# Tracking Interfacial Instability using Level Set Method

A Project Report Submitted

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*by*


**Aditya & Sunny Narzary**

(Roll No. 200123003 & 200123062)


*to the*


**DEPARTMENT OF MATHEMATICS**

**INDIAN INSTITUTE OF TECHNOLOGY GUWAHATI**

**GUWAHATI - 781039, INDIA**


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# CERTIFICATE

This is to certify that the work contained in this project report entitled **Tracking Interfacial Instability using Level Set Method** submitted by **Aditya** (Roll No.: 200123003) & **Sunny Narzary** (Roll No.: 200123062) to the Department of Mathematics, Indian Institute of Technology Guwahati towards partial requirement of Bachelor of Technology in Mathematics and Computing has been carried out by them under my supervision.

It is also certified that, along with literature survey, a few extra simulation studies have been carried out by the students under the project.

Turnitin Similarity: __ %

Guwahati - 781 039 | Dr. Satyajit Pramanik
August 2023 | Project Supervisor

# ABSTRACT

In the course of our research, we have devised an innovative strategy for augmenting fundamental properties, such as velocity, within the framework of level set methods. This enhancement is achieved through the solution of a specialized mathematical construct known as a biharmonic equation. Distinctively, our approach obviates the necessity for comprehensive interface intricacies, rendering its implementation straightforward. We provide the flexibility of employing either a direct solver or a conjugate gradient solver, all while minimizing the demand for extensive matrix operations. Additionally, we introduce a novel technique, termed the fast Poisson preconditioner, to expedite computational processes.

Our method has undergone rigorous testing across a spectrum of diverse problem domains, revealing its exceptional proficiency in generating seamlessly accurate extensions proximate to interfaces. Furthermore, it exhibits remarkable compatibility with systems characterized by symmetrical or periodic patterns, underscoring its adaptability and effectiveness in addressing a wide array of complex challenges.

# Contents

**List of Figures** **vi**

**List of Tables** **vii**

**1 Introduction** **1**

1.1 Background . . . . . . . . . . . . . . . . . . . . . . . . . . . 1

1.2 Objectives . . . . . . . . . . . . . . . . . . . . . . . . . . . 2

1.3 Scope of the Study . . . . . . . . . . . . . . . . . . . . . . . 2

1.4 Structure of the Thesis . . . . . . . . . . . . . . . . . . . . . 2

**2 Literature Review** **3**

2.1 Interface Instability . . . . . . . . . . . . . . . . . . . . . . . 3

2.2 Level Set Method . . . . . . . . . . . . . . . . . . . . . . . . 3

2.3 Previous Studies on Level Set Method . . . . . . . . . . . . . 4

**3 Methodology** **5**

3.1 Problem Statement . . . . . . . . . . . . . . . . . . . . . . . 5

3.2 Solving the Biharmonic Equation . . . . . . . . . . . . . . . . 6

3.3 Computation of Biharmonic Extension . . . . . . . . . . . . . 6

3.3.1 Example 1, two-dimensional peanut . . . . . . . . . . 7

3.3.2 Example 2, Circular Domain from a 3d perspective . 8

3.4 Summary . . . . . . . . . . 8

**4 Formulation of Interface** **10**

4.1 Initial Value Formulation . . . . . . . . . . 10

4.1.1 Finite Difference Approximation . . . . . . . . . . 11

4.1.2 Simulations . . . . . . . . . . 11

**Bibliography** **13**

# List of Figures

3.1 after 24 frames . . . . . . . . . . . . . . . . . . . . . . . . . . . . 7

3.2 after 374 frames . . . . . . . . . . . . . . . . . . . . . . . . . . . . 7

3.3 Initial . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 8

3.4 after some time . . . . . . . . . . . . . . . . . . . . . . . . . . . . 8

4.1 This simulation ran with a constant $F$ and was used as a test subject to see if the program was working as intended, we have taken $F = 1$, which means a constant force acting in the *outward* direction . . . . . . . . . . . . . . . . . . . . . . . . 12

4.2 This simulation started with 5 dots around the domain $(0,1)\times(0,1)$, and a variable $F = sin(x) \cdot cos(y)$ . . . . . . . . . . . . 12

4.3 This simulation started with 5 dots around the domain $(0,1)\times(0,1)$, and a variable $F = sin(x) \cdot cos(y) + 0.5$ . . . . . . . . . 12

# List of Tables

# Chapter 1

# Introduction

## 1.1 Background

The study of interface instability is a critical aspect across various scientific and engineering disciplines, particularly in understanding dynamic phenomena such as fluid flows, material interactions, and biological processes. Interfaces play a pivotal role in determining the behaviour of these systems, and their instability can lead to complex and intriguing patterns.

In this context, the Level Set Method emerges as a powerful tool for tracking and simulating interface dynamics. This method provides a versatile framework for representing and evolving interfaces in a computational environment. In this research, we aim to dive deeper into the intricacies of the Level Set Method and explore different coding methods to enhance our understanding of how it can be employed to track interface instability.

## 1.2 Objectives

The primary objective of this study is to comprehend the theoretical foundations of the Level Set Method, explore and implement various coding methods for Level Sets, and conduct simulations that elucidate the workings of Level Sets in tracking interface instability.

## 1.3 Scope of the Study

This research is limited to a comprehensive exploration of the Level Set Method and its application in tracking interface instability. The focus is on exploring and understanding the mathematical underpinnings of the Level Set Method, implementing different coding methods, and conducting simulations.

## 1.4 Structure of the Thesis

This thesis is organized into several chapters to provide a logical flow of information. Chapter 2 reviews relevant literature on interface instability and the Level Set Method. Chapter 3 details the methodology employed, including the coding methods and simulation setups. Chapter 4 presents the results of the simulations, and Chapter 5 provides a discussion and interpretation of these results. Finally, Chapter 6 concludes the thesis, summarizing key findings and suggesting avenues for future research.

# Chapter 2

# Literature Review

## 2.1 Interface Instability

Interface instability is a phenomenon that has been extensively studied in various scientific and engineering disciplines. The behaviour of interfaces, where different materials or phases meet, is crucial in understanding dynamic processes such as fluid flows, material interactions, and biological systems. The literature on interface instability provides a foundation for comprehending the complexities and implications of unstable interfaces.

## 2.2 Level Set Method

In the study of natural phenomena like fluid flows, solidification, or biological growth, it's essential to keep track of moving boundaries or interfaces. This means we need a way to simulate how these boundaries change over time. The level set method is a powerful tool that helps us do just that. Unlike other methods that explicitly define these boundaries, the level set method uses a clever trick: it represents them indirectly using a special function called the level set function. The theoretical foundations of the Level Set Method involve the use of a scalar function to implicitly define the interface, making it particularly well-suited for problems involving interface dynamics.

## 2.3 Previous Studies on Level Set Method

Several researchers have explored the use of the Level Set Method specifically in the context of tracking interface instability.

In a 2019 paper titled, **Numerical investigation of controlling interfacial instabilities in non-standard Hele-Shaw configurations** by Liam C. Morrow, Timothy J. Moroney, and Scott W. McCue, the authors conducted an experiment involving a Hele-Shaw cell, consisting of two parallel plates separated by a small gap—filled with a viscous fluid. By introducing air through a minute aperture at the centre of the plate, intriguing finger-like patterns, recognized as viscous fingering, appeared. Following this experimental phase, the researchers complemented their findings with computer simulations. Through these simulations, they not only successfully replicated the observed behaviour but when you kept the parameters like flow rate, gap, and angle between plates the same they also gained the capability to predict the resultant patterns,

This paper mentioned using a technique known as Biharmonic Extension used for calculating Level Sets, for which they referenced a 2017 paper titled, **Extending fields in a level set method by solving a biharmonic equation** by Timothy J. Moroney , Dylan R. Lusmore, Scott W. McCue, D.L. Sean McElwain, understanding and implementing this technique of calculating Level Sets using the Biharmonic Extension will be our goal for this semester.

# Chapter 3

# Methodology

## 3.1 Problem Statement

Our core challenge involves extending known values of a vector field, denoted as $F$, to encompass the entire domain meaningfully. To simplify, we narrow our focus to a scalar field, representing $F$ or a general scalar field. Known values of this scalar field are distributed across nodes within a finite difference grid in $\Omega$. Our goal is to construct a function $f : D \to \mathbb{R}$ that is smooth and satisfies the following conditions:

- The Biharminic Equation.

$$\nabla^4 f = 0 \quad \forall\ k \in D \setminus \{1, \ldots, N\} \tag{3.1}$$

- The value of the function on known grid points should remain unchanged.

$$f(x_k) = f_k \quad \forall\ k \in \{1, \ldots, N\} \tag{3.2}$$

- The Boundary condition should be true.

$$f = 0 \text{ and } \nabla^2 f = 0 \text{ on } \partial D \tag{3.3}$$

This requirement drives our mission to create a well-behaved function connecting known values $f_k$ at various grid nodes, ensuring continuity and precision in the extension process.

For the purpose of implementing the Biharmonic Equation in a Discrete setting, Numerically, we simply solve and get a Discrete Biharmonic Stencil.

$$\begin{aligned} 20f_{i,j} - 8f_{i-1,j} - 8f_{i,j+1} - 8f_{i,j-1} - 8f_{i+1,j} \\ +2f_{i+1,j+1} + 2f_{i-1,j-1} + 2f_{i+1,j-1} + 2f_{i-1,j+1} \\ +f_{i+2,j} + f_{i-2,j} + f_{i,j+2} + f_{i,j-2} = 0 \end{aligned} \tag{3.4}$$

## 3.2 Solving the Biharmonic Equation

We solve the discrete biharmonic equation at each node in $D \backslash \Omega$. Stencils near the domain boundary are modified as per boundary conditions (3). When a stencil crosses the interface, referenced values on the other side (the values $f_k$ for extension) are known. This construction results in a smooth, continuous extension in both value and derivative.

The key advantage is that it doesn't require knowing the interface location or field values on the interface, simplifying implementation.

## 3.3 Computation of Biharmonic Extension

The simulation done in this report is conducted in the JavaScript language, using the p5.js library, which has allowed us to easily run the code on any machine as long as there is a browser, and makes the code easy to share with the p5.js web editor, one can just click the link and see the simulation running.

### 3.3.1 Example 1, two-dimensional peanut

The graphic is of a two-dimensional peanut defined as,

$$f(x,y) = sin(x)cos(y) \tag{3.5}$$

in the region

$$\Omega = \{(x,y) \in D | min(\ (x-0.8)^2 + y^2, (x+0.8)^2 + y^2) < 1\ )\} \tag{3.6}$$

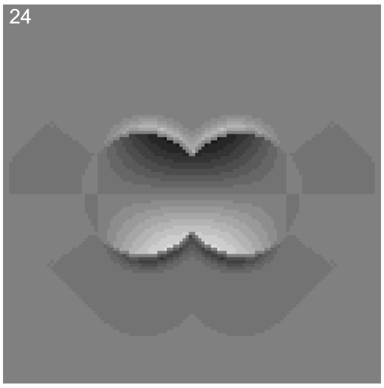


Figure 3.1: after 24 frames

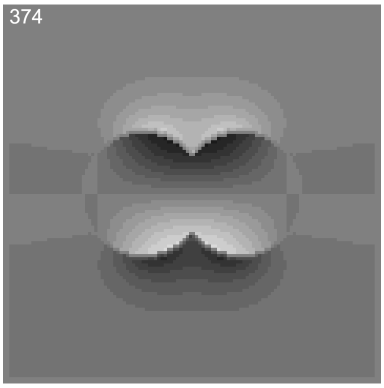


Figure 3.2: after 374 frames

The colour white represents a value of 1 and black for -1, rest of the values take values in colour in between linearly,

### 3.3.2 Example 2, Circular Domain from a 3d perspective

The graphic is of a two-dimensional peanut defined as,

$$f(x, y) = sin(x)cos(y) \tag{3.7}$$

in the region

$$\Omega = \{(x, y) \in D|\ x^2 + y^2 < 2\ )\} \tag{3.8}$$

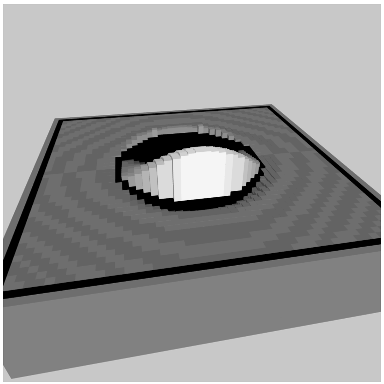

Figure 3.3: Initial

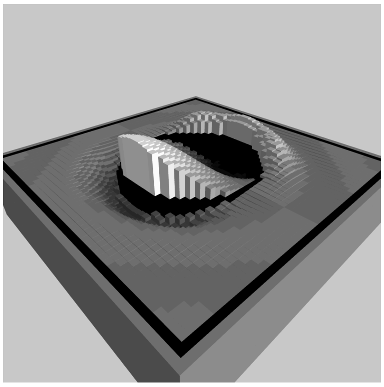

Figure 3.4: after some time

The colour white represents a value of 1 and black for -1, rest of the values take values in colour in between linearly, the block in solid black are the boundary points

## 3.4 Summary

This chapter has outlined the methodology employed in the study, encompassing the theoretical foundations of the Level Set Method, coding implementations, simulation setups, and validation procedures. The subsequent chapter will present the results obtained through the simulations and provide an in-depth analysis of the interface instability phenomena observed.

# Chapter 4

# Formulation of Interface

In this section, we created a simple model for the formulation of interfaces, this is not the complete Tracking of Interface Instability, but a program that can simulate the evolution of an interface given an interface speed $F$.

## 4.1 Initial Value Formulation

In order to derive an equation of motion for the level set function $\phi$, we first require that the level set value of a particle on the front with path $x(t)$ must always be zero, and hence

$$\phi(x(t), t) = 0. \tag{4.1}$$

By the chain rule,

$$\phi_t + \nabla\phi(x(t), t) \cdot x'(t) = 0 \tag{4.2}$$

where $F$ supplies speed in the outward normal direction

$$x'(t) \cdot n = F \quad \text{where, } n = \nabla\phi / |\nabla\phi| \tag{4.3}$$

finally, we arrive at

$$\phi_t + F|\nabla\phi| = 0 \qquad \text{given} \quad \phi(x, t = 0) \tag{4.4}$$

### 4.1.1 Finite Difference Approximation

$$\phi_t + F|\nabla\phi| = 0 \tag{4.5}$$

$$\frac{\phi_{ij}^{n+1} - \phi_{ij}^{n}}{\Delta t} + F|\nabla_{ij}\phi_{ij}^{n}| = 0 \tag{4.6}$$

$$\frac{\phi_{ij}^{n+1} - \phi_{ij}^{n}}{\Delta t} + F\left\{\left(\frac{\phi_{i+1,j}^{n} - \phi_{i-1,j}^{n}}{2h}\right)^2 + \left(\frac{\phi_{i,j+1}^{n} - \phi_{i,j-1}^{n}}{2k}\right)^2\right\} = 0 \tag{4.7}$$

### 4.1.2 Simulations

Figure 4.1: This simulation ran with a constant $F$ and was used as a test subject to see if the program was working as intended, we have taken $F = 1$, which means a constant force acting in the *outward* direction

Figure 4.2: This simulation started with 5 dots around the domain $(0, 1) \times (0, 1)$, and a variable $F = sin(x) \cdot cos(y)$

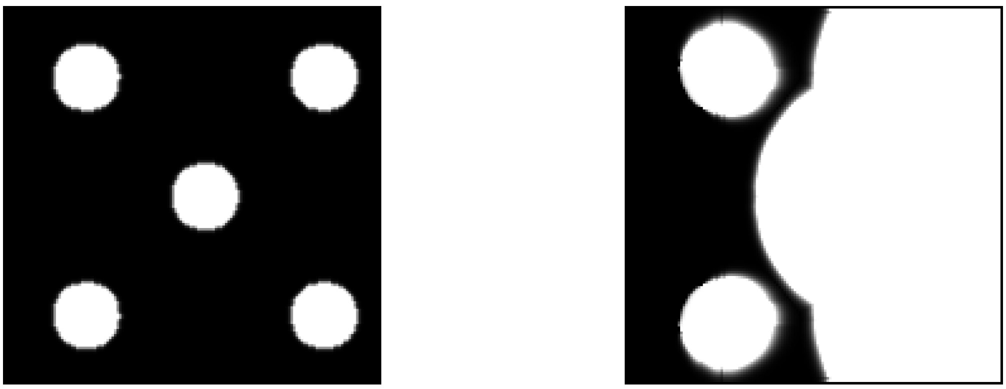

Figure 4.3: This simulation started with 5 dots around the domain $(0, 1) \times (0, 1)$, and a variable $F = sin(x) \cdot cos(y) + 0.5$

# Bibliography


[1] Timothy J. Moroney, Dylan R. Lusmore, Scott W. McCue, and D.L. Sean McElwain. Extending fields in a level set method by solving a biharmonic equation. *Journal of Computational Physics*, 343(5):170–185, 2017.

[2] Liam C. Morrow, Timothy J. Moroney, and Scott W. McCue. Numerical investigation of controlling interfacial instabilities in non-standard hele-shaw configurations.

[3] J.A. Sethian. *Level Set Methods and Fast Marching Methods: Evolving interfaces in computational geometry, fluid mechanics, computer vision and materials science*. Second Edition. Cambridge University Press, 1999.