Solutions to some exercises and problems

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Abstract

Solutions to some exercises and problems from Stein and Shakarchi's Fourier Analysis. The book by Y. Ketznelson, "An introduction of Harmonic Analysis" (2nd corrected edition) is referred to frequently.
Chapter 1: The Genesis of Fourier Analysis
Chapter 2: Basic Properties of Fourier Series
Chapter 3: Convergence of Fourier Series
Chapter 4: Some applications of Fourier Series
Chapter 5: The Fourier transform on $\mathbb{R}$
Chapter 6: The Fourier transform on $\mathbb{R}^d$
Chapter 7: Finite Fourier Analysis
Chapter 8: Dirichlet's Theorem

Chapter 1 The Genesis of Fourier Analysis

1. (Exercise 8) Suppose F is a function on (a, b) with two continuous derivatives. Show that whenever x and $x + h$ belong to (a, b), one may write

$$F(x+h) = F(x) + hF'(x) + \frac{h^2}{2}F''(x) + h^2\phi(h),$$

where $\phi(h) \to 0$ as $h \to 0$.
Deduce that

$$\frac{F(x+h) + F(x-h) - 2F(x)}{h^2} \to F''(x) \text{ as } h \to 0.$$

Proof.
Firstly one has

$$F(x+h) - F(x) = \int_x^{x+h} F'(y)\,dy.$$

Then $F'(y) = F'(x) + (y-x)F''(x) + (y-x)\psi(y-x)$, where ψ is continuous and $\psi(h) \to 0$ as $h \to 0$. Therefore

$$F(x+h) - F(x) = F'(x)h + F''(x)\frac{h^2}{2} + \int_0^h u\psi(u)\,du.$$

By mean-value theorem

$$\int_0^h u\psi(u)\,du = \psi(\xi)\int_0^h u\,du = h^2\frac{\psi(\xi)}{2} = h^2\phi(h)$$

where ξ is between 0 and h, and $\phi(h) \to 0$ as $h \to 0$ by the continuity of ψ.
From above, one easily gets

$$\frac{F(x+h) + F(x-h) - 2F(x)}{h^2} - F''(x) = \phi(h) + \phi(-h) \to 0 \text{ as } h \to 0.$$

Remark. Suppose F is periodic. The function $\zeta(x,h) = \frac{F(x+h)+F(x-h)-2F(x)}{h^2} - F''(x)$ is continuous for all x, h. And there exists M such that

$$|F(x+h) + F(x-h) - 2F(x)| \le Mh^2$$

for all x, h (since F, F'', ζ are all periodic in x).

2. (Exercise 10) Show that the expression of the Laplacian

$$\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is given in polar coordinates by the formula

$$\triangle = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

Also prove that

$$\left|\frac{\partial u}{\partial x}\right|^2 + \left|\frac{\partial u}{\partial y}\right|^2 = \left|\frac{\partial u}{\partial r}\right|^2 + \frac{1}{r^2}\left|\frac{\partial u}{\partial \theta}\right|^2.$$

Solution.
From $x = r\cos\theta, y = r\sin\theta$, we get

$$\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix},$$

hence

$$\begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\frac{1}{r}\sin\theta & \frac{1}{r}\cos\theta \end{pmatrix}.$$

By Chain Rule,

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}\\ &= \frac{\partial u}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial u}{\partial \theta}\sin\theta.\end{aligned}$$

By Chain Rule and Product Rule,

$$\begin{aligned}&\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\cos\theta\right)\\ =\ &\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\cos\theta\right)\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial r}\cos\theta\right)\frac{\partial \theta}{\partial x}\\ =\ &\frac{\partial^2 u}{\partial r^2}\cos^2\theta + \frac{1}{r}\frac{\partial u}{\partial r}\sin^2\theta - \frac{1}{r}\frac{\partial^2 u}{\partial\theta\partial r}\cos\theta\sin\theta\end{aligned}$$

Similarly,

$$\begin{aligned}&\frac{\partial}{\partial x}\left(-\frac{1}{r}\frac{\partial u}{\partial \theta}\sin\theta\right)\\ =\ &\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\sin^2\theta + \frac{2}{r^2} + \frac{2}{r^2}\frac{\partial u}{\partial \theta}\sin\theta\cos\theta - \frac{1}{r}\frac{\partial^2 u}{\partial r\partial\theta}\cos\theta\sin\theta.\end{aligned}$$

Since $\frac{\partial^2 u}{\partial r\partial\theta} = \frac{\partial^2 u}{\partial\theta\partial r}$, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial r^2}\cos^2\theta + \frac{1}{r}\frac{\partial u}{\partial r}\sin^2\theta + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}\sin^2\theta + \frac{2}{r^2}\frac{\partial u}{\partial\theta}\sin\theta\cos\theta - \frac{2}{r}\frac{\partial^2 u}{\partial r\partial\theta}\cos\theta\sin\theta.$$

Similarly,

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial u}{\partial\theta}\cos\theta,$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2}\sin^2\theta + \frac{1}{r}\frac{\partial u}{\partial r}\cos^2\theta + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}\cos^2\theta - \frac{2}{r^2}\frac{\partial u}{\partial\theta}\sin\theta\cos\theta + \frac{2}{r}\frac{\partial^2 u}{\partial r\partial\theta}\cos\theta\sin\theta.$$

Consequently,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}.$$

Assume the general case that u is complex-valued. Then

$$\left|\frac{\partial u}{\partial x}\right|^2 = \left|\frac{\partial u}{\partial r}\right|^2\cos^2\theta + \frac{1}{r^2}\left|\frac{\partial u}{\partial\theta}\right|^2\sin^2\theta - \frac{2}{r}\Re\{\overline{\frac{\partial u}{\partial r}}\frac{\partial u}{\partial\theta}\cos\theta\sin\theta\}$$

and

$$\left|\frac{\partial u}{\partial y}\right|^2 = \left|\frac{\partial u}{\partial r}\right|^2\sin^2\theta + \frac{1}{r^2}\left|\frac{\partial u}{\partial\theta}\right|^2\cos^2\theta + \frac{2}{r}\Re\{\overline{\frac{\partial u}{\partial r}}\frac{\partial u}{\partial\theta}\cos\theta\sin\theta\},$$

hence

$$\left|\frac{\partial u}{\partial x}\right|^2 + \left|\frac{\partial u}{\partial y}\right|^2 = \left|\frac{\partial u}{\partial r}\right|^2 + \frac{1}{r^2}\left|\frac{\partial u}{\partial\theta}\right|^2.$$

3. (Added in) Let $f(x)$ be an odd 2π-periodic function. For $n \in \mathbb{Z}$ define

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt$$

Prove that

$$c{-}ne^{-inx} + c_n e^{inx} = A_n \sin nx$$

where

$$A_n = \frac{2}{\pi}\int_0^{\pi} f(t)\sin nt\,dt.$$

Proof.

$$\begin{aligned}
& c{-}ne^{-inx} + c_n e^{inx} \\
= \ & \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{int}e^{-inx} + f(t)e^{-int}e^{inx}\,dt \\
= \ & \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)(e^{-in(x-t)} + e^{in(x-t)})\,dt \\
= \ & \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)2\cos n(x-t)\,dt \\
= \ & \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)(\cos nx\cos nt + \sin nx \sin nt)\,dt \\
= \ & \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nx \sin nt\,dt \text{ (since } f(t)\cos nt \text{ is odd)} \\
= \ & \sin nx\frac{2}{\pi}\int_0^{\pi} f(t)\sin nt\,dt \text{ (since } f(t)\sin nt \text{ is even)} \\
= \ & A_n \sin nx
\end{aligned}$$

4. (Exercise 11) Show that if $n \in \mathbb{Z}$ the only solutions of the differential equation

$$r^2F''(r) + rF'(r) - n^2F(r) = 0,$$

which are twice differentiable when $r > 0$, are given by linear combinations of r^n and r^{-n} when $n \neq 0$, and 1 and $\log r$ when $n = 0$.

Solution.

If $F(r)$ solves the equation, write $g(r) = F(r)/r^n$. Then g is twice differentiable, $F(r) = r^n g(r)$ and

$$F'(r) = nr^{n-1}g(r) + r^n g'(r),$$

$$F''(r) = n(n-1)r^{n-2}g(r) + nr^{n-1}g'(r) + nr^{n-1}g'(r) + r^n g''(r).$$

Then

$$r^2F''(r){+}rF'(r){-}n^2F(r) = r^{n+1}(rg''(r){+}(2n{+}1)g'(r)) = r^{n+1}(2ng'(r){+}(rg'(r))') = 0,$$

hence

$$2ng(r) + rg'(r) = c.$$

It follows that $g(r)$ is a linear combination of r^{-2n} and 1 if $n \neq 0$, and $\log r$ and 1 if $n = 0$. The result follows.

5. (Problem 1) Consider the Dirichlet problem illustrated in Figure 11. More precisely, we look for a solution of the steady-state heat equation $\triangle u = 0$ in the rectangle $R = \{(x, y) : 0 \leq x \leq \pi, 0 \leq y \leq 1\}$ that vanishes on the vertical sides of R, and so that

$$u(x, 0) = f_0(x) \text{ and } u(x, 1) = f_1(x),$$

where f_0 and f_1 are initial data which fix the temperature distribution on the horizontal sides of the rectangle.
Use separation of variables to show that if f_0 and f_1 have Fourier expansions

$$f_0(x) = \sum_{k=1}^{\infty} A_k \sin kx \text{ and } f_1(x) = \sum_{k=1}^{\infty} B_k \sin kx,$$

then

$$u(x, y) = \sum_{k=1}^{\infty} \left(\frac{\sinh k(1-y)}{\sinh k} A_k + \frac{\sinh ky}{\sinh k} B_k \right) \sin kx.$$

Compare this result with the solution of the Dirichlet problem in the strip obtained in Problem 3, Chapter 5.
Solution.
Consideration of basic solution of the form $A(x)B(y)$ yields

$$A(x) = \sin kx, \;\; B(y) = c_k \sinh ky + d_k \cosh ky.$$

Setting $y = 0, y = 1$ and comparing coefficients give

$$d_k = A_k, \;\; c_k \sinh k + d_k \cosh k = B_k.$$

Solving for d_k, c_k and using the identity $\sinh(a - b) = \sinh a \cosh b - \sinh b \cosh a$, we find

$$c_k \sinh ky + d_k \cosh ky = \frac{\sinh k(1-y)}{\sinh k} A_k + \frac{\sinh ky}{\sinh k} B_k.$$

Chapter 2

1. (Exercise 1) Suppose f is 2π-periodic and integrable on any finite interval. Prove that if $a, b \in \mathbb{R}$, then

$$\int_a^b f(x)\, dx = \int_{a+2\pi}^{b+2\pi} f(x)\, dx = \int_{a-2\pi}^{b-2\pi} f(x)\, dx.$$

Also prove that

$$\int_{-\pi}^{\pi} f(x+a)\,dx = \int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi+a}^{\pi+a} f(x)\,dx.$$

Proof.
For the first two identities, substitute $u = x - 2\pi$ and $u = x + 2\pi$ respectively, and note that $f(x \pm 2\pi) = f(x)$. For the second equalities, firstly we have $\int_{-\pi}^{\pi} f(x+a)\,dx = \int_{-\pi+a}^{\pi+a} f(x)\,dx$ by substitution $u = x + a$. Then by applying the first set of equalities, we get

$$\int_{-\pi+a}^{-\pi} f(x)\,dx = \int_{\pi+a}^{\pi} f(x)\,dx,$$

so

$$\int_{-\pi+a}^{\pi+a} f(x)\,dx = \int_{-\pi+a}^{-\pi} f(x)\,dx + \int_{-\pi}^{\pi} f(x)\,dx + \int_{\pi}^{\pi+a} f(x)\,dx = \int_{-\pi}^{\pi} f(x)\,dx.$$

2. (Exercise 2) In this exercise we show how the symmetries of a function imply certain properties of its Fourier coefficients. Let f be a 2π-periodic Riemann integrable function on $\mathbb{R}$.
(a) Show that the Fourier series of the function f can be written as

$$f(\theta) \sim \hat{f}(0) + \sum_{n\geq 1} [\hat{f}(n) + \hat{f}(-n)] \cos n\theta + i[\hat{f}(n) - \hat{f}(-n)] \sin n\theta.$$

(b) Prove that if f is even, then $\hat{f}(n) = \hat{f}(-n)$, and we get a cosine series.
(c) Prove that if f is odd, then $\hat{f}(n) = -\hat{f}(-n)$, and we get a sine series.
(d) Suppose that $f(\theta + \pi) = f(\theta)$ for all $\theta \in \mathbb{R}$. Show that $\hat{f}(n) = 0$ for all odd n.
(e) Show that f is real-valued if and only if $\overline{\hat{f}(n)} = \hat{f}(-n)$ for all n.
Proof.

$$\begin{aligned}
& \hat{f}(n)e^{int} + \hat{f}(-n)e^{-int} \\
= \; & (\hat{f}(n) + \hat{f}(-n))\frac{1}{2}(e^{int} + e^{-int}) + i(\hat{f}(n) - \hat{f}(-n))\frac{1}{2i}(e^{int} - e^{-int}) \\
= \; & A_n \cos nt + B_n \sin nt
\end{aligned}$$

where $A_n = \hat{f}(n)+\hat{f}(-n) = \frac{1}{2\pi}\int_0^{2\pi} f(s)(e^{-ins}+e^{ins})\,ds = \frac{1}{\pi}\int_0^{2\pi} f(s)\cos ns\,ds$ and $B_n = i(\hat{f}(n)-\hat{f}(-n)) = i\frac{1}{2\pi}\int_0^{2\pi} f(s)(e^{-ins}-e^{ins})\,ds = \frac{1}{\pi}\int_0^{2\pi} f(s)\sin ns\,ds$.
If f is real, then A_n, B_n are real, which is true iff $\overline{\hat{f}(n)} = \hat{f}(-n)$ for all n.
If f is odd, then $f(s)\cos ns$ is odd, so $A_n = 0$. If f is even, then $f(s)\sin ns$ is odd, so $B_n = 0$.
If $f(\theta + \pi) = f(\theta)$ for all $\theta \in \mathbb{R}$, then

$$\int_0^{2\pi} f(t)e^{-int}\,dt = \int_0^{\pi} f(t)e^{-int}\,dt + \int_{\pi}^{2\pi} f(t)e^{-int}\,dt$$

and

$$\int_{\pi}^{2\pi} f(t)e^{-int}\,dt = \int_0^{\pi} f(u+\pi)e^{-inu}e^{-in\pi}\,du = -\int_0^{\pi} f(t)e^{-int}\,dt$$

for all odd n.

3. (Exercise 4) Consider the 2π-periodic odd function defined on $[0,\pi]$ by $f(\theta) = \theta(\pi - \theta)$.
(a) Draw the graph of f.
(b) Computer the Fourier coefficients of f, and show that

$$f(\theta) = \frac{8}{\pi} \sum_{k \text{ odd } \geq 1} \frac{\sin k\theta}{k^3}$$

Solution.
Checked with Maple and found that it is correct.

4. (Exercise 6) Let f be the function defined on $[-\pi,\pi]$ by $f(\theta) = |\theta|$.
(a) Draw the graph of f.
(b) Calculate the Fourier coefficients of f, and show that ... $O(1/n^2)$.
(c) What is the Fourier series of f in terms of sines and cosines?
(d) Taking $\theta = 0$, prove that

$$\sum_{n \text{ odd } \geq 1} \frac{1}{n^2} = \frac{\pi^2}{8} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Solution.
Since it is an even function, by Exercise 2, $\hat{f}(n) = \hat{f}(-n)$. We find $\hat{f}(n) + \hat{f}(-n) = 0$, if n is even, and $-\frac{2}{\pi n^2}$ if n is odd. So the series is

$$\frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ odd } \geq 1} \frac{\cos n\theta}{n^2}.$$

Assuming that the series converges at $\theta = 0$, we find the first part of (d). It is easy to see that $\sum_{n \text{ even } \geq 1} \frac{1}{n^2} = \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^2}$. Combining this with what we got for the odd sum, we get the second part of (d).

5. (Exercise 7) Suppose $\{a_n\}_{n=1}^N$ and $\{b_n\}_{n=1}^N$ are two finite sequences of complex numbers. Let $B_k = \sum_{n=1}^k b_n$ denote the partial sums of the series $\sum b_n$ with the convention $B_0 = 0$.
(a) Prove the **summation by parts** formula

$$\sum_{n=M}^{N} a_n b_n = a_N B_N - a_M B_{M-1} - \sum_{n=M}^{N-1} (a_{n+1} - a_n) B_n.$$

(b) Deduce from this formula Dirichlet's test for convergence of a series : if the partial sums of the series $\sum b_n$ are bounded, and $\{a_n\}$ is a sequence of real numbers that decreases monotonically to 0, then $\sum a_n b_n$ converges.
Proof. (a)

$$\begin{aligned}
& \sum_{n=M}^{N} a_n b_n \\
= & \sum_{n=M}^{N} a_n (B_n - B_{n-1}) \\
= & \sum_{n=M}^{N} a_n B_n - \sum_{n=M}^{N} a_n B_{n-1} \\
= & \sum_{n=M}^{N} a_n B_n - \sum_{n=M-1}^{N-1} a_{n+1} B_n \\
= & a_n B_N + \sum_{n=M}^{N-1} a_n B_n - a_M B_{M-1} - \sum_{n=M}^{N-1} a_{n+1} B_n \\
= & a_N B_N - a_M B_{M-1} - \sum_{n=M}^{N-1} (a_{n+1} - a_n) B_n
\end{aligned}$$

(b) Suppose $|B_n| \leq B$ for all n. Then for $N > M$

$$|\sum_{n=M}^{N} a_n b_n| \leq B(a_N + a_M + \sum_{n=M}^{N-1} (a_n - a_{n+1})) = 2Ba_M$$

proving that the series is Cauchy and hence converges.

6. (Exercise 8) Verify that $\frac{1}{2i} \sum_{n \neq 0} \frac{e^{inx}}{n}$ is the Fourier series of the 2π-periodic **sawtooth** function illustrated in Figure 6, defined by $f(0) = 0$, and

$$f(x) = \begin{cases} -\pi/2 - x/2 & \text{if } -\pi < x < 0 \\ \pi/2 - x/2 & \text{if } 0 < x < \pi. \end{cases}$$

Note that this function is not continuous. Show that nevertheless, the series converges for every x (by which we mean, as usual, that the symmetric partial sums of the series converge). In particular, the value of the series at the origin, namely 0, is the average of the values of $f(x)$ as x approaches the origin from the left and the right.
Solution.
Checked with Maple and found it correct. $b_n = \sin nx$,

$$B_n = \sum_{k=1}^{n} b_k = \frac{\sin(nx/2)\sin((n+1)x/2)}{\sin(x/2)}$$

(to prove write $\sin nx$ as $\frac{1}{2i}(e^{inx} - e^{-inx})$)) and $|B_n| \leq \csc(|x|/2)$ for $|x| < \pi, x \neq 0$. $1/n$ is decreasing to 0. So series converges for $x \neq 0$. At $x = 0$, all the symmetric sums are 0, so series converges to 0. It must converge to the function since its Cesaro means converge to the function.

7. (Exercise 9) Let $f(x) = \chi_{[a,b]}(x)$ be the characteristic function of the interval $[a,b] \subset [-\pi, \pi]$, that is,

$$\chi_{[a,b]}(x) = \begin{cases} 1 & \text{if } x \in [a,b] \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that the Fourier series of f is given by

$$f(x) \sim \frac{b-a}{2\pi} + \sum_{n\neq 0} \frac{e^{-ina} - e^{-inb}}{2\pi in} e^{inx}.$$

The sum extends over all positive and negative integers excluding 0.
(b) Show that if $a \neq -\pi$ or $b \neq \pi$ and $a \neq b$, then the Fourier series does not converge absolutely for any x.
(c) However, prove that Fourier series converges at every point x. What happens if $a = -\pi$ and $b = \pi$?
Solution.
(a) Straightforward.
(b) $|e^{-ina} - e^{-inb}|^2 = 1 + 1 - 2\cos n(b-a) = 4\sin^2(n(b-a)/2)$, thus $|e^{-ina} - e^{-inb}| = 2|\sin n\theta_0|$, where

$$0 < \theta_0 = \frac{b-a}{2} < \frac{\pi}{2}.$$

The series

$$\sum_{n=1}^{\infty} \frac{|\sin n\theta_0|}{n}$$

diverges because ....
(c) $a_n = 1/n$, $b_n = \sin(na) - \sin(nb)$ which has bounded partial sums by the solution of Exercise 8. So series converges by Exercise 7. If $a = -\pi$ and $b = \pi$, then the Fourier series is 1.

8. (Exercise 10) Suppose f is a periodic function of period 2π which belongs to the class C^k. Show that

$$\hat{f}(n) = O(1/|n|^k) \text{ as } |n| \to \infty.$$

Solution.
Integrate by parts k times as in Corollary 2.4.

9. (Exercise 11) Suppose that $\{f_k\}_{k=1}^{\infty}$ is a sequence of Riemann integrable functions on the interval $[0,1]$ such that

$$\int_0^1 |f_k(x) - f(x)|\,dx \to 0 \text{ as } k \to \infty.$$

Show that $\hat{f}_k(n) \to \hat{f}(n)$ uniformly in n as $k \to \infty$.
Proof.
Use
$$|\hat{g}(n)| \leq \int_0^1 |g(x)|\, dx$$
for all n.

10. (Exercise 12) Prove that if a series of complex numbers $\sum c_n$ converges to s, then $\sum c_n$ is Cesaro summable to s.
Proof.
Let the sequence of partial sums be $s_n, n = 1, 2, \cdots$. First assume that $s = 0$. Let $\epsilon > 0$. Choose N_1 such that $|s_n| < \frac{\epsilon}{2}$ for all $n \geq N_1$. Choose $N > N_1$ such that
$$\frac{\sum_{k=1}^{N_1} |s_k|}{n} < \frac{\epsilon}{2}$$
for all $n \geq N$. Then for all $n \geq N$,
$$\begin{aligned} & |\frac{s_1 + \cdots + s_n}{n}| \\ \leq & \frac{|s_1| + \cdots + |s_n|}{n} \\ = & \frac{|s_1 + \cdots + |s_{N_1}|}{n} + \frac{|s_{N_1+1} + \cdots + |s_n|}{n} \\ < & \frac{\epsilon}{2} + \frac{n\epsilon/2}{n} \\ = & \epsilon \end{aligned}$$
This proves that $\sum c_n$ is Cesaro summable to 0. Notice that we prove above that if a_n is *any* sequence converging to 0, then the sequence $\sigma_n = \frac{a_1+\cdots+a_n}{n}$ converges to 0.
Now suppose that $s \neq 0$. Then sequence $t_n = s_n - s$ converges to 0. So by the proof above $(t_1 + \cdots + t_n)/n = (s_1 + \cdots + s_n)/n - s$ converges to 0, i.e $(s_1 + \cdots + s_n)/n$ converges to s.

11. (Exercise 13) The purpose of this exercise is to prove that Abel summability is more general than the standard or Cesaro methods of summation.
(a) Show that if the series $\sum_{n=1}^{\infty} c_n$ of complex numbers converges to a finite limit s, then the series is Abel summable to s.
(b) However, show that there exist series which are Abel summable, but that do not converge.
(c) Argue similarly to prove that if a series $\sum_{n=1}^{\infty} c_n$ is Cesaro summable to σ, then it is Abel summable to σ.
(d) Give an example of a series that is Abel summable but not Cesaro summable.
The results above can be summarized by the following implications about series:
convergent $\Rightarrow$ Cesaro summable $\Rightarrow$ Able summable,

and the fact that none of the arrows can be reversed.
Solution:
In what follows, $0 \le r < 1$; s_n denotes the sequence of partial sums of $\sum_{n=1}^{\infty} c_n$.
(a). First assume that $s = 0$. For $0 \le r < 1$ consider $t_n = \sum_{k=1}^{n} s_k r^k$. We have $rt_n = \sum_{k=1}^{n} s_k r^{k+1} = \sum_{k=2}^{n+1} s_{k-1} r^k$. Since $s_1 = c_1$ and $s_k - s_{k-1} = c_k$, we get $(1-r)t_n = \sum_{k=1}^{n} c_k r^k - s_n r^{n+1}$, hence $\sum_{k=1}^{n} c_k r^k = (1-r)\sum_{k=1}^{n} s_k r^k + s_n r^{n+1}$. Since c_k, s_k are bounded sequences (in fact, they converge to 0) and $0 \le r < 1$, both series $\sum_{k=1}^{\infty} c_k r^k$ and $\sum_{k=1}^{\infty} s_k r^k$ are absolutely convergent, and $\sum_{k=1}^{\infty} c_k r^k = (1-r)\sum_{k=1}^{\infty} s_k r^k$. Let $\epsilon > 0$. Choose K such that $|s_k| < \epsilon$ for all $k \ge K$. Then

$$|\sum_{k=1}^{\infty} c_k r^k| \le (1-r)\sum_{k=1}^{K-1} M r^k + (1-r)\epsilon \frac{r^K}{1-r} \le (1-r)\sum_{k=1}^{K-1} M r^k + \epsilon$$

It follows that

$$\limsup_{r\to 1-} |\sum_{k=1}^{\infty} c_k r^k| \le \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have $\limsup_{r\to 1-} |\sum_{k=1}^{\infty} c_k r^k| \le 0$ which is equivalent to $\lim_{r\to 1-} \sum_{k=1}^{\infty} c_k r^k = 0$.
Next assume that $s \neq 0$. Define $d_n = c_n - s/2^n$. Then $\sum_{n=1}^{\infty} d_n = 0$, and $\sum_{n=1}^{\infty} d_n r^n = \sum_{n=1}^{\infty} c_n r^n - sr/(2-r)$. Thus by our proof above, $\lim_{r\to 1-} \sum_{n=1}^{\infty} c_n r^n - s = 0$, i.e. $\lim_{r\to 1-} \sum_{n=1}^{\infty} c_n r^n = s$.
(b). The series $\sum_{n=1}^{\infty} (-1)^n$ does not converge, but $\lim_{r\to 1-} \sum_{n=1}^{\infty} (-1)^n r^n = \lim_{r\to 1-} \frac{-r}{1+r} = -1/2$, i.e. $\sum_{n=1}^{\infty} (-1)^n$ is Abel summable to $-1/2$.
(c). Write

$$\sigma_n = \frac{s_1 + \cdots + s_n}{n}$$

and $\tau_n = n\sigma_n$. Since $\tau_n - \tau_{n-1} = s_n$, by the same argument as in (a), we have

$$(1-r)\sum_{k=1}^{\infty} \tau_k r^k = \sum_{k=1}^{\infty} s_k r^k,$$

hence

$$(1-r)^2 \sum_{k=1}^{\infty} \sigma_k k r^k = \sum_{k=1}^{\infty} c_k r^k$$

from the proof of (a). Assume that $\sigma = 0$. Let $\epsilon > 0$. Choose N such that $|\sigma_k| < \epsilon$ for all $n \ge N$, and B be a bound for $|\sigma_k|, k = 1, 2, \cdots$. Then

$$|\sum_{k=1}^{\infty} c_k r^k| \le (1-r)^2 \sum_{k=1}^{N-1} Bkr^k + (1-r)^2 \epsilon \frac{r^N((1-r)N + r)}{(1-r)^2} \le (1-r)^2 \sum_{k=1}^{N-1} Bkr^k + \epsilon r^N((1-r)N + r)$$

where we have use the fact that $\sum_{k=N}^{\infty} kr^k = \frac{r^N((1-r)N+r)}{(1-r)^2}$. As in the proof of (a), this implies that $\lim_{r\to 1-} \sum_{k=1}^{\infty} c_k r^k = 0$. For the case $\sigma \neq 0$, consider $d_n = c_n - \sigma/2^n$. Since $\sum_{n=1}^{\infty} c_n$ is Cesaro summable to σ and $\sum_{n=1}^{\infty} \sigma/2^n = \sigma$ (and hence Cesaro summable to σ), we have $\sum_{n=1}^{\infty} d_n$ Cesaro summable to 0. Thus by our proof above and the same argument as in the proof of (a), the result follows.

(d). Consider the series $\sum_{n=1}^{\infty}(-1)^n n$. It is Abel summable to $-1/4$ since

$$\sum_{n=1}^{\infty}(-1)^n n r^n = \frac{-r}{(1+r)^2}.$$

Note that

$$\sigma_n - \frac{n-1}{n}\sigma_{n-1} = \frac{a_n}{n}.$$

Thus for a Cesaro summable series $\sum_{n=1}^{\infty} a_n$, $\lim_{n\to\infty} \frac{a_n}{n}$ must be 0. This proves that $\sum_{n=1}^{\infty}(-1)^n n$ is not Cesaro summable.

12. (Exercise 14) This exercise deals with a theorem of Tauber which says that under an additional condition on the coefficients c_n, the above arrows can be reversed.

(a) If $\sum c_n$ is Cesaro summable to σ and $c_n = o(1/n)$ (that is, $nc_n \to 0$), then $\sum c_n$ converges to σ.

(b) The above statement holds if we replace Cesaro summable by Abel summable.

Proof.

(a). Denote $(n-1)c_n$ by t_n. Since $t_n = \frac{n-1}{n} nc_n$ and $nc_n \to 0$ we have $t_n \to 0$. Now

$$s_n - \sigma_n = s_n - \frac{s_1 + \cdots + s_n}{n} = \frac{(s_n - s_1) + \cdots + (s_n - s_n)}{n} = \frac{t_2 + \cdots + t_n}{n}$$

and it is immediate that $s_n - \sigma_n \to 0$.

(b). Let $r = 1 - \frac{1}{N}$. We have

$$|\sum_{n=1}^{N} c_n - \sum_{n=1}^{N} c_n r^n| \le \sum_{n=1}^{N} |c_n| \left(1 - (1 - \frac{1}{N})^n\right) \le \sum_{n=1}^{N} |c_n| \frac{n}{N}$$

where we have used $(1-\frac{1}{N})^n \ge 1 - \frac{n}{N}$ for $1 \le n \le N$, which can be proved easily by induction on n. Also if $|c_n n| < \epsilon$ for all $n \geq N$, then

$$\begin{aligned}
& |\sum_{n=1}^{\infty} c_n r^n - \sum_{n=1}^{N} c_n r^n| \\
\leq & \sum_{n=N+1}^{\infty} |c_n|(1-\frac{1}{N})^n \\
\leq & \sum_{n=N+1}^{\infty} \frac{n|c_n|}{N}(1-\frac{1}{N})^n \\
\leq & \sum_{n=N+1}^{\infty} \frac{\epsilon}{N}(1-\frac{1}{N})^n \\
= & \epsilon(1-\frac{1}{N})^{N+1} \to \frac{\epsilon}{e}
\end{aligned}$$

as $N \to \infty$.

13. (Exercise 15) Prove that the Fejer kernel is given by

$$F_N(x) = \frac{1}{N}\frac{\sin^2(Nx/2)}{\sin^2(x/2)}.$$

Proof.
Recall that $NF_N(x) = D_0(x)+\cdots+D_{N-1}(x)$ where $D_n(x)$ is the Dirichlet kernel. Write $\omega = e^{ix}$. Then

$$\begin{aligned}
& D_n(x) \\
= & \omega^{-n} + \cdots + \omega^{-1} + 1 + \omega + \cdots + \omega^n \\
= & (\omega^{-n} + \cdots + \omega^{-1}) + (1 + \omega + \cdots + \omega^n) \\
= & \omega^{-1}\frac{\omega^{-n}-1}{\omega^{-1}-1} + \frac{1-\omega^{n+1}}{1-\omega} \\
= & \frac{\omega^{-n}-1}{1-\omega} + \frac{1-\omega^{n+1}}{1-\omega} \\
= & \frac{\omega^{-n}-\omega^{n+1}}{1-\omega}
\end{aligned}$$

So

$$
\begin{aligned}
& NF_N(x) \\
= \quad & \sum_{n=0}^{N-1} \frac{\omega^{-n} - \omega^{n+1}}{1-\omega} \\
= \quad & \frac{1}{1-\omega}\left(\sum_{n=0}^{N-1} \omega^{-n} - \sum_{n=0}^{N-1} \omega^{n+1}\right) \\
= \quad & \frac{1}{1-\omega}\left(\frac{\omega^{-N}-1}{\omega^{-1}-1} - \omega\frac{1-\omega^N}{1-\omega}\right) \\
= \quad & \frac{1}{1-\omega}\left(\frac{\omega^{-N+1}-\omega}{1-\omega} - \omega\frac{1-\omega^N}{1-\omega}\right) \\
= \quad & \omega\frac{\omega^{-N}-2+\omega^N}{(1-\omega)^2} \\
= \quad & \frac{1}{(\omega^{-1/2})^2}\frac{(\omega^{N/2}-\omega^{-N/2})^2}{(1-\omega)^2} \\
= \quad & \frac{(\omega^{N/2}-\omega^{-N/2})^2}{(\omega^{1/2}-\omega^{-1/2})^2} \\
= \quad & \frac{-4\sin^2(Nx/2)}{-4\sin^2(x/2)} \\
= \quad & \frac{\sin^2(Nx/2)}{\sin^2(x/2)}.
\end{aligned}
$$

Therefore

$$F_N(x) = \frac{1}{N}\frac{\sin^2(Nx/2)}{\sin^2(x/2)}.$$

14. (Exercise 16) The Weierstrass approximation theorem states: Let f be a continuous function on the closed and bounded interval $[a,b] \subset \mathbb{R}$. Then, for any $\epsilon > 0$, there exists a polynomial P such that

$$\sup_{x\in[a,b]} |f(x) - P(x)| < \epsilon.$$

Proof.
Let $\epsilon > 0$. We may extend f to a continuous $(c-a)$-periodic function where $b \le c$. By Corollary 5.4 of Fejer's theorem, there exists a trigonometric polynomial $Q = \sum_{n=M}^{N} a_n e^{inx}$ such that $|Q(x) - f(x)| < \epsilon/2$ for all x. For each $n, M \le n \le N$, there exists a polynomial $p_n(x)$ such that $|a_n e^{inx} - p_n(x)| < \epsilon/2N$ for all $x \in [a,c]$. Then $P = p_M + \cdots + p_N$ is a polynomial in x that satisfies the requirement.

15. (Exercise 17) In Section 5.4 we proved that the Abel means of f converge to f at all points of continuity, that is,

$$\lim_{r\to 1} A_r(f)(\theta) = \lim_{r\to 1}(P_r * f)(\theta) = f(\theta), \text{ with } 0 < r < 1,$$

whenever f is continuous at θ. In this exercise, we will study the behavior of $A_r(f)(\theta)$ at certain points of discontinuity.
An integrable function is said to have a **jump discontinuity** at θ if the two limits

$$\lim_{h\to 0, h>0} f(\theta+h) = f(\theta^+) \text{ and } \lim_{h\to 0, h<0} f(\theta+h) = f(\theta^-)$$

exist.
(a) Prove that if f has a jump discontinuity at θ, then

$$\lim_{r\to 1} A_r(f)(\theta) = \frac{f(\theta^+)+f(\theta^-)}{2}, \text{ with } 0 \le r < 1.$$

(b) Using a similar argument, show that if f has a jump discontinuity at θ, the Fourier series of f at θ is Cesaro summmable to $\frac{f(\theta^+)+f(\theta^-)}{2}$.
Proof.
Since $P_r(\theta) = P_r(-\theta)$, and $\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)\,d\theta = 1$, we have $\frac{1}{2\pi}\int_{-\pi}^{0} P_r(\theta)\,d\theta = \frac{1}{2\pi}\int_{0}^{\pi} P_r(\theta)\,d\theta = 1/2$. Suppose f has a jump discontinuity at θ. Let $\epsilon > 0$ be given. Choose $\delta > 0$ so that $0 < h < \delta$ implies $|f(\theta - h) - f(\theta-)| < \epsilon$ and $|f(\theta+h) - f(\theta+)| < \epsilon$. Let M be such that $|f(y)| \le M$ for all y. Then

$$\begin{aligned}
& \left|(f * P_r)(\theta) - \frac{f(\theta^+)+f(\theta^-)}{2}\right| \\
= & \left|\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(y) f(\theta-y)\,dy\right) - \frac{f(\theta^+)+f(\theta^-)}{2}\right| \\
\le & \frac{1}{2\pi}\int_{-\pi}^{0} P_r(y)|f(\theta-y) - f(\theta^+)|\,dy + \frac{1}{2\pi}\int_{0}^{\pi} P_r(y)|f(\theta-y) - f(\theta^-)|\,dy \\
\le & \frac{1}{2\pi}\int_{-\delta<y<0} P_r(y)|f(\theta-y) - f(\theta^+)|\,dy + \frac{1}{2\pi}\int_{0<y<\delta} P_r(y)|f(\theta-y) - f(\theta^-)|\,dy \\
& + \frac{1}{2\pi} 2M \int_{\delta\le|y|\le\pi} P_r(y)\,dy \\
\le & \frac{\epsilon}{2} + \frac{\epsilon}{2} + \frac{M}{\pi}\int_{\delta\le|y|\le\pi} P_r(y)\,dy
\end{aligned}$$

Therefore, recalling that $\lim_{r\to 1}\int_{\delta\le|y|\le\pi} P_r(y)\,dy = 0$,

$$\limsup_{r\to 1}\left|(f * P_r)(\theta) - \frac{f(\theta^+)+f(\theta^-)}{2}\right| \le \epsilon$$

Since $\epsilon > 0$ is arbitrary, we have

$$\lim_{r\to 1}(f * P_r)(\theta) = \frac{f(\theta^+)+f(\theta^-)}{2}.$$

(b) Since the Fejer kernel $F_n(\theta)$ is even and positive, we also have $\frac{1}{2\pi}\int_{-\pi}^{0} F_n(\theta)\,d\theta = \frac{1}{2\pi}\int_{0}^{\pi} F_n(\theta)\,d\theta = 1/2$, for all n. Repeat the above argument.

Remark One can replace $\frac{f(\theta^+)+f(\theta^-)}{2}$ by $\lim_{h\to 0}\frac{f(\theta+h)+f(\theta-h)}{2}$ and obtain a more general statement of Exercise 17. (See Katznelson's book.)

16. (Exercise 18) If $P_r(\theta)$ denotes the Poisson kernel, show that the funciton

$$u(r,\theta) = \frac{\partial P_r}{\partial \theta},$$

defined for $0 \le r < 1$ and $\theta \in \mathbb{R}$, satisfies:
(i) $\triangle u = 0$ in the disc.
(ii) $\lim_{r\to 1} u(r,\theta) = 0$ for each θ.
However, u is not identically zero.
Solution.
We have

$$u(r,\theta) = \sum_{n=-\infty}^{\infty} r^{|m|} i n e^{in\theta}.$$

Each summand is Laplacian (see chapter 1, separation of variable), and the series and its derivatives converge uniformly on $r < \rho$ for any $0 < \rho < 1$, hence the infinite sum is also Laplacian.

$$u(r,\theta) = \frac{2r(r^2-1)\sin\theta}{(1-2r\cos\theta + r^2)^2}$$

If $\theta \neq 0$, then $1 - 2\cos\theta + 1 \neq 0$, so the limit is 0 as $r \to 1$. If $\theta = 0$, then $u(r,0) = 0$ and the limit is trivially 0. Note that

$$u(x,y) = \frac{2y(x^2+y^2-1)}{((1-x)^2+y^2)^2}$$

and $u(1-\epsilon,\epsilon) \to -\infty$ as $\epsilon \to 0+$. This implies that u does not converge to 0 uniformly as $r \to 1$.

17. (Exercise 19) Solve Laplace's equation $\triangle u = 0$ in the semi infinite strip

$$S = \{(x,y) : 0 < x < 1, 0 < y\},$$

subject to the following boundary conditions

$$\begin{cases} u(0,y) = 0 & \text{when } 0 \le y, \\ u(1,y) = 0 & \text{when } 0 \le y, \\ u(x,0) = f(x) & \text{when } 0 \le x \le 1 \end{cases}$$

where f is a given function, with of course $f(0) = f(1) = 0$. Write

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)$$

and expand the general solution in terms of the special solutions given by

$$u_n(x,y) = e^{-n\pi y}\sin(n\pi x).$$

Express u as an integral involving f, analogous to the Poisson integral formula (6).
Solution.
By considering the odd extension of f and following the derivation of Poisson's kernel with $e^{-\pi y}$ and $e^{i\pi t}$ replacing r and e^{it}, respectively, we obtain

$$u(x,y)=\frac{1}{2}\int_{-1}^{1} f(t)Q_y(x-t)\,dt$$

where

$$Q_y(t)=\frac{1-e^{-2\pi y}}{1-2e^{-\pi y}\cos\pi t+e^{-2\pi y}}.$$

or, using the fact that f is odd, we have the alternate form

$$u(x,y)=\frac{1}{2}\int_{0}^{1} f(t)Q(x,t)\,dt$$

where

$$Q(x,t)=\frac{1-e^{-2\pi y}}{1-2e^{-\pi y}\cos\pi(x-t)+e^{-2\pi y}}-\frac{1-e^{-2\pi y}}{1-2e^{-\pi y}\cos\pi(x+t)+e^{-2\pi y}}.$$

18. (Exercise 20) Consider the Dirichlet problem in the annulus defined by $\{(r,\theta):\rho<r<1\}$, where $0<\rho<1$ is the inner radius. The problem is to solve

$$\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}=0$$

subject to the boundary conditions

$$\begin{cases} u(1,\theta)=f(\theta),\\ u(\rho,\theta)=g(\theta),\end{cases}$$

where f and g are given continuous functions.
Arguing as we have previously for the Dirichlet problem in the disc, we can hope to write

$$u(r,\theta)=\sum c_n(r)e^{in\theta}$$

with $c_n(r)=A_nr^n+B_nr^{-n}, n\neq 0$. Set

$$f(\theta)\sim\sum a_ne^{in\theta} \text{ and } g(\theta)\sim\sum b_ne^{in\theta}.$$

We want $c_n(1)=a_n$ and $c_n(\rho)=b_n$. This leads to the solution

$$\begin{aligned} u(r,\theta) &= \sum_{n\neq 0}\left(\frac{1}{\rho^n-\rho^{-n}}\right)[((\rho/r)^n-(r/\rho)^n)a_n+(r^n-r^{-n})b_n]e^{in\theta}\\ &\quad +a_0+(b_0-a_0)\frac{\log r}{\log\rho}. \end{aligned}$$

Show that as a result we have

$$u(r,\theta) - (P_r * f)(\theta) \to 0 \text{ as } r \to 1 \text{ uniformly in } \theta,$$

and

$$u(r,\theta) - (P_{\rho/r} * g)(\theta) \to 0 \text{ as } r \to \rho \text{ uniformly in } \theta.$$

Solution.
Note that $\frac{r^n - r^{-n}}{\rho^n - \rho^{-n}} = \frac{r^{2n}-1}{\rho^{2n}-1}(\rho/r)^n \to 0$ and $\frac{(\rho/r)^n - (r/\rho)^n}{\rho^n - \rho^{-n}} = \frac{(\rho/r)^{2n}-1}{\rho^{2n}-1} r^n \to 0$ as $n \to \infty$.
The series converges because for instance

$$\frac{r^{2n}-1}{\rho^{2n}-1}(\rho/r)^n \le \frac{1+r^2}{1-\rho^2}(\rho/r)^n.$$

And the last result follows from, for instance,

$$\begin{aligned}\left(\frac{(\rho/r)^{2n}-1}{\rho^{2n}-1} - 1\right) r^n &= \left(\frac{(\rho/r)^{2n} - \rho^{2n}}{\rho^{2n}-1}\right) r^n \\ &\le 2[(\rho/r)^{2n} - \rho^{2n}] r^n\end{aligned}$$

and

$$\sum_{n=1}^{\infty} [(\rho/r)^{2n} - \rho^{2n}] r^n = \frac{\rho^2/r}{1-\rho^2/r} - \frac{\rho^2 r}{1-\rho^2 r}$$

which approaches 0 as $r \to 1$. **Details are to be given.**

19. (Problem 2) Let D_N denote the Dirichlet kernel

$$D_N(\theta) = \sum_{n=N}^{N} e^{ik\theta} = \frac{\sin(N+1/2)\theta}{\sin(\theta/2)},$$

and define

$$L_N = \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_N(\theta)|\, d\theta.$$

(a) Prove that

$$L_N \ge c \log N$$

for some constant $c > 0$. A more careful estimate gives

$$L_N = \frac{4}{\pi^2}\log N + O(1).$$

(See Katznelson Chapter 2, Exercise 1.1 for section p.50.)
(b) Prove the following as a consequence: for each $n \ge 1$, there exists a continuous function f_n such that $|f_n| \le 1$ and $|S_n(f_n)(0)| \ge c' \log n$.
Solution.

(a) Note that $\frac{x}{\sin x} \geq 1$ for x in the interval $[-\pi/2, \pi/2]$. It follows that for $\theta \in [-\pi, \pi]$,

$$|D_N(\theta)| \geq 2\frac{|\sin(N+1/2)\theta|}{\theta}.$$

Then

$$\begin{aligned}
&\int_{-\pi}^{\pi} |D_N(\theta)|\, d\theta \\
\geq\ & 4\int_0^{\pi} \frac{|\sin(N+1/2)\theta|}{\theta}\, d\theta \\
=\ & 4\int_0^{(N+1/2)\pi} \frac{|\sin\theta|}{\theta}\, d\theta \\
\geq\ & 4\int_0^{N\pi} \frac{|\sin\theta|}{\theta}\, d\theta \\
=\ & 4\sum_{k=0}^{N-1} \int_{k\pi}^{(k+1)\pi} \frac{|\sin\theta|}{\theta}\, d\theta \\
\geq\ & 4\sum_{k=0}^{N-1} \frac{1}{(k+1)\pi} \int_{k\pi}^{(k+1)\pi} |\sin\theta|\, d\theta \\
=\ & \frac{8}{\pi}\sum_{k=0}^{N-1} \frac{1}{k+1} \\
=\ & \frac{8}{\pi}\log(N+1) \\
\geq\ & \frac{8}{\pi}\log N
\end{aligned}$$

Therefore $L_N \geq c\log N$.
(b) The function g_n which is equal to 1 when D_n is positive and -1 when D_n is negative has the desired property but is not continuous. Approximate g_n in the integral norm (in the sense of Lemma 3.2) by continuous functions h_k satisfying $|h_k| \leq 1$.

20. (Problem 3) Littlewood provided a refinement of Tauber's theorem:
(a) If $\sum c_n$ is Abel summable to s and $c_n = O(1/n)$, then $\sum c_n$ converges to s.
(b) As a consequence, if $\sum c_n$ is Cesàro summable to s and $c_n = O(1/n)$, then $\sum c_n$ converges to s.
These results may be applied to Fourier series. By Exercise 17, they imply that if f is an integrable function that satisfies $\hat{f}(\nu) = O(1/|\nu|)$, then:
(i) If f is continuous at θ, then

$$S_N(f)(\theta) \to f(\theta) \text{ as } N \to \infty.$$

(ii) If f has a jump discontinuity at θ, then

$$S_N(f)(\theta) \to \frac{f(\theta^+) + f(\theta^-)}{2} \text{ as } N \to \infty.$$

(iii) If f is continuous on $[-\pi, \pi]$, then $S_N(f) \to f$ uniformly.
For the simpler assertion (b), hence of proof of (i),(ii),and (iii), see Problem 5 in Chapter 4.
Solution.

Lemma 1 Let f be a C^2 real function on $[0, 1)$. Suppose as $x \to 1$, $f(x) = o(1), f''(x) = O\left(\frac{1}{(1-x)^2}\right)$, then $f'(x) = o\left(\frac{1}{1-x}\right)$.

Proof. Choose $C > 0$ such that $|(1-x)^2 f''(x)| \le C$ for all x. Let $\epsilon > 0$. Choose a δ, $0 < \delta < \min\{1/2, \epsilon/(4C)\}$. Choose $\eta < 1$ such that $|f(x)| < (1/4)\delta\epsilon$ for all $x > \eta$. We claim that $|(1-x)f'(x)| < \epsilon$ for all $x > \eta$. Let $x > \eta$. With $x' = x + \delta(1-x)$ we have

$$f(x') = f(x) + \delta(1-x)f'(x) + \frac{1}{2}\delta^2(1-x)^2 f''(\zeta)$$

for some $x < \zeta < x'$.
Since $\delta < 1/2$, we have $1 - x \le 2(1-\zeta)$, so that $|(1-x)^2 f''(\zeta)| \le 4|(1-\zeta)^2 f''(\zeta)| \le 4C$. Therefore

$$\begin{aligned} & |(1-x)f'(x)| \\ = \;& \left|\frac{f(x') - f(x)}{\delta} - \frac{1}{2}\delta(1-x)^2 f''(\zeta)\right| \\ \le \;& \frac{|f(x')| + |f(x)|}{\delta} + \frac{1}{2}\delta(1-x)^2 |f''(\zeta)| \\ < \;& \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon \end{aligned}$$

Lemma 2 Let $\sum_{n=0}^{\infty} a_n x^n$ be a real power series with $a_n \ge 0$ for all n. Suppose

$$\lim_{x \to 1^-} (1-x) \sum_{n=0}^{\infty} a_n x^n = 1.$$

Then for any integrable function $g(t)$ on $[0, 1]$

$$\lim_{x \to 1^-} (1-x) \sum_{n=0}^{\infty} a_n x^n g(x^n) = \int_0^1 g(t)\, dt.$$

Proof. First prove for functions of the type x^k. Then for any polynomial; then for any continuous function; then for any integral function.

Corollary 1 Let $\sum_{n=0}^{\infty} a_n x^n$ be a real power series with $a_n \ge 0$ for all n. Suppose

$$\lim_{x \to 1^-} (1-x) \sum_{n=0}^{\infty} a_n x^n = 1.$$

Then

$$\lim_{N \to \infty} \frac{\sum_{n=0}^{N} a_n}{N} = 1.$$

Proof. Apply the above lemma to the function $g(t) = 0$ for $t < e^{-1}, 1/t$ otherwise, and let $x = e^{-1/N}$.

Finishing the solution of Problem 3

We shall write $f(x) \sim g(x)$ to mean $\frac{f(x)}{g(x)} \to 1$ as $x \to 1$, and $f(n) \sim g(n)$ to mean $\frac{f(n)}{g(n)} \to 1$ as $n \to \infty$.

(a) We may assume that $s = 0$ (see Exercise 13, Chapter 2), i.e. we assume that $f(x) = \sum_{n=0}^{\infty} a_n x^n \to 0$ as $x \to 1$. Then $f''(x) = O(1/(1-x)^2)$ because

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = O(\sum_{n=2}^{\infty}(n-1)x^{n-2}) = O(1/(1-x)^2).$$

So by Lemma 1,

$$f'(x) = o\left(\frac{1}{1-x)}\right).$$

Suppose $|na_n| \leq c$. Then

$$\sum_{n=1}^{\infty}(1 - \frac{na_n}{c})x^{n-1} = \frac{1}{1-x} - \frac{f'(x)}{c} \sim \frac{1}{1-x}$$

Since $1 - \frac{na_n}{c} \geq 0$, Corollary 1 implies that

$$\sum_{k=1}^{n}(1 - \frac{ka_k}{c}) \sim n$$

or what is the same

$$\sum_{k=1}^{n} ka_k = o(n).$$

Write $w_n = \sum_{k=1}^n ka_k, w_0 = 0$. So $w_n/n \to 0$ as $n \to \infty$. Then

$$\begin{aligned}
& f(x) - a_0 \\
= & \sum_{n=1}^{\infty} a_n x^n \\
= & \sum_{n=1}^{\infty} \frac{w_n - w_{n-1}}{n} x^n \\
= & \sum_{n=1}^{\infty} \frac{w_n}{n} x^n - \sum_{n=1}^{\infty} \frac{w_{n-1}}{n} x^n \\
= & \sum_{n=1}^{\infty} \frac{w_n}{n} x^n - \sum_{n=1}^{\infty} \frac{w_n}{n+1} x^{n+1} \\
= & \sum_{n=1}^{\infty} w_n \left(\frac{x^n}{n} - \frac{x^{n+1}}{n+1} \right) \\
= & \sum_{n=1}^{\infty} w_n \left(\frac{x^n - x^{n+1}}{n+1} + \frac{x^n}{n(n+1)} \right) \\
= & (1-x) \sum_{n=1}^{\infty} w_n \frac{x^n}{n+1} + \sum_{n=1}^{\infty} \frac{w_n}{n(n+1)} x^n
\end{aligned}$$

Since $f(x) \to 0$ and the first term in the last sum approaches 0 as $x \to 1$, we get

$$\lim_{x \to 1} \sum_{n=1}^{\infty} \frac{w_n}{n(n+1)} x^n = -a_0$$

Since $\frac{w_n}{n(n+1)} = o(1/n)$, by the regular Tauberian theorem

$$\sum_{n=1}^{\infty} \frac{w_n}{n(n+1)} = -a_0.$$

Now

$$\begin{aligned}
& \sum_{n=1}^{N} \frac{w_n}{n(n+1)} \\
= & \sum_{n=1}^{N} w_n \left(\frac{1}{n} - \frac{1}{n+1} \right) \\
= & \sum_{n=1}^{N} \frac{w_n - w_{n-1}}{n} - \frac{w_N}{N+1} \\
= & \sum_{n=1}^{N} a_n - \frac{w_N}{N+1}.
\end{aligned}$$

Letting $N \to \infty$ we get $\sum_{n=1}^{\infty} a_n = -a_0$, i.e. $\sum_{n=0}^{\infty} a_n = 0$.

Chapter 3

1. (Exercise 1) Show that the first two examples of inner product spaces, namely $\mathbb{R}^d$ and $\mathbb{C}^d$, are complete.

2. (Exercise 2) Prove that the vector space $\ell^2(\mathbb{Z})$ is complete.

3. (Exercise 3) Construct a sequence of integrable functions $\{f_k\}$ on $[0, 2\pi]$ such that
$$\lim_{k\to\infty} \frac{1}{2\pi} \int_0^{2\pi} |f_k(t)|^2 \, dt = 0$$
but $\lim_{k\to\infty} f_k(t)$ fails to exist for any t.

4. (Exercise 4) (In (c), use the fact that f is continuous except possibly on a set of measure 0.)

5. (Exercise 5) Let
$$f(t) = \begin{cases} 0 & \text{for } t = 0 \\ \log(1/t) & \text{for } 0 < t \leq 2\pi, \end{cases}$$
and define a sequence of functions in $\mathcal{R}$ by
$$f_n(t) = \begin{cases} 0 & \text{for } 0 \leq t \leq 1/n \\ f(t) & \text{for } 1/n < t \leq 2\pi. \end{cases}$$
Prove that $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathcal{R}$. However, f does not belong to $\mathcal{R}$.
Solution.
$$\int (\log t)^2 \, dt = t(\log t)^2 - 2t\log t + 2t$$
and $\lim_{t\to 0} t\log t = 0, \lim_{t\to 0} t\log^2 t = 0$.

6. (Exercise 6) Consider the sequence $\{a_k\}_{k=-\infty}^{\infty}$ defined by
$$a_k = \begin{cases} 1/k & \text{if } k \geq 1 \\ 0 & \text{if } k \leq 0. \end{cases}$$
Note that $\{a_k\} \in l^2(\mathbb{Z})$, but that no Riemann integrable function has k^{th} Fourier coefficient equal to a_k for all k.
Solution.
Let $M = \sup_{-\pi\leq t\pi} |f(t)|$. Recall that
$$A_r(f)(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t) f(t) \, dt.$$
So
$$|A_r(f)(\theta)| \leq M \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t) \, dt = M$$

Also

$$A_r(f)(\theta) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{in\theta} r^{|n|}.$$

So

$$A_r(f)(0) = \sum_{n=-\infty}^{\infty} \hat{f}(n) r^{|n|}.$$

It follows that if f is a Riemann integrable function that has k^{th} Fourier coefficient equal to a_k for all k, then

$$\sum_{n=1}^{\infty} \frac{r^n}{n}$$

is bounded for $0 \le r < 1$. This is a contradiction since the sum of the series is $-\log(1-r)$.

7. (Exercise 7) Show that the trigonometric series

$$\sum_{n \ge 2} \frac{\sin nx}{\log n}$$

converges for every x, yet it is not the Fourier series of a Riemann integrable function.
The same is true for $\sum \frac{\sin nx}{n^\alpha}$ for $0 < \alpha < 1$, but the case $1/2 < \alpha < 1$ is more difficult. See Problem 1.
Solution.
Apply Parseval's identity.(A Riemann integrable function is in L^2.) Series converges because the partial sums of $\sum \sin nx$ is bounded; see Exercise 9. Postpone the case $1/2 < \alpha < 1$ to Problem 1.

8. (Exercise 8) Exercise 6 in Chapter 2 dealt with the sums

$$\sum_{n \text{ odd } \ge 1} \frac{1}{n^2} = \frac{\pi^2}{8} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Similar sums can be derived using the methods of this chapter.
(a) Let f be the function defined on $[-\pi, \pi]$ by $f(\theta) = |\theta|$. Use Parseval's identity to find the sums of the following two series:

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^4}.$$

In fact, they are $\frac{\pi^4}{96}$ and $\frac{\pi^4}{90}$, respectively.
(b) Consider the 2π-periodic odd function defined on $[0, \pi]$ by $f(\theta) = \theta(\pi - \theta)$. Show that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^6} = \frac{\pi^6}{960} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}.$$

Remark. The general expression when k is even for $\sum_{n=1}^{\infty} 1/n^k$ in terms of π^k is given in Problem 4. However, finding a formula for the sum $\sum_{n=1}^{\infty} 1/n^3$, or more generally $\sum_{n=1}^{\infty} 1/n^k$ with k odd, is a famous unresolved question.

Solution. The Fourier series for $f(\theta) = |\theta|$ is

$$\frac{\pi}{2} - \frac{2}{\pi} \sum_{n \text{ odd } \geq 1} \frac{e^{inx} + e^{-inx}}{n^2}.$$

By Parseval's identity,

$$\frac{\pi^2}{4} + \frac{4}{\pi^2} \sum_{n \text{ odd } \geq 1} \frac{2}{n^4} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\theta|^2 \, d\theta = \frac{\pi^2}{3}$$

from which we get

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}.$$

The sum $\sum_{n=1}^{\infty} \frac{1}{n^4}$ is obtained from solving for x in the equation

$$x = \frac{\pi^4}{96} + \frac{x}{16}$$

which yields $x = \frac{\pi^4}{90}$.

The Fourier series for the function $f(\theta) = \theta(\pi - \theta)$ is

$$\frac{8}{\pi} \sum_{k \text{ odd } \geq 1} \frac{\sin k\theta}{k^3} = -i\frac{4}{\pi} \sum_{k \text{ odd } \geq 1} \frac{e^{ik\theta} - e^{-ik\theta}}{k^3}.$$

By Parseval's identity,

$$\frac{16}{\pi^2} \sum_{n=0}^{\infty} \frac{2}{(2n+1)^6} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \theta^2 (1-\theta)^2 \, d\theta = \frac{1}{30}\pi^4,$$

from which we get

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^6} = \frac{\pi^6}{960}.$$

Solving x from

$$x = \frac{1}{64}x + \frac{\pi^6}{960}$$

we get $x = \frac{\pi^6}{945}$.

9. (Exercise 9) Show that for α not an integer, the Fourier series of

$$\frac{\pi}{\sin \pi\alpha} e^{i(\pi - x)\alpha}$$

on $[0, 2\pi)$ is given by

$$\sum_{-\infty}^{\infty} \frac{e^{inx}}{n+\alpha}.$$

Apply Parseval's formula to show that

$$\sum_{-\infty}^{\infty} \frac{1}{(n+\alpha)^2} = \frac{\pi^2}{(\sin \pi\alpha)^2}.$$

Solution
Straightforward checking.

10. (Exercise 10) Consider the example of a vibrating string which we analyzed in Chapter 1. The displacement $u(x,t)$ of the string at time t satisfies the wave equation

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, c^2 = \tau/\rho.$$

The string is subject to the initial conditions

$$u(x,0) = f(x) \text{ and } \frac{\partial u}{\partial t}(x,0) = g(x),$$

where we assume that $f \in C^1$ and g is continuous. We define the total energy of the string by

$$E(t) = \frac{1}{2}\rho \int_0^L \left(\frac{\partial u}{\partial t}\right)^2 dx + \frac{1}{2}\tau \int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx.$$

The first term corresponds to the "kinetic energy" of the string (in analogy with $(1/2)mv^2$, the kinetic energy of a particle of mass m and velocity v), and the second term corresponds to its "potential energy."
Show that the total energy of the string is conserved, in the sense that $E(t)$ is constant. Therefore,

$$E(t) = E(0) = \frac{1}{2}\rho \int_0^L g(x)^2\, dx + \frac{1}{2}\rho \int_0^L f'(x)^2\, dx.$$

Solution. We have

$$\begin{aligned}
E'(t) &= \rho \int_0^L \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2}\, dx + \tau \int_0^L \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial t}\, dx \\
&= \rho \int_0^L \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2}\, dx - \tau \int_0^L \frac{\partial^2 u}{\partial^2 x}\frac{\partial u}{\partial t}\, dx \\
&= 0
\end{aligned}$$

where we have used integration by parts, and that $\frac{\partial u}{\partial t}(0,t) = \frac{\partial u}{\partial t}(L,t) = 0$ for all t.

11. (Exercise 11) The inequalities of Wirtinger and Poincare establish a relationship between the norm of a function and that of its derivative.
(a) If f is T-periodic, continuous, and piecewise C^1 with $\int_0^T f(t)\,dt = 0$, show that

$$\int_0^T |f(t)|^2\,dt \le \frac{T^2}{4\pi^2}\int_0^T |f'(t)|^2\,dt,$$

with equality if and only if $f(t) = A\sin(2\pi t/T) + B\cos(2\pi t/T)$.
(b) If f is aa above and g is just C^1 and T-periodic, prove that

$$\left|\int_0^T \overline{f(t)}g(t)\,dt\right|^2 \le \frac{T^2}{4\pi^2}\int_0^T |f(t)|^2\,dt\int_0^T |g'(t)|^2\,dt.$$

(c) For any compact interval $[a,b]$ and any continuously differentiable function f with $f(a) = f(b) = 0$, show that

$$\int_a^b |f(t)|^2\,dt \le \frac{(b-a)^2}{\pi^2}\int_a^b |f'(t)|^2\,dt.$$

Discuss the case of equality, and prove that the constant $(b-a)^2/\pi^2$ cannot be improved.

Solution.

(a) The condition $\int_0^T f(t)\,dt = 0$ implies that $\hat{f}(0) = 0$. The continuity of f guarantees that $\hat{f'}(n) = \frac{2\pi i n}{T}\hat{f}(n)$. Indeed, write τ for $\frac{2\pi}{T}$. Then for $n \neq 0$,

$$\begin{aligned}\hat{f}(n) &= \frac{1}{T}\int_0^T f(t)e^{-in\tau t}\,dt \\ &= \frac{1}{T}(f(0^+) - f(T^-))\frac{1}{in\tau} + \frac{1}{in\tau}\frac{1}{T}\int_0^T f'(t)e^{-in\tau t}\,dt \\ &= \frac{T}{2\pi i n}\hat{f'}(n)\end{aligned}$$

Therefore, by Parseval's identity,recalling that $\hat{f}(0) = 0$,

$$\begin{aligned}\int_0^T |f(t)|^2\,dt &= T\sum_{|n|>0}|\hat{f}(n)|^2 \\ &= \frac{T^3}{4\pi^2}\sum_{|n|>0}\frac{|\hat{f'}(n)|^2}{n^2} \\ &\le \frac{T^3}{4\pi^2}\sum_{|n|>0}|\hat{f'}(n)|^2 \\ &= \frac{T^3}{4\pi^2}\frac{1}{T}\int_0^T |f'(t)|^2\,dt \\ &= \frac{T^2}{4\pi^2}\int_0^T |f'(t)|^2\,dt\end{aligned}$$

From the above inequalities, we see that equality holds if and only if $\hat{f}(n) = 0$ for all $n \geq 2$. This means that, writing a_n for $\hat{f}(n)$, $f(x) = a_1 e^{i\tau x} + a_{-1}e^{-i\tau x}$ which simplifies to $A\sin(\tau x) + B\cos(\tau x)$.
Remark. It is clear from the proof above that in the absence of the condition $\int_0^T f(t)\,dt = 0$, the inequality in (a) is

$$\sum_{|n|>0} |a_n|^2 \leq \frac{T^2}{4\pi^2}\sum_{|n|>0}|b_n|^2$$

where $a_n = \hat{f}(n), b_n = \hat{f}'(n)$.
(b) Let $a_n = \hat{f}(n), b_n = \hat{g}(n), c_n = \hat{g'}(n)$. Then

$$\begin{aligned}
&\left|\int_0^T \overline{f(t)}g(t)\,dt\right|^2 \\
= \;& T|\sum_{|n|\geq 0} a_n\overline{b_n}|^2 \\
= \;& T|\sum_{|n|>0} a_n\overline{b_n}|^2 \\
\leq \;& (T\sum_{|n|>0}|a_n|^2)(\sum_{|n|>0}|b_n|^2) \\
\leq \;& \int_0^T |f(t)|^2\,dt \frac{T^2}{4\pi^2}\int_0^T |g'(t)|\,dt
\end{aligned}$$

(c) Extend f to a function on $[a, 2b-a]$ such that $f(b+h) = -f(b-h)$ for $0 \leq h \leq b-a$ and then extend it so that it is $T = 2(b-a)$-periodic. It is easy to see that now $f(b+h) = -f(b-h)$ and $f'(b+h) = f'(b-h)$ for all h. Then $\int_0^T f(x)\,dx = \int_a^{2b-a} f(x)\,dx = 0$, $\int_0^T |f(x)|\,dx = \int_a^{2b-a}|f(x)|\,dx = 2\int_a^b |f(x)|\,dx$, and $\int_0^T |f'(x)|\,dx = 2\int_a^b |f'(x)|\,dx$. Check that f so extended is also C^1 on $\mathbb{R}$. Therefore by (a), we have

$$\begin{aligned}
&\int_a^b |f(x)|\,dx \\
= \;& \frac{1}{2}\int_0^T |f(x)|\,dx \\
\leq \;& \frac{1}{2}\frac{2^2(b-a)^2}{4\pi^2}\int_0^T |f'(x)|\,dx \\
= \;& \frac{(b-a)^2}{\pi^2}\frac{1}{2}\int_0^T |f'(x)|\,dx \\
= \;& \frac{(b-a)^2}{\pi^2}\int_a^b |f'(x)|\,dx
\end{aligned}$$

For the function $f(x)$ that we define, its translation $g(x) = f(x+b)$ is an odd function. So $\hat{g}(n) = e^{inb}\hat{f}(n)$, and the Fourier series for g is of the form

$$\sum_{n=1}^{\infty} A_n \sin \frac{2\pi n x}{2(b-a)} = \sum_{n=1}^{\infty} A_n \sin \frac{\pi n x}{b-a}.$$

Thus the Fourier series for f is of the form

$$\sum_{n=1}^{\infty} A_n \sin \frac{\pi n(x-b)}{b-a}.$$

According the inequality in (c) is an equality iff $f(x)$ is of the form $A \sin \frac{\pi(x-b)}{b-a}$. (The book says $A \sin \frac{\pi(x-a)}{b-a}$; this is equivalent since $\sin \frac{\pi(x-b)}{b-a} = -\sin \frac{\pi(x-a)}{b-a}$).

12. (Exercise 12) Prove that $\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$.
Proof.
We have $\int_{-\pi}^{\pi} D_N(t)\, dt = 2\pi$. So

$$\int_{-\pi}^{\pi} \frac{\sin(N+1/2)x}{\sin(x/2)}\, dx = 2\pi$$

Write $\csc(x/2)$ as $\csc(x/2) - 2/x + 2/x$. $\lim_{x\to 0} \csc(x/2) - 2/x = 0$, so it has a removable discontinuity at 0 on the interval $[-\pi, \pi]$. By Lebesgue-Riemann lemma we get

$$\int_{-\pi}^{\pi} \frac{2\sin(N+1/2)x}{x}\, dx \to 2\pi \text{ as } N \to \infty.$$

This yields

$$\int_0^{\pi} \frac{\sin(N+1/2)x}{x}\, dx \to \frac{\pi}{2} \text{ as } N \to \infty.$$

By change of variable, we get

$$\int_0^{(N+1/2)\pi} \frac{\sin x}{x}\, dx \to \frac{\pi}{2} \text{ as } N \to \infty.$$

Since $\int_{N\pi}^{(N+1/2)\pi} \frac{\sin x}{x}\, dx \to 0$ as $N \to \infty$, (use Mean-Value Theorem), we get

$$\int_0^{N\pi} \frac{\sin x}{x}\, dx \to \frac{\pi}{2} \text{ as } N \to \infty.$$

By MVT,

$$\int_{N\pi}^{(N+1)\pi} \frac{|\sin x|}{x}\, dx \le \frac{1}{N},$$

so for any $t > \pi$ there exists $N > 0$ such that

$$\left| \int_{N\pi}^{t} \frac{\sin x}{x}\, dx \right| \le \frac{1}{N}.$$

It follows that
$$\int_0^\infty \frac{\sin x}{x}\,dx = \lim_{N\to\infty}\int_0^{N\pi} \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

13. (Exercise 13) Suppose that f is periodic and of class C^k. Show that
$$\hat{f}(n) = o(1/|n|^k),$$
that is, $|n|^k \hat{f}(n)$ goes to 0 as $|n| \to \infty$. This is an improvement over Exercise 10 in Chapter 2.
Solution.
We have (see p.43)
$$2\pi\hat{f}(n) = \frac{1}{(in)^k}\int_0^{2\pi} f^{(k)}(\theta)e^{-in\theta}\,d\theta.$$
So
$$2\pi(in)^k\hat{f}(n) = \int_0^{2\pi} f^{(k)}(\theta)e^{-in\theta}\,d\theta.$$
Now use Lebesgue-Riemann lemma.

14. (Exercise 14) Prove that the Fourier series of a continuously differentiable function f on the circle is absolutely convergent.
Proof.
From $\hat{f}(n) = \frac{1}{in}\hat{f'}(n)$, we get, by Cauchy-Schwarz inequality and Parseval's identity,
$$\begin{aligned}
& \sum_{n=-\infty}^{\infty} |\hat{f}(n)| \\
= \quad & \sum_{n=-\infty}^{\infty} \frac{1}{|n|}|\hat{f'}(n)| \\
\leq \quad & (2\frac{\pi^2}{6})^{1/2}(\sum_{n=-\infty}^{\infty} |\hat{f'}(n)|^2)^{1/2} \\
= \quad & (2\frac{\pi^2}{6})^{1/2}\frac{1}{2\pi}\int_{-\pi}^{\pi} |f'(t)|^2\,dt < \infty.
\end{aligned}$$

15. (Add in) Prove that the Fourier series of a 2π periodic absolutely continuous function whose derivative (exists a.e.) in $[0, 2\pi]$ is square integrable (in particular, Riemann integrable), is absolutely convergent. (Note that Exercise 16 below shows that Lipschitz condition alone is enough. But derivative of a Lipschitz function is bounded. And Lipschitz functions are precisely functions representable as integral of a bounded measurable function.)
Proof. Use the proof in Exercise 14. Note that the integration by parts formula is valid for absolutely continuous functions.

16. (Exercise 15) Let f be a 2π-periodic and Riemann integrable on $[-\pi, \pi]$.
(a) Show that

$$\hat{f}(n) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x + \frac{\pi}{n})e^{-inx}\,dx$$

hence

$$\hat{f}(n) = \frac{1}{4\pi}\int_{-\pi}^{\pi} [f(x) - f(x + \frac{\pi}{n})]e^{-inx}\,dx.$$

(b) Now assume that f satisfies a Hölder condition of order α, namely

$$|f(x+h) - f(x)| \le C|h|^{\alpha}$$

for some $0 < \alpha \le 1$, some $C > 0$, and all x, h. Use part (a) to show that

$$\hat{f}(n) = O(1/|n|^{\alpha}).$$

(c) Prove that the above result cannot be improved by showing that the function

$$f(x) = \sum_{k=0}^{\infty} 2^{-k\alpha}e^{i2^k x},$$

where $0 < \alpha < 1$, satisfies

$$|f(x+h) - f(x)| \le C|h|^{\alpha},$$

and $\hat{f}(N) = 1/N^{\alpha}$ whenever $N = 2^k$.

Solution.

Note that for any real x, $|1 - e^{ix}| \le |x|$ (to prove, just note that LHS is $2|\sin(x/2)|$ and $|\sin a| \le |a|$ for all real number a).

(a).

$$\begin{aligned}
& \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x + \frac{\pi}{n})e^{-inx}\,dx \\
= & \frac{1}{2\pi}\int_{-\pi+\frac{\pi}{n}}^{\pi+\frac{\pi}{n}} f(u)e^{-inu}e^{i\pi}\,du \\
= & -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(u)e^{-inu}\,du \\
= & -\hat{f}(n)
\end{aligned}$$

(b)

$$\begin{aligned}
& |\hat{f}(n)| \\
\le & \frac{1}{4\pi}\int_{-\pi}^{\pi} |f(x) - f(x + \frac{\pi}{n})|\,dx \\
\le & \frac{1}{4\pi}2\pi C\frac{\pi^{\alpha}}{|n|^{\alpha}} \\
= & \frac{C_1}{|n|^{\alpha}}
\end{aligned}$$

(c).

$$\begin{aligned}
& |f(x+h) - f(x)| \\
= \ & |\sum_{k=0}^{\infty} 2^{-k\alpha} e^{i2^k(x+h)} - \sum_{k=0}^{\infty} 2^{-k\alpha} e^{i2^k x}| \\
\leq \ & \sum_{k=0}^{\infty} 2^{-k\alpha} |e^{i2^k h} - 1| \\
\leq \ & \sum_{2^k \leq 1/|h|} 2^{-k\alpha} 2^k |h| + \sum_{2^k > 1/|h|} 2^{-k\alpha} 2
\end{aligned}$$

The second sum is easily seen to be less than $2|h|^\alpha$. The first sum is 0 if $|h| > 1$. So assume $|h| \leq 1$. Let l be the unique nonnegative integer such that $2^{-l-1} < |h| \leq 2^{-l}$. Then the first sum is

$$\sum_{k=0}^{l} (2^k|h|)^{1-\alpha} |h|^\alpha \leq |h|^\alpha \sum_{k=0}^{l} (2^{k-l})^{1-\alpha} \leq \frac{1}{1 - 2^{\alpha-1}} |h|^\alpha.$$

Since the series converges uniformly, the coefficient of e^{inx} is $\hat{f}(n)$ for all n.

17. (Exercise 16) Let f be a 2π-periodic function which satisfies a Lipschitz condition with constant K; that is

$$|f(x) - f(y)| \leq K|x - y| \text{ for all } x, y.$$

This is simply the Hölder condition with $\alpha = 1$, so by the previous exercise, we see that $\hat{f}(n) = O(1/|n|)$. Since the harmonic series $\sum 1/n$ diverges, we cannot say anything (yet) about the absolute convergence of the Fourier series of f. The outline below actually proves that the Fourier series of f converges absolutely and uniformly.
(a) For every positive h we define $g_h(x) = f(x+h) - f(x-h)$. Prove that

$$\frac{1}{2\pi} \int_0^{2\pi} |g_h(x)|^2 \, dx = \sum_{n=-\infty}^{\infty} 4|\sin nh|^2 |\hat{f}(n)|^2,$$

and show that

$$\sum_{n=-\infty}^{\infty} |\sin nh|^2 |\hat{f}(n)|^2 \leq K^2 h^2.$$

(b) Let p be a positive integer. By choosing $h = \pi/2^{p+1}$, show that

$$\sum_{2^{p-1} < |n| \leq 2^p} |\hat{f}(n)|^2 \leq \frac{K^2 \pi^2}{2^{2p+1}}.$$

(c) Estimate $\sum_{2^{p-1} < |n| \leq 2^p} |\hat{f}(n)|$, and conclude that the Fourier series of f converges absolutely, hence uniformly.

(d) In fact, modify the argument slightly to prove Bernstein's theorem: If f satisfies a Hölder condition of order $\alpha > 1/2$, then the Fourier series of f converges absolutely.

Solution.

(a). The Fourier coefficients of the translated function $f(x+h)$ is $e^{inh}\hat{f}(n)$. So

$$\hat{g_h}(n) = (e^{inh} - e^{-inh})\hat{f}(n) = 2i \sin nh \hat{f}(n).$$

The first equation in (a) follows from Parseval's identity. Since $|g_h(x)| \leq K[x+h-(x-h)] = 2K|h|$, the second inequality in (a) follows.

(b). $h = \pi/2^{p+1}$ and $2^{p-1} < |n| \leq 2^p$ imply $\pi/4 < |n|h \leq \pi/2$, hence $|\sin nh|^2 \geq 1/2$. Thus

$$\frac{1}{2}\sum_{2^{p-1}<|n|\leq 2^p} |\hat{f}(n)|^2 \leq \sum_{n=-\infty}^{\infty} |\sin nh|^2 |\hat{f}(n)|^2 \leq K^2 h^2 = \frac{K^2\pi^2}{2^{2(p+1)}}$$

from which it follows

$$\sum_{2^{p-1}<|n|\leq 2^p} |\hat{f}(n)|^2 \leq \frac{K^2\pi^2}{2^{2p+1}}.$$

(c). By Cauchy-Schwarz inequality, we have

$$\begin{aligned} & \sum_{2^{p-1}<|n|\leq 2^p} |\hat{f}(n)| \\ \leq \ & (\sum_{2^{p-1}<|n|\leq 2^p} 1^2)^{1/2} (\sum_{2^{p-1}<|n|\leq 2^p} |\hat{f}(n)|^2)^{1/2} \\ \leq \ & (2^p - 2^{p-1})^{1/2} \frac{K\pi}{2^{p+1/2}} \\ \leq \ & \frac{K\pi}{(\sqrt{2})^p} \end{aligned}$$

Therefore

$$\sum_{1\leq|n|<\infty} |\hat{f}(n)| \leq \sum_{1\leq p<\infty} \frac{K\pi}{(\sqrt{2})^p} < \infty$$

(d). Modifications are given below:

The Fourier coefficients of the translated function $f(x+h)$ is $e^{inh}\hat{f}(n)$. So

$$\hat{g_h}(n) = (e^{inh} - e^{-inh})\hat{f}(n) = 2i \sin nh \hat{f}(n).$$

The first equation in (a) follows from Parseval's identity. Since $|g_h(x)| \leq K|x+h-(x-h)|^\alpha = 2^\alpha K|h|^\alpha$, the second inequality in (a) becomes

$$\sum_{n=-\infty}^{\infty} |\sin nh|^2 |\hat{f}(n)|^2 \leq 2^{2(\alpha-1)} K^2 |h|^{2\alpha}.$$

$h = \pi/2^{p+1}$ and $2^{p-1} < |n| \leq 2^p$ imply $\pi/4 < |n|h \leq \pi/2$, hence $|\sin nh|^2 \geq 1/2$. Thus

$$\frac{1}{2} \sum_{2^{p-1}<|n|\leq 2^p} |\hat{f}(n)|^2 \leq \sum_{n=-\infty}^{\infty} |\sin nh|^2 |\hat{f}(n)|^2 \leq 2^{2(\alpha-1)} K^2 |h|^{2\alpha} = \frac{K^2 \pi^{2\alpha}}{2^{2(\alpha p+1)}}$$

from which it follows

$$\sum_{2^{p-1}<|n|\leq 2^p} |\hat{f}(n)|^2 \leq \frac{K^2 \pi^{2\alpha}}{2^{2\alpha p+1}}.$$

By Cauchy-Schwarz inequality, we have

$$\begin{aligned} & \sum_{2^{p-1}<|n|\leq 2^p} |\hat{f}(n)| \\ \leq\ & \Big(\sum_{2^{p-1}<|n|\leq 2^p} 1^2\Big)^{1/2} \Big(\sum_{2^{p-1}<|n|\leq 2^p} |\hat{f}(n)|^2\Big)^{1/2} \\ \leq\ & (2^p - 2^{p-1})^{1/2} \frac{K\pi^\alpha}{2^{\alpha p+1/2}} \\ \leq\ & \frac{K\pi}{(2^{\alpha-1/2})^p} \end{aligned}$$

Therefore, if $\alpha > 1/2$, then

$$\sum_{1\leq|n|<\infty} |\hat{f}(n)| \leq \sum_{1\leq p<\infty} \frac{K\pi}{(2^{\alpha-1/2})^p} < \infty$$

Remark. The condition $\alpha > 1/2$ is sharp. See Katznelson's book, p. 32.

18. (Exercise 17) If f is a bounded monotonic function on $[-\pi, \pi]$, then

$$\hat{f}(n) = O(1/|n|).$$

19. (Exercise 18) Here are a few things we have learned about the decay of Fourier coefficients:
(a) if f is of class C^k, then $\hat{f}(n) = o(1/|n|^k)$;
(b) if f is Lipschitz, then $\hat{f}(n) = O(1/|n|)$;
(c) if f is monotonic, then $\hat{f}(n) = O(1/|n|)$;
(d) if f satisfies a Hölder condition with exponent α where $0 < \alpha < 1$, then $\hat{f}(n) = O(1/|n|^\alpha)$;
(e) if f is merely Riemann integrable, then $\sum |\hat{f}(n)|^2 < \infty$ and therefore $\hat{f}(n) = o(1)$.
Nevertheless, show that the Fourier coefficients of a continuous function can tend to 0 arbitrarily slowly by proving that for every sequence of nonnegative real numbers $\{\epsilon_k\}$ converging to 0, there exists a continuous

function f such that $|\hat{f}(n)| \geq \epsilon_n$ for infinitely many values of n.
Solution.
Choose a subsequence $\{\epsilon_{n_k}\}$ such that $\sum_k \epsilon_{n_k} < \infty$, e.g., $\epsilon_{n_k} < 1/2^k$. Then the function $\sum_{k=1}^{\infty} \epsilon_{n_k} e^{in_k x}$, is such a function, because it is absolutely (and hence uniformly) convergent.

20. (Exercise 19) Give another proof that the sum $\sum_{0<|n|\leq N} e^{inx}/n$ is uniformly bounded in N and $x \in [-\pi, \pi]$ by using the fact that

$$\frac{1}{2i} \sum_{0<|n|\leq N} \frac{e^{inx}}{n} = \sum_{n=1}^{N} \frac{\sin nx}{n} = \frac{1}{2} \int_0^x (D_N(t) - 1)\, dt,$$

where D_N is the Dirichlet kernel. Now use the fact that $\int_0^\infty \frac{\sin t}{t}\, dt < \infty$ which was proved in Exercise 12.

21. (Exercise 20) Let $f(x)$ denote the sawtooth function defined by $f(x) = (\pi - x)/2$ on the interval $(0, 2\pi)$ with $f(0) = 0$ and extended by periodicity to all of $\mathbb{R}$. The Fourier series of f is

$$f(x) \sim \frac{1}{2i} \sum_{n\neq 0} \frac{e^i nx}{n} = \sum_{n=1}^{\infty} \frac{\sin nx}{n},$$

and f has a jump discontinuity at the origin with

$$f(0^+) = \frac{\pi}{2}, f(0^-) = -\frac{\pi}{2}, \text{ and hence } f(0^+) - f(0^-) = \pi.$$

Show that

$$\lim_{N\to\infty} \max_{0<x\leq \pi/N} S_N(f)(x) - \frac{\pi}{2} = \int_0^\pi \frac{\sin t}{t}\, dt - \frac{\pi}{2} \approx 0.08949\pi,$$

which is roughly 9% of the jump π. This result is a manifestation of Gibbs's phenomenon which states that near a jump discontinuity, the Fourier series of a function overshoots (or undershoots) it by approximately 9% of the jump.

22. (Problem 1) For each $0 < \alpha < 1$ the series

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n^\alpha}$$

converges for every x but is not the Fourier series of a Riemann integrable function.
(a) If the **conjugate Dirichlet kernel** is defined by

$$\tilde{D}_N(x) = \sum_{|n|\leq N} \text{sign}(n) e^{inx} \text{ where } \text{sign}(n) = \begin{cases} 1 & \text{if } n > 0 \\ 0 & \text{if } n = 0 \\ -1 & \text{if } n < 0, \end{cases}$$

then show that

$$\tilde{D}_N(x) = i\frac{\cos(x/2) - \cos((N+1/2)x)}{\sin(x/2)},$$

and

$$\int_{-\pi}^{\pi} |\tilde{D}_N(x)|\, dx \le c\log N, \text{ for } N \ge 2$$

(b) As a result, if f is Riemann integrable, then

$$(f * \tilde{D})(0) = O(\log N).$$

(c) In the present case, this leads to

$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} = O(\log N),$$

which is a contradiction.
Solution.
Let $\omega = e^{ix}$.

$$\begin{aligned}
& \tilde{D}_N(x) \\
= & \sum_{k=1}^{n} \omega^k - \sum_{k=1}^{n} \omega^{-k} \\
= & \omega\frac{1-\omega^n}{1-\omega} - \omega^{-1}\frac{1-\omega^{-n}}{1-\omega^{-1}} \\
= & \omega\frac{1-\omega^n}{1-\omega} + \frac{1-\omega^{-n}}{1-\omega} \\
= & \frac{\omega + 1 - \omega^{n+1} - \omega^{-n}}{1-\omega} \\
= & \frac{\omega^{1/2} + \omega^{-1/2} - \omega^{n+1/2} - \omega^{-n-1/2}}{\omega^{-1/2} - \omega^{1/2}} \\
= & \frac{\cos(x/2) - \cos((N+1/2)x)}{\sin(x/2)} i
\end{aligned}$$

Note that by trigonometric identity, this is

$$\frac{2\sin(\frac{N+1}{2}x)\sin(\frac{N}{2}x)}{\sin(x/2)} i.$$

Note that $t/\sin t \le \pi/2$ for $t \in [-\pi/2, \pi/2]$, so for $x \in [-\pi, \pi]$,

$$|\tilde{D}_N(x)| \le 4\frac{|\sin(\frac{N}{2}x)|}{x}$$

Then

$$
\begin{aligned}
& \int_{-\pi}^{\pi} |\tilde{D}_N(\theta)|\, d\theta \\
\leq\; & 8\int_0^{\pi} \frac{|\sin(N/2)\theta|}{\theta}\, d\theta \\
=\; & 8\int_0^{(N/2)\pi} \frac{|\sin\theta|}{\theta}\, d\theta \\
=\; & 8\sum_{k=0}^{N-1}\int_{k\pi/2}^{(k+1)\pi/2} \frac{|\sin\theta|}{\theta}\, d\theta \\
=\; & 8\sum_{k=1}^{N-1}\int_{k\pi/2}^{(k+1)\pi/2} \frac{|\sin\theta|}{\theta}\, d\theta + \int_0^{\pi/2}\frac{\sin\theta}{\theta}\, d\theta \\
\leq\; & 8\frac{2}{\pi}\sum_{k=1}^{N-1}\frac{1}{k}\int_{k\pi/2}^{(k+1)\pi/2} |\sin\theta|\, d\theta + \int_0^{\pi/2}\frac{\sin\theta}{\theta}\, d\theta \\
=\; & \frac{16}{\pi}\sum_{k=1}^{N-1}\frac{1}{k} + \int_0^{\pi/2}\frac{\sin\theta}{\theta}\, d\theta \\
=\; & \frac{16}{\pi}\sum_{k=2}^{N-1}\frac{1}{k} + \frac{16}{\pi} + \int_0^{\pi/2}\frac{\sin\theta}{\theta}\, d\theta \\
\leq\; & \frac{16}{\pi}\log N + c_1 \\
\leq\; & c\log N
\end{aligned}
$$

where c is chosen such that $c/2 \geq \frac{16}{\pi}$ and $(c/2)\log 2 \geq c_1$, so that $c\log N = c/2\log N + c/2\log N \geq \frac{16}{\pi}\log N + c_1$ for all $N \geq 2$.
(b),(c). If f is such a Riemann integrable function, then $\hat{f}(n) = 1/n^{\alpha}$, $\hat{f}(-n) =$

$-1/n^\alpha$ for $n > 0$, and $\hat{f}(0) = 0$. So

$$\begin{aligned}
(f * \tilde{D}_N)(0) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\tilde{D}_N(0-t)\,dt \\
&= -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\tilde{D}_N(t)\,dt \\
&= -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\sum_{|n|\le N}\operatorname{sign}(n)e^{int}\,dt \\
&= -\sum_{|n|\le N}\operatorname{sign}(n)\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{int}\,dt \\
&= -\sum_{|n|\le N}\operatorname{sign}(n)\hat{f}(n) \\
&= -2\sum_{n=1}^{N}\frac{1}{n^\alpha}
\end{aligned}$$

Since $f(t)$ is bounded, we also have by (a),

$$\left|\int_{-\pi}^{\pi} f(t)\tilde{D}_N(t)\,dt\right| \le M\int_{-\pi}^{\pi}|\tilde{D}_N(t)|\,dt = O(\log N).$$

Thus $\sum_{n=1}^{N}\frac{1}{n^\alpha} = O(\log N)$; this is impossible (since $\sum_{n=1}^{N}\frac{1}{n^\alpha}$ grows like $N^{1-\alpha}$, see my ApostolStudyNote, p. 25).

23. (Problem 2)

24. (Problem 3) Let α be a complex number not equal to an integer.
(a) Calculate the Fourier series of the 2π-periodic function defined on $[-\pi,\pi]$ by $f(x) = \cos(\alpha x)$.
(b) Prove the following formulas due to Euler:

$$\sum_{n=1}^{\infty}\frac{1}{n^2-\alpha^2} = \frac{1}{2\alpha^2} - \frac{\pi}{2\alpha\tan(\alpha\pi)}.$$

For all $u \in \mathbb{C} - \pi\mathbb{Z}$,

$$\cot u = \frac{1}{u} + 2\sum_{n=1}^{\infty}\frac{u}{u^2-n^2\pi^2}.$$

(c) Show that for all $\alpha \in \mathbb{C} - \mathbb{Z}$ we have

$$\frac{\alpha\pi}{\sin(\alpha\pi)} = 1 + 2\alpha^2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2-\alpha^2}.$$

(d) For all $0 < \alpha < 1$, show that

$$\int_0^\infty \frac{t^{\alpha-1}}{t+1}\,dt = \frac{\pi}{\sin(\alpha\pi)}.$$

Solution.
From Maple, for all n,

$$\hat{f}(n) = (-1)^{n+1} \frac{\alpha \sin(\alpha\pi)}{\pi(n^2 - \alpha^2)}.$$

It follows that for $-\pi \leq x \leq \pi$,

$$\cos(\alpha x) = \frac{\sin(\alpha\pi)}{\alpha\pi} + \frac{2\alpha \sin(\alpha\pi)}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\cos nx}{n^2 - \alpha^2}.$$

Evaluating at $x = \pi$ yields the first equation in part (b). The second equation follows directly from the first by substituting u for $\alpha\pi$.
(c) follows by substituting $x = 0$ in the Fourier series.
(d). Split the integral as $\int_0^1 + \int_1^\infty$ and change variables $t = 1/u$ in the second integral, one gets

$$\int_0^\infty \frac{t^{\alpha-1}}{t+1}\, dt = \int_0^1 \frac{t^{\alpha-1}}{t+1}\, dt + \int_0^1 \frac{t^{(1-\alpha)-1}}{t+1}\, dt.$$

Claim: For $0 < \gamma < 1$,

$$\int_0^1 \frac{t^{\gamma-1}}{t+1}\, dt = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k+\gamma}.$$

Proof.
For each $0 < r < 1$, the series $t^{\gamma-1} \sum_{k=0}^{\infty} (-1)^k t^k$ converges to $\frac{t^{\gamma-1}}{t+1}$ uniformly on $[0, r]$. So

$$\int_0^r \frac{t^{\gamma-1}}{t+1}\, dt = \sum_{k=0}^{\infty} (-1)^k \frac{r^{\gamma+k}}{k+\gamma}$$

Since

$$\begin{aligned} & \int_0^1 \frac{t^{\gamma-1}}{t+1}\, dt \\ = & \lim_{r\to 1^-} \int_0^r \frac{t^{\gamma-1}}{t+1}\, dt \\ = & \lim_{r\to 1^-} \sum_{k=0}^{\infty} (-1)^k \frac{r^{\gamma+k}}{k+\gamma} \\ = & \lim_{r\to 1^-} \sum_{k=0}^{\infty} (-1)^k \frac{r^k}{k+\gamma} \end{aligned}$$

The series $\sum_{k=0}^{\infty} (-1)^k \frac{1}{k+\gamma}$ is Abel summable and hence convergent to $\int_0^1 \frac{t^{\gamma-1}}{t+1}\, dt$, by Littlewood's theorem since $(-1)^k 1/(k+\gamma) = O(1/k)$. Q.E.D.

Therefore

$$
\begin{aligned}
&\int_0^1 \frac{t^{\alpha-1}}{t+1}\,dt + \int_0^1 \frac{t^{(1-\alpha)-1}}{t+1}\,dt \\
= \quad & \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1-\alpha} + \sum_{k=0}^{\infty} \frac{(-1)^k}{k+\alpha} \\
= \quad & \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k-\alpha} + \frac{1}{\alpha} + \sum_{k=1}^{\infty} \frac{(-1)^k}{k+\alpha} \\
= \quad & \frac{1}{\alpha} + 2\alpha \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2-\alpha^2} \\
= \quad & \frac{\pi}{\sin(\alpha\pi)}
\end{aligned}
$$

by (c).

25. (Problem 4) In this problem, we find the formula for the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^k}$$

where k is any even integer. These numbers are expressed in terms of the Bernoulli numbers; the related Bernoulli polynomials are discussed in the next problem.
Define the **Bernoulli numbers** B_n by the formula

$$\frac{z}{e^z-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n.$$

(a) Show that $B_0 = 1, B_1 = -1/2, B_2 = 1/6, B_3 = 0, B_4 = -1/30$, and $B_5 = 0$.
(b) Show that for $n \geq 1$ we have

$$B_n = -\frac{1}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{k} B_k.$$

(c) By writing

$$\frac{z}{e^z-1} = 1 - \frac{z}{2} + \sum_{n=2}^{\infty} \frac{B_n}{n!} z^n,$$

show that $B_n = 0$ if n is odd and > 1. Also prove that

$$z \cot z = 1 + \sum_{n=1}^{\infty} \frac{2^{2n} B_{2n}}{(2n)!} z^{2n}.$$

(d) The **zeta function** is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \text{ for all } s > 1.$$

Deduce from the result in (c), and the expression for the cotangent function obtained in the previous problem, that

$$x \cot x = 1 - 2 \sum_{m=1}^{\infty} \frac{\zeta(2m)}{\pi^{2m}} x^{2m}.$$

(e) Conclude that

$$2\zeta(2m) = (-1)^{m+1} \frac{(2\pi)^{2m}}{(2m)!} B_{2m}.$$

Solution.
(a) Get these numbers directly from $(1 + z/2! + z^2/3! + \cdots)^{-1}$ by long division.
(b).

$$\begin{aligned} 1 &= \frac{z}{e^z - 1} \frac{e^z - 1}{z} \\ &= \left(\sum_{n=0}^{\infty} \frac{B_n}{n!} z^n \right) \left(\sum_{n=0}^{\infty} \frac{1}{(n+1)!} z^n \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{B_k}{k!} \frac{1}{(n-k+1)!} \right) t^n \end{aligned}$$

from which it follows that $B_0 = 1$, and for $n \geq 1$,

$$\sum_{k=0}^{n} \frac{B_k}{k!} \frac{1}{(n-k+1)!} = 0,$$

i.e.

$$\begin{aligned} \frac{B_n}{n!} &= -\sum_{k=0}^{n-1} \frac{B_k}{k!} \frac{1}{(n-k+1)!} \\ &= -\frac{1}{(n+1)!} \sum_{k=0}^{n-1} \binom{n+1}{k} B_k \end{aligned}$$

(c). Since $B_0 = 1, B_1 = -\frac{1}{2}$, we have

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{n=2}^{\infty} \frac{B_n}{n!} z^n.$$

Now,

$$\begin{aligned}
&\frac{z}{e^z-1}+\frac{z}{2} \\
= \quad & z\left(\frac{1}{e^z-1}+\frac{1}{2}\right) \\
= \quad & \frac{z}{2}\left(\frac{e^z+1}{e^z-1}\right) \\
= \quad & \frac{z}{2}\left(\frac{e^{z/2}+e^{-z/2}}{e^{z/2}-e^{-z/2}}\right) \\
= \quad & \frac{z}{2}\coth\frac{z}{2}
\end{aligned}$$

is an even function, so $B_{2n+1}=0$ for all $n \geq 1$. Note that

$$\coth(iz)=\frac{e^{iz/2}+e^{-iz/2}}{e^{iz/2}-e^{-iz/2}}=\frac{2\cos z}{2i\sin z}$$

so $\cot z = i\coth(iz)$. Therefore

$$\begin{aligned}
z\cot z = iz\coth(iz) \quad &= \quad 1+\sum_{n=1}^{\infty}\frac{B_{2n}}{(2n)!}(i2z)^{2n} \\
&= \quad \sum_{n=0}^{\infty}(-1)^n\frac{2^{2n}B_{2n}}{(2n)!}z^{2n}
\end{aligned}$$

From Problem 3 (b), we have for $0<x<\pi$,writing r_n for $\frac{x}{n\pi}$,

$$\begin{aligned}
&x\cot x = 1-2\sum_{n=1}^{\infty}\frac{x^2}{n^2\pi^2-x^2} \\
= \quad & 1-2\sum_{n=1}^{\infty}\frac{r_n^2}{1-r_n^2} \\
= \quad & 1-2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}r_n^{2m} \\
= \quad & 1-2\sum_{m=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{1}{n^{2m}}\right)\frac{x^{2m}}{\pi^{2m}} \\
= \quad & 1-2\sum_{m=1}^{\infty}\frac{\zeta(2m)}{\pi^{2m}}x^{2m}
\end{aligned}$$

Comparing coefficients of $x\cot x$ in both formulas, we get (e).
Remark. Ross Tang has discovered explicit formulas for Bernoulli numbers and Euler numbers in

`http://www.voofie.com/content/117/an-explicit-formula-for-the-euler-zigzag-numbers-updown-numbers-from-power-series/#Explicit_Formula_for_Euler_number`

His formula is

$$B_{2n} = \frac{2n}{2^{2n} - 4^{2n}} \sum_{k=1}^{2n} \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^j (k-2j)^{2n}}{2^k i^k k},$$

$$E_{2n} = i \sum_{k=1}^{2n+1} \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^j (k-2j)^{2n+1}}{2^k i^k k}.$$

Update: Simpler formulas are already in

`http://en.wikipedia.org/wiki/Bernoulli_number`

Perhaps it is a good project to try to find explicit formulas for Bernoulli polynomials.

From the above solution, we get the following series expansion for $\cot x$ and $\coth x$:

$$\cot z = \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^n \frac{2^{2n} B_{2n}}{(2n)!} z^{2n-1},$$

$$\coth z = \sum_{n=1}^{\infty} \frac{2^{2n} B_{2n}}{(2n)!} z^{2n-1}.$$

I have yet to figure out the proofs of the next four formulas:

$$\tan z = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{4^n (4^n - 1) B_{2n}}{(2n)!} z^{2n-1} = z + \frac{z^3}{3} + \frac{2z^5}{15} + \frac{17z^7}{315} + \cdots,$$

$$\tanh z = \sum_{n=1}^{\infty} \frac{4^n (4^n - 1) B_{2n}}{(2n)!} z^{2n-1} = z - \frac{z^3}{3} + \frac{2z^5}{15} - \frac{17z^7}{315} + \cdots,$$

$$\sec z = \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{(2n)!} z^{2n},$$

$$\operatorname{sech} z = \sum_{n=0}^{\infty} \frac{E_{2n}}{(2n)!} z^{2n},$$

where E_{2n} are the Euler's numbers. (The generating function for E_{2n} is $\operatorname{sech} z = \frac{2}{e^z + e^{-z}}$).

Update: First note that

$$2\coth(2z) - \coth z = \tanh z$$

from which we get the series for $\tanh z$. Then use $\tan z = -i \tanh(iz)$ to get $\tan z$ series. Other series that use Bernoulli numbers are:

$$\frac{z}{\sin z}, \frac{z}{\sinh z}, \log \frac{\sin z}{z}, \log(\cos z), \log \frac{\tan z}{z}, \cdots,$$

but I have not looked into this assertion.
Atkinson (American Math Monthly, vol. 93, no. 5,1986, p. 387-; this paper in pdf form is in my computer under the name TangentSeries.pdf) has discovered that the above series for tangent and secant can be read off from the sides of the following triangle (only seven rows are shown):

$$\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 0 & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & 0 & & & & & \\
 & & & & 0 & & 1 & & 2 & & 2 & & & & \\
 & & & 5 & & 5 & & 4 & & 2 & & 0 & & & \\
 & & 0 & & 5 & & 10 & & 14 & & 16 & & 16 & & \\
 & 61 & & 61 & & 56 & & 46 & & 32 & & 16 & & 0 & \\
. & & . & & . & & . & & . & & . & & . & & .
\end{array}$$

The eighth row is $0, 61, 122, 178, 224, 256, 272, 272$.

26. (Problem 5) Define the **Bernoulli polynomials** $B_n(x)$ by the formula

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} z^n.$$

(a) The functions $B_n(x)$ are polynomials in x and

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}.$$

Show that $B_0(x) = 1, B_1(x) = x - 1/2, B_2(x) = x^2 - x + 1/6$, and $B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$.
(b) If $n \geq 1$, then

$$B_n(x+1) - B_n(x) = nx^{n-1},$$

and if $n \geq 2$, then

$$B_n(0) = B_n(1) = B_n.$$

(c) Define $S_m(n) = 1^m + 2^m + \cdots + (n-1)^m$. Show that

$$(m+1)S_m(n) = B_{m+1}(n) - B_{m+1}.$$

(d) Prove that the Bernoulli polynomials are the only polynomials that satisfy (i) $B_0(x) = 1$, (ii) $B_n'(x) = nB_{n-1}(x)$ for $n \geq 1$, (iii) $\int_0^1 B_n(x)\,dx = 0$ for $n \geq 1$, and show that from (b) one obtains

$$\int_m^{x+1} B_n(t)\,dt = x^n.$$

(e) Calculate the Fourier series of $B_1(x)$ to conclude that for $0 < x < 1$ we have

$$B_1(x) = x - \frac{1}{2} = \frac{-1}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2\pi kx)}{k}.$$

Integrate and conclude that

$$B_{2n}(x) = (-1)^{n+1}\frac{2(2n)!}{(2\pi)^{2n}}\sum_{k=1}^{\infty}\frac{\cos(2\pi kx)}{k^{2n}},$$

$$B_{2n+1}(x) = (-1)^{n+1}\frac{2(2n+1)!}{(2\pi)^{2n+1}}\sum_{k=1}^{\infty}\frac{\sin(2\pi kx)}{k^{2n+1}}.$$

Finally, show that for $0 < x < 1$,

$$B_n(x) = -\frac{n!}{(2\pi i)^n}\sum_{k\neq 0}\frac{e^{2\pi ikx}}{k^n}.$$

We observe that the Bernoulli polynomials are, up to normalization (i.e. requiring $\int_0^1 B_n(x)\,dx = 0$), successive integrals (antiderivatives) of the sawtooth function $x - 1/2, 0 < x < 1$.
Solution.

$$\begin{aligned}
& \frac{ze^{xz}}{e^z-1} = \frac{z}{e^z-1}e^{xz} \\
= & \sum_{n=0}^{\infty}\frac{B_n}{n!}z^n\sum_{n=0}^{\infty}\frac{x^n}{n!}z^n \\
= & \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{B_k}{k!}\frac{x^{n-k}}{(n-k)!}\right)z^n \\
= & \sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{k=0}^{n}\binom{n}{k}B_kx^{n-k}\right)z^n
\end{aligned}$$

Therefore

$$B_n(x) = \sum_{k=0}^{n}\binom{n}{k}B_kx^{n-k}.$$

$$\begin{aligned}
& \frac{ze^{(x+1)z}}{e^z-1} - \frac{ze^{xz}}{e^z-1} \\
= & \frac{ze^{xz}}{e^z-1}(e^z-1) \\
= & ze^{xz} \\
= & \sum_{n=0}\frac{x^n}{n!}z^{n+1} \\
= & \sum_{n=1}\frac{x^{n-1}}{(n-1)!}z^n
\end{aligned}$$

Therefore comparing coefficients of $z^n, n \geq 1$ we get

$$\frac{B_n(x+1)}{n!} - \frac{B_n(x)}{n!} = \frac{x^{n-1}}{(n-1)!}$$

and hence

$$B_n(x+1) - B_n(x) = nx^{n-1},$$

So for $n \geq 2$, $B_n(1) - B_n(0) = n0^{n-1} = 0$,i.e. $B_n(1) = B_n(0) = B_n$. (Note that by definition of $B_n(x)$, we have $B_n(0) = B_n$.)
For $m \geq 1$,

$$\begin{aligned} & (m+1)\sum_{k=1}^{n-1} k^m \\ = & \sum_{k=1}^{n-1} B_{m+1}(k+1) - B_{m+1}(k) \\ = & B_{m+1}(n) - B_{m+1}(1) \\ = & B_{m+1}(n) - B_{m+1} \end{aligned}$$

Note that this yields a formula for the sum

$$1^m + 2^m + \cdots + n^m = \frac{1}{m+1}(B_{m+1}(n+1) - B_{m+1}).$$

Write $F(x,z)$ for $\frac{ze^{xz}}{e^z-1}$. Since

$$\frac{\partial}{\partial x}F(x,z) = zF(x,z)$$

, comparing coefficient of z^n on both sides, we get

$$\frac{B_n'(x)}{n!} = \frac{B_{n-1}(x)}{(n-1)!},$$

hence $B_n'(x) = nB_{n-1}(x)$. Next,

$$\int_0^1 B_n(x)\,dx = \frac{1}{n+1}\int_0^1 B_{n+1}'(x)\,dx = \frac{1}{n+1}(B_{n+1}(1) - B_{n+1}(0)) = 0.$$

From (b),

$$\int_x^{x+1} B_n(t)\,dt = \frac{1}{n+1}\int_x^{x+1} B_{n+1}'(t)\,dt = \frac{1}{n+1}(B_{n+1}(x+1) - B_{n+1}(x)) = x^n.$$

From Maple, one gets

$$B_1(x) = x - \frac{1}{2} = \frac{-1}{\pi}\sum_{k=1}^{\infty}\frac{\sin(2\pi kx)}{k}.$$

$B_2(x)$ is an antiderivative of $2B_1(x)$, so

$$B_2(x) = \frac{-2}{\pi}\sum_{k=1}^{\infty}\frac{-\cos(2\pi kx)}{2\pi k^2} + C$$

But $\int_0^1 B_2(x)\,dx = 0$ and $\int_0^1 \cos(2\pi kx)\,dx = 0$, so $C = 0$, and

$$B_2(x) = \frac{2}{2\pi^2}\sum_{k=1}^{\infty}\frac{\cos(2\pi kx)}{2\pi k^2} = \frac{2\cdot 2!}{(2\pi)^2}\sum_{k=1}^{\infty}\frac{\cos(2\pi kx)}{k^2}.$$

Next, arguing the same manner, $B_3(x)$ is an antiderivative of $3B_2(x)$, and

$$B_3(x) = \frac{2\cdot 3!}{(2\pi)^3}\sum_{k=1}^{\infty}\frac{\sin(2\pi kx)}{k^3}.$$

And then

$$B_4(x) = -\frac{2\cdot 4!}{(2\pi)^4}\sum_{k=1}^{\infty}\frac{\cos(2\pi kx)}{k^4}.$$

and so on, and we get

$$B_{2n}(x) = (-1)^{n+1}\frac{2(2n)!}{(2\pi)^{2n}}\sum_{k=1}^{\infty}\frac{\cos(2\pi kx)}{k^{2n}},$$

$$B_{2n+1}(x) = (-1)^{n+1}\frac{2(2n+1)!}{(2\pi)^{2n+1}}\sum_{k=1}^{\infty}\frac{\sin(2\pi kx)}{k^{2n+1}}.$$

Writing $\sin, \cos$ in terms of e powers, and noting that $i^{2m} = (-1)^m$, we get the last formula

$$B_n(x) = -\frac{n!}{(2\pi i)^n}\sum_{k\neq 0}\frac{e^{2\pi ikx}}{k^n}.$$

Chapter 4

1. (Problem 5) Let f be a Riemann integrable function on the interval $[-\pi, \pi]$. We define the generalized delayed means of the Fourier series of f by

$$\sigma_{N,K} = \frac{S_N + \cdots + S_{N+K-1}}{K}.$$

Note that in particular

$$\sigma_{0,N} = \sigma_N, \sigma_{N,1} = S_N \text{ and } \sigma_{N,N} = \triangle_N,$$

where $\triangle_N$ are the specific delayed means used in Section 3.
(a) Show that

$$\sigma_{N,K} = \frac{1}{K}((N+K)\sigma_{N+K} - N\sigma_N),$$

and

$$\sigma_{N,K} = S_N + \sum_{N+1 \le |j| \le N+K-1} \left(1 - \frac{|j| - N}{K}\right) \hat{f}(j) e^{ij\theta}.$$

From this last expression for $\sigma_{N,K}$ conclude that

$$|\sigma_{N,K} - S_M| \le \sum_{N+1 \le |j| \le N+K-1} |\hat{f}(j)|$$

for all $N \le M < N + K$.
(b) Use one of the above formulas and Fejer's theorem to show that with $N = kn$ and $K = n$, then

$$\sigma_{kn,n}(f)(\theta) \to f(\theta) \text{ as } n \to \infty$$

whenever f is continuous at θ, and also

$$\sigma_{kn,n}(f)(\theta) \to \frac{f(\theta^+) + f(\theta^-)}{2} \text{ as } n \to \infty$$

at a jump discontinuity (refer to the preceding chapters and their exercises for the appropriate definitions and results). In the case when f is continuous on $[-\pi, \pi]$, show that $\sigma_{kn,n}(f) \to f$ uniformly as $n \to \infty$.
(c) Using part (a), show that if $\hat{f}(j) = O(1/|j|)$ and $kn \le m < (k+1)n$, we get

$$|\sigma_{kn,n} - S_m| \le \frac{C}{k} \text{ for some constant } C > 0.$$

(d) Suppose that $\hat{f}(j) = O(1/|j|)$. Prove that if f is continuous at θ then

$$S_N(f)(\theta) \to f(\theta) \text{ as } N \to \infty,$$

and if f has a jump discontinuity at θ then

$$S_N(f)(\theta) \to \frac{f(\theta^+) + f(\theta^-)}{2} \text{ as } N \to \infty.$$

Also, show that if f is continuous on $[-\pi, \pi]$, then $S_N(f) \to f$ uniformly.
(e) The above arguments show if $\sum c_n$ is Cesàro summable to s and $c_n = O(1/n)$, then $\sum c_n$ converges to s. This is a weak version of Littlewood's theorem (Problem 3, Chapter 2).
Solution.

(a). $\sigma_{N,K} = \frac{1}{K}((N+K)\sigma_{N+K} - N\sigma_N)$ is straightforward. Write a_j for $\hat{f}(j)e^{ij\theta}$. We have

$$\begin{aligned}
& \sigma_{N,K} \\
= \; & \frac{1}{K}[S_N + (S_N + a_{N+1} + a_{-N-1}) + \cdots + (S_N + \sum_{N+1\le|j|\le N+K-1} a_j)] \\
= \; & S_N + \frac{1}{K}\sum_{l=1}^{K-1}\sum_{N+1\le|j|\le N+l} a_j \\
= \; & S_N + \frac{1}{K}\sum_{N+1\le|j|\le N+K-1} a_j \sum_{l=|j|-N}^{K-1} 1 \\
= \; & S_N + \frac{1}{K}\sum_{N+1\le|j|\le N+K-1} (K+N-|j|)a_j \\
= \; & S_N + \sum_{N+1\le|j|\le N+K-1} \left(1 - \frac{|j|-N}{K}\right) a_j
\end{aligned}$$

For $N \le M < N+K$, $\sigma_{N,K} - S_M = \sum_{N+1\le|j|\le N+K-1} b_j a_j$, where b_j is either $1 - \frac{|j|-N}{K}$ or $-\frac{|j|-N}{K}$, hence $|\sigma_{N,K} - S_M| \le \sum_{N+1\le|j|\le N+K-1} |a_j|$.
(b). We have

$$\sigma_{kn,n} = \frac{1}{n}[(k+1)n\sigma_{(k+1)n} - kn\sigma_{kn}] = (k+1)\sigma_{(k+1)n} - k\sigma_{kn}$$

Let A be either $f(\theta)$ or $\frac{f(\theta^+)+f(\theta^-)}{2}$. Since $\sigma_{(k+1)n}, \sigma_{kn}$ approach the same value A, as $n \to \infty$, we see that $\sigma_{kn,n} \to A$. Uniform convergence statement is also clear from $\sigma_{kn,n} = (k+1)\sigma_{(k+1)n} - k\sigma_{kn}$. (I think I have to elaborate this last sentence.)
(c). Suppose $|\hat{f}(j)| = |a_j| \le C_1/|j|$ for some constant $C_1 > 0$. From the last part of (a), for $kn \le m < (k+1)n$,

$$\begin{aligned}
& |\sigma_{kn,n} - S_m| \\
\le \; & C_1 \sum_{kn+1\le|j|\le(k+1)n-1} \frac{1}{|j|} \\
\le \; & 2C_1\frac{n-1}{kn+1} \\
\le \; & 2C_1\frac{1}{k} = \frac{C}{k}
\end{aligned}$$

(d). Let $\epsilon > 0$. Fix a positive integer k such that $\frac{C}{k} < \epsilon$. Choose N such that

$$|\sigma_{kn,n} - A| < \epsilon$$

for all $n \geq N$. Then for all $m \geq kN$, we have $kn \leq m < (k+1)n$ for some $n \geq N$, and

$$|S_m - A| \leq |S_m - \sigma_{kn,n}| + |\sigma_{kn,n} - A| < 2\epsilon.$$

This proves that $S_m \to A$ as $m \to \infty$.
(e)Note that except for the uniform convergence part, the only properties of the series $\sum_j a_j$ that we use in the above proof are that it is Cesaro summable to A and that $a_j = O(1/|j|)$.

Chapter 5 The Fourier Transform on $\mathbb{R}$

1. (Exercise 1) Corollary 2.3 in Chapter 2 leads to the following simplified version of the Fourier inversion formula. Suppose f is a continuous function supported on an interval $[-M, M]$, whose Fourier transform $\hat{f}$ is of moderate decrease.
(a) Fix L with $L/2 > M$, and show that $f(x) = \sum a_n(L) e^{2\pi i n x/L}$ where

$$a_n(L) = \frac{1}{L}\int_{-L/2}^{L/2} f(x) e^{-2\pi i n x/L}\, dx = \frac{1}{L}\hat{f}(n/L).$$

Alternatively, we may write $f(x) = \delta \sum_{n=-\infty}^{\infty} \hat{f}(n\delta) e^{2\pi i n \delta x}$ with $\delta = 1/L$.
(b) Prove that if F is continuous and of moderate decrease, then

$$\int_{-\infty}^{\infty} F(\xi)\, d\xi = \lim_{\delta \to 0, \delta > 0} \delta \sum_{n=-\infty}^{\infty} F(\delta n).$$

(c) Conclude that $f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi}\, d\xi$.
Solution.
(a)

$$\sum_{n=-\infty}^{\infty} a_n(L) e^{2\pi i n x/L} = \frac{1}{L}\sum_{n=-\infty}^{\infty} \hat{f}(\frac{n}{L}) e^{2\pi i n x/L}$$

Let B, C be the constants defined as in Remark 1 below. Since $|\hat{f}(\frac{n}{L})| \leq \frac{B}{n^2}$, the above series is absolutely convergent. Hence the series converges uniformly (and absolutely) to $f(x)$ by Corollary 2.3 in Chapter 2.
(b). Given $\epsilon > 0$, choose $N > C$ such that $\int_{|x|>N} |F(x)|\, dx < \epsilon/4$ and $\sum_{|n|>N} \frac{B}{n^2} < \epsilon/4$. Choose $0 < \delta_1 < 1$ such that for all $0 < \delta < \delta_1$,

$$|\int_{-N}^{N} F(x)\, dx - \delta \sum_{|n| \leq N/\delta} F(n\delta)| < \frac{\epsilon}{2}.$$

This is possible because $\delta \sum_{|n| \leq N/\delta} F(n\delta)$ is almost a Riemann sum. (Fill in the details.) Since

$$\delta | \sum_{|n| > N/\delta} F(\delta n)| \leq \sum_{|n|>N} |F(\delta n)| \leq \sum_{|n|>N} \frac{B}{n^2}$$

the result follows.
(c). Apply (b) to $F(\xi) = \hat{f}(\xi)e^{2\pi i\xi x}$ and use (a).
Remark 1. From the book's definition, a function f defined on $\mathbb{R}$ is said to be of moderate decrease if f is continuous and there exists a constant $A > 0$ so that

$$|f(x)| \le \frac{A}{1+x^2} \text{ for all } x \in \mathbb{R}.$$

Note that this is equivalent to saying that f is continuous and there are positive constants B, C such that

$$|f(x)| \le \frac{B}{x^2} \text{ for all } |x| \ge C.$$

Proof. Suppose f satisfies the first condition. Since for all $x \neq 0$

$$\frac{A}{1+x^2} = \frac{A}{1+x^2/2+x^2/2} \le \frac{2A}{x^2}$$

we can choose $B = 2A, C = 1$ in the second statement.
Now suppose f satisfies the second condition. Let $D = \max\{1, C\}$. Then

$$\frac{B}{x^2} = \frac{2B}{x^2+x^2} \le \frac{2B}{1+x^2} \text{ for } |x| \ge D.$$

Let $M = \max\{(1+x^2)|f(x)| : |x| \le D\}$. Then we can let $A = \max\{2B, M\}$ in the first condition.
Remark 2. Let $f(x) = 1, -1 \le x \le 1$, zero elsewhere. Then the Fourier transform of f is $\frac{\sin(2\pi\xi}{\pi\xi}$, which is not integrable. Modifying f, making it continuous, we let f_n be the even extension of the following function

$$g(x) = 1 \text{ for } 0 \le x \le \frac{n-1}{n}, \text{ and } -n(x-1) \text{ for } \frac{n-1}{n} \le x \le 1$$

and zero for $x \ge 1$. The Fourier transform of $f_n, n \ge 1$, is

$$\frac{n(1-2\cos^2(\pi\xi)+\cos(\frac{n-1}{n}2\pi\xi))}{2\pi^2\xi^2}$$

which is of moderate decrease. A continuous with compact support such it Fourier transform is not of moderate decrease is in Exercise 3. **It remains to see an example of continuous function of compact support such that its Fourier transform is not integrable.** From sci.math, cronopio said Terry Tao seems to say that the function $(1-\log(1-|x|))^{-a}, |x| < 1$, zero elsewhere, ($0 < a < 1$ is a fixed number) is an example. See http://mathoverflow.net/questions/43550/is-the-fourier-transform-of-1-1-log1-x2-supported-in-1-1-integrable
Existence of such function also follows from closed graph theorem, see the link in the above link.

2. (Exercise 2) Let f and g be the functions defined by

$$f(x) = \chi_{[-1,1]}(x) = \begin{cases} 1 & \text{if } |x| \leq 1, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and } g(x) = \begin{cases} 1 - |x| & \text{if } |x| \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Although f is not continuous, the integral defining its Fourier transform still make sense. Show that

$$\hat{f}(\xi) = \frac{\sin 2\pi\xi}{\pi\xi} \text{ and } \hat{g}(\xi) = \left(\frac{\sin \pi\xi}{\pi\xi}\right)^2,$$

with the understanding that $\hat{f}(0) = 2$ and $\hat{g}(0) = 1$.
Solution. Check with Maple and found to agree.

3. (Exercise 3) The following exercise illustrates the principle that the decay of $\hat{f}$ is related to the continuity properties of f.
(a) Suppose that f is a function of moderate decrease on $\mathbb{R}$ whose Fourier transform $\hat{f}$ is continuous and satisfies

$$\hat{f}(\xi) = O\left(\frac{1}{|\xi|^{1+\alpha}}\right) \text{ as } |\xi| \to \infty$$

for some $0 < \alpha < 1$. Prove that f satisfies a Holder condition of order α, that is, that

$$|f(x+h) - f(x)| \leq M|h|^{\alpha} \text{ for some } M > 0 \text{ and all } x, h \in \mathbb{R}.$$

(b) Let f be a continuous function on $\mathbb{R}$ which vanishes for $|x| \geq 1$, with $f(0) = 0$, and which is equal to $1/\log(1/|x|)$ for all x in a neighborhood of the origin. Prove that $\hat{f}$ is not of moderate decrease. In fact, there is no $\epsilon > 0$ so that $\hat{f}(\xi) = O(1/|\xi|^{1+\epsilon})$ as $|\xi| \to \infty$.
Solution.
By remarks in section 1.7, since both f and $\hat{f}$ are of moderate decrease, the inverse Fourier transform of $\hat{f}$ is f. Thus

$$f(x+h) - f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi x}(e^{2\pi i \xi h} - 1)\, d\xi.$$

The condition on $\hat{f}$ is equivalent to

$$|\hat{f}(\xi)| \leq \frac{A}{1 + |\xi|^{1+\alpha}} \text{ for all } \xi \in \mathbb{R}$$

for some $A > 0$. Thus (note that $|e^{2\pi i\xi h} - 1| = 2|\sin(\pi\xi h)|$)

$$\begin{aligned}
\left|\frac{f(x+h)-f(x)}{h^\alpha}\right| &\le \frac{1}{|h|^\alpha}\int_{-\infty}^{\infty}\frac{A|e^{2\pi i\xi h}-1|}{1+|\xi|^{1+\alpha}}\,d\xi \\
&\le \frac{4A}{|h|^\alpha}\int_0^\infty \frac{|\sin(\pi\xi h)|}{1+\xi^{1+\alpha}}\,d\xi \\
&= \frac{4A}{\pi}\int_0^\infty \frac{|\sin(u)|}{|h|^{1+\alpha}+u^{1+\alpha}}\,du \\
&\le \frac{4A}{\pi}\left(\int_0^1 \frac{|\frac{\sin(u)}{u}|}{u^\alpha}\,du + \int_1^\infty \frac{1}{u^{1+\alpha}}\,du\right) \\
&\le \frac{4A}{\pi}\left(\int_0^1 \frac{1}{u^\alpha}\,du + \int_1^\infty \frac{1}{u^{1+\alpha}}\,du\right) \\
&< \infty
\end{aligned}$$

(b). We have

$$\frac{|f(h)-f(0)|}{|h|^\epsilon} = \frac{1}{-|h|^\epsilon \log|h|} \to \infty$$

as $h \to 0$ for any fixed $\epsilon > 0$. So by (a), $\hat{f}$ is not of moderate decrease.

4. (Exercise 4) Examples of compactly supported functions in $\mathcal{S}(\mathbb{R})$ are very handy in many applications in analysis. Some examples are:
(a) Suppose $a < b$, and f is the function such that $f(x) = 0$ if $x \le a$ or $x \ge b$ and
$$f(x) = e^{-1/(x-a)}e^{-1/(b-x)} \text{ if } a < x < b.$$
Show that f is indefinitely differentiable on $\mathbb{R}$.
(b) Prove that there exists an indefinitely differentiable function F on $\mathbb{R}$ such that $F(x) = 0$ if $x \le a$, $F(x) = 1$ if $x \ge b$, and F is strictly increasing on $[a, b]$.
(c) Let $\delta > 0$ be so small that $a + \delta < b - \delta$. Show that there exists an indefinitely differentiable function g such that g is 0 if $x \le a$ or $x \ge b$, g is 1 on $[a + \delta, b - \delta]$, and g is strictly monotonic on $[a, a + \delta]$ and $[b - \delta, b]$.
Solution. Note that the graph of $-1/(x-a) - 1/(b-x)$ is symmetric about the line $x = (a+b)/2$ with maximum value of $-4/(b-a)$ at the point $x = (a+b)/2$.
For (b) consider $F(x) = c\int_{-\infty}^x f(t)\,dt$ where c is the reciprocal of $\int_{-\infty}^{\infty} f(t)\,dt$.

5. (Exercise 5) Suppose f is continuous and of moderate decrease.
(a) Prove that $\hat{f}$ is continuous and $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$.
(b) Show that if $\hat{f}(\xi) = 0$ for all ξ, then f is identically 0.
Solution.

$$\begin{aligned}|\hat{f}(\xi+h)-\hat{f}(\xi)| &= \left|\int_{-\infty}^{\infty} f(x)e^{-2\pi i\xi x}(e^{-2\pi ihx}-1)\,dx\right| \\ &\leq \int_{-\infty}^{\infty} |f(x)||e^{-2\pi ihx}-1|\,dx \\ &= \int_{-\infty}^{\infty} |f(x)|2|\sin 2\pi hx|\,dx \\ &\leq 2A\int_{-\infty}^{\infty} \frac{|\sin 2\pi hx|}{1+x^2}\,dx\end{aligned}$$

Let $\epsilon > 0$. Choose $N > 0$ such that $2A\int_{|x|>N} \frac{1}{1+x^2}\,dx < \epsilon/2$. Since $\frac{|\sin 2\pi hx|}{1+x^2} \to 0$ uniformly on $[-N, N]$ as $h \to 0$, there exists $\delta > 0$ such that $2A\int_{-N}^{N} \frac{|\sin 2\pi hx|}{1+x^2}\,dx < \epsilon/2$ for $|h| < \delta$. This proves that $\hat{f}$ is continuous.
Note: Lebesgue dominated convergence theorem could be applied to yield a simpler proof.
(b).

$$\begin{aligned}\hat{f}(\xi) &= \frac{1}{2}\left(\int_{-\infty}^{\infty} f(x)e^{-2\pi i\xi x} - \int_{-\infty}^{\infty} f(x)e^{-2\pi i(\xi x+1/2)}\,dx\right) \\ &= \frac{1}{2}\int_{-\infty}^{\infty}(f(x)-f(x-\frac{1}{2\xi}))e^{-2\pi i\xi x}\,dx\end{aligned}$$

Now the result follows from applying Lebesgue dominated convergence theorem; without it, the proof is harder.
(b). If f, g are of moderate decrease, then the function $f(x)g(y)e^{-2\pi ixy}$ is integrable over $\mathbb{R}^2$. Then it follows from Fubini's theorem that

$$\int f(x)\hat{g}(x)\,dx = \int g(x)\hat{f}(x)\,dx.$$

Suppose $\hat{f}(x) = 0$ for all x. For any t and any $\delta > 0$, $K_\delta(t-x)$ as a function of x is in $\mathcal{S}(\mathbb{R})$, so by Corollary 1.10 it is equal to the Fourier transform $\hat{g_\delta}$ for some g_δ in $\mathcal{S}(\mathbb{R})$. So $\int f(x)\hat{g_\delta}(x)\,dx = \int f(x)K_\delta(t-x)\,dx = 0$. Letting $\delta \to 0$, we get $f(t) = 0$.

6. (Exercise 8) Prove that if f is continuous, of moderate decrease, and $\int_{-\infty}^{\infty} f(y)e^{-y^2}e^{2xy}\,dy = 0$ for all $x \in \mathbb{R}$, then $f = 0$.
Proof.
Let $g(x) = e^{-x^2}$. Then $(f*g)(x) = \int_{-\infty}^{\infty} f(y)e^{-(x-y)^2}\,dy = e^{-x^2}\int_{-\infty}^{\infty} f(y)e^{-y^2}e^{2xy}\,dy = 0$ for all x. This implies that $\widehat{f*g}(\xi) = \hat{f}(\xi)\hat{g}(\xi) = \hat{f}(\xi)\sqrt{\pi}e^{-\pi^2\xi^2} = 0$ for all ξ. So $\hat{f} = 0$. By Theorem 1.9, $f = 0$.

7. (Exercise 12) Show that the function defined

$$u(x,t) = \frac{x}{t}\mathcal{H}_t(x)$$

satisfies the heat equation for $t > 0$ and $\lim_{t\to 0} u(x,t) = 0$ for every x, but u is not continuous at the origin. **Proof.**
Maple shows that

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} = \frac{x(x^2-6t)}{8\sqrt{\pi}t^{7/2}} e^{x^2/4t}.$$

It is clear that $\lim_{t\to 0^+} u(x,t) = 0$ for all x. If we approach the origin along the parabola $x^2 = t$, we get

$$\lim_{x\to 0} \frac{1}{x^2\sqrt{4\pi}} e^{-1/4} = \infty.$$

8. (Add in) Suppose f is an even function such that both f and $\hat{f}$ are of moderate decrease. Then $\hat{f}$ is also even and the Fourier transform of $\hat{f}(x)$ is $f(\xi)$.
Proof.
The $\int$ below denotes $\int_{-\infty}^{\infty}$. We have

$$\hat{f}(-\xi) = \int f(x)e^{2\pi i x\xi}\,dx = \int f(-x)e^{-2\pi i x\xi}\,dx = \int f(x)e^{-2\pi i x\xi}\,dx = \hat{f}(\xi)$$

so $\hat{f}$ is even.
Since

$$f(x) = \int \hat{f}(t)e^{2\pi i x t}\,dt$$

we have

$$\int \hat{f}(t)e^{-2\pi i \xi t}\,dt = \int \hat{f}(-t)e^{2\pi i \xi t}\,dt = \int \hat{f}(t)e^{2\pi i \xi t}\,dt = f(\xi)$$

proving that the Fourier transform of $\hat{f}(x)$ is $f(\xi)$.

9. (Exercise 15) This exercise provides another example of periodization.
(a) Apply the Poisson summation formula to the function $\hat{g}(x)$ and its Fourier transform in Exercise 2 to obtain

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+\alpha)^2} = \frac{\pi^2}{(\sin\pi\alpha)^2}$$

whenever α is real, but not equal to an integer.
(b) Prove as a consequence that

$$\sum_{n=-\infty}^{\infty} \frac{1}{n+\alpha} = \frac{\pi}{\tan\pi\alpha} \tag{1}$$

whenever α is real but not equal to an integer.
Solution.

By the last item, the Fourier transform of $h(x) = \hat{g}(x)$ is $g(x)$ in Exercise 2. By Poisson summation formula

$$\sum_{n=-\infty}^{\infty} \frac{\sin^2(\pi(n+\alpha))}{\pi^2(n+\alpha)^2} = \sum_{n=-\infty}^{\infty} g(n)e^{2\pi i n\alpha} = 1.$$

(If α is an integer, then the equality is reduced to trivial $1 = 1$.) Since $\sin(n\pi + \pi\alpha) = (-1)^n \sin(\pi\alpha)$, the result follows.
(b). Assume that $0 < \alpha < 1$. We have from (a), for $n \neq 0$,

$$\int_0^\alpha \frac{1}{(-n+x)^2} + \frac{1}{(n+x)^2}\, dx = -\left(\frac{1}{-n+\alpha} + \frac{1}{n+\alpha}\right)$$

and since $\lim_{x\to 0} \frac{\pi^2}{(\sin \pi x)^2} - \frac{1}{x^2} = \pi^2/3$, it has a removable discontinuity at $x = 0$ and $\lim_{x\to 0} -\frac{\pi}{\tan \pi x} + \frac{1}{x} = 0$, we have

$$\int_0^\alpha \frac{\pi^2}{(\sin \pi x)^2} - \frac{1}{x^2}\, dx = -\frac{\pi}{\tan \pi\alpha} + \frac{1}{\alpha}.$$

Therefore

$$\frac{1}{\alpha} + \sum_{n=1}^{\infty}\left(\frac{1}{n+\alpha} + \frac{1}{-n+\alpha}\right) = \frac{\pi}{\tan \pi\alpha}.$$

Evaluating at $\alpha = 1/2$ yields 0; this is correct since

$$\sum_{n=1}^{\infty} \frac{4}{4n^2-1} = \sum_{n=1}^{\infty} \frac{2}{2n-1} - \frac{2}{2n+1}$$

is a telescoping series whose sum is 2.
Now we have to prove for any α, a noninteger. Let $n_1 = \lfloor \alpha \rfloor$. Then $\alpha = n_1 + \alpha_1$ for some $0 < \alpha_1 < 1$. Then

$$\sum_{n=-\infty}^{\infty} \frac{1}{n+\alpha} = \sum_{n=-\infty}^{\infty} \frac{1}{n+\alpha_1} = \frac{\pi}{\tan \pi\alpha_1} = \frac{\pi}{\tan \pi\alpha}.$$

10. (Exercise 19) The following is a variant of the calculation of $\zeta(2m) = \sum_{n=1}^{\infty} 1/n^{2m}$ found in Problem 4, Chapter 3.
(a) Apply the Poisson summation formula to $f(x) = t/(\pi(x^2+t^2))$ and $\hat{f}(\xi) = e^{-2\pi t|\xi|}$ where $t > 0$ in order to get

$$\frac{1}{\pi}\sum_{n=-\infty}^{\infty} \frac{t}{t^2+n^2} = \sum_{n=-\infty}^{\infty} e^{-2\pi t|n|}.$$

(b) Prove the following identity:

$$\frac{1}{\pi}\sum_{n=-\infty}^{\infty} \frac{t}{t^2+n^2} = \frac{1}{\pi t} + \frac{2}{\pi}\sum_{m=1}^{\infty}(-1)^{m+1}\zeta(2m)t^{2m-1}, 0 < t < 1$$

as well as

$$\sum_{n=-\infty}^{\infty} e^{-2\pi t|n|} = \frac{2}{1-e^{-2\pi t}} - 1, 0 < t < 1.$$

(c) Use the fact that

$$\frac{z}{e^z-1} = 1 - \frac{z}{2} + \sum_{m=1}^{\infty} \frac{B_{2m}}{(2m)!} z^{2m},$$

where B_k are the Bernoulli numbers to deduce from the above formula,

$$2\zeta(2m) = (-1)^{m+1} \frac{(2\pi)^{2m}}{(2m)!} B_{2m}.$$

Solution.
(a) By Lemma 2.4, the Fourier transform of $f(x) = t/(\pi(x^2+t^2))$ is $\hat{f}(\xi) = e^{-2\pi t|\xi|}$. Then by Poisson summation formula with $x = 0$ ($(x+n)^2 = n^2$), we get (a).
(b).

$$\begin{aligned}
\frac{1}{\pi}\sum_{n=-\infty}^{\infty} \frac{t}{t^2+n^2} &= \frac{1}{\pi t} + \frac{1}{\pi}\sum_{n\neq 0} \frac{t}{t^2+n^2} \\
&= \frac{1}{\pi t} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{t}{t^2+n^2} \\
&= \frac{1}{\pi t} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{t/n^2}{1+(t/n)^2} \\
&= \frac{1}{\pi t} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{t}{n^2} \sum_{m=0}^{\infty} (-1)^m \frac{t^{2m}}{n^{2m}} \\
&= \frac{1}{\pi t} + \frac{2}{\pi}\sum_{n=1}^{\infty} \sum_{m=0}^{\infty} (-1)^m \frac{t^{2m+1}}{n^{2(m+1)}} \\
&= \frac{1}{\pi t} + \frac{2}{\pi}\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-1)^{m+1} \frac{t^{2m-1}}{n^{2m}} \\
&= \frac{1}{\pi t} + \frac{2}{\pi}\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (-1)^{m+1} \frac{t^{2m-1}}{n^{2m}} \\
&= \frac{1}{\pi t} + \frac{2}{\pi}\sum_{m=1}^{\infty} (-1)^{m+1} \zeta(2m) t^{2m-1}
\end{aligned}$$

Next,

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} e^{-2\pi t|n|} &= 1 + 2\sum_{n=1}^{\infty} e^{-2\pi t n} \\
&= 1 + 2\frac{e^{-2\pi t}}{1 - e^{-2\pi t}} \\
&= \frac{1 + e^{-2\pi t}}{1 - e^{-2\pi t}} \\
&= \frac{2 - (1 - e^{-2\pi t})}{1 - e^{-2\pi t}} \\
&= \frac{2}{1 - e^{-2\pi t}} - 1
\end{aligned}$$

Dividing by 2 on both sides, we get

$$\frac{1}{2\pi t} + \frac{1}{\pi}\sum_{m=1}^{\infty}(-1)^{m+1}\zeta(2m)t^{2m-1} = \frac{1}{1 - e^{-2\pi t}} - \frac{1}{2}.$$

Letting $z = -2\pi t$ and multiplying both sides by $-z$, we get

$$1 + \frac{1}{\pi}\sum_{m=1}^{\infty}(-1)^{m+1}\zeta(2m)\frac{2\pi z^{2m}}{(2\pi)^{2m}} = \frac{z}{e^z - 1} + \frac{z}{2}.$$

Comparing with

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{m=1}^{\infty}\frac{B_{2m}}{(2m)!}z^{2m},$$

we get

$$2\zeta(2m) = (-1)^{m+1}\frac{(2\pi)^{2m}}{(2m)!}B_{2m}.$$

11. (Exercise 23) The Heisenberg uncertainty principle can be formulated in terms of the operator $L = -\frac{d^2}{dx^2} + x^2$, which acts on Schwartz functions by the formula

$$L(f) = -\frac{d^2 f}{dx^2} + x^2 f.$$

This operator, sometimes called the **Hermite operator**, is the quantum analogue of the harmonic oscillator. Consider the usual inner product on $\mathcal{S}$ given by

$$(f, g) = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx \text{ whenever } f, g \in \mathcal{S}.$$

(a) Prove that the Heisenberg uncertainty principle implies

$$(Lf, f) \geq (f, f) \text{ for all } f \in \mathcal{S}.$$

This is usually denoted by $L \geq I$.
(b) Consider the operators A and A^* defined on $\mathcal{S}$ by

$$A(f) = \frac{df}{dx} + xf \text{ and } A^*(f) = -\frac{df}{dx} + xf.$$

The operators A and A^* are sometimes called the **annihilation** and **creation** operators, respectively. Prove that for all $f, g \in \mathcal{S}$ we have
(i) $(Af, g) = (f, A^*g)$,
(ii) $(Af, Af) = (A^*Af, f) \geq 0$,
(iii) $A^*A = L - I$.
In particular, this again shows that $L \geq I$.
(c) Now for $t \in \mathbb{R}$, let

$$A_t(f) = \frac{df}{dx} + txf \text{ and } A_t^*(f) = -\frac{df}{dx} + txf.$$

Use the fact that $(A_t^* A_t f, f) \geq 0$ to give another proof of the Heisenberg uncertainty principle which says that whenever $\int_{-\infty}^{\infty} |f(x)|^2 = 1$ then

$$\left(\int_{-\infty}^{\infty} x^2 |f(x)|^2\right)\left(\int_{-\infty}^{\infty} \left|\frac{df}{dx}\right|^2 dx\right) \geq \frac{1}{4}.$$

Solution.
(a) By dividing f by $\sqrt{(f, f)}$ if necessary, we may assume that $(f, f) = 1$. We need to prove that $(Lf, f) \geq 1$. Now

$$(Lf, f) = -(f'', f) + (x^2 f, f) = (f', f') + (x^2 f, f) = \int |f'|^2 + \int x^2 |f|^2 = 4\pi^2 \int \xi^2 |\hat{f}|^2 + \int x^2 |f|^2$$
$$\geq 4\pi(\int \xi^2 |\hat{f}|^2)^{1/2})(\int x^2 |f|^2)^{1/2} \geq 4\pi \frac{1}{4\pi} = 1$$

where we have used integration by parts and theorem 4.1 (Heisenberg uncertainty principle).
(b) Direct checking. Use integration by parts. Note: $AA^* = L + I$.
(c) Similar to (b), we also have $(A_t^* A_t f, f) \geq 0$ for all t. When this is written out, we get $t^2(x^2 f, f) - t(f, f) - (f'', f) = t^2(x^2 f, f) - t + (f', f') \geq 0$ for all t. It follows that

$$1 - 4(f', f')(x^2 f, f) \leq 0$$

from which the result follows.

12. (Problem 1) The equation

$$x^2 \frac{\partial^2 u}{\partial x^2} + ax\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t}$$

with $u(x, 0) = f(x)$ for $0 < x < \infty$ and $t > 0$ is a variant of the heat equation which occurs in a number of applications. To solve it, make the change of variables $x = e^{-y}$ so that $-\infty < y < \infty$. Set $U(y,t) = u(e^{-y}, t)$ and $F(y) = f(e^{-y})$. Then the problem reduces to the equation

$$\frac{\partial^2 U}{\partial y^2} + (1-a)\frac{\partial U}{\partial y} = \frac{\partial U}{\partial t},$$

with $U(y,0) = F(y)$. This can be solved like the usual heat equation (the case $a = 1$) by taking the Fourier transform in the y variable. One must then compute the integral

$$\int_{-\infty}^{\infty} e^{(-4\pi^2\xi^2 + (1-a)2\pi i\xi)t} e^{2\pi i\xi\nu}\, d\xi.$$

Show that the solution of the original problem is then given by

$$u(x,t) = \frac{1}{(4\pi t)^{1/2}} \int_0^{\infty} e^{(-\log(\nu/x) + (1-a)t)^2/(4t)} f(\nu)\frac{d\nu}{\nu}.$$

Solution.
I have checked that what it says up to the new differential equation is correct.
Taking the Fourier transform w.r.t. y of the new equation, we get

$$\frac{\partial \hat{U}}{\partial t} = (-4\pi^2\xi^2 + (1-a)2\pi i\xi)\hat{U}.$$

Solve, using $\hat{U}(\xi, 0) = \hat{F}(\xi)$, to get

$$\hat{U}(\xi,t) = \hat{F}(\xi)e^{(-4\pi^2\xi^2 + (1-a)2\pi i\xi)t}.$$

It remains to find the inverse Fourier transform of $e^{(-4\pi^2\xi^2+(1-a)2\pi i\xi)t}$ and then write $U(y,t)$ as a convolution. We have

$$\int_{-\infty}^{\infty} e^{(-4\pi^2\xi^2 + (1-a)2\pi i\xi)t} e^{2\pi i\xi y}\, d\xi = \int_{-\infty}^{\infty} e^{-4\pi^2\xi^2 t} e^{((1-a)t+y)2\pi i\xi}\, d\xi$$

which is equal to the inverse transform of $e^{-4\pi^2 t\xi^2}$ evaluated at $(1-a)t+y$. Using $\mathcal{F}(e^{-\pi y^2}) = e^{-\pi\xi^2}$ and $\mathcal{F}(\delta f(\delta y)) = \hat{f}(\delta^{-1}\xi)$ with $\delta = 1/\sqrt{4\pi t}$, we find that the required inverse transform is

$$\frac{1}{\sqrt{4\pi t}} e^{-y^2/4t}$$

evaluating at $(1-a)t + y$ yields

$$\frac{1}{\sqrt{4\pi t}} e^{-(y+(1-a)t)^2/(4t)}$$

Therefore

$$U(y,t) = \int_{-\infty}^{\infty} F(\mu)\frac{1}{\sqrt{4\pi t}} e^{-(y-\mu+(1-a)t)^2/(4t)}\, d\mu.$$

Make a substitution $\mu = -\log\nu$, and substitute $y = -\log x$, we get the result.

13. (Problem 2) The **Black-Scholes** equation from finance theory is

$$\frac{\partial V}{\partial t} + rs\frac{\partial V}{\partial s} + \frac{\sigma^2 s^2}{2}\frac{\partial^2 V}{\partial s^2} - rV = 0, 0 < t < T, \tag{2}$$

subject to the "final" boundary condition $V(s,T) = F(s)$. An appropriate change of variables reduces this to the equation in Problem 1. Alternatively, the substitution $V(s,t) = e^{ax+b\tau}U(x,\tau)$ where $x = \log s, \tau = \frac{\sigma^2}{2}(T-t), a = \frac{1}{2} - \frac{r}{\sigma^2}$, and $b = -\left(\frac{1}{2} + \frac{r}{\sigma^2}\right)^2$ reduces (2) to the one-dimensional heat equation with the initial condition $U(x,0) = e^{-ax}F(e^x)$. Thus a solution to the Black-Scholes equation is

$$V(s,t) = \frac{e^{-r(T-t)}}{\sqrt{2\pi\sigma^2(T-t)}}\int_0^\infty e^{-\frac{(\log(s/s^*)+(r-\sigma^2/2)(T-t))^2}{2\sigma^2(T-t)}}F(s^*)\,\frac{ds^*}{s^*}.$$

Solution. Multiplying the equation by e^{-rt}. Define $V_1 = e^{-rt}V$. Then the equation becomes

$$\frac{\partial V_1}{\partial t} + rs\frac{\partial V_1}{\partial s} + \frac{\sigma^2 s^2}{2}\frac{\partial^2 V_1}{\partial s^2} = 0.$$

Let $t_1 = -(\sigma^2/2)t$. Then $\frac{\partial V_1}{\partial t} = \frac{\partial V_1}{\partial t_1}(-\sigma^2/2)$, and upon dividing by $-\sigma^2/2$ on both sides we get

$$\frac{\partial V_1}{\partial t_1} = \frac{2r}{\sigma^2}s\frac{\partial V_1}{\partial s} + s^2\frac{\partial^2 V_1}{\partial s^2}.$$

This is the equation in Problem 1.
For the alternative way, compute $\frac{\partial V}{\partial t}, \frac{\partial V}{\partial s}$ and $\frac{\partial^2 V}{\partial s^2}$, replacing $1/s$ by e^x and canceling out $e^{ax+b\tau}$, we get

$$-\frac{\sigma^2}{2}(bU+\frac{\partial U}{\partial \tau})+r(aU+\frac{\partial U}{\partial x})+\frac{\sigma^2}{2}(a(a-1)U+(2a-1)\frac{\partial U}{\partial x}+\frac{\partial^2 U}{\partial x^2})-rU = 0.$$

Since $r + \frac{\sigma^2}{2}(2a-1) = 0, -\frac{\sigma^2}{2}b + ra + \frac{\sigma^2}{2}a(a-1) - r = 0$, the coefficients of U and $\frac{\partial U}{\partial x}$ are zero. Upon canceling out $\frac{\sigma^2}{2}$, we get the heat equation $\frac{\partial U}{\partial \tau} = \frac{\partial^2 U}{\partial x^2}$. Since $F(e^x) = F(s) = V(s,T) = e^{ax}U(x,0)$, the initial condition for the heat equation is $U(x,0) = e^{-ax}F(e^x)$. From the formula for solution of heat equation, we get

$$U(x,\tau) = \frac{1}{\sqrt{4\pi\tau}}\int_{-\infty}^{\infty} e^{-ay}F(e^y)e^{-(x-y)^2/(4\tau)}\,dy.$$

Making a change of variable $y = \log s^*$ and substitute $x = \log s$, we find

$$U(\log s,\tau) = \frac{1}{\sqrt{4\pi\tau}}\int_0^\infty e^{-a\log s^*}F(s^*)e^{-(\log(s/s^*)^2/(4\tau)}\,\frac{ds^*}{s^*}.$$

and

$$V(s,t) = \frac{1}{\sqrt{4\pi\tau}} \int_0^\infty e^{a\log s + b\tau} e^{-a\log s^*} F(s^*) e^{-(\log(s/s^*)^2/(4\tau)} \frac{ds^*}{s^*}.$$

Adding up the exponents of e, we get

$$\begin{aligned} & a\log(s/s^*) + b\tau - (\log(s/s^*)^2/(4\tau) \\ = & -\frac{1}{4\tau}[\log^2(s/s^*) - 4a\tau\log(s/s^*) - 4b\tau^2] \\ = & -\frac{1}{4\tau}(\log(s/s^*) - 2a\tau)^2 + (a^2+b)\tau \end{aligned}$$

Since $4\tau = 2\sigma^2(T-t)$, $-2a\tau = (r - \frac{\sigma^2}{2})(T-t)$ and $(a^2+b)\tau = -r(T-t)$, we are done.

14. (Problem 4) If g is a smooth function on $\mathbb{R}$, define the formal power series

$$u(x,t) = \sum_{n=0}^\infty g^{(n)}(t) \frac{x^{2n}}{(2n)!} \tag{3}$$

(a) Check formally that u solves the heat equation.
(b) For $a > 0$, consider the function defined by

$$g(t) = \begin{cases} e^{-t^{-a}} & \text{if } t > 0 \\ 0 & \text{if } t \le 0. \end{cases}$$

One can show that there exists $0 < \theta < 1$ depending on a so that

$$|g^{(k)}(t)| \le \frac{k!}{(\theta t)^k} e^{-\frac{1}{2}t^{-a}} \text{ for } t > 0.$$

(c) As a result, for each x and t the series (3) converges; u solves the heat equation; u vanishes for $t = 0$; and u satisfies the estimate $|u(x,t)| \le Ce^{c|x|^{2a/(a-1)}}$ for some constants $C, c > 0$.
(d) Conclude that for every $\epsilon > 0$ there exists a nonzero solution to the heat equation which is continuous for $x \in \mathbb{R}$ and $t \ge 0$, which satisfies $u(x,0) = 0$ and $|u(x,t)| \le Ce^{c|x|^{2+\epsilon}}$.

15. (Problem 7) The **Hermite functions** $h_k(x)$ are defined by the generating identity

$$\sum_{k=0}^\infty h_k(x) \frac{t^k}{k!} = e^{-(x^2/2 - 2tx + t^2)}. \tag{4}$$

(a) Show that an alternate definition of the Hermite functions is given by the formula

$$h_k(x) = (-1)^k e^{x^2/2} \left(\frac{d}{dx}\right)^k e^{-x^2}. \tag{5}$$

Conclude from the above expression that each $h_k(x)$ is of the form $P_k(x)e^{-x^2/2}$, where P_k is a polynomial of degree k. In particular, the Hermite functions belong to the Schwartz space and $h_0(x) = e^{-x^2/2}, h_1(x) = 2xe^{-x^2/2}$.
(b) Prove that the family $\{h_k\}_{k=0}^{\infty}$ is complete in the sense that if f is a Schwartz function, and

$$(f, h_k) = \int_{-\infty}^{\infty} f(x)h_k(x)\, dx = 0 \text{ for all } k \geq 0,$$

then $f = 0$.
(c) Define $h_k^*(x) = h_k((2\pi)^{1/2}x)$. Then

$$\widehat{h_k^*}(\xi) = (-i)^k h_k^*(\xi).$$

Therefore, each h_k^* is an eigenfunction for the Fourier transform.
(d) Show that h_k is an eigenfunction for the operator defined in Exercise 23, and in fact, prove that

$$Lh_k = (2k+1)h_k.$$

In particular, we conclude that the functions h_k are mutually orthogonal for the L^2 inner product on the Schwartz space.
(e) Finally, show that $\int_{-\infty}^{\infty} h_k(x)^2\, dx = \pi^{1/2}2^k k!$.
Solution.
(a) Note that $e^{-(x^2/2-2tx+t^2)} = e^{x^2/2}e^{-(t-x)^2}$. Note that

$$\left(\frac{d}{dx}\right)^k e^{-x^2} = p_k(x)e^{-x^2}$$

for some degree k polynomial $p_k(x)$ which is odd if k is odd, and even if k is even. Therefore the k partial derivative of $e^{-(t-x)^2}$ w.r.t. t, evaluated at $t = 0$ is $p_k(-x)e^{-x^2} = (-1)^k p_k(x)e^{-x^2}$. The formula then follows from Taylor expansion.
(b) Under the hypothesis, we have by equation (4)

$$\int_{-\infty}^{\infty} f(x)e^{-(x^2/2-2tx+t^2)}\, dx = e^{-t^2}\int_{-\infty}^{\infty} f(x)e^{-x^2/2+2tx}\, dx$$

for all t. By Exercise 8, $f = 0$.
(c) From equation (4), we get

$$\begin{aligned} & \sum_{k=0}^{\infty} h_k(\sqrt{2\pi}x)\frac{t^k}{k!} \\ = \quad & \exp(-(\pi x^2 - 2t\sqrt{2\pi}x + t^2)) \\ = \quad & \exp(t^2)\exp(-\pi(x - \sqrt{2/\pi}t)^2) \end{aligned}$$

Taking Fourier transform of both sides, we get

$$\begin{aligned}\sum_{k=0}^{\infty}\widehat{h_k^*}(\xi)\frac{t^k}{k!} &= \exp(t^2)\exp(-2\sqrt{2\pi}it\xi)\exp(-\pi\xi^2)\\ &= \exp(-(\pi\xi^2-2(-it)\sqrt{2\pi}\xi+(-it)^2))\\ &= \sum_{k=0}^{\infty}h_k^*(\xi)\frac{(-it)^k}{k!}\end{aligned}$$

Comparing both sides, we get the result.
(d) From the RHS of (4), we have

$$\begin{aligned}&\sum_{k=0}^{\infty}L(h_k(x))\frac{t^k}{k!} = L(e^{-(x^2/2-2tx+t^2)})\\ =&\ -\frac{d^2}{dx^2}e^{-(x^2/2-2tx+t^2)}+x^2e^{-(x^2/2-2tx+t^2)}\\ =&\ -\frac{d}{dx}e^{-(x^2/2-2tx+t^2)}(-x+2t)+x^2e^{-(x^2/2-2tx+t^2)}\\ =&\ -e^{-(x^2/2-2tx+t^2)}((-x+2t)^2-1)+x^2e^{-(x^2/2-2tx+t^2)}\\ =&\ e^{-(x^2/2-2tx+t^2)}(1-4t^2+4tx)\\ =&\ e^{-(x^2/2-2tx+t^2)}+(-4t^2+4tx)e^{-(x^2/2-2tx+t^2)}\\ =&\ \sum_{k=0}^{\infty}h_k(x)\frac{t^k}{k!}+2t\frac{d}{dt}e^{-(x^2/2-2tx+t^2)}\\ =&\ \sum_{k=0}^{\infty}h_k(x)\frac{t^k}{k!}+\sum_{k=0}^{\infty}2kh_k(x)\frac{t^k}{k!}\\ =&\ \sum_{k=0}^{\infty}(2k+1)h_k(x)\frac{t^k}{k!}\end{aligned}$$

from which we get $Lh_k = (2k+1)h_k$. By Exercise 23, $L = I + A^*A$, so it is Hermitian. Therefore the functions h_k are mutually orthogonal.
(e) Multiply both sides of equation (4) by $h_k(x)$ and integrate. On the LHS, using the orthogonality property, we get

$$\frac{t^k}{k!}\int_{-\infty}^{\infty}[h_k(x)]^2\,dx.$$

On the RHS, using equation (5) and integration by parts k times, we get

$$(-1)^k(-1)^k\int_{-\infty}^{\infty}e^{-x^2}\left(\frac{d}{dx}\right)^k e^{2tx-t^2} = (2t)^k\int_{-\infty}^{\infty}e^{-(x-t)^2}\,dx = 2^kt^k\sqrt{\pi}.$$

Therefore result.

Chapter 6 The Fourier Transform on $\mathbb{R}^d$

1. (Exercise 1) Suppose that R is a rotation in the plane $\mathbb{R}^2$, and let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

denote its matrix with respect to the standard basis vectors $e_1 = (1, 0)$ and $e_2 = (0, 1)$.
(a) Write the conditions $R^t = R^{-1}$ and $\det(R) = \pm 1$ in terms of equations in a, b, c, d.
(b) Show that there exists $\phi \in \mathbb{R}$ such that $a + ib = e^{i\phi}$.
(c) Conclude that if R is proper, then it can be expressed as $z \mapsto ze^{-i\phi}$, and if R is improper, then it takes the form $z \mapsto \bar{z}e^{i\phi}$, where $\bar{z} = x - iy$.
Solution.
(a) Write $D = \det(R)$. Then $D = \pm 1$ and the conditions $R^t = R^{-1}$ implies

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} d/D & -b/D \\ -c/D & a/D \end{pmatrix},$$

i.e. we have $D = ad - bc = \pm 1, a = dD, b = -cD, c = -bD, d = aD$.
(b) From (a) we have $D = ad - bc = a^2D - (-b^2D) = (a^2 + b^2)D$ which implies that $a^2 + b^2 = 1$. Thus (b) follows.
(c). Suppose $D = 1$. Then $ze^{-i\phi} = (x + iy)(a - ib) = (ax + by) + i(-bx + ay) = (ax + by) + i(cx + dy)$.
If $D = -1$, then $\bar{z}e^{i\phi} = (x - iy)(a + ib) = ax + by + i(bx - ay) = ax + by + i(cx + dy)$. This proves (c).

2. (Exercise 2) Suppose $R : \mathbb{R}^3 \to \mathbb{R}^3$ is a proper rotation.
(a) Show that $p(t) = \det(R - tI)$ is a polynomial of degree 3, and prove that there exists $\gamma \in S^2$ (where S^2 denotes the unit sphere in $\mathbb{R}^3$) with

$$R(\gamma) = \gamma.$$

(b) If $\mathcal{P}$ denotes the plane perpendicular to γ and passing through the origin, show that

$$R : \mathcal{P} \to \mathcal{P},$$

and that this linear map is a rotation.
Solution.
(a). $p(t)$ is clearly a polynomial of degree 3. (The coefficient of t^3 is -1.). $p(0) = \det(R) > 0$. Since $\lim_{t\to\infty} p(t) = -\infty$, we see that there exists $\lambda > 0$ such that $p(\lambda) = 0$. So $R - \lambda I$ is singular, and its kernel is nontrivial.

Chapter 7 Finite Fourier Analysis

1. (Exercise 1.) Let f be a function on the circle. For each $N \geq 1$ the discrete Fourier coefficients of f are defined by

$$a_N(n) = \frac{1}{N}\sum_{k=0}^{N-1} f(e^{2\pi ik/N})e^{-2\pi ikn/N} = \frac{1}{N}\sum_{k=0}^{N-1} f(e^{2\pi ik/N})\overline{e_n(k)}, \text{ for } n \in \mathbb{Z}.$$

We also let

$$a(n) = \int_0^1 f(e^{2\pi ix})e^{-2\pi inx}\,dx$$

denote the ordinary Fourier coefficients of f.
(a) Show that $a_N(n) = a_N(n+N)$.
(b) Prove that if f is continuous, then $a_N(n) \to a(n)$ as $N \to \infty$.
Proof.
(a) is easy since $e^{-2\pi i} = 1$.
(b) Note that $a_N(n)$ is the Riemann sum of the integral $a(n)$ with partition: $k/N, k = 0, \cdots, N$, and choice of points: $k/N, k = 0, \cdots, N-1$.

2. (Exercise 2) If f is a C^1 function on the circle, prove that $|a_N(n)| \leq c/|n|$ whenever $0 < |n| \leq N/2$.
Proof. Let l be an integer. Then

$$\begin{aligned}
& a_N(n)[1 - e^{2\pi iln/N}] \\
= \;& \frac{1}{N}\sum_{k=0}^{N-1} f(e^{2\pi ik/N})e^{-2\pi ikn/N} - \frac{1}{N}\sum_{k=0}^{N-1} f(e^{2\pi ik/N})e^{-2\pi i(k-l)n/N} \\
= \;& \frac{1}{N}\sum_{k=0}^{N-1} f(e^{2\pi ik/N})e^{-2\pi ikn/N} - \frac{1}{N}\sum_{k=-l}^{N-l-1} f(e^{2\pi i(k+l)/N})e^{-2\pi ikn/N} \\
= \;& \frac{1}{N}\sum_{k=0}^{N-1} f(e^{2\pi ik/N})e^{-2\pi ikn/N} - \frac{1}{N}\sum_{k=0}^{N-1} f(e^{2\pi i(k+l)/N})e^{-2\pi ikn/N} \\
= \;& \frac{1}{N}\sum_{k=0}^{N-1} [f(e^{2\pi ik/N}) - f(e^{2\pi i(k+l)/N})]e^{-2\pi ikn/N}
\end{aligned}$$

$f \in C^1$ implies f is Lipschitz. Let M be its Lipschitz constant. Then from the above equations,

$$|a_N(n)||1 - e^{2\pi iln/N}| \leq M|1 - e^{2\pi il/N}|$$

Let l be the integer closest to $\frac{N}{2n}$ (in case of ambiguity, choose either integer). Then $|l - N/2n| \leq 1/2$, from which we get

$$\left|\frac{ln}{N} - \frac{1}{2}\right| \leq \frac{|n|}{2N} \leq \frac{1}{4}.$$

Thus

$$|1 - e^{2\pi il/N}| \leq |2\pi l/N| = |2\pi ln/N|/|n| \leq \frac{3\pi}{2|n|}.$$

On the other hand, since

$$\left|\frac{ln}{N} - \frac{1}{2}\right| \leq \frac{1}{4}$$

implies

$$\frac{\pi}{2} \leq \frac{2\pi ln}{N} \leq \frac{3\pi}{2},$$

we have

$$|1 - e^{2\pi i ln/N}| \geq \sqrt{2}.$$

Therefore for $0 < |n| \leq N/2$,

$$|a_N(n)| \leq \frac{3\pi M}{2\sqrt{2}|n|}.$$

3. (Exercise 3) By a similar method, show that if f is a C^2 function on the circle, then

$$|a_N(n)| \leq \frac{c}{|n|^2}, \text{ whenever } 0 < |n| \leq N/2.$$

As a result, prove the inversion formula for $f \in C^2$,

$$f(e^{2\pi ix}) = \sum_{n=-\infty}^{\infty} a(n) e^{2\pi inx}$$

from its finite version.

Solution.

We have

$$\begin{aligned}
& a_N(n)[2 - e^{2\pi iln/N} - e^{-2\pi iln/N}] \\
= \; & \frac{1}{N} \sum_{k=0}^{N-1} [2f(e^{2\pi ik/N}) - f(e^{2\pi i(k+l)/N}) - f(e^{2\pi i(k-l)/N})] e^{-2\pi ikn/N} \\
\leq \; & \frac{1}{N} \sum_{k=0}^{N-1} M |e^{2\pi i(k+l)/N} - e^{2\pi ik/N}|^2 \\
= \; & M|e^{2\pi il/N} - 1|^2
\end{aligned}$$

Then do as we did in Exercise 2 to choose l to reach the conclusion.

For the second part, let N be odd, then for a fixed k,

$$\begin{aligned}
& \sum_{|n|<N/2} a_N(n) e^{2\pi ikn/N} \\
= \; & \frac{1}{N} \sum_{|n|<N/2} \sum_{j=0}^{N-1} f(e^{2\pi ij/N}) e^{-2\pi ijn/N} e^{2\pi ikn/N} \\
= \; & \frac{1}{N} \sum_{j=0}^{N-1} f(e^{2\pi ij/N}) \sum_{|n|<N/2} e^{-2\pi ijn/N} e^{2\pi ikn/N} \\
= \; & f(e^{2\pi ik/N})
\end{aligned}$$

where the last equality follows because $\sum_{|n|<N/2} e^{-2\pi ijn/N} e^{2\pi ikn/N} = 0$ if $j \neq k$ and equal to N if $j = k$. Note that I have not used the condition $|a_N(n)| \leq c/|n|^2$. (to be continued.....).

4. (Exercise 4) Let e be a character on $G = \mathbb{Z}(N)$, the additive group of integers modulo N. Show that there exists a unique $0 \le \ell \le N-1$ so that

$$e(k) = e_\ell(k) = e^{2\pi i \ell k/N} \text{ for all } k \in \mathbb{Z}(N).$$

Conversely, every function of this type is a character on $\mathbb{Z}(N)$. Deduce that $e_\ell \mapsto \ell$ defines an isomorphism from $\hat{G}$ to G.
Proof.
Let the N roots of unit be $1, \zeta, \zeta^2, cdots, \zeta^{N-1}$, where $\zeta = e^{2\pi i/N}$.
Let e be a character in $\mathbb{Z}(N)$. We have $1 = e(0) = e(1+1+\cdots+1) = e(1)^N$, so $e(1) = \zeta^l$ for some $0 \le l \le N-1$. Then for each $n \in \mathbb{Z}(N)$,

$$e(n) = e(1+1+\cdots+1) = e(1)^n = \zeta^{ln}$$

proving that $e = e_l$. $e_l e_m = e_{l+m} \mapsto l + m$.

5. (Exercise 5) Show that all characters on S^1 are given by

$$e_n(x) = e^{inx} \text{ with } n \in \mathbb{Z},$$

and check that $e_n \mapsto n$ defines an isomorphism from $\hat{S^1}$ to $\mathbb{Z}$.
Solution.
Since $F(0) \neq 0$ and F is continuous, there exists $\delta > 0$ such that $c = \int_0^\delta F(y)\,dy \neq 0$. Then

$$cF(x) = \int_0^\delta F(y)F(x)\,dy = \int_0^\delta F(y+x)\,dy = \int_x^{\delta+x} F(y)\,dy$$

for all x. Differentiating both sides,

$$cF'(x) = F(\delta + x) - F(x) = F(\delta)F(x) - F(x) = (F(\delta) - 1)F(x).$$

It follows that $F(x) = e^{Ax}$ for some constant A. Since $\pi + \pi = 0$ in S^1, we have $1 = F(0) = F(\pi + \pi) = e^{2\pi A}$. Therefore $A = in$ for some integer n.

6. (Exercise 6) Prove that all characters on $\mathbb{R}$ take the form

$$e_\xi(x) = e^{2\pi \xi x} \text{ woth } \xi \in \mathbb{R},$$

and that $e_\xi \mapsto \xi$ defines an isomorphism from $\hat{\mathbb{R}}$ to $\mathbb{R}$.
Solution.
Using the same argument in Exercise 5, $F(x) = e^{Ax}$ for some complex number A. Since $|F(x)| = 1$, A must be pure imaginary. So $A = 2\pi i\xi$ for some real number ξ. $e_\xi e_\eta = e_{\xi+\eta} \mapsto \xi + \eta$.

7. (Exercise 8) Suppose that $P(x) = \sum_{n=1}^N a_n e^{2\pi inx}$.
(a) Show by using the Parseval identities for the circle and $\mathbb{Z}(N)$, that

$$\int_0^1 |P(x)|^2\,dx = \frac{1}{N}\sum_{j=1}^N |P(j/N)|^2.$$

(b) Prove the reconstruction formula

$$P(x) = \sum_{j=1}^{N} P(j/N)K(x - (j/N))$$

where

$$K(x) = \frac{e^{2\pi ix}}{N}\frac{1 - e^{2\pi iNx}}{1 - e^{2\pi ix}} = \frac{1}{N}(e^{2\pi ix} + e^{2\pi i2x} + \cdots + e^{2\pi iNx}).$$

Observe that P is completely determined by the values $P(j/N)$ for $1 \leq j \leq N$. Note also that $K(0) = 1$, and $K(j/N) = 0$ whenever j is not congruent to 0 modulo N.

Solution.

Considering $P(x)$ as a 1-periodic function, we have by Parseval identity: $\int_0^1 |P(x)|^2\, dx = \sum_{n=1}^N |a_n|^2$. On the other hand, considering $f(j) = P(j/N) = \sum_{n=1}^N a_n e^{2\pi inj/N}$ as a function on $\mathbb{Z}(N)$, Parseval identity yields $\sum_{n=1}^N |a_n|^2 = \|f\|^2 = \frac{1}{N}\sum_{j=1}^N |f(j)|^2 = \frac{1}{N}\sum_{j=1}^N |P(j/N)|^2$.

8. (Exercise 12) Suppose that G is a finite abelian group and $e : G \to \mathbb{C}$ is a function that satisfies $e(x + y) = e(x)e(y)$ for all $x, y \in G$. Prove that either e is identically 0, or e never vanishes. In the second case, show that for each x, $e(x) = e^{2\pi ir(x)}$ for some $r = r(x) \in \mathbb{Q}$ of the form p/q, where $q = |G|$.

   Proof.

   Since $e(0) = e(0 + 0) = e(0)e(0)$, we see that either $e(0) = 0$ or $e(0) = 1$. In the first case, $e(x) = e(x + 0) = e(x)e(0) = 0$ for all $x \in G$. In the second case, $1 = e(0) = e(x + (-x)) = e(x)e(-x)$, proving that $e(x) \neq 0$ and $e(-x) = e(x)^{-1}$.

   Let $N = |G|$. For any $x \in G$, $Nx = x + x + \cdots + x = 0$, so $1 = e(0) = e(x)^N$, proving that $e(x)$ is equal to ζ^k for some $k = 0, \cdots, N - 1$, where $\zeta = e^{2\pi i/N}$.

9. (Exercise 13) In analogy with ordinary Fourier series, one may interpret finite Fourier expansions using convolutions as follows. Suppose G is a finite abelian group, 1_G its unit, and V the vector space of complex-valued functions on G.

   (a) The convolution of two functions f and g in V is defined for each $a \in G$ by

   $$(f * g)(a) = \frac{1}{|G|}\sum_{b\in G} f(b)g(a \cdot b^{-1}).$$

   Show that for all $e \in \hat{G}$ one has $\widehat{(f * g)}(e) = \hat{f}(e)\hat{g}(e)$.

   (b) Use Theorem 2.5 to show that

   $$\sum_{e\in\hat{G}} e(c) = 0 \text{ whenever } c \in G \text{ and } c \neq 1_G.$$

(c) As a result of (b), show that the Fourier series $Sf(a) = \sum_{e \in \hat{G}} \hat{f}(e)e(a)$ of a function $f \in V$ take the form

$$Sf = f * D,$$

where D is defined by

$$D(c) = \sum_{e \in \hat{G}} e(c) = \begin{cases} |G| & \text{if } c = 1_G, \\ 0 & \text{otherwise.} \end{cases}$$

Since $f * D = f$, we recover the fact that $Sf = f$. Loosely speaking, D corresponds to a "Dirac delta function"; it has unit mass

$$\frac{1}{|G|} \sum_{c \in G} D(c) = 1,$$

and (4) says that this mass is concentrated at the unit element in G. Thus D has the same interpretation as the "limit" of a family of good kernels. (See Section 4, Chapter 2.)
Note. The function D reappears in the next chapter as $\delta_1(n)$.
Solution.
Let $N = |G|$.

$$\begin{aligned} & \widehat{f * g}(e) \\ = \ & \frac{1}{N} \sum_{x \in G} (f * g)(x) \overline{e(x)} \\ = \ & \frac{1}{N^2} \sum_{x \in G} \sum_{b \in G} f(b) g(xb^{-1}) \overline{e(x)} \\ = \ & \frac{1}{N^2} \sum_{b \in G} f(b) \sum_{x \in G} g(xb^{-1}) \overline{e(x)} \\ = \ & \frac{1}{N^2} \sum_{b \in G} f(b) \sum_{y \in G} g(y) \overline{e(by)} \\ = \ & \frac{1}{N^2} \sum_{b \in G} f(b) \overline{e(b)} \sum_{y \in G} g(y) \overline{e(y)} \\ = \ & \hat{f}(e) \hat{g}(e) \end{aligned}$$

(b). If $c \neq 1_G$, then there exists $e' \in \hat{G}$ such that $e'(c) \neq 1$.[This can be proved in two ways. Suppose $e(c) = 1$ for all $e \in \hat{G}$. Let H be the cyclic subgroup generated by c. Since $c \neq 1_G$, we have $|H| > 1$ so $|G/H| < |G|$. Each $e \in \hat{G}$ induces a character in G/H, and different e's induce different characters. This is impossible since there are exactly $|G/H|$ characters on G/H.
Another proof is to use the fact that G is isomorphic to a direct product of cyclic groups $G1 \times G2 \times \cdots \times Gn$ where $|G1| | |G2| | \cdots | |Gn|$. Let for each i

between 1 and n g_i be a generator of G_i, then $c = \sum_i (a_i * g_i)$. Since $c \neq 0$ not all integers a_i can be zero, let us say it's k. Now the map defined by $g_i \to 1$ for $i \neq k$ and $g_k \to z$, with $z = e^{2\pi i/N}$ where $N = |G_k|$, defines a character that is not 1 on c (it is z^{a_k}).]
Now back to the proof. We have $e' \sum_{e \in \hat{G}} e = \sum_{e \in \hat{G}} e'e = \sum_{e \in \hat{G}} e$ since $\hat{G}$ is a group. So $e'(c) \sum_{e \in \hat{G}} e(c) = \sum_{e \in \hat{G}} e(c)$. Since $e'(c) \neq 1$, we get $\sum_{e \in \hat{G}} e(c) = 0$.

$$
\begin{aligned}
Sf(a) &= \sum_{e \in \hat{G}} \hat{f}(e) e(a) \\
&= \sum_{e \in \hat{G}} \frac{1}{N} \sum_{x \in G} f(x) \overline{e(x)} e(a) \\
&= \frac{1}{N} \sum_{x \in G} f(x) \sum_{e \in \hat{G}} e(a) e(x)^{-1} \\
&= \frac{1}{N} \sum_{x \in G} f(x) \sum_{e \in \hat{G}} e(a) e(x^{-1}) \\
&= \frac{1}{N} \sum_{x \in G} f(x) \sum_{e \in \hat{G}} e(ax^{-1}) \\
&= (f * D)(a)
\end{aligned}
$$