conclude that $||\alpha \cdot T||_* = |\alpha| \cdot ||T||_*$. Therefore $||\cdot||_*$ satisfies positive homogeneity.

3. It was shown in the text that any bounded linear functional satisfies the continuity property. To prove the converse, suppose a linear functional is unbounded. Then for any index n there exists $f_n \in X$ such that

$$T(f_n) > n^2 \cdot ||f_n||$$

Since $T(0) = 0$, $||f_n|| > 0$ for all n. For each n, define the function

$$\tilde{f}_n = \frac{f_n}{n \cdot ||f_n||}$$

Then $||\tilde{f}_n|| = n^{-1}$ by positive homogeneity of $||\cdot||$, so $\left\{\tilde{f}_n\right\} \to 0$ in X. But by linearity of T, we have

$$T(\tilde{f}_n) = \frac{T(f_n)}{n \cdot ||f_n||} > n$$

for all n. Therefore $|T(\tilde{f}_n)| \not\to T(0)$, so T does not satisfy the continuity property.

4. Suppose T is bounded. Then

$$|T(g) - T(h)| = |T(g-h)| \leq ||T||_* \cdot ||g-h||$$

so T is Lipschitz and the Lipschitz constant c_* must satisfy $c_* \leq ||T||_*$. If c is any constant that satisfies the Lipschitz condition, then

$$|T(f)| = |T(f) - T(0)| \leq c \cdot ||f - 0|| = c \cdot ||f||$$

Thus $||T||_* \leq c$, so $||T||_* \leq c_*$. We conclude that $||T||_* = c_*$.

Conversely, suppose T is Lipschitz with Lipschitz constant c_*. Then

$$|T(f)| = |T(f) - T(0)| \leq c_* \cdot ||f - 0|| = c_* \cdot ||f||$$

Thus T is bounded and $||T||_* \leq c_*$. If M is any constant that satisfies the boundedness condition, then

$$|T(g) - T(h)| = |T(g-h)| \leq M \cdot ||g-h||$$

Thus $c_* \leq M$, so $c_* \leq ||T||_*$. We conclude that $||T||_* = c_*$.

5. Assume $1 \leq p < \infty$ and let $\mathcal{F}^p$ denote the functions in $L^p(E)$ that vanish outside a bounded set. Pick $f \in L^p(E)$ and $\epsilon > 0$. By the Monotone Convergence Theorem,

$$\int_{E \cap [-n,n]} |f|^p \to \int_E |f|^p$$

Since $\int_E |f|^p < \infty$, we can pick n so that

$$\int_E |f|^p - \int_{E\cap[-n,n]} |f|^p < \epsilon$$

We then have

$$\int_E \left|f - f\cdot\chi_{[-n,n]}\right|^p = \int_{E\sim[-n,n]} |f|^p = \int_E |f|^p - \int_{E\cap[-n,n]} |f|^p < \epsilon$$

Since $f\cdot\chi_{[-n,n]} \in \mathcal{F}^p$, $\mathcal{F}^p$ is dense in $L^p(E)$.

Now suppose $p = \infty$. Let $\epsilon = 1/2$ and $g = 1$. Pick $f \in \mathcal{F}^\infty$. Then there exists $M \geq 0$ such that $f = 0$ outside $[-M, M]$, so

$$||f - g||_\infty \geq 1 > \epsilon$$

Therefore $\mathcal{F}^\infty$ is not dense in $L^\infty(\mathbf{R})$.

6. The Riesz Representation Theorem follows from Theorem 5. To show that Theorem 5 holds for $p = 1$, let T denote a bounded linear functional on $L^1([a, b])$. Define $\Phi(x) = T(\chi_{[a,x)})$. Then for any $[c, d] \subseteq [a, b]$, we have

$$|\Phi(d) - \Phi(c)| = |T(\chi_{[c,d)})| \leq ||T||_* \cdot ||\chi_{[c,d)}||_1 = ||T||_* \cdot |d - c|$$

Therefore $\Phi(\cdot)$ is Lipschitz on $[a, b]$, so $\Phi(\cdot)$ is absolutely continuous on $[a, b]$ by Proposition 7 of Chapter 6. The rest of the proof of Theroem 5 can then be followed.

7. We first prove a few preliminary results. Let

$$\bar{\ell} = \{a = (a_1, a_2, \cdots) \in \ell^\infty \mid \text{there exists an index } n \text{ such that } a_k = 0 \text{ for all } k \geq n.\}$$

Claim 1: $\bar{\ell}$ is dense in ℓ^p for $1 \leq p < \infty$.

Proof: Suppose $b \in \ell^p$ for $1 \leq p < \infty$. Then for any $\epsilon > 0$, there exists an index n such that

$$\sum_{k=n}^{\infty} |b_k|^p < \epsilon^p$$

Define

$$a_k = \begin{cases} b_k & \text{if } k < n \\ 0 & \text{if } k \geq n \end{cases}$$

and let $a = (a_1, a_2, \cdots)$. Then $a \in \bar{\ell}$ and

$$||b - a||_p = \left(\sum_{k=n}^{\infty} |b_k|^p\right)^{1/p} < \epsilon$$

$\square$

Claim 2: Fix $1 \leq p < \infty$ and let q denote the conjugate of p. Let $b = (b_1, b_2, \cdots)$ denote a sequence.

Suppose there exists an $M \geq 0$ for which

$$\left|\sum_{k=1}^{\infty} a_k \cdot b_k\right| \leq M \cdot ||a||_p$$

for all sequences $a \in \bar{\ell}$. Then $b \in \ell^q$ and $||b||_q \leq M$.

Proof: Let $a_k = \text{sgn}(b_k) \cdot |b_k|^{q-1}$ and define

$$\bar{a}_k = \begin{cases} a_k & \text{if } k \leq n \\ 0 & \text{if } k > n \end{cases}$$

Then

$$\sum_{k=1}^{n} |b_k|^q = \sum_{k=1}^{n} a_k \cdot b_k \leq M \cdot ||\bar{a}||_p \tag{1}$$

But since p and q are conjugates, we know that

$$||\bar{a}||_p = \left(\sum_{k=1}^{n} |b_k|^{p(q-1)}\right)^{1/p} = \left(\sum_{k=1}^{n} |b_k|^q\right)^{1/p}$$

Rearranging (1), we obtain

$$\left(\sum_{k=1}^{n} |b_k|^q\right)^{1/q} \leq M$$

Since the expression holds for all n, we can conclude

$$||b||_q \leq M$$

$\square$

Claim 3: Let $1 \leq p < \infty$ and q denote the conjugate of p. Fix $c \in \ell^q$ and define the functional

$$T(a) = \sum_{k=1}^{\infty} a_k \cdot c_k \text{ for all } a \in \ell^p$$

Then T is a bounded, linear functional on ℓ^p with $||T||_* = ||c||_q$.

Proof: If $c = 0$, the results holds trivially. So assume $c \neq 0$ throughout. By Problem 12 of Chapter 7, we have

$$|T(a)| \leq \sum_{k=1}^{\infty} |a_k \cdot c_k| \leq ||a||_p \cdot ||c||_q \tag{2}$$

Thus T is bounded. Since T is bounded, for all a and b in ℓ^p and α and β in $\mathbf{R}$ we have

$$\begin{aligned}
T(\alpha\cdot a+\beta\cdot b) &= \sum_{k=1}^{\infty}(\alpha\cdot a_k+\beta\cdot b_k)\cdot c_k \\
&= \sum_{k=1}^{\infty}(\alpha\cdot a_k\cdot c_k+\beta\cdot b_k\cdot c_k) \\
&= \alpha\cdot\sum_{k=1}^{\infty}a_k\cdot c_k+\beta\cdot\sum_{k=1}^{\infty}b_k\cdot c_k \\
&= \alpha\cdot T(a)+\beta\cdot T(b)
\end{aligned}$$

Therefore T is linear. By (2), $||T||_* \leq ||c||_q$. Define

$$c_k^* = ||c||_q^{1-q}\cdot \operatorname{sgn}(c_k)\cdot |c_k|^{q-1}$$

Then $||c^*||_p = 1$ and

$$T(c^*) = \sum_{k=1}^{\infty} c_k^*\cdot c_k = ||c||_q^{1-q}\cdot\sum_{k=1}^{\infty}|c_k|^q = ||c||_q$$

Thus $||T||_* \geq ||c||_q$. □

Riesz Representation Theorem for the Dual of ℓ^p: Let $1\leq p<\infty$ and q the conjugate of p. For each $b\in\ell^q$, define the bounded linear functional $\mathcal{R}_b$ on ℓ^p as

$$\mathcal{R}_b(a) = \sum_{k=1}^{\infty} a_k\cdot b_k \text{ for all } a\in\ell^p$$

For each bounded linear functional T on ℓ^p, there is a unique sequence $b\in\ell^q$ for which

$$\mathcal{R}_b = T \text{ and } ||T||_* = ||b||_q$$

Proof: Let T denote a bounded linear functional on ℓ^p. Define

$$e_k(n) = \begin{cases} 1 & \text{if } k=n \\ 0 & \text{otherwise}\end{cases}$$

and let $e(n) = (e_1(n), e_2(n), \cdots)$. Define

$$b_n = T(e(n))$$

and let $b=(b_1,b_2,\cdots)$. Pick $a=(a_1,a_2,\cdots)\in\bar{\ell}$. Then there exists an index n such that $a_k=0$ for all $k\geq n$. We therefore have

$$\sum_{k=1}^{\infty}a_k\cdot b_k = \sum_{k=1}^{n}a_k\cdot b_k = \sum_{k=1}^{n}a_k\cdot T(e(k)) = T\left(\sum_{k=1}^{n}a_k\cdot e(k)\right) = T(a) \tag{3}$$

by linearity of T. This expression implies

$$\left|\sum_{k=1}^{\infty} a_k \cdot b_k\right| = |T(a)| \leq ||T||_* \cdot ||a||_p$$

Therefore $b \in \ell^q$ by Claim 2. Since $\mathcal{R}_b = T$ on $\bar{\ell}$, $\mathcal{R}_b = T$ on ℓ^p by Claim 1 and Proposition 3. By Claim 3, $||T||_* = ||\mathcal{R}_b||_* = ||b||_q$. If $R_{b'} = T$ on ℓ^p, then

$$b'_k = \mathcal{R}_{b'}(e(k)) = T(e(k)) = \mathcal{R}_b(e(k)) = b_k$$

for all k. Thus $b' = b$, so b is unique.

8. We first prove some preliminary results (notation introduced in the solution to Problem 7 is re-used).

Claim 1: $\bar{\ell}$ is dense in c_0.

Proof: Let b denote a sequence in c_0 and fix $\epsilon > 0$. Since $\{b_k\} \to 0$, there exists an index k such that

$$|b_k| < \frac{\epsilon}{2} \text{ for all } k \geq n$$

Define

$$a_k = \begin{cases} b_k & \text{if } k < n \\ 0 & \text{if } k \geq n \end{cases}$$

Then $a \in \bar{\ell}$ and

$$||a - b||_\infty = \sup_{k \geq n} |b_k| \leq \frac{\epsilon}{2} < \epsilon$$

□

Claim 2: Fix $b \in \ell_1$ and define the functional

$$T(a) = \sum_{k=1}^{\infty} a_k \cdot b_k \text{ for all } a \in c_0$$

Then T is a bounded, linear functional on c_0 with $||T||_* = ||b||_1$.

Proof: To show that T is bounded, observe that for any $a \in c_0$ we have

$$|T(a)| \leq \sum_{k=1}^{\infty} |a_k| \cdot |b_k| \leq ||a||_\infty \cdot \sum_{k=1}^{\infty} |b_k| = ||b||_1 \cdot ||a||_\infty$$

Linearity of $T(\cdot)$ was shown in Problem 7. Therefore T is a bounded linear functional on c_0 with $||T||_* \leq ||b||_1$. To show that $||T||_* = ||b||_1$, suppose we define $a_k = \text{sgn}(b_k)$ and

$$a_k(n) = \begin{cases} a_k & \text{if } k \leq n \\ 0 & \text{if } k > n \end{cases}$$

Then

$$\sum_{k=1}^{n} |b_k| = \sum_{k=1}^{n} a_k \cdot b_k = T(a(n))$$

Since $||a(n)||_\infty = 1$, we must have

$$||b||_1 = \sup_n \; T(a(n)) \leq \sup_n \; \big(||T||_* \cdot ||a(n)||_\infty\big) = ||T||_*$$

$\square$

Claim 3: Let

$$\bar{c} = \{a = (a_1, a_2, \cdots) \in \ell^\infty \mid \text{ there exists a number } \bar{a} \text{ such that } a_n = \bar{a} \text{ for all } n\}$$

If T is a bounded linear functional on $\bar{c}$, then there exists a real number α such that $T(a) = \alpha \cdot \bar{a}$.

Proof: Pick $a \in \bar{c}$. By linearity, $T(a) = \bar{a} \cdot T(e)$ where $e = (1, 1, \cdots)$ denotes the sequence of containing only ones. Since T is bounded, $T(e) \leq ||T||_* \cdot ||e||_\infty = ||T||_* < \infty$. Therefore setting $\alpha = T(e)$ yields the desired result. $\square$

The Dual of c_0 and c: For each $b \in \ell^1$, define the bounded linear functional $\mathcal{R}_b$ on c_0 as

$$\mathcal{R}_b(a) = \sum_{k=1}^{\infty} a_k \cdot b_k \text{ for all } a \in c_0$$

For each bounded linear functional T_0 on c_0, there is a unique sequence $b \in \ell^1$ for which

$$\mathcal{R}_b = T_0 \text{ and } ||T_0||_* = ||b||_1$$

For each bounded linear functional T on c, there is a unique sequence $b \in \ell^1$ and a real number α for which

$$T(a) = \alpha \cdot \bar{a} + \mathcal{R}_b(a - \bar{a}) \text{ for all } a \in c \tag{4}$$

where $\bar{a} = \lim_{n\to\infty} a_n$.

Proof: Let T_0 denote a bounded, linear operator on c_0. In Problem 7, we showed how to define a sequence $b = (b_1, b_2, \cdots)$ such that

$$T_0(a) = \sum_{k=1}^{\infty} a_k \cdot b_k \text{ for all } a \in \bar{\ell} \tag{5}$$

Since $\bar{\ell}$ is dense in c_0 by Claim 1, (5) must hold on c_0 by Proposition 3. By Claim 2, $||T_0||_* = ||b||_1$. Uniqueness follow by the same argument used in the solution to Problem 7.

Now let T denote a bounded, linear operator on c. For any $a \in c$, we have

$$T(a) = T(a - \bar{a} \cdot e) + T(\bar{a} \cdot e)$$

by linearity. Since $a - \bar{a} \cdot e \in c_0$ and $\bar{a} \cdot e \in \bar{c}$, we can write this expression as

$$T(a) = T_0(a - \bar{a} \cdot e) + \bar{T}(\bar{a} \cdot e)$$

where T_0 denotes the restriction of T to c_0 and $\bar{T}$ denotes the restriction of T to $\bar{c}$. The representation in (4) then follows from the result for c_0 and Claim 3.

9. For any $f \in C[a,b]$, we have

$$|T(f)| = |f(x_0)| \leq \max_{x \in [a,b]} |f(x)| = ||f||_{\max}$$

Therefore T is bounded.

Let g denote a function of bounded variation on $[a,b]$ and let $P = \{y_0, y_1, \cdots, y_k\}$ denote a partition of $[a,b]$. Let

$$S(P,f,g) = \sum_{i=1}^{k} f(c_i) \cdot |g(y_i) - g(y_{i-1})|$$

where $y_i \leq c_i \leq y_{i-1}$. We seek a function g such that for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|S(P,f,g) - f(x_0)| < \epsilon$$

whenever $y_i - y_{i-1} < \delta$ for $i = 1, \cdots, k$.

Fix $\epsilon > 0$. By Theorem 23 of Chapter 1, there exists a $\delta > 0$ such that $|f(x) - f(x')| < \epsilon$ if $|x' - x| < \delta$. Suppose $x_0 \in (a,b]$ and let $g = \chi_{[x_0,b]}$. There exists a unique index i^* such that

$$y_{i^*-1} < x_0 \leq y_{i^*}$$

This implies

$$|g(y_i) - g(y_{i-1})| = \begin{cases} 1 & \text{if } i = i^* \\ 0 & \text{otherwise} \end{cases}$$

so that

$$|S(P,f,g) - f(x_0)| = |f(c_{i^*}) - f(x_0)|$$

Thus if $y_i - y_{i-1} < \delta$ for $i = 1, \cdots, k$, then $|S(P,f,g) - f(x_0)| < \epsilon$. Moreover $TV(g) = 1 < \infty$, so g is of bounded variation. A similar argument can be used for $g = \chi_{(a,b]}$ when $x_0 = a$.

10. By the Extreme Value Theorem, there exists an index x_0 such that $f(x_0) = ||f||_{\max}$. By Problem 9, there exists a function g with $TV(g) = 1$ such that $\int_a^b f dg = f(x_0) = ||f||_{\max}$.

11. Let x, y belong to $[a,b]$. Then

$$\begin{aligned} |\Phi(x) - \Phi(y)| &= |T(g_x) - T(g_y))| \\ &= |T(g_x - g_y))| \\ &\leq ||T||_* \cdot ||g_x - g_y||_{\max} \end{aligned}$$

Let $\underline{z} = \min\{x,y\}$ and $\overline{z} = \max\{x,y\}$ Then

$$g_x(z) - g_y(z) = \begin{cases} 0 & \text{for } a \leq z \leq \underline{z} \\ z - \underline{z} & \text{for } \underline{z} < z \leq \overline{z} \\ \overline{z} - \underline{z} & \text{for } \overline{z} < z \leq b \end{cases}$$

But since $\overline{z} - \underline{z} = |x-y|$, we can conclude

$$|\Phi(x) - \Phi(y)| \leq ||T||_* \cdot |x-y|$$

Therefore Φ is Lipschitz.

8.2 Weak Sequential Convergence in L^p

12. For all $1 \leq p < \infty$, we have
$$\int_0^1 |f_n|^p = 1$$
Therefore $||f_n||_p = 1$ for all n, so $||f_n||_p \not\to 0$. Thus $||f_n||$ does not converge strongly to 0 in $L_p[0,1]$

13. Fix $1 < p < \infty$. We can write f_n as
$$f_n(x) = \frac{\alpha+\beta}{2} - \frac{\alpha-\beta}{2} \cdot (-1)^k \text{ for } k/2^n \leq x < (k+1)/2^n,\ 0 \leq k < 2^n - 1$$
Let $f = \frac{\alpha+\beta}{2}$ on $[0,1]$. For any $x \in [0,1]$, we have
$$\left| \int_0^x (f_n - f) \right| \leq \frac{|\alpha-\beta|}{2^{n+1}}$$
which implies
$$\lim_{n\to\infty} \left[\int_0^x f_n \right] = \int_0^x f$$
We also have
$$\int_0^1 |f_n|^p \leq \big(|\alpha| + |\beta|\big)^p$$
Thus $||f_n||_p \leq |\alpha| + |\beta|$ for all n, so $\{f_n\}$ is bounded in $L_p[0,1]$. Since $||f||_p = \frac{\alpha+\beta}{2} < \infty$, we can apply Theorem 11 to conclude that $f_n \rightharpoonup f$.

If $\alpha \neq \beta$, $|f_n - f_m|$ takes the value $|\alpha-\beta|$ on a set of measure 1/2 whenever $n \neq m$. Therefore $||f_n - f_m||_p \geq \frac{|\alpha-\beta|}{2^{1/p}}$ for $n \neq m$, which implies that no subsequence of $\{f_n\}$ is Cauchy in $L^p[0,1]$. Hence no subsequence of $\{f_n\}$ can converge strongly in $L_p[0,1]$.

14. Since h is continuous, it is integrable over $[0,T]$. Define
$$M = \int_0^T |h|$$
Since h is periodic, $\left|\int_c^d h\right| \leq M$ for any real numbers c and d. Therefore for any $x \in [a,b]$, we have
$$\left| \int_a^x f_n \right| = \frac{1}{n} \left| \int_{na}^{nx} h \right| \leq \frac{M}{n}$$
which implies
$$\lim_{n\to\infty} \left[\int_a^x f_n \right] = 0 \tag{1}$$

Let

$$\bar{h} = \max_{x\in[0,T]} |h(x)|$$

We then have

$$\int_a^b |f_n|^p = \frac{1}{n}\int_{na}^{nb} |h|^p \leq (b-a)\cdot \bar{h}^p$$

We can conclude that f_n is bounded in $L^p[a,b]$, so $f_n \rightharpoonup 0$ by Theorem 11 if $1 < p < \infty$.

It remains to show the result holds for $p = 1$. If $\bar{h} = 0$, then $f_n = 0$ and the result follows trivially. So assume $\bar{h} > 0$. Fix $g \in L^\infty[a,b]$. If g is a step function, (1) can be used to conclude

$$\lim_{n\to\infty}\int_a^b g\cdot f_n = 0$$

Now suppose g is not a step function. Since $[a,b]$ is bounded, $g \in L^1[a,b]$. By Proposition 10 of Chapter 7, for any $\epsilon > 0$ there exists a step function $\bar{g} \in L^1[a,b]$ such that $\int_a^b |g-\bar{g}| < \epsilon/\bar{h}$. We then have

$$\begin{aligned}\left|\int_a^b g\cdot f_n\right| &\leq \left|\int_a^b \bar{g}\cdot f_n\right| + \int_a^b |g-\bar{g}|\cdot|f_n| \\ &< \left|\int_a^b \bar{g}\cdot f_n\right| + \epsilon\end{aligned}$$

which implies

$$\lim_{n\to\infty}\left|\int_a^b g\cdot f_n\right| \leq \epsilon$$

Since ϵ can be made arbitrarily small, we can conclude

$$\lim_{n\to\infty}\int_a^b g\cdot f_n = 0$$

Therefore $f_n \rightharpoonup 0$ by Proposition 6.

15. If $||f_0||_p = 0$, then $f_n = 0$ for all n and the result holds trivially. So suppose $||f_0||_p > 0$ and fix $g \in L^q(\mathbf{R})$. By Theorem 11 of Chapter 7, for any $\epsilon > 0$ there exists a step function $g_0 \in L^q(\mathbf{R})$ that vanishes outside a bounded interval and that satisfies

$$||g-g_0||_q < \frac{\epsilon}{||f_0||_p}$$

We then have

$$\begin{aligned}\left|\int_{\mathbf{R}} f_n\cdot g\right| &\leq \int_{\mathbf{R}} |f_n\cdot(g-g_0)| + \left|\int_{\mathbf{R}} f_n\cdot g_0\right| \\ &\leq ||f_n||_p\cdot||g-g_0||_q + \left|\int_{\mathbf{R}} f_n\cdot g_0\right| \\ &< \epsilon + \left|\int_{-\infty}^{\infty} f_0(x)\cdot g_0(x+n)dx\right| \qquad (2)\end{aligned}$$

Let $g_n(x) = g_0(x+n)$. Then

$$\lim_{n\to\infty} g_n(x) = 0$$

for all $x \in \mathbf{R}$ and $||g_n||_q = ||g_0||_q$ for all n. Therefore $\{g_n\} \rightharpoonup 0$ in $L^q(\mathbf{R})$ by Theorem 12, so

$$\lim_{n\to\infty} \int_{\mathbf{R}} f_0 \cdot g_n = 0$$

by Proposition 6. Taking limits of (2), we obtain

$$\lim_{n\to\infty} \left| \int_{\mathbf{R}} f_n \cdot g \right| \le \epsilon$$

Since ϵ can be made arbitrarily small, we must have

$$\lim_{n\to\infty} \int_{\mathbf{R}} f_n \cdot g = 0$$

Since g was an arbitrary member of $L^q(\mathbf{R})$, $\{f_n\} \rightharpoonup 0$ in $L^p(\mathbf{R})$.

The bottom of page 167 in the text provides an example of a function $f_0 \in L^1(\mathbf{R})$ for which $\{f_n\}$ does not converge weakly to 0 in $L^1(\mathbf{R})$.

16. By linearity of integration, we have

$$\int_E (f_n - f)^2 = \int_E f_n^2 - 2 \cdot \int_E f \cdot f_n + \int_E f^2$$

Taking limits, we obtain

$$\lim_{n\to\infty} \int_E (f_n - f)^2 = 0$$

Therefore $\{f_n\} \to f$ in $L^2(E)$.

17. Let $f_n = n \cdot \chi_{[0,1/n^2)}$ and $f = 0$. Then $\{f_n\} \to f$ pointwise a.e. on $[0,1]$ and $\{f_n\} \rightharpoonup f$ in $L^2[0,1]$ by Theorem 12. Thus $\{f_n\}$ satisfies properties (i) and (ii). However $||f_n||_2 = 1$ for all n, so no subsequence of $\{||f_n||_2\}$ converges to $||f||_2 = 0$ (property (iii)). By Corollary 13, no subsequence of $\{f_n\}$ converges strongly to f.

18. Since $\{||f_n||\}$ is unbounded, we can pick an index n_1 such that $||f_{n_1}|| \ge 3$. Now choose an index $n_2 > n_1$ such that $||f_{n_2}|| \ge 2 \cdot 3^2$. Notice that if no such index exists, $\{||f_n||\}$ would be bounded by $\max\{||f_1||, \cdots, ||f_{n_1}||, 2 \cdot 3^2\}$. Continuing in this way, we can construct a subsequence $\{f_{n_k}\}$ that satisfies $||f_{n_k}|| \ge \alpha_k$ for all k.

Now suppose $||f_n|| \ge \alpha_n$ for all n. Then

$$\alpha := \liminf_{n\to\infty} \frac{||f_n||}{\alpha_n} \ge 1$$

By Problem 38 of Chapter 1, there exists a subsequence of $\{||f_n||/\alpha_n\}$ that convergences to α.

Lastly, suppose $\{||f_n||/\alpha_n\} \to \alpha \in [1, \infty]$. Let $g_n = \alpha_n/||f_n|| \cdot f_n$. By the homogeneity property of norms, $||g_n|| = \alpha_n = n \cdot 3^n$. If $\alpha < \infty$, let $g = f/\alpha$ and choose $T \in X^*$. Since $\{f_n\} \rightharpoonup f$, we have

$$\lim_{n\to\infty} T(g_n) = \lim_{n\to\infty} \frac{\alpha_n}{||f_n||} \cdot T(f_n) = \frac{T(f)}{\alpha} = T(g)$$

If $\alpha = \infty$, let $g = 0$. In this case $\{\alpha_n/||f_n||\} \to 0$, so we have

$$\lim_{n\to\infty} T(g_n) = \lim_{n\to\infty} \frac{\alpha_n}{||f_n||} \cdot T(f_n) = 0 = T(g)$$

Therefore $\{g_n\} \rightharpoonup g$ in X.

19. Suppose $\{\zeta_n\} \rightharpoonup \zeta$. Let T^k denote the bounded linear transformation on ℓ^p that returns the k-th element of the sequence. Then

$$\lim_{n\to\infty} \zeta_n^k = \lim_{n\to\infty} T^k(\zeta_n) = T^k(\zeta) = \zeta^k$$

Conversely, suppose $\{\zeta_n\} \to \zeta$ component-wise. Let T denote a bounded linear functional on ℓ^p. Since $\{\zeta_n\}$ is bounded in ℓ^p, there exists a real number $M \geq 0$ such that

$$||\zeta_n||_p \leq M \text{ for all } n$$

By Problem 7, there exists a unique sequence $\beta \in \ell^q$ such that

$$T(\alpha) = \sum_{k=1}^{\infty} \alpha^k \cdot \beta^k \text{ for all } \alpha \in \ell^p$$

Since $\beta \in \ell^q$, for any $\epsilon > 0$ there exists an index N such that

$$\sum_{k=N+1}^{\infty} |\beta^k|^q < \left(\frac{\epsilon}{M}\right)^q$$

Applying Hölder's Inequality, we can derive

$$\begin{aligned}
|T(\zeta_n) - T(\zeta)| &\leq \sum_{k=1}^{\infty} \left|(\zeta_n^k - \zeta^k) \cdot \beta^k\right| \\
&= \sum_{k=1}^{N} \left|(\zeta_n^k - \zeta^k) \cdot \beta^k\right| + \sum_{k=N+1}^{\infty} \left|(\zeta_n^k - \zeta^k) \cdot \beta^k\right| \\
&\leq \sum_{k=1}^{N} \left|(\zeta_n^k - \zeta^k) \cdot \beta^k\right| + \left(\sum_{k=N+1}^{\infty} \left|\zeta_n^k - \zeta^k\right|^p\right)^{1/p} \cdot \left(\sum_{k=N+1}^{\infty} \left|\beta^k\right|^q\right)^{1/q} \\
&\leq \sum_{k=1}^{N} \left|(\zeta_n^k - \zeta^k) \cdot \beta^k\right| + ||\zeta_n||_p \cdot \frac{\epsilon}{M} \\
&\leq \sum_{k=1}^{N} \left|(\zeta_n^k - \zeta^k) \cdot \beta^k\right| + \epsilon
\end{aligned}$$

Taking limits as $n \to \infty$, we obtain

$$\lim_{n\to\infty} |T(\zeta_n) - T(\zeta)| \leq \epsilon$$

Since ϵ can be made arbitrarily small, we must have

$$\lim_{n\to\infty} T(\zeta_n) = T(\zeta)$$

Therefore $\{\zeta_n\} \rightharpoonup \zeta$ in ℓ^p.

20. For any $T \in (L^{p_1}[0,1])^*$ and $f \in L^{p_2}[0,1]$, we have

$$|T(f)| \leq ||T||_* \cdot ||f||_{p_1} \leq ||T||_* \cdot ||f||_{p_2}$$

by Corollary 3 of Chapter 7. Thus $T \in (L^{p_2}[0,1])^*$, so $(L^{p_1}[0,1])^* \subseteq (L^{p_2}[0,1])^*$. We conclude that $\{f_n\} \rightharpoonup f$ in $L^{p_2}[0,1]$ implies $\{f_n\} \rightharpoonup f$ in $L^{p_1}[0,1]$.

To see that the converse does not hold, suppose $p_1 > 1$. Define

$$f_n = n^{1/p_1} \cdot \chi_{[0,1/n)}$$

and $f = 0$. Then

$$||f_n||_{p_1} = 1, \qquad ||f_n||_{p_2} = n^{p_2/p_1 - 1}$$

so $\{f_n\}$ is in $L^{p_2}[0,1]$ and $L^{p_1}[0,1]$. For any measurable subset A of $[0,1]$, we have

$$\left| \int_A f_n \right| \leq n^{1/p_1 - 1}$$

which implies

$$\lim_{n\to\infty} \int_A f_n = 0 = \int_A f$$

By Theorem 10, $\{f_n\} \rightharpoonup f$ in $L^{p_1}[0,1]$. However, $\lim_{n\to\infty} ||f_n||_{p_2} = \infty$, so $\{f_n\} \not\rightharpoonup f$ in $L^{p_2}[0,1]$ by Theorem 7.

21. Suppose $p > 1$ and fix $T \in (\ell^p)^*$. By Problem 7, there exists a unique sequence $b \in \ell^q$ for which

$$T(a) = \sum_{k=1}^{\infty} a_k \cdot b_k$$

for all $a \in \ell^p$. But since

$$\sum_{n=1}^{\infty} |b_n|^q < \infty$$

we must have

$$0 = \lim_{n\to\infty} |b_n| = \lim_{n\to\infty} |T(e_n)|$$

Therefore $\{e_n\} \rightharpoonup 0$ in ℓ^p. But $||e_n||_p = 1$ for all n, so no subsequence of $\{e_n\}$ can converge strongly to 0 in ℓ^p.

Now suppose $p = 1$ and consider the bounded linear functional

$$T(a) = \sum_{k=1}^{\infty} (-1)^k \cdot a_k$$

Then

$$T(e_n) = (-1)^n$$

which does not converge. Therefore $\{e_n\}$ cannot converge weakly in ℓ^1.

22. **The Radon-Riesz Theorem for** ℓ^p: Let $1 < p < \infty$. Suppose $\{a_n\} \rightharpoonup a$ in ℓ^p. Then

$$\{a_n\} \to a \text{ in } \ell^p \text{ if and only if } \lim_{n\to\infty} ||a_n||_p = ||a||_p$$

Proof for $p = 2$: Strong convergence implies convergence of the norms by the Triangle Inequality.

It remains to show that weak convergence and convergence of the norms implies strong convergence in ℓ^2. Suppose $\{a_n\}$ is a sequence in ℓ^2 for which

$$\{a_n\} \to a \text{ in } \ell^2 \text{ and } \lim_{n\to\infty} \sum_{k=1}^{\infty} (a_n^k)^2 = \sum_{k=1}^{\infty} (a^k)^2$$

For each n, we have

$$\sum_{k=1}^{\infty} (a_n^k - a^k)^2 = \sum_{k=1}^{\infty} \left((a_n^k)^2 - 2 \cdot a_n^k \cdot a^k + (a^k)^2 \right)$$

But $a \in \ell^2 = \ell^q$, so

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_n^k \cdot a^k = \sum_{k=1}^{\infty} (a^k)^2$$

by weak convergence. Therefore

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} (a_n^k - a^k)^2 = 0$$

Thus $\{a_n\} \to a$ in ℓ^2.

23. For any $x \in [a, b]$, define

$$g_x = \begin{cases} \chi_{[x,b]} & \text{if } x > a \\ \chi_{(a,b]} & \text{if } x = a \end{cases}$$

Let

$$T_x(h) = \int_a^b h(u) dg_x(u) \text{ for all } h \in C[a, b]$$

By the example on page 156, T_x is a bounded linear functional. Since $\{f_n\} \rightharpoonup f$, we must have

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} T_x(f_n) = T_x(f) = f(x)$$

24. For any $x \in [a,b]$, let $g_x = \chi_{[a,x]}$, Define

$$T_x(h) = \int_a^b h \cdot g_x \text{ for all } h \in L^\infty[a,b]$$

Following the remark on page 161, T_x is a bounded, linear functional on $L^\infty[a,b]$. Since $\{f_n\} \rightharpoonup f$, we have

$$\lim_{n\to\infty} \int_a^x f_n = \lim_{n\to\infty} T_x(f_n) = T_x(f) = \int_a^x f_n$$

25. (i) Suppose $\{f_n\} \rightharpoonup f$ and $\{f_n\} \rightharpoonup g$ in X. Pick $T \in X^*$ such that $T(f-g) = ||f-g||$. Then

$$||f-g|| = T(f-g) = T(f) - T(g) = \lim_{n\to\infty} T(f_n) - \lim_{n\to\infty} T(f_n) = 0$$

Since $||f-g|| = 0$, $f-g=0$. Thus $f=g$.

(ii) Suppose $\{f_n\} \rightharpoonup f$ in X. Pick $T \in X^*$ such that $T(f) = ||f||$ and $||T||_* = 1$. Since

$$T(f_n) \leq ||T||_* \cdot ||f_n|| = ||f_n||$$

for all n, we have

$$||f|| = |T(f)| = \lim_{n\to\infty} |T(f_n)| \leq \liminf_{n\to\infty} ||f_n||$$

26. Suppose by way of contradiction that $\{||f_n||_p\}_n$ is unbounded. By the argument in Problem 18, by taking a subsequence we can assume without loss that $\{||f_n||_p/\alpha_n\} \to \alpha \in [1,\infty]$ where $\alpha_n = n \cdot 3^n$. Now define $g_n = \alpha_n/||f_n||_p \cdot f_n$, and pick $g \in L^q(E)$. Then

$$\int_E g_n \cdot g = \frac{\alpha_n}{||f_n||_p} \int_E f_n \cdot g$$

Since $\{\alpha_n/||f_n||_p\}$ and $\{\int_E f_n \cdot g\}$ are bounded, $\{\int_E g_n \cdot g\}$ is bounded. We also have $||g_n||_p = \alpha_n$. It was shown in the proof of Theorem 7 that if $||g_n||_p = \alpha_n$, we can construct a function $g \in L^q$ that satisfies

$$\left| \int_E g \cdot g_n \right| \geq n/2$$

for all n. But this contradicts the boundedness of $\{\int_E g \cdot g_n\}$.

8.3 Weak Sequential Compactness

27. Without loss of generality, assume $b=1$ and $a=0$. Define

$$f_n(x) = \begin{cases} 0 & \text{if } x \in \left[0, \dfrac{1}{n+1}\right) \\[2ex] n \cdot (n+1) \cdot x - n & \text{if } x \in \left[\dfrac{1}{n+1}, \dfrac{1}{n}\right) \\[2ex] 1 & \text{if } x \in \left[\dfrac{1}{n}, 1\right] \end{cases}$$

Then $||f_n||_{\max} = 1$ for all n, so $\{f_n\}$ is bounded in $C[a,b]$. For a fixed n, we have

$$f_n\left(\frac{1}{n+1}\right) = 0$$

and

$$f_m\left(\frac{1}{n+1}\right) = 1$$

for all $m > n$. Therefore $||f_n - f_m||_{\max} = 1$ for all $m > n$, so no subsequence of $\{f_n\}$ can be Cauchy.

28. The example sequence in Problem 21 is bounded in ℓ^p for all p but fails to have any strongly convergent subsequence.

29. If E contains a non-degenerate interval, we can reuse the example on page 173 to construct a sequence bounded in $L^1(E)$ that does not contain a weakly convergent subsequence.

    Now suppose $E_0 \subseteq E$ has measure zero. Then for any sequence $\{f_n\}$ in $L^1(E_0)$,

$$\int_{E_0} g \cdot f_n = 0 = \int_{E_0} g$$

    for all $g \in L^\infty(E_0)$. Therefore $\{f_n\} \rightharpoonup 0$ in $L^\infty(E_0)$ by Proposition 6.

30. Suppose $\{T_n\} \to T$ with respect to the $||\cdot||_*$ norm. Then for any $\epsilon > 0$, there exists N such that $||T_n - T||_* < \epsilon$ for all $n \geq N$. But then for any $f \in X$ with $||f|| \leq 1$, we have

$$|T_n(f) - T(f)| \leq ||T_n - T||_* \cdot ||f|| < \epsilon \quad \text{for all } n \geq N$$

    Therefore $\lim_{n\to\infty} T_n(f) = T(f)$ uniformly on $\{f \in X \mid ||f|| \leq 1\}$.

    Conversely, suppose $\lim_{n\to\infty} T_n(f) = T(f)$ uniformly on $\{f \in X \mid ||f|| \leq 1\}$. Then for any $\epsilon > 0$, there exists N such that $|T_n(f) - T(f)| < \epsilon$ for all $n \geq N$ and for all $f \in X$ with $||f|| \leq 1$. But this implies

$$||T_n - T||_* = \sup\{(T_n - T)(f) \mid f \in X, ||f|| \leq 1\} \leq \epsilon$$

    for all $n \geq N$. We can therefore conclude that $\{T_n\} \to T$ with respect to the $||\cdot||_*$ norm.

31. Fix $\epsilon \in (0,1)$ and $\delta > 0$. Choose n such that $1/n < \delta$ and let $A = [0, 1/n]$. Then $m(A) < \delta$ but $\int_A f_n = 1 > \epsilon$. Therefore $\{f_n\}$ is not uniformly integrable.

32. $L^\infty(E)$ may not be separable, so Helley's Theorem need not apply.

33. Let q denote the conjugate of p. Let $\{a_n\}$ be a bounded sequence in ℓ^p. For each n, define the functional T_n on ℓ^q by

$$T_n(b) = \sum_{k=1}^{\infty} a_n^k \cdot b^k \text{ for } b \in \ell^q$$

    In the solution to Problem 7, we showed that T_n is a bounded, linear functional on ℓ^q with $||T_n||_* = ||a_n||_p$. Since $\{a_n\}$ is bounded in ℓ^p, $\{T_n\}$ is bounded in $(\ell^q)^*$. By Helley's Theorem, there is a subsequence $\{T_{n_j}\}$ and $T \in (\ell^q)^*$ such that

$$\lim_{j\to\infty} T_{n_j}(b) = T(b) \text{ for all } b \in \ell^q$$

In the solution to Problem 7, we showed that there exists an $a \in \ell^p$ such that

$$T(b) = \sum_{k=1}^{\infty} a^k \cdot b^k \text{ for all } b \in \ell^q$$

But this implies

$$\lim_{j \to \infty} \sum_{k=1}^{\infty} a_{n_j}^k \cdot b^k = \sum_{k=1}^{\infty} a^k \cdot b^k \text{ for all } b \in \ell^q$$

Again applying the Riesz Representation Theorem proven in Problem 7, we can conclude that $\{a_{n_j}\} \rightharpoonup a$ in ℓ^p.

34. Let $X = C[0,1]$. X is a separable space. By the Example on page 156, the functional

$$T_n(g) = \int_0^1 g(x) df_n(x)$$

is in X^* for all n. By assumption, there exists a real number $M \geq 0$ such that $TV(f_n) \leq M$ for all n. We can therefore conclude

$$|T_n(g)| \leq ||g||_{\max} \cdot TV(f_n) \leq M \cdot ||g||_{\max} \text{ for all } g \in X \text{ and all } n$$

But then by Helley's Theorem, there is a subsequence $\{T_{n_k}\}$ of $\{T_n\}$ and $T \in X^*$ for which

$$\lim_{k \to \infty} T_{n_k}(g) = T(g) \text{ for all } g \in X \tag{1}$$

By the Remark on page 161, there exists a function f of bounded variation such that

$$T(g) = \int_0^1 g(x) \cdot df(x) \text{ for all } g \in X$$

The expression in (1) therefore implies

$$\lim_{k \to \infty} \int_0^1 g(x) df_{n_k}(x) = \int_0^1 g(x) df(x) \text{ for all } g \in X$$

Thus $\left\{\int_0^1 g(x) df_{n_k}(x)\right\}$ is Cauchy for each $g \in X$.

35. (i) Assume without loss that $M \geq 1$. Fix $g \in X$ and $\epsilon > 0$. Since $\mathcal{S}$ is dense in X, we can pick $g' \in \mathcal{S}$ such that

$$||g - g'|| < \frac{\epsilon}{3 \cdot M}$$

Since $\{T_n(g')\}$ is Cauchy, we can pick an index N such that

$$|T_n(g') - T_m(g')| < \frac{\epsilon}{3}$$

for all $n, m \geq N$. Then

$$\begin{aligned}|T_n(g) - T_m(g)| &\leq |T_n(g) - T_n(g')| + |T_n(g') - T_m(g')| + |T_m(g) - T_m(g')| \\ &\leq ||T_n||_* \cdot ||g - g'|| + \frac{\epsilon}{3} + ||T_m||_* \cdot ||g - g'|| \\ &\leq 2 \cdot M \cdot ||g - g'|| + \frac{\epsilon}{3} \\ &< \epsilon\end{aligned}$$

for all $n, m \geq N$. Therefore $\{T_n(g)\}$ is Cauchy.

(ii) Pick α, β in $\mathbf{R}$ and g, g' in X. Since T_n is linear for all n, we must have

$$T_n(\alpha \cdot g + \beta \cdot g') = \alpha \cdot T_n(g) + \beta \cdot T_n(g') \text{ for all } n$$

Taking limits, we conclude that

$$T(\alpha \cdot g + \beta \cdot g') = \alpha \cdot T(g) + \beta \cdot T(g')$$

Therefore T is linear. Since each T_n is bounded, we also know that

$$|T_n(g)| \leq ||T_n||_* \cdot ||g|| \leq M \cdot ||g|| \text{ for all } n$$

Taking limits, we conclude that

$$|T(g)| \leq M \cdot ||g||$$

for all n. Therefore T is bounded.

36. Let $f_n = 2^{n+1} \cdot \chi_{(2^{-(n+1)}, 2^{-n}]}$ and define

$$T_n(g) = \int_0^1 g \cdot f_n \text{ for all } g \in L^\infty[0,1]$$

Since

$$|T_n(g)| = \left| \int_0^1 g \cdot f_n \right| \leq ||g||_\infty \cdot \int_0^1 |f_n| = ||g||_\infty$$

the sequence $\{T_n\}$ is bounded. Now suppose there exists a subsequence $\{T_{n_k}\}$ such that $\{T_{n_k}(g)\}$ converges for all $g \in L^\infty[0,1]$. Define the function

$$g_0(x) = \begin{cases} (-1)^k & \text{if } x \in (2^{-(n_k+1)}, 2^{-n_k}] \\ 0 & \text{otherwise} \end{cases}$$

Then $g_0 \in L^\infty[0,1]$, so

$$T_{n_k}(g_0) = (-1)^k \text{ for all } k$$

Clearly $T_{n_k}(g_0)$ does not converge, a contradiction.[1]

37. (i) $\{f_n\} \rightharpoonup f$ in $L^p(E)$ because strong convergence implies weak convergence.

[1]This solution was found at `https://math.stackexchange.com/a/3388752`.

By the Riesz-Fischer Theorem, a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ converges pointwise a.e. on E to f.

By Proposition 3 of Chapter 5 $\{f_{n_k}\}$ also converges in measure to f on E.

(ii) For $p = 1$, we can use the example on page 169 to construct a sequence that converges weakly in $L^p(E)$ but for which no subsequence converges strongly in $L^p(E)$. For $1 < p < \infty$, we can use the example in Problem 13.

Since both sequences in these examples are bounded, no subsequence can converge pointwise a.e. on $L^p(E)$ by the Bounded Convergence Theorem.

Since no subsequence can converge pointwise, no subsequence can converge in measure to f by Theorem 4 of Chapter 5.

(iii) Suppose $f_n = n^{1/p} \cdot \chi_{[0,1/n]}$. Then $f_n \to 0$ pointwise a.e. on $[0,1]$, but $||f_n||_p = 1$ for all n. Therefore no subsequence of $\{f_n\}$ can converge strongly to 0 on $[0,1]$.

For $p = 1$, it was shown on page 173 that the above sequence contains no weakly convergent subsequence. For $1 < p < \infty$, weak convergence follows from Theorem 12.

Convergence in measure on E follows by Proposition 3 of Chapter 5.

(iv) Suppose $f_n = n^{1/p} \cdot \chi_{[0,1/n]}$ and $f = 0$. By Proposition 3 of Chapter 5, $\{f_n\}$ converges in measure to f. However $||f_n||_p = 1$ for all n, so no subsequence of $\{f_n\}$ converges in $L^p[0,1]$ to 0.

For $p = 1$, it was shown on page 173 that the above sequence contains no weakly convergent subsequence. For $1 < p < \infty$, weak convergence of a subsequence follows from Theorem 4 of Chapter 5 and by Theorem 12.

A subsequence converges pointwise a.e. on E to f by Theorem 4 of Chapter 5.

8.4 The Minimization of Convex Functionals

38. In Problem 21, it was shown that $\{e_n\} \rightharpoonup 0$ in ℓ^p but no subsequence converges strongly to 0 in ℓ^p for $1 < p < \infty$.

Let $1 < q < \infty$ denote the conjugate of p. Then

$$\left|\left|\frac{1}{N}\sum_{n=1}^{N} e_n\right|\right|_p = \frac{1}{N^{1/q}} \to 0$$

Therefore the arithmetic means of $\{e_n\}$ converge strongly to 0 in ℓ^p.

39. By replacing each a_n with $a_n - a$, we can assume without loss that $a = 0$. Fix $\epsilon > 0$ and pick N such that $|a_n| < \epsilon$ for all $n \geq N$. Let $A = \left|\sum_{n=1}^{N} a_n\right|$. Then

$$\left|\frac{1}{n}\sum_{i=1}^{n} a_n\right| \leq \frac{A}{n} + \frac{1}{n}\sum_{i=N+1}^{n} |a_n| < \frac{A}{n} + \epsilon$$

for all $n \geq N$. But this expression implies

$$\limsup_{n\to\infty} \left| \frac{1}{n} \sum_{i=1}^{n} a_n \right| \leq \epsilon$$

Since ϵ can be made arbitrarily small, we can conclude

$$\lim_{n\to\infty} \left| \frac{1}{n} \sum_{i=1}^{n} a_n \right| = 0$$

40. **The Banach-Saks Theorem for ℓ^2**: Suppose $\{a_n\} \rightharpoonup a$ in ℓ^2. Then there is a subsequence $\{a_{n_k}\}$ for which the sequence of arithematic means converges strongly to a in ℓ^2.

    Proof: By simply replacing integrals with infinite series, the proof can proceed in the same way as the proof for the Banach-Saks Theorem given in the text.

41. By the Example on page 176, K is a closed, bounded, convex subset of $L^p[a,b]$. Linear functionals are also convex. Therefore by Theorem 17, there exists a minimizing function $f_0 \in K$. Since $f \in K$ implies $-f \in K$, we must also have

    $$T(f) = -T(-f) \leq -T(f_0) \text{ for all } f \in K$$

    Thus $|T(f)| \leq |T(f_0)|$ for all $f \in L^p[a,b]$. But this implies

    $$\frac{|T(f)|}{||f||_p} = \left| T\left(\frac{f}{||f||_p}\right) \right| \leq |T(f_0)|$$

    for any non-zero f in K. Therefore

    $$|T(f)| \leq |T(f_0)| \cdot ||f||_p \text{ for all } f \in L^p[0,1]$$

    We conclude that T is bounded. By the Riesz Representation Theorem, there exists $g \in L^q[a,b]$ such that

    $$T(f) = \int_a^b f \cdot g \text{ for all } f \in L^p[a,b]$$

    By Proposition 2, we know that $T(-g^*) = -||T||_*$ But the characterization of f_0 from Theorem 17 tells us that

    $$\begin{aligned} T(f_0) &= \inf\{T(f) \mid f \in L^p[a,b] \text{ and } ||f||_p \leq 1\} \\ &= -\sup\{T(f) \mid f \in L^p[a,b] \text{ and } ||f||_p \leq 1\} \\ &= -||T||_* \end{aligned}$$

    Therefore $-g^*$ satisfies

    $$T(-g^*) = T(f_0) \leq T(f) \text{ for all } f \in K$$

    Thus $-g^*$ is a minimizer of T on K.

42. We first prove a preliminary uniform integrability result.

Claim. Suppose $\{f_n\} \to f$ in $L^p(E)$ for some $p \in [1, \infty)$. Then $\{|f_n|^p\}$ is uniformly integrable.

Proof: Fix $\epsilon > 0$. Let $A \subseteq E$ be a measurable set. Since

$$|f_n|^p \leq 2^p \cdot (|f_n - f|^p + |f|^p)$$

we have

$$\int_A |f_n|^p \leq 2^p \cdot \int_A |f_n - f|^p + 2^p \cdot \int_A |f|^p$$

Since $|f|^p$ is integrable, there exists $\delta_0 > 0$ such that

$$\text{if } m(A) < \delta_0, \text{ then } \int_A |f|^p < \frac{\epsilon}{2^{p+1}}$$

Since $\{f_n\} \to f$ in $L^p(E)$, there exists an index N such that

$$\int_E |f_n - f|^p < \frac{\epsilon}{2^{p+1}} \text{ for all } n \geq N$$

By Proposition 24 of Chapter 4, there exists $\delta_1 > 0$ such that

$$\text{if } m(A) < \delta_1, \text{ then } \int_A |f_n - f|^p < \frac{\epsilon}{2^{p+1}} \text{ for } 1 \leq n < N$$

Let $\delta = \min\{\delta_0, \delta_1\}$. Combining the above results, we can conclude

$$\text{if } m(A) < \delta, \text{ then } \int_A |f_n|^p < \epsilon \text{ for all } n$$

Suppose $\{f_n\} \to f$ in L^{p_1}. Then $\{||f_n||_{p_1}\} \to ||f||_{p_1} < \infty$, so there exists $M > 0$ such that

$$\int_E |f_n|^{p_1} \leq M \text{ for all } n$$

Since

$$|\varphi \circ f_n|^{p_2} \leq |c_1 + c_2 \cdot |f_n|^{p_1/p_2}|^{p_2} \leq 2^{p_2} \cdot (|c_1|^{p_2} + |c_2|^{p_2} \cdot |f_n|^{p_1}) \tag{1}$$

we know that

$$\begin{aligned}\int_E |\varphi \circ f_n|^{p_2} &\leq 2^{p_2} \cdot \left(|c_1|^{p_2} \cdot m(E) + |c_2|^{p_2} \cdot \int_E |f_n|^{p_1}\right) \\ &\leq 2^{p_2} \cdot (|c_1|^{p_2} \cdot m(E) + |c_2|^{p_2} \cdot M) \\ &< \infty\end{aligned}$$

Therefore $\varphi \circ f_n \in L^{p_2}(E)$ for all n. By the same argument, $\varphi \circ f$ is also in $L^{p_2}(E)$.

We next show that $\{|\varphi \circ f_n - \varphi \circ f|^{p_2}\}$ is uniformly integrable. Fix $\epsilon > 0$ and let A denote a measurable subset of E. Since

$$\begin{aligned} |\varphi \circ f_n - \varphi \circ f|^{p_2} &\leq (|\varphi \circ f_n| + |\varphi \circ f|)^{p_2} \\ &\leq 2^{p_2} \cdot (|\varphi \circ f_n|^{p_2} + |\varphi \circ f|^{p_2}) \\ &\leq 4^{p_2} \cdot (|c_1|^{p_2} + |c_2|^{p_2} \cdot |f_n|^{p_1}) + 2^{p_2} \cdot |\varphi \circ f|^{p_2} \end{aligned}$$

we know that

$$\int_A |\varphi \circ f_n - \varphi \circ f|^{p_2} \leq 4^{p_2} \cdot \left(|c_1|^{p_2} \cdot m(A) + |c_2|^{p_2} \cdot \int_A |f_n|^{p_1} \right) + 2^{p_2} \cdot \int_A |\varphi \circ f|^{p_2}$$

By the above claim, $\{|f_n|^{p_1}\}$ is uniformly integrable. Therefore there exists $\delta_0 > 0$ such that

$$\text{if } m(A) < \delta_0, \text{ then } \int_A |f_n|^{p_1} < \frac{\epsilon}{3 \cdot |4 \cdot c_2|^{p_2}}$$

Since $|\varphi \circ f|^{p_2}$ is integrable, there exists $\delta_1 > 0$ such that

$$\text{if } m(A) < \delta_1, \text{ then } \int_A |\varphi \circ f|^{p_2} < \frac{\epsilon}{3 \cdot 2^{p_2}}$$

Let $\delta = \min\left\{ \frac{\epsilon}{3 \cdot |4 \cdot c_1|^{p_2}}, \delta_0, \delta_1 \right\}$. Then

$$\text{if } m(A) < \delta, \text{ then } \int_A |\varphi \circ f_n - \varphi \circ f|^{p_2} < \epsilon$$

as desired.

We are now ready to show

$$\int_E |\varphi \circ f_n - \varphi \circ f|^{p_2} \to 0$$

Fix $\epsilon > 0$. For any $\kappa > 0$, define

$$E_n(\kappa) = \{x \in E : |f_n(x)| < \kappa \text{ and } |f(x)| < \kappa\}$$

Since φ is continuous on $\mathbf{R}$, φ is uniformly continuous on the closed and bounded interval $[-\kappa, \kappa]$. Therefore there exists $\delta(\kappa) > 0$ such that for all $x \in E_n(\kappa)$,

$$\text{if } |f_n(x) - f(x)| \leq \delta(\kappa), \text{ then } |(\varphi \circ f_n)(x) - (\varphi \circ f)(x)| < \left(\frac{\epsilon}{2 \cdot m(E)} \right)^{1/p_2}$$

Now define

$$A_n(\kappa) = \{x \in E_n(\kappa) : |f_n(x) - f(x)| < \delta(\kappa)\}$$

Then

$$\begin{aligned}\int_E |\varphi \circ f_n - \varphi \circ f|^{p_2} &= \int_{E\sim A_n(\kappa)} |\varphi \circ f_n - \varphi \circ f|^{p_2} + \int_{A_n(\kappa)} |\varphi \circ f_n - \varphi \circ f|^{p_2} \\ &\leq \int_{E\sim A_n(\kappa)} |\varphi \circ f_n - \varphi \circ f|^{p_2} + \frac{m(A_n(\kappa))}{m(E)} \cdot \frac{\epsilon}{2} \\ &\leq \int_{E\sim A_n(\kappa)} |\varphi \circ f_n - \varphi \circ f|^{p_2} + \frac{\epsilon}{2}\end{aligned}$$

for all n and κ. Since $\{|\varphi \circ f_n - \varphi \circ f|^{p_2}\}$ is uniformly integrable, there exists $\eta > 0$ such that

$$\text{if } m(E \sim A_n(\kappa)) < \eta \text{ for all } n \geq N, \text{ then } \int_{E\sim A_n(\kappa)} |\varphi \circ f_n - \varphi \circ f|^{p_2} < \frac{\epsilon}{2} \text{ for all } n \geq N$$

Now

$$\begin{aligned}&m(E \sim A_n(\kappa)) \leq \\ &\qquad m(\{x \in E : |f_n(x)| \geq \kappa\}) + m(\{x \in E : |f(x)| \geq \kappa\}) + m(\{x \in E : |f_n(x) - f(x)| \geq \delta(\kappa)\})\end{aligned}$$

By Markov's Inequality,

$$\begin{aligned}&m(\{x \in E : |f_n| \geq \kappa\}) \leq \frac{1}{\kappa^{p_1}} \cdot \int_E |f_n|^{p_1} \leq \frac{M}{\kappa^{p_1}} \text{ for all } n \\ &m(\{x \in E : |f| \geq \kappa\}) \leq \frac{1}{\kappa^{p_1}} \cdot \int_E |f|^{p_1} \\ &m(\{x \in E : |f_n(x) - f(x)| \geq \delta_n(\kappa)\}) \leq \frac{1}{\delta(\kappa)^{p_1}} \cdot \int_E |f_n(x) - f(x)|^{p_1}\end{aligned}$$

Choose κ_0 such that

$$\kappa_0 > \frac{\max\{M^{1/p_1}, ||f||_{p_1}\}}{(\eta/3)^{1/p_1}}$$

Since $\{f_n\} \to f$ in L^{p_1}, there exists an index N such that

$$\int_E |f_n(x) - f(x)|^{p_1} < \frac{\eta \cdot \delta(\kappa_0)^{p_1}}{3} \text{ for all } n \geq N$$

Thus

$$m(E \sim A_n(\kappa_0)) < \eta \text{ for all } n \geq N$$

so

$$\int_E |\varphi \circ f_n - \varphi \circ f|^{p_2} < \epsilon \text{ for all } n \geq N$$

as desired.

43. Define

$$T(g) = ||g - f_0||_p \text{ for all } g \in L^p(E)$$

Then for any λ in $[0,1]$ and g, h in $L^p(E)$, we have

$$\begin{aligned} T(\lambda \cdot g + (1-\lambda)\cdot h) &= ||\lambda \cdot g + (1-\lambda)\cdot h||_p \\ &\leq ||\lambda \cdot g||_p + ||(1-\lambda)\cdot h||_p \\ &\leq \lambda \cdot ||g||_p + (1-\lambda)\cdot ||h||_p \\ &= \lambda \cdot T(g) + (1-\lambda)\cdot T(h) \end{aligned}$$

Therefore T is convex. Now suppose $\{g_n\} \to g$ in $L^p(E)$. Then

$$\begin{aligned} |T(g_n) - T(g)| &= \big| ||g_n - f_0||_p - ||g - f_0||_p \big| \\ &\leq ||g_n - g||_p \\ &\to 0 \end{aligned}$$

Therefore T is continuous. By Theorem 17, there exists $g_0 \in C$ such that

$$||g_0 - f_0||_p = T(g_0) \leq T(g) = ||g - f_0||_p \text{ for all } g \in C$$

44. (i) By direct computation, it is straight-forward to show that

$$\int_0^x f_n = \begin{cases} \left(x - \dfrac{k}{2^n}\right)\cdot\left(1 - 2^{n+1}\right) & \text{if } \dfrac{k}{2^n} \leq x < \dfrac{k}{2^n} + \dfrac{1}{2^{2n+1}} \\ x - \dfrac{k+1}{2^n} & \text{if } \dfrac{k}{2^n} + \dfrac{1}{2^{2n+1}} \leq x < \dfrac{k+1}{2^n} \end{cases}$$

for $0 \leq k \leq 2^n - 1$. Therefore

$$\left| \int_0^x f_n \right| \leq \frac{1}{2^n} - \frac{1}{2^{2n+1}} \leq \frac{1}{2^n} \text{ for all } x \in [0,1] \text{ and all } n$$

which implies

$$\lim_{n\to\infty} \int_0^x f_n = 0 = \int_0^x f \text{ for all } x \in [0,1]$$

(ii) Let E_n denote the set of values for which $f_n = 1$ on E. Then

$$m(E_n) = 2^n \cdot \left(\frac{1}{2^n} - \frac{1}{2^{2n+1}}\right) = 1 - \frac{1}{2^{n+1}}$$

From the Fréchet Inequality, we know that

$$m\left(\bigcap_{n=1}^N E_n\right) \geq n - \sum_{n=1}^N \frac{1}{2^{n+1}} - (n-1) = \frac{1}{2} + \left(\frac{1}{2}\right)^{N+1}$$

Therefore

$$m(E) = \lim_{N\to\infty} m\left(\bigcap_{n=1}^N E_n\right) \geq \frac{1}{2}$$

(iii) Direct computation reveals that

$$||f_n||_1 = 2 \cdot \left(1 - \frac{1}{2^{n+1}}\right) \leq 2 \text{ for all } n$$

Therefore $\{||f_n||_1\}$ is a bounded sequence. Since E is measurable set and

$$\lim_{n\to\infty} \int_E f_n = m(E) > 0 = \int_E f$$

we know that $\{f_n\}$ cannot converge weakly to f in $L^1[0,1]$ by Theorem 10. Theorem 11 is not violated because it does not apply when $p = 1$.

(iv) As in part (iii), $\{f_n\}$ cannot converge weakly to f by Theorem 10.

Now by direct computation, we have

$$\begin{aligned} ||f_n||_p &= \left(1 - \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} \cdot (2^{n+1} - 1)^p\right)^{1/p} \\ &\geq \frac{2^{n+1} - 1}{2^{(n+1)/p}} \\ &\geq 2^{(n+1)/q} \end{aligned}$$

where q denotes the conjugate of p. Since $2^{(n+1)/q}$ is unbounded, $\{||f_n||_p\}$ is not a bounded sequence. Therefore Theorem 11 need not apply.

45. Let $\{g_n\}$ denote the sequence of Radamacher functions defined in the example on page 163. It was shown in that example that $\{g_n\}$ is in $L^2[0,1]$ but does not contain any Cauchy subsequence. Fix $\epsilon > 0$. By Theorem 12 of Chapter 7, for each n there exists a continuous real-valued function f_n on $[0,1]$ such that

$$||f_n - g_n||_2 \leq \epsilon$$

Let $\{f_{n_k}\}$ denote a subsequence of $\{f_n\}$. Fix an index N. Since $\{g_{n_k}\}$ is not Cauchy, there exists $n_k, n_j \geq N$ such that

$$||g_{n_j} - g_{n_k}||_2 > 3 \cdot \epsilon$$

But by the Triangle Inequality, we have

$$\begin{aligned} ||f_{n_k} - f_{n_j}||_2 &\geq ||f_{n_k} - g_{n_j}||_2 - ||f_{n_j} - g_{n_j}||_2 \\ &\geq ||g_{n_j} - g_{n_k}||_2 - ||f_{n_k} - g_{n_k}||_2 - ||f_{n_j} - g_{n_j}||_2 \\ &> \epsilon \end{aligned}$$

Therefore $\{f_{n_k}\}$ is not Cauchy. We conclude that $\{f_n\}$ does not contain a Cauchy subsequence.