Notes on the Triangulation conjecture

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Abstract

This notes is about the development of The triangulation conjecture, mainly of the work by Galewski& Stern in 1980s, and the work of Manolescu, published in 2013. The content is mainly referred to a lecture notes of Bourbaki, and the Lecture notes by manolescu. Besides, I added some of my own understanding, and added more topics like the topology of 4-dimensional topology and Poincaré conjecture.

Keywords: Triangulation conjecture, Homology theory, Characteristic classes, Geometric topology

Remark: this is an unfinished translation as English version, while the Chinese version can be viewed by URL

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1 Introduction

The Triangulation Conjecture is one of the most important problems in the field of geometric topology since the last century, it connected a mainline of researching, where new methods invented and the subject hit the prosperity.

One triangulable space(more precisely, the simplicially triangulated space), corresponding to one simplicial complex, which is a powerful subject to study topology. The simplicial complex is a space with combinatorial structures, can be also senn as a finite sets, with structures on subsets, regulating which $k < n$ dimensional sets is the "face" of n dimensional sets. Intuitively, the simplicial complex just like cubic, with vertices, edges, and faces determined. This viewpoint can help us to define some topological invariants directly.

The manifolds are another type of topological space with fine properties. They are locally homeomorphic to Euclidean spaces, i.e. have a very good local property. The manifolds are widespread, for example, spheres, Möbius bond, projective planes, etc. The local euclidean property enables us to do calculus on manifolds. However, this kind of local property brings almost no global information. If a manifold have smooth structure, then we can do calculus globally. The good news is, if the dimension of manifolds less or equal to 3, then it has one unique smooth structure. Therefore, some pure topological problems can be surprisingly solved by methods of calculus and differential functions, like 3-dimensional Poincaré solved by Perelman, using Ricci flow, a kind of Partial differential equations methods.

To triangulate a manifold is a powerful method to study the global properties. Like the Euler characteristic, is initially introduced by polyhedra, then generalized to high dimensional polyhera and tiangulable manifolds. We can prove that, for a given manifolds, it's Euler characteristic is independent of the ways of triangulation, i.e., it is a topological invariant. In this instance, we care more about the existence of triangulation. Once exist, the invariants are computed by the triangulation. Relative to the pure topological methods, this kind of computation is more easier.

With the help of triangulation on topological space, we can easily derive the "simplicial homology" on it. One more general homology is singular homology, which can be applied to all topological spaces, but hard to do concrete computation. The simplicial homology can only applied to simplicial complex, i.e. the trangulated space. However, besides the existence of triangulation, people were also caring about the uniqueness(in the sense of "have a same refining"). But some special topological spaces don't have a unique triangulation, others even can't be triangulated. Though the defect that can't be applied to all spaces, the simplicial homology still plays an important role, because of its intuitive definition and directive computation, which enable us to drive amounts of properties on "fine spaces".

When the existence of triangulation on every topological spaces had been proven wrong, people cared whether it is exist or unique on manifolds, which are more "regular" spaces. The ideal condition is, when a topological space was claimed to be a manifold, it was also a simplicial complex, then it obtains both good local properties and good global properties.

So people conjestured:

Conjecture 1.1. Every topological manifold is homeomorphic to a simplicial complex.