Assignment 6 - solutions

February 6, 2022

```
[ ]: import numpy as np
     import matplotlib.pyplot as plt

     plt.rcParams['figure.figsize'] = (15, 7)
     np.set_printoptions(formatter={'float': lambda x: f"{x:0.3f}"})
```

1 Question 1

Let π be the fraction of the \$1,000,000.00 invested in the risky asset and r be the return from the riskless asset. Then, the wealth of the portfolio after a year is:

$$W = \pi(1+x) + (1-\pi)(1+r)$$

where x is the annual returns on the portfolio. Since $x \sim N\left(\mu, \sigma^2\right)$, we know that $W \sim N\left(1+r+\pi(\mu-r), \pi^2\sigma^2\right)$. Therefore, we have:

- $\mathbb{E}\left(U(W)\right) = \mathbb{E}\left(W - \frac{\alpha}{2}W^2\right) = 1+r+\pi(\mu-r) - \frac{\alpha}{2}\left(\left(1+r+\pi(\mu-r)\right)^2 + \pi^2\sigma^2\right)$
- $x_{\text{CE}} = U^{-1}\left(1+r+\pi(\mu-r) - \frac{\alpha}{2}\left(\left(1+r+\pi(\mu-r)\right)^2 + \pi^2\sigma^2\right)\right)$
- $\pi_A = \mu - U^{-1}\left(1+r+\pi(\mu-r) - \frac{\alpha}{2}\left(\left(1+r+\pi(\mu-r)\right)^2 + \pi^2\sigma^2\right)\right)$

To find the optimal value for z, given α, we need to maximize x_{CE}:

$$\frac{\partial x_{\text{CE}}}{\partial \pi} = \frac{(1-\alpha r)(\mu - r) - 2\pi\alpha\left((\mu-r)^2+\sigma^2\right)}{1-\alpha U^{-1}\left(1+r+\pi(\mu-r) - \frac{\alpha}{2}\left(\left(1+r+\pi(\mu-r)\right)^2 + \pi^2\sigma^2\right)\right)}$$

$$\therefore \left.\frac{\partial x_{\text{CE}}}{\partial \pi}\right|_{\pi=\pi^*} = 0 \iff (1-\alpha r)(\mu-r) - 2\pi^*\alpha\left((\mu-r)^2+\sigma^2\right) = 0$$

$$\therefore \pi^* = \frac{(1-\alpha r)(\mu-r)}{2\alpha\left((\mu-r)^2+\sigma^2\right)} \implies z^* = \$1,000,000 \times \frac{(1-\alpha r)(\mu-r)}{2\alpha\left((\mu-r)^2+\sigma^2\right)}$$

The first equality comes from the fact that:

$$\left(f^{-1}(x)\right)' = \frac{1}{f'\left(f^{-1}(x)\right)}$$

I now find the corresponding value of z^* for every value of α between 0 and 15 and plot the result:

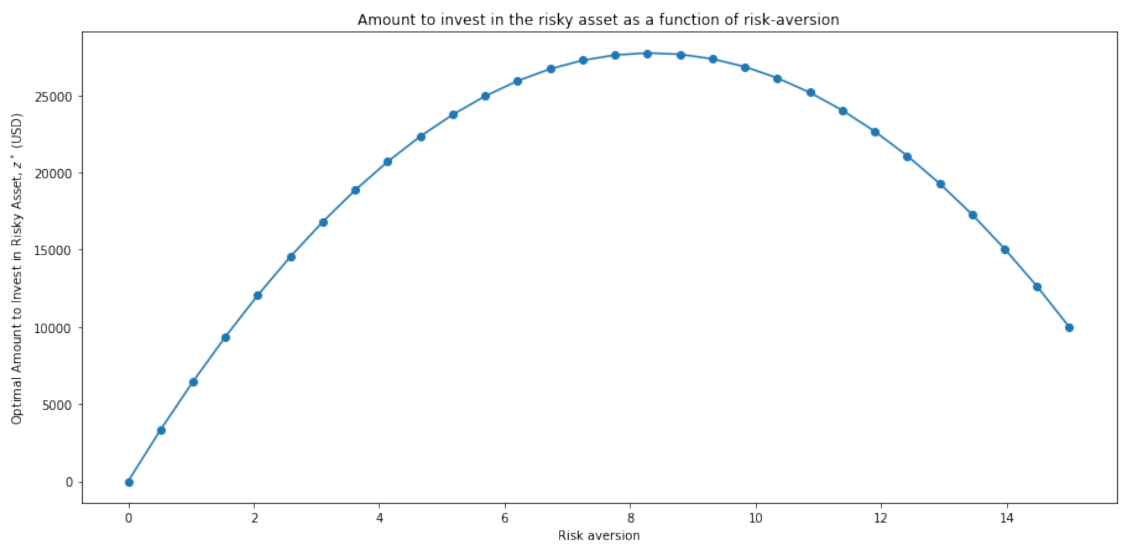
```
[ ]: mu = 0.14 # annual returns of 25%
     sigma2 = 0.4 ** 2 # risk
     rf = 0.06 # riskfree return of 7%
```

```
z_star = lambda mu, sigma2, rf, a : 1e6 * (1 - a * rf) * (mu - rf) / 2 * a *␣
↪((mu - rf) ** 2 + sigma2)

alpha = np.linspace(0.0, 15, 30)
z = z_star(mu, sigma2, rf, alpha)
```

[ ]:
```
plt.plot(alpha, z)
plt.scatter(alpha, z)

plt.title("Amount to invest in the risky asset as a function of risk-aversion")
plt.xlabel("Risk aversion")
plt.ylabel("Optimal Amount to Invest in Risky Asset, $z^*$ (USD)")
plt.show()
```



2 Question 3: Kelly's Criterion

After betting $f \cdot W_0$ amount in the game, we can get the returns of either α or $-\beta$. Therefore, the wealth you can have after the game is:

$$W_1 = \begin{cases} W_0\left(1 + f \cdot \alpha\right), & \text{with probability } p \\ W_0\left(1 - f \cdot \beta\right), & \text{with probability } 1 - p \end{cases}$$

Therefore, the utility obtained after playing the game becomes:

$$\log\left(W_1\right) = \log\left(W_0\right) + \begin{cases} \log\left(1 + f \cdot \alpha\right), & \text{with probability } p \\ \log\left(1 - f \cdot \beta\right), & \text{with probability } 1 - p \end{cases}$$

Accordingly, the expected utility after the game becomes:

$$\mathbb{E}\left(\log W_1\right) = \log W_0 + p \cdot \log\left(1 + f \cdot \alpha\right) + (1-p) \cdot \log\left(1 - f \cdot \beta\right)$$

To find the optimal fraction, f^*, we set the first derivative of the above function to 0 and solve for f:

$$\begin{aligned}
\frac{\partial \mathbb{E}\left(\log W_1\right)}{\partial f} &= \frac{p \cdot \alpha}{1 + \alpha \cdot f} - \frac{(1-p) \cdot \beta}{1 - \beta \cdot f} \\
\left.\frac{\partial \mathbb{E}\left(\log W_1\right)}{\partial f}\right|_{f=f^*} &= 0 \implies p\alpha\left(1 - \beta f^*\right) = \beta(1-p)\left(1 + \alpha f^*\right) \\
\therefore f^* &= \frac{p}{\beta} - \frac{1-p}{\alpha}
\end{aligned}$$

This makes sense intuitively, because if the probability of getting positive returns (α) is higher, you should invest more of your wealth and vice versa. On the other hand, if α becomes smaller, f^* also becomes smaller, meaning that you shouldn't invest more since the risk isn't worth the returns. Similar reasoning for β concludes this question.