《偏微分方程》**部分**习题参考答案

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写在前面

北师大数科院的《偏微分方程》课程通常在大三上学期开设，部分教师会使用保继光、朱汝金编著的《偏微分方程》教材，此教材习题数量较多，且部分题目难度较大，又缺少官方的参考答案，**本人在学习过程中遇到了不少困难**，因此在该课程结束后，**决定将做过的习题的答案整理出来**，供师弟师妹们参考.

此参考答案中的习题主要来自于**2024-2025学年秋季学期**的**课后作业**，仅涵盖了第三、四、五章的一部分课后习题，部分解答由我自行给出，部分来自于习题课上助教师兄/师姐所讲内容. 由于本人水平有限，并且本文完全由个人整理，没有后期校对，**难免存在错误**（尤其是涉及到计算的部分），所以**仅供参考**.

本文**完全开源**，latex源代码的地址为https://cn.overleaf.com/4468797959rpxvhqcsfstv#e98dd8，可以直接编辑，**欢迎大家修订、补充**，但请勿用于盈利.

1　第三章 双曲型方程

本章部分课后习题计算量较大，但本人计算能力较差，尽管整理答案时又算了一遍，仍不能保证百分百正确，计算结果仅供参考（强调）.

1.

（1）直接由d'Alembert公式可得，$u(x,t)=\frac{1}{2}(\sin\pi(x+t)+\sin\pi(x-t))$.

（2）直接由d'Alembert公式可得，$u(x,t)=\frac{1}{2}\left(\mathrm{e}^{-(x-t)^2}+\mathrm{e}^{-(x+t)^2}\right)$.

（3）直接由d'Alembert公式可得，$u(x,t)=\frac{1}{2}\left[\ln(1+(x-at)^2)+\ln(1+(x+at)^2)\right]+2t$.

（5）由叠加原理得，$u(x,t)=V^{(1)}(x,t)+V^{(2)}(x,t)$，其中$V^{(1)}$，$V^{(2)}$分别满足

$$\begin{cases}\frac{\partial^2 V^{(1)}}{\partial t^2}-a^2\frac{\partial^2 V^{(1)}}{\partial x^2}=x\\ V^{(1)}(x,0)=0\\ \frac{\partial V^{(1)}}{\partial t}(x,0)=0\end{cases}$$

$$\begin{cases}\frac{\partial^2 V^{(2)}}{\partial t^2}-a^2\frac{\partial^2 V^{(2)}}{\partial x^2}=0\\ V^{(2)}(x,0)=0\\ \frac{\partial V^{(2)}}{\partial t}(x,0)=3\end{cases}$$

对$V^{(1)}$，考虑如下方程:

$$\begin{cases}\frac{\partial^2 W}{\partial t^2}-a^2\frac{\partial^2 W}{\partial x^2}=0, \quad t>\tau, x\in\mathbb{R}\\ W(x,\tau)=0, \quad x\in\mathbb{R}\\ \frac{\partial W}{\partial t}(x,\tau)=x, \quad x\in\mathbb{R}\end{cases}$$

由d'Alembert公式得$W(x,t;\tau)=x(t-\tau)$，从而$V^{(1)}(x,t)=\int_0^t x(t-\tau)\mathrm{d}\tau=\frac{xt^2}{2}$，对于$V^{(2)}$，直接由d'Alembert公式得$V^{(2)}(x,t)=3t$. 综上，$u(x,t)=3t+\frac{xt^2}{2}$.

（7）由叠加原理得，$u(x,t)=V^{(1)}(x,t)+V^{(2)}(x,t)$，其中$V^{(1)}$，$V^{(2)}$分别满足

$$\begin{cases}\frac{\partial^2 V^{(1)}}{\partial t^2}-a^2\frac{\partial^2 V^{(1)}}{\partial x^2}=\mathrm{e}^x\\ V^{(1)}(x,0)=0\\ \frac{\partial V^{(1)}}{\partial t}(x,0)=0\end{cases}$$

$$\begin{cases}\frac{\partial^2 V^{(2)}}{\partial t^2}-a^2\frac{\partial^2 V^{(2)}}{\partial x^2}=0\\ V^{(2)}(x,0)=5\\ \frac{\partial V^{(2)}}{\partial t}(x,0)=x^2\end{cases}$$

对$V^{(1)}$，考虑如下方程:

$$\begin{cases}\frac{\partial^2 W}{\partial t^2}-a^2\frac{\partial^2 W}{\partial x^2}=0, \quad t>\tau, x\in\mathbb{R}\\ W(x,\tau)=0, \quad x\in\mathbb{R}\\ \frac{\partial W}{\partial t}(x,\tau)=\mathrm{e}^x, \quad x\in\mathbb{R}\end{cases}$$

第（5）题同理得$V^{(1)}(x,t)=\int_0^t W(x,t;\tau)\mathrm{d}\tau$，$V^{(2)}$可由d'Alembert公式直接求出，综上有$u(x,t)=\frac{1}{2a^2}(\mathrm{e}^{x-at}+\mathrm{e}^{x+at})-\frac{\mathrm{e}^x}{a^2}+\frac{a^2t^3}{3}+x^2t+5$

2. 原方程的古典解形如$u(x,t) = f(x-at)+g(x+at)$，不妨设$d_1 < d_2$，$c_1 < c_2$，$P_i = (x_i, t_i) \in \mathbb{R}^2$，且$P_1$是$x+at=c_2$和$x-at=d_2$的交点，则有

$$u(P_1) = f(x_1-at_1)+g(x_1+at_1) = f(d_2)+g(c_2),$$

$$u(P_2) = f(x_2-at_2)+g(x_2+at_2) = f(d_1)+g(c_2),$$

$$u(P_3) = f(x_3-at_3)+g(x_3+at_3) = f(d_1)+g(c_1),$$

$$u(P_4) = f(x_4-at_4)+g(x_4+at_4) = f(d_2)+g(c_1).$$

所以$u(P_1)+u(P_3) = f(d_1)+f(d_2)+g(c_1)+g(c_2) = u(P_2)+u(P_4)$.

3. 设$u(x,y) = f(x)+g(y)$，则代入定解条件可知

$$\begin{cases} u(1,y) = f(1)+g(y) = \cos y \\ u(x,0) = f(x)+g(0) = x^2 \end{cases}$$

第一式中令$y=0$或第二式中令$x=1$可知$f(1)+g(0)=1$.所以两式相加可得

$$f(x)+g(y)+f(1)+g(0) = \cos y + x^2.$$

因此有$u(x,y) = f(x)+g(y) = \cos y + x^2 - 1$.

4. $a_{11} = a_{12} = 1$，$a_{22} = -3$，得到特征方程为$\mathrm{d}y^2 - 2\mathrm{d}x\mathrm{d}y - 3\mathrm{d}x^2 = 0$，求解得到两组特征曲线

$$3x - y = c_1,$$

$$x + y = c_2.$$

令$\xi = 3x-y$，$\eta = x+y$，则有

$$u_x = 3u_\xi + u_\eta,$$

$$u_y = -u_\xi + u_\eta,$$

$$u_{xx} = 9u_{\xi\xi} + 6u_{\xi\eta} + u_{\eta\eta},$$

$$u_{xy} = -3u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta},$$

$$u_{yy} = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}.$$

从而原方程可化简为$u_{\xi\eta} = 0$，即$u(x,y) = f(\xi)+g(\eta) = f(3x-y)+g(x+y)$，根据定解条件可以得到

$$\begin{cases} u(x,0) = f(3x)+g(x) = \sin x \\ u_y(x,0) = -f'(3x)+g'(x) = x \end{cases}$$

再对第二式积分得到

$$\begin{cases} f(3x)+g(x) = \sin x \\ -\frac{1}{3}f(3x)+g(x) = \frac{x^2}{2}+c \end{cases}$$

解得

$$\begin{cases} f(x)=\frac{3}{4}\sin\frac{x}{3}-\frac{1}{24}x^2-\frac{3c}{4} \\ g(x)=\frac{1}{4}\sin x+\frac{3}{8}x^2+\frac{3c}{4} \end{cases}$$

于是原方程的解为$u(x,y)=\frac{3}{4}\sin(x-\frac{y}{3})+\frac{1}{4}\sin(x+y)-\frac{1}{24}(3x-y)^2+\frac{3}{8}(x+y)^2$.

5. 设$u(x,t)=f(x-at)+g(x+at)$，则由定解条件得

$$\begin{cases} u(at,t)=f(0)+g(2at)=\varphi(t) \\ u(-at,t)=f(-2at)+g(0)=\psi(t) \end{cases}$$

即：

$$\begin{cases} f(t)=\psi(-\frac{t}{2a})-g(0) \\ g(t)=\varphi(\frac{t}{2a})-f(0) \end{cases} \tag{1}$$

从而$u(x,t)=\varphi(\frac{x+at}{2a})+\psi(\frac{at-x}{2a})-f(0)-g(0)$. 在(1)中令$t=0$可知$f(0)+g(0)=\varphi(0)=\psi(0)$，故$u(x,t)=\varphi(\frac{x+at}{2a})+\psi(\frac{at-x}{2a})-\psi(0)$.

6. 令$y=\sqrt{t}$，则

$$u_t=\frac{1}{2\sqrt{t}}u_y=\frac{1}{2y}u_y$$

$$u_{tt}=-\frac{1}{4y^3}u_y+\frac{1}{4y^2}u_{yy}$$

所以原方程可化为$u_{yy}-a^2u_{xx}=0$，所以有$u(x,t)=f(x-ay)+g(x+ay)=f(x-a\sqrt{t})+g(x+a\sqrt{t})$.
根据初始条件得，$u(x,0)=f(x)+g(x)=x^2$. 又因为$u_t=\frac{a}{2\sqrt{t}}(-f'(x-a\sqrt{t})+g'(x+a\sqrt{t}))$，
所以$u_t(x,0)=\lim\limits_{t\to 0^+}\dfrac{a}{2\sqrt{t}}(-f'(x-a\sqrt{t})+g'(x+a\sqrt{t}))$.
因为$u_t(x,0)$有界且$\lim\limits_{t\to 0+}\dfrac{a}{2\sqrt{t}}=+\infty$，所以一定有$-f'(x)+g'(x)=0$，即$f(x)-g(x)=c$. 综上，

$$\begin{cases} f(x)+g(x)=x^2 \\ f(x)-g(x)=c \end{cases}$$

解得

$$\begin{cases} f(x)=\frac{1}{2}x^2+\frac{c}{2} \\ g(x)=\frac{1}{2}x^2-\frac{c}{2} \end{cases}$$

所以有$u(x,t)=x^2+a^2t$.

7. 设$u(x,t)=F(x-t)+G(x+t)$，代入定解条件可得

$$u(x,x)=F(0)+G(2x)=\varphi(x)$$

$$u(x,f(x))=F(x-f(x))+G(x+f(x))$$

整理得到

$$\begin{cases} G(x)=\varphi(\frac{x}{2})-F(0) \\ F(x-f(x))=\psi(x)-G(x+f(x))=\psi(x)-\varphi(\frac{x+f(x)}{2})+F(0) \end{cases}$$

余下只需求$F(x)$，记$h(x)=x-f(x)$，由于$t=f(x)$是位于$x=-t$和$x=t$之间的光滑曲线，所以h也光滑，又因为$f'(x)\neq 1$，所以$h'(x)\neq 0$，$\forall x\in(-t,t)$，故h的反函数h^{-1}存在. 于是可以得到

$$F(x)=\psi(h^{-1}(x))-\varphi(\frac{h^{-1}(x)+f(h^{-1}(x))}{2})+F(0)$$

所以$u(x,t)=\psi(h^{-1}(x-t))-\varphi(\frac{h^{-1}(x-t)+f(h^{-1}(x-t))}{2})+\varphi(\frac{x+t}{2})$.

8. 设$u(x,t)=f(x-at)+g(x+at)$，由定解条件得，

$$\begin{cases} u(0,t)=f(-t)+g(t)=A\sin\frac{\omega t}{a}, & t\geq 0 \\ u(x,0)=f(x)+g(x)=0, & x\geq 0 \\ u_t(x,0)=-af'(x)+ag'(x)=0, & x\geq 0 \end{cases}$$

(1) 当$x\geq 0$时，由上面第二、三式得$f(x)=-\frac{c}{2}$，$g(x)=\frac{c}{2}$，即$u(x,t)=0$

(2) 当$0\leq x<at$时，有$at-x>0$，由上面第一式得$f(x-at)+g(at-x)=A\sin\frac{\omega(at-x)}{a}$，从而

$$\begin{aligned} u(x,t)&=f(x-at)+g(at-x)-g(at-x)+g(x+at) \\ &=A\sin\frac{\omega(at-x)}{a}-\frac{c}{2}+\frac{c}{2} \\ &=A\sin\frac{\omega(at-x)}{a} \end{aligned}$$

10. 设$u(x,t)=f(x-at)+g(x+at)$，当$x\geq at$时，解可由d'Alembert公式直接求出，所以只需求$x<at$时的解. 代入边界条件得$f'(-at)+g'(at)+bf(-at)+bg(at)=0$，即

$$g'(x)+bg(x)=-f'(-x)-bf(-x)\ \ (x\geq 0).$$

注意到$g'(x)+bg(x)=\mathrm{e}^{-bx}[\mathrm{e}^{bx}g(x)]'$，$-f'(-x)-bf(-x)=\mathrm{e}^{bx}[\mathrm{e}^{-bx}f(-x)]'$，所以有

$$\mathrm{e}^{-bx}[\mathrm{e}^{bx}g(x)]'=\mathrm{e}^{bx}[\mathrm{e}^{-bx}f(-x)]'$$

整理后两边积分得

$$f(-x)=\mathrm{e}^{bx}\left(\int_0^x \mathrm{e}^{-2bs}(\mathrm{e}^{bs}g(s))'\mathrm{d}s+C\right)\ \ (x>0) \tag{2}$$

再根据d'Alembert公式可知当$x>0$时有

$$f(x)=\frac{1}{2}\varphi(x)-\frac{1}{2a}\int_0^x\psi(s)\mathrm{d}s-\frac{c}{2}$$

$$g(x)=\frac{1}{2}\varphi(x)+\frac{1}{2a}\int_0^x\psi(s)\mathrm{d}s+\frac{c}{2}$$

其中$c=g(0)-f(0)$，对(2)式分部积分然后将g的表达式代入(2)式即得当$x>0$时有

$$\begin{aligned} f(-x)&=\mathrm{e}^{bx}\left[\mathrm{e}^{-bx}g(x)-g(0)+2b\int_0^x\mathrm{e}^{-bt}g(t)\mathrm{d}t+C\right] \\ &=\mathrm{e}^{bx}\left[\frac{1}{2}\mathrm{e}^{-bx}\left(\varphi(x)+\frac{1}{a}\int_0^x\psi(s)\mathrm{d}s+c\right)-\frac{1}{2}\varphi(0)-\frac{c}{2}+b\int_0^x\mathrm{e}^{-bt}\left(\varphi(t)+\frac{1}{a}\int_0^t\psi(s)\mathrm{d}s+c\right)\mathrm{d}t+C\right] \end{aligned}$$

即

$$
f(x)=\begin{cases}\frac{1}{2}\varphi(x)-\frac{1}{2a}\int_0^x\psi(s)\mathrm{d}s-\frac{c}{2},\ \ x\geq 0\\ \mathrm{e}^{-bx}\left[\frac{1}{2}\mathrm{e}^{bx}\left(\varphi(-x)+\frac{1}{a}\int_0^{-x}\psi(s)\mathrm{d}s+c\right)-\frac{1}{2}\varphi(0)-\frac{c}{2}+b\int_0^{-x}\mathrm{e}^{-bt}\left(\varphi(t)+\frac{1}{a}\int_0^t\psi(s)\mathrm{d}s+c\right)\mathrm{d}t+C\right],\ \ x<0\end{cases}
$$

代入$u(x,t)=f(x-at)+g(x+at)$即得$x<at$时的形式解，余下只需证明上述形式解在相容条件下是古典解，即证明f在原点处二阶连续可导，即证$f_-(0)=f(0)$，$f_-'(0)=f'(0)$和$f_-''(0)=f''(0)$在相容条件下成立，求导后代入相容条件不难验证.

15. (3)代入课本公式可得

$$
u(x,y,z,t)=\frac{\partial}{\partial t}\left[\frac{1}{4\pi a^2t}\int_{S_{at}(\boldsymbol{x})}(y_1^3+y_2^2y_3)\mathrm{d}\sigma_{\mathrm{at}}\right]
$$

令$\boldsymbol{\xi}=\boldsymbol{y}-\boldsymbol{x}$即$(\xi,\eta,\zeta)=(y_1-x,y_2-y,y_3-z)$得

$$
u(x,y,z,t)=\frac{\partial}{\partial t}\left[\frac{1}{4\pi a^2t}\int_{S_{at}(\mathbf{0})}(x+\xi)^3+(y+\eta)^2(z+\zeta)\mathrm{d}\sigma_{\mathrm{at}}\right]
$$

令

$$
\begin{cases}\xi=at\cos\theta\sin\varphi\\ \eta=at\sin\theta\sin\varphi\\ \zeta=at\cos\varphi\end{cases}
$$

再代入上式得

$$
\begin{aligned}
u(x,y,z,t)&=\frac{\partial}{\partial t}\left[\frac{1}{4\pi a^2t}\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi}\left((x+at\cos\theta\sin\varphi)^3+(y+at\sin\theta\sin\varphi)^2(z+at\cos\varphi)\right)a^2t^2\sin\varphi\mathrm{d}\varphi\right]\\
&=x^3+y^2z+3a^2t^2x+a^2t^2z
\end{aligned}
$$

(8)代入课本公式得

$$
u(x,y,z,t)=\frac{1}{4\pi a^2}\int_{B_{at}(\boldsymbol{x})}\frac{(y_1^2+y_2^2+y_3^2)\mathrm{e}^{t-a^{-1}|\boldsymbol{y}-\boldsymbol{x}|}}{|\boldsymbol{y}-\boldsymbol{x}|}\mathrm{d}\boldsymbol{y}
$$

令$\boldsymbol{\xi}=\boldsymbol{y}-\boldsymbol{x}$，以及$\xi=r\cos\theta\sin\varphi$，$\eta=r\sin\theta\sin\varphi$，$\zeta=r\cos\varphi$ 代入原式得

$$
\begin{aligned}
u(x,y,z,t)&=\frac{1}{4\pi a^2}\int_0^{at}\mathrm{d}r\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi}\frac{(r^2+x^2+y^2+z^2)\mathrm{e}^{t-a^{-1}r}}{r}r^2\sin\varphi\mathrm{d}\varphi\\
&=a^2(6\mathrm{e}^t-t^3-3t^2-6t-6)+(x^2+y^2+z^2)(\mathrm{e}^t-t-1)
\end{aligned}
$$

（也可利用第16题结论）

16. 令$w(r,t)=ru(r,t)$，计算得$w_r=u+ru_r$，$w_{rr}=2u_r+ru_{rr}$，以及

$$
u_x=u_rr_x=u_r\frac{x}{r}
$$

$$
u_{xx}=u_r(\frac{1}{r}-\frac{x^2}{r^3})+\frac{x^2}{r^2}u_{rr}
$$

同理可求得u_{yy},u_{zz}，于是得到

$$\Delta u=u_{xx}+u_{yy}+u_{zz}=\frac{2}{r}u_r+u_{rr}=\frac{1}{r}w_{rr}$$

所以原问题化为

$$\begin{cases}w_{tt}-a^2u_{rr}=0, & r>0,t>0\\ w(r,0)=r\varphi(r), & r\geq 0\\ w_t(r,0)=r\psi(r), & r\geq 0\end{cases}$$

根据初始条件，可以对$r\varphi(r)$和$r\psi(r)$分别在$\mathbb{R}$上做奇延拓，得到

$$\Phi(x)=\begin{cases}r\varphi(r), & r\geq 0\\ r\varphi(-r), & r<0\end{cases}$$

$$\Psi(r)=\begin{cases}r\psi(r), & r\geq 0\\ r\psi(-r), & r<0\end{cases}$$

根据d'Alembert公式得$w(r,t)=\frac{1}{2}\left[\Phi(r+at)+\Phi(r-at)\right]+\frac{1}{2a}\int_{r-at}^{r+at}\Psi(s)\mathrm{d}s$（或者不做奇延拓，直接对$r\geq at,r<at$分情况讨论，即课本38页步骤），即

$$w(r,t)=\begin{cases}\frac{1}{2}\left[(r-at)\varphi(r-at)+(r+at)\varphi(r+at)\right]+\frac{1}{2a}\int_{r-at}^{r+at}\rho\psi(\rho)\mathrm{d}\rho, & r\geq at\\ \frac{1}{2}\left[(r-at)\varphi(at-r)+(r+at)\varphi(r+at)\right]+\frac{1}{2a}\int_{at-r}^{r+at}\rho\psi(\rho)\mathrm{d}\rho, & r\geq at\end{cases}$$

再根据$u=\frac{w}{r}$即得结论.（课本原题结论有误，只考虑到了$r\geq at$的情况）

17. 令$v(x,y,z,t)=\mathrm{e}^{\frac{c}{a}z}u(x,y,t)$，则$u=\mathrm{e}^{-\frac{c}{a}z}v$，且原问题化为

$$\begin{cases}v_{tt}-a^2\Delta v=0, & (x,y,z)\in\mathbb{R}^3,t>0\\ v(x,y,z,0)=\varphi(x,y)\mathrm{e}^{\frac{c}{a}z}, & (x,y,z)\in\mathbb{R}^3\\ v_t(x,y,z,0)=\psi(x,y)\mathrm{e}^{\frac{c}{a}z}, & (x,y,z)\in\mathbb{R}^3\end{cases}$$

根据书上公式解得

$$v(x,y,z,t)=\frac{\partial}{\partial t}\left[\frac{1}{4\pi a^2t}\int_{S_{at}(x,y,z)}\varphi(\xi,\eta)\mathrm{e}^{\frac{c}{a}\zeta}\mathrm{d}\sigma_{at}\right]+\frac{1}{4\pi a^2t}\int_{S_{at}(x,y,z)}\psi(\xi,\eta)\mathrm{e}^{\frac{c}{a}\zeta}\mathrm{d}\sigma_{at}$$

又因为$u(x,y,t)$与z取值无关，所以不妨令$z=0$，得

$$u(x,y,t)=\frac{\partial}{\partial t}\left[\frac{1}{4\pi a^2t}\int_{S_{at}(x,y,0)}\varphi(\xi,\eta)\mathrm{e}^{\frac{c}{a}\zeta}\mathrm{d}\sigma_{at}\right]+\frac{1}{4\pi a^2t}\int_{S_{at}(x,y,0)}\psi(\xi,\eta)\mathrm{e}^{\frac{c}{a}\zeta}\mathrm{d}\sigma_{at}$$

18. (2)根据书上公式有$u(x,y,t)=\frac{1}{2\pi a}\int_{B_{at}(0)}\frac{\xi+\eta+x+y}{\sqrt{a^2t^2-(\xi^2+\eta^2)}}\mathrm{d}\xi\mathrm{d}\eta$，注意到$\frac{\xi+\eta}{\sqrt{a^2t^2-(\xi^2+\eta^2)}}$是奇函数，所以有

$$\begin{aligned}u(x,y,t)&=\frac{1}{2\pi a}\int_{B_{at}(0)}\frac{x+y}{\sqrt{a^2t^2-(\xi^2+\eta^2)}}\mathrm{d}\xi\mathrm{d}\eta\\&=\frac{x+y}{2\pi a}\int_0^{2\pi}\mathrm{d}\theta\int_0^{at}\frac{r}{\sqrt{a^2t^2-r^2}}\mathrm{d}r\\&=(x+y)t\end{aligned}$$

(7)**法一**： 根据书上公式有

$$
\begin{aligned}
u(x,y,t)=&\frac{\partial}{\partial t}\left[\frac{1}{2\pi}\int_{B_t(x,y)}\frac{2y_1^2-y_2^2}{\sqrt{t^2-[(y_1-x)^2+(y_2-y)^2]}}\mathrm{d}y_1\mathrm{d}y_2\right]+\frac{1}{2\pi}\int_{B_t(x,y)}\frac{2y_1^2-y_2^2}{\sqrt{t^2-[(y_1-x)^2+(y_2-y)^2]}}\mathrm{d}y_1\mathrm{d}y_2+\\
&\frac{1}{2\pi}\int_0^t\mathrm{d}r\int_{B_r(x,y)}\frac{(t-r)\sin y_2}{\sqrt{r^2-[(y_1-x)^2+(y_2-y)^2]}}\mathrm{d}y_1\mathrm{d}y_2\\
=&I_1'(t)+I_1(t)+I_2(t)
\end{aligned}
$$

其中

$$
\begin{aligned}
I_1(t)&=\frac{1}{2\pi}\int_{B_t(0)}\frac{2(\xi+x)^2-(\eta+y)^2}{\sqrt{t^2-(\xi^2+\eta^2)}}\mathrm{d}\xi\mathrm{d}\eta\\
&=\frac{1}{2\pi}\int_0^t\frac{r\mathrm{d}r}{\sqrt{t^2-r^2}}\int_0^{2\pi}[2(r\cos\theta+x)^2-(r\sin\theta+y)^2]\mathrm{d}\theta\\
&=\frac{1}{2\pi}\left[2\pi(2x^2-y^2)\int_0^t\frac{r\mathrm{d}r}{\sqrt{t^2-r^2}}\int_0^{2\pi}\frac{r^3}{2\sqrt{t^2-r^2}}\mathrm{d}r\right]\\
&=(2x^2-y^2)t+\frac{1}{3\pi}t^3
\end{aligned}
$$

从而$I_1'(t)=2x^2-y^2+\frac{1}{\pi}t^2$，下面只需计算$I_2(t)$

$$
\begin{aligned}
2\pi I_2&=\int_0^t\mathrm{d}r\int_{B_r(0)}\frac{(t-r)\sin(\eta+y)}{\sqrt{r^2-(\xi^2+\eta^2)}}\mathrm{d}\xi\mathrm{d}\eta\\
&=\int_0^t(t-r)I_3\mathrm{d}r
\end{aligned}
$$

其中

$$
\begin{aligned}
I_3(t)&=\int_{-r}^r\sin(\eta+y)\mathrm{d}\eta\int_{-\sqrt{r^2-\eta^2}}^{\sqrt{r^2-\eta^2}}\frac{1}{\sqrt{r^2-(\xi^2+\eta^2)}}\mathrm{d}\eta\\
&=\pi\int_{-r}^r\sin(\eta+y)\mathrm{d}\eta\\
&=\pi[\cos(r-y)-\cos(r-y)]
\end{aligned}
$$

于是得到$I_2(t)=t\sin y-\sin t\sin y$

法二：

根据叠加原理，容易得到下面命题

命题 1. 对于二维波动方程得$Cauchy$问题

$$
\begin{cases}
u_{tt}-a^2\Delta u=f_1(x)+g_1(y)\\
u(x,y,0)=f_2(x)+g_2(y)\\
u_t(x,y,0)=f_3(x)+g_3(y)
\end{cases},
$$

其古典解可以表示为下面两个一维波动方程$Cauchy$问题古典解之和

$$
\begin{cases}
v_{tt}-a^2v_{xx}=f_1(x)\\
v(x,0)=f_2(x)\\
v_t(x,0)=f_3(x)
\end{cases},
$$

$$\begin{cases} w_{tt} - a^2 w_{yy} = g_1(y) \\ w(y,0) = g_2(y) \\ w_t(y, 0 = g_3(y)) \end{cases}.$$

根据上面命题，只需分别计算两个一维波动方程Cauchy问题即可

19. 考虑一维波动方程Cauchy问题

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0 \\ u(x,0) = \varphi(x) \\ u_t(x,0) = \psi(x) \end{cases} \tag{3}$$

和三维波动方程的Cauchy问题

$$\begin{cases} u_{tt} - a^2 \Delta u = 0 \\ u(x,y,z,0) = \varphi(x) \\ u_t(x,y,z,0) = \psi(x) \end{cases} \tag{4}$$

容易得到问题(4)的古典解为

$$u(x,y,z,t) = \frac{\partial}{\partial t}\left[\frac{1}{4\pi a^2 t}\int_{S_{at}(x,y,z)} \varphi(\xi)\mathrm{d}\xi\right] + \frac{1}{4\pi a^2 t}\int_{S_{at}(x,y,z)} \psi(\xi)\mathrm{d}\xi$$

注意到问题(3)的解$u(x,t)$一定是问题(4)的解，并且与y,z取值无关，于是在上式中令$y=z=0$再对积分变量做球坐标变换（令$\xi = x + at\cos\phi, \eta = y + at\sin\theta\sin\phi, \zeta = z + at\cos\theta\sin\phi$）得到

$$\begin{aligned} u(x,t) &= \frac{\partial}{\partial t}\left[\frac{1}{4\pi a^2 t}\int_0^{2\pi}\int_0^{\pi} \varphi(x+at\cos\phi)a^2t^2\sin\phi\mathrm{d}\phi\mathrm{d}\theta\right] + \frac{1}{4\pi a^2 t}\int_0^{2\pi}\int_0^{\pi}\psi(x+at\cos\theta)a^2t^2\sin\phi\mathrm{d}\phi\mathrm{d}\theta \\ &= \frac{1}{2}\frac{\partial}{\partial t}\left[\int_0^{\pi} t\varphi(x+at\cos\phi)\sin\phi\mathrm{d}\phi\right] + \frac{1}{2}\int_0^{\pi} t\psi(x+at\cos\phi)\sin\phi\mathrm{d}\phi \\ &= \frac{1}{2a}\frac{\partial}{\partial t}\left[\int_{x-at}^{x+at}\varphi(s)\mathrm{d}\right] + \frac{1}{2a}\int_{x-at}^{x+at}\psi(s)\mathrm{d}s \\ &= \frac{1}{2}[\varphi(x-at)+\varphi(x+at)] + \frac{1}{2a}\int_{x-at}^{x+at}\psi(s)\mathrm{d}s \end{aligned}$$

即为d'Alembert公式.

20. 由于$u_z(x,y,0,t) = 0$，所以对φ和ψ关于z做偶延拓，得到

$$\bar{\varphi}(x,y,z) = \begin{cases} \varphi(x,y,z), & z \geq 0 \\ \varphi(x,y,-z), & z < 0 \end{cases}$$

$$\bar{\psi}(x,y,z) = \begin{cases} \psi(x,y,z), & z \geq 0 \\ \psi(x,y,-z), & z < 0 \end{cases}$$

于是如下Cauchy问题

$$\begin{cases} u_{tt}-a^2\Delta u=0 \\ u(x,y,0)=\bar{\varphi}(x,y,z) \\ u_t(x,y,0)=\bar{\psi}(x,y,z) \end{cases}$$

有古典解

$$u(x,y,z,t)=\frac{\partial}{\partial t}\left[\frac{1}{4\pi a^2 t}\int_{S_{at}(\boldsymbol{x})}\bar{\varphi}(\xi,\eta,\zeta)\mathrm{d}\sigma_{at}\right]+\frac{1}{4\pi a^2 t}\int_{S_{at}(\boldsymbol{x})}\bar{\psi}(\xi,\eta,\zeta)\mathrm{d}\sigma_{at}$$

记$S_{at}(\boldsymbol{x})$在xOy平面上方部分为$S_{at}^{+}(\boldsymbol{x})$，下方部分为$S_{at}^{-}(\boldsymbol{x})$，则原问题的解为

$$\begin{aligned} u(x,y,z,t)=&\frac{\partial}{\partial t}\left[\frac{1}{4\pi a^2 t}\int_{S_{at}^{+}(\boldsymbol{x})}\varphi(\xi,\eta,\zeta)\mathrm{d}\sigma_{at}+\int_{S_{at}^{-}(\boldsymbol{x})}\varphi(\xi,\eta,-\zeta)\mathrm{d}\sigma_{at}\right]+ \\ &\frac{1}{4\pi a^2 t}\left(\int_{S_{at}^{+}(\boldsymbol{x})}\psi(\xi,\eta,\zeta)\mathrm{d}\sigma_{at}+\int_{S_{at}^{-}(\boldsymbol{x})}\psi(\xi,\eta,-\zeta)\mathrm{d}\sigma_{at}\right) \end{aligned}$$

21. 题目有误，改为“试证明$\lim\limits_{t\to+\infty}\dfrac{u}{t^\alpha}=C$，并计算常数$C$的值”

原方程的解为$u(\boldsymbol{x},t)=\dfrac{1}{4\pi a^2 t}\displaystyle\int_{S_{at}(0)}\varphi(\boldsymbol{x}+\boldsymbol{\xi})\mathrm{d}\sigma_{at}$ 于是

$$\begin{aligned} \lim_{t\to+\infty}\frac{u(\boldsymbol{x},t)}{t^\alpha}&=\lim_{t\to+\infty}\frac{1}{4\pi a^2 t^{\alpha+1}}\int_{S_1(0)}\varphi(\boldsymbol{x}+at\boldsymbol{\xi})a^2t^2\mathrm{d}\sigma_1 \\ &=\lim_{t\to+\infty}\frac{1}{4\pi}\int_{S_1(0)}\frac{\varphi(\boldsymbol{x}+at\boldsymbol{\xi})}{|\boldsymbol{x}+at\boldsymbol{\xi}|^{\alpha-1}}\frac{|\boldsymbol{x}+at\boldsymbol{\xi}|^{\alpha-1}}{t^{\alpha-1}}\mathrm{d}\sigma_1 \end{aligned}$$

注意到当t充分大时$\frac{\varphi(\boldsymbol{x}+at\boldsymbol{\xi})}{|\boldsymbol{x}+at\boldsymbol{\xi}|^{\alpha-1}}$和$\frac{|\boldsymbol{x}+at\boldsymbol{\xi}|^{\alpha-1}}{t^{\alpha-1}}$在$S_1(0)$上均有界，根据Lebesgue控制收敛定理，极限和积分可以换序，于是

$$\begin{aligned} \lim_{t\to+\infty}\frac{u}{t^\alpha}&=\frac{1}{4\pi}\int_{S_1(0)}Aa^{\alpha-1}\mathrm{d}\sigma_1 \\ &=Aa^{\alpha-1} \end{aligned}$$

22.(1)设原方程有非零解$u(x,t)=X(x)T(t)$，则有$\frac{X''(x)}{X(x)}=\frac{T''(t)}{a^2T(t)}=-\lambda$，于是得到

$$\begin{cases} X''(x)+\lambda X(x)=0 \\ X(0)=X(l)=0 \end{cases} \tag{5}$$

以及

$$T''(t)+a^2\lambda T(t)=0 \tag{6}$$

根据课本第53页的讨论，特征值问题(5)有通解$X(x)=c_1\cos\sqrt{\lambda}x+c_2\sin\sqrt{\lambda}x$，代入边界条件容易得到

$$c_1=0,\ \ c_2\sin\sqrt{\lambda}l=0$$

进一步得到

$$\sqrt{\lambda}l=k\pi\Rightarrow\lambda=\lambda_k=\frac{(k\pi)^2}{l^2},\ \ k=1,2,...$$

从而解得问题(5)的解为$X_k(x)=c_k\sin\dfrac{k\pi x}{l}$，$k=1,2,...$，代入(6)解得$T_k(t)=A_k\cos\dfrac{k\pi at}{l}+B_k\sin\dfrac{k\pi at}{l}$，所以原问题有形式解

$$u(x,t)=\sum_{k=1}^{+\infty}\left(A_k\cos\frac{k\pi at}{l}+B_k\sin\frac{k\pi at}{l}\right)\sin\frac{k\pi x}{l}$$

现只需求A_k,B_k，根据初始条件得

$$\begin{aligned}A_k&=\frac{2}{l}\int_0^l\sin^3\frac{\pi\xi}{l}\sin\frac{k\pi\xi}{l}\mathrm{d}\xi\\&=\frac{2}{\pi}\int_0^\pi\sin^3t\sin(kt)\mathrm{d}t\\&=\frac{1}{\pi}\int_0^\pi(1-\cos 2t)\sin t\sin(kt)\mathrm{d}t\\&=\frac{1}{\pi}\mathrm{Im}\left(\int_0^\pi(\mathrm{e}^{\mathrm{i}kt}\sin t-\mathrm{e}^{\mathrm{i}kt}\cos 2t\sin t)\mathrm{d}t\right)\end{aligned}$$

其中

$$\int_0^\pi\mathrm{e}^{\mathrm{i}kt}\sin t\mathrm{d}t=\begin{cases}\frac{\cos k\pi+1}{1-k^2}, & k\neq 1\\ \frac{\pi\mathrm{i}}{2}, & k=1\end{cases}$$

$$\begin{aligned}\int_0^\pi\mathrm{e}^{\mathrm{i}kt}\cos 2t\sin t\mathrm{d}t&=\frac{1}{2}\int_0^\pi(\sin 3t-\sin t)\mathrm{e}^{\mathrm{i}kt}\mathrm{d}t\\&=\begin{cases}-\dfrac{\pi\mathrm{i}}{4}, & k=1\\ \dfrac{\pi\mathrm{i}}{4}, & k=3\\ (\cos k\pi+1)\left(\dfrac{3}{9-k^2}-\dfrac{1}{1-k^2}\right) & ,k\neq 1,3\end{cases}\end{aligned}$$

所以$A_k=\begin{cases}\frac{3}{4}, & k=1\\ -\frac{1}{4}, & k=3\\ 0, & k\neq 1,3\end{cases}$，类似地，

$$B_k=\frac{2}{k\pi a}\int_0^l\xi(l-\xi)\sin\frac{k\pi\xi}{l}\mathrm{d}\xi=\frac{(4-4\cos k\pi)l^3}{k^4\pi^4a}$$

所以原问题的解为

$$u(x,t)=\frac{3}{4}\cos\frac{\pi at}{l}\sin\frac{k\pi x}{l}-\frac{1}{4}\cos\frac{3\pi at}{l}\sin\frac{3\pi x}{l}+\sum_{k=1}^{+\infty}\frac{(4-4\cos k\pi)l^3}{k^4\pi^4a}\sin\frac{k\pi at}{l}\sin\frac{k\pi x}{l}$$

(4)类似上一题有

$$u(x,t)=\sum_{k=1}^{+\infty}\left(A_k\cos\frac{k\pi at}{l}+B_k\sin\frac{k\pi at}{l}\right)\sin\frac{k\pi x}{l}$$

根据初始条件有

$$\begin{aligned}A_k&=\frac{2}{l}\left(\int_0^{\frac{l}{2}}\xi\sin\frac{k\pi\xi}{l}\mathrm{d}\xi+\int_{\frac{l}{2}}^l(l-\xi)\sin\frac{k\pi\xi}{l}\mathrm{d}\xi\right)\\&=\frac{4l\sin\frac{k\pi}{2}}{k^2\pi^2}\end{aligned}$$

并且$B_k=0$.

23.(2)**法一**：根据齐次化原理，首先考虑如下问题

$$\begin{cases} w_{tt}-a^2w_{xx}=0, & 0<x<l, t>\tau \\ w(x,\tau)=0, & 0\le x\le l \\ w_t(x,\tau)=b\sinh x, & 0\le x\le l \\ w(0,t)=w(l,t)=0, & t\ge\tau \end{cases} \tag{7}$$

令$s=t-\tau$，$v(x,s)=w(x,t)=w(x,s+\tau)$，与前面题目类似可得

$$w(x,t)=v(x,s)=\sum_{k=1}^{\infty}\left(A_k\cos\frac{k\pi s}{l}+B_k\sin\frac{k\pi s}{l}\right)\sin\frac{k\pi x}{l}$$

$$A_k=\frac{2}{l}\int_0^l 0\sin\frac{k\pi x}{l}\mathrm{d}x=0$$

$$\begin{aligned} B_k&=\frac{2}{l}\int_0^l b\sinh x\sin\frac{k\pi x}{l}\mathrm{d}x \\ &=\frac{b}{k\pi a}\int_0^l(\mathrm{e}^x-\mathrm{e}^{-x})\sin\frac{k\pi x}{l}\mathrm{d}x \\ &=\mathrm{Im}\left(\frac{b}{k\pi a}\int_0^l(\mathrm{e}^x-\mathrm{e}^{-x})\mathrm{e}^{\frac{k\pi x\mathrm{i}}{l}}\mathrm{d}x\right) \\ &=\frac{bl(\mathrm{e}^{-l}-\mathrm{e}^l)(-1^k)}{a(k^2\pi^2+l^2)} \end{aligned}$$

从而容易求出$w(x,t;\tau)$，再由$u(x,t)=\int_0^t w(x,t;\tau)\mathrm{d}\tau$即可求出$u(x,t)$

法二： 参考课本58页的方法，将$b\sinh x$关于特征函数系$\{\sin\frac{k\pi x}{l}\}_{k=1}^{\infty}$展开成Fourier级数，得到

$$\begin{aligned} f_k(t)&=\frac{2}{l}\int_0^l b\sinh x\sin\frac{k\pi x}{l}\mathrm{d}x \\ &=\frac{k\pi b(\mathrm{e}^{-l}-\mathrm{e}^l)\cos k\pi}{l^2+k^2\pi^2} \end{aligned}$$

$$b\sinh x=f_k(t)\sin\frac{k\pi x}{l}$$

设原方程有形式解$u(x,t)=\sum\limits_{k=1}^{\infty}u_k(t)\sin\dfrac{k\pi x}{l}$，则根据初始条件有

$$\sum_{k=1}^{\infty}\left(u_k''(t)+\frac{a^2k^2\pi^2}{l^2}\right)\sin\frac{k\pi x}{l}=\sum_{k=1}^{\infty}f_k(t)\sin\frac{k\pi x}{l}$$

$$\sum_{k=1}^{\infty}u_k(0)\sin\frac{k\pi x}{l}=0$$

$$\sum_{k=1}^{\infty}u_k'(0)\sin\frac{k\pi x}{l}=0$$

所以原问题化为

$$\begin{cases} u_k''(t)+\frac{a^2k^2\pi^2}{l^2}u_k(t)=f_k(t)\\ u_k(0)=0\\ u_k''(0)=0\end{cases}$$

解常微分方程得到

$$u(x,t)=\sum_{k=1}^{\infty}\frac{2bl^2(-1)^{k+1}}{k\pi a^2(l^2+k^2\pi^2)}(1-\cos\frac{k\pi at}{l})\sinh l\sin\frac{k\pi x}{l}$$

(3)设原问题解为$u(x,t)=X(x)T(t)$，则类似前面题目可知$X(x)=c_1\cos\sqrt{\lambda}x+c_2\sin\sqrt{\lambda}x$，有边界条件得

$$X(0)=c_1=0$$

$$X'(l)=c_2\sqrt{\lambda}\cos\sqrt{\lambda}l=0\Rightarrow\lambda=\lambda_k=(\frac{\pi}{2l}+\frac{k\pi}{l})^2,\ \ k=0,1,2,...$$

于是得到

$$X_k(x)=c_k\sin\frac{(\pi+2k\pi)x}{2l},\ \ k=0,1,2,...$$

$$T_k(t)=A_k\cos\frac{a(\pi+2k\pi)t}{2l}+B_k\sin\frac{a(\pi+2k\pi)t}{2l}$$

（此处k取值范围是$\{0,1,2,...\}$，可以取到0是因为要使$\{X_k\}$关于内积$(f,g)=\dfrac{2}{l}\displaystyle\int_0^l f(x)g(x)\mathrm{d}x$构成一组正交规范基，具体可参考泛函分析或傅里叶分析教材）

从而原问题有形式解

$$u(x,t)=\sum_{k=0}^{+\infty}\left(A_k\cos\frac{a(\pi+2k\pi)t}{2l}+B_k\sin\frac{a(\pi+2k\pi)t}{2l}\right)\sin\frac{(\pi+2k\pi)x}{2l},3$$

根据初始条件可知

$$A_k=\frac{2}{l}\int_0^l\xi\sin\frac{(\pi+2k\pi)\xi}{2l}\mathrm{d}\xi=\frac{2\sin[(\frac{\pi}{2}+k\pi)l^2]}{(\frac{\pi}{2}+k\pi)^2l},$$

$$B_k=0.$$

24.(1)(b)

设原方程有解$u(x,t)=X(x)T(t)$，则有

$$X(x)T''(t)-a^2X''(x)T(t)+X(x)T(t)=0$$

两边同时除以$a^2X(x)T(t)$得到

$$\frac{T''(t)}{a^2T(t)}+1=\frac{X''(x)}{X(x)}:=-\lambda$$

即

$$\begin{cases} X''(x)+\lambda X(x)=0\\ T''(t)+(a^2\lambda+1)T(t)=0\end{cases}$$

由边界条件得$X'(0)=X'(l)=0$，类似前面题目可以求出

$$X(x)=X_k(x)=c_k\cos\frac{k\pi x}{l}$$

$$T(t)=T_k(t)=A_k\cos\left(\sqrt{1+\frac{k^2\pi^2a^2}{l^2}}t\right)+B_k\sin\left(\sqrt{1+\frac{k^2\pi^2a^2}{l^2}}t\right)$$

于是原方程有形式解

$$u(x,t)=\sum_{k=0}^{\infty}\left[A_k\cos\left(\sqrt{1+\frac{k^2\pi^2a^2}{l^2}}t\right)+B_k\sin\left(\sqrt{1+\frac{k^2\pi^2a^2}{l^2}}t\right)\right]\cos\frac{k\pi x}{l}$$

根据初始条件得

$$u(x,0)=\sum_{k=0}^{\infty}A_k\cos\frac{k\pi x}{l}=\cos\frac{k\pi x}{l}$$

$$u_t(x,0)=\sum_{k=0}^{\infty}B_k\sqrt{1+\frac{k^2\pi^2a^2}{l^2}}\cos\frac{k\pi x}{l}=0$$

容易求得$A_k=\begin{cases}1, & k=1\\ 0, & k\neq 1\end{cases}$以及$B_k=0$，所以原问题的解为 $u(x,t)=\cos\left(\sqrt{1+\frac{k^2\pi^2a^2}{l^2}}t\right)\cos\frac{k\pi x}{l}$

(2)（题目有误，将题目中的l改为π）（设$u(x,t)=X(x)T(t)$并代入原方程然后两边同时除以$X(x)T(t)$可得

$$\begin{cases}X''(x)+\lambda X(x)=0\\ T''(t)+2T'(t)+(a^2\lambda+1)T(t)=0\end{cases}$$

再由边界条件得$X(0)=X(\pi)=0$，于是解得

$$X(x)=X_k(x)=c_k\sin kx$$

$$T(t)=T_k(t)=A_k\mathrm{e}^{-t}\cos kat+B_k\mathrm{e}^{-t}\sin kat$$

即原问题有形式解

$$u(x,t)=\sum_{k=1}^{\infty}(A_k\mathrm{e}^{-t}\cos kat+B_k\mathrm{e}^{-t}\sin kat)\sin kx$$

根据初始条件有$u(x,0)=\sum_{k=1}^{\infty}A_k\sin kx=\pi x-x^2$，于是得到

$$A_k=\frac{2}{l}\int_0^l(\pi x-x^2)\sin kx\mathrm{d}x=\frac{4(1-\cos k\pi)}{k^3\pi}$$

再由$u_t(x,t)=\sum\limits_{k=1}^{\infty}[A_k(-\mathrm{e}^{-t}\cos kat-ka\mathrm{e}^{-t}\sin kat)+B_k(-\mathrm{e}^{-t}\sin kat+\mathrm{e}^{-t}\cos kat)]\sin kx$得

$$u_t(x,0)=\sum_{k=1}^{\infty}(-A_k+kaB_k)\sin kx$$

从而有$B_k=\dfrac{A_k}{k\pi}=\dfrac{4(1-\cos k\pi)}{k^4\pi}$.因此，原问题的解为

$$u(x,t)=\sum_{k=1}^{\infty}\frac{4(1-\cos k\pi)\mathrm{e}^{-t}}{k^3\pi}(\cos kat+\sin kat)\sin kx$$

(5)根据齐次化原理，考虑如下问题

$$\begin{cases}w_{tt}-w_{xx}-4w=0, & t>\tau, 0<x<\pi\\ w(x,\tau)=0, w_t(x,\tau)=2\sin^2 x, & 0\le x\le\pi\\ w_x(0,t)=w(\pi,t)=0, & t\ge\tau\end{cases}$$

为了简化计算，再令$s=t-\tau$，$v(x,s)=w(x,t)=w(x,s+\tau)$，则上述问题可化为

$$\begin{cases} v_{ss}-v_{xx}-4v=0, & s>0, 0<x<\pi \\ v(x,0)=0, v_s(x,0)=2\sin^2 x, & 0\le x\le \pi \\ v_x(0,s)=v(\pi,s)=0, & s>0 \end{cases}$$

设$v(x,s)=X(x)T(s)$,则有$X(x)T''(s)-X''(x)T(s)-4X(x)T(s)=0$，两边同时除以$X(x)T(s)$得

$$\begin{cases} X''(x)+\lambda X(x)=0 \\ T''(s)+(\lambda-4)T(s)=0 \end{cases}$$

根据边界条件容易求得$\lambda=\lambda_k=(k+\frac{1}{2})^2$，$X(x)=X_k(x)=c_k\cos(k+\frac{1}{2})x$，$k=0,1,2...$，下面求$T$.
当$k=0,1$时，$\lambda_k-4<0$，此时$T(s)$有通解$T_k(s)=A_k\mathrm{e}^{s\sqrt{4-\lambda_k}}+B_k\mathrm{e}^{-s\sqrt{4-\lambda_k}}$，即

$$T_0(s)=A_0\mathrm{e}^{\frac{\sqrt{15}}{2}s}+B_0\mathrm{e}^{-\frac{\sqrt{15}}{2}s}$$

$$T_1(s)=A_1\mathrm{e}^{\frac{\sqrt{7}}{2}s}+B_1\mathrm{e}^{-\frac{\sqrt{7}}{2}s}$$

当$k\ge 2$时，$T(s)$有通解$T_k(s)=A_k\cos\sqrt{\lambda_k-4}s+B_k\sin\sqrt{\lambda_k-4}$ 于是有

$$\begin{aligned} w(x,t)=v(x,s)=&(A_0\mathrm{e}^{\frac{\sqrt{15}}{2}s}+B_0\mathrm{e}^{-\frac{\sqrt{15}}{2}s})\cos\frac{1}{2}x+(A_1\mathrm{e}^{\frac{\sqrt{7}}{2}s}+B_1\mathrm{e}^{-\frac{\sqrt{7}}{2}s})\cos\frac{3}{2}x+ \\ &\sum_{k=0}^{\infty}(A_k\cos\sqrt{\lambda_k-4}s+B_k\sin\sqrt{\lambda_k-4})\cos(k+\frac{1}{2})x \end{aligned}$$

然后代入初始条件得

$$\begin{cases} A_0+B_0=A_1+B_1=0 \\ A_k=0,\ \ k\ge 2 \\ \dfrac{\sqrt{15}}{2}A_0-\dfrac{\sqrt{15}}{2}B_0=\dfrac{2}{\pi}\displaystyle\int_0^{\pi}\cos\frac{1}{2}x\cdot 2\sin^2 x\mathrm{d}x \\ \dfrac{\sqrt{7}}{2}A_1-\dfrac{\sqrt{7}}{2}B_1=\dfrac{2}{\pi}\displaystyle\int_0^{\pi}\cos\frac{3}{2}x\cdot 2\sin^2 x\mathrm{d}x \\ B_k\sqrt{\lambda_k-4}=\dfrac{2}{\pi}\displaystyle\int_0^{\pi}\cos(k+\frac{1}{2})x\cdot 2\sin^2 x\mathrm{d}x \end{cases}$$

解出A_k,B_k即得$w(x,t)$，最后再根据$u(x,t)=\displaystyle\int_0^t w(x,t;\tau)\mathrm{d}\tau$即得$u(x,t)$**（实在太长了，懒得敲了）**（敲到后面发现有的式子比这个还要长，2025.4.25）

(7)令$u=\mathrm{e}^{-x}v+xt$，则有

$$u_t=\mathrm{e}^{-x}v_t+x,$$

$$u_{tt}=\mathrm{e}^{-x}v_{tt},$$

$$u_x=-\mathrm{e}^{-x}v+\mathrm{e}^{-x}v_x+t,$$

$$u_{xx}=\mathrm{e}^{-x}v-2\mathrm{e}^{-x}v_x+\mathrm{e}^{-x}v_{xx},$$

$$u(x,0)=\mathrm{e}^{-x}v(x,0)=\mathrm{e}^{-x}\sin x\Rightarrow v(x,0)=\sin x,$$

$$u_t(x,0)=\mathrm{e}^{-x}v_t(x,0)+x=x\Rightarrow v_t(x,0)=0,$$

$$v(0,t)=v(\pi,t)=0.$$

于是原问题转化为

$$\begin{cases} v_{tt}-v_{xx}-3v_t+v=0, & t>0, 0<x<\pi \\ v(x,0)=\sin x, v_t(x,0)=0, & 0\le x\le\pi \\ v(0,t)=v(\pi,t)=0, & t\ge 0 \end{cases}$$

设$v(x,t)=X(x)T(t)$，则有$XT''-X''T-3XT'+XT=0$，两边同时除以$X(x)T(t)$得

$$\begin{cases} X''(x)+\lambda X(x)=0 \\ T''(t)-3T'(t)+(\lambda+1)T(t)=0 \end{cases}$$

根据边界条件容易求得$\lambda=\lambda_k=k^2$，$X(x)=X_k(x)=\sin kx$，于是有

$$v(x,t)=\sum_{k=1}^{\infty}T_k(t)\sin kx$$

代入初始条件得

$$\begin{cases} \sum\limits_{k=1}^{\infty}T_k(0)\sin kx=\sin x \\ \sum\limits_{k=1}^{\infty}T_k'(0)\sin x=0 \end{cases}$$

所以

$$\begin{cases} T_k(0)=\begin{cases} 1, & k=1 \\ 0, & k\ge 2 \end{cases} \\ T_k'(0)=0,\ \ k=1,2,... \end{cases} \tag{8}$$

再求解$T_k''-3T_k'+(k^2+1)T_k=0$，为此，求解$\mu^2-3\mu+k^2+1=0$得其特征值为

$$\mu_k=\begin{cases} \frac{1}{2}(3\pm\sqrt{5-4k^2}), & k=1 \\ \frac{1}{2}(3\pm \mathrm{i}\sqrt{4k^2-5}), & k\ge 2 \end{cases}$$

所以其通解为

$$T_k(t)=\begin{cases} A_k\mathrm{e}^{2t}+B_k\mathrm{e}^{t}, & k=1 \\ \mathrm{e}^{\frac{3}{2}}\left(A_k\cos\frac{\sqrt{4k^2-5}}{2}t+B_k\sin\frac{4k^2-5}{2}t\right), & k\ge 2 \end{cases}$$

再根据初始条件(8)得

$$T_k(t)=\begin{cases} 2\mathrm{e}^{t}-\mathrm{e}^{2t}, & k=1 \\ 0, & k\ge 2 \end{cases}$$

所以解得$v(x,t)=(2\mathrm{e}^t-\mathrm{e}^{2t})\sin x$，即$u(x,t)=\mathrm{e}^{-x}(2\mathrm{e}^t-\mathrm{e}^{2t})\sin x+xt$

25.(1)设$u(x,y,t)=X(x)Y(y)T(t)$，则有$XYT''-a^2(X''YT+XY''T)=0$，两边同时除以XYT再根据边界条件、初始条件即得

$$\begin{cases} X''(x)+\mu X(x)=0 \\ X(0)=X(l) \end{cases}$$

$$\begin{cases} Y''(y)+(\lambda-\mu)Y(y)=0 \\ Y(0)=Y(l)=0 \end{cases}$$

$$\begin{cases} T''(t)+a^2\lambda T(t)=0 \\ T'(0)=0 \end{cases}$$

分别求得$X(x)=X_k(x)=\sin\frac{k\pi x}{l}$，$Y(y)=Y_m(y)=\sin\frac{m\pi x}{l}$，$T(t)=T_{km}(t)=A_{km}\cos a\sqrt{\lambda_{km}}t$，$k,m=1,2,...$，所以有

$$u(x,y,t)=\sum_{k,m=1}^{\infty}A_{km}\sin\frac{k\pi x}{l}\sin\frac{m\pi y}{l}\cos a\sqrt{\lambda_{km}}t$$

根据初始条件得

$$\sum_{k,m=1}^{\infty}A_{km}\sin\frac{k\pi x}{l}\sin\frac{m\pi y}{l}=A\sin\frac{\pi x}{l}\sin\frac{\pi y}{l}$$

从而$A_{km}=\begin{cases} A, & k=m=1 \\ 0, & otherwise \end{cases}$，故$u(x,y,t)=A\sin\dfrac{\pi x}{l}\sin\dfrac{\pi y}{l}\cos\dfrac{\sqrt{2}a\pi t}{l}$.

28. 要证明古典解的唯一性，只需证明原问题对应的齐次问题的古典解只有零解，即证

$$\begin{cases} u_{tt}-a^2\Delta u=0, & \boldsymbol{x}\in\Omega \\ u(\boldsymbol{x},0)=u_t(\boldsymbol{x},0)=0, & \boldsymbol{x}\in\overline{\Omega} \\ \dfrac{\partial u}{\partial\boldsymbol{\nu}}+\dfrac{c}{b}u=0, & \boldsymbol{x}\in\partial\Omega \end{cases} \tag{9}$$

的古典解只有零解.(9)中第一式两边同时乘以u_t得$u_t(u_{tt}-a^2\Delta u)=0$，从而$\int_\Omega u_t(u_{tt}-a^2\Delta u)\mathrm{d}\boldsymbol{x}=0$，同时注意到

$$\int_\Omega u_tu_{tt}\mathrm{d}\boldsymbol{x}=\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega\frac{1}{2}u_t^2\mathrm{d}\boldsymbol{x}\right)$$

$$\int_\Omega u_tu_{x_1x_1}\mathrm{d}\boldsymbol{x}=\int_{\partial\Omega}u_tu_{x_1}\mathrm{d}x_2\mathrm{d}x_3-\int_\Omega u_{x_1t}u_{x_1}\mathrm{d}\boldsymbol{x}$$

$$\int_\Omega u_tu_{x_2x_2}\mathrm{d}\boldsymbol{x}=\int_{\partial\Omega}u_tu_{x_2}\mathrm{d}x_3\mathrm{d}x_1-\int_\Omega u_{x_2t}u_{x_2}\mathrm{d}\boldsymbol{x}$$

$$\int_\Omega u_tu_{x_3x_3}\mathrm{d}\boldsymbol{x}=\int_{\partial\Omega}u_tu_{x_3}\mathrm{d}x_1\mathrm{d}x_2-\int_\Omega u_{x_3t}u_{x_3}\mathrm{d}\boldsymbol{x}$$

因此

$$\begin{aligned}\int_\Omega u_t(u_{tt}-a^2\Delta u)\mathrm{d}\boldsymbol{x}&=\int_\Omega u_tu_{tt}\mathrm{d}\boldsymbol{x}-a^2\sum_{i=1}^{3}\int_\Omega u_tu_{x_ix_i}\mathrm{d}\boldsymbol{x}\\&=\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega\frac{1}{2}u_t^2\mathrm{d}\boldsymbol{x}\right)-a^2\left(\int_{\partial\Omega}u_t\frac{\partial u}{\partial\boldsymbol{\nu}}\mathrm{d}S\right)+a^2\sum_{i=1}^{3}\int_\Omega u_{x_i}u_{x_it}\mathrm{d}\boldsymbol{x}\\&=\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega\frac{1}{2}u_t^2\mathrm{d}\boldsymbol{x}\right)+\frac{a^2c}{b}\left(\int_{\partial\Omega}u_tu\mathrm{d}S\right)+\frac{\mathrm{d}}{\mathrm{d}t}\left(a^2\sum_{i=1}^{3}\int_\Omega\frac{1}{2}u_{x_i}^2\mathrm{d}\boldsymbol{x}\right)\\&=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega u_t^2\mathrm{d}\boldsymbol{x}\right)+\frac{a^2c}{2b}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\partial\Omega}u^2\mathrm{d}S\right)+\frac{a^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega|\nabla u|^2\mathrm{d}\boldsymbol{x}\right)\\&=\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\int_\Omega(u_t^2+a^2|\nabla u|^2)\mathrm{d}\boldsymbol{x}+\frac{a^2c}{2b}\int_{\partial\Omega}u^2\mathrm{d}S\right]\end{aligned}$$

令$E(t)=\dfrac{1}{2}\displaystyle\int_\Omega(u_t^2+a^2|\nabla u|^2)\mathrm{d}\boldsymbol{x}+\dfrac{a^2c}{2b}\int_{\partial\Omega}u^2\mathrm{d}S$，则有

$$E'(t)=\int_\Omega u_t(u_{tt}-a^2\Delta u)\mathrm{d}\boldsymbol{x}=0$$

所以$E(t)\equiv E(0)=0$，又因为$u_t^2\geq 0$，$|\nabla u|^2\geq 0$，$u^2\geq 0$，$a,b,c>0$，因此$u\equiv u_t\equiv|\nabla u|\equiv 0$，即(9)的古典解只有零解，从而原问题的古典解唯一.

31. 要证解的唯一性，只需证原问题对应的齐次问题的古典解只有零解.对$\forall t>0,\forall x_0>t$，令$E(t)=\displaystyle\int_0^{x_0-at}u_t^2+a^2u_x^2\mathrm{d}x$，于是有

$$\begin{aligned}
E'(t)&=2\int_0^{x_0-at}(u_tu_{tt}+a^2u_xu_{xt})\mathrm{d}x-a[u_t^2(x_0-at,t)+a^2u_x^2(x_0-at,t)]\\
&=2\int_0^{x_0-at}u_tu_{tt}\mathrm{d}x+2a^2u_xu_t|_0^{x_0-at}-2a^2\int_0^{x_0-at}u_tu_{xx}\mathrm{d}x-a[u_t^2(x_0-at,t)+a^2u_x^2(x_0-at,t)]\\
&=2\int_0^{x_0-at}u_t(u_{tt}-a^2u_{xx})\mathrm{d}x+2a^2[u_x(x_0-at,t)u_t(x_0-at,t)-u_x(0,t)u_t(0,t)]-a[u_t^2(x_0-at,t)+a^2u_x^2(x_0-at,t)]\\
&=2a^2u_x(x_0-at,t)u_t(x_0-at,t)-a[u_t^2(x_0-at,t)+a^2u_x^2(x_0-at,t)]\\
&\leq 0
\end{aligned}$$

因此$E(t)$关于t单调不增，故$E(t)\leq E(0)=0$，又因为$E(t)\geq 0$，所以$E(t)\equiv 0$，从而$u_t\equiv u_x\equiv 0$，即得$u\equiv 0$，故原问题对应的齐次问题的古典解只有零解.

32. 构造辅助函数$v(y)$满足$v''(y)=-\frac{b}{a^2}v(y)$.（这样的$v$存在，因为$v''+\frac{b}{a^2}v=0$总有非零解），再令$\tilde{u}(x,y,t)=u(x,t)v(y)$，则

$$\tilde{u}_{tt}=u_{tt}v,\quad \tilde{u}_{xx}=u_{xx}v,\quad \tilde{u}_{yy}=-\frac{b}{a^2}uv$$

从而

$$\tilde{u}_{tt}-a^2\Delta\tilde{u}=(u_{tt}-a^2u_{xx}+bu)v=f(x,t)v(y)$$

$$\tilde{u}(x,y,0)=\varphi(x)v(y)$$

$$\tilde{u}_t(x,y,0)=\psi(x)v(y)$$

再记$\tilde{f}=fv$，$\tilde{\varphi}=\varphi v$，$\tilde{\psi}=\psi v$，则$\tilde{u}$是如下Cauchy问题的解

$$\begin{cases}\tilde{u}_{tt}-a^2\Delta\tilde{u}=\tilde{f}(x,y,t)\\ \tilde{u}(x,y,0)=\tilde{\varphi}(x,y)\\ \tilde{u}_t(x,y,0)=\tilde{\psi}(x,y)\end{cases}$$

只需证明该问题古典解的唯一性即可.

35. (1)

$$\begin{aligned}
E'(t)&=\int_\Omega[u_tu_{tt}+a^2(u_xu_{xt}+u_yu_{yt}+u_zu_{zt})]\mathrm{d}\boldsymbol{x}\\
&=\int_\Omega u_tu_{tt}\mathrm{d}\boldsymbol{x}+a^2\int_\Omega(u_xu_{xt}+u_yu_{yt}+u_zu_{zt})\mathrm{d}\boldsymbol{x}
\end{aligned}$$

由分部积分得

$$\int_{\Omega} u_x u_{xt} \mathrm{d}\boldsymbol{x} = \int_{\partial\Omega} u_x u_t \mathrm{d}y\mathrm{d}z - \int_{\Omega} u_{xx} u_t \mathrm{d}\boldsymbol{x}$$
$$\int_{\Omega} u_y u_{yt} \mathrm{d}\boldsymbol{x} = \int_{\partial\Omega} u_y u_t \mathrm{d}z\mathrm{d}x - \int_{\Omega} u_{yy} u_t \mathrm{d}\boldsymbol{x}$$
$$\int_{\Omega} u_z u_{zt} \mathrm{d}\boldsymbol{x} = \int_{\partial\Omega} u_z u_t \mathrm{d}x\mathrm{d}y - \int_{\Omega} u_{zz} u_t \mathrm{d}\boldsymbol{x}$$

根据边界条件知$u_t(x,y,z,t)$在$\partial\Omega$上恒为0，从而

$$E'(t) = \int_{\Omega} u_t(u_{tt} - a^2\Delta u)\mathrm{d}\boldsymbol{x} = -\alpha\int_{\Omega} u_t^2 \mathrm{d}\boldsymbol{x} \leq 0$$

故$E(t)$随t增加而不增加.

(2)与前面题目同理，只需证明原问题对应的齐次问题的古典解只有零解

36. 原问题中第一式两边同时乘以u_t得

$$u_t\left(u_{tt} - \sum_{i=1}^{3}\frac{\partial}{\partial x_i}(p_i(\boldsymbol{x})u_{x_i}) + c^2 u\right) = 0$$

于是

$$\begin{aligned}
&\int_{\Omega} u_t\left(u_{tt} - \sum_{i=1}^{3}\frac{\partial}{\partial x_i}(p_i(\boldsymbol{x})u_{x_i}) + c^2 u\right)\mathrm{d}\boldsymbol{x} \\
&= \int_{\Omega} u_t(u_{tt} + c^2 u)\mathrm{d}\boldsymbol{x} - \sum_{i=1}^{3}\int_{\Omega}\frac{\partial}{\partial x_i}(p_i(\boldsymbol{x})u_{x_i})u_t \mathrm{d}\boldsymbol{x} \\
&\overset{(1)}{=} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}(u_t^2 + c^2u^2)\mathrm{d}\boldsymbol{x} - \sum_{i=1}^{3}\left[\int_{\partial\Omega} p_i(\boldsymbol{x})u_{x_i}u_t\nu^i \mathrm{d}S - \int_{\Omega} p_i(\boldsymbol{x})u_{x_i}u_{x_it}\mathrm{d}\boldsymbol{x}\right] \\
&\overset{(2)}{=} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}(u_t^2 + c^2u^2)\mathrm{d}\boldsymbol{x} + \sum_{i=1}^{3}\int_{\Omega} p_i(\boldsymbol{x})u_{x_i}u_{x_it}\mathrm{d}\boldsymbol{x} \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}(u_t^2 + c^2u^2)\mathrm{d}\boldsymbol{x} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{3}\int_{\Omega} p_i(\boldsymbol{x})u_{x_i}^2 \mathrm{d}\boldsymbol{x}\right) \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{\Omega}\left(u_t^2 + c^2u^2 + \sum_{i=1}^{3} p_i(\boldsymbol{x})u_{x_i}^2\right)\mathrm{d}\boldsymbol{x}\right] \\
&= 0
\end{aligned}$$

上式中(1)用到了分部积分，其中ν^i表示$\partial\Omega$的单位外法向量$\boldsymbol{\nu}$的第i个分量. (2)则用到了边界条件，由于u在$\partial\Omega$上恒为0，所以u_t在$\partial\Omega$也等于零. 现令$E(t) = \int_{\Omega}\left(u_t^2 + c^2u^2 + \sum_{i=1}^{3} p_i(\boldsymbol{x})u_{x_i}^2\right)\mathrm{d}\boldsymbol{x}$，不难发现$E(t) \geq 0$且$E'(t) \equiv 0$，因此$E(t) \equiv E(0)$，即

$$\begin{aligned}
E(t) &\equiv E(0) \\
&= \int_{\Omega}\left(\psi(\boldsymbol{x})^2 + \sum_{i=1}^{3} p_i(\boldsymbol{x})\varphi_{x_i}(\boldsymbol{x})^2 + c^2\varphi(\boldsymbol{x})^2\right)\mathrm{d}\boldsymbol{x} \\
&\leq C\int_{\Omega}\left(\psi^2 + |\nabla\psi|^2 + c^2\varphi^2\right)\mathrm{d}\boldsymbol{x}
\end{aligned}$$

又因为$p_i \geq a^2 > 0$，所以

$$E(t) = \int_\Omega \left(u_t^2 + c^2u^2 + \sum_{i=1}^{3} p_i(\boldsymbol{x}) u_{x_i}^2 \right) \mathrm{d}\boldsymbol{x} \geq \int_\Omega \left(u_t^2 + c^2u^2 + a^2|\nabla u|^2 \right) \mathrm{d}\boldsymbol{x}$$

故$\int_\Omega \left(u_t^2 + c^2u^2 + a^2|\nabla u|^2 \right) \mathrm{d}\boldsymbol{x} \leq M \int_\Omega \left(\psi^2 + |\nabla\psi|^2 + \varphi^2 \right) \mathrm{d}\boldsymbol{x}$.

38. 与35(1)同理可知

$$E'(t) = \int_\Omega u_t(u_{tt} - a^2\Delta u)\mathrm{d}\boldsymbol{x} + a^2 \int_{\partial\Omega} u_x u_t \mathrm{d}y\mathrm{d}z + u_y u_t \mathrm{d}z\mathrm{d}x + u_z u_t \mathrm{d}x\mathrm{d}y$$

由于在Ω上有$u_{tt} - a^2\Delta u = 0$，在Γ_0上有$u(x,y,z,t) = 0$，所以

$$E(t) = \int_{\Gamma_1} u_x u_t \mathrm{d}y\mathrm{d}z + u_y u_t \mathrm{d}z\mathrm{d}x + u_z u_t \mathrm{d}x\mathrm{d}y$$

根据Γ_1上的边界条件以及第二类曲面积分的定义可知

$$\begin{aligned}
E(t) &= -\frac{a^2}{\sigma} \int_{\Gamma_1} u_x \frac{\partial u}{\partial \boldsymbol{n}} \mathrm{d}y\mathrm{d}z + u_y \frac{\partial u}{\partial \boldsymbol{n}} \mathrm{d}z\mathrm{d}x + u_z \frac{\partial u}{\partial \boldsymbol{n}} \mathrm{d}x\mathrm{d}y \\
&= -\frac{a^2}{\sigma} \int_{\Gamma_1} \frac{\partial u}{\partial \boldsymbol{n}} \nabla u \cdot \boldsymbol{n} \mathrm{d}S \\
&= -\frac{a^2}{\sigma} \int_{\Gamma_1} \left(\frac{\partial u}{\partial \boldsymbol{n}} \right)^2 \mathrm{d}S \\
&\leq 0
\end{aligned}$$

因此$E(t)$随t单调不增，进一步，只需证明原问题对应的齐次问题古典解只有零解即可．事实上，令$\psi = \varphi = 0$得原问题对应的齐次问题，此时$E(t) \equiv E(0) = 0$.

40. (1)记$f(t) = \int_{x_1}^{x_2} (u_t^2 + a^2u_x^2)\mathrm{d}x - \int_{x_1-at}^{x_2+at} [\psi(x)^2 + a^2\varphi'(x)^2]\mathrm{d}x$，则

$$\begin{aligned}
f'(t) &= 2\int_{x_1}^{x_2} (u_t u_{tt} + a^2 u_x u_{xt})\mathrm{d}x - a\left[\psi(x_2+at)^2 + a^2\varphi'(x_2+at)^2\right] - a\left[\psi(x_1-at)^2 + a^2\varphi'(x_1-at)^2\right] \\
&= 2a^2\int_{x_1}^{x_2} (u_t u_{xx} + u_x u_{xt})\mathrm{d}x - a\left[\psi(x_2+at)^2 + a^2\varphi'(x_2+at)^2\right] - a\left[\psi(x_1-at)^2 + a^2\varphi'(x_1-at)^2\right] \\
&= 2a^2(u_x u_t)\Big|_{(x_1,t)}^{(x_2,t)} - a\left[\psi(x_2+at)^2 + a^2\varphi'(x_2+at)^2\right] - a\left[\psi(x_1-at)^2 + a^2\varphi'(x_1-at)^2\right]
\end{aligned}$$

根据d'Alembert公式有

$$u(x,t) = \frac{1}{2}\left[\varphi(x+at) + \varphi(x-at)\right] + \frac{1}{2a}\int_{x-at}^{x+at} \psi(s)\mathrm{d}s$$

于是可以得到

$$\begin{aligned}
u_x(x,t) &= \frac{1}{2}[\varphi'(x+at) + \varphi'(x-at)] + \frac{1}{2a}[\psi(x+at) - \psi(x-at)] \\
u_t(x,t) &= \frac{a}{2}[\varphi'(x+at) - \varphi'(x-at)] + \frac{1}{2}[\psi(x+at) + \psi(x-at)]
\end{aligned}$$

经计算得

$$u_x u_t = \frac{a}{2}\left[\left(a\varphi'(x+at) + \psi(x+at) \right)^2 - \left(a\varphi'(x-at) - \psi(x-at) \right)^2 \right]$$

从而

$$
\begin{aligned}
f'(t)=&2a^2(u_xu_t)(x_2,t)-2a^2(u_xu_t)(x_1,t)-a\left[\psi(x_2+at)^2+a^2\varphi'(x_2+at)^2\right]-a\left[\psi(x_1-at)^2+a^2\varphi'(x_1-at)^2\right]\\
=&-\frac{a}{2}\Big[\left(a\varphi'(x_2+at)-\psi(x_2+at)\right)^2+\left(a\varphi'(x_2-at)-\psi(x_2-at)\right)^2+\left(a\varphi'(x_1+at)-\psi(x_1+at)\right)^2\\
&+\left(a\varphi'(x_1-at)-\psi(x_1-at)\right)^2\Big]\\
\leq&0
\end{aligned}
$$

因此$f(t)\leq f(0)$，又初始条件知$f(0)=0$，从而$f(t)\leq 0$，原题得证.
(2)在(1)中，由x_1,x_2的任意性，令$x_2\to+\infty$，$x_1\to-\infty$即得结论.
(3)令$E(t)=\int_{x_1-at}^{x_2+at}(u_t^2+a^2u_x^2)\mathrm{d}x$，则

$$
\begin{aligned}
E'(t)&=\int_{x_1-at}^{x_2+at}(2u_tu_{tt}+2a^2u_xu_{xt})\mathrm{d}x+a(u_t^2+a^2u_x^2)(x_2+at,t)+a(u_t+a^2u_x)(x_1-at,t)\\
&=2a^2\int_{x_1-at}^{x_2+at}(u_tu_{xx}+u_xu_{xt})\mathrm{d}x+a(u_t^2+a^2u_x^2)(x_2+at,t)+a(u_t+a^2u_x)(x_1-at,t)\\
&=2a^2(u_xu_t)(x_2+at,t)-2a^2(u_xu_t)(x_1-at,t)+a(u_t^2+a^2u_x^2)(x_2+at,t)+a(u_t+a^2u_x)(x_1-at,t)\\
&=a(u_t+au_x)^2(x_2+at,t)+a(u_t-au_x)^2(x_1-at)\\
&\geq 0
\end{aligned}
$$

因此$E(t)\geq E(0)=\int_{x_1}^{x_2}(\psi(x)+a^2\varphi'(x)^2)\mathrm{d}x$，令$x_2\to+\infty$，$x_1\to-\infty$得

$$
\int_{-\infty}^{+\infty}(u_t^2+a^2u_x^2)\mathrm{d}x\geq\int_{-\infty}^{+\infty}(\psi(x)^2+a^2\varphi'(x)^2)\mathrm{d}x
$$

再结合第(2)问得结论成立.

41.题干有误，“$\varphi,\psi\in C_0^\infty(\mathbb{R})$”改为“$\varphi,\psi\in C_c^\infty(\mathbb{R})$”，即具有紧支集的光滑函数.
(1)由40(3)知结论成立.
(2)根据d’Alembert公式有

$$
u(x,t)=\frac{1}{2}\left[\varphi(x+at)+\varphi(x-at)\right]+\frac{1}{2a}\int_{x-at}^{x+at}\psi(s)\mathrm{d}s
$$

于是可以得到

$$
\begin{aligned}
u_x(x,t)&=\frac{1}{2}[\varphi'(x+at)+\varphi'(x-at)]+\frac{1}{2a}[\psi(x+at)-\psi(x-at)]\\
u_t(x,t)&=\frac{a}{2}[\varphi'(x+at)-\varphi'(x-at)]+\frac{1}{2}[\psi(x+at)+\psi(x-at)]
\end{aligned}
$$

经计算得

$$
2(u_t^2-a^2u_x^2)=-a^2\varphi'(x+at)\varphi'(x-at)+\psi(x+at)\psi(x-at)+a\varphi'(x+at)\psi(x-at)-a\varphi'(x-at)\psi(x+at)
$$

由于φ,ψ具有紧支集，所以当t充分大时，$u_t-a^2u_x=0$对$\forall x\in\mathbb{R}$成立，从而$k(t)=p(t)$.

2 第四章 抛物型方程

1.(1)若$\alpha=0$，则$\mathscr{F}[f](y)=0$.

若$\alpha>0$，则令$g(x)\begin{cases}\pi, & |x|\le\alpha\\ 0, & |x|>\alpha\end{cases}$ 由例4.1.1知，$\mathscr{F}[g](y)=2\pi f(y)$，即$f(x)=\frac{1}{2\pi}\mathscr{F}[g](x)$，因此

$$
\begin{aligned}
\mathscr{F}[f](y)&=\int_{-\infty}^{+\infty}\frac{1}{2\pi}\mathscr{F}[g](x)\mathrm{e}^{-ixy}\mathrm{d}x\\
&=\mathscr{F}^{-1}\mathscr{F}[g](-y)\\
&=g(-y)\\
&=g(y)
\end{aligned}
$$

当$\alpha<0$时，令$g(x)\begin{cases}\pi, & |x|\le-\alpha\\ 0, & |x|>-\alpha\end{cases}$，同理可得$\mathscr{F}[f](y)=g(y)$

(2)

$$
\begin{aligned}
\mathscr{F}[f](y)&=\int_{-\infty}^{+\infty}\mathrm{e}^{-\eta x^2}\mathrm{e}^{-\mathrm{i}xy}\mathrm{d}x\\
&=\int_{-\infty}^{+\infty}\mathrm{e}^{-\eta x^2}\cos yx\mathrm{d}x-\mathrm{i}\int_{-\infty}^{+\infty}\mathrm{e}^{-\eta x^2}\sin yx\mathrm{d}x\\
&\triangleq I_1(y)-\mathrm{i}I_2(y)
\end{aligned}
$$

由于$\eta>0$，所以根据含参变量反常积分求导定理有，

$$
\begin{aligned}
I_1'(y)&=-\int_{-\infty}^{+\infty}x\mathrm{e}^{-\eta x^2}\sin yx\mathrm{d}x\\
&=\frac{1}{2\eta}\mathrm{e}^{-\eta x^2}\sin yx\Big|_{-\infty}^{+\infty}-\frac{y}{2\eta}I_1\\
&=-\frac{y}{2\eta}I_1
\end{aligned}
$$

所以$I_1(y)=C\mathrm{e}^{-\frac{y^2}{4\eta}}$，由于$I_1(0)=\int_{-\infty}^{+\infty}\mathrm{e}^{-\eta x^2}\mathrm{d}x=\sqrt{\frac{\pi}{\eta}}$，故$I_1(y)=\sqrt{\frac{\pi}{\eta}}\mathrm{e}^{\frac{y^2}{4\eta}}$，再根据函数奇偶性显然有$I_2(y)=0$，因此$\mathscr{F}[f](y)=\sqrt{\frac{\pi}{\eta}}\mathrm{e}^{\frac{y^2}{4\eta}}$.或参考例4.1.7.

(3)参考例4.1.8，4.1.9.

(4)

$$
\begin{aligned}
\mathscr{F}[f](y)&=\int_{-\infty}^{+\infty}\mathrm{e}^{-\alpha|x|}\mathrm{e}^{-\mathrm{i}xy}\mathrm{d}x\\
&=\int_{-\infty}^{0}\mathrm{e}^{(\alpha-\mathrm{i}y)x}\mathrm{d}x+\int_{0}^{+\infty}\mathrm{e}^{(-\alpha-\mathrm{i}y)x}\mathrm{d}x\\
&=\frac{1}{\alpha+\mathrm{i}y}+\frac{1}{\alpha-\mathrm{i}y}\\
&=\frac{2\alpha}{\alpha^2+y^2}
\end{aligned}
$$

(6)

$$\begin{aligned}\mathscr{F}[f](y) &= \int_{-\infty}^{+\infty} \cos x \mathrm{e}^{-\alpha|x|-\mathrm{i}xy}\mathrm{d}x \\ &= \int_{0}^{+\infty} \cos x \mathrm{e}^{(-\alpha-\mathrm{i}y)x}\mathrm{d}x + \int_{-\infty}^{0} \cos x \mathrm{e}^{\alpha-\mathrm{i}y}\mathrm{d}x \\ &= \frac{\alpha+\mathrm{i}y}{1+(\alpha+\mathrm{i}y)^2} + \frac{\alpha-\mathrm{i}y}{1+(\alpha-\mathrm{i}y)^2}\end{aligned}$$

(9)

$$\begin{aligned}\mathscr{F}[f](y) &= \int_{-\alpha}^{\alpha} \sin\lambda_0 x \cdot \mathrm{e}^{-\mathrm{i}xy}\mathrm{d}x \\ &= \frac{2\mathrm{i}(\lambda_0\cos\lambda_0\alpha\cdot\sin\alpha y - y\sin\lambda_0\alpha\cos\alpha y)}{\lambda_0^2-y^2}\end{aligned}$$

3.(1)

$$\begin{aligned}\mathscr{F}[f(-x)](y) &= \int_{-\infty}^{+\infty} f(-x)\mathrm{e}^{-\mathrm{i}xy}\mathrm{d}x \\ &= \int_{-\infty}^{+\infty} f(t)\mathrm{e}^{\mathrm{i}ty}\mathrm{d}t \\ &= \int_{-\infty}^{+\infty} f(t)\mathrm{e}^{-\mathrm{i}t(-y)}\mathrm{d}t \\ &= \hat{f}(-y)\end{aligned}$$

(2)

$$\begin{aligned}\mathscr{F}[f(\alpha x)][y] &= \int_{-\infty}^{+\infty} f(\alpha x)\mathrm{e}^{-\mathrm{i}xy}\mathrm{d}x \\ &= \int_{-\infty}^{+\infty} f(t)\mathrm{e}^{-\mathrm{i}\frac{t}{\alpha}y}\mathrm{d}(\frac{t}{\alpha}) \\ &= \frac{1}{|\alpha|}\int_{-\infty}^{+\infty} f(t)\mathrm{e}^{-\mathrm{i}t\frac{y}{\alpha}}\mathrm{d}t \\ &= \frac{1}{|\alpha|}\hat{f}(\frac{y}{\alpha})\end{aligned}$$

(3)题干有误，待证等式右端ω改为y.

$$\begin{aligned}\mathscr{F}[f(x)\cos\omega_0 x](y) &= \int_{-\infty}^{+\infty} f(x)\cos\omega_0 x\cdot\mathrm{e}^{-\mathrm{i}xy}\mathrm{d}x \\ &= \frac{1}{2}\int_{-\infty}^{+\infty} f(x)\mathrm{e}^{-\mathrm{i}xy}(\mathrm{e}^{\mathrm{i}\omega_0 x}+\mathrm{e}^{-\mathrm{i}\omega_0 x}) \\ &= \frac{1}{2}\int_{-\infty}^{+\infty} f(x)\mathrm{e}^{-\mathrm{i}x(y-\omega_0)}\mathrm{d}x + \frac{1}{2}\int_{-\infty}^{+\infty} f(x)\mathrm{e}^{-ix(y+\omega_0)}\mathrm{d}x \\ &= \frac{1}{2}[\hat{f}(y-\omega_0)+\hat{f}(y+\omega_0)]\end{aligned}$$

(4)

$$\begin{aligned}\mathscr{F}[f(x)\sin\omega_0 x](y) &= \int_{-\infty}^{+\infty} f(x)\sin\omega_0 x\cdot\mathrm{e}^{-\mathrm{i}xy}\mathrm{d}x \\ &= \frac{1}{2\mathrm{i}}\int_{-\infty}^{+\infty} f(x)\mathrm{e}^{-\mathrm{i}xy}(\mathrm{e}^{\mathrm{i}\omega_0 x}-\mathrm{e}^{-\mathrm{i}\omega_0 x})\mathrm{d}x \\ &= \frac{1}{2\mathrm{i}}[\hat{f}(y-\omega_0)-\hat{f}(y+\omega_0)]\end{aligned}$$

(5)题干有误，待证等式右端$\pm$改为$\mp$.

$$\begin{aligned}\mathscr{F}[f(x\mp x_0)](y)&=\int_{-\infty}^{+\infty}f(x\mp x_0)\mathrm{e}^{-\mathrm{i}xy}\mathrm{d}x\\&=\int_{-\infty}^{+\infty}f(t)\mathrm{e}^{-\mathrm{i}(t\pm x_0)y}\mathrm{d}t\\&=\mathrm{e}^{\mp\mathrm{i}x_0y}\hat{f}(y)\end{aligned}$$

5. 注意到$g(x)=g(0)+\int_0^x f(\xi)\mathrm{d}\xi$，由于$f\in L^1$，所以$\lim\limits_{x\to+\infty}g(x)$存在且是常数，同理可证$\lim\limits_{x\to-\infty}g(x)$也是常数. 又因为$g\in L^1$，所以$\lim\limits_{|x|\to\infty}g(x)=0$，于是

$$\begin{aligned}\mathscr{F}[g](y)&=\int_{-\infty}^{+\infty}g(x)\mathrm{e}^{-\mathrm{i}xy}\mathrm{d}x\\&=\frac{-1}{\mathrm{i}y}\mathrm{e}^{-\mathrm{i}xy}g(x)\Big|_{-\infty}^{+\infty}+\frac{1}{\mathrm{i}y}\int_{-\infty}^{+\infty}g'(x)\mathrm{e}^{-\mathrm{i}xy}\mathrm{d}x\\&=\frac{1}{\mathrm{i}y}\int_{-\infty}^{+\infty}f(x)\mathrm{e}^{-\mathrm{i}xy}\mathrm{d}x\\&=\frac{1}{\mathrm{i}y}\hat{f}(y)\end{aligned}$$

注：在上面的证明中，仅由g的连续性和绝对可积性并不能推出$\lim\limits_{|x|\to\infty}g(x)=0$，因此证明$\lim\limits_{x\to\infty}g(x)$存在是必要的. 考虑如下反例：

令$\phi_n(x)=\begin{cases}2n^2(x-n)+1, & x\in[n-\frac{1}{2n^2},n]\\ -2n^2(x-n)+1, & x\in[n,n+\frac{1}{2n^2}]\\ 0, & x\notin[n,n+1]\end{cases}$以及$g(x)=\sum\limits_{n=1}^{\infty}\phi_n(x)$，则容易证明$g$连续，并且$\left\{\sum\limits_{n=1}^{N}\phi_n(x)\right\}_{N=1}^{\infty}$是非负递增函数列，所以根据单调收敛定理可知

$$\int_{-\infty}^{+\infty}|g(x)|\mathrm{d}x=\sum_{n=1}^{\infty}\int_{-\infty}^{+\infty}\phi_n(x)\mathrm{d}x=\sum_{n=1}^{\infty}\frac{1}{2n^2}<\infty$$

即$g\in L^1$，但显然$\lim\limits_{x\to+\infty}g(x)$不存在.

7.(2)对未知函数关于变量x做Fourier变换得$\hat{u}(\xi,y)=\mathscr{F}[u](\xi)$，从而

$$\mathscr{F}[u_x](\xi)=\mathrm{i}\xi\hat{u}(\xi,y)$$

$$\mathscr{F}[u_{xx}](\xi)=-\xi^2\hat{u}(\xi,y)$$

于是原方程化为

$$\begin{cases}-\xi^2\hat{u}+\hat{u}_{yy}=0, & \xi\in\mathbb{R},y>0\\ \hat{u}(\xi,0)=\hat{\varphi}(\xi), & \xi\in\mathbb{R}\end{cases}$$

得$\hat{u}$通解为$\hat{u}(\xi,y)=C_1(\xi)\mathrm{e}^{|\xi|y}+C_2(\xi)\mathrm{e}^{-|\xi|y}$，同时注意到

$$|\hat{u}(\xi,y)|=|\int_{-\infty}^{+\infty}u(x,y)\mathrm{e}^{-\mathrm{i}x\xi}\mathrm{d}x|\le\int_{-\infty}^{+\infty}|u(x,y)|\mathrm{d}x<+\infty,\quad\forall\xi,y\in\mathbb{R}$$

因此$C_1(\xi)\equiv0$，再代入初值条件即得$C_2(\xi)=\hat{\varphi}(\xi)$，从而$\hat{u}(\xi,y)=\hat{\varphi}(\xi)\mathrm{e}^{-|\xi|y}$，两边同时做Fourier逆变换并根据卷积的性质得

$$u(x,y)=\varphi(x)*\mathscr{F}^{-1}[\mathrm{e}^{-|\xi|y}](x,y)$$

其中

$$\begin{aligned}\mathscr{F}^{-1}[\mathrm{e}^{-|\xi|y}](x,y)&=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{e}^{\mathrm{i}\xi x}\mathrm{e}^{-|\xi|y}\mathrm{d}\xi\\&=\frac{1}{2\pi}\int_{0}^{+\infty}\mathrm{e}^{(\mathrm{i}x-y)\xi}\mathrm{d}\xi+\frac{1}{2\pi}\int_{-\infty}^{0}\mathrm{e}^{(\mathrm{i}x+y)\xi}\mathrm{d}\xi\\&=\frac{y}{\pi(x^2+y^2)}\end{aligned}$$

于是根据卷积的定义得

$$u(x,y)=\frac{1}{\pi}\int_{-\infty}^{+\infty}\varphi(s)\frac{y}{(x-s)^2+y^2}\mathrm{d}s$$

(5)对未知函数关于x,y,z做Fourier变换，原方程化为

$$\begin{cases}\hat{u}_t+\mathrm{i}(\xi^2+\eta^2+\zeta^2)\hat{u}=0, & (\xi,\eta,\zeta)\in\mathbb{R}^3,t>0\\ \hat{u}(\xi,\eta,\zeta,0)=\hat{\varphi}(\xi,\eta,\zeta), & (\xi,\eta,\zeta)\in\mathbb{R}^3\end{cases}$$

解得$\hat{u}(\xi,\eta,\zeta,t)=\hat{\varphi}(\xi,\eta,\zeta)\mathrm{e}^{-\mathrm{i}(\xi^2+\eta^2+\zeta^2)t}$，两边做Fourier逆变换得到$u(x,y,z,t)=\varphi*\mathscr{F}^{-1}[\mathrm{e}^{-\mathrm{i}(\xi^2+\eta^2+\zeta^2)t}](x,y,z)$，受例4.1.7启发，不难证明以下结论：令$A=\frac{1}{4\mathrm{i}t}$，则$\mathrm{e}^{-\mathrm{i}(\xi^2+\eta^2+\zeta^2)t}=\left(\sqrt{\frac{A}{\pi}}\right)^3\mathscr{F}[\mathrm{e}^{-A(x^2+y^2+z^2)}](\xi,\eta,\zeta)$. 于是记$g(x,y,z)=\mathrm{e}^{-A(x^2+y^2+z^2)}$即得

$$\begin{aligned}u(x,y,z,t)&=\left(\sqrt{\frac{A}{\pi}}\right)^3\varphi*g\\&=\left(\frac{1}{4\pi it}\right)^{\frac{3}{2}}\int_{\mathbb{R}^3}\varphi(x-\xi,y-\eta,z-\zeta)\mathrm{e}^{-\frac{\xi^2+\eta^2+\zeta^2}{4\mathrm{i}t}}\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}\zeta\end{aligned}$$

11.(1)由Poisson公式得

$$u(x,t)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\sin(x+2a\sqrt{t}\eta)\cdot\mathrm{e}^{-\eta^2}\mathrm{d}\eta$$

为此，考虑含参变量积分$I(b)=\displaystyle\int_{-\infty}^{+\infty}\sin(b\eta+c)\cdot\mathrm{e}^{-\eta^2}\mathrm{d}\eta$，根据积分号下求导定理得

$$\begin{aligned}I'(b)&=\int_{-\infty}^{+\infty}x\cos(b\eta+c)\cdot\mathrm{e}^{-\eta^2}\mathrm{d}\eta\\&=-\frac{1}{2}\cos(b\eta+c)\cdot\mathrm{e}^{-\eta^2}\Big|_{-\infty}^{+\infty}-\frac{b}{2}\int_{-\infty}^{+\infty}\sin(b\eta+c)\cdot\mathrm{e}^{-\eta^2}\mathrm{d}\eta\\&=-\frac{b}{2}I(b)\end{aligned}$$

因此得到$I(b)=C\mathrm{e}^{-\frac{b^2}{4}}$，代入$I(0)=\sin c\int_{-\infty}^{+\infty}\mathrm{e}^{-\eta^2}\mathrm{d}\eta=\sqrt{\pi}\sin c$得

$$I(b)=\sqrt{\pi}\sin c\cdot\mathrm{e}^{-\frac{b^2}{4}}$$

令$b=2a\sqrt{t}$，$c=x$即得$u(x,t)=\sin x\cdot\mathrm{e}^{-a^2t}$.

13. 根据Poisson公式得

$$u(x,t)=\frac{1}{2ah\sqrt{\pi t}}\int_{-h}^{h}\mathrm{e}^{-\frac{(x-\xi)^2}{4a^2t}}\mathrm{d}\xi$$

由积分第一中值定理知，存在$s\in(-h,h)$使得$u(x,t)=\frac{1}{2ah\sqrt{\pi t}}\mathrm{e}^{-\frac{(x-s)^2}{4a^2t}}\cdot 2h=\frac{1}{a\sqrt{\pi t}}\mathrm{e}^{-\frac{(x-s)^2}{4a^2t}}$，令$h\to0^+$则有$s\to0$，故结论成立.

14. 直接代入验证$u=u_1u_2$满足方程和初值条件即可.

15. 根据叠加原理， 有$u(x,t)=\sum_{i=1}^{n}u_i(x,t)$，其中$u_i$满足问题

$$\begin{cases}\partial_t u_i + a^2(\partial_{xx}u_i+\partial_{yy}u_i)=0, & (x,y)\in\mathbb{R}^2, t>0\\ u_i(x,y,0)=\alpha_i(x)\beta_i(y), & (x,y)\in\mathbb{R}^2\end{cases}$$

再根据第14题结论可知

$$u_i(x,y,t)=\frac{1}{\pi}\int_{-\infty}^{+\infty}\alpha_i(x+2a\sqrt{t}\xi)\mathrm{e}^{-\xi^2}\mathrm{d}\xi\int_{+\infty}^{+\infty}\beta_i(y+2a\sqrt{t}\eta)\mathrm{e}^{-\eta^2}\mathrm{d}\eta$$

因此

$$u(x,y,t)=\frac{1}{\pi}\sum_{i=1}^{n}\int_{-\infty}^{+\infty}\alpha_i(x+2a\sqrt{t}\xi)\mathrm{e}^{-\xi^2}\mathrm{d}\xi\int_{+\infty}^{+\infty}\beta_i(y+2a\sqrt{t}\eta)\mathrm{e}^{-\eta^2}\mathrm{d}\eta$$

17. 做变换$w=\mathrm{e}^{-t^2}u$，则有

$$\begin{cases}w_t=-2tw+\mathrm{e}^{-t^2}u_t\\ w_{xx}=\mathrm{e}^{-t^2}u_{xx}\end{cases}$$

故$u_t=\mathrm{e}^{t^2}(w_t+2tw)$，$u_{xx}=\mathrm{e}^{t^2}w_{xx}$，从而原问题化为

$$\begin{cases}w_t-a^2w_{xx}=f(x,t)\mathrm{e}^{-t^2}, & x\in\mathbb{R}, t>0\\ w(x,0)=0, & x\in\mathbb{R}\end{cases}$$

为此，考虑如下问题

$$\begin{cases}v_t-a^2v_{xx}=0, & x\in\mathbb{R}, t>\tau\\ v(x,0)=f(x,\tau)\mathrm{e}^{-\tau^2}, & x\in\mathbb{R}\end{cases}$$

根据Poisson公式得该问题的解为$v(x,t;\tau)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}f(x+2a\sqrt{t-\tau}\eta,\tau)\mathrm{e}^{-(\eta^2+\tau^2)}\mathrm{d}\eta$，因此

$$u(x,t)=\frac{\mathrm{e}^{t^2}}{\sqrt{\pi}}\int_0^t\mathrm{e}^{-\tau^2}\mathrm{d}\tau\int_{-\infty}^{+\infty}f(x+2a\sqrt{t-\tau}\eta)\mathrm{e}^{-\eta^2}\mathrm{d}\eta$$

23.(2)设$u(x,t)=X(x)T(t)$，代入原方程得$-\lambda=\frac{X''}{X}=\frac{T'}{a^2T}$，再根据边界条件得$X(0)=X(\pi)=0$，于是

$$\begin{cases}X''+\lambda X=0\\ X(0)=X(\pi)=0\end{cases}$$

$$T'+a^2\lambda T=0$$

容易得到X的通解为$X(x)=c_1\cos\sqrt{\lambda}x+c_2\sin\sqrt{\lambda}x$，代入边界条件易知$c_1=0$，$\lambda=\lambda_k=k^2$，从而$X(x)=X_k(x)=c_k\sin kx$，$k=1,2,....$

同理T有通解$T(t)=T_k(t)=C_k\mathrm{e}^{-a^2k^2t}$，故原问题有形式解

$$u(x,t)=\sum_{k=1}^{\infty}c_k\mathrm{e}^{-a^2k^2t}\sin kx$$

根据初始条件得$u(x,0)=\sum\limits_{k=1}^{\infty}c_k\sin kx=\sin x$，所以

$$c_k=\begin{cases}1, & k=1\\ 0, & otherwise\end{cases}$$

从而原问题的解为$u(x,t)=\mathrm{e}^{-a^2t}\sin x$.

24. 做变换$v(x,t)=u(x,t)-U_1-\frac{x}{l}(U_2-U_1)$，则原问题化为

$$\begin{cases}v_t-a^2v_{xx}=0, & x\in(0,l),t>0\\ v(x,0)=\varphi(x)-U_1-\frac{x}{l}U_2:=\tilde{\varphi}(x), & x\in[0,l]\\ v(0,t)=v(l,t)=0, & t\geq 0\end{cases}$$

令$v(x,t)=X(x)T(t)$容易得到

$$\begin{cases}X''+\lambda X=0\\ X(0)=X(l)=0\end{cases}$$

$$T'+a^2\lambda T=0$$

与前面题目同理可知$X(x)=X_k(x)=c_k\sin\frac{k\pi x}{l}$，$T_k(t)=C_k\mathrm{e}^{-\frac{a^2k^2\pi^2}{l^2}t}$，所以原问题有形式解

$$v(x,t)=\sum_{k=1}^{\infty}c_k\mathrm{e}^{-\frac{a^2k^2\pi^2}{l^2}t}\sin\frac{k\pi x}{l}$$

再根据初始条件可知

$$c_k=\frac{2}{l}\int_0^l\tilde{\varphi}(x)\sin\frac{k\pi x}{l}\mathrm{d}x$$

故

$$v(x,t)=\frac{2}{l}\sum_{k=1}^{\infty}\mathrm{e}^{-\frac{a^2k^2\pi^2}{l^2}t}\sin\frac{k\pi x}{l}\int_0^l\tilde{\varphi}(\xi)\sin\frac{k\pi\xi}{l}\mathrm{d}\xi$$

$$u(x,t)=v(x,t)+U_1+\frac{x}{l}(U_2-U_1)$$

令$t\to+\infty$可得$\mathrm{e}^{-\frac{a^2k^2\pi^2}{l^2}t}\to 0$从而$v(x,t)\to 0$，所以（当Fourier级数绝对收敛时，求和和极限可以换序）

$$\lim_{t\to+\infty}u(x,t)=U_1+\frac{x}{l}(U_2-U_1)$$

29. 做变换$w=\mathrm{e}^{-\frac{b}{a^2}u}$，则

$$w_t=-\frac{b}{a^2}\mathrm{e}^{-\frac{b}{a^2}u}u_t$$

$$w_{x_i}=-\frac{b}{a^2}\mathrm{e}^{-\frac{b}{a^2}u}u_{x_i}$$

$$u_{x_ix_i}=\frac{b^2}{a^4}\mathrm{e}^{-\frac{b}{a^2}u}u_{x_i}^2-\frac{b}{a^2}\mathrm{e}^{-\frac{b}{a^2}}u_{x_ix_i}$$

代入原方程则有

$$\begin{cases} w_t - a^2\Delta u = 0, & x\in\mathbb{R}^n, t>0 \\ w(x,0) = \mathrm{e}^{-\frac{b}{a^2}\varphi(x)}, & x\in\mathbb{R}^n \end{cases}$$

根据多元函数的Fourier变换容易得到n维热传导方程的Poisson公式，于是

$$w(x,t) = \frac{1}{(4\pi a^2 t)^{n/2}}\int_{\mathbb{R}^n} \mathrm{e}^{-\frac{b}{a^2}\varphi(\xi)}\mathrm{e}^{-\frac{|x-\xi|^2}{4a^2t}}\mathrm{d}\xi$$

35.(1)

①若在Ω_T内有$v_t - a^2\Delta v < 0$

反证法，假设v在$\overline{\Omega}_T$的最大值在Ω_T内取得，即$\exists P_0=(x_0,t_0)\in\Omega_T$使得$v(x_0,t_0)=\max\limits_{\overline{\Omega}_T} v(x,t) = \max\limits_{\Omega_T} v(x,t)$.

由于v在P_0处取得最大值，所以

$$v_{x_i}(P_0) = 0, v_{x_ix_i}(P_0)\le 0,\ \ \forall i\in\{1,...,n\}$$

并且如果$t_0\ne T$，则$v_t(P_0)=0$；如果$t_0 = T$，则$v_t(P_0)\ge 0$，总之，一定有

$$v_t(P_0)\ge 0$$

从而$v_t(P_0) - a^2\Delta v(P_0)\ge 0$，与$v_t - a^2\Delta v<0$矛盾，故结论得证.

②若在Ω_T内有$v_t - a^2\Delta v \le 0$

$\forall\varepsilon>0$，令$u(x,t) = v(x,t)-\varepsilon t$，则有

$$u_t - a^2\Delta u = v_t - a^2\Delta v - \varepsilon < 0,\ \ (x,t)\in\Omega_T$$

根据①中的讨论可知$\max\limits_{\overline{\Omega}_T} u = \max\limits_{\partial_p\Omega_T} u$，从而

$$\max_{\overline{\Omega}_T} v = \max_{\overline{\Omega}_T}(u+\varepsilon t) \le \max_{\overline{\Omega}_T} u + \varepsilon T = \max_{\partial_p\Omega_T} u + \varepsilon T \le \max_{\partial_p\Omega_T} v + \varepsilon T$$

根据ε的任意性，令$\varepsilon\to 0^+$，得

$$\max_{\overline{\Omega}_T} v \le \max_{\partial_p\Omega_T} v$$

又因为$\partial_p\Omega_T\subset\overline{\Omega}_T$，所以

$$\max_{\overline{\Omega}_T} v = \max_{\partial_p\Omega_T} v$$

综上，结论成立.

(2)根据v的定义得

$$v_t = \phi'(u)u_t$$

$$v_{x_i} = \phi'(u)u_{x_i}$$

$$v_{x_ix_i} = \phi''(u)u_{x_i}^2 + \phi'(u)u_{x_ix_i}$$

因此

$$\begin{aligned} v_t - a^2\Delta v &= \phi'(u)u_t - a^2\sum_i[\phi''(u)u_{x_i}^2 + \phi'(u)u_{x_ix_i}] \\ &= \phi'(u)(u_t - a^2\Delta u) - a^2\phi''(u)|Du|^2 \end{aligned}$$

因为u满足热传导方程，所以$v_t - a^2\Delta v = -a^2\phi''(u)|Du|^2$，又因为$\phi$是（下）凸函数，所以$\phi'' \geq 0$，因此

$$v_t - a^2\Delta v \leq 0$$

即$v = \phi(u)$是方程的下解.

(3)记$Lv = v_t - a^2\Delta v$，则显然L是线性算子，因此

$$\begin{aligned}
Lv &= a^2 L(|Du|^2) + L(u_t^2) \\
&= a^2 \sum_i L(u_{x_i}^2) + L(u_t^2)
\end{aligned}$$

其中

$$\begin{aligned}
L(u_t^2) &= 2u_t u_{tt} - a^2 \sum_i \frac{\partial^2}{\partial x_i^2}(u_t^2) \\
&= 2u_t u_{tt} - 2a^2 \sum_i \frac{\partial}{\partial x_i}(u_t u_{tx_i}) \\
&= 2u_t u_{tt} - 2a^2 \sum_i (u_{tx_i}^2 + u_t u_{tx_i x_i}) \\
&= 2u_t \frac{\partial}{\partial t}(u_t - a^2\Delta u) - 2a^2 \sum_i u_{tx_i}^2 \\
&= -2a^2 \sum_i u_{tx_i}^2 \\
&\leq 0
\end{aligned}$$

$$\begin{aligned}
L(u_{x_i}^2) &= 2u_{x_i} u_{x_i t} - a^2 \sum_j \frac{\partial^2}{\partial x_j^2}(u_{x_i}^2) \\
&= 2u_{x_i} u_{x_i t} - 2a^2 \sum_j \frac{\partial}{\partial x_j}(u_{x_i} u_{x_i x_j}) \\
&= 2u_{x_i} u_{x_i t} - 2a^2 \sum_j (u_{x_i x_j}^2 + u_{x_i} u_{x_i x_j x_j}) \\
&= 2u_{x_i} \frac{\partial}{\partial x_i}(u_t - a^2\Delta u) - 2a^2 \sum_j u_{x_i x_j}^2 \\
&= -2a^2 \sum_j u_{x_i x_j^2} \\
&\leq 0
\end{aligned}$$

因此$Lv \leq 0$，即v是方程的下解.

36. 令$w = \mathrm{e}^{-t}u$，则代入原方程可以得到

$$w_t - a^2 w_{xx} = 0, \quad (x,t) \in Q_T$$

根据极值原理以及u的非负性可知

$$\max_{\overline{Q}_T} w = \max_{\partial_p Q_T} w = \max_{\partial_p Q_T} \mathrm{e}^{-t}u \leq \max_{\partial_p Q_T} u \leq M$$

所以
$$w = \mathrm{e}^{-t}u \le M, \quad (x,t) \in \overline{Q}_T$$
即$u \le M\mathrm{e}^t$，$(x,t) \in \overline{Q}_T$.

37. 设u, v满足比较原理的条件，对任意$\varepsilon > 0$，令$w = u - v - \varepsilon t$，现假设$w$的最大值在$Q_T$内取得，即存在$P_0 = (x_0, t_0) \in Q_T$使得$\max\limits_{\overline{Q}_T} w = w(P_0)$，则
$$w_x(P_0) = u_x(P_0) - v_x(P_0) = 0 \Rightarrow u_x(P_0) = v_x(P_0)$$
$$w_{xx}(P_0) = u_{xx}(P_0) - v_{xx}(P_0) \le 0$$
并且当$t_0 \ne T$时，$w_t(P_0) = 0$；当$t_0 = T$时，$w_t(P_0) \ge 0$，总之，$w_t(P_0) \ge 0$，从而$u_t(P_0) - v_t(P_0) - \varepsilon \ge 0$. 于是
$$\begin{aligned} Lu(P_0) - Lv(P_0) - \varepsilon &= u_t(P_0) - v_t(P_0) + v_{xx}(P_0) - u_{xx}(P_0) + |u_x(P_0)| - |v_x(P_0)| - \varepsilon \\ &= u_t(P_0) - v_t(P_0) - \varepsilon - [u_{xx}(P_0) - v_{xx}(P_0)] \\ &\ge 0 \end{aligned}$$
故$Lu(P_0) - Lv(P_0) \ge \varepsilon$，与$Lu - Lv \le 0, (x,t) \in Q_T$的条件矛盾，因此假设不成立，即
$$\max_{\overline{Q}_T} w = \max_{\partial_p Q_T} w$$
设$\max\limits_{\partial_p Q_T} w = w(x', t')$，其中$(x', t') \in \partial_p Q_T$，则根据条件有$u(x', t') - v(x', t') \le 0$，于是对任意$(x,t) \in \overline{Q}_T$，有
$$w(x,t) = u(x,t) - v(x,t) - \varepsilon t \le w(x', t') = u(x', t') - v(x', t') - \varepsilon t' \le -\varepsilon t'$$
故
$$u(x,t) \le v(x,t) + \varepsilon t - \varepsilon t' \le v(x,t) + \varepsilon(T - t')$$
由ε的任意性，令$\varepsilon \to 0^+$得$u(x,t) \le v(x,t)$，即比较原理成立.

38. 设u, v满足比较原理的条件，即
$$\begin{cases} Lu \le Lv, & x \in Q_T \\ u \le v, & x \in \partial_p Q_T \end{cases}$$
令$w = u - v$，则
$$Lu - Lv = w_t - w_{xx} + (u^2 + uv + v^2)w$$
假设w的最大值在Q_T内取得，即存在$P_0 = (x_0, t_0) \in Q_T$使得$\max\limits_{\overline{Q}_T} w = w(x_0, t_0)$，则类似上一题可知$w_t(P_0) \ge 0$，$w_{xx}(P_0) \le 0$. ①如果$w(P_0) \le 0$

则在$\overline{Q}_T$上有$w = u - v \le w(P_0) \le 0$。从而$u \le v$，结论得证.

②如果$w(P_0) > 0$

则由$w_t(P_0) \ge 0$，$w_{xx}(P_0) \le 0$得
$$\begin{aligned} (Lu - Lv)(P_0) &= w_t(P_0) - w_{xx}(P_0) + (u^2 + v^2 + uv)(P_0) \cdot w(P_0) \\ &\ge (u^2 + uv + v^2)(P_0) \cdot w(P_0) \end{aligned}$$

同时注意到$u^2+v^2+uv\geq 2|uv|+uv\geq 0$，当且仅当$u=v=0$时取等号. 又因为此时$w(P_0)=u(P_0)-v(P_0)>0$，故$u(P_0)\neq v(P_0)$，从而

$$(Lu-Lv)(P_0)>0$$

与条件矛盾，从而假设不成立.

综上，$\max\limits_{\overline{Q}_T} w=\max\limits_{\partial_p Q_T} w\leq 0$，从而在$\overline{Q}_T$上有$u\leq v$，即比较原理成立.

39. 令$v=u_t$，则有

$$\begin{cases} v_t-a^2v_{xx}=f_t(x,t) \\ v(x,0)=f(x,0)+a^2u_{xx}(x,0)=f(x,0)+a^2\varphi''(x) \\ v(0,t)=v(l,t)=0 \end{cases}$$

由课本定理4.11知

$$\begin{aligned} \max_{\overline{Q}_T}|v| &\leq T\cdot\sup_{Q_T}|f(x,t)|+\max_{[0,l]}|f(x,0)+a^2\varphi''(x)| \\ &\leq T\cdot\max_{\overline{Q}_T}|f(x,t)|+\max_{[0,l]}|f(x,0)|+a^2\max_{[0,l]}|\varphi''(x)| \\ &\leq (T+1)\max_{\overline{Q}_T}|f(x,t)|+a^2\max_{[0,l]}|\varphi''(x)| \\ &= (T+1)\|f\|_{C^1(\overline{Q}_T)}+a^2\|\varphi''\|_{C[0,l]} \end{aligned}$$

40.(1)由初始条件和边界条件可知$\varphi(0)=\varphi(l)=0$，于是

$$\varphi(x)\leq x\|\varphi\|_{C^1}$$

事实上，令$h(x)=\pm x\|\varphi\|_{C^1}-\varphi(x)$再求导不难证明上式.

现令$C=\|\varphi\|_{C^1}$，以及$v(x,t)=Cx$，于是在Q_T上有$Lu=Lv=0$，在∂_pQ_T上有$u\leq v$，根据比较原理，在$\overline{Q}_T$上成立

$$|u(x,t)|\leq v(x,t)=Cx$$

所以$\forall x\in(0,l],\forall t\in[0,T]$，有

$$|\frac{u(x,t)-u(0,t)}{x}|=|\frac{u(x,t)}{x}|\leq\|\varphi\|_{C^1}$$

由Lagrange中值定理知存在$\xi\in(0,x)$使得$u_x(\xi,t)=\frac{u(x,t)}{x}$从而$|u_x(\xi,t)|\leq\|\varphi\|_{C^1}$，令$x\to 0^+$得到$|u_x(0,t)|\leq\|\varphi\|_{C^1}$，$\forall t\in[0,T]$，即$\max\limits_{[0,T]}|u_x(0,t)|\leq\|\varphi\|_{C^1}$.

同理可证$\max\limits_{[0,T]}|u_x(l,t)|\leq\|\varphi\|_{C^1}$.

(2)令$v=u_x$，则

$$\begin{cases} v_t-a^2v_{xx}=0 \\ v(x,0)=\varphi'(x) \\ v(0,t)=u_x(0,t) \\ v(l,t)=u_x(l,t) \end{cases}$$

由定理4.11得

$$\max_{\overline{Q}_T}|v(x,t)|\le\max\{\max_{[0,l]}|\varphi'(x)|,\max_{[0,T]}|u_x(0,t)|,\max_{[0,T]}|u_x(l,t)|\}$$

再由(1)知$\max\limits_{[0,T]}|u_x(0,t)|$，$\max\limits_{[0,T]}|u_x(l,t)|\le\|\varphi\|_{C^1}$，所以

$$\max_{\overline{Q}_T}|u_x(x,t)|\le\|\varphi\|_{C^1}$$

42.(1)

$$\begin{aligned}\mathscr{L}[f](p)&=\int_0^{+\infty}\mathrm{e}^{-(2+p)t}\mathrm{d}t\\&=\frac{1}{p+2}\end{aligned}$$

(2)

$$\begin{aligned}\mathscr{L}[f](p)&=\int_0^{+\infty}\mathrm{e}^{-(p+4)t}\cos 4t\mathrm{d}t\\&=\frac{1}{p+4}-\frac{4}{p+4}\int_0^{+\infty}\mathrm{e}^{-(p+4)t}\sin 4t\mathrm{d}t\\&=\frac{1}{p+4}-\frac{16}{(p+4)^2}\mathscr{L}[f](p)\end{aligned}$$

故$\mathscr{L}[f](p)=\dfrac{p+4}{16+(p+4)^2}$

(9)考虑

$$\begin{aligned}\mathscr{L}[\frac{f(t)}{t}](p)&=\mathscr{L}[\int_0^t\mathrm{e}^{-3\tau}\sin 2\tau\mathrm{d}\tau](p)\\&=\frac{1}{p}\mathscr{L}[\mathrm{e}^{-3t}\sin 2t](p)\\&=\frac{1}{p}\int_0^{+\infty}\mathrm{e}^{-(p+3)t}\sin 2t\mathrm{d}t\\&=\frac{2}{(p^2+6p+13)p}\end{aligned}$$

若记$F(p)=\mathscr{L}[f](p)$，则

$$\int_p^{+\infty}F(\eta)\mathrm{d}\eta=\frac{2}{p(p^2+6p+13)}$$

于是

$$F(p)=-\frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{2}{p(p^2+6p+13)}\right)=\frac{6p^2+24p+26}{p^2(p^2+6p+13)^2}$$

43.(1)$F(p)\mathrm{e}^{pt}=\dfrac{\mathrm{e}^{pt}(p+3)}{(p+1)(p-3)}$有两个奇点$p_1=-1,p_2=3$，易知

$$\mathrm{Res}(F(p)\mathrm{e}^{pt},p_1)=-\frac{1}{2}\mathrm{e}^{t}$$

$$\mathrm{Res}(F(p)\mathrm{e}^{pt},p_2)=\frac{3}{2}\mathrm{e}^{3t}$$

所以$\mathscr{L}^{-1}[F](t)=-\dfrac{1}{2}\mathrm{e}^t+\dfrac{3}{2}\mathrm{e}^{3t}$.

3 第五章 椭圆型方程

学习本章时上课参考的书籍还包括周蜀林编写的《偏微分方程》以及Evans编写的《Partial Differential Equations》，部分课后作业来自这两本书，由于本人精力有限，并且部分作业的手写稿已经遗失（我也不知道为什么找不到了），在此不再一一整理，仅整理本书中部分习题的答案；此外，由于学习过程中参考的书籍较多，部分符号可能与本书不一致，后文中也不再一一说明，如有疑问，还请大家自行查阅相关资料（orz）.

1. 做球坐标变换

$$\begin{cases} x = r\cos\theta\sin\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\varphi \end{cases}$$

则有

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \arctan\frac{y}{x} \\ \varphi = \arccos\frac{z}{r} \end{cases}$$

代入计算即可.

2.做柱坐标变换

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

则有

$$\begin{cases} y = \sqrt{x^2+y^2} \\ \theta = \arctan\frac{y}{x} \\ z = z \end{cases}$$

代入计算即可.

5. 根据第二题知

$$\begin{aligned} \Delta(r^n\sin n\theta) &= \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial(r^n\sin n\theta)}{\partial r}) + \frac{1}{r^2}\frac{\partial^2(r^n\sin n\theta)}{\partial\theta^2} \\ &= n^2r^{n-2}\sin n\theta - n^2r^{n-2}\sin n\theta \\ &= 0 \end{aligned}$$

6.令$u = v + \frac{1}{4}\mathrm{e}^{-r^2}$，则$\Delta v = -\frac{1}{2}\mathrm{e}^{-r^2}$. 令$v(x) = \varphi(|x|) = \varphi(r)$代入方程有

$$\varphi'' + \frac{2}{r}\varphi' = -\frac{1}{2}\mathrm{e}^{-r^2}$$

从而

$$[r\varphi(r)]'' = -\frac{1}{2}r\mathrm{e}^{-r^2}$$

积分得

$$r\varphi(r) = \frac{1}{4}\int_0^r \mathrm{e}^{-t^2}\mathrm{d}t + Cr + D$$

令$r=0$得$D=0$，再令$r\to+\infty$得$C=0$. 令$\varphi(0)=\lim_{r\to 0}\varphi(r)=\frac{1}{4}$，容易验证$\varphi\in C^\infty[0,+\infty)$，代入$u$即得方程的解.

7. 注意到

$$\Delta\left[|x|^{2-n}u(\frac{x}{|x|^2})\right] = |x|^{2-n}\Delta[u(\frac{x}{|x|^2})] + 2D(|x|^{2-n})\cdot D(u(\frac{x}{|x|^2})) + \Delta(|x|^{2-n})u(\frac{x}{|x|^2})$$

记$u_j=\frac{\partial u}{\partial x_j}$，$r=|x|, y=\frac{x}{|x|^2}$，经计算有

$$\frac{\partial}{\partial x_i}(|x|^{2-n}) = (2-n)x_i|x|^{-n}$$

$$\frac{\partial^2}{\partial x_i^2}(|x|^{2-n}) = (2-n)|x|^{-n} - n(2-n)x_i^2|x|^{-2-n}$$

$$\Delta(|x|^{2-n}) = n(2-n)|x|^{-n} - n(2-n)|x|^{-2-n}|x|^2 = 0$$

$$\frac{\partial u(y)}{\partial x_i} = \sum_j u_j\frac{\partial y_j}{\partial x_i} = \sum_j u_j\frac{-2x_ix_j}{r^4} + \frac{u_i}{r^2}$$

$$\frac{\partial u_j(y)}{\partial x_i} = \sum_k u_{jk}\frac{\partial y_k}{\partial x_i} = \sum_k u_{jk}\frac{-2x_ix_k}{r^4} + \frac{u_{ij}}{r^2}$$

$$\frac{\partial^2 u(y)}{\partial x_i^2} = \sum_j\left(\frac{\partial u_j(y)}{\partial x_i}\frac{-2x_ix_i}{r^4} - 2u_j\frac{x_j(r^2-4x_i^2)}{r^6}\right) - 2u\frac{x_iu_i}{r^4} + \frac{\frac{\partial u_i(y)}{\partial x_i}r^2 - 2x_iu_i}{r^4}$$

$$\begin{aligned}
\Delta(u(y)) &= \sum_i \frac{\partial^2 u(y)}{\partial x_i^2} \\
&= \sum_i\sum_j\sum_k u_{jk}\frac{4x_i^2x_jx_k}{r^8} - \sum_i\sum_k u_{ik}\frac{2x_ix_k}{r^4} - \sum_j\sum_j u_{ij}\frac{2x_ix_j}{r^6} \\
&+ \sum_i\frac{u_{ii}}{r^4} - 2\sum_i\sum_j u_j\frac{x_j(r^2-4x_i^2)}{r^6} - 4\sum_i\frac{x_iu_i}{r^4} \\
&= 2(2-n)\sum_i\frac{x_iu_i}{r^4}
\end{aligned}$$

$$2D(|x|^{2-n})\cdot D(u(\frac{x}{|x|^2})) = 2(2-n)\sum_i\frac{x_iu_i}{r^{n+2}}$$

代入计算即得结论.（太痛苦了）

8. 对方程两侧积分并应用Green公式得

$$\int_\Omega \Delta u\mathrm{d}x = \int_\Omega f\mathrm{d}x = \int_{\partial\Omega}\frac{\partial u}{\partial n}\mathrm{d}S$$

所以得到一个必要条件为

$$\int_{\partial\Omega}\frac{\partial u}{\partial n}\mathrm{d}S = \int_\Omega f\mathrm{d}x$$

10. 将$u(x)=\varphi(|x|)$代入方程有

$$\begin{cases}(n-1)\varphi'(|x|)+|x|\varphi''(|x|)=0, & |x|>1\\ \varphi(|x|)=0, & |x|=1\end{cases}$$

解ODE$(n-1)\varphi'(r)+r\varphi''(r)=0$并带入边界条件$\varphi(1)=0$得

$$\varphi(r)=\begin{cases}C(r-1), & n=1\\ C\ln r, & n=2\\ C(\frac{1}{x^{n-2}}-1), & n\geq 3\end{cases}$$

代入u即得原问题的解.

11.(1)

①若u在$\partial\Omega$上取非负最大值，则

$$\max_{\bar{\Omega}} u(x)=u(x)|_{x\in\partial\Omega}=0$$

从而$u(x)\leq 0,\forall x\in\bar{\Omega}$,于是$\max\limits_{\bar{\Omega}}|u(x)|=-\min\limits_{\Omega}u(x)$，记为$-u(\tilde{x}),\tilde{x}\in\Omega$. 于是得到

$$u_{x_i}(\tilde{x})=0,\ \ u_{x_ix_i}(\tilde{x})\geq 0,\ \ \forall i$$

$\Rightarrow c(\tilde{x})u(\tilde{x})=f(\tilde{x})+\Delta u(\tilde{x})\geq f(\tilde{x})$
$\Rightarrow -u(\tilde{x})\leq -\dfrac{1}{c(\tilde{x})}f(\tilde{x})\leq \dfrac{1}{c(\tilde{x})}|f(\tilde{x})|\leq \dfrac{1}{c_0}|f(\tilde{x})|$
$\Rightarrow \max\limits_{\bar{\Omega}}|u(x)|=-u(\tilde{x})\leq \dfrac{1}{c_0}|f(\tilde{x})|\leq \dfrac{1}{c_0}\sup\limits_{\Omega}|f(x)|$
②若u在Ω内取非负最大值
设$\max\limits_{\bar{\Omega}}u(x)=u(\tilde{x})$，$\tilde{x}\in\Omega$，则

$$u_{x_i}(\tilde{x}),\ \ u_{x_ix_i}(\tilde{x})\leq 0,\ \ \forall i$$

于是有$c(\tilde{x})u(\tilde{x})=f(\tilde{x})+\Delta u(\tilde{x})\leq f(\tilde{x})\leq |f(\tilde{x})|\leq \sup\limits_{\Omega}|f(x)|$，从而

$$u(\tilde{x})\leq \frac{1}{c_0}\sup_{\Omega}|f(x)|$$

此时若$u(x)$在Ω内还有小于0的最小值，记为$\min\limits_{\bar{\Omega}}u(x)=u(x')<0$，$x'\in\Omega$，则

$$u_{x_i}(x')=0,\ \ u_{x_ix_i}(x')>0,\ \ \forall i$$

于是$c(x')u(x')=f(x')+\Delta u(x')\geq f(x')$，从而

$$-u(x')\leq \frac{1}{c(x')}|f(x')|\leq \frac{1}{c_0}\sup_{\Omega}|f(x)|$$

综上，有$\max\limits_{\bar{\Omega}}|u(x)|\leq \dfrac{1}{c_0}\sup\limits_{\Omega}|f(x)|$.
(2)记Ω的直径为d.
令$w(x)=\dfrac{F}{2n}(d^2-|x|^2)\pm u(x)$，其中$F=\sup\limits_{\Omega}|f(x)|$，则有

$$-\Delta w+c(x)w(x)=F\pm f(x)+c(x)\frac{F}{2n}(d^2-|x|^2)\geq c(x)\frac{F}{2n}(d^2-|x|^2)$$

不妨设原点$O\in\Omega$，则$|x|\le d$，从而$Lw\ge 0$. 由弱极值原理，w的非正最小值在$\partial\Omega$取得，故$w(x)\ge 0$，可得

$$|u(x)|\le\frac{c(x)F}{2n}(d^2-|x|^2)\le\frac{c(x)Fd^2}{2n}\le\frac{md^2}{2n}\sup_{\Omega}|f(x)|$$

其中$c(x)\le m$.

(3)考虑方程

$$\begin{cases}-\Delta u-2u=0, & x\in\Omega\\ u(x)=0, & x\in\partial\Omega\end{cases}$$

其中$\Omega=\{(x,y)\in\mathbb{R}^2:0<x+y<\pi\}$.

显然$u(x,y)=\sin(x+y)$是方程的解，但$\sup\limits_{\bar\Omega}|u(x)|=1>\sup\limits_{\Omega}|f(x)|=0$.

12.(1) 记$M=\max\{\frac{1}{c_0}\sup\limits_{\Omega}|f(x)|,\frac{1}{\alpha_0}\max\limits_{\partial\Omega_1}|\varphi_1(x)|,\max\limits_{\partial\Omega}|\varphi_2(x)|\}$，令$w(x)=M\pm u(x)$，则有

$$\begin{cases}Lw=-\Delta w+c(x)w(x)=Mc(x)\pm f(x)\ge 0, & \forall x\in\Omega\\ w(x)=M\pm\varphi_2(x)\ge 0, & \forall x\in\partial\Omega_2\\ \dfrac{\partial w}{\partial n}+\alpha w=\alpha(x)M\pm\varphi_1(x)\ge 0, & \forall x\in\partial\Omega_1\end{cases}\tag{10}$$

根据弱极值原理，若w有负最小值，则必在$\partial\Omega$上取得，再根据(10)中第二式可知负最小值一定在$\partial\Omega_1$上取得，记$\min\limits_{\bar\Omega}w(x)=w(x_0)<0$，$x_0\in\partial\Omega_1$，则$\frac{\partial w}{\partial n}(x_0)\le 0$，于是

$$(\frac{\partial w}{\partial n}+\alpha w)|_{x=x_0}\le\alpha(x_0)w(x_0)<0$$

与(10)中第三式矛盾，所以w最小值非负，即$w(x)=M\pm u(x)\ge 0$，从而$|u(x)|\ge M$.

(2)题干有误，修改为“$\partial\Omega_1$满足内球条件”

只需令$f=\varphi_1=\varphi_2\equiv 0$然后证明相应问题的古典解只有零解即可.

根据弱极值原理，u的非负最大值和非正最小值均在$\partial\Omega=\partial\Omega_1\cup\partial\Omega_2$上取得，

① 若均在$\partial\Omega_2$上取得，则$\max u=\min u=0$，所以$u\equiv 0$显然成立.

② 若均在$\partial\Omega_1$上取得，则记$\max u(x)=u(x')\ge 0$，$\min u(x)=u(x'')\le 0$，于是

$$\frac{\partial u}{\partial n}(x')\ge 0,\frac{\partial u}{\partial n}(x'')\le 0,$$

$$\alpha(x')u(x')\ge 0,\alpha(x'')u(x'')\le 0$$

再根据$\partial\Omega_1$上的边界条件知$u(x')=u(x'')=0$，所以$u(x)\equiv 0$.

③ 若在$\partial\Omega_2$取最小值，在$\partial\Omega_1$取最大值，则显然有$u(x)\ge 0,\forall x\in\bar\Omega$，设$\max u(x)=u(x_0)\ge 0$，其中$x_0\in\partial\Omega_1$，则存在球$B\subset\Omega$使得$\partial B\cap\partial\Omega_1=\{x_0\}$. 假设$u$不为常数，则在$B$内有$u(x)<u(x_0)$，于是根据Hopf引理可知$\frac{\partial u}{\partial n}(x_0)>0$，从而

$$\frac{\partial u}{\partial n}(x_0)+\alpha(x_0)u(x_0)>0,$$

矛盾，所以u为常函数，记$u(x)\equiv C$，则由$u|_{\partial\Omega_2}\equiv 0$可知$u\equiv 0$.

④ 若在$\partial\Omega_2$取最大值，在$\partial\Omega_1$取最小值，则只需考虑$-u$，与③同理可证.

13. 题干有误，应在方程的第一项前加负号.

记$Mu=-\sum\limits_{i,j=1}^{n}a_{ij}(x)u_{x_ix_j}+\sum\limits_{i=1}^{n}b_i(x)u_{x_i}$，则$Lu=Mu+c(x)u$，首先证明弱极值原理成立，即若$u\in C^2(\Omega)\cap C(\bar{\Omega})$满足$Lu\le 0$，则$\max\limits_{\bar{\Omega}}u\le\max\limits_{\partial\Omega}u^+$.

① 若$Mu<0$，假设存在$x_0\in\Omega$使得$u(x_0)=\max_{\bar{\Omega}}u(x)$，则有

$$u_{x_i}(x_0)=0,\ \ i=1,2,...,n$$

由于$\boldsymbol{A}=(a_{ij}(x))$是正定矩阵，所以存在正交矩阵$\boldsymbol{O}=(o_{ij})$使得

$$\boldsymbol{O}\boldsymbol{A}\boldsymbol{O}^{\boldsymbol{T}}=\mathrm{diag}(d_1,...,d_n)$$

其中$d_1,...,d_n>0$.

现记$y=x_0+\boldsymbol{O}(x-x_0)$，则$x-x_0=\boldsymbol{O}^{\boldsymbol{T}}(y-x_0)$，因此

$$u_{x_i}=\sum_{k=1}^{n}u_{y_k}o_{ki},$$

$$u_{x_ix_j}=\sum_{k,l=1}^{n}u_{y_k}u_{y_l}o_{ki}o_{lj}$$

因此在$x=x_0$处有，

$$\begin{aligned}\sum_{i,j}a_{ij}(x)u_{x_ix_j}&=\sum_{i,j}\sum_{k,l}a_{ij}u_{y_ky_l}o_{ki}o_{lj}\\&=\sum_{k}d_ku_{y_ky_k}\\&\le 0\end{aligned}$$

此时$Mu(x_0)\ge 0$，矛盾，故假设不成立，结论得证.

② 设$Lu\le 0$，若$u\le 0$在$\bar{\Omega}$内成立，则$\max\limits_{\bar{\Omega}}u\le\max\limits_{\partial\Omega}u^+$显然成立，若不然，由$u$的连续性，令$V:=\{x\in\Omega|u(x)>0\}$为$\Omega$中的非空开集.

对$\forall x\in V$，有$Mu=Lu-c(x)u(x)<-c(x)u(x)<0$，由①得

$$\max_{\bar{V}}u=\max_{\partial V}u=\max_{\partial\Omega}u^+$$

由V的定义可知$\max\limits_{\bar{\Omega}}u=\max\limits_{\bar{V}}u$，从而$\max\limits_{\bar{\Omega}}u=\max\limits_{\partial\Omega}u^+$.

下证强极值原理，设$Lu\le 0$，若u在Ω内达到非负最大值M，设$U=\{x\in\Omega|u(x)=M\}$，现只需证$U=\Omega$. 假设$u\not\equiv M$，则令$V=\{x\in\Omega|u(x)<M\}$. 选取$y\in V$使得$\mathrm{dist}(y,U)<\mathrm{dist}(y,\partial\Omega)$.

设B是以y为球心且含于V的最大球，则存在$x_0\in U$且$x_0\in\partial\Omega$. 由Hopf引理知，$\frac{\partial u}{\partial v}(x_0)>0$，但由于$u$在$x_0$取到最大值，所以$\nabla u=0$，从而$\frac{\partial u}{\partial v}(x_0)=0$，矛盾.

14. 题干有误，待证不等式右端应为$\max\{|l|,\max|\varphi(x)|\}$.

首先需要以下命题,

命题 2. 设$f \in C(\bar{\Omega})$，$\varphi \in C(\partial\Omega)$，$u \in C^2(\Omega) \cap C(\bar{\Omega})$是*Dirichlet*问题

$$\begin{cases} -\Delta u + cu = f, \quad x \in \Omega \\ u|_{\partial\Omega} = \varphi \end{cases}$$

的解. 若$c(x) \geq 0$，则

$$\max_{\Omega} |u(x)| \leq \max_{\partial\Omega} |\varphi(x)| + C \max_{\Omega} |f(x)|,$$

其中C是依赖于维数n和$\mathrm{diam}(\Omega)$的正常数.

Proof. 证明与保继光、朱汝金版《偏微分方程》的定理5.7完全类似. □

下面证明本题.

对$\forall y \in \Omega$，总存在$R_1 > 0$使得$y \in B(0, R_1)$，又因为$\lim\limits_{|x| \to \infty} |u(x)| = l$，所以对$\forall \varepsilon > 0$，存在$R_2 > 0$使得当$|x| \geq R_2$时有$|u(x)| < |l| + \varepsilon$，且$\Omega_0 \subset B(0, R_2)$，现取$R = \max\{R_1, R_2\}$，则

$$\Omega_0 \subset B(0, R), \max_{\partial B(0,R)} |u(x)| < |l| + \varepsilon.$$

在$B(0, R) \backslash \Omega_0$上考虑方程，则根据上面命题有

$$\max_{B(0,R) \backslash \Omega_0} |u(x)| \leq \max\{|l| + \varepsilon, \max_{\partial\Omega} |\varphi(x)|\}$$

从而$|u(y)| \leq \max\{|l| + \varepsilon, \max_{\partial\Omega} |\varphi(x)|\}$，即对$\forall y \in \Omega, \forall \varepsilon > 0$，根据$y$和$\varepsilon$的任意性，关于$y$取上确界，并令$\varepsilon \to 0^+$即得结论.

15. 反证法，假设$\max\limits_{\bar{\Omega}} |u(x)| > \dfrac{1}{\alpha_0} \max\limits_{\partial\Omega} |\varphi(x)|$. 设$x_0 \in \bar{\Omega}$使得$|u(x_0)| = \max\limits_{\bar{\Omega}} |u(x)|$.

① 如果$u(x_0) > 0$，则$\max\limits_{\bar{\Omega}} u(x) = u(x_0)$.

若$x_0 \in \Omega$，则$u_{x_i x_i}(x_0) \leq 0$从而$\Delta u(x_0) \leq 0$，于是$u(x_0)^3 = \Delta u(x_0) \leq 0$，与$u(x_0) > 0$矛盾.

若$x_0 \in \partial\Omega$，则$\dfrac{\partial u}{\partial n}(x_0) \geq 0$，从而

$$\varphi(x_0) \geq \alpha(x_0) u(x_0) \geq \alpha_0 u(x_0) = \alpha_0 \max_{\bar{\Omega}} |u(x)| > \max_{\partial\Omega} |\varphi(x)|,$$

矛盾.

② 如果$u(x_0) < 0$，同理可证.

③ 如果$u(x_0) = 0$，则$u(x) \equiv 0$，结论显然成立.

综上，结论得证.

17. 令$\varphi(r)=\dfrac{1}{4\pi r^2}\displaystyle\int_{S(0,r)}u(x)\mathrm{d}S=\dfrac{1}{4\pi}\int_{S(0,1)}u(rx)\mathrm{d}S$，于是

$$\begin{aligned}\varphi'(r)&=\frac{1}{4\pi}\int_{S(0,1)}\nabla u(rx)\cdot x\mathrm{d}S\\&=\frac{1}{4\pi r^2}\int_{S(0,r)}\nabla u(x)\cdot\frac{x}{r}\mathrm{d}S\\&=\frac{1}{4\pi r^2}\int_{S(0,r)}\frac{\partial u}{\partial n}(x)\mathrm{d}S\\&=\frac{1}{4\pi r^2}\int_{B(0,r)}\Delta u(x)\mathrm{d}x\\&=\frac{1}{4\pi r^2}\int_{B(0,r)}f(x)\mathrm{d}x\\&\geq 0\end{aligned}$$

于是$\varphi(r)$单调递增. 再令

$$\begin{aligned}\psi(r)&=\frac{1}{r^3}\int_{B(0,r)}u(x)\mathrm{d}x\\&=\frac{1}{r^3}\int_0^r\int_{S(0,t)}u(x)\mathrm{d}S\mathrm{d}t\\&=\frac{4\pi}{r^3}\int_0^r t^2\varphi(t)\mathrm{d}t\end{aligned}$$

于是

$$\begin{aligned}\psi'(r)&=\frac{4\pi r^2}{r^3}\varphi(r)-\frac{12\pi}{r^4}\int_0^r t^2\varphi(r)\mathrm{d}t\\&=-\frac{4\pi}{r}\left(\frac{3}{r^3}\int_0^r t^2\varphi(t)\mathrm{d}t-\varphi(r)\right)\\&\geq-\frac{4\pi}{r}\left(\frac{3}{r^3}\varphi(r)\int_0^r t^2\mathrm{d}t-\varphi(r)\right)\\&=0\end{aligned}$$

所以$\psi(r)$单调递增，从而$\psi(r)\leq\psi(R)$.

18. $|\nabla u|^2=u_x^2+u_y^2+u_z^2$，从而

$$\frac{\partial}{\partial x}|\nabla u|^2=2(u_{xx}+u_{xy}+u_{xz}),$$
$$\frac{\partial^2}{\partial x^2}|\nabla u|^2=2(u_{xxx}+u_{xxy}+u_{xxz}).$$

同理可求$\frac{\partial^2}{\partial y^2}|\nabla u|^2$和$\frac{\partial^2}{\partial z^2}|\nabla u|^2$，于是

$$\begin{aligned}\Delta|\nabla u|^2&=2(u_{xxx}+u_{yyy}+u_{zzz}+u_{xxy}+u_{xxz}+u_{yyx}+u_{yyz}+u_{zzx}+u_{zzy})\\&=2[(\Delta u)_x+(\Delta u)_y+(\Delta u)_z]\\&=0\end{aligned}$$

根据极值原理，其最大值在边界处取得.

24. 题干缺条件，还应要求c_i有界.

先证明$u_i(x) \geq 0$.

若存在$x_0 \in \Omega$，使得$u_i(x_0) = \min_{\bar{\Omega}} u(x)$，则有

$$\frac{\partial u_i}{\partial x_j} = 0, \frac{\partial^2 u_i}{\partial x_j^2} \geq 0$$

从而$\Delta u_i(x_0) = c_i(x_0)u_i(x_0) \geq 0$，又因为$c_i(x_0) \geq 0$，所以$u_i(x_0) \geq 0$，即$\min u_i(x) \geq 0$.

若u_i在$\partial\Omega$取得最小值，则由$g_i \geq 0$可知$\min u(x) \geq 0$.

综上，总有$u_i(x) \geq 0$，$i = 1, 2$.

现令$w(x) = u_1(x) - u_2(x)$，则

$$\Delta w = \Delta u_1 - \Delta u_2 = c_1 u_1 - c_2 u_2$$

从而$-\Delta w + c_1 w = (c_2 - c_1)u_2 \geq 0$，根据弱极值原理（课本定理5.2），得

$$\min_{\bar{\Omega}} w(x) \geq \min_{\partial\Omega} w^-(x).$$

再由$w|_{\partial\Omega} = g_1 - g_2 \geq 0$可知$\min_{\bar{\Omega}} w(x) \geq 0$，因此$u_1(x) \geq u_2(x)$.

25. 反证法，假设存在$x_0 \in \Omega$使得$u(x_0) < 0$，则由$u|_{\partial\Omega} \equiv 0$可知$\min_{\bar{\Omega}} u(x)$在$\Omega$内取得且小于零，不妨设其为$u(x_0)$，于是根据$u$的连续性可知存在$u$的邻域$U$使得在$U$上恒有$u(x) < 0$，从而在$U$上有

$$-\Delta u = f(u) = 0$$

根据强极值原理知在U上有

$$u(x) \equiv c < 0.$$

现只需证明在Ω上有$u \equiv c < 0$从而根据$u \in C(\bar{\Omega})$和$u|_{\bar{\Omega}} \equiv 0$推出矛盾.

为此，设$A = \{x \in \Omega | u(x) = c\}$，则只需证明$A$既是$\Omega$中的开集又是$\Omega$中的闭集.

先证开集，$\forall y \in A$，由于$u(y) = c < 0$，所以一定存在$\delta > 0$使得$B(y, \delta) \subset \Omega$上有$u(x) < \frac{c}{2} < 0$，因此在$B(y, \delta)$上有$-\Delta u(x) = f(u) = 0$，根据强极值原理可知在$B(y, \delta)$上$u(x) \equiv u(y) = c$，从而$B(y, \delta) \subset A$，即$A$是开集.

再证闭集，对任意$y \in \bar{A}$，一定存在A中的点列$\{y_k\}_{k \in \mathbb{N}}$使得$\lim\limits_{k \to \infty} y_k = y$，又因为$u$的连续性，所以

$$u(y) = \lim_{k \to \infty} u(y_k) = c$$

即$y \in A$，所以A既是Ω的开集又是Ω的闭集，从而$A = \Omega = \{x | u(x) = c\}$，与$u \in C(\bar{\Omega})$矛盾，所以假设不成立.

注:

（1）上面证明过程中所说的A的开闭指的是在Ω的子空间拓扑下的.

（2）上面证明用到了如下结论，详见拓扑学教材.

命题 3. 若X是连通拓扑空间，则X的既开又闭的子集只有X和$\varnothing$.

（3）关于$A = \Omega$的证明，也可以进一步利用Ω的连通性（欧氏空间中连通和道路连通是等价的），有如下证明:

由于Ω是道路连通的，所以对$y \in A$和任意$x' \in \Omega$，存在道路将其连通，即存在连续映射$\gamma : [0, 1] \to \Omega$使得$\gamma(0) = y$，$\gamma(1) = x'$.现令

$$B = \{t \in [0, 1] : u(\gamma(t)) = u(y)\}$$

现只需证$l=\sup B=1$.反证法，假设$l<1$，则记$x_l=\gamma(l)$，由$u(\gamma(t))$的连续性可知

$$u(x_l)=u(y)=c$$

又因为$x_l\in\Omega$，所以存在$\delta>0$使得$B(x_l,\delta)\subset\Omega$，再在此邻域上应用强极值原理得在$B(x_l,\delta)$上有$u\equiv c$，再利用$\gamma$得连续性得，存在$\epsilon>0$使得$l+\epsilon<1$且$\gamma([l-\epsilon,l+\epsilon])\subset B(x_l,\delta)$，于是

$$u(\gamma(l+\epsilon))=c$$

与$l=\sup B$矛盾，故$l=1$，即$u(x')=c$，再由$x'\in\Omega$的任意性得$u|_\Omega\equiv c$.

27. 假设$\max\limits_{\bar{\Omega}}|u(x)|>1$，则存在$x_0\in\bar{\Omega}$使得$|u(x_0)|=\max|u(x)|>1$. 由于

$$\max_{\partial\Omega}|g(x)|=\max_{\partial\Omega}|u(x)|\le 1$$

所以有$x_0\in\Omega$. 如果$u(x_0)>1$，则$u(x_0)=\max\limits_{\bar{\Omega}}u(x)$，从而

$$\frac{\partial^2 u}{\partial x_i^2}(x_0)\le 0$$

于是

$$-\Delta u(x_0)=u(x_0)-u(x_0)^3=u(x_0)[1-u(x_0)^2]<0$$

与$u(x_0)>1$矛盾.
如果$u(x_0)<-1$，则$u(x_0)=\min_{\bar{\Omega}}u(x)$，同理可证. 综上，有$\max\limits_{\bar{\Omega}}|u(x)|\le 1$.

29. 题干有误，应为“$A:\Omega\to\mathbb{R}^n$”.
不妨设$0\in\Omega$，并且由于A有界，所以可以令$M=\sup_\Omega|A(x)|+1$，对$\forall\varepsilon>0$，构造辅助函数$w(x)=u(x)+\varepsilon(\mathrm{e}^{Md}-\mathrm{e}^{Mx_1})$，于是对任意$x\in\Omega$

$$\begin{aligned}-\Delta w+A\cdot Dw&=-\Delta u+A\cdot Du-\Delta(\varepsilon(\mathrm{e}^{Md}-\mathrm{e}^{Mx_1}))+A\cdot D(\varepsilon(\mathrm{e}^{Md}-\mathrm{e}^{Mx_1}))\\&=f(x)+\varepsilon M^2\mathrm{e}^{Mx_1}-\varepsilon a_1(x)M\mathrm{e}^{Mx_1}\\&=f(x)+\varepsilon M\mathrm{e}^{Mx_1}(M-a_1(x))\\&>0\end{aligned}$$

在$\partial\Omega$上，有$w(x)=g(x)+\varepsilon(\mathrm{e}^{Md}-\mathrm{e}^{Mx_1})>0$，现断言$w|_\Omega\ge 0$. 事实上，假设该断言不成立，则$\exists x_0\in\Omega$使得$w(x_0)=\min_{\bar{\Omega}}w(x)<0$，所以

$$w_{x_i}(x_0)=0,\ \ w_{x_ix_i}(x_0)\ge 0$$

从而$-\Delta w(x_0)+A\cdot Dw(x_0)\le 0$，矛盾，故假设不成立.
因此在$\bar{\Omega}$上，有$w\ge 0$，即

$$u(x)\ge -\varepsilon(\mathrm{e}^{Md}-\mathrm{e}^{Mx_1})$$

令$\varepsilon\to 0^+$即得结论.

35. 为此，只需证明原问题证明的齐次问题的有界解只有零解即可，现设$u(x)$是原问题对应齐次问题的有界解，则

$$\begin{cases} -\Delta u = 0, & x \in \Omega \\ u = 0, & x \in \partial\Omega \setminus \{x_0\} \end{cases}$$

由于$u(x)$有界，所以可设$|u(x)| \le M$，而对于给定的$x_0 \in \partial\Omega$以及任意的$\varepsilon > 0$，由于$\lim\limits_{x \to x_0} \dfrac{\varepsilon}{|x - x_0|^{n-2}} = +\infty$（因为$n \ge 3$），所以一定存在$\delta > 0$使得$\forall x \in B(x_0, \delta) \cap \bar{\Omega}$都有$\dfrac{\varepsilon}{|x - x_0|^{n-2}} > M$. 考虑辅助函数$w(x) = \dfrac{\varepsilon}{|x - x_0|^{n-2}} \pm u(x)$，于是在$B(x_0, \delta) \cap \bar{\Omega}$内，有

$$w(x) \ge \frac{\varepsilon}{|x - x_0|^{n-2}} - |u(x)| > M - |u(x)| \ge 0$$

现令$\Omega' = \Omega \setminus B(x_0, \frac{\delta}{2})$，则不难发现$w|_{\partial\Omega'} \ge 0$，同时经计算可以得到在$\Omega'$内$\Delta w = 0$，于是

$$\begin{cases} -\Delta w = 0, & x \in \Omega' \\ w(x) \ge 0, & x \in \partial\Omega' \end{cases}$$

根据极值原理可知$w|'_{\Omega} \ge 0$，从而在整个Ω内都有

$$|u(x)| \le \frac{\varepsilon}{|x - x_0|^{n-2}}$$

令$\varepsilon \to 0^+$即得结论.

37. 证明方法与第35题类似，当$n \ge 3$时，由于u, v有界，所以对任意$\varepsilon > 0$，一定存在$\delta > 0$使得在$B(x_0, \delta) \cap \Omega$内有$\dfrac{\varepsilon}{|x - x_0|^{n-2}} > |u(x) - v(x)|$，构造辅助函数$w(x) = \dfrac{\varepsilon}{|x - x_0|^{n-2}} \pm (u(x) - v(x))$，令$\Omega' = \Omega \setminus B(x_0, \frac{\delta}{2})$，则有

$$\begin{cases} -\Delta w = 0, & x \in \Omega' \\ w(x) \ge 0, & x \in \partial\Omega' \end{cases}$$

根据极值原理可知在Ω'上有$w(x) \ge 0$，从而在整个Ω上有$w(x) \ge 0$，即

$$\frac{\varepsilon}{|x - x_0|^{n-2}} \ge |u(x) - v(x)|$$

令$\varepsilon \to 0^+$即得结论.

当$n = 2$时，由于u, v有界,所以$\forall \varepsilon > 0$，存在$\delta' > 0$使得在$B(x_0, \delta')$内有$-\varepsilon \ln(|x - x_0|) > |u(x) - v(x)|$，构造辅助函数$w(x) = -\varepsilon \ln(|x - x_0|) \pm (u(x) - v(x))$，令$\Omega'' = \Omega \setminus B(x_0, \frac{\delta}{2})$，则有

$$\begin{cases} -\Delta w = 0, & x \in \Omega'' \\ w(x) \ge 0, & x \in \partial\Omega'' \end{cases}$$

根据极值原理可知在Ω''上有$w(x) \ge 0$，从而在整个Ω上有$w(x) \ge 0$，即

$$-\varepsilon \ln(|x - x_0|) \ge |u(x) - v(x)|$$

令$\varepsilon \to 0^+$即得结论.

38. 法一：当$n\geq 3$时，令$\phi(t)=\dfrac{1}{\omega_n t^{n-1}}\displaystyle\int_{\partial B_t}u(x)\mathrm{d}S(x)=\dfrac{1}{\omega_n}\int_{\partial B_1}u(ty)\mathrm{d}S(y)$，则

$$\begin{aligned}\phi'(t)&=\frac{1}{\omega_n}\int_{\partial B_1}Du(ty)\cdot y\mathrm{d}S(y)\\&=\frac{1}{\omega_n t^{n-1}}\int_{\partial B_t}Du(x)\cdot\frac{x}{t}\mathrm{d}S(x)\\&=\frac{1}{\omega_n t^{n-1}}\int_{\partial B_t}\frac{\partial u}{\partial\nu}\mathrm{d}S(x)\\&=\frac{1}{\omega_n t^{n-1}}\int_{B_t}\Delta u(x)\mathrm{d}x\\&=-\frac{1}{\omega_n t^{n-1}}\int_{B_t}f(x)\mathrm{d}x\end{aligned}$$

对任意$\varepsilon\in(0,r)$有

$$\begin{aligned}\phi(r)-\phi(\varepsilon)&=\int_\varepsilon^r\phi'(t)\mathrm{d}t\\&=-\int_\varepsilon^r\frac{1}{\omega_n t^{n-1}}\int_{B_t}f(x)\mathrm{d}x\mathrm{d}t\\&=\frac{1}{\omega_n(n-2)t^{n-2}}\int_{B_t}f(x)\mathrm{d}x\bigg|_\varepsilon^r-\frac{1}{\omega_n(n-2)t^{n-2}}\int_\varepsilon^r\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{B_t}f(x)\mathrm{d}x\right)\mathrm{d}t\\&=\frac{1}{\omega_n(n-2)t^{n-2}}\int_{B_t}f(x)\mathrm{d}x\bigg|_\varepsilon^r-\frac{1}{\omega_n(n-2)t^{n-2}}\int_\varepsilon^r\int_{\partial B_t}f(x)\mathrm{d}S(x)\mathrm{d}t\\&=\frac{1}{\omega_n(n-2)}\left(\frac{1}{r^{n-2}}\int_{B_r}f(x)\mathrm{d}x-\frac{1}{\varepsilon^{n-2}}\int_{B_\varepsilon}f(x)\mathrm{d}x-\int_\varepsilon^r\int_{\partial B_t}\frac{f(x)}{t^{n-2}}\mathrm{d}S(x)\mathrm{d}t\right)\\&=\frac{1}{\omega_n(n-2)}\left(\frac{1}{r^{n-2}}\int_{B_r}f(x)\mathrm{d}x-\frac{1}{\varepsilon^{n-2}}\int_{B_\varepsilon}f(x)\mathrm{d}x-\int_{B_r\setminus B_\varepsilon}\frac{f(x)}{|x|^{n-2}}\mathrm{d}x\right)\end{aligned}$$

同时注意到

$$\begin{aligned}\left|\frac{1}{\varepsilon^{n-2}}\int_{B_\varepsilon}f(x)\mathrm{d}x\right|&\leq\frac{1}{\varepsilon^{n-2}}\int_{B_\varepsilon}|f(x)|\mathrm{d}x\\&\leq\frac{1}{\varepsilon^{n-2}}\cdot M\cdot\frac{1}{n}\cdot\omega_n\varepsilon^n\\&=\frac{M\omega_n\varepsilon^2}{n}\\&\to 0,\quad(\varepsilon\to 0^+)\end{aligned}$$

以及$\phi(\varepsilon)=\dfrac{1}{\omega_n}\displaystyle\int_{\partial B_1}u(ty)\mathrm{d}S(y)\to u(0),(\varepsilon\to 0^+)$，于是令$\varepsilon\to 0^+$得

$$\phi(r)-u(0)=\frac{1}{\omega_n(n-2)}\int_{B_r}\left(\frac{1}{r^{n-2}}-\frac{1}{|x|^{n-2}}\right)f(x)\mathrm{d}x$$

又注意到$\phi(r)=\dfrac{1}{\omega_n r^{n-1}}\displaystyle\int_{\partial B_r}\varphi(x)\mathrm{d}S(x)$，于是原式得证.

当$n=2$时，只需注意到$\omega_n=2\pi r$以及$\displaystyle\int_\varepsilon^r t^{1-n}\mathrm{d}t=\ln r-\ln\varepsilon$则同理可证.

法二：类似Green函数的构造，在Green恒等式中令$x=0$（可参考周蜀林或Evans书）

$$u(0)=\int_{\partial B_r}\left(\Phi(y)\frac{\partial u}{\partial\nu}(y)-u(y)\frac{\partial\Phi}{\partial\nu}(y)\right)\mathrm{d}S(y)-\int_{B_r}\Phi(y)\Delta u(y)\mathrm{d}y\tag{11}$$

由于$\frac{\partial u}{\partial \nu}$在球面上的取值未知，所以根据格林公式$\displaystyle\int_\Omega u\Delta v - v\Delta u \mathrm{d}x = \int_{\partial\Omega} u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu}\mathrm{d}S$，希望能找到某个函数$\phi$满足

$$\begin{cases} -\Delta\phi = 0, & x \in B_r \\ \phi(x) = \Phi(x), & x \in \partial B_r \end{cases}$$

于是有

$$-\int_{B_r}\phi(x)\Delta u(x)\mathrm{d}x = \int_{\partial B_r} u\frac{\partial \phi}{\partial \nu} - \phi\frac{\partial u}{\partial \nu}\mathrm{d}S$$

从而

$$\int_{\partial B_r}\Phi(x)\frac{\partial u}{\partial \nu}\mathrm{d}S = \int_{B_r}\phi(x)\Delta u(x)\mathrm{d}x + \int_{\partial B_r} u\frac{\partial \phi}{\partial \nu}\mathrm{d}S \tag{12}$$

代入(11)式即可替换关于$\Phi(x)\frac{\partial u}{\partial \nu}$的项，现求解$\phi$.

因为

$$\Phi(x) = \begin{cases} \frac{1}{(n-2)\omega_n |x|^{n-2}}, & n \geq 3 \\ -\frac{1}{2\pi}\ln|x|, & n = 2 \end{cases}$$

所以Φ在∂B_r上为常数，根据极值原理可知ϕ也是常数

$$\phi(x) = \begin{cases} \frac{1}{(n-2)\omega_n r^{n-2}}, & n \geq 3 \\ \frac{1}{2\pi}\ln r, & n = 2 \end{cases}$$

代入(12)后再代入(11)整理即得结论.

39. 根据$n = 3$时球上的Poisson公式得

$$u(x) = \frac{R - |x|^2}{4\pi R}\int_{\partial B_R}\frac{u(y)}{|x-y|^3}\mathrm{d}S(y)$$

从而

$$\frac{\partial u}{\partial x_i} = -\frac{2x_i}{4\pi R}\int_{\partial B_R}\frac{u(y)}{|x-y|^3}\mathrm{d}S(y) + \frac{R^2 - |x|^2}{4\pi R}\int_{\partial B_R} u(y)\left[-\frac{3}{2}\left(\sum (x_i - y_i)^2\right)^{-\frac{5}{2}}\cdot 2(x_i - y_i)\right]\mathrm{d}S(y)$$

$$\frac{\partial u}{\partial x_i}(0) = \frac{R}{4\pi}\int_{\partial B_R} 3u(y)\cdot R^{-5}y_i\mathrm{d}S(y)$$

所以

$$\begin{aligned} \left|\frac{\partial u}{\partial x_i}(0)\right| &\leq \frac{R}{4\pi}\cdot 3\max_{\partial \bar{B}_R}|u(y)|\cdot R^{-4}\cdot 4\pi R^2 \\ &\leq \frac{3}{R}\max_{\bar{B}_R}|u(x)|. \end{aligned}$$

原式得证.

41. 定义$\displaystyle\phi(r) = \frac{1}{\omega_n r^{n-1}}\int_{\partial B(\hat{x},y)} u(y)\mathrm{d}S(y) = \frac{1}{\omega_n}\int_{\partial B(0,1)} u(\hat{x} + rz)\mathrm{d}S(z)$，则

$$\phi'(r) = \frac{1}{\omega_n}\int_{\partial B(0,1)} Du(\hat{x} + rz)\cdot z\mathrm{d}S(z)$$

当$\Delta u = 0$时，根据Green公式得

$$\begin{aligned}\phi'(r) &= \frac{1}{\omega_n r^{n-1}}\int_{\partial B(\hat{x},r)} Du(y)\cdot\frac{y-x}{r}\mathrm{d}S(y)\\ &= \frac{1}{\omega_n r^{n-1}}\int_{\partial B(\hat{x},r)} \frac{\partial u}{\partial \nu}\mathrm{d}S(y)\\ &= \frac{1}{\omega_n r^{n-1}}\int_{B(\hat{x},r)} \Delta u(y)\mathrm{d}S(y)\\ &= 0\end{aligned}$$

所以$\phi(r)$是常函数，又因为$\lim\limits_{r\to 0^+}\phi(r) = \lim\limits_{r\to 0^+}\frac{1}{\omega_n}\int_{\partial B(0,1)} u(\hat{x}+rz)\mathrm{d}S(z) = u(\hat{x})$，所以$\phi(r)\equiv u(\hat{x})$，因此

$$\begin{aligned}\frac{n}{\omega_n r^n}\int_{B(\hat{x},r)} &= \frac{n}{\omega_n r^n}\int_0^r \omega_n t^{n-1}\phi(t)\mathrm{d}t\\ &= \frac{n}{\omega_n r^n}\int_0^r \omega_n t^{n-1}u(\hat{x})\mathrm{d}t\\ &= u(\hat{x})\end{aligned}$$

$\Delta u \le 0$和≥ 0时同理可证.

44.

(1)记$M = \max\limits_{\bar{\Omega}} v(x)$.

若$\max\limits_{\bar{\Omega}} v(x)$在边界上取得，则结论一定成立.

若$\max\limits_{\bar{\Omega}} v(x)$在$\Omega$内部取得，则存在$x_0\in\Omega$使得$v(x_0) = M$，则存在$r>0$使得$B(x_0,r)\subset\Omega$,并且由第41题可知

$$M = v(x_0) \le \frac{n}{\omega_n r^n}\int_{B(x_0,r)} v(y)\mathrm{d}y$$

又因为$v(y)\le M$，所以在$B(x_0,r)$内有$v(x)\equiv M$. 记$A = \{x\in\Omega | v(x) = M\}$，现断言$A = \Omega$，故只需证$A$在$\Omega$中既是开集又是闭集，根据$v$的连续性，闭集是显然的，现证明开集，对任意的$x\in A$，存在$\delta>0$使得$B(x,\delta)\subset\Omega$，再根据第41题可知

$$M = \frac{n}{\omega_n r^n}\int_{B(x,\delta)} v(y)\mathrm{d}y$$

又因为$v(y)\le M$，所以在$B(x,\delta)$上也有$v(y)\equiv M$，即$B(x,\delta)\subset A$，从而A既开又闭，$A=\Omega$，同样有$\max\limits_{\bar{\Omega}} v(x) = \max\limits_{\partial\Omega} v(x)$. 综上，结论得证.

注：（敲完之后才发现的）上面的证明与第25题的思路类似，但请注意不同之处，第25题的题目条件中明确说明Ω是$\mathbb{R}^n$中的区域，即连通开集，但本题的题目条件中并未对Ω做任何限制（不知道是出题人是否有意为之），所以不能用Ω的连通性与开性，故**上面的证明是错的**. 正确的证明如下：

记$M = \max\limits_{\bar{\Omega}} v(x)$.

若$\max\limits_{\bar{\Omega}} v(x)$在边界上取得，则结论一定成立.

若$\max\limits_{\bar{\Omega}} v(x)$在$\Omega$内部取得，则存在$x_0\in\Omega$使得$v(x_0) = M$. 设$\Omega$的包含$x_0$的连通分支为$\Omega_1$，则用类似上面的证明或25题注记中的方法，可知在$\Omega_1$上有$v(x)\equiv M$，再由$v$的连续性知在$\bar{\Omega}_1$上也有$v(x)\equiv M$，为了说明$M = \max\limits_{\partial\Omega} v(x)$，现只需证$\partial\Omega_1\cap\partial\Omega\ne\varnothing$.

事实上，假设$\partial\Omega_1\cap\partial\Omega = \varnothing$，则$\partial\Omega_1\subset\mathring{\Omega}$，于是对任意的$x\in\partial\Omega_1$，一定存在其开邻域$U$使得$U\subset\Omega$，再由边界的定

义知$U\cap\Omega_1\neq\varnothing$，于是有$\Omega_1\cup U$也是连通的，但由于$\Omega_1$是连通分支（极大连通子集），所以一定有$U\cup\Omega_1\subset\Omega_1$，从而$U\subset\Omega_1$，即$x\in\mathring{\Omega}_1$，这与$x\in\partial\Omega_1$矛盾，故假设不成立.
因此存在$x'\in\partial\Omega$使得$v(x')=M=\max\limits_{\bar{\Omega}}v(x)$，从而$\max\limits_{\bar{\Omega}}v(x)=\max\limits_{\partial\Omega}v(x)$，结论得证.
(2)

$$v_{x_i}=\Phi'(u)u_{x_i}$$

$$v_{x_ix_i}=\Phi''(u)(u_{x_i})^2+\Phi'(u)u_{x_ix_i}$$

所以

$$\Delta v=\Phi''(u)|Du|^2+\Phi'(u)\Delta u$$

因为u调和，所以$\Delta u=0$，又因为Φ是凸函数，所以$\Phi''\geq 0$，于是

$$\Delta v=\Phi''(u)|Du|^2\geq 0$$

因此v是下调和函数.
(3)经计算

$$v(x)=\sum_{i=1}^n(u_{x_i})^2$$

$$v_{x_j}=\sum_{i=1}^n 2u_{x_i}\cdot u_{x_ix_j}$$

$$v_{x_jx_j}=2\sum_{i=1}^n[(u_{x_ix_j})^2+u_{x_i}\cdot u_{x_ix_ix_j}]$$

由于$\Delta u=0$所以

$$\frac{\partial}{\partial x_j}(\Delta u)=\sum_{i=1}^n u_{x_ix_ix_j}=0$$

因此

$$\Delta v=2\sum_{j=1}^n\sum_{i=1}^n(u_{x_ix_j})^2\geq 0$$

即v是下调和函数.

50.

$$G_1(M,M_0)=\Phi(M_0-M)-g_1(M,M_0)$$

$$G(M,M_0)=\Phi(M_0-M)-g(M,M_0)$$

其中g_1,g分别满足

$$\begin{cases}-\Delta g_1(M,M_0)=0, & M\in\Omega_1\\ g_1(M,M_0)=\Phi(M_0-M), & M\in\partial\Omega_1\end{cases}$$

$$\begin{cases}-\Delta g(M,M_0)=0, & M\in\Omega\\ g(M,M_0)=\Phi(M_0-M), & M\in\partial\Omega\end{cases}$$

因此当$M\in\partial\Omega_1$时，$G_1(M,M_0)=0,G(M,M_0)\geq 0$，从而$g_1(M,M_0)\geq g(M,M_0)$，$M\in\partial\Omega_1$.再根据

$$\begin{cases}-\Delta(g-g_1)(M,M_0)=0, & M\in\Omega_1\\ (g-g_1)(M,M_0)\leq 0, & M\in\partial\Omega_1\end{cases}$$

再根据极值原理可知$g(M,M_0)\leq g_1(M,M_0)$，$M\in\Omega_1$，从而$G_1\leq G$.

51. 首先有$u(x,y)=\dfrac{y}{\pi}\displaystyle\int_{-\infty}^{+\infty}\frac{\varphi(z)}{(z-x)^2+y^2}\mathrm{d}z$

(2)

$$\begin{aligned}u(x,y)=&\frac{y}{\pi}\int_a^b\frac{1}{(z-x)^2+y^2}\mathrm{d}z\\ =&\frac{1}{\pi}\int_{\frac{a-x}{y}}^{\frac{b-x}{y}}\frac{1}{z^2+1}\mathrm{d}z\\ =&\frac{1}{\pi}(\arctan\frac{b-x}{y}-\arctan\frac{a-x}{y})\end{aligned}$$

(3)法一：

$$\begin{aligned}u(x,y)&=\frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{1}{1+z^2}\cdot\frac{1}{(z-x)^2+y^2}\mathrm{d}z\\ &=\frac{y}{\pi}(\frac{1}{1+x^2}*\frac{1}{y^2+x^2})\\ &=\frac{y}{\pi}\mathscr{F}^{-1}[\mathscr{F}(\frac{1}{1+x^2})\cdot\mathscr{F}(\frac{1}{x^2+y^2})]\\ &=\frac{y+1}{x^2+(y+1)^2}\end{aligned}$$

法二：

$$u(x,y)=\frac{y}{\pi}\cdot 2\pi\mathrm{i}\left[\mathrm{Res}\left(\frac{1}{1+z^2}\cdot\frac{1}{(z-x)^2+y^2},\mathrm{i}\right)+\mathrm{Res}\left(\frac{1}{1+z^2}\cdot\frac{1}{(z-x)^2+y^2},x+y\mathrm{i}\right)\right]$$

法三：对方程和边界条件分别关于x做Fourier变换

$$\begin{cases}-\xi^2\hat{u}(\xi,y)+\hat{u}_{yy}(\xi,y)=0, & (\xi,y)\in\mathbb{R}_+^2\\ \hat{u}(\xi,y)=\hat{\varphi}(\xi), & \xi\in\mathbb{R}\end{cases}$$

从而

$$\begin{aligned}\hat{u}(\xi,y)&=\hat{\varphi}(\xi)\mathrm{e}^{-|\xi|y}\\ &=\pi\mathrm{e}^{-|\xi|(y+1)}\end{aligned}$$

所以

$$\begin{aligned}u(x,y)&=\mathscr{F}^{-1}[\hat{u}](x,y)\\ &=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\pi\mathrm{e}^{-(y+1)|\xi|}\mathrm{e}^{\mathrm{i}x\xi}\mathrm{d}\xi\\ &=\frac{1}{2(\mathrm{i}x-y-1)}\mathrm{e}^{(\mathrm{i}x-y-1)\xi}\Big|_0^{+\infty}+\frac{1}{2(\mathrm{i}x+y+1)}\mathrm{e}^{\mathrm{i}x+y+1}\xi\Big|_{-\infty}^0\\ &=\frac{y+1}{(y+1)^2+x^2}\end{aligned}$$

53. 记$B=B(0,1)$，将φ关于y做奇延拓

$$\psi(x,y)=\begin{cases}\varphi(x,y), & (x,y)\in\partial B^+\cap\{y>0\}\\ -\varphi(x,-y), & (x,y)\in\partial B^-\cap\{y<0\}\end{cases}$$

由φ的性质知$\psi\in C(\partial B)$.
设$w\in C^2(B)\cap C(\bar{B})$满足方程

$$\begin{cases}-\Delta w=0, & (x,y)\in B\\ w(x,y)=\psi(x,y), & (x,y)\in\partial B\end{cases}$$

根据球上的Poisson公式得

$$w(x,y)=\frac{1-x^2-y^2}{2\pi}\int_{\partial B}\frac{\psi(\xi,\eta)}{(x-\xi)^2+(y-\eta)^2}\mathrm{d}S(\xi,\eta)$$

根据Schwarz反射定理（第八题）知在$\bar{B}^+$上有$u=w$，从而当$(x,y)\in B^+$时

$$u(x,y)=\frac{1-x^2-y^2}{2\pi}\left[\int_{\partial B\cap\{y>0\}}\frac{\varphi(\xi,\eta)}{(x-\xi)^2+(y-\eta)^2}\mathrm{d}S+\int_{\partial B\cap\{y<0\}}\frac{-\varphi(\xi,-\eta)}{(x-\xi)^2+(y-\eta)^2}\mathrm{d}S\right]$$

4 哈哈哈哈哈哈哈

我敲完了——2025.4.27

乱七八糟的，大家凑活看吧