Question 1

Let us denote the following variables:

- c_i is the cash left at the end of day i
- y_i is the amount borrowed from the other bank at the end of day i
- π_i is the amount invested in the risky asset at the end of day i
- dw_i is the amount of withdrawal unfulfilled on day i
- d_i is the total amount deposited in the bank on day i
- w_i is the total amount withdrawn from the bank on day i

Since I need to keep track of how much **unfulfilled** withdrawals to pay on day t, I will need to know w_{t-1}. I also need to know what the current cash amount is, c_t, as well as the amount to be invested in the risky asset on day $t-1$, denoted as π_{t-1}, since this will result in some returns the next day. Lastly, I also keep track of the amount borrowed yesterday, y_{t-1}, since I have to make that payment today. Therefore, my state space is:

$$\mathcal{S} = \{\,(c_t, dw_{t-1}, \pi_{t-1}, y_{t-1}) \mid c_t, w_{t-1}, \pi_{t-1}, y_{t-1} \geq 0\}$$

The actions we have to decide are how much to borrow from the other bank at the end of day t and how much to invest in the risky asset:

$$\mathcal{A} = \{\,(\pi_t, y_t) \mid \pi_t, y_t \geq 0\}$$

The transitions of this MDP are as follows:

$$c_t = c_{t-1} + y_t - (1+R)y_{t-1} - \pi_t + (1+x)\pi_{t-1} + (d_t - w_t - dw_{t-1}) - K\cot\left(\frac{\pi c_t}{2C}\right) \cdot I_{c_t \leq C}$$

where x is the distribution of the returns from the risky asset, I_A is the indicator function of the event A. For this MDP, the reward per day would simply be the utility of the assets less liabilities, $c_t - c_{t-1}$. Furthermore, d_t and w_t are not specified but will be inputted by the user to model the flow of deposits and withdrawals over the horizon. These variables along with x introduce a degree of randomness into the MDP.

To find the optimal value function and policy, we will need to use ADP techniques, owing to the fact that the state-action spaces are continuous (and hence, huge to implement with DP techniques).

Question 2

We need to maximize:

$$g(S) = p \cdot \int_S^\infty (x - S) \cdot f(x)\, dx + h \cdot \int_{-\infty}^S (S - x) \cdot f(x) dx$$

We see that:

$$\begin{aligned}
g(S) &= p \cdot \int_S^\infty (x - S) \cdot f(x)\, dx + h \cdot \int_{-\infty}^S (S - x) \cdot f(x) dx \\
\therefore g'(S) &= -p \int_S^\infty f(x)\, dx + h \int_{-\infty}^S f(x)\, dx \quad \text{(using Leibniz's rule)} \\
\therefore g'(S) &= -p \cdot (1 - F(S)) + h \cdot F(S) \quad \text{(here } F \text{ is the CDF of } x\text{)}
\end{aligned}$$

By setting the derivative to 0 and solving for S^*, we get:

$$\begin{aligned}
p \cdot F(S^*) + h \cdot F(S^*) &= p \\
\therefore F(S^*) &= \frac{p}{p+h} \\
\therefore S^* &= F^{-1}\left(\frac{p}{p+h}\right)
\end{aligned}$$

As a confirmation, the second derivative is:

$$g''(S) = p \cdot f(S) + h \cdot f(S) \geq 0$$

as $f(\cdot)$ is the PDF of x and hence, is non-negative. This is similar to holding a portfolio of p calls and h puts, all with the same strike price of $K = S$.