2024 秋季学期偏微分方程期中考试

命题人:　　　　**考试时间**: 2024 **年** 11 **月** 14 **日** 8:00-9:　　　　**整理人**: ***Aut***

1.(10 points) Write down an explicit formula for a function u solving the initial value problem

$$\begin{cases} u_t + b\cdot Du + cu = 0 & \text{in } \mathbb{R}^n\times(0,\infty) \\ u = g & \text{on } \mathbb{R}^n\times\{t=0\}. \end{cases}$$

Here, $c\in\mathbb{R}$ and $b\in\mathbb{R}^n$ are constants.
(10 分) 写出下面初值问题解的具体表达式

$$\begin{cases} u_t + b\cdot Du + cu = 0 & \text{in } \mathbb{R}^n\times(0,\infty) \\ u = g & \text{on } \mathbb{R}^n\times\{t=0\}. \end{cases}$$

其中，$c\in\mathbb{R}$ 和 $b\in\mathbb{R}^n$ 是常数.

2.(10 points) Use Poisson's formula for the ball to prove

$$r^{n-2}\frac{r-|x|}{(r+|x|)^{n-1}}u(0)\leqslant u(x)\leqslant r^{n-2}\frac{r+|x|}{(r-|x|)^{n-1}}u(0),\quad x\in B^0(0,r)$$

whenever u is positive and harmonic in $B^0(0,r)$.
(10 分) 利用球的 Poisson 公式证明

$$r^{n-2}\frac{r-|x|}{(r+|x|)^{n-1}}u(0)\leqslant u(x)\leqslant r^{n-2}\frac{r+|x|}{(r-|x|)^{n-1}}u(0),\quad x\in B^0(0,r)$$

其中 u 在开球 $B^0(0,r)$ 中是正的并且是调和的.

3.(30 points) Recall that the Poisson’s kernel for the ball $B(0,r)$ is given by

$$K(x,y)=\frac{r^2-|x|^2}{n\alpha(n)r}\cdot\frac{1}{|x-y|^n},\quad x\in B^0(0,r), y\in\partial B(0,r).$$

(1) Show that for each fixed $y\in\partial B(0,r)$, the mapping $x\mapsto K(x,y)$ is harmonic in $B^0(0,r)$.
(2) Define $v(x)=\int_{\partial B(0,r)}K(x,y)dS(y), x\in B^0(0,r)$. Show that $v_{x_i}(y)=\int_{\partial B(0,r)}K_{x_i}(x,y)dS(y)$, $x\in B^0(0,r)$ and then conclude that v is harmonic in $B^0(0,r)$.
(3) Show that $v\equiv 1$ in $B^0(0,r)$.
(30 分) 回顾球 $B(0,r)$ 上的 Poisson 核由下列表达式给出

$$K(x,y)=\frac{r^2-|x|^2}{n\alpha(n)r}\cdot\frac{1}{|x-y|^n},\quad x\in B^0(0,r), y\in\partial B(0,r).$$

(1) 证明对于每个固定的 $y\in\partial B(0,r)$，映射 $x\mapsto K(x,y)$ 在开球 $B^0(0,r)$ 中是调和的.
(2) 定义 $v(x)=\int_{\partial B(0,r)}K(x,y)dS(y), x\in B^0(0,r)$. 证明 $v_{x_i}(y)=\int_{\partial B(0,r)}K_{x_i}(x,y)dS(y)$, $x\in B^0(0,r)$ 并且得出 v 在 $B^0(0,r)$ 中是调和的.
(3) 证明 v 在 $B^0(0,r)$ 中恒等于 1.

4.(10 points) Suppose u is smooth and solves $u_t - \Delta u = 0$ in $\mathbb{R}^n \times (0,\infty)$. Show that $u_\lambda(x,t) := u(\lambda x, \lambda^2 t)$ also solves the heat equation for each $\lambda \in \mathbb{R}$.
(10 分) 假设 u 是光滑的并且满足 $\mathbb{R}^n \times (0,\infty)$ 上的热方程 $u_t - \Delta u = 0$. 证明对每一个 $\lambda \in \mathbb{R}, u_\lambda(x,t) := u(\lambda x, \lambda^2 t)$ 也满足热方程.

5.(20 points) Assume $n = 1$ and $u(x,t) = v(\frac{x}{\sqrt{t}})$. Show $u_t = u_{xx}$ if and only if

$$v'' + \frac{z}{2} v' = 0. \tag{$\star$}$$

Show that the general solution of $(\star)$ is

$$v(z) = c \int_0^z \mathrm{e}^{-s^2/4} ds + d.$$

(20 分) 令 $n = 1$ 以及 $u(x,t) = v(\frac{x}{\sqrt{t}})$. 证明 $u_t = u_{xx}$ 当且仅当

$$v'' + \frac{z}{2} v' = 0. \tag{$\star$}$$

证明 $(\star)$ 的一般解是

$$v(z) = c \int_0^z \mathrm{e}^{-s^2/4} ds + d.$$

6. (20 points) (1) Show that the general solution of the PDE $u_{xy} = 0$ is $u(x,y) = F(x) + G(y)$ for arbitrary functions F, G.
(2) Using the change of variables $\xi = x + t, \eta = x - t$, show $u_{tt} - u_{xx} = 0$ if and only if $u_{\xi\eta} = 0$.
(20 分)(1) 证明 PDE $u_{xy} = 0$ 的一般解是 $u(x,y) = F(x) + G(y)$, 其中 F, G 为任意函数.
(2) 利用变量替换 $\xi = x + t, \eta = x - t$, 证明 $u_{tt} - u_{xx} = 0$ 当且仅当 $u_{\xi\eta} = 0$.