北京师范大学 2023-2024 学年第一学期期末考试试卷（A 卷）

课程名称：<u>概率论</u>　　　　任课老师姓名：[redacted]

卷面总分：____ 分　　考试时长：120 分钟　　考试类别：闭卷 ☒ 开卷 ☐ 其他 ☐

院（系[redacted]

姓名：[redacted]

题号	一	二	三	四	五	六	总分
得分							

阅卷老师（签字）：______________

（注意：可以承认并使用问题 $1, \cdots, k$ 的结果来回答第 $k+1$ 题。）

1. 基础题：陈述中心极限定理 Central Limit Theorem

2. Let $\{X_n\}_{n\geq 1}$ be a sequence of i.i.d. random variables taking values in $\mathbb{N}^*$ such that

$$\mathbf{P}(X_1 \geq n) = \frac{1}{n}, \forall n \in \mathbb{N}^*.$$

Let $S_n = X_1 + \cdots + X_n$ for any $n \geq 1$.

   (a) Determine the law of X_1.

   (b) Prove that $X_1 \notin L^1(\mathbf{P})$. So the Law of large numbers can not be applied to S_n/n.

   (c) We are going to prove that $\frac{S_n}{n \ln n}$ converges in probability to 1。

      i. For any $n \geq 1$, we define

$$S'_n = \sum_{k=1}^{n} X_k 1_{X_k \leq n \ln n}.$$

      Prove that $\mathbf{P}(S_n \neq S'_n) \to 0$ as $n \to \infty$.

      ii. Prove that

$$\sum_{j=1}^{n} \frac{1}{j} - \ln n$$

      converges to some real limit γ as $n \to \infty$.

      iii. Prove that

$$\frac{\mathrm{Var}(S'_n)}{(\mathbf{E}[S'_n])^2} \to 0.$$

$\frac{1}{n} - \ln\frac{n}{n-1}$

$\sum_{j=1}^{n}\left(\frac{1}{j} - \ln\frac{j}{j-1}\right) + \ln\left(1+\frac{1}{j-1}\right)$

$\ln\frac{n}{n+1} \leq \frac{1}{n} \leq \ln\frac{n}{n-1}$

iv. Deduce that $\frac{S'_n}{n \ln n}$ converges in probability to 1.

v. Prove that $\frac{S_n}{n \ln n}$ converges in probability to 1.

(d) Next, we prove that $\frac{S_n}{n \ln n}$ does not converge a.s.

i. Prove that

$$\sum_{n \geq 1} \mathbf{P}(X_n \geq cn \ln n) = \infty, \forall c > 0.$$

ii. Deduce that for any $c > 0$,

$$\mathbf{P}(\limsup_{n \to \infty} \frac{X_n}{n \ln n} \geq c) = 1.$$

iii. Then verify that $\mathbf{P}$-a.s.,

$$\limsup_n \frac{S_n}{n \ln n} = \infty.$$

iv. Find out a subsequence m_k such that $\mathbf{P}$-a.s.,

$$\lim_{k \to \infty} \frac{S_{m_k}}{m_k \ln m_k} = 1.$$

3. Let X and Y be independent random variables, X having the standard normal distribution, and Y having the $\chi^2(n)$ distribution.

(a) Show that

$$T_n = \frac{X}{\sqrt{Y/n}}$$

has density function

$$f(t) = \frac{1}{\sqrt{\pi n}} \frac{\Gamma(\frac{1}{2}(n+1))}{\Gamma(\frac{1}{2}n)} \left(1 + \frac{t^2}{n}\right)^{-\frac{1}{2}(n+1)}.$$

(b) Prove that T_n converges in law to $N(0,1)$.

备注： The density function of $\chi^2(r)$ distribution is

$$f(x) = \frac{(\frac{1}{2})^{\frac{r}{2}}}{\Gamma(\frac{r}{2})} x^{\frac{r}{2}-1} e^{-\frac{x}{2}} 1_{\mathbb{R}_+}(x).$$

$\frac{\sum_{i=1}^{n} i}{n}$

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（注意：可以承认并使用问题 $1, \cdots, k$ 的结果来回答第 $k+1$ 题。）

1. 基础题：State Fatou's Lemma and prove it.

2. We consider a matrix A of size $n \times n$, given by

$$A = (A_{i,j})_{1 \le i,j \le n}$$

where $A_{i,j}$ are real-valued i.i.d random variables.

(a) If $\mathbf{P}(A_{i,j} = 1) = \mathbf{P}(A_{i,j} = -1) = \frac{1}{2}$, compute $\mathbf{E}[\det(A)]$ and $\mathbf{E}[\det(A^2)]$;

(b)* If $\mathbf{P}(A_{i,j} = 1) = 1 - \mathbf{P}(A_{i,j} = 0) = p \in (0,1)$, compute $\mathbf{E}[\det(A)]$ and $\mathbf{E}[\det(A^2)]$.

3. The random variable X' is said to be obtained from the random variable X by truncation at the point a if

$$X' = X 1_{X \le a}$$

Express the law of X' in terms of the law of X.

4. Let $\{X_k\}_{k \ge 1}$ be a sequence of i.i.d. random variables with Exponential distribution of parameter $\lambda > 0$. This means that

$$\mathbf{P}_{X_1}(dx) = \lambda e^{-\lambda x} 1_{x>0} dx$$

(a) Determine the law of X_1^2.

(b) Compute $\mathbf{E}[X_1]$ and $\mathbf{Var}(X_1)$.

(c) Prove that

$$\mathbf{P}(X_1 = X_2) = 0.$$

(d) Deduce that

$$\mathbf{P}(X_1 \geq X_2 \geq X_3 \geq \cdots \geq X_N) = \frac{1}{N!}, \forall N \geq 2.$$

(e) Let

$$M_n = \max_{1 \leq k \leq n} X_k.$$

Compute the distribution function of M_n.

5. Let X be a random variable which takes values in the interval $[-M, M]$ only. Show that

$$\mathbf{P}(|X| \geq a) \geq \frac{\mathbf{E}[|X|] - a}{M - a}$$

if $0 \leq a \leq M$.

$0 \leq a < M$

$\mathbb{P}(|X| \geq a) \leq \frac{E[|X|]}{a}.$

$M\mathbb{P}(|X| \geq a) - a\mathbb{P}(|X| \geq a)$

$\geq M\,\mathbb{P}(|X| \geq a) - E[|X|]$

6. The real number m is called a median of the random variable X if

$$\mathbf{P}(X < m) \leq \frac{1}{2} \leq \mathbf{P}(X \leq m).$$

$\mathbb{P}(a \leq |X| \leq M)$

Show that every random variable has at least one median.

$\mathbb{P}\left((|X| - a) \leq M - a\right)$

$\geq \frac{E[|X| - a]}{M - a}.$