

assignment6_solutions

February 3, 2022

```
[ ]: import numpy as np
import matplotlib.pyplot as plt

plt.rcParams['figure.figsize'] = (15, 7)
np.set_printoptions(formatter={'float': lambda x: f"{x:0.3f}"})
```

1 Question 1

Let π be the fraction of the \$1,000,000.00 invested in the risky asset and r be the return from the riskless asset. Then, the wealth of the portfolio after a year is:

$$W = \pi(1 + x) + (1 - \pi)(1 + r)$$

where x is the annual returns on the portfolio. Since $x \sim N(\mu, \sigma^2)$, we know that $W \sim N(1 + r + \pi(\mu - r), \pi^2\sigma^2)$. Therefore, we have:

$$\begin{aligned} - \mathbb{E}(U(W)) &= \mathbb{E}\left(W - \frac{\alpha}{2}W^2\right) = 1 + r + \pi(\mu - r) - \frac{\alpha}{2}\left((1 + r + \pi(\mu - r))^2 + \pi^2\sigma^2\right) \\ - x_{\text{CE}} &= U^{-1}\left(1 + r + \pi(\mu - r) - \frac{\alpha}{2}\left((1 + r + \pi(\mu - r))^2 + \pi^2\sigma^2\right)\right) \\ - \pi_A &= \mu - U^{-1}\left(1 + r + \pi(\mu - r) - \frac{\alpha}{2}\left((1 + r + \pi(\mu - r))^2 + \pi^2\sigma^2\right)\right) \end{aligned}$$

To find the optimal value for z , given α , we need to maximize x_{CE} :

$$\begin{aligned} \frac{\partial x_{\text{CE}}}{\partial \pi} &= \frac{(1 - \alpha r)(\mu - r) - 2\pi\alpha((\mu - r)^2 + \sigma^2)}{1 - \alpha U^{-1}\left(1 + r + \pi(\mu - r) - \frac{\alpha}{2}\left((1 + r + \pi(\mu - r))^2 + \pi^2\sigma^2\right)\right)} \\ \therefore \frac{\partial x_{\text{CE}}}{\partial \pi} \Big|_{\pi=\pi^*} &= 0 \iff (1 - \alpha r)(\mu - r) - 2\pi^*\alpha((\mu - r)^2 + \sigma^2) = 0 \\ \therefore \pi^* &= \frac{(1 - \alpha r)(\mu - r)}{2\alpha((\mu - r)^2 + \sigma^2)} \implies z^* = \$1,000,000 \times \frac{(1 - \alpha r)(\mu - r)}{2\alpha((\mu - r)^2 + \sigma^2)} \end{aligned}$$

The first equality comes from the fact that:

$$(f^{-1}(x))' = \frac{1}{f'(f^{-1}(x))}$$

I now find the corresponding value of z^* for every value of α between 0 and 15 and plot the result:

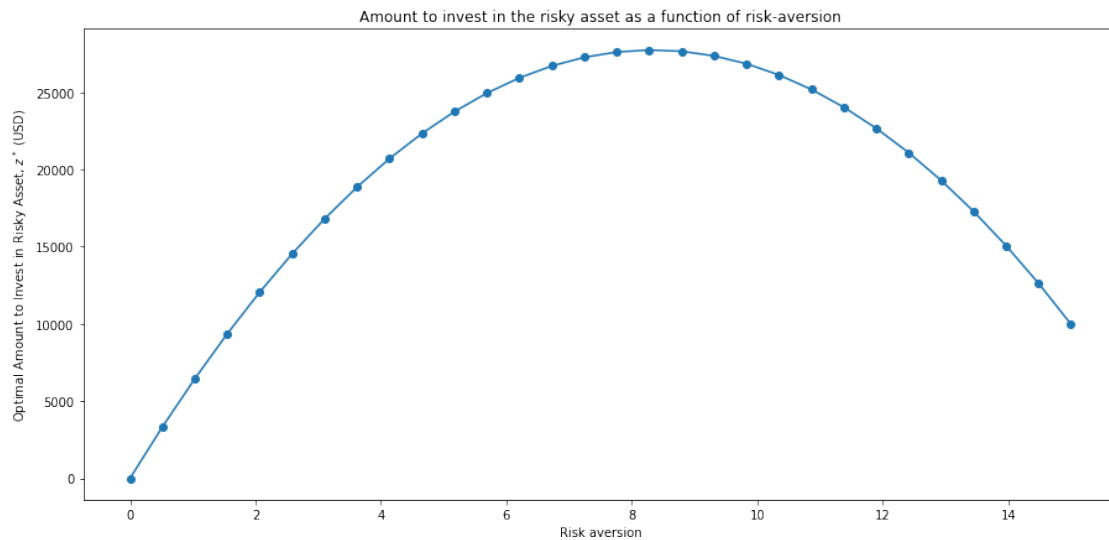
```
[ ]: mu = 0.14 # annual returns of 25%
sigma2 = 0.4 ** 2 # risk
rf = 0.06 # riskfree return of 7%
```

```
z_star = lambda mu, sigma2, rf, a : 1e6 * (1 - a * rf) * (mu - rf) / 2 * a * \
    ((mu - rf) ** 2 + sigma2)
```

```
alpha = np.linspace(0.0, 15, 30)
z = z_star(mu, sigma2, rf, alpha)
```

```
[ ]: plt.plot(alpha, z)
plt.scatter(alpha, z)

plt.title("Amount to invest in the risky asset as a function of risk-aversion")
plt.xlabel("Risk aversion")
plt.ylabel("Optimal Amount to Invest in Risky Asset, $z^*$ (USD)")
plt.show()
```



2 Question 3: Kelly's Criterion

After betting $f \cdot W_0$ amount in the game, we can get the returns of either α or $-\beta$. Therefore, the wealth you can have after the game is:

$$W_1 = \begin{cases} W_0 (1 + f \cdot \alpha), & \text{with probability } p \\ W_0 (1 - f \cdot \beta), & \text{with probability } 1 - p \end{cases}$$

Therefore, the utility obtained after playing the game becomes:

$$\log(W_1) = \log(W_0) + \begin{cases} \log(1 + f \cdot \alpha), & \text{with probability } p \\ \log(1 - f \cdot \beta), & \text{with probability } 1 - p \end{cases}$$

Accordingly, the expected utility after the game becomes:

$$\mathbb{E}(\log W_1) = \log W_0 + p \cdot \log(1 + f \cdot \alpha) + (1 - p) \cdot \log(1 - f \cdot \beta)$$

To find the optimal fraction, f^* , we set the first derivative of the above function to 0 and solve for f :

$$\begin{aligned} \frac{\partial \mathbb{E}(\log W_1)}{\partial f} &= \frac{p \cdot \alpha}{1 + \alpha \cdot f} - \frac{(1 - p) \cdot \beta}{1 - \beta \cdot f} \\ \left. \frac{\partial \mathbb{E}(\log W_1)}{\partial f} \right|_{f=f^*} &= 0 \implies p\alpha(1 - \beta f^*) = \beta(1 - p)(1 + \alpha f^*) \\ \therefore f^* &= \frac{p}{\beta} - \frac{1 - p}{\alpha} \end{aligned}$$

This makes sense intuitively, because if the probability of getting positive returns (α) is higher, you should invest more of your wealth and vice versa. On the other hand, if α becomes smaller, f^* also becomes smaller, meaning that you shouldn't invest more since the risk isn't worth the returns. Similar reasoning for β concludes this question.