Assignment3-solutions

February 5, 2022

[ ]:
```
import sys
sys.path.append("../") # to access code from rl.distribution, etc.

from dataclasses import dataclass
import numpy as np
from typing import Mapping, Dict, Tuple
from rl.distribution import Categorical, FiniteDistribution, Distribution
from rl.markov_process import FiniteMarkovProcess, FiniteMarkovRewardProcess
from rl.markov_decision_process import FiniteMarkovDecisionProcess
from rl.policy import FiniteDeterministicPolicy
import matplotlib.pyplot as plt
import itertools

np.set_printoptions(formatter={'float': lambda x: "{0:.3f}".format(x)})
plt.rcParams['figure.figsize'] = (25, 10)
```

1 Question 1:

With a deterministic policy, we know that:

$$\pi(s) = \begin{cases} 1, & \text{if } a = \pi_D(s) \\ 0, & \text{otherwise} \end{cases}$$

Therefore, the MDP Bellman equations become:

$$\begin{aligned}
V^{\pi_D}(s) &= R^{\pi_D}\left(s, \pi_D(s)\right) + \gamma \cdot \sum_{s' \in \mathcal{N}} P^{\pi_D}\left(s, \pi_D(s), s'\right) \cdot V^{\pi_D}(s') \\
V^{\pi_D}(s) &= Q^{\pi_D}\left(s, \pi_D(s)\right) \\
Q^{\pi_D}\left(s, \pi_D(s)\right) &= R^{\pi_D}\left(s, \pi_D(s)\right) + \gamma \cdot \sum_{s' \in \mathcal{N}} P^{\pi_D}\left(s, \pi_D(s), s'\right) \cdot V^{\pi_D}(s') \\
Q^{\pi_D}\left(s, \pi_D(s)\right) &= R^{\pi_D}\left(s, \pi_D(s)\right) + \gamma \cdot \sum_{s' \in \mathcal{N}} P^{\pi_D}\left(s, \pi_D(s), s'\right) \cdot Q^{\pi_D}\left(s', \pi_D(s')\right)
\end{aligned}$$

2 Question 2:

We see that:

$$
\begin{aligned}
V(s) &= \mathbb{E}\left(\sum_{t=1}^{\infty} G_t \,\middle|\, S_t = s\right)\\
\therefore V(s) &= \mathbb{E}\left(\sum_{t=1}^{\infty} \left(\frac{1}{2}\right)^{t-1} R_i \,\middle|\, S_t = s\right)\\
\therefore V(s) &= \sum_{t=1}^{\infty} \left(\frac{1}{2}\right)^{t-1} \mathbb{E}\left(R_i \,|\, S_t = s\right)\\
\therefore V(s) &= \sum_{t=1}^{\infty} \left(\frac{1}{2}\right)^{t-1} \cdot [(1-a)\cdot a + (1+a)\cdot(1-a)]\\
\therefore V(s) &= (1-a)(1+2a)\cdot \sum_{t=0}^{\infty} \left(\frac{1}{2}\right)^{i}\\
\therefore V(s) &= 2(1-a)(1+2a)
\end{aligned}
$$

The optimal value function can be found as shown below:

$$
\begin{aligned}
V^*(s) &= \max_{a\in[0,1]} 2(1-a)(1+2a) = 2\left(1-\frac{1}{4}\right)\left(1+2\cdot\frac{1}{4}\right) = \frac{9}{4}\\
\pi_D(s) &= \arg\max_{a\in[0,1]} 2(1-a)(1+2a) = \frac{1}{4}
\end{aligned}
$$

Question 3: - The state space is: $\mathcal{S} = \{0, 1, \cdots, n\}$ where n represents the lilypad - The action space is: $\mathcal{A} = \{'A', 'B'\}$ - The reward for all states is:

$$
R(s) = \begin{cases} 0, & \text{if } s \neq n\\ 1, & \text{otherwise} \end{cases}
$$

since we want to find the **probability** of escaping the pond.

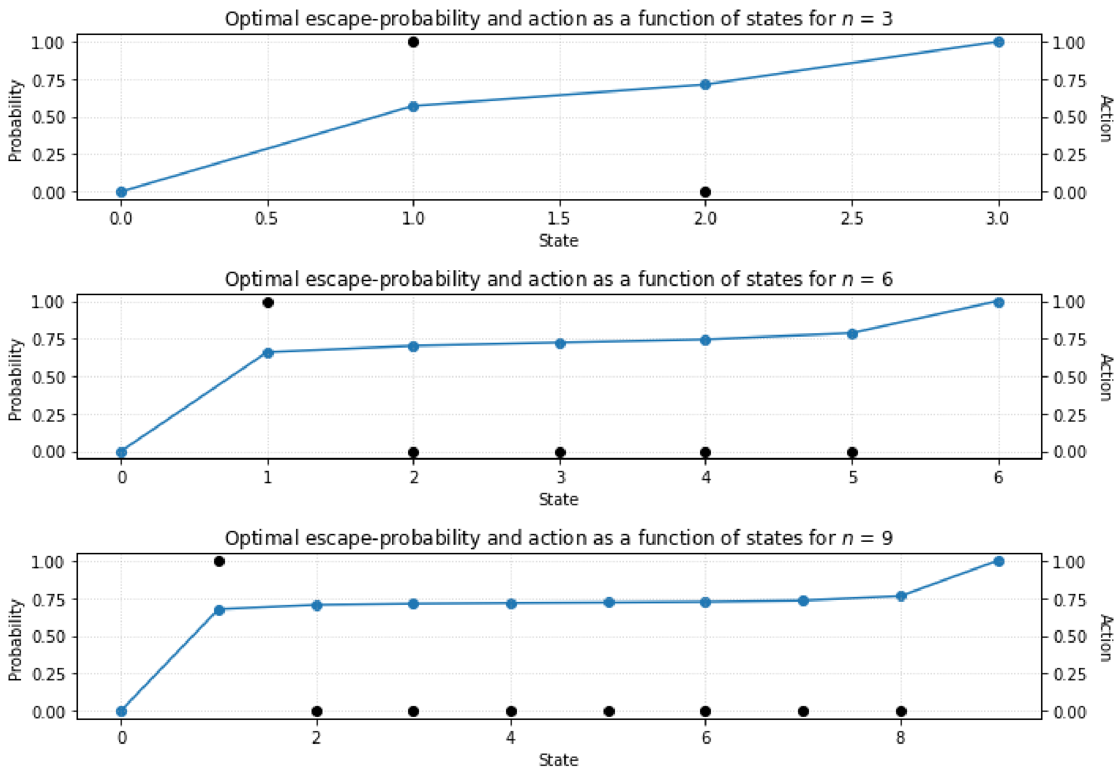
[ ]:
```
@dataclass(frozen = True)
class Position:
        position : int

ActionMapping = Mapping[Position, Mapping[str, Categorical[Tuple[Position,
 ↪float]]]]

class FrogEscape(FiniteMarkovDecisionProcess[Position, str]):
        def __init__(self, length : int):
                self.length = length # resembles n in the problem
                super().__init__(self.get_action_transition_reward_map())

        def get_action_transition_reward_map(self) -> ActionMapping:
                d : Dict[Position, Dict[str, Categorical[Tuple[Position,
 ↪float]]]] = {}
```

```
        for state in range(1, self.length):
            state = Position(position = state)

            inner_dict : Dict[str, Categorical[Tuple[Position,␣
↪float]]] = {}

            # if the frog croaks sound A...
            dict_A = {}

            dict_A[(Position(position = state.position - 1), 0.0)]␣
↪= state.position / self.length
            if state.position + 1 == self.length: # if the frog␣
↪escapes, reward = 1.0, else 0.0
                dict_A[(Position(position = self.length), 1.0)]␣
↪= 1 - state.position / self.length

            else:
                dict_A[(Position(position = state.position +␣
↪1), 0.0)] = 1 - state.position / self.length

            inner_dict['A'] = Categorical(dict_A)

            # if the frog croaks sound B... if the frog escapes,␣
↪reward = 1.0, else 0.0
            dict_B = {(Position(position = i), 0.0) : 1. / self.
↪length for i in range(self.length) if i != state.position}
            dict_B[(Position(position = self.length), 1.0)] = 1. /␣
↪self.length
            inner_dict['B'] = Categorical(dict_B)

            d[state] = inner_dict

        return d
```

2.0.1 For different lengths of the river, I create an instance of the Markov decision process FrogEscape and then find the optimal value function and policy amongst all the possible 2^{n-1} policies possible, and plot the optimal escape probabilities and the optimal policies, as a function of the length of the river:

[ ]:
```
lengths = range(3, 10, 3)
fig, axes = plt.subplots(len(lengths), figsize = (10, 7))
for j, length in enumerate(lengths):
    puzzle : FiniteMarkovDecisionProcess[Position, str] = FrogEscape(length =␣
↪length)

    # generate all possible combinations of actions for a given position
```

```
    outcomes = [[(pos, act) for act in ['A', 'B']] for pos in range(1, length)]
    optimal_policy = None # will store the optimal policy
    optimal_value = None # will store the optimal value function
    max_val = - np.inf # will be used to find the optimal value function

    # then form the cartesian product of all those lists
    for i in itertools.product(*outcomes):
        my_dict : Dict[Position, str] = {Position(position = pos) : act for␣
↪pos, act in i}
        policy : FiniteDeterministicPolicy[Position, str] =␣
↪FiniteDeterministicPolicy(my_dict)
        implied_mrp : FiniteMarkovRewardProcess[Position] = puzzle.
↪apply_finite_policy(policy)
        val = implied_mrp.get_value_function_vec(gamma = 1.0)

        # store the optimal policy
        if np.linalg.norm(val, ord = 1) > max_val:
            optimal_policy = my_dict
            optimal_value = val
            max_val = np.linalg.norm(val, ord = 1)

    # plot the optimal escape-probabilities
    optimal_value = [0.] + [i for i in optimal_value] + [1.0]
    axes[j].plot(range(length + 1), optimal_value)
    axes[j].scatter(range(length + 1), optimal_value)

    ax2 = axes[j].twinx()
    policy = [0.0 if act == 'A' else 1.0 for act in optimal_policy.values()]
    ax2.scatter(range(1, length), policy, color = 'black')

    axes[j].set_title(f"Optimal escape-probability and action as a function of␣
↪states for $n$ = {length}")
    axes[j].set_xlabel("State")
    axes[j].set_ylabel("Probability")
    ax2.set_ylabel("Action", rotation = -90, labelpad=15)
    axes[j].grid(alpha = 0.6, linestyle = ':')

fig.tight_layout()
plt.show()
```



3 Question 4:

Since $s' \sim N\left(s, \sigma^2\right)$, we know that $e^{as'} \sim \text{lognormal}\left(a\mu, a^2\sigma^2\right)$ and therefore, we need to minimize the following:

$$\min_{a \in \mathbb{R}} \mathbb{E}\left(e^{as'}\right) = \exp a\mu + \frac{a^2\sigma^2}{2}$$

By letting $f(a) = \exp a\mu + \frac{a^2\sigma^2}{2}$, we see that:

$$f'(a) = \exp a\mu + \frac{a^2\sigma^2}{2} \cdot \left(\mu + a\sigma^2\right)$$
$$f''(a) = \exp a\mu + \frac{a^2\sigma^2}{2} \cdot \left[\left(\mu + a\sigma^2\right)^2 + \sigma^2\right] \geq 0 \, \forall \, a \in \mathbb{R}$$

Therefore, we know that $f(a)$ does have a global minimum, which is achieved at the following optimal policy:

$$a^* = -\frac{\mu}{\sigma^2}$$

The corresponding optimal value function is:

$$f\left(a^*\right) = \exp -\frac{\mu}{2\sigma^2}$$