# 2024 春季学期——实变函数期中考试

注意事项：

- 闭卷测试。禁止使用计算器、手机等电子设备。请提前静音手机。
- 间隔座位。邻桌无人。
- 测试时间：8：00－9：45 am. 不可提前交卷。
- 学号姓名。答题纸上明确标注学号和姓名。请将学生卡作为准考证放在桌面左上角，以供查验。

**Exercise 1** *Let $E$ be a measurable set in $\mathbb{R}$.*

1. *For a function $f : E \to \mathbb{R}$ which is measurable, recall* **Lusin's Theorem**.
2. *Now take a function $g : E \to \mathbb{R}$. Suppose that for every $n \in \mathbb{N}$, there exists a closed set $F_n \subset E$ with $m(E \sim F_n) < \frac{1}{n}$, such that $g$ is continuous on $F_n$. Prove that $g$ is a measurable function.*

**Exercise 2** *Let $f : E \to \mathbb{R}_+$ be a measurable non-negative function with $E$ a measurable set in $\mathbb{R}$. Assume that*

$$\int_E f dm < \infty.$$

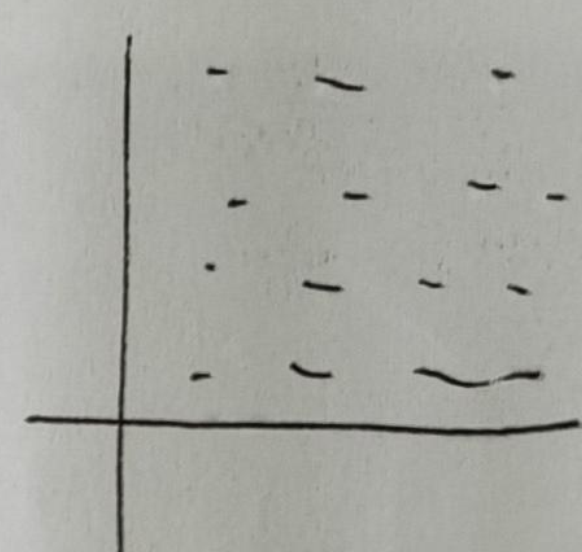
1. *For any $t \geq 0$, let*

$$F(t) := \int_{E\cap[-t,t]} f(x)m(dx)$$

*Prove that $t \mapsto F(t)$ is a continuous function on $\mathbb{R}_+$.*

2. *Is $F$ uniformly continuous? Verify your answer.*
3. *Show that there exists $B \subset E$ such that*

$$\int_B f dm = \frac{e^{2024}}{\pi^{2024}} \int_E f dm.$$



**Exercise 3** *Show that there exist two disjoint sets $A, B$, such that $\mathbb{N}^2 = A \bigcup B$, and any straight line parallel to the $x$-axis intersects $A$ at most finitely many times, any straight line parallel to the $y$-axis intersects $B$ at most finitely many times.*

$\Sigma\, m(A\quad) < \infty$

**Exercise 4** 1. *Verify that the Cantor set $C$ is*

$$C = \left\{ \sum_{k=1}^{\infty} \frac{a_k}{3^k} \mid a_k \in \{0, 2\}, \forall k \in \mathbb{N} \right\}$$

2. *Prove that $C$ is equipotent to $P(\mathbb{N})$, the collection of all subsets of $\mathbb{N}$.*

$2^{\mathbb{N}}.$

3. Let

$$A := \left\{ x \in [0,1] \mid x = \sum_{k=1}^{n} \frac{a_k}{3^k} \text{ where } n \in \mathbb{N}, \text{ and } a_k \in \{0,2\}, \forall 1 \leq k \leq n \right\}$$

*Prove that A is countable.*

4. *Prove that C is not countable.*

5. *Prove that C is measurable and* $m(C) = 0$.

6. Let

$$B := \left\{ \sum_{k=1}^{\infty} \frac{a_k}{2^{2k}} \mid a_k \in \{0,2\}, \forall k \in \mathbb{N} \right\}$$

*Prove that B is measurable and* $m(B) = 0$. $4^k$

**Exercise 5** *Consider a sequence of measurable subsets* $B_n$ *in* $[0,1]$ *such that*

$$\inf_{n \in \mathbb{N}} m(B_n) > 0.$$

1. *Prove that* $\limsup_n B_n$ *is measurable and that*

$$m(\limsup_n B_n) > 0.$$

2. *Deduce that there exists one point* $x \in [0,1]$ *satisfying*

$$\sum_{n=1}^{\infty} \chi_{B_n}(x) = \infty.$$