Notes on the Triangulation Conjecture

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Contents

In the end of this notes, I will introduce the Poincaré Conjecture as a supplement. To see more detailed explanation, see this survey .

1 Introduction

The Triangulation Conjecture has been one of the most important problems in the field of geometric topology since the last century, serving as a connecting thread for a major line of research.

A space that can be triangulated (or simplicially triangulated) corresponds to a simplicial complex. Simplicial complexes are powerful tools for studying topological problems. They are topological spaces with a combinatorial structure, or can be viewed as finite sets with a combinatorial structure between subsets, dictating which $(n-1)$-dimensional spaces are "faces" of which n-dimensional spaces. Intuitively, like a tetrahedron, it is a figure where relationships between faces, edges, and vertices are prescribed. This helps us directly define certain invariants.

Manifolds are another class of well-behaved topological spaces. They are locally homeomorphic to Euclidean space, meaning they have good local structure. Manifolds are ubiquitous, such as spheres, Möbius strips, projective spaces, etc. The property of being locally Euclidean gives us the opportunity to perform calculus on more general spaces. However, this local property yields almost no feedback on the global information of the manifold. If a manifold has a smooth structure, we can perform global calculus. The good news is that topological manifolds of dimension less than or equal to three possess a unique smooth structure. Therefore, purely topological problems like the 3-dimensional Poincaré Conjecture can be solved using tools from analysis and equations; Perelman solved the 3-dimensional Poincaré Conjecture using Ricci flow, a tool from partial differential equations.

Triangulating a manifold is a powerful method for studying its global properties. For instance, the Euler characteristic was initially derived from polyhedra, later generalized to higher-dimensional polyhedra, and then to triangulable manifolds. We can prove that for a given manifold, its Euler characteristic is independent of the method of triangulation; that is, it is a topological invariant. In this context, we are more concerned with whether a triangulation of a manifold exists: once it exists, we can calculate well-defined topological quantities using the triangulation structure. Compared to purely topological methods, such calculations are often simpler.

Furthermore, with a triangulation on a topological space, it is easy to calculate its simplicial homology. One widely applied homology theory is singular homology, which applies to all topological spaces, but specific calculations are often complex. Simplicial homology can only be used for topological spaces where a triangulation exists. Mathematicians are also concerned with whether triangulations are essentially unique (i.e., in the sense of having a common subdivision). Some special topological spaces have triangulations that are not essentially unique, or may not even possess a triangulation. Despite these defects, simplicial homology still holds an important place in topology. Because its definition is more intuitive and calculation more direct, it facilitates deriving conclusions on spaces with nice properties.

After the existence and uniqueness of triangulations on general topological spaces were disproven, mathematicians became concerned with the existence and uniqueness of triangulations on manifolds, which have better regularity. The ideal situation is that a topological space is simultaneously a manifold and a simplicial complex (i.e., the manifold can be triangulated), so that it possesses both good local properties and global properties.

Thus, mathematicians conjectured:

Conjecture 1 (Triangulation Conjecture). *A topological manifold is homeomorphic to a simplicial complex.*

This problem was first proposed by Kneser in 1926, and it was not until 2013 that Manolescu formally resolved this problem [Man15].

Historically, the resolution of this problem proceeded by dimension; in fact, the research methods for $n = 2, 3, 4, \geqslant 5$ are all different.

The cases for $n \leqslant 3$ were solved earliest, and encouragingly, the Triangulation Conjecture is correct in these cases.

When the case for $n \geqslant 4$ was highly open, mathematicians settled for the next best thing. Instead of limiting themselves to the most general topological manifolds (the topological category), they studied the behavior of the Triangulation Conjecture in the P.L. (Piecewise Linear) category and the smooth category, which have better regularity.

The Triangulation Conjecture is correct in all dimensions within the smooth category. Manifolds in the P.L. category possess combinatorial triangulations, which are stronger than simplicial triangulations, so the conjecture is also correct there. In the topological category, when $n \geqslant 4$, there are examples where P.L. structures do not exist. The Kirby-Siebenmann class, a tool for studying the existence of P.L. structures, can be composed with the Bockstein homomorphism to study simplicial triangulations.

When mathematicians constructed 4-dimensional non-triangulable manifolds, thereby disproving the 4-dimensional Triangulation Conjecture, they dared not assert the correctness of the conclusion for $\geqslant 5$ due to the exotic behavior of dimension 4 (e.g., $\mathbb{R}^n$ has a unique smooth structure for $n \neq 4$, while for $n = 4$ there are uncountably many smooth structures). The Triangulation Conjecture for $n \geqslant 5$ remained unresolved for a long time following the progress of Galewski & Stern in the 1980s, until Manolescu constructed Pin(2)-equivariant Seiberg-Witten Floer homology in the 21st century. This revealed more symmetry and disproved the conjecture for $n \geqslant 5$.

2 Triangulation and Combinatorial Triangulation

First, we give the definition of manifolds in different categories and the definition of simplicial complexes, and then describe the concepts of triangulation and combinatorial triangulation of manifolds.

Definition 1. *Let M^n be a Hausdorff space with a countable topological basis. If for all $x \in X$, there exists a neighborhood U of x and a homeomorphism $\phi : U \to \mathbb{R}^n$, then M is called an n-dimensional topological manifold, and (U, ϕ) is a coordinate chart at point x.*

Definition 2. *If M^n is a topological manifold, a collection of coordinate charts $\{U_i, \phi_i\}_{i\in I}$ such that $\bigcup_i U_i = M$ is called an atlas of M. The continuous maps $\phi_i \circ \phi_j^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ are called transition maps. If all transition maps are smooth, then M is called a smooth manifold, and its maximal atlas is called a smooth structure on M (referring here to the equivalence class under smooth homeomorphism).*

Definition 3. *If M^n is a topological manifold and the transition maps are P.L. (Piecewise Linear) maps, then M is called a P.L. manifold (Piecewise Linear manifold), and its maximal atlas is called a P.L. structure on M.*

Definition 4. (V, S) *is called an abstract simplicial complex if* V *is a set of vertices and* $S \subset \mathcal{P}(V)$*, such that if* $\sigma \in S$ *and* $\tau \subset \sigma$*, then* $\tau \in S$*. For all* $\sigma \in S$ *with* $|\sigma| = d$*, if we replace* σ *with a d-dimensional simplex* Δ^d*, this is called the geometric realization of* (V, S)*, denoted as* K*, and referred to as a simplicial complex.* $|K| = \bigsqcup_{\Delta^d \in K} \Delta^d$ *is called the underlying space of* K*.*

Hereafter, we do not distinguish between the abstract simplicial complex and its corresponding simplicial complex, denoting it as $K = (V, S)$.

The star of a simplex $\tau \in S$ is

$$st(\tau) = \{\sigma \in S | \tau \subset \sigma\}$$

The closure of $S' \subset S$ is

$$Cl(S') = \{\tau \in S | \tau \subset \sigma \in S'\}$$

The link of a simplex $\tau \in S$ is

$$lk(\tau) = \{\sigma \in Cl(st(\tau)) | \tau \cap \sigma = \varnothing\}$$

A triangulation that only requires the topological manifold to be homeomorphic to a simplicial complex is called a simplicial triangulation:

Definition 5 (Simplicial Triangulation / Triangulation)**.** *An n-dimensional topological manifold* M *is triangulable if and only if there exists a homeomorphism* $\varphi : M^n \to |K|$*, where* K *is a simplicial complex.*

To better study the triangulation of manifolds, we need to introduce the concept of combinatorial triangulation. In this paper, if "triangulation" is used without the adjective "combinatorial," it refers to simplicial triangulation.

If we require more "combinatorial structure" on the simplicial complex, i.e., requiring it to also be "locally Euclidean," we have combinatorial triangulation:

Definition 6 (Combinatorial Triangulation)**.** *An n-dimensional topological manifold* M *is combinatorially triangulable if and only if there exists a homeomorphism* $\varphi : M^n \to |K|$*, where* K *is a simplicial complex, and for all* $A \in K$*,* $lk(A)$ *is P.L. homeomorphic to the standard sphere.*

In fact, a manifold has a combinatorial triangulation if and only if it is a P.L. manifold.

Combinatorial triangulation is a stronger type of triangulation than simplicial triangulation, but its regularity is weaker than differential conditions, while its properties are closer to simplicial triangulation. One can study simplicial triangulation on the basis of combinatorial triangulation.

An example of a space with a simplicial triangulation but no combinatorial triangulation is the double suspension of the Poincaré homology sphere P^3, $\Sigma^2 P$. By the Double Suspension Theorem, $\Sigma^2 P \cong S^5$, so there exists a triangulation on $\Sigma^2 P$; however, the link at its cone points is ΣP, which is not P.L. homeomorphic to S^4 (it is not even a manifold), so it has no P.L. structure, i.e., it is not combinatorially triangulable.

In fact, there is an obstruction to the existence of a combinatorial triangulation on a topological manifold M—the Kirby-Siebenmann class $\Delta(M)$. By composing the Bockstein homomorphism δ once in the cohomology sequence, we obtain the obstruction to the existence of a simplicial triangulation, $\delta(\Delta(M))$.

Regarding the uniqueness of triangulation, there was an important conjecture: the Hauptvermutung (Main Conjecture of Combinatorial Topology), see [ARC96] for details.

Conjecture 2 (Hauptvermutung for Polyhedra (Topological Spaces))**.** *Any two triangulations of a triangulable topological space have combinatorially equivalent subdivisions.*

Or equivalently characterized as: If two simplicial complexes are homeomorphic, then they are P.L. homeomorphic, and the homeomorphism is homotopic to the P.L. homeomorphism between them.

The Hauptvermutung for polyhedra was disproven by Milnor in 1961. Consequently, we are curious whether triangulations are unique on spaces with stronger regularity, such as manifolds:

Conjecture 3 (Hauptvermutung for Manifolds)**.** *If two P.L. manifolds M^n and N^n are homeomorphic, then they are P.L. homeomorphic, and the homeomorphism is homotopic to the P.L. homeomorphism between them.*

The Hauptvermutung for manifolds was eventually disproven as well.

If two triangulations of a triangulable manifold have combinatorially equivalent subdivisions, then these two triangulations are said to be essentially unique.

We will introduce later that for smooth manifolds, all their triangulations are essentially unique.

An example of a manifold with two distinct triangulation (simplicial triangulation) structures is the double suspension of the Poincaré homology sphere P^3. As mentioned above, $\Sigma^2 P = S^5$, and S^5 has a standard P.L. structure obtained from the standard P.L. structure of S^3 by double suspension. However, since $S^3 \ncong P^3$, the P.L. structure on P^3, after double suspension, yields a triangulation structure on S^5 (as mentioned before, this cannot be a P.L. structure) via the Double Suspension Theorem. This triangulation is not combinatorially equivalent to the standard P.L. structure of S^5, otherwise it would contradict $S^3 \ncong P^3$. Thus, S^5 has two inequivalent triangulation structures: one induced by the double suspension of S^3, and one induced by the double suspension of P^3. This does not contradict the correctness of the P.L. Poincaré Conjecture for $n = 5$, which states that there is only a unique P.L. structure on S^5.

3 History of the Development of the Triangulation Conjecture

We now consider the problem of the Triangulation Conjecture to be completely solved by Manolescu in 2013 [Man15]. Let us first give the answers to the Triangulation Conjecture in various categories and dimensions [Man24]:

3.1 Do all smooth manifolds have triangulations?

The regularity of smooth manifolds is sufficient, so the answer here is affirmative.

Cairns in 1935 [Cai35] and Whitehead in 1940 [Whi40] proved that: Any smooth manifold has an essentially unique P.L. structure, and therefore it is triangulable.

3.2 Do all topological manifolds have triangulations?

- For $n = 0, 1$, Correct, it is trivial;
- For $n = 2$, Correct, Radó proved in 1925 that any 2-dimensional surface has a P.L. structure, and thus has a triangulation;
- For $n = 3$, Correct, Moise proved in 1952 that any 3-dimensional manifold has a smooth structure, and thus has a triangulation;
- For $n = 4$, Incorrect, in 1990, Casson applied Casson invariants to the E_8 manifold constructed by Freedman, demonstrating that it cannot be triangulated;
- For $n \geqslant 5$, Galewski & Stern in 1980 and Matumoto in 1978 reduced the existence of triangulation to the split exactness problem of an exact sequence. This problem was finally disproven by Manolescu in 2013 using Pin(2)-equivariant Seiberg-Witten Floer homology.

3.3 Are all topological manifolds P.L. manifolds (is there a combinatorial triangulation)?

- For $n \leqslant 3$, Correct, reason as above;
- For $n = 4$, Incorrect, the 4-dimensional E_8 manifold constructed by Freedman in 1982 has no piecewise linear structure;
- For $n \geqslant 5$, Incorrect, Kirby-Siebenmann constructed an obstruction to the existence of P.L. structures on a topological manifold M, namely the Kirby-Siebenmann class $\Delta(M) \in H^4(M; \mathbb{Z}/2)$. For $n \geqslant 5$, let $M^n = E_8 \times T^{n-4}$, then $\Delta(M) \neq 0$, so M is a manifold with no combinatorial triangulation.

For the categories to which manifolds belong, we have the following inclusion relations (Figure 1): Regularity increases from topological manifolds, to simplicially triangulable manifolds, to P.L. manifolds (combinatorially triangulable manifolds), to smooth manifolds. When $n \leqslant 3$, all categories are equivalent, i.e., there exists a unique smooth structure and P.L. structure on a topological manifold, and these structures determine each other. When $n \leqslant 6$, the P.L. structure of a manifold determines a unique smooth structure, whereas for $n = 7$, a smooth structure exists on a P.L. manifold, but it may not be unique [Mil11].

4 Kirby-Siebenmann's Work on Combinatorial Triangulation

4.1 Homology Cobordism Group Θ_3

Two oriented n-dimensional manifolds M_1^n, M_2^n are called oriented cobordant, denoted as $M_1^n \sim M_2^n$, if there exists an $(n+1)$-dimensional oriented manifold with boundary W^{n+1}

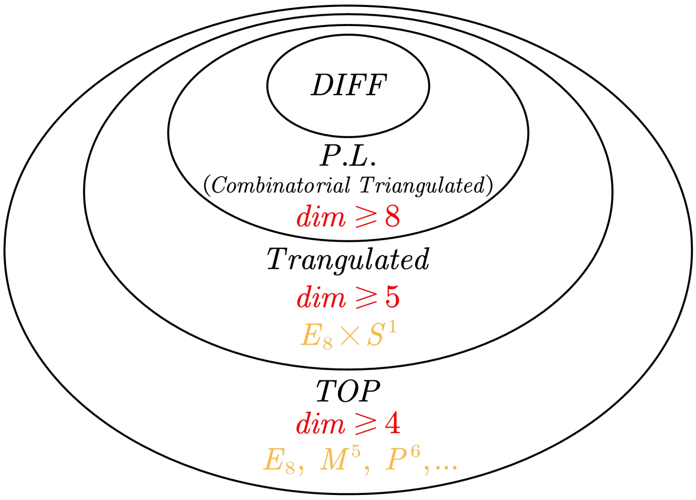


Figure 1: Relationships of Manifold Classifications

such that $\partial W^{n+1} \cong M_1^n \sqcup (-M^n)$. In this case, the manifold W^{n+1} is called a cobordism between M_1^n and M_2^m. When M_2^n is the empty manifold, M_1^n is said to be null-cobordant.

The oriented cobordism relation $\sim$ defined above is indeed an equivalence relation. Reflexivity follows from the trivial cobordism $W^{n+1} = M^n \times I$, symmetry follows from reversing the orientation of W^{n+1}, and transitivity follows from gluing along common boundaries.

Definition 7. *Let $[M^n]$ denote the oriented cobordism equivalence class of M^n. The set of equivalence classes is denoted by Ω_n. Define an addition operation in Ω_n:*

$$[M_1^n] + [M_2^n] := [M_1^n \sqcup M_2^n],$$

then Ω_n becomes an Abelian group. Its zero element is the equivalence class of null-cobordant manifolds. Ω_n is called the oriented cobordism group of n-dimensional manifolds.

It can be verified that for two trivial cobordisms $M_1^n \times I, M_2^n \times I$, by performing boundary connected sum along $M_1 \times 0$ and $M_2 \times 0$, we obtain an $(n+1)$-dimensional oriented manifold W, where $\partial W = (M_1 \# M_2) \sqcup -(M_1 \sqcup M_2)$. Thus $[M_1 \# M_2] = [M_1^n] + [M_2^n]$, so we can also use connected sum to define addition in Ω_n.

Through Thom's cobordism theory or 3-dimensional topological surgery, it can be proven that $\Omega_3 = 0$, meaning all oriented 3-dimensional compact manifolds are boundaries of some oriented compact 4-dimensional manifold.

Definition 8. *The homology cobordism group Θ_3 is the set of equivalence classes of integer homology spheres under the cobordism relation. That is, $\forall Y \in \Theta_3$, $H_*(Y;\mathbb{Z}) = H_*(S^3;\mathbb{Z})$, and $Y_1 \sim Y_2 \in \Theta_3 \Leftrightarrow \exists X^4, \partial X = -Y_1 \cup Y_2, H_*(X;\mathbb{Z}) = H_*(S^3 \times [0,1];\mathbb{Z})$. $[S^3]$ is regarded as the identity element of the group, $Y \mapsto -Y$ is the inverse mapping, and $(Y_1, Y_2) \mapsto Y_1 \# Y_2$ is addition, thus forming an Abelian group.*

Unlike the simple conclusion that $\Omega_3 = 0$, Θ_3 has the requirement of "homology cobordism" added to its definition, making its structure much more complex. Since $\Omega_3 = 0$, any two 3-dimensional homology spheres are cobordant. However, only when this cobordism is a "homology cobordism" (analogous to the "trivial cobordism" $S^3 \times I$) are two homology spheres considered to belong to the same equivalence class in the homology cobordism group.

To study its structure, we can use homomorphisms from it to other Abelian groups, such as the Rokhlin homomorphism:

Definition 9. *The Rokhlin homomorphism* $\mu : \Theta_3 \to \mathbb{Z}/2$, $\mu(Y) = \dfrac{\sigma(W)}{8} \mod 2$, *where* W *is a smooth compact spin 4-manifold with* $\partial W = Y$, *and* $\sigma(W)$ *is the signature of the intersection form of* W.

For example, we have $\mu(S^3) = 0$, $\mu(P) = 1$, where P is the Poincaré homology sphere. It can be viewed as the boundary of a 4-manifold obtained by E_8-plumbing (called the E_8 manifold). Its intersection form matrix is

$$\begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 \end{pmatrix}$$

This matrix is a negative definite matrix of order 8 (in some of Manolescu's papers, the E_8 matrix is taken as positive definite, with no essential difference), so $\sigma(E_8) = 8$, and $\mu(P) = 1$.

Using Rokhlin's theorem, we can prove that μ is indeed a homomorphism:

Theorem 1 (Rokhlin)**.** *For a smooth closed spin 4-manifold* X, $\sigma(X)$ *is divisible by 16.*

If Y^3 serves as the boundary for 4-dimensional smooth spin manifolds W_1, W_2, let $X = W_1 \cup_Y W_2$. Then X is a closed 4-dimensional spin manifold. In the homology sequence induced by the commutative diagram of inclusion maps

$$\begin{tikzcd} & W_1 \arrow[dr] & \\ Y \arrow[ur] \arrow[dr] & & X \\ & W_2 \arrow[ur] & \end{tikzcd}$$

we have $H^2(X) = H^2(W_1) \bigoplus H^2(W_2)$. Therefore, the signature of the intersection form of X equals the sum of the signatures of the intersection forms of W_1 and W_2. By Rokhlin's theorem, $\sigma(W_1) + \sigma(W_2) = \sigma(X) \equiv 0 \mod 16$, so $\sigma(W_1)/8 \equiv \sigma(W_2)/8 \mod 2$. This means the value of $\mu(Y)$ is independent of the choice of W. Similarly, it can be proven that the value of μ is independent of the choice of representative in the equivalence class of the homology cobordism group. Thus, μ is a well-defined map on Θ_3. Furthermore, if compact spin manifolds with boundary satisfy $\partial W_1 = Y_1, \partial W_2 = Y_2$, then $Y_1 \# Y_2 = \partial(W_1 \# W_2)$, so $\mu(Y_1 \# Y_2) = \mu(Y_1) + \mu(Y_2)$, which proves that μ is a homomorphism.

Also, since μ is a surjective homomorphism, this shows that Θ_3 is not a trivial group.

Due to the existence of the Rokhlin homomorphism, we know $|\Theta_3| \geqslant 2$. For a time, mathematicians hoped that μ was an isomorphism from Θ_3 to $\mathbb{Z}/2$. However, when Donaldson introduced gauge theory tools to study 4-manifolds, Furuta used them to prove that Θ_3 is not finitely generated, i.e., it has a $\mathbb{Z}$ subgroup. Later, it was discovered that it even has a $\mathbb{Z}$ subgroup as a direct summand, and even a $\mathbb{Z}^\infty$ as a direct summand, and furthermore $\Theta_3^H/\mathbb{Z}^\infty$ also has a $\mathbb{Z}^\infty$ as a direct summand.

4.2 Kirby-Siebenmann class

A principal G-bundle is a fiber bundle with fiber being a topological group G, and G acting freely on the fibers.

For a given group G, one can construct a principal G-bundle $EG \to BG$ such that any principal G-bundle $P \to B$ with a paracompact base space B can be pulled back from $EG \to BG$. That is, there exists a bundle map $f : B \to BG$ inducing an isomorphism on fibers, and $P = f^*(EG)$. Additionally, the total space EG is contractible. A principal G-bundle satisfying these conditions is called a universal G-bundle.

The classifying space of principal G-bundles is the base space BG of the universal bundle $EG \to BG$. The meaning of classification is that for any topological space X, the homotopy equivalence classes of principal G-bundles on it correspond one-to-one with the homotopy classes of continuous maps from X to BG.

For example, the infinite-dimensional Grassmannian manifold can serve as the classifying space for principal O_k-bundles (since k-dimensional real vector bundles can be reduced to O_k bundles, the Grassmannian can also serve as the classifying space for all k-dimensional real vector bundles).

The Grassmannian manifold $Gr(k,n)$ is the quotient space of the orthonormalized Stiefel manifold $V^0(k,n)$, i.e., the set of all orthonormal frames in $\mathbb{R}^n$, where two orthonormal frames are equivalent if they span the same k-dimensional subspace in $\mathbb{R}^n$. $Gr(k,n)$ is the set parameterizing all k-dimensional subspaces in $\mathbb{R}^n$. By definition, $V^0(k,n) \to Gr(k,n)$ is a principal $O(k)$ bundle. The canonical embedding of Euclidean space from low dimension to high dimension induces embeddings $Gr(k,n), V^0(k,n)$ as $n \to \infty$. Let $Gr(k,\infty) = \lim\limits_{n\to\infty} Gr(k,n)$. The resulting infinite-dimensional manifold $Gr(k,\infty)$ is the classifying space for $O(k)$, i.e.,

$$EO(k) = V^0(k,\infty) \to BO(k) = Gr(k,\infty)$$

is the universal bundle for principal $O(k)$ bundles.

If we select a representative element of the fiber on $V^0(k,\infty)$ for each point on $Gr(k,\infty)$ and replace this fiber with the k-dimensional subspace spanned by the orthonormal frame of the representative element, we obtain the canonical vector bundle γ^k. $\gamma^k \to Gr(k,\infty)$ is the universal bundle for k-dimensional vector bundles.

Similarly, we define $\mathrm{TOP}(n), \mathrm{PL}(n), \mathrm{Diff}(n)$ as the groups of all origin-preserving self-homeomorphisms, self-P.L. homeomorphisms, and self-diffeomorphisms on $\mathbb{R}^n$, respectively. Obviously, there are inclusion maps $\mathrm{TOP}(n) \to \mathrm{TOP}(n+1)$, and similarly for the other two sequences of groups. Letting $n \to \infty$, we obtain three groups $\mathrm{TOP}, \mathrm{PL}, \mathrm{Diff}$, with inclusion maps $\mathrm{Diff} \to \mathrm{PL} \to \mathrm{TOP}$. Let their classifying spaces be $B\mathrm{TOP}, B\mathrm{PL}, B\mathrm{Diff}$. Then there are inclusion maps $B\mathrm{Diff} \to B\mathrm{PL} \to B\mathrm{TOP}$.

Generally, if $H \subset G$ is a subgroup, then EG is a contractible space under free H action, and $EG \to EG/H$ is a principal H bundle. Thus EG/H can be viewed as the classifying space BH. Therefore, the natural map $\psi : EG/H \to EG/G$ is actually a model for $BH \to BG$, and the fiber of ψ equals G/H.

Therefore, we can identify the fibers of $B\mathrm{Diff} \to B\mathrm{PL}$ and $B\mathrm{PL} \to B\mathrm{TOP}$ with PL/Diff and TOP/PL, respectively.

Thus, for any topological manifold X, there exists a canonical map $f : X \to B\mathrm{TOP}$. The existence of a PL structure on X is equivalent to the problem of lifting f to a map $F : X \to B\mathrm{PL}$. Furthermore, the existence of $G : X \to B\mathrm{Diff}$ determines whether X has a smooth structure.

The above lifting problems can be studied through obstruction theory, specifically obstruction classes in cohomology rings. The homotopy types of the fibers of $B\mathrm{Diff} \to B\mathrm{PL}$ and $B\mathrm{PL} \to B\mathrm{TOP}$ play an important role in the research.

For an n-dimensional smooth manifold X, there exists a canonical map $f : X \to B\mathrm{GL}_n(\mathbb{R})$. Whether a lift $F : X \to B\mathrm{GL}_n^+(\mathbb{R})$ exists determines if X is orientable. Since $\mathrm{GL}_n^+(\mathbb{R}) \subset \mathrm{GL}_n(\mathbb{R})$ is one of the two connected components, and $\mathrm{GL}_n(\mathbb{R})/\mathrm{GL}_n^+(\mathbb{R}) = \mathbb{Z}/2$, the fiber of $B\mathrm{GL}_n^+(\mathbb{R}) \to B\mathrm{GL}_n(\mathbb{R})$ is a $K(\mathbb{Z}/2, 0)$ space. By standard obstruction theory, the obstruction class determining the existence of orientation lies in $H^1(X; \mathbb{Z}/2)$. It can be proven that this is exactly the Stiefel-Whitney class $w_1(X)$.

Another example is the double covering $\mathrm{Spin}(n) \to \mathrm{SO}(n)$, which induces a fiber bundle $B\mathrm{Spin}(n) \to B\mathrm{SO}(n)$ with fiber $B(\mathbb{Z}/2)$, which is a $K(\mathbb{Z}/2, 1)$ space (can be taken as $\mathbb{R}P^\infty$). The existence of a spin structure on X depends on whether the canonical map $f : X \to B\mathrm{SO}(n)$ (fixing a Riemannian metric and orientation on X) can be lifted to $B\mathrm{Spin}(n)$. By obstruction theory, the obstruction to the existence of a spin structure lies in $H^2(X; \mathbb{Z}/2)$. It can be proven that this is exactly the Stiefel-Whitney class $w_2(X)$.

As mentioned earlier, for a general topological manifold M, the problem of the existence of a P.L. structure is the map lifting problem shown in the diagram:

$$\begin{tikzcd} & B\mathrm{PL} \arrow[d, "\psi"] \\ M \arrow[r, "f"] \arrow[ur, dashed] & B\mathrm{TOP} \end{tikzcd}$$

$\Delta(M)$ is defined as the obstruction to lifting $f : M \to B\mathrm{TOP}$ to $B\mathrm{PL}$. It has the following property:

Theorem 2. *If M^n is a topological manifold and $n \geqslant 5$, then M has a P.L. structure if and only if $\Delta(M) = 0 \in H^4(M; \mathbb{Z}/2)$. Furthermore, if $\Delta(M) = 0$, the inequivalent P.L. structures on M can be parameterized by $H^3(M; \mathbb{Z}/2)$.*

Similar to the previous examples of orientation and spin, here we discuss the homotopy type of the fiber TOP/PL of $B\mathrm{PL} \to B\mathrm{TOP}$. In [KS77], it was proven that TOP/PL is a $K(\mathbb{Z}/2, 3)$ space, so the obstruction class lies in $H^4(M; \mathbb{Z}/2)$.

Generally, it is difficult to express the specific form of $\Delta(M)$. For special cases: If the topological manifold M has a triangulation, we can give an expression for $\Delta(M)$.

For simplicity, let us first consider the orientable case: Let an oriented topological manifold M^n have a fixed triangulation K (not necessarily a combinatorial triangulation). Let

$$c(K) = \sum_{\sigma \in K^{n-4}} [lk(\sigma)] \in H_{n-4}(M, \Theta_3) \cong H^4(M, \Theta_3)$$

By Poincaré duality, this can be viewed as $c(K) \in H^4(M;\Theta_3)$.

Using the surjective homomorphism μ, we can construct a short exact sequence:

$$0 \to \ker(\mu) \xrightarrow{\iota} \Theta_3 \xrightarrow{\mu} \mathbb{Z}/2 \to 0$$

This induces a long exact sequence with variable coefficients:

$$\begin{aligned} \cdots \to H^4(M;\Theta_3) &\xrightarrow{\mu} H^4(M;\mathbb{Z}/2) \xrightarrow{\delta} H^5(M;\ker\mu) \to \cdots \\ c(K) &\mapsto \Delta(M) \end{aligned}$$

That is, $\Delta(M) = \mu(c(K))$.

When K is a combinatorial triangulation, $c(K) = 0$. On the other hand, $\mu(c(K)) = 0$ implies that there exists some combinatorial triangulation on M (possibly different from K). This shows that $\mu(c(K))$ is indeed the obstruction to the existence of a P.L. structure, i.e., $\Delta(M) = \mu(c(K))$.

For the case where M is non-orientable, we can use Poincaré duality with local coefficients to similarly obtain $c(K) \in H^4(M;\Theta_3)$.

However, this construction depends on the choice of triangulation K, so it is not suitable for directly studying the problem of simplicial triangulation. But in the work of Galewski & Stern, a "Universal 5-manifold" is constructed, where this comes into play using proof by contradiction.

5 Galewski & Stern's Work for $n \geqslant 5$

5.1 Steenrod Squares

For the short exact sequence

$$0 \to \mathbb{Z}/2 \xrightarrow{\times 2} \mathbb{Z}/4 \xrightarrow{r} \mathbb{Z}/2 \to 0$$

where r is the $\mod 2$ homomorphism, it induces a long exact sequence

$$\cdots \xrightarrow{\beta} H^i(M;\mathbb{Z}/2) \xrightarrow{\times 2} H^i(M;\mathbb{Z}/4) \xrightarrow{r} H^i(M;\mathbb{Z}/2) \xrightarrow{\beta} H^{i+1}(M;\mathbb{Z}/2) \xrightarrow{\times 2}$$

where M is a fixed topological manifold. The Bockstein homomorphism β here is the first Steenrod square Sq^1, i.e., $\mathrm{Sq}^1 = \beta : H^k(M;\mathbb{Z}/2) \to H^{k+1}(M;\mathbb{Z}/2)$.

5.2 Equivalence Conditions for Triangulation when $n \geqslant 5$

Galewski & Stern [GS80] and Matumoto [Mat76] gave a cohomology obstruction to the existence of triangulation (simplicial triangulation), and this obstruction is precisely the Bockstein homomorphism $\delta : H^4(M;\mathbb{Z}/2) \to H^5(X, \ker(\mu))$ composed with the Kirby-Siebenmann class $\Delta(M)$, i.e., $\delta(\Delta(M))$.

Similar to the introduction of Δ, the obstruction to triangulation is also obtained by studying classifying spaces and their fibrations. Galewski & Stern constructed a classifying space $B\mathrm{TRI}$. Whether a topological manifold X can be triangulated depends on whether $X \to B\mathrm{TOP}$ can be lifted to $X \to B\mathrm{TRI}$. Skipping the details of the proof, we ultimately have the theorem:

Theorem 3. *There exists a triangulation on a topological manifold M of dimension $n \geqslant 5$ if and only if $\delta(\Delta(M)) \in H^5(M; \ker(\mu))$ is 0. If $\delta(\Delta(M)) = 0$, distinct triangulations on M can be parameterized by $H^4(M; \ker(\mu))$.*

Using this theorem, we can deduce:

Theorem 4. *If the exact sequence*

$$0 \to \ker(\mu) \xrightarrow{\iota} \Theta_3 \xrightarrow{\mu} \mathbb{Z}/2 \to 0 \tag{1}$$

is split exact, then triangulations exist on all manifolds of dimension $n \geqslant 5$.

Proof. If the short exact sequence (1) is split exact, then $\exists \varphi : \mathbb{Z}/2 \to \Theta_3$ s.t. $\mu \circ \varphi = \text{id}$. Thus $\delta = \delta(\mu \circ \varphi) = (\delta \circ \mu) \circ \varphi = 0$. Therefore $\delta(\Delta(M)) \equiv 0$, $\forall M^n$, $n \geqslant 5$. Hence, by Theorem 3, triangulations exist on all manifolds of dimension $n \geqslant 5$. □

In fact, the converse of Theorem 4 also holds.

Galewski & Stern constructed a "universal 5-manifold" N^5 in their 1979 paper [GS79], satisfying $\mathrm{Sq}^1(\Delta(N)) \neq 0$. The original construction will be detailed in 5.3.

Using this condition, we can prove the converse of Theorem 4:

Theorem 5. *If the "universal 5-manifold" N^5 can be triangulated, then the exact sequence (1) is split exact.*

To prove this theorem, we supply a conclusion from homology theory:

Lemma 1. *The short exact sequence (1) is split exact $\Longleftrightarrow$ there exists a 3-dimensional integer homology sphere Y such that $\mu(Y) = 1$ and $2[Y] = 0 \in \Theta_3$ (i.e., $Y \# Y$ is the boundary of an integer homology disk W^4).*

Proof. $\Longrightarrow$: If the short exact sequence (1) splits, then $\exists \varphi : \mathbb{Z}/2 \to \Theta_3$ s.t. $\mu \circ \varphi = \text{id}$. Let $[Y] = \varphi(1)$, then $\mu(Y) = \mu(\varphi(1)) = 1$, and $2[Y] = 2\varphi(1) = \varphi(2) = 0$.

$\Longleftarrow$: Let $\varphi : \mathbb{Z}/2 \to \Theta_3$, $\varphi(1) = [Y]$, $\varphi(0) = 0$. Since $2[Y] = 0$, φ is a homomorphism, and $\mu \circ \varphi = \text{id}$. Thus (1) splits. □

Proof of Theorem 5. The proof uses contradiction. Assume the short exact sequence 1 does not split. Then by 1, $\forall [Y] \in \Theta$ satisfying $\mu(Y) = 1$, we have $2[Y] \neq 0$, i.e., Θ_3 contains no element of order 2.

Let Θ be the group generated by all 3-dimensional links in a given triangulation of N. It is a subgroup of the 3-dimensional homology cobordism group Θ_3. Let $i : \Theta \hookrightarrow \Theta_3$ be the inclusion map.

Since the triangulation on N contains finitely many 3-dimensional links, Θ can be written as a direct sum of finitely many cyclic groups, $\Theta = \langle h_1 \rangle \oplus \cdots \oplus \langle h_k \rangle$, where each term is either a free cyclic group or a finite cyclic group of prime power order.

We define a map $\gamma : \Theta \to \mathbb{Z}/4$. We only need to define it on $\{h_i\}_{i=1}^k$: 1' If $\mu(h_i) = 0$, let $\gamma(h_i) = 0$; 2' If $\mu(h_i) = 1$ and $\langle h_i \rangle \cong \mathbb{Z}$, let $\gamma(h_i) = \langle h_i \rangle \mod 4$; 3' If $\mu(h_i) = 1$ and the order of h_i is p^m, since μ is a homomorphism and the order of $\mu(h_i) \in \mathbb{Z}/2$ is 2, then $p = 2$. Also, since $\Theta \subset \Theta_3$ contains no element of order 2, $m \geqslant 2$. We can also let $\gamma(h_i) = \langle h_i \rangle \mod 4$. By definition, $\mu \circ i = r \circ \gamma$.

The obstruction determining whether a P.L. structure exists on N is denoted by $c(N) \in H^4(N;\Theta_3)$, and by definition, there exists $c'(N) \in H^4(N;\Theta)$ such that $i(c'(N)) = c(N)$. Thus

$$\mathrm{Sq}^1(\mu(c(N))) = \mathrm{Sq}^1(\mu(i(c'(N)))) = \mathrm{Sq}^1(r(\gamma(c'(N)))) = 0$$

since $\mathrm{Sq}^1 \circ r = 0$. Also since $\mu(c(N)) = \Delta(N)$, we obtain $\mathrm{Sq}^1(\Delta(N)) = 0$, contradicting $\mathrm{Sq}^1(\Delta(N)) \neq 0$. □

The contrapositive of Theorem 5 states that if the exact sequence (1) does not split, then N^5 cannot be triangulated. If we let $M^n = N^5 \times T^{n-5}$, we also have $\mathrm{Sq}^1(\Delta(M)) \neq 0$. Similar to the previous conclusion, we have:

Theorem 6. *If the exact sequence (1) is not split exact, then in every dimension $n \geqslant 5$, there exists a manifold that cannot be triangulated.*

The contrapositive of this theorem is the converse of Theorem 4.

Thus we have the following conclusions:

$$\begin{aligned} \text{All manifolds for } n \geqslant 5 \text{ are triangulable} &\Longleftrightarrow \text{Exact sequence 1 splits} && (2)\\ &\Longleftrightarrow \exists [Y] \in \Theta_3 \text{ s.t. } 2[Y] = 0,\ \mu(Y) = 1 && (3)\\ &\Longleftrightarrow \text{Triangulation exists on } N^5 && (4) \end{aligned}$$

5.3 Construction of the Universal 5-manifold

The following construction process is essentially a translation of Galewski & Stern's 1979 article published in "Geometric Topology" [GS79]: *A UNIVERSAL 5-MANIFOLD WITH RESPECT TO SIMPLICIAL TRIANGULATIONS*. The original text had numerous typos, and in this article, I have corrected the errors I could identify.

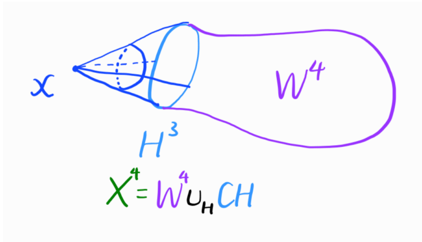
Let's briefly review the conclusion obtained using the "universal 5-manifold": if we construct a **closed** 5-dimensional topological manifold N such that $Sq^1(\Delta(N)) \neq 0$, then a triangulation exists on it if and only if triangulations exist for all manifolds with $n \geqslant 5$. This is why it is called a "universal 5-manifold". Since the number of vertices in a triangulation is finite, compactness is a necessary condition for the existence of a triangulation on a manifold, so we need to make a closed manifold here.

Lemma 2. *An n-dimensional cell complex (i.e., simplicial complex where the link of every vertex is an $(n-1)$-dimensional homology sphere) K with $n \geqslant 5$, $K \neq \varnothing$, is an n-dimensional topological manifold $|K|$ if and only if the links at the vertices of $|K|$ are all simply connected.*

This is a corollary of Cannon & Edwards' double suspension theorem. (I haven't yet researched how double suspension implies this lemma; the double suspension theorem says that the double suspension of a 3-dimensional homology sphere is homeomorphic to the 5-sphere.)

Next, we geometrically construct a closed 5-dimensional topological manifold N satisfying $Sq^1(\Delta(N)) \neq 0$. I will use numerous illustrations to assist the discussion. Objects constructed in different steps will be indicated in different colors, and the dimension of geometric bodies will be marked with superscript ^{n} where possible.

Take H^3 to be any oriented 3-dimensional P.L. homology sphere. It is the boundary of an oriented parallelizable 4-dimensional P.L. manifold W with $\sigma(W) = 8$. Let $X = W \cup_H CH$, where CH is the topological cone on H, and let x be the cone point of CH.



Next, we paste a P.L. 1-handle $D^3 \times [0,1]$ onto $(CH \times 0 \cup CH \times 1) \subset X \times [0,1]$.

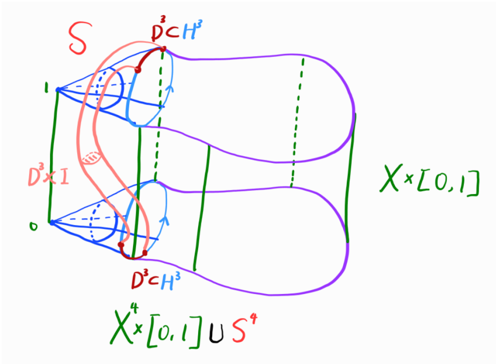


Figure 2: Note the choice of orientation for the two H's, and the connection method of D^3

Let $S = CH \times 0 \cup_{D^3 \subset H^3} D^3 \times I \cup_{D^3 \subset H^3} CH \times 1$. The way we paste the handle must ensure $\partial S = H \# H$ (note it is not $H \# - H$; our connected sum here is the connected sum of two homology spheres with the same orientation).

It can be seen that ∂S is also a homology sphere, and precisely $H \# H$, and S it itself is a homology disk $\approx \mathbb{D}^4$. To avoid confusion, we use `\mathbb{S}` to represent the standard sphere/ball, i.e., $\mathbb{S}$.

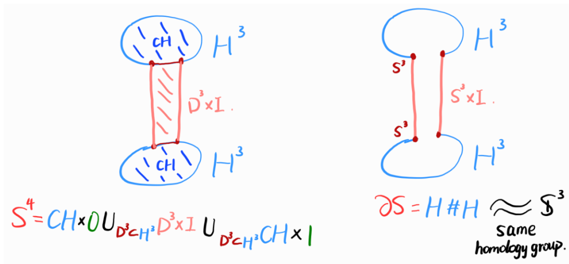
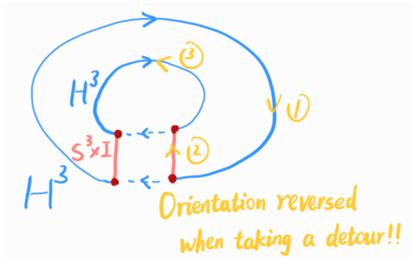
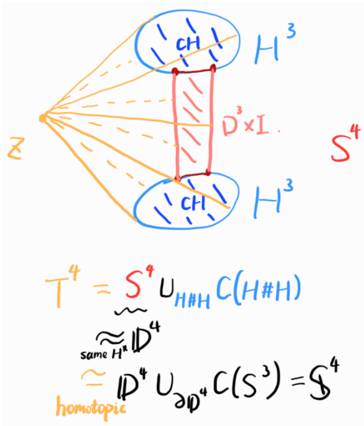


Figure 3: Here, for convenience of drawing, the orientation issue is not highlighted

The reason we emphasize that the connected sum is not $H \# - H$ is that the latter preserves orientation compatibility across the boundary of the connected sum, while the former reverses orientation when crossing the boundary. Utilizing this property, and the fact that they are connected to opposite sides of $X \times [0,1]$, we create a non-orientable geometric body $X \times [0,1] \cup S$. That is, its first Stiefel–Whitney class $w_1(X \times [0,1] \cup S) = 1 \in H^1(X \times [0,1] \cup S; \mathbb{Z}/2\mathbb{Z})$. Subsequent embellishments from the construction will maintain w_1 non-zero.



Next, construct a topological cone $T = S \cup_{H\#H} C(H\#H)$ on the boundary of S.



Since S is a homology disk, forming a topological cone on its boundary yields T as a 4-dimensional homology sphere. In fact, T is a 4-dimensional homotopy sphere.

Let $Y = X \times I \cup (S \cup_{H\#H} C(H\#H))$, and let z be the cone point of $C(H\#H)$. (Here Y is not a manifold; it is the union of a 5-dimensional manifold along its boundary with a 4-dimensional manifold, so Y is merely a simplicial complex, or polyhedron.) The polyhedron Y contains the sub-polyhedron $T = S \cup_{H\#H} C(H\#H)$.

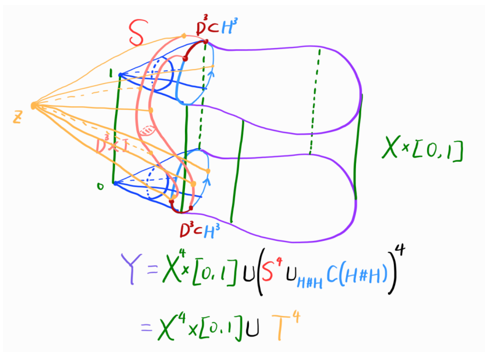
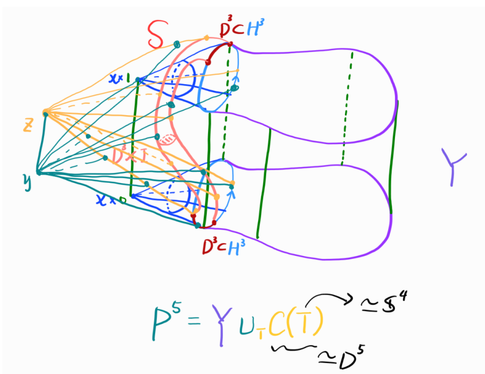


Figure 4: Note that Y is formed by union of parts with different dimensions, so it cannot be a manifold

Let $P = Y \cup_T CT$, and let y be the cone point of CT. This yields P as a 5-dimensional simplicial complex.



Here two cones are drawn simultaneously, which is dazzling on paper. We try to regain geometric intuition using the property that T is a homotopy sphere; the cone on it is simply a homotopy disk of one higher dimension.

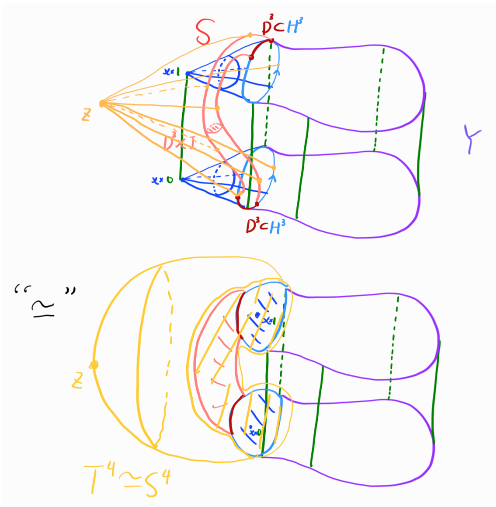
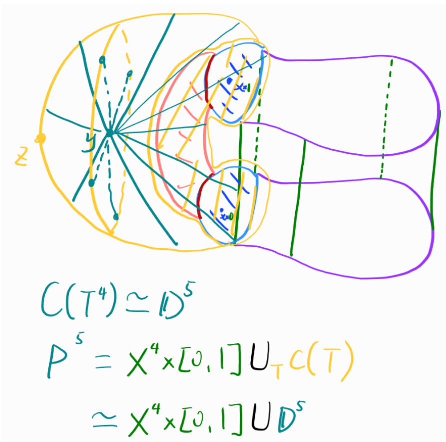


Figure 5: Wrapping a 3-dimensional homology circle $H\#H$ with a 4-dimensional spherical membrane

Thus, forming a topological cone on the homotopy sphere T intuitively corresponds to filling the interior of the sphere.



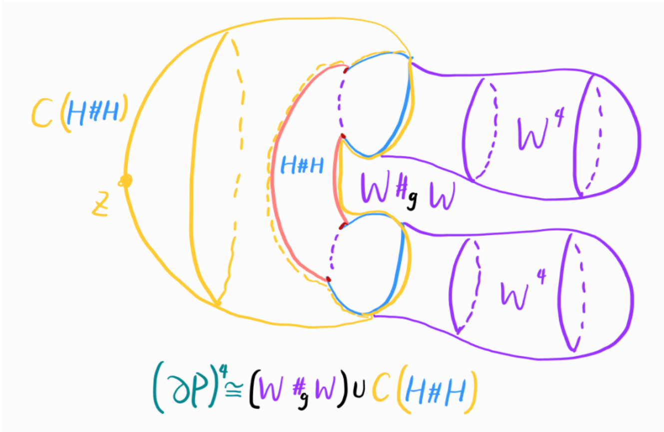
And ∂P is P.L. homeomorphic to $W\#_g W \cup C(H\#H)$, where g denotes the connected sum along the boundary. (There is a typo in the original text, I temporarily use g as a substitute) Because of the following calculation:

$$\partial P \text{ "=" } X^4 \times \{0,1\} \triangle T$$
$$= \left((W^4 \cup_H CH) \times \{0,1\} \right) \triangle \left(S^4 \cup_{H\#H} C(H\#H) \right)$$
$$\cong (W \#_g W) \cup C(H\#H)$$

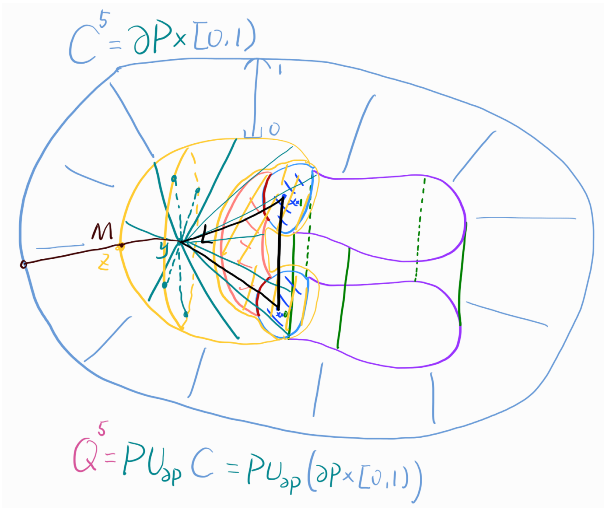
g denote "boundary connected sum". thus $H\#H \subset W\#_g W$

Here I use the symmetric difference symbol Δ loosely for brevity. Strictly speaking, it should be the "union of the two" minus the "interior of the intersection of the two".

Intuitively, ∂P looks like this:



Since W is a parallelizable manifold, all characteristic classes on it vanish, and rank-2 characteristic classes on $C(H\#H)$ also all vanish. Thus all Stiefel-Whitney classes of ∂P are 0. Next, add an exterior collar $C = \partial P \times [0, 1)$ along ∂P to obtain a 5-dimensional simplicial complex Q.



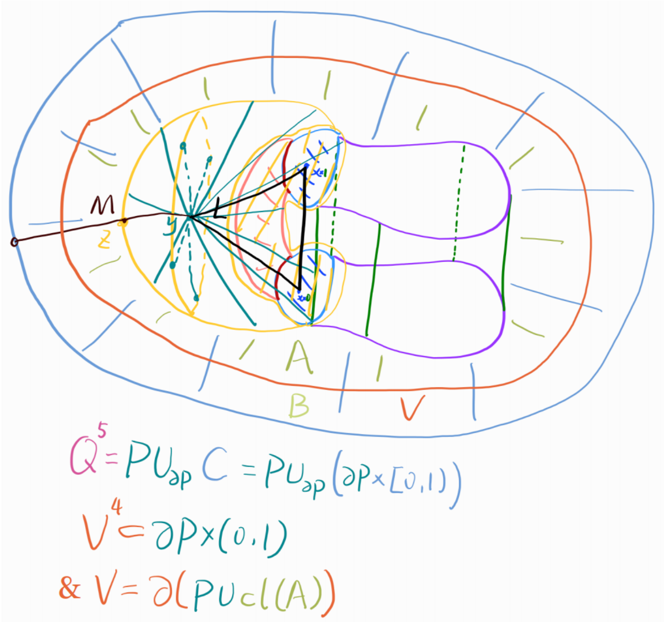
We first observe that those 4-dimensional links not PL homeomorphic to $\mathbb{S}^4$, such as the links of z, y, $x \times 0$, and $x \times 1$, are simply connected (I only understood that the link of y is T^4 which is a homotopy sphere, thus $\pi_1 = 0$). Therefore, by Lemma 2, Q is a triangulated 5-dimensional manifold.

We next observe that the 3-dimensional links on Q not P.L. homeomorphic to $\mathbb{S}^3$ are all sub-polyhedra on Q, such as $L = x \times [0, 1] \cup y * (x \times \{0, 1\}) \cong \mathbb{S}^1$ and $M = y * z \cup z \times [0, 1)$. The links of 1-simplices in L are all P.L. homeomorphic to H, and the links of 1-simplices in M are all P.L. homeomorphic to $H\#H$. Since $\mu(H\#H) = 0$, and all other links are standard spheres, naturally μ is also 0. By Siebenmann's Theorem [C], there must exist a PL structure Σ on $Q - L$. Here Σ is not consistent with the polyhedral structure of Q.

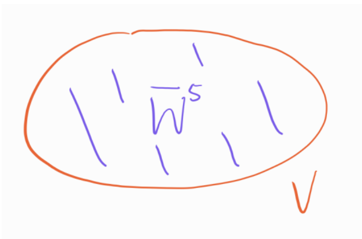
Theorem 7 (Siebenmann's Theorem [C])**.** *If a boundaryless topological manifold $W^n, n \geqslant 5$ is triangulable but has no P.L. structure, then there exists a 3-dimensional homology sphere M^3 with Rokhlin invariant $\mu(M^3) = 1$, such that the suspension $\Sigma^{n-3}M^3$ is homeomorphic to $\mathbb{S}^\kappa$.*

Our construction is nearly complete, but the manifold Q is currently open. We need to "trim and patch" it into a closed manifold:

We can now use P.L. transversality relative to $\Sigma|_{\partial P \times (0,1)}$ to obtain a compact connected oriented 4-dimensional submanifold $V \subset \partial P \times (0, 1)$. Its normal bundle is trivial, and it separates $\partial P \times [0,1)$ into two parts, A and B. Without loss of generality assume $A \supset \partial P$. Then $P \cup cl[A]$ is a topological manifold, and $\partial(P \cup cl[A]) = V$.



Since the normal bundle of V is trivial, all Stiefel-Whitney classes are 0, so there exists a 5-dimensional P.L. manifold $\overline{W}$ such that $V = \overline{W}$.



Finally, we define $N^5 = P \cup_{\partial P} cl[A] \cup_V \overline{W}$.

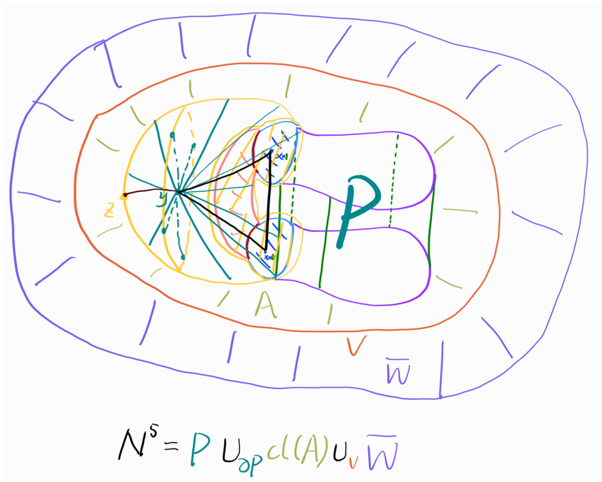


Figure 6: $\overline{W}$ is actually compact with boundary V; for convenience, we draw it outside.

Since there is a P.L. structure Σ on $Q - L$, $N - L$ is also a PL manifold. Therefore, the Poincaré dual of $\Delta(N)$ (the obstruction to the existence of P.L. structure, clearly dominated by L here) is L. Also, the Poincaré dual of $w_1(N)$ (obstruction to orientation; since $w_1(X \times [0,1] \cup S)$ is non-zero, $w_1(N)$ is also non-zero, caused by the co-oriented connected sum $H\#H$) restricted to P is $X \times \frac{1}{2}$ (this can be thought of as migrating this homology obstruction from the intersection line of the $H\#H$ connected sum to the cross-section of the product space; removing this cross-section from N makes it an orientable manifold).

By the definition of Wu class, for any manifold M and $x \in H^{n-k}(M)$, the Wu class v_k satisfies $v_k \smile x = Sq^k(x) \in H^n(M)$, where Sq^k is the Steenrod squaring operator: $H^m(M) \to H^{m+k}(M)$, satisfying $Sq^0 = id_{H^m(M)}$, so $v_0 = 1$. Wu Wenjun's theorem gives the relationship between Wu classes and Stiefel-Whitney classes:

Theorem 8 (Wu)**.** $Sq(v) = w$

Here $Sq = \sum_{k=0}^{\infty} Sq^k$, $v = \sum_{k=0}^{\infty} v_k$, $w = \sum_{k=0}^{\infty} w_k$, and $w_0 = 1$. Thus $w_1 = v_1 + Sq^1(1)$.

By the Cartan formula $Sq^k(a \smile b) = \sum_{i+j=k} Sq^i(a) \smile Sq^j(b)$, we know $Sq^1(a) = Sq^1(1 \smile a) = Sq^0(1) \smile Sq^1(a) + Sq^1(1) \smile Sq^0(a) = Sq^1(a) + Sq^1(1) \smile a$, so $Sq^1(1) = 0$, and $w_1 = v_1$. (This proof can also be considered from the naturalness of Sq, i.e., there exists a trivial continuous function $f : M \to pt$, so $Sq^1(1_M) = Sq^1(f^*1_{pt}) = f^*(Sq^1(1_{pt}))$, while $Sq^1(1_{pt}) \in H^1(pt; \mathbb{Z}/2) = 0$, so $Sq^1(1_M) = 0$.)

In summary, $Sq^1(x) = v_1 \smile x = w_1 \smile x$.

Thus $Sq^1(\Delta(N)) = w_1(N) \smile \Delta(N)$. However, $\langle w_1(N) \smile \Delta(N), [N]\rangle \neq 0$ is the intersection number of L and $X \times \frac{1}{2}$ ($L \cap (X \times \frac{1}{2}) \neq \varnothing$), so $Sq^1(\Delta(N)) \neq 0$. That is, N is the required 5-dimensional manifold.

To summarize the construction idea above: We want to make a 5-dimensional manifold N where $Sq^1(\Delta(N))$ is a non-zero element in the 5th cohomology group. From calculations regarding characteristic classes, it is the cup product of the 1st Stiefel-Whitney class $w_1 \in H^1(N; \mathbb{Z}/2)$ and the Kirby-Siebenmann class $\Delta(N) \in H^4(N; \mathbb{Z}/2)$. Since w_1 is the obstruction to orientation and Δ is the obstruction to the existence of a P.L. structure, geometrically, we need to find a 4-dimensional closed submanifold (corresponding to w_1) and a 1-dimensional closed submanifold (corresponding to Δ) in a non-orientable manifold without a P.L. structure. These are elements in homology classes, and removing them gives the manifold orientation and a P.L. structure respectively. Thus they are the geometric obstructions for both, and their Poincaré duals are the elements in the corresponding cohomology classes. During the construction, we repeatedly form topological cones, introducing many cone points. Cone points are often not manifold points (their links may not be spheres), so we need to add collars to "hide" the cone points. However, this results in an open manifold, so we perform some "trimming and patching" surgeries to obtain a closed manifold.

5.3.1 A More Direct Construction

[Man24] gives another example where $\mathrm{Sq}^1(X) \neq 0$. Leveraging the theory of intersection forms of 4-manifolds, its construction is more direct (though this theory was published later than Galewski & Stern's construction), coming from Kronheimer:

Let $X = *(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2})$ be a simply connected 4-manifold with intersection form

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \text{ congruent to } -\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Here '$*$' indicates it is another simply connected closed 4-manifold with the same intersection form as $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$, homotopy equivalent to it but not homeomorphic.

Freedman's series of conclusions in 1982 [Fre82] proved the existence of such manifolds: Generally, if an intersection form is odd (if x^TQx is always even, Q is called an even form, otherwise odd), there are exactly two homeomorphism types of topological manifolds with it as an intersection form, and at most one of them admits a smooth structure. The presence or absence of a smooth structure is distinguished by the Kirby-Siebenmann invariant Δ. Since $\mathbb{CP}^2\#\overline{\mathbb{CP}^2}$ has a natural smooth structure, X is the other homeomorphism type without a smooth structure, and $\Delta(X) \neq 0$.

Freedman's work also showed that since there exists a congruence transformation from the intersection form of X to its negative matrix, there exists an orientation-reversing homeomorphism $f : X \to X$. Let M^5 be the mapping torus of f. That is,

$$M = (X \times I)/(x, 0) \sim (f(x), 1).$$

Since $\Delta(X) \neq 0 \in H^4(X; \mathbb{Z}/2) = \mathbb{Z}/2$, we have $\Delta(M) \neq 0$. Also, the mapping torus glues the boundaries of the cylinder "with the same orientation" (relative to the base, but reversing fiber orientation), making M a non-orientable manifold (similar to the Klein bottle), so $w_1(M) \neq 0$. Similar to the conclusion in Galewski & Stern's construction, we have:

$$Sq^1\Delta(M) = \Delta(M) \smile w_1(M) \neq 0.$$

6 Manolescu's Work

Manolescu's work uses techniques from gauge theory. Specifically, it is a type of Floer homology called Pin(2)-equivariant Seiberg-Witten Floer homology. Gauge theory is the study of specific elliptic partial differential equations, first appearing in physics to reflect strong and weak interactions between particles. In the 1980s, Donaldson pioneered the application of gauge theory to low-dimensional topology. Floer homology, constructed from gauge theory, is an invariant of 3-manifolds useful in studying cobordism. (A cobordism between two 3-manifolds Y, Y' is a 4-manifold with initial boundary Y and final boundary Y'.) Atiyah called Floer homology a Topological Quantum Field Theory (TQFT). The main property of TQFT is that a cobordism from Y to Y' induces a map between the corresponding invariants of the two 3-manifolds (in this case, a map between their Floer homologies). For standard homology theories, we need an actual map (not a cobordism) between Y and Y' to obtain a map between homologies. Different types of Floer homology (e.g., Seiberg-Witten, Heegaard Floer) are primary tools for studying cobordisms between 3-manifolds, and resolving the Triangulation Conjecture is just one application.

6.1 Brief Summary of the Proof

From the work of Galewski-Stern and Matumoto, we know that whether the Triangulation Conjecture holds is equivalent to whether the short exact sequence 1 splits. Splitting is equivalent to $\exists [Y] \in \Theta_3$, $2[Y] = 0$, $\mu(Y) = 1$.

If we want to disprove the Triangulation Conjecture, we simply need to show that such a $[Y]$ does not exist. We just need to find a lift from the Rokhlin homomorphism

$\mu : \Theta_3 \to \mathbb{Z}/2$ to $M : \Theta_3 \to \mathbb{Z}$, i.e.,

$$\begin{tikzcd} & \mathbb{Z} \arrow[d, "\text{mod } 2"] \\ \Theta_3 \arrow[ur, dashed, "M"] \arrow[r, "\mu"] & \mathbb{Z}/2 \end{tikzcd}$$

Equivalently, we need to find an invariant $m(Y) \in \mathbb{Z}$ for an (oriented) integer homology sphere Y satisfying:

1. $m(Y)$ is a homology cobordism invariant, thus inducing (descending to) a map $M : \Theta_3 \to \mathbb{Z}$;
2. The mod 2 reduction of $m(Y)$ is $\mu(Y)$;
3. m satisfies $m(Y_1 \# Y_2) = m(Y_1) + m(Y_2)$, so M is a group homomorphism.

Thus, if $\mu(Y) = 1$, the order of $[Y]$ cannot be 2.

So far, we have not found an invariant satisfying all three conditions above. The Casson invariant $\lambda(Y)$ is a lift of $\mu(Y)$, but it is not a homology cobordism invariant. The Frøyshov invariant $h(Y)$, the map δ derived from Monopole Floer homology, and the Ozsvath-Szabo correction term $d(Y)$ derived from Heegaard Floer homology are all homomorphisms from Θ_3 to $\mathbb{Z}$, but they are not lifts of μ.

However, Manolescu used Pin(2)-equivariant theory to find **homology cobordism invariants** (invariant under homology cobordism relation) α, β, γ. Although they do not satisfy the above conditions (β is not a homomorphism; for example, let $Y = \Sigma(2, 3, 11)$ be the Brieskorn sphere, we have $\beta(Y) = 0$, but $\beta(Y \# Y) = 1$), β has the property: $\beta(-Y) = -\beta(Y)$. This is sufficient to disprove the Triangulation Conjecture.

Theorem 9.

$$0 \to \ker(\mu) \xrightarrow{\iota} \Theta_3 \xrightarrow{\mu} \mathbb{Z}/2 \to 0$$

is not split exact.

Proof. Proof by contradiction. Assume the exact sequence splits. Then there exists $[Y] \in \Theta_3, 2[Y] = 0$, and $\mu(Y) = 1 \in \mathbb{Z}/2$. Then $\beta(Y) \in \mathbb{Z}$ is an odd number. We have $\beta(-Y) = -\beta(Y)$. Since the order of Y is 2, Y and $-Y$ are homology cobordant, meaning $\beta(Y) = \beta(-Y)$. This implies $\beta(Y) = 0$, a contradiction. □

Although the invariant β constructed by Manolescu solved the Triangulation Conjecture, it does not satisfy $\beta(Y_1 \# Y_2) = \beta(Y_1) + \beta(Y_2)$, meaning it is not a homomorphism. Whether there exists a homomorphism $M : \Theta_3 \to \mathbb{Z}$ that is a lift of μ remains an open problem.

6.2 Kronheimer-Mrowka's Construction Method

First, we introduce some basic concepts: Monopole Floer homology consists of three finitely generated graded $\mathbb{F}[U]$-modules, where $deg(U) = -2$, meaning U acting on an element of the graded group reduces the degree by 2; $\mathbb{F} = \mathbb{Z}/2$. The definition of Monopole Floer homology requires the use of a $spin^c$ structure on the manifold. Specifically, it is a topological invariant defined on any orientable closed 3-manifold (orientable closed 3-manifolds have a unique *spin* structure, and thus a unique *spin*^{c} structure). Disproving the Triangulation Conjecture only requires its properties on integer homology spheres. Later we will prove that the three functions α, β, γ on integer homology spheres derived from it are invariants under the cobordism relation, and thus can be viewed as maps $\Theta \to \mathbb{Z}$.

Therefore, for brevity of construction, we only consider the Monopole Floer homology of integer homology spheres Y, namely $\widecheck{HM}(Y), \widehat{HM}(Y)$, and $\overline{HM}(Y)$. They correspond respectively to the homology of a **manifold with boundary of infinite dimension**, the homology relative to the boundary, and the homology of the boundary. Thus, it fits into the exact triangle:

$$\begin{tikzcd} \widecheck{HM}(Y) \arrow[rr, "j_*"] & & \widehat{HM}(Y) \arrow[dl, "p_*"] \\ & \overline{HM}(Y) \arrow[ul, "i_*"] & \end{tikzcd}$$

In fact, the corresponding cohomology groups of these homologies are isomorphic to the homology groups of $-Y$.

Regarding the construction of this infinite-dimensional manifold with boundary, there are two methods. One is given by Kronheimer and Mrowka [KM07]; Francesco Lin's two articles [Lin16][Lin17] also introduce how to construct Seiberg-Witten Floer homology using this method, with the idea of using "real blow-up" to handle reducible critical points. The other is given by Manolescu adopting Furuta's "finite dimensional approximation" method, with the idea of applying finite-dimensional Morse theory to infinite dimensions.

We will outline the proof idea of the first method and detail the construction process of the second method in the next section.

The infinite-dimensional manifold with boundary constructed by the first method is $\mathcal{B}_0^\sigma \times ES^1/S^1$. The process and reasoning are as follows:

First, we give some basic concepts to be used.

For an integer homology sphere Y, let g be a Riemannian metric on Y, and take the Levi-Civita connection ∇ on TY. Consider a trivial $\mathbb{C}^2$ bundle $S \to Y$. Define the action of TY on S:

$$\rho : TY \to \mathfrak{su}(S) \subset \mathrm{End}(S)$$

mapping an orthonormal frame $\{e_1, e_2, e_3\}$ of TY to Pauli matrices, i.e.,

$$\rho(e_1) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \rho(e_2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \rho(e_3) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

Using the canonical isomorphism $TY \cong T^*Y$ given by the metric and complex linear extension, we can induce a map, still denoted as ρ,

$$\rho : T^*Y \otimes \mathbb{C} \to \mathfrak{sl}(S) \subset \mathrm{End}(S)$$

Let A be a *spin* connection on $S \to Y$, i.e., the covariant derivative ∇_A satisfies $\nabla_A(\rho(v)\phi) = \rho(\nabla v)\phi + \rho(v)\nabla_a\phi$, where $v \in \Gamma(TY)$, and $\phi \in \Gamma(S)$ is a spinor.

The trivialization of TY provides a trivial connection A_0, and a *spin* connection can be written as $A = A_0 + a$, where $a \in \Omega^1(Y; i\mathbb{R})$.

The configuration space $\mathcal{C}(Y)$ consists of $(a, \phi) \in \Omega^1(Y; i\mathbb{R}) \oplus \Gamma(S)$, where $A_0 + a$ is a *spin* connection on S, and $\phi \in \Gamma(S)$ is a spinor.

Let the gauge group be $\mathcal{G}(Y) = \{f : Y \to S^1\}$. It acts naturally on $\mathcal{C}(Y)$:

$$f \cdot (a, \phi) = (a - f^{-1}\mathrm{d}f, f \cdot \phi).$$

When $\phi \neq 0$, this action is free. For $(a, 0)$, it has a stabilizer subgroup isomorphic to S^1, namely the constant maps $Y \to S^1$. Therefore, fixing a base point $y_0 \in Y$, the based gauge group $\mathcal{G}_0(Y) := \{f : Y \to S^1, f(y_0) = 1\}$ acts freely on $\mathcal{C}(Y)$. Thus we can define the infinite-dimensional manifold

$$\mathcal{B}_0(Y) = \mathcal{C}(Y)/\mathcal{G}_0(Y)$$

By definition, the S^1-action on $\mathcal{B}_0(Y)$ is free at points other than reducibles.

Define the Dirac operator $\partial\!\!\!/ : \Gamma(S) \to \Gamma(S)$,

$$\partial\!\!\!/(\phi) = \sum_{i=1}^{3} \rho(e_i)\nabla_{e_i}\phi$$

More generally, for a *spin* connection A, we can define the "twisted Dirac operator" $\partial\!\!\!/_A := \rho \circ \nabla_A$, i.e.,

$$\Gamma(S) \xrightarrow{\nabla_A} \Gamma(T^* \times S) \xrightarrow{\rho} \Gamma(S)$$

.

Next, define the Chern-Simons-Dirac functional $\mathrm{CSD} : \mathcal{C}(Y) \to \mathbb{R}$,

$$\mathrm{CSD}(a, \phi) = \frac{1}{2}\int_Y (\langle \phi, \partial\!\!\!/\phi + \rho(A)\phi \rangle \mathrm{d}\nu - a \wedge \mathrm{d}a)$$

where $\mathrm{d}\nu$ is the volume element of the Riemannian metric on Y.

Since Y is an integer homology sphere, CSD is a gauge invariant, meaning its value is invariant under the action of $\mathcal{G}(Y)$. Thus it can be viewed as an S^1-invariant functional on $\mathcal{B}_0(Y)$.

Similar to Morse homology, the critical points of CSD here will serve as generators for Monopole Floer homology, and solutions to the flow equation $\dot{x} = -\operatorname{grad}\mathrm{CSD}(x(t))$ will serve as boundary maps.

The gradient of CSD is

$$\operatorname{grad}\mathrm{CSD}(a, \phi) = (*\mathrm{d}a - 2\rho^{-1}((\phi \otimes \phi^*)_0), \partial\!\!\!/\phi + \rho(a)\phi) \in \mathcal{C}(S),$$

where ϕ^* denotes the dual section, so $\phi \otimes \phi^*$ is an endomorphism on S, and $(\phi \otimes \phi^*)_0$ denotes its traceless part. The critical points of CSD are determined by the Seiberg-Witten equations:

$$\widetilde{SW}(a, \phi) = \begin{cases} *\mathrm{d}a - 2\rho^{-1}(\phi \otimes \phi^*)_0 = 0, \\ \partial\!\!\!/\phi + \rho(a)\phi \end{cases}$$

However, the current objects cannot be realized as simple Morse homology due to problems such as:

1. Limit points on $\mathcal{B}_0/S^1$ are not isolated, so the S-W equations need perturbation to obtain isolated critical points;

2. The Hessian matrix of CSD at critical points may have infinitely many positive eigenvalues and infinitely many negative eigenvalues, so the index (number of negative eigenvalues) of critical points is not well-defined.

3. Reducible critical points are not manifold points of $\mathcal{B}_0/S^1$.

The first two problems can be overcome by standard methods of Floer theory. To solve the third problem, there are options: (a) ignore reducible critical points; (b) use the "real blow-up" operation. However, ignoring reducible critical points brings many problems: for instance, the result will not be a diffeomorphism invariant, and much topological information (such as the key information to disprove the Triangulation Conjecture) lies in the reducible critical points.

The real blow-up operation was pioneered by Kronheimer-Mrowka. Consider $\mathcal{C}^\sigma(Y) := \mathcal{C}(Y) \times \mathbb{R}^{\geqslant 0}$. The map $\pi : \mathcal{C}^\sigma(Y) \to \mathcal{C}(Y)$, $(a, \psi, s) \mapsto (a, s \cdot \psi)$ is the Blow-down map. Let $\mathcal{B}_0^\sigma = \pi(\mathcal{C}^\sigma(Y))/\mathcal{G}_0(Y)$ be an infinite-dimensional manifold with boundary.

For a general compact Lie group G, if it acts on space X, one can perform the Borel construction: define the equivariant cohomology of G as $H_*^G(X; \mathbb{F}) := H_*(X \times_G EG; \mathbb{F})$, where $EG \to BG$ is the universal principal G-bundle. Thus EG is a contractible space with free G action, and $X \times_G EG = (X \times EG)/G$ is the orbit space of the G action, where the action of $g \in G$ on the product space is defined as $g \cdot (x, e) = (gx, eg^{-1})$. Since $BG = EG/G$, there exists a fibration

$$\begin{tikzcd} X \arrow[r, hook] & X \times_G EG \arrow[d, "\pi"] \\ & BG \end{tikzcd}$$

Specifically, if G acts freely on X, then $X \times EG/G = X/G$.

And from this construction, the homology graded group $H_*^G(X; \mathbb{F})$ is a module over the cohomology ring $H_G^*(pt; \mathbb{F}) := H^*(BG; \mathbb{F})$, where the action of the cohomology ring on the homology group is the cap product.

Specifically, when $G = S^1$, $BG^1 = \mathbb{C}P^\infty$, and $H^*(\mathbb{C}P^\infty, \mathbb{F}) \cong \mathbb{F}[U]$, with $\deg(U) = -2$. The cobordism relation induces homomorphisms between $\mathbb{F}[U]$ modules.

In this case, we obtain $\widecheck{HM} = H_*^{S^1}(\mathcal{B}_0^\sigma(Y); \mathbb{F})$, which is isomorphic to $\mathbb{F}[U, U^{-1}]/\mathbb{F}[U]$. It is an infinitely long single chain, called a "tower". The degree of the lowest non-trivial homology group in the sequence is an invariant of homology cobordism. Dividing it by 2 yields a surjective homomorphism $\delta : \Theta_3 \to \mathbb{Z}$, and $\delta(Y) \equiv \mu(Y) \mod 2$. However, the chain complex here is not a topological invariant; it depends on the choice of metric g. In fact, δ here is not a lift of μ over $\mathbb{Z}$.

Although S^1-equivariant Monopole Floer homology failed to bring the invariant we needed, if we consider the Pin(2) action, the Seiberg-Witten equations will display more symmetry and yield the required invariants.

$\mathrm{Pin}(2) := \mathrm{S}^1 \cup \mathrm{jS}^1 \subset \mathbb{C} \cup \mathrm{j}\mathbb{C} \subset \mathbb{H}$. Its action on $\mathcal{B}_0(Y)$ is defined as follows: S^1 acts on $\mathbb{C}^2$ by complex multiplication, while j acts on $(v_1, v_2) \in \mathbb{C}^2$ to give $(-\bar{v}_2, -\bar{v}_1)$, and $j \cdot (a, \phi) := (-a, \phi j)$.

Similar to the S^1-equivariant case, $H_*^{\mathrm{Pin}(2)}(\mathcal{B}_0^\sigma(Y); \mathbb{F}) := H_*(\mathcal{B}_0^\sigma(Y) \times_{\mathrm{Pin}(2)} E\mathrm{Pin}(2); \mathbb{F})$, $\widecheck{HS} = H_*^{\mathrm{Pin}(2)}(\mathcal{B}_0^\sigma(Y); \mathbb{F})$, and we can further obtain $\widehat{HS}, \overline{HS}$. To avoid repetition, we will formally prove (using the Leray spectral sequence) in the next section that $H^*_{\mathrm{Pin}(2)}(pt; \mathbb{F}) =$

$H^*(B\mathrm{Pin}(2);\mathbb{F}) \cong \mathbb{F}[v,q]/(q^3)$, where $\deg(v) = -4, \deg(q) = -1$. Thus $\widecheck{HS}$ is a module over $\mathcal{R} = \mathbb{F}[v,q]/(q^3)$.

Here we have a Gysin sequence linking the two types of Monopole Floer homology: on $\widecheck{HM}$, regarding v as U^2 and q as the 0 map, we have

$$\cdots \to \widecheck{HS}(Y) \xrightarrow{\cdot q} \widecheck{HS}(Y) \to \widecheck{HM}(Y) \to \widecheck{HS}(Y) \to \cdots$$

which is a sequence of graded modules over $\mathcal{R}$.

As an $\mathbb{F}[v]$-module, $\widecheck{HS}(Y)$ has 3 infinitely long towers, linked to each other by multiplication by q. The degrees of their lowest non-trivial homology groups are directly related to $\mu(Y)$. Let the degrees of the lowest non-trivial homology groups of the three sequences be A, B, C. Then $\alpha = \frac{A}{2}, \beta = \frac{B-1}{2}, \gamma = \frac{C-2}{2}$ are invariants of Y, satisfying $\alpha \geqslant \beta \geqslant \gamma$, and $\equiv \mu(Y) \mod 2$. They are also homology cobordism invariants, i.e., can be viewed as maps $\Theta_3 \to \mathbb{Z}$. Furthermore, they satisfy $\alpha(-Y) = -\gamma(Y), \beta(-Y) = -\beta(Y), \gamma(-Y) = -\alpha(Y)$. The specific proof will be shown in the next section.

6.3 Manolescu's Construction Method

Manolescu adopted Furuta's "finite dimensional approximation" method, applying finite-dimensional Morse theory to infinite dimensions. The process of finite dimensional approximation is "uniformly convergent," so this method works.

The previous construction process is consistent with Kronheimer and Mrowka's approach. Here we pick up from after the construction of the Seiberg-Witten equations:

6.3.1 Seiberg-Witten Equations in Coulomb Gauge

Define the (global) Coulomb slice

$$V := \ker(d^*) \oplus \Gamma(S) \subset \mathcal{C}(Y, \mathfrak{s})$$

where $\mathfrak{s}$ is the unique *spin*^{c} structure on the homology sphere Y, and $d^* : \Omega^k(Y) \to \Omega^{k-1}(Y)$ is the codifferential operator, related to the Hodge star operator and the exterior differential operator by $d^*\omega = (-1)^{n(k+1)+1} * d * \omega$.

We can view V as the quotient space of the "normalized gauge group" $\mathcal{G}_0 \subset \mathcal{G}$ action,

$$\mathcal{G}_0 = \{u : Y \to S^1 \big| u = e^{\xi}, \xi : Y \to i\mathbb{R}, \int_Y \xi = 0\}$$

Since Y is an integer homology sphere, we have the Hodge decomposition

$$\Omega^1(Y) = \ker(d) \oplus \ker(d^*)$$

Fix $(a, \phi) \in V$. Let $\pi_V : T_{(a,\phi)}\mathcal{C}(Y, \mathfrak{s}) \to V$ be the linear projection such that the kernel of the projection is tangent to the $\mathcal{G}_0$ orbit. Let $T\mathcal{G}_0$ be the tangent space of the $\mathcal{G}_0$ orbit, so $\ker \pi_V \subset T\mathcal{G}_0$.

Recall the previously defined Seiberg-Witten equations

$$\widetilde{SW}(a, \phi) = \begin{cases} *\mathrm{d}a - 2\rho^{-1}(\phi \otimes \phi^*)_0 = 0, \\ \partial\!\!\!/\phi + \rho(a)\phi \end{cases}$$

Now let $SW := \pi_V \circ \widetilde{SW} : V \to V$.

Using the S^1 action $e^{i\theta} : (a, \phi) \mapsto (a, e^{i\theta}\phi)$, we obtain a bijection

$$\{\text{Flow lines determined by } \widetilde{SW}\}/\mathcal{G} \overset{1:1}{\longleftrightarrow} \{\text{Flow lines determined by } SW\}/S^1$$

Further let $\pi^{elc} : T_{(a,\phi)}\mathcal{C}(Y, \mathfrak{s}) \to T^{\perp}\mathcal{G}_0$, having $\ker \pi^{elc} \subset T\mathcal{G}_0$. The image of π^{elc} is called the extended local Coulomb slice K^{elc}, which is the orthogonal complement of the $\mathcal{G}_0$ orbit.

On the Coulomb slice V, the SW equation can be written as the sum of 1 linear part and 1 continuous part

$$SW = l + c$$

where $l, c : V \to V$ are defined as

$$l(a, \phi) = (*da, \not{\partial}\phi) \tag{5}$$

$$c(a, \phi) = \pi_V \circ (-2\rho^{-1}(\phi \otimes \phi^*)_0, \rho(a)\phi) \tag{6}$$

Let $V_{(k)}$ be the L^2_k completion of V, where $k \gg 0 \in \mathbb{N}$. Here we take $k > 5$, then $l : V_{(k)} \to V_{(k-1)}$ is a linear, self-dual, Fredholm operator, while $c : V_{(k)} \to V_{(k-1)}$ is a compact operator.

Below is the compactness theorem for the Seiberg-Witten equations suitable for the Coulomb gauge.

Theorem 10. *Fix $k > 5$. There exists $R > 0$ such that all critical points of SW and flow lines between critical points are contained in the ball $B(R) \subset V_{(k)}$.*

6.3.2 Finite Dimensional Approximation

Seiberg-Witten Floer homology is similar to the Morse homology of SW on V. However, compared to finding a generic perturbation on the Seiberg-Witten equations to realize transversality conditions, it is more convenient to use the method of finite dimensional approximation.

In the previous construction, V is an infinite-dimensional space. Its finite dimensional approximation is

$$V^{\mu}_{\lambda} = \bigoplus_{\lambda < \zeta < \mu} V(\zeta), \lambda \ll 0 \ll \mu$$

where $V(\zeta)$ is the eigenspace for eigenvalue ζ.

We can then replace $SW = l + c$ with

$$l + p^{\mu}_{\lambda}c : V^{\mu}_{\lambda} \to V^{mu}_{\lambda}$$

where $p^{\mu}_{\lambda} : V \to V^{\mu}_{\lambda}$ is the L^2 projection. Thus $SW^{\mu}_{\lambda} := l + p^{\mu}_{\lambda}c$ is a vector field on V^{μ}_{λ}.

For finite dimensional approximation, there is the following compactness theorem:

Theorem 11. *There exists $R > 0$ such that for all $\mu \gg 0 \gg \lambda$, all critical points of SW^{μ}_{λ} in $B(2R)$, and all flow lines connecting critical points in $B(2R)$ lie in a smaller ball $B(R)$.*

The idea of the proof is to use the fact that in $B(2R)$, $l + p^{\mu}_{\lambda}c$ converges uniformly to $l + c$, so the previous theorem can be used.

6.3.3 Conley Index

In the finite-dimensional case, the Morse homology of a compact manifold is exactly the usual homology. Now we have to deal with the non-compact space $B(2R) \subset V_\lambda^\mu$. In this case, Morse homology is the homology on the Conley index.

For a fixed m-dimensional manifold M and a flow $\{\phi_t\}$, the Conley index can be defined on an isolated invariant set S of $\{\phi_t\}$.

Definition 10. *For a subset $A \subseteq M$, define*

$$\mathrm{Inv}A = \{x \in M \big| \phi_t(x) \in A, \forall t \in \mathbb{R}\}$$

Definition 11. *If a compact set $S \subset M$ satisfies $S = \mathrm{Inv}A \subseteq \mathrm{Int}A$, where A is a compact neighborhood of S, then S is called an isolated invariant set of M.*

Definition 12. *For an isolated invariant set S, the Conley index $I(S) \coloneqq N/L$, where $L \subseteq N \subseteq M$, and L, N are compact sets satisfying*

1. $\mathrm{Inv}(N - L) = S \subset \mathrm{Int}(N - L)$
2. $\forall x \in N$, *if* $\exists t > 0$ *such that* $\phi_t(x) \notin N$, *then* $\exists \tau \in [0, t)$ *such that* $\phi_\tau(x) \in L$
3. $x \in L, t > 0$, $\phi_{[0,t]}(x) \subset N \Rightarrow \phi_{[0,t]}(x) \subset L$

That is to say, all flow lines exiting N must pass through L.

Now, we take $A = B(2R)$, so $S = \mathrm{Inv}A$. From the previous theorem, S is the union of all isolated points and flow lines therein. Next, N can be taken as a manifold with boundary, and $L \subset \partial N$ is a codimension 0 submanifold of ∂N, so L itself also has a boundary.

It can be proven that if the flow lines satisfy the Morse-Smale condition, then the Morse homology of $B(2R)$ is isomorphic to the reduced singular cohomology of $I(S)$.

6.3.4 Seiberg-Witten Floer Homology

Now we define the S^1-equivariant Seiberg-Witten Floer homology as an S^1-equivariant Borel homology,

$$SWFH_*^{S^1}(Y) \coloneqq \tilde{H}_{*+shift}^{S^1}(I_\lambda^\mu), \quad \mu \gg 0 \gg \lambda$$

Here I_λ^μ is the Conley index of $S^\mu \subset V_\lambda^\mu$, and the degree of the graded group is shifted by an amount depending on λ, μ. The shift amount will be specified below.

In fact, the above constructions are all Pin(2)-equivariant as well, because on a 3-dimensional homology sphere, the *spin*^{c} structure is a *spin* structure.

Therefore, we can obtain Pin(2)-equivariant Seiberg-Witten Floer homology. Specifically, we take coefficients in $\mathbb{F} = \mathbb{Z}/2$:

$$SWFH_*^{\mathrm{Pin}(2)}(Y; \mathbb{F}) \coloneqq \tilde{H}_{*+shift}^{\mathrm{Pin}(2)}(I_\lambda^\mu; \mathbb{F}), \quad \mu \gg 0 \gg \lambda$$

6.3.5 Invariant Properties of SWFH and Determination of the Shift

Although λ, μ in the definition can vary, when they are large enough, the defined Floer homologies $SWFH_*^{S^1}(Y)$ and $SWFH_*^{\mathrm{Pin}(2)}(Y)$ are invariants of Y.

Consider the flow equation $\dot{x} = -SW_\lambda^\mu(x(t))$ determined by $SW_\lambda^\mu = l + p_\lambda^\mu c : V_\lambda^\mu \to V_\lambda^\mu$, and examine the change in the Conley index I_λ^μ as μ, λ change. If we change $\mu \rightsquigarrow \mu' > \mu \gg 0$, we have the decomposition

$$\begin{array}{ccccc} V_\lambda^{\mu'} & = & V_\lambda^\mu & \oplus & V_\mu^{\mu'} \\ \vdots & & \vdots & & \vdots \\ l + p_\lambda^{\mu'} c & \rightsquigarrow & l + p_\lambda^\mu & \oplus & l + p_\mu^{\mu'} c \end{array}$$

and $l + p_\mu^{\mu'}$ depends almost entirely on the linear part l.

And the Conley index remains invariant under deformation, meaning if we have a family of flows $\varphi(s),\ s \in [0,1]$, such that

$$S(s) = \mathrm{Inv}(\text{determined by } \varphi(S) \text{ on } B(R)) \subset \mathrm{Int}B(R),\ s \in [0,1]$$

then $I(S(0)) \simeq I(S(1))$.

At this point, let $\varphi(0)$ be $l + p_\lambda^{\mu'}$. It can be deformed to $\varphi(1)$, defined as the direct sum of the flow of $l + p_\lambda^\mu$ and the linear flow l on $V_\mu^{\mu'}$. Thus we obtain

$$I_\lambda^{\mu'} = I(S(0)) = I(S(1)) = I_\lambda^\mu \wedge I_\mu^{\mu'}(l)$$

where $I_\mu^{\mu'}(l)$ is the Conley index of the linear flow $\dot{x} = -l(x)$ on $V_\mu^{\mu'}$.

Since l has only positive eigenvalues on $V_\mu^{\mu'}$, it implies

$$I_\mu^{\mu'}(l) = S^{(\text{Morse index})} = S^0$$

Thus $I_\lambda^{\mu'} = I_\lambda^\mu,\ \mu, \mu' \gg 0$.

On the other hand, when changing the lower bound λ of negative eigenvalues, the Conley index is

$$I_{\lambda'}^\mu = I_\lambda^\mu \wedge I_{\lambda'}^\lambda(l)$$

where $I_{\lambda'}^\lambda(l) = S^{|\lambda - \lambda'|}$. Therefore, the shift amount is the dimension of V_λ^0, i.e.,

$$\tilde{H}^{S^1}_{*+\dim V_\lambda^0}(I_\lambda^\mu)$$

When $\mu \gg 0 \gg \lambda$, it is independent of the values of λ, μ. The conclusion for Pin(2) is the same.

To show that *SWFH* is a topological invariant, we should also prove that it is independent of the choice of Riemannian metric g. Fix μ, λ. When g is perturbed, e.g., continuously changing from g_0 to g_1, the dimension of V_λ^μ does not change. However, the dimension of V_λ^0 may change. This change is determined by the "spectral flow" of the linear operator l, which counts with sign the number of eigenvalues crossing zero as g changes.

For the linear part $l = (*d, \not{\partial})$, since $H^1 = 0$, $*d$ has no zero eigenvalues. However, $\not{\partial}$ has spectral flow. Choose a spin 4-manifold W with boundary (Y, g), and attach a cylindrical end to the boundary, i.e., $W^4 \cup_Y Y \times [0,1]$. Then the spectral flow of $\not{\partial}$ is determined by the following formula:

$$\begin{aligned} SF(\not{\partial}) &= n(Y, g_0) - n(Y, g_1) \\ &= 2 \text{ index}(\not{\partial}) \text{ on } \ Y \times [0,1] \end{aligned}$$

Here $\text{index}(\not{\partial}) = \ker \not{\partial} - \text{coker}\not{\partial}$ represents the Atiyah-Singer index of $\not{\partial}$.

$$n(Y,g) = -2 \ (\text{index}_{\mathbb{C}}(\not{D}_W) + \frac{\sigma(W)}{8}) \in 2\mathbb{Z}$$

Thus $n(Y,g) \equiv 2\mu \mod 4$.

So we finally obtain

$$SWFH_*^{S^1}(Y) := \tilde{H}^{S^1}_{*+\dim V^0_\lambda - n(Y,g)}(I^\mu_\lambda)$$

and

$$SWFH_*^{\mathrm{Pin}(2)}(Y;\mathbb{F}) := \tilde{H}^{\mathrm{Pin}(2)}_{*+\dim V^0_\lambda - n(Y,g)}(I^\mu_\lambda;\mathbb{F})$$

are topological invariants of Y.

Similarly, generalized homology theories can be constructed, such as K-theory $K_*^{\mathrm{Pin}(2)}$, or Borel homology $\tilde{H}_*^G$, where G is any subgroup of Pin(2).

6.3.6 Seiberg-Witten Floer Stable Homotopy Type

In fact, the above construction yields an invariant stronger than homology groups, namely the Pin(2)-equivariant stable homotopy type *SWF*. We will explain its relationship with *SWFH* later.

Definition 13. *Without requiring equivariance, a suspension spectrum is* (X,n)*, where* X *is a topological space and* $n \in \mathbb{Z}$*. We consider* (X,n) *as formally de-suspending* X n *times, i.e.,*

$$(X,n) = \Sigma^{-n}X$$

And the n-th suspension of X is

$$\Sigma^n X = S^n \wedge X.$$

Let $[X,Y]$ denote the homotopy class of pointed maps. We can define a category where objects and morphisms are:

$$\begin{aligned} Obj &= (X,n) \\ Mor &= [(X,n),(Y,m)] = \begin{cases} \lim\limits_{\substack{N\to\infty \\ N-n\in\mathbb{Z}}} [\Sigma^{N-n}X, \Sigma^{N-n}Y], & m-n \in \mathbb{Z} \\ 0 & m-n \notin \mathbb{Z} \end{cases} \end{aligned}$$

We can similarly define Pin(2)-equivariant suspension spectra.

Since Pin(2) can be viewed as two symmetric S^1 connected by reflection j, it has the following irreducible representations:

$$\begin{cases} \mathbb{R} & \text{trivial action} \\ \tilde{\mathbb{R}} & \begin{cases} j & \text{multiplication by } -1 \\ S^1 & \text{trivial action} \end{cases} \\ \mathbb{H} & \text{Pin(2) left multiplication} \end{cases}$$

Therefore, a Pin(2)-equivariant suspension spectrum is a quadruple $(X, n_{\mathbb{R}}, n_{\tilde{\mathbb{R}}}, n_{\mathbb{H}})$.

The finite dimensional approximation of V can be decomposed as

$$V_\lambda^\mu \cong \tilde{\mathbb{R}}^a \oplus \mathbb{H}^b$$

where $\tilde{\mathbb{R}}$ is the component for the sign representation, and $\mathbb{H}$ is the component for the spinor representation.

Therefore, we define the Seiberg-Witten Floer equivariant spectrum of Y as

$$SWF(Y) := \Sigma^{\mathbb{H}^{\frac{n(Y,g)}{4}}} \Sigma^{-V_\lambda^0} I_\lambda^\mu$$

For a linear space V representing a group G, $\Sigma^V X = X \wedge S^V$, where S^V denotes the one-point compactification of V, which naturally has a G action, so $\Sigma^V X$ also has a natural G action.

It satisfies

$$H^{\mathrm{Pin}(2)}(SWF(Y); \mathbb{F}) = SWFH^{\mathrm{Pin}(2)}(Y; \mathbb{F}).$$

6.3.7 Homology Cobordism Invariants α, β, γ

We will use $SWFH^{\mathrm{Pin}(2)}(Y; \mathbb{F})$ to construct a map $\beta : \Theta_3^H \to \mathbb{Z}$ satisfying

1. $\beta(-Y) = -\beta(Y)$
2. $\beta(Y) \equiv \mu(Y) \mod 2$

In this process, we will also obtain two other maps $\alpha, \gamma : \Theta_3^H \to \mathbb{Z}$.

In the KM construction process, we mentioned the Borel homology group $H_*^{\mathrm{Pin}(2)}(X)$, which can be viewed as a module over the cohomology ring (Borel cohomology) $H^*_{\mathrm{Pin}(2)}(pt) = H^*(B\mathrm{Pin}(2)) \cong \mathbb{F}[v, q]/(q^3)$. Below we give the complete proof: For $\mathrm{Pin}(2) \subset SU(2)$, the inclusion map i induces a fibration

$$\begin{tikzcd} \mathrm{Pin}(2) \arrow[r, hook, "i"] & SU(2) \arrow[d, "\psi"] \\ & \mathbb{RP}^2 \end{tikzcd}$$

Here ψ is the quotient map of the Hopf fibration with the involution on S^2 (e.g., the antipodal map). This fibration continues to induce another fibration:

$$\begin{tikzcd} \mathbb{RP}^2 \arrow[r, hook] & B\mathrm{Pin}(2) \arrow[d] \\ & BSU(2) = \mathbb{HP}^\infty \end{tikzcd}$$

The cohomology ring of $\mathbb{RP}^2$ is generated by a generator q, acting as shown:

$$\begin{tikzcd} \mathbb{F} \arrow[r, bend left, "q"] & \mathbb{F} \arrow[r, bend left, "q"] & \mathbb{F} \end{tikzcd}$$

The cohomology ring of $BSU(2) = \mathbb{HP}^\infty$ is generated by a generator v, acting as shown:

$$\begin{tikzcd}
\mathbb{F} \arrow[rrrr, bend left, "v"] & 0 & 0 & 0 & \mathbb{F} \arrow[rrrr, bend left, "v"] & 0 & 0 & 0 & \mathbb{F} & \cdots
\end{tikzcd}$$

Therefore, by the Leray-Serre spectral sequence of the above fibration, $B\mathrm{Pin}(2)$ has the form:

$$\begin{tikzcd}
\mathbb{F} \arrow[rrrr, bend left, "v"] \arrow[r, bend right, "q"'] & \mathbb{F} \arrow[rrrr, bend left, "v"] \arrow[r, bend right, "q"'] & \mathbb{F} \arrow[rrrr, bend left, "v"] & 0 & \mathbb{F} \arrow[r, bend right, "q"'] & \mathbb{F} \arrow[r, bend right, "q"'] & \mathbb{F} & 0 & \cdots
\end{tikzcd}$$

Thus we obtain the isomorphism $H^*(B\mathrm{Pin}(2)) \cong \mathbb{F}[v,q]/(q^3)$, where $\deg v = 4, \deg q = 1$.

So the degrees of action of this cohomology ring on the homology graded group are $\deg q = -1, \deg v = -4$.

Now we derive the three "towers" in the homology graded group sequence:

Let $(I^\mu_\lambda)^{S^1}$ denote the fixed point set of I^μ_λ under the action of $S^1 \subset \mathrm{Pin}(2)$, which contains points in the reducible locus $\{(a,\phi)\big|\phi = 0\}$.

On this locus, SW generates a linear flow determined by $*da$.

Thus from this perspective, $(I^\mu_\lambda)^{S^1} = S^{\dim V^0_\lambda}$ is a sphere.

Also since

$$SWF(Y) := \Sigma^{\mathbb{H}^{\frac{n(Y,g)}{4}}} \Sigma^{-V^0_\lambda} I^\mu_\lambda$$

therefore

$$(SWF(Y))^{S^1} = S^{n(Y,g)}$$

Intuitively, $SWF(Y)$ consists of a reducible part $S^{n(Y,g)}$ and an irreducible part composed of some free cells. Thus we have

$$\text{Sphere} \subset SWF(F) \to SWF(Y)/\text{Sphere} \circlearrowleft \mathrm{Pin}(2) \text{ acts freely}$$

Therefore

$$SWFH^{\mathrm{Pin}(2)}_*(Y;\mathbb{F}) = \tilde{H}^{\mathrm{Pin}(2)}_*(SWF(Y);\mathbb{F})$$

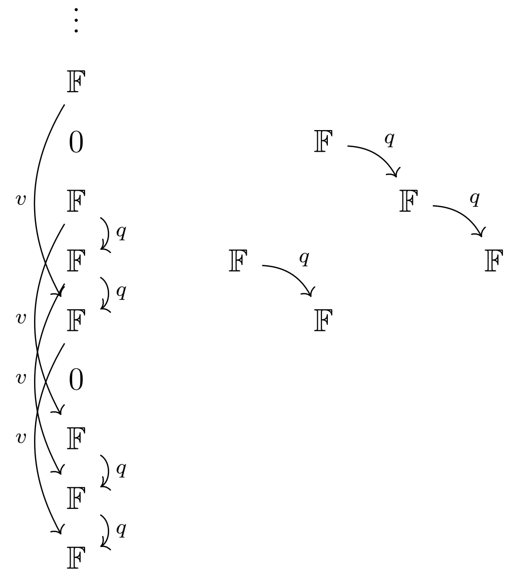
is a module over $\mathbb{F}[q,v]/(q^3)$.

For the equivariant cohomology ring, by the Localization Theorem, we obtain

$$V^{-1}\tilde{H}^*_{\mathrm{Pin}(2)}(SWF(Y);\mathbb{F}) = V^{-1}\tilde{H}^*_{\mathrm{Pin}(2)}(S^{n(Y,g)};\mathbb{F})$$

Note that $\tilde{H}^*_{\mathrm{Pin}(2)}(S^{n(Y,g)};\mathbb{F}) = H^{*-n(Y,g)}(B\mathrm{Pin}(2);\mathbb{F})$.

Since both homology and cohomology here take field coefficients, Borel homology is simply the dual of Borel cohomology. Additionally, we can apply the localization theorem to Borel homology, thus obtaining the complex structure of $SWFH^{\mathrm{Pin}(2)}_*$, i.e., the equivariant cellular structure of the Conley index, in the following form:



Where the finite part can be a vector space of any dimension, connected by v, q actions or ∂; additionally, there exists an infinitely long homology sequence composed of three infinitely long towers acted on by v. The infinite-dimensional part corresponds to the S^1-fixed point set of $SWF(Y)$, and the finite part corresponds to free cells.

Since $(SWF(Y))^{S^1} = S^{n(Y,g)}$, and $n(Y,g) \equiv 2\mu \mod 4$, we know:

- The degrees of all groups on the first tower from the bottom are $2\mu \mod 4$.
- The degrees of all groups on the second tower are $2\mu + 1 \mod 4$.
- The degrees of all groups on the third tower are $2\mu + 2 \mod 4$.

Now take the lowest degrees of the three towers as $A, B, C \in \mathbb{Z}$ respectively. Then we can construct

$$\alpha = \frac{A}{2}, \beta = \frac{B-1}{2}, \gamma = \frac{C-2}{2}$$

as invariants of Y, and $\alpha, \beta, \gamma \equiv \mu \mod 2$.

Furthermore, due to the module structure (i.e., q cannot map a 0 element to a non-zero element), we must have $\alpha \geqslant \beta \geqslant \gamma$.

Next we prove that they are indeed homology cobordism invariants.

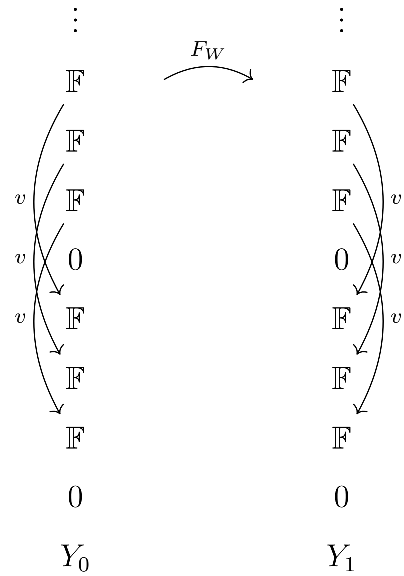
Let W^4 be a smooth oriented $spin(4)$ cobordism, with $b_1(W) = 0$, and $\partial W = (-Y_0) \cup Y_1$ (in our actual application, we only care about the case where Y_0, Y_1 are homology spheres). Consider the SW equations on W, and performing finite-dimensional approximation on the solution space, we will get conclusions similar to the 3D case. The final result is that we will obtain a stable equivariant map between two suspension spectra:

$$\Psi_W : \Sigma^{m\mathbb{H}} SWF(Y_0) \to \Sigma^{n\tilde{R}} SWF(Y_1)$$

Here $m\mathbb{H}$ is the direct product of m quaternion type representations, $n\tilde{\mathbb{R}}$ represents the direct product of n sign representations. And

$$m = \frac{-\sigma(W)}{9} = index(\not{D}), \quad n = b_2^+(W) = index(d^+)$$

Now, when W is a homology cobordism between homology spheres Y_0, Y_1, there exists a unique $spin(4)$ structure on W, and $b_1(W) = 0$, $m = n = 0$. Let F_W be the module homomorphism induced by Ψ_W between Pin(2)-equivariant *SWFH*, with the form



By equivariant localization, when $k \gg 0$, F_W is an isomorphism, and F_W is a module map, i.e., there is a commutative diagram

$$\begin{tikzcd} \mathbb{F} \arrow[r, "F_W"] \arrow[d, "v"] & \mathbb{F} \arrow[d, "v"] \\ \mathbb{F} \arrow[r, "F_W"] & \mathbb{F} \end{tikzcd}$$

Therefore, we must have

$$\alpha(Y_1) \geqslant \alpha(Y_0)$$
$$\beta(Y_1) \geqslant \beta(Y_0)$$
$$\gamma(Y_1) \geqslant \gamma(Y_0)$$

From the module map from $SWFH(Y_1)$ to $SWFH(Y_0)$, we can get inequalities in the other direction, so we have

$$\alpha(Y_1) = \alpha(Y_0)$$
$$\beta(Y_1) = \beta(Y_0)$$
$$\gamma(Y_1) = \gamma(Y_0)$$

This shows that α, β, γ are homology cobordism invariants.

6.3.8 Duality

So far we have proved that β is a homology cobordism invariant equal to $\mu \mod 2$. We are one step away from the invariant we need, which is that it should satisfy $\beta(-Y) = -\beta(Y)$.

For this, we are concerned with the changes in the topological invariant *SWFH* caused by reversing the orientation of the 3-dimensional homology sphere (Y, g) to become

$(-Y, g)$. At this time, the flow line equation determined by the Seiberg-Witten equations changes

$$\dot{x} = -SW(x(t)) \rightsquigarrow \dot{x} = SW(x(t))$$

For the finite dimensional approximation V_λ^μ, reversing orientation corresponds to the space pair (N, L_+) and (N, L_-) constructing the Conley index, such that N is a codimension 0 submanifold of V_λ^μ, and

$$L_+ \cup L_- = \partial N, \quad \partial L_+ = \partial L_- = L_+ \cap L_-$$

Since there is an embedding $X \subset V_\lambda^\mu \times \mathbb{R} = \mathbb{R}^{n+1}$ such that

$$X \simeq N/L_+, \quad \mathbb{R}^{d+1} - X \simeq N/L_-$$

by Alexander Duality, we obtain

$$\tilde{H}_*(N/L_+) = \tilde{H}^{d-*}(N/L_-) \tag{7}$$

Here $d = \dim(V_\lambda^\mu) = \dim(N)$.

However, in G-equivariant $SWFH$, we need a conclusion similar to 7. Before that, we introduce a weaker duality isomorphism theorem for stable homotopy versions, called Spanier-Whitehead duality.

Without requiring equivariance first, consider a suspension spectrum, i.e., the formal suspension of a topological space X:

$$\mathcal{Z} = (X, k) = \Sigma^{-k} X$$

and there is an embedding map $X \hookrightarrow S^N,\ N \gg 0$.

Definition 14. *The Spanier-Whitehead dual of* $\Sigma^{-k}X$ *is*

$$D(\Sigma^{-k}X) := \Sigma^k(\Sigma^{-(N-1)}(S^N - X))$$

By definition, $D(S^k) = S^{-k} = (S^0, k)$, and for two elements $\mathcal{Z}, \mathcal{W}$ in the suspension spectrum, D commutes with wedge product and smash product, i.e., $D(\mathcal{Z} \vee \mathcal{W}) = D(\mathcal{Z} \vee D(\mathcal{W}),\ D(\mathcal{Z} \wedge W) = D(\mathcal{Z}) \wedge D(\mathcal{W})$. By Alexander duality, we get

$$\tilde{H}_k(\mathcal{Z}) = \tilde{H}^{-k}(D(\mathcal{Z}))$$

For the equivariant case, there is a similar equivariant Spanier-Whitehead duality.

Definition 15. *Let G be a Lie group, X be a G-space, W be a representation of G. For some representation V of G, there is an embedding map $X \hookrightarrow V^+$. Then the Spanier-Whitehead dual on $\Sigma^{-W}X$ is*

$$D(\Sigma^{-W}X) := \Sigma^W(\Sigma^{-V}\Sigma^{\mathbb{R}}(V^+ - X)).$$

For the two Conley index space pairs with opposite orientations of V_λ^μ mentioned earlier, they respectively generate the Seiberg-Witten Floer spectra of Y and $-Y$, satisfying the Pin(2)-equivariant duality $D(SWF(Y)) = SWF(-Y)$.

However, for the equivariant case, the $SWFH$ of the dual spaces $\tilde{H}_*^G(\mathcal{Z})$ and $\tilde{H}_G^{-*}(D\mathcal{Z})$ may not be isomorphic. This is because the homology of the former has infinite non-trivial graded groups only in the positive direction, while the latter has them only in the negative direction.

For this reason, we need to introduce the concept of co-Borel homology (instead of dual cohomology of Borel homology).

Definition 16. *The co-Borel homology of an equivariant suspension spectrum is defined as*

$$c\tilde{H}_*^G(\mathcal{Z}) = \tilde{H}_G^{-*}(D\mathcal{Z})$$

where $\mathcal{Z} = \Sigma^{-V} X$*.*

Borel homology and co-Borel homology are linked by Tate homology.

Definition 17. *The Tate homology of* $\mathcal{Z} = \Sigma^{-V} X$ *is*

$$t\tilde{H}_*^G(\mathcal{Z}) = c\tilde{H}_*^G(\widetilde{EG} \wedge \mathcal{Z})$$

where $\widetilde{EG}$ *is the unreduced suspension of* EG*.*

An important property of Tate homology is

$$t\tilde{H}_*^G(\mathcal{Z}) = 0, \quad \text{if } G \text{ acts freely on } \mathcal{Z}.$$

We will use this property later to simplify Tate homology, leaving only the homology on the fixed points of the G action.

Additionally, Borel, co-Borel, and Tate homology satisfy the Tate-Swan exact sequence:

$$\cdots \to \tilde{H}^G_{n-\dim G}(\mathcal{Z}) \to c\tilde{H}^G_n(\mathcal{Z}) \to t\tilde{H}^G_n(\mathcal{Z}) \to \tilde{H}^G_{n-\dim G-1}(\mathcal{Z}) \to \cdots$$

When $G = S^1$, $\mathcal{Z} = SWF(Y)$, with Y being a homology sphere,

$$t\tilde{H}_*^{S^1}(\mathcal{Z}) = t\tilde{H}_*^{S^1}(\text{fixed point set of } S^1 \text{ action on } \mathcal{Z}) = t\tilde{H}_*^{S^1}(\text{Sphere}) = \mathbb{Z}[U, U^{-1}]$$

where $\deg U = -2$.

Similarly, when $G = \mathrm{Pin}(2)$, we have

$$\begin{aligned} t\tilde{H}_*^{\mathrm{Pin}(2)}(SWF(Y);\mathbb{F}) =& t\tilde{H}_*^{\mathrm{Pin}(2)}(\text{fixed point set of } S^1 \text{ action}; \mathbb{F}) \\ =& t\tilde{H}_*^{\mathrm{Pin}(2)}(\text{Sphere}) \\ =& \mathbb{F}[q, v, v^{-1}]/(q^3) \end{aligned}$$

Recalling the definition of co-Borel homology and $D(SWF(Y)) = SWF(-Y)$, we have $c\tilde{H}_*^{\mathrm{Pin}(2)}(SWF(Y);\mathbb{F}) = \tilde{H}^{-*}_{\mathrm{Pin}(2)}(SWF(-Y);\mathbb{F})$. Putting this into the Tate-Swan exact sequence, we obtain

$$\begin{gathered} \cdots \to \tilde{H}^{\mathrm{Pin}(2)}_{n-2}(SWF(Y);\mathbb{F}) \to \tilde{H}^{-n}_{\mathrm{Pin}(2)}(SWF(-Y);\mathbb{F}) \to \\ t\tilde{H}^{\mathrm{Pin}(2)}_n(SWF(Y);\mathbb{F}) \to \tilde{H}^{\mathrm{Pin}(2)}_{n-3}(SWF(Y);\mathbb{F}) \to \cdots \end{gathered}$$

Therefore, the element of degree $n-2$ in the first tower from the bottom in $SWFH(Y)$ corresponds to the element of degree $-n$ in the third tower from the top in the cohomology of $SWFH(-Y)$, which further corresponds to the element of degree $-n$ in the third tower from the bottom in $SWFH(-Y)$. Similarly, the second tower from the bottom in $SWFH(Y)$ corresponds to the second tower from the bottom in $SWFH(-Y)$, with the degree changing from $n-2$ to $-n$; the third tower from the bottom in $SWFH(Y)$ corresponds to the first tower from the bottom in $SWFH(-Y)$, with the degree changing from $n-2$ to $-n$.

Thus we obtain

$$\begin{aligned} \gamma(-Y) &= -\alpha(Y) \\ \beta(-Y) &= -\beta(Y) \\ \alpha(-Y) &= -\gamma(Y) \end{aligned}$$

7 Applications of 4-Dimensional Topology and Gauge Theory

2-dimensional topological closed manifolds can be completely classified via triangulation and computation of homology groups. 3-dimensional topological closed manifolds have a unique smooth structure. In the 1980s, Thurston proposed the Geometrization Conjecture, which was proven by Perelman in the early 21st century, also achieving geometric classification.

However, the classification of 4-dimensional closed manifolds is much more difficult. On one hand, many 4-dimensional topological manifolds do not admit smooth structures; on the other hand, even within the smooth category, classification cannot be achieved.

Theorem 12 (Markov[Mar58])**.** *There is no algorithm capable of distinguishing whether two 4-dimensional closed manifolds are smoothly homeomorphic.*

This is because the fundamental group $\pi_1(X)$ of a smooth 4-manifold X^4 can realize any finitely presented group $G = \langle S|R\rangle$, where S is the set of generators and R is the set of relations. However, Adyan and Rubin proved in 1955 that there is no algorithm to determine whether a finitely presented group is the trivial group, so classification of finitely presented groups is impossible, and thus classification of smooth 4-manifolds is impossible.

If we circumvent the classification obstacles caused by the complexity of the fundamental group, for example by considering simply connected manifolds, i.e., manifolds with trivial fundamental group $\pi_1 = 0$, we can obtain very rich conclusions.

For a closed simply connected oriented 4-manifold X, its $H_0 = \mathbb{Z}, \pi_1 = 0$. By the Hurewicz Theorem, we get $H_1 = 0$. Then by the Universal Coefficient Theorem and Poincaré duality, we know $H_4 = \mathbb{Z}, H_3 = 0, H_2 = \mathbb{Z}^b, b \geqslant 0$.

Therefore, using the dual pairing of generators of H_2, we can define a symmetric bilinear "intersection form"

$$\begin{aligned} Q_X : \mathbb{Z}^b \times \mathbb{Z}^b &\to \mathbb{Z} \\ (\xi, \eta) &\mapsto \langle \xi, D(\eta)\rangle \end{aligned}$$

where $D : H_2(X) \to H^2(X)$ is the Poincaré duality isomorphism.

From the duality pairing property, the determinant of the matrix of Q_X is ± 1. That is, Q_X is unimodular.

As early as the 1940s and 50s, mathematicians proved that intersection forms can achieve the classification of 4-manifolds in the sense of homotopy.

Theorem 13. *Let X be a closed simply connected oriented 4-manifold. Then the intersection form Q_X determines the homotopy type of X.*

Next, we wish to know which unimodular symmetric quadratic forms can be realized as the intersection form of some topological 4-manifold or smooth 4-manifold.

The theory of quadratic forms in linear algebra tells us that over $\mathbb{R}$, Q_X is congruent to $m\langle 1\rangle \oplus n\langle -1\rangle$. Let $b_2^+(X) = m, b_2^-(X) = n$ be the positive and negative indices of inertia. Then the signature of the intersection form is $\sigma(X) = b_2^+(X) - b_2^-(X)$, and clearly the Euler characteristic $\chi(X) = 2 + b_2(X) = 2 + b_2^+(X) + b_2^-(X)$.

Over $\mathbb{Z}$, symmetric unimodular bilinear forms can be divided into "definite" (positive definite $m = 0$ or negative definite $n = 0$) and "indefinite" ($m, n > 0$). They can also be divided into even and odd types. If $\forall x \in \mathbb{Z}, Q_X(x, x)$ is even, Q_X is called even; otherwise, it is called odd.

Indefinite forms have a complete algebraic classification. Odd indefinite forms are congruent to $m\langle 1\rangle \oplus n\langle -1\rangle$, $m, n > 0$. Even indefinite forms are of the form $p\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus qE_8$, $p > 0, q \in \mathbb{Z}$.

Here the matrix of E_8 can be written as

$$\begin{pmatrix}
-2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -2 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -2
\end{pmatrix}$$

However, definite unimodular bilinear forms have no direct classification. Examples like $n\langle 1\rangle$ are diagonalizable, while $E_8, E_8 \oplus E_8, D_{16}^+$, Leech lattice, etc., are non-diagonalizable examples.

Another algebraic result is that if Q_X is even, then its signature must be divisible by 8. The first non-trivial important theorem restricting intersection forms was given by Rokhlin in 1952.

Theorem 14. *If X is a closed spin smooth 4-manifold (e.g., a manifold that is simply connected and Q_X is even), then the signature of X is divisible by 16.*

From this theorem, we immediately get a corollary: since the signature of E_8 is 8, E_8 cannot be realized as the intersection form of a closed simply connected oriented smooth 4-manifold.

Since the theorem gives a necessary condition, we cannot use it to judge whether $E_8 \oplus E_8$ can be realized as the intersection form of a smooth 4-manifold.

In fact, intersection forms can realize not only homotopy classification, as in Theorem 13; Freedman gave a series of stronger conclusions in 1982:

Theorem 15 (Freedman[Fre82]).

- *For any unimodular symmetric bilinear form Q, there exists a simply connected closed topological 4-manifold X such that $Q_X \cong Q$;*
- *If Q is an even form, the corresponding manifold X is unique up to homeomorphism;*
- *If Q is an odd form, there are exactly two corresponding homeomorphism types, and at most one is smoothable. (One has Kirby-Siebenmann invariant 1, one has 0; KS invariant is 0 if and only if the topological manifold has an $\mathbb{R}^n$ vector bundle, analogous to a "tangent bundle").*

This not only demonstrates that all unimodular symmetric quadratic forms can be realized as intersection forms of some simply connected closed 4-manifold, but also gives their classification.

For example, there exists a 4-dimensional simply connected closed manifold M_{E_8} whose intersection form is E_8. By Rokhlin's theorem, it does not admit a smooth structure.

As another example, the intersection form of the smooth manifold $\mathbb{CP}^2$ is the odd type $\langle 1\rangle$. Therefore, there exists a "fake" 2-dimensional complex projective plane, denoted $*\mathbb{CP}^2$, which admits no smooth structure and is not homeomorphic to $\mathbb{CP}^2$. By Theorem 13, it is homotopy equivalent to $\mathbb{CP}^2$.

Since the intersection form of a 4-dimensional homotopy sphere is trivial, which is an even form, Freedman's theorem directly implies the 4-dimensional Topological Poincaré Conjecture: a 4-dimensional homotopy sphere is homeomorphic to the 4-dimensional sphere.

Starting from the 1970s, mathematicians introduced tools from gauge theory into the study of 4-dimensional topology. Donaldson in 1983 gave an application that shocked the mathematical community, namely Donaldson's Diagonalization Theorem:

Theorem 16 (Donaldson[Don83])**.** *If the intersection form Q_X of a closed smooth simply connected 4-manifold is definite, then $Q_X \cong n\langle 1\rangle, n \in \mathbb{Z}$.*

Through this theorem, we can not only show that E_8 cannot be realized as the intersection form of a smooth 4-manifold, but also that $E_8 \oplus E_8$ cannot. That is, $M_{E_8}\#M_{E_8}$ admits no smooth structure.

Another important conclusion in 4D topology is the existence of exotic smooth structures on $\mathbb{R}^4$. One way to construct this is to take $X = \mathbb{CP}^2\#9\overline{\mathbb{CP}^2}$. Then $Q_X = \langle 1\rangle \oplus 9\langle -1\rangle = (-E_8)\oplus\langle -1\rangle \oplus \langle 1\rangle$. Take α as the generator of the last $\langle 1\rangle$. By Freedman's theorem, there exists $\Sigma \cong \mathbb{S}^2$ representing α, and by Donaldson's Diagonalization Theorem, $\Sigma \cong \mathbb{S}^2$ is not smooth (otherwise $(-E_8)\oplus\langle 1\rangle$ would be diagonalizable, but it represents a smooth manifold, contradiction). Take U as a neighborhood of Σ. Then U can be embedded in $\mathbb{CP}^2$. Thus $\mathbb{CP}^2\backslash\Sigma \cong B^4$, which is homeomorphic but not diffeomorphic to $\mathbb{R}^4$.

Based on this idea, Gompf proved that there are infinitely many smooth structures on $\mathbb{R}^4$ [Gom85], and Taubes proved there are uncountably many [Tau87].

8 The Poincaré Conjecture

Finally, we introduce the Poincaré Conjecture. This is another very important problem driving the development of geometric topology. Its research path is similar to that of the Triangulation Conjecture, proceeding by category and dimension. Interestingly, the 3-dimensional triangulation problem was solved very early, while the 3-dimensional Poincaré Conjecture was only solved in this century by Perelman; the smooth triangulation problem was proven early, but the smooth Poincaré Conjecture remains unresolved. That is, the problem of the existence of exotic spheres in different dimensions, especially the existence of 4-dimensional exotic spheres, is a highly open problem.

In 1904, Poincaré proposed the conjecture:

Conjecture 4 (Poincaré Conjecture)**.** *Let M be a closed 3-dimensional manifold. If M is simply connected, then M is homeomorphic to S^3.*

For a century, this conjecture remained unresolved. But people turned to studying the analogue of the Poincaré Conjecture in other dimensions, namely the Generalized Poincaré Conjecture. This is correct for $n \geqslant 4$.

Theorem 17 (Generalized Poincaré Conjecture)**.** *Let M be a closed n-dimensional manifold. If M is homotopy equivalent to S^n, then M is homeomorphic to S^n.*

Since a closed 3-dimensional manifold is simply connected if and only if it is homotopy equivalent to S^3 (we give a short proof at the end of this section), the Generalized Poincaré Conjecture is the original Poincaré Conjecture when $n = 3$.

1. When $n = 1$, correct, because a closed curve is necessarily homeomorphic to S^1;
2. When $n = 2$, correct, by the classification theorem of closed surfaces, a simply connected closed surface must be S^2;
3. When $n = 3$, correct. In 2003, Perelman used Ricci flow to prove Thurston's Geometrization Conjecture (i.e., any closed 3-manifold can be decomposed along 2-spheres into pieces, each assigned one of eight homogeneous geometric structures; since a simply connected closed manifold cannot be decomposed, it can only have spherical geometry, i.e., S^3), thereby solving the Poincaré Conjecture;
4. When $n = 4$, correct. In 1982, Freedman developed the topological h-cobordism theory for 4-manifolds, which, combined with conclusions on intersection forms of 4-manifolds, can provide a proof (smooth h-cobordism theory can only be used for $n \geqslant 5$);
5. When $n \geqslant 5$, correct. In 1961, Smale used h-cobordism theory to give a proof, although the case for $n = 5$ requires Freedman's topological h-cobordism theory.

Conjecture 5 (Smooth Poincaré Conjecture)**.** *Let M be a closed n-dimensional manifold. If M is homotopy equivalent to S^n, then M is diffeomorphic to S^n.*

For a review of the Smooth Poincaré Conjecture, one can refer to Guozhen Wang's article [WX17].

In sufficiently high dimensions, all odd-dimensional spheres possess exotic smooth structures. Specifically, the only odd-dimensional spheres with a unique smooth structure are S^1, S^3, S^5, S^{61}.

More than half of the even dimensions have been proven to possess exotic spheres; it is conjectured that they exist in the remaining even dimensions as well [BMQ23].

And there is a conjecture:

Conjecture 6. *For spheres of dimension greater than 4, the only ones with a unique smooth structure are $S^5, S^6, S^{12}, S^{56}, S^{61}$.*

Research on the theory of exotic spheres is currently progressing rapidly, and people believe this conjecture is correct.

Regarding the existence of 4-dimensional exotic spheres, although it is a highly open problem, people tend to believe that 4-dimensional exotic spheres do exist. Because we have discovered that 4-dimensional space possesses "wild" properties: for example, $\mathbb{R}^4$ has infinitely many smooth structures that are not mutually diffeomorphic.

Conjecture 7 (P.L. Poincaré Conjecture)**.** *Let M be a closed n-dimensional manifold. If M is homotopy equivalent to S^n, then M is P.L. homeomorphic to S^n.*

The P.L. Poincaré Conjecture has been solved for all dimensions except 4. That is, the P.L. Poincaré Conjecture is correct for $n \neq 4$ [Buo0s]. Since smooth structures are equivalent to P.L. structures for $n \leqslant 6$, the 4-dimensional P.L. Poincaré Conjecture is equivalent to the existence of 4-dimensional exotic spheres.

Proposition 1. *Let M^3 be a closed 3-dimensional manifold. Then M is simply connected $\Longleftrightarrow$ M is homotopy equivalent to S^3.*

Proof. $\Leftarrow$: $\pi_1(M) = \pi_1(S^3) = 0$, so M is simply connected;

$\Rightarrow$: If M is simply connected, its connected orientable covering is the trivial covering, meaning M is an orientable manifold, so $H_3(M) = \mathbb{Z}$. Also since $H_1(M)$ is the abelianization of $\pi_1(M)$, $\pi_1(M) = 0$ implies $H_1(M) = 0$. By the Universal Coefficient Theorem, $H^1(M) = 0$. Then by Poincaré Duality, $H_2(M) \cong H^1(M) = 0$. By the Hurewicz Theorem, $\pi_2(M) \cong H_2(M) = 0$, and consequently $\pi_3(M) \cong H_3(M) \cong \mathbb{Z}$. This means a generator of $\pi_3(M)$ can be determined by a map $S^3 \to M$ of degree 1, inducing an isomorphism between H_3 and π_3. Furthermore, there exists a map from S^3 to M (regarded as simply connected simplicial complexes) that induces isomorphisms on all homology groups. By Whitehead's Theorem, this map is a homotopy equivalence. $\square$

The idea for the necessity part of the proposition comes from [Hat04].

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